A coprime action version of a solubility criterion of Deskins

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Abstract

Let A and G be finite groups of relatively prime orders and suppose that A acts on G via automorphisms. We demonstrate that if G has a maximal A-invariant subgroup M that is nilpotent and the Sylow 2-subgroup of M has class at most 2, then G is soluble. This result extends, in the context of coprime action, a solubility criterion given by W.E. Deskins.

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1 Introduction

In [2], in the course of a study of the lattice of subinvariant subgroups in a finite group, W.E. Deskins provided an interesting solubility criterion concerning maximal subgroups: When a finite group G contains a maximal subgroup M that is nilpotent of class less than 3, then G is soluble. This result is similar to a theorem of B. Huppert, which originally appeared in [6], except in the case in which M has a Sylow 2-subgroup of class 2. The criterion of Deskins was also in line with a theorem announced by Thompson: If a finite group G has a maximal subgroup that is nilpotent of odd order, then G is soluble. The crucial tool of Deskins's work, which allowed him to extend the nilpotence class to 2 instead of 1 (abelian), was the First Theorem of Grün (see for instance [5, IV.3.4]), which is an application of the transfer theory into a Sylow subgroup. Precisely, Grün's Theorem was used so as to obtain the existence of a normal complement to the maximal subgroup M.

In this paper we study such results in the context in which a finite group A with $(|A|, |G|) = 1$ acts on G. We ask whether the existence of a maximal A-invariant subgroup in G (which needs not be a maximal subgroup) satisfying the same conditions as in Deskins's theorem must imply the solubility of G. We give an affirmative answer.

Theorem. Let G and A be finite groups of coprime orders and assume that A acts on G by automorphisms. If G has a maximal A-invariant subgroup that is nilpotent with a Sylow 2-subgroup of class less than 3, then G is soluble.

At first sight, there seems only to be a subtle difference from Deskins's theorem, but there exists a great distinction between our development and Deskins's approach. It is not possible to use Grün's Theorem in the setting of a coprime action, and instead, we appeal to the Classification of the Finite Simple Groups. We point out that the authors have already obtained a coprime action version of the Thompson's aforementioned result [1, Theorem B]. This is not done by employing the Classification, but by transferring into the setting of coprime action results like the Glauberman-Thompson criterion for p-nilpotence. In fact, this result will be used in the proof of our theorem.

We denote by $\pi(G)$ the set of primes dividing the order of a group G. The rest of the notation is standard and all groups are supposed to be finite.

2 Preliminaries

We start with an elementary observation that is needed for the inductive arguments.

Lemma 2.1. *Let P be a finite p-group of class 2. If $A \trianglelefteq P$, then the class of A and P/A is less than or equal to* 2.

We require the following theorem of Wielandt.

Theorem 2.2 (IV.7.3, [5]). *Let H be a Hall π-subgroup of a group G which is not a Sylow subgroup of G. Suppose that for every $p \in \pi$ and for every Sylow p-subgroup H_p of H, we have $\mathbf{N}_G(H_p) = H$. Then H has a normal π-complement in G.*

We also recall the Thompson subgroup. If p is prime and P is a p-group, the Thompson subgroup $\mathbf{J}(P)$ is the subgroup generated by all abelian subgroups of P of maximal order. It is immediate that $\mathbf{J}(P)$ and $\mathbf{Z}(\mathbf{J}(P))$ are characteristic in P, and hence, these subgroups are left invariant by every automorphism acting on P, so in particular, by every group acting coprimely on P. As we said in the Introduction, in order to prove our result we need to use the celebrated Glauberman-Thompson p-nilpotence criterion.

Theorem 2.3 (Theorem 8.3.1, [4]). *Let P be a Sylow p-subgroup of a finite group G, where p is an odd prime. If $\mathbf{N}_G(\mathbf{Z}(\mathbf{J}(P)))$ is p-nilpotent, then G is p-nilpotent.*

As mentioned in the Introduction, we appeal to the Classification of the Finite Simple Groups. Precisely, we need to determine all non-abelian simple finite groups whose Sylow 2-subgroups are self-normalising as well as all those simple groups whose Sylow 2-subgroups have nilpotence class at most 2. Such groups have been classified by Kondrat'ev [7] and by Gilman and Gorenstein [3], respectively, so we can gather the list of those simple groups satisfying both conditions in the next result.

Theorem 2.4. *Let G be a finite non-abelian simple group and P a Sylow 2-subgroup of G. If $\mathbf{N}_G(P) = P$ and P has class at most 2, then $G \cong \mathrm{PSL}(2, q)$, where $q \equiv 7, 9 \pmod{16}$.*

Proof. This is a consequence of combining the main result of [7] and Theorems 7.1 and 7.4 of [4]. □

We will also need to know the structure of the Sylow normalisers in $\mathrm{PSL}(2, q)$, especially for odd primes.

Lemma 2.5. *Let $G = \mathrm{PSL}(2, q)$, where q is a power of prime p and $d = (2, q + 1)$. Let $r \in \pi(G)$ and $R \in \mathrm{Syl}_r(G)$.*

(1) *If $r = p$, then $\mathbf{N}_G(R) = R \rtimes C_{\frac{q-1}{d}}$ is a dihedral group;*

(2) *If $2 \neq r \mid \frac{q+1}{d}$, then $\mathbf{N}_G(R) = C_{\frac{q+1}{d}} \rtimes C_2$;*

(3) *If $2 \neq r \mid \frac{q-1}{d}$, then $\mathbf{N}_G(R) = C_{\frac{q-1}{d}} \rtimes C_2$;*

(4) *Assume $p \neq r = 2$.*

(4.1) *If $q \equiv \pm 1 (mod\ 8)$, then $\mathbf{N}_G(R) = R$;*

(4.2) *If $q \equiv \pm 3 (mod\ 8)$, then $\mathbf{N}_G(R) = (C_2 \times C_2) \rtimes C_3$.*

Proof. This follows from [5, Theorem 2.8.27]. □

3 Proof of the Theorem

Proof. We study a minimal counter-example. Suppose then that G is a minimal counter-example to the theorem and let M be the nilpotent maximal A-invariant subgroup of G with a Sylow 2-subgroup of class less than 3. We divide the proof into the following steps.

Step 1. We can assume that M is a Hall subgroup of G and that M does not contain any A-invariant normal subgroup of G.

If M contains a non-trivial A-invariant normal subgroup N of G, then by taking into account Lemma 2.1, G/N satisfies the hypotheses of the theorem, so G/N is soluble by minimality, and consequently, G is soluble for N being nilpotent. Henceforth, it can be assumed M does not contain any A-invariant normal subgroup of G.

Suppose that there exists a prime $p \in \pi(M)$ such that the Sylow p-subgroup of M is not a Sylow p-subgroup of G. Then by elementary coprime action properties there exists an A-invariant Sylow p-subgroup G_p of G and an A-invariant Sylow p-subgroup M_p of M with $M_p < G_p$. Since M is nilpotent, we have $M < \mathbf{N}_G(M_p)$. Also, $\mathbf{N}_G(M_p)$ is A-invariant. By the maximality of M we get $\mathbf{N}_G(M_p) = G$, that is $M_p \trianglelefteq G$, a contradiction with the above paragraph. This shows that $M_p = G_p$, or equivalently, M is a Hall subgroup of G.

Step 2. We can assume that M is a Sylow 2-subgroup of G.

Suppose that M is not a Sylow subgroup of G. For every prime $p \in \pi(M)$ we take P an A-invariant Sylow p-subgroup of M. Then $M \leq \mathbf{N}_G(P)$ and by maximality of M and Step 1, it follows that $\mathbf{N}_G(P) = M$. Thus, we can apply Theorem 2.2, so there exists a normal complement K of M in G. Clearly, K is A-invariant. Now let us consider the action of MA on K. Since the orders of MA and K are coprime, we get that K has a MA-invariant Sylow q-subgroup Q. Therefore $MQ \leq G$ is A-invariant, and by the maximality of M, we have $G = MQ$. However, Q and G/Q are soluble, so we deduce that G is soluble as well, a contradiction. This shows that M is a Sylow p-subgroup of G for some prime p.

Next we prove that $p = 2$. Assume that $p \neq 2$. Let $J = \mathbf{J}(M)$, the Thompson's subgroup of M, and $Z = \mathbf{Z}(J)$. Note that Z and $\mathbf{N}_G(Z)$ are A-invariant by the observation made before Theorem 2.3. Since by Step 1, Z is not normal in G, we have $M \leq \mathbf{N}_G(Z) < G$. By the maximality of M, we get $M = \mathbf{N}_G(Z)$, so in particular it is a p-subgroup. Then G is p-nilpotent by Theorem 2.3, that is, G has a normal p-complement, say L, which is obviously A-invariant too. This means that $G = ML$ with $M \cap L = 1$. The rest of the proof of this step consists in proving that L is a q-group for some prime q. Indeed, take Q an A-invariant Sylow q-subgroup of L for some prime q. The Frattini argument gives $G = \mathbf{N}_G(Q)L$. Now, the Schur-Zassenhaus Theorem assures that $\mathbf{N}_L(Q)$ has complements in $\mathbf{N}_G(Q)$ that are conjugate in $\mathbf{N}_G(Q)$. Since A acts on the set of complements, Glauberman's Lemma (for instance [8, Theorem 6.2.2]) implies that there exists an A-invariant complement X of $\mathbf{N}_L(Q)$ in $\mathbf{N}_G(Q)$. As a result, $G = X\mathbf{N}_L(Q)L = XL$, so X is an A-invariant complement of L in G. Again by Glauberman's Lemma, we know that the A-invariant complements of L are conjugate in the fixed point subgroup $\mathbf{C}_G(A)$, so in particular, $X = M^c$ for some $c \in \mathbf{C}_G(A)$. We conclude that X is a maximal A-invariant subgroup of G. However, X normalizes Q and by maximality of X, we get $G = XQ$. This forces $L = Q$, as wanted. As a consequence, G is soluble by Burnside p^aq^b Theorem, a contradiction. Hence $p = 2$ and M is a Sylow 2-subgroup of G.

Step 3. We can assume that M has nilpotence class 2.

Suppose on the contrary, that the class of M is not 2, so by hypothesis M is abelian. As $\mathbf{N}_G(M) = M$ by the maximality of M, we have $M \leq \mathbf{Z}(\mathbf{N}_G(M))$. We can apply then Burnside normal p-complement Theorem for $p = 2$ (for instance [5, 2.2.6]), and we conclude that G has a normal 2-complement. Now Feit-Thompson Theorem implies that G is soluble, a contradiction.

Step 4. Final contradiction.

Let N be a minimal A-invariant normal subgroup of G. We can assume that N is not soluble; otherwise by Step 1, N is not contained in M, and by maximality we obtain $NM = G$. As a consequence, G would be soluble and the proof is finished. Therefore, we can write $N = S_1 \times \ldots \times S_n$ where S_i are isomorphic non-abelian simple groups (possibly $n = 1$). Put $S = S_1$, $B = \mathbf{N}_A(S)$ and let T be a transversal of B in A. Now, as M is self-normalising in G for being maximal, then $M \cap S$ is self-normalising in S and it has class at most 2 by Lemma 2.1. Then by applying Theorem 2.4, we obtain $S \cong \mathrm{PSL}(2, q)$ with $q \equiv 7, 9 \pmod{16}$. We distinguish separately these two cases. If $q \equiv 9 \pmod{16}$, with $q > 9$, then we can certainly choose an odd prime $r \mid (q-1)/2$ and R to be a B-invariant Sylow r-subgroup of S. By Lemma 2.5(3), we know that $|\mathbf{N}_S(R)| = q + 1$, so $\mathbf{N}_S(R)$ has odd index in S and contains properly a Sylow 2-subgroup of S. Analogously, if $q \equiv 7 \pmod{16}$, with $q > 7$, there exists an odd prime $r \mid (q + 1)/2$ and we take R to be a B-invariant Sylow r-subgroup of S. Again by Lemma 2.5(2), we know that $|\mathbf{N}_S(R)| = (q-1)$, so $\mathbf{N}_S(R)$ has odd index in S and hence, it contains properly a Sylow 2-subgroup of S. In both cases, we put $R_0 = \prod_{t\in T} R^t$, which is an A-invariant Sylow r-subgroup of N because A acts transitively on the S_i. We deduce that $|N : \mathbf{N}_N(R_0)| = |S : \mathbf{N}_S(R)|^n$ is odd too. Now, by the Frattini argument, $G = N\mathbf{N}_G(R_0)$ and thus, $|G : \mathbf{N}_G(R_0)| = |N : \mathbf{N}_N(R_0)|$. We conclude that $\mathbf{N}_G(R_0)$ properly contains an A-invariant Sylow 2-subgroup of G, contradicting the maximality of M.

Finally, suppose that $S \cong \mathrm{PSL}(2, 9)$ or $\mathrm{PSL}(2, 7)$. Both groups contain $\{2, 3\}$-Hall subgroups, which are isomorphic to the symmetric group S_4. We remark that these subgroups are not all conjugate in S. If this were the case, then Glauberman's Lemma would provide an A-invariant Hall $\{2, 3\}$-subgroup, against the maximality of M. But this is not the case and we give the following alternative argument. The Sylow 2-subgroups of S are dihedral groups of order 8. Now, $M \cap N$ is an A-invariant Sylow 2-subgroup of N, which is the direct product of n copies of such a B-invariant dihedral group, say D, of S. Let K be the cyclic group of order 4 of D, which is also B-invariant for being characteristic, and let $K_0 = \prod_{t\in T} K^t$. It is easily seen that K_0 is A-invariant because A is acting transitively on the factors. Moreover, since K is characteristic in D, then K_0 is characteristic in $M \cap N$, so $K_0 \trianglelefteq M$, that is, $M \leq \mathbf{N}_G(K_0)$. On the other hand, in both cases $S \cong \mathrm{PSL}(2, 9)$ or $\mathrm{PSL}(2, 7)$, we have that K is normalised by an element of order 3 lying in S, so the same occurs with K_0 and N. We conclude that $\mathbf{N}_G(K_0)$ is an A-invariant subgroup that contains properly M. Again this contradicts the maximality of M. □

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