Modelling, mathematical analysis and numerical simulation of problems related to counterparty risk and CVA

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Ph.D. Thesis

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Los abajo firmantes hacen constar que son directores de la Tesis Doctoral titulada **"Modelling, mathematical analysis and numerical simulation of problems related to counterparty risk and CVA"** desarrollada por Beatriz Salvador Mancho, cuya firma también se incluye, dentro del programa de doctorado **"Métodos Matemáticos y Simulación Numérica en Ingeniería y Ciencias Aplicadas"** en el Departamento de Matemáticas (Universidade da Coruña), dando su consentimiento para su presentación y posterior defensa.

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Abstract

This thesis deals with the modelling, mathematical analysis and numerical solution of partial differential equation (PDE) problems for pricing European and American options when considering counterparty risk. Several valuation adjustments are considered, the most important one being the credit value adjustment (CVA).

In the modelling, the intensity of default from each risky counterparty plays a relevant role. In the present work we analyze two situations. In the first one constant intensities of default are assumed, leading to PDE models with one spatial dimension. In the second setting stochastic intensities are assumed, although only one counterparty can default so that PDE models with two spatial variables are deduced. Thus, Cauchy–boundary value PDE problems are posed for European options, while complementarity problems govern the pricing of American options.

The two more usual choices for the mark–to–market value, risk–free and risky derivative values, lead to linear and nonlinear PDE problems, respectively. The mathematical analysis of the nonlinear models is one of the main achievements of this work, thus obtaining the existence and uniqueness of solution for the different problems.

For the numerical solution, a method of characteristics jointly with a fixed point iteration and finite elements are used. In the case of American options, an augmented Lagrangian active set method is additionally applied. Also, the equivalent formulations in terms of expectations have been posed and numerically solved by means of appropiate Monte Carlo techniques. Finally, we show illustrative results of the performance of the models and numerical methods that have been implemented.

Resumen

Esta tesis se centra en el modelado, análisis matemático y resolución numérica de problemas de ecuaciones en derivadas parciales para opciones europeas y americanas con riesgo de contrapartida. Se consideran diferentes valoraciones de ajustes, el más importante de los cuales es el riesgo de contrapartida (CVA).

En el modelado, la intensidad de quiebra de cada contraparte juega un papel importante. En el presente trabajo consideramos dos situaciones. En la primera se asumen intensidades de quiebra constantes, lo cual da lugar a modelos unidimensionales. En el segundo escenario se consideran intensidades de quiebra estocásticas, pero solo una contraparte puede quebrar, obteniéndose un modelo de EDPs bidimensional. Se obtiene así un problema de valor inicial y de contorno regido por EDPs para las opciones europeas, mientras que la valoración de opciones americanas está gobernada por problemas de complementariedad.

Las dos opciones más habituales del valor de mercado en el instante de quiebra (valores sin riesgo y con riesgo) conducen a EDPs lineales y no lineales, respectivamente. El análisis matemático de los modelos no lineales es uno de los principales logros de este trabajo, obteniéndose la existencia y unicidad de solución.

Para la solución numérica, se combinan métodos de características, punto fijo y elementos finitos. En el caso de las opciones americanas, el problema discretizado es resuelto mediante un método de lagrangiano aumentado. Se han planteado también formulaciones equivalentes en términos de esperanzas, que han sido resueltas mediante técnicas adecuadas de Monte Carlo. Finalmente se muestran resultados del comportamiento de los modelos y de los métodos numéricos implementados.

Resumo

Esta tese céntrase no modelado, análise matemática e solución numérica de problemas de ecuacións en derivadas parciais para opcións europeas e americanas con risco de contrapartida. Considéranse diferentes valoracións de axustes, o máis importante dos cales é o risco de contrapartida (CVA).

No modelado, a intensidade de quebra de cada contraparte xoga un papel importante. No presente traballo consideramos dúas situacións. Na primeira asúmense intensidades de quebra constantes, o cal dá lugar a modelos unidimensionais. No segundo escenario considéranse intensidades de quebra estocásticas, pero só unha contraparte pode quebrar, obténdose un modelo de EDPs bidimensional. Obtense así un problema de valor inicial e de contorno rexido por EDPs para as opcións europeas, mentres que a valoración de opcións americanas está gobernada por problemas de complementariedade.

As dúas opcións máis habituais do valor de mercado no instante de quebra (valores sen risco e con risco) conducen a EDPs lineais e non lineais, respectivamente. A análise matemática dos modelos non lineais é un dos principais logros deste traballo, obténdose a existencia e unicidade de solución.

Para a solución numérica, combínanse métodos de características, punto fixo e elementos finitos. No caso das opcións americanas, o problema discretizado é resolto mediante un método de lagrangiano aumentado. Propónse tamén formulacións equivalentes en termos de esperanzas, que son resoltas mediante técnicas adecuadas de Monte Carlo. Finalmente móstranse resultados do comportamento dos modelos e dos métodos numéricos implementados.

Abbreviations and notations

XVA	Total Value Adjustment
CVA	Credit Value Adjustment
FVA	Funding Value Adjustment
DVA	Debit Value Adjustment
CollVA	Collateral Value Adjustment
FCA	Funding Cost Adjustment
FBA	Funding Benefit Adjustment
CDS	Credit Default Swap
EONIA	Euro Over Night Index Average
EURIBOR	Euro Interbank Offered Rate
PDE	Partial Differential Equation
SDE	Stochastic Differential Equation

$\widehat{V}$	Derivative value considering counterparty risk
V	Risk–free derivative value
U	Total value adjustment
M	Mark–to–market close out
$G(S), H(S)$	Payoff functions
K	Strike
S_t	Underlying asset
h_t	Spread of the investor
W_t, W_t^S	Brownian motions in the dynamics of S_t

W_t^h	Brownian motion in the dynamics of h_t
$\sigma(t), \sigma^S(t)$	Volatilities in the dynamics of S_t
$\sigma^h(t)$	Volatility in the dynamics of h_t
ρ	Correlation between S and h
$q(t)$	Asset dividend yield
$M^h(t)$	Market price of investor
J_t^B	Default state at time t from counterparty B
J_t^C	Default state at time t from counterparty C
J_t^I	Default state at time t from the investor I
P_B	Counterparty B zero recovery bond price
P_C	Counterparty C zero recovery bond price
P_{1_t}	Issued junior bond
P_{2_t}	Issued senior bond
P_D	Total B bond position
X_t	Value of the collateral at time of default
r_{P_B}	Yield associated to P_B
r_{P_C}	Yield associated to P_C
r_R	Repo rate, the rate paid for the underlying asset in a repurchase agreement
r	Risk–free interest rate
r_F	Funding rate from the issuer
r_1	Yield for P_{1_t}
r_2	Yield for P_{2_t}
r_X	Rate paid on collateral
s_X	Spread between the rate paid on unsecured funding and the risk–free rate
$c(t)$	Accrual rate
R_B	Recovery rate from B
R_C	Recovery rate from C
R_1	Recovery rate of P_{1_t}
R_2	Recovery rate of P_{2_t}

Π_t	Portfolio at time t
$\gamma(t)$	Units of cash in the portfolio
γ_P	Cost of the portfolio
γ_{P_B}	Cost of the B's bond
γ_{P_C}	Cost of the C's bond
γ_F	Funding account
γ_R	Repo account
B_F	Bonds in which the cash amount is invested or borrowed
λ_B	Intensity of default from B
λ_C	Intensity of default from C
$\Delta\widehat{V}_B$	Variation of the derivative value in case of B's default
$\Delta\widehat{V}_C$	Variation of the derivative value in case of C's default
$\Delta(t)$	Units of the underlying asset in the portfolio
$\alpha_B(t)$	Units of P_{B_t} in the portfolio
$\alpha_C(t)$	Units of P_{C_t} in the portfolio
$\alpha(t)$	Units of net present value in the portfolio
$\beta(t)$	Units of cash in collateral accounts in the portfolio
$\gamma(t)$	Units of a long term Credit Default Swap
$\epsilon(t)$	Units of a short term Credit Default Swap
$\Omega(t)$	Units of bonds in the portfolio
$g_B(M_t, X_t)$	Value of the derivative including collateral when B makes default
$g_C(M_t, X_t)$	Value of the derivative including collateral when C makes default
h_e	Hedge error
t	Backward time
τ	Forward time
T	Maturity time
$D_{i,i-1}$	Discounting factor

N_T	Number of time steps
N_S	Number of nodes of the asset S discretization
N_h	Number of nodes of the spread h discretization
N_P	Number of paths in Monte Carlo simulations
N_I	Number of secondary paths in Monte Carlo simulations
$\cdot^+$	$Z^+ = \max(Z, 0)$
$\cdot^-$	$Z^- = \min(Z, 0)$

Introduction

Since 2007 crisis, when important financial entities went bankrupt, the counterparty risk has become an important ingredient that needs to be taken into account in all financial contracts. It can be described as the risk to each party of a contract that the counterparty will not live up to its contractual obligations. Different institutions and financial analysts consider that the crisis was due to mistakes made in the financial system, namely in the management of the risk. The complexity of the financial derivatives and the consideration of a low probability of default were two of the factors that led to the crisis. As a consequence, a review of the counterparty risk consideration has been addressed.

The two parts of a financial contract are usually named as the investor (or the buyer) and the hedger (or the seller). Nevertheless, both counterparties will buy or sell different assets, playing the role of buyer or seller at each situation. From the point of view of the seller, the risk neutral value of a derivative can be adjusted by the following items:

- It is reduced by the existence of funding costs, in the case the latter takes part (Funding Cost Adjustment, or FCA).
- It is increased in the case its value produces liquidity for the entity (Funding Benefit Adjustment, or FBA).
- It is reduced by the necessary costs to compensate the credit risk due to the counterparty (Credit Value Adjustment, or CVA).

- If a bilateral counterparty risk is assumed, the derivative value is increased by its potential benefits due to the issuer probability of default and the issuer has not to face its contractual responsibilities, when those are positive for the issuer (Debit Value Adjustment, or DVA).

- It is increased by the cost of borrowing the collateral (Collateral Value Adjustment, or CollVA).

The FCA and the FBA can be merged and the sum of them is known as FVA (Funding Value Adjustment), which is understood as the correction to the risk–free price to account for the funding costs. The presence of FVA in the adjustment is reasonable in the case of non–collateralized trades; however, when a collateral is posted to fully cover the counterparty risk then the FVA reduces to zero. In this sense, FVA is given by the difference of price between non–collateralized and fully collateralized contracts (see [45]). CVA represents the price to mitigate counterparty credit risk on a trade and the concept was first introduced in [47, 33, 24]. However, as no parts in the contract are risk–free, then DVA is the price of the hedging used to mitigate the own credit risk and from the other counterparty is understood as a CVA. DVA was first introduced in [24] to account for the presence of two risky counterparties and the consideration of DVA allows to agree on the price by both traders (symmetric prices). However, a long controversy exists about the consideration of DVA, and the same happens with FVA (see [31, 32, 37, 18] for different views on FVA).

Thus, including counterparty risk in the pricing of derivatives represents an important change in the existent risk–free pricing models. In particular, in this setting nonlinear partial differential equations (PDE) models can be deduced, and have to be mathematically analyzed and solved by means of suitable numerical methods. The main goal of the present thesis concerns the computing of European and American options prices, accounting for all the associated cash flows that come from the derivative itself, the act of hedging, the default risk management and the funding costs. Following the usual terminology, we will refer to the total value of these adjustments as XVA, which in terms of the previously introduced notations is defined by:

$$\mathrm{XVA} = \mathrm{DVA} - \mathrm{CVA} + (\mathrm{FBA} - \mathrm{FCA}) + \mathrm{CollVA} = \mathrm{DVA} - \mathrm{CVA} + \mathrm{FVA} + \mathrm{CollVA}\,.$$

Thus, we pose PDE models for the derivative value, $\widehat{V}$, from the point of view of the seller, when the trade takes place between two risky counterparties. More precisely, we focus on the case of European and American vanilla options. We use hedging arguments to derive the extensions to the Black–Scholes PDE in the presence of bilateral jump–to–default model and include funding considerations into the financing of the hedged positions.

Firstly, we consider a framework with constant intensity of default for the counterparties, then a model depending on one stochastic factor, the asset price is obtained. Nevertheless, the behaviour of the probability of default, from each counterparty that takes part in a contract, is not always constant. Thus, in a second part we model the XVA associated with a contract where the intensity of default from the counterparties is stochastic. As a result a model depending on two stochastic factors, the asset price and the spread, is posed.

Actually, nowadays there are three main methodologies to include funding costs, collateral and credit risk in the pricing of derivatives. A first approach follows the seminal papers by [45] and [15], that obtain PDE formulations by means of suitable hedging arguments and the use of Itô's lemma for jump–diffusion processes. In [45] funding costs are introduced while in [15] both funding costs and bilateral counterparty credit risk are considered. This approach is also followed in [27] in the more general setting of stochastic spreads, in which three underlying stochastic factors are involved. Moreover, in [27] the solution is also equivalently written in terms of expectations. A second approach follows the initial ideas in [12] to include DVA by means of expectations, and extend it to the collateralized, close–out and funding costs in [42, 13]. A third approach is based on backward stochastic differential equations introduced in [21] and [22]. In all previous papers, only the case of European derivatives is addressed. More recently, Borovykh *et al.* pose the problem in terms of a forward backward stochastic differential equation and solve a problem on Bermudan options [9].

In this thesis, we follow the methodology introduced by [15] and [27], where the XVA is given in terms of partial differential equations. Moreover, we also extend the previous results to American options. It is well known that European and American options are among the most popular derivative products on assets. In both contracts, the holder has the right (but not the obligation) to buy or sell an asset at a price that has been agreed with the counterparty. While European options can only be exercised by the holder at the end of the maturity period, the holder of an American option can exercise it at any moment along this period.

Taking into account such dissimilarity —according to the modelled, European or American, option— different problems are obtained. The total value adjustment associated to a European option contract is modelled by initial–boundary value problems associated to partial differential equations. However, for an American option the related XVA is obtained solving complementarity problems. Both of them are posed in terms of the mark–to–market price. Throughout this thesis, two possible values for such mark–to–market are considered, the risk–free derivative value or the derivative value including counterparty risk. The first choice leads to linear partial differential equations for European options, or linear complementarity problems in the case of American options. For the second one, nonlinear PDE problems are posed.

As we have mentioned, the most common methodology to compute the XVA is posed in terms of expectations. In order to write the XVA following such methodology, Feynman–Kac theorem is applied on the partial differential equations and complementarity problems previously obtained. As a result, we can also write the XVA in terms of expectations. For European options, classical Monte Carlo techniques will be applied, jointly with a fixed point scheme for nonlinear problems. For American options, the methodology introduced by Longstaff and Schwartz [38] and Glasserman [28] to obtain the risk–free derivative value will be extended to include the counterparty risk in the derivatives pricing.

In order to obtain a numerical solution of the different problems, some numerical methods previously introduced in [3] are applied. As we could expect, the numerical results obtained solving the partial differential equations (or the analogous complementarity problems) and those deduced from the Monte Carlo techniques show a similar behaviour.

The outline of this thesis is as follows.

In Chapter 1 the mathematical model to price the total value adjustment for European options is posed as a Cauchy problem. Constant intensity of default from each part of the contract is considered, then a model depending on one stochastic factor is deduced. Using a hedging strategy and applying Itô's lemma, the PDE models are derived. Next, the mathematical analysis to obtain the existence and uniqueness of a solution for the model is described. Moreover, some numerical methods are proposed to solve the problem. Finally, some examples showing the obtained results by solving the PDE problems and by implementing the alternative Monte Carlo simulation techniques are presented.

In Chapter 2, we study the total value adjustment for American options. Then, linear and nonlinear complementarity problems are posed. As in the previous chapter, a model depending on the asset price is deduced. Moreover, the augmented Lagrangian active set algorithm is introduced to solve the discretized obstacle problem. Additionally, the Longstaff–Schwartz methodology is extended in order to price American options considering counterparty risk. Finally, the results obtained by solving the complementarity problem, or by implementing the adapted Longstaff–Schwartz technique are presented.

In Chapter 3, the total value adjustment for European options is also modelled. Nevertheless, the main difference with Chapter 1 comes from the behaviour of the intensity of default. In this case, a stochastic behaviour is considered. As a result, a model depending on two stochastic factors, the spread and the asset price is deduced. The mathematical analysis to prove the existence of the unique solution for the PDEs is developed. Finally, the numerical methods and the associated numerical results are also included.

In Chapter 4, the American options considering counterparty risk are introduced. A similar framework as in Chapter 3 is considered. Thus, linear and nonlinear complementarity problems depending on two stochastic factors are deduced. We will study the existence and uniqueness of solution of the problem. Finally, we describe how to solve the model, and we present some examples to show the obtained numerical results by Longstaff–Schwartz techniques and solving the complementarity problem.

Chapter 1

One stochastic factor model for European options with XVA

1.1 Introduction

In this first chapter, we focus on European options. More precisely, a contract between two defaultable counterparties is considered. The departure point in this model is the consideration of a contract between two counterparties with constant intensity of default. As a result, the derivative value including counterparty risk, is modelled by a Cauchy problem depending on one stochastic factor, the asset price.

We follow the approach based in hedging arguments and the use of Itô's lemma for jump diffusion processes to obtain partial differential equations (PDE) formulations. Thus, after recalling the hedging strategy proposed for European–style derivatives, different kinds of PDEs arise depending on the assumptions on the mark–to–market value at default [15]. Thus, if this mark–to–market value is equal to the risk–free derivative then a linear PDE that involves the value of the risk–free derivative is obtained. However, if the mark–to–market value is given by the risky derivative, then a nonlinear PDE is obtained. In the linear case, the equivalent expression of the solution in terms of expectations can be solved. In the nonlinear case, this equivalent expression takes the form of a nonlinear integral equation and numerical methods are also required.

Moreover, different adjustments are included in the trade of the derivative, thus leading to different models. A first model includes adjustments for a non collateralized contract, i.e, only CVA, DVA and FVA are considered. Nevertheless, in a second model the CollVA is taken into account in the XVA.

We also prove the existence of the unique solution of the obtained nonlinear problem. With this aim, the methodology introduced by Henry [30] is followed.

In order to solve the resulting PDEs for both choices of the mark–to–market at default, we propose a set of numerical techniques. For this purpose, we truncate the unbounded asset domain and pose original suitable conditions at the boundaries of the resulting finite domain, following some ideas in [19] also taken from [23]. After truncation, we propose a time discretization based on the method of characteristics combined with a finite element discretization in the asset variable. For the case leading to a nonlinear PDE a fixed point iteration algorithm is proposed. Moreover, to obtain the XVA from the equation in terms of expectations, Monte Carlo techniques are applied.

The plan of the chapter is the following. In Section 1.2, some one stochastic factor models from the literature to price European–style options in the presence of counterparty credit risk are described. More precisely, first counterparty credit risk and funding costs are considered, while in a second step the collateral is added to the previous model. In Section 1.3 the existence and uniqueness of solution for the problems modelled in Section 1.2 are proved. Section 1.4 is devoted to the description of different numerical methods that are proposed to solve the linear and nonlinear PDE models stated in Section 1.2. Particularly, the domain truncation to pose the PDE problem in a bounded domain requires the consideration of appropriate and original boundary conditions. In Section 1.5 we present the Monte Carlo technique to estimate the XVA. Finally, in Section 1.6 we present and discuss the numerical results for different examples. Most of the contents presented in this chapter are included in [4].

1.2 Mathematical model

In this section, we deduce the Cauchy problems that model the total value adjustment associated to European options considering counterparty risk. Different models are introduced depending on the adjustments taken into account in the contract between both counterparties. Finally, we deduce the model for the total value adjustment.

1.2.1 Pricing with counterparty credit risk and funding costs

Following [15] we model the derivative value by considering different adjustments on the value of the corresponding risk–free derivative, i.e. a derivative without counterparty risk. In particular, bilateral default risk and funding costs are taken into account. More precisely, we consider two counterparties, the seller B and the buyer C, and the following assets associated to the trading [15]:

- Counterparty B zero recovery bond price, P_B, with yield r_{P_B}.
- Counterparty C zero recovery bond price, P_C, with yield r_{P_C}.
- Underlying asset with no default risk.

Due to the involved risks, the stock and the bond prices are modelled as stochastic processes satisfying the following stochastic differential equations (SDEs):

$$\begin{aligned} dP_{B_t} &= r_{P_B}(t) P_{B_t} dt - P_{B_t} dJ_t^B \\ dP_{C_t} &= r_{P_C}(t) P_{C_t} dt - P_{C_t} dJ_t^C \\ dS_t &= r_R(t) S_t dt + \sigma(t) S_t dW_t \,, \end{aligned} \tag{1.1}$$

where W_t is a Wiener process, and J_t^B and J_t^C are two independent jump processes that change from 0 to 1 on default of B and C, respectively.

Next, we consider a derivative trade where both counterparties can default. From the point of view of the seller, the value of this derivative at time t is denoted by $\widehat{V}_t = \widehat{V}(t, S_t, J_t^B, J_t^C)$ and it depends on the spot value of the asset, S_t, and on the default states at time t, J_t^B and J_t^C, of the seller B and buyer C, respectively. The value of the same derivative when the trade takes place between two default free counterparties is denoted by $V_t = V(t, S_t)$.

Since the trade takes place between defaultable counterparties, we need to incorporate some technical issues around close–outs. In this chapter it is assumed that the close–out mark–to–market can only take two possible values, namely the value of the risk–free derivative or the one of the defaultable derivative. The value of the defaultable derivative, $\widehat{V}(t, S_t, J_t^B, J_t^C)$, includes adjustments —such as CVA, DVA and FCA— into valuation whereas the value of the derivative without default risk, $V(t, S_t)$, does not include any counterparty adjustment. Moreover, we assume a setting such that the function $V(t, S)$ can be computed using a Black–Scholes model.

The conditions of the risky value upon default of the issuer or the counterparty are:

- if counterparty B defaults first,

$$\widehat{V}(t, S_t, 1, 0) = M^+(t, S_t) + R_B M^-(t, S_t) \tag{1.2}$$

- if counterparty C defaults first,

$$\widehat{V}(t, S_t, 0, 1) = R_C M^+(t, S_t) + M^-(t, S_t)\,, \tag{1.3}$$

where $R_B \in [0, 1]$ and $R_C \in [0, 1]$ represent the recovery rates on the derivatives positions of parties B and C, respectively, and M represents the close–out mark–to–market value.

In order to deduce the value of the credit risky derivative, we hedge the derivative with a self–financing portfolio Π which covers all underlying risk factors of the model. Recall that we want to compute the XVA from the point of view of the seller, B.

Thus, we have:

$$-\widehat{V}_t = \Pi_t.$$

Let us assume that the portfolio Π_t at time t consists of:

- $\Delta(t)$ units of the underlying asset S_t.
- $\alpha_B(t)$ units of P_{B_t}.
- $\alpha_C(t)$ units of P_{C_t}.
- $\gamma(t)$ units of cash, which is made up of a financing amount, needed to buy a position in C's bond and a repo amount, such that the portfolio value at time t hedges out the value of the derivative contract to the seller. Furthermore, the following issues need to be pointed out:
  1. The cost of the portfolio is denoted by γ_P, whereas the amount which is necessary to buy a position in B's bond or the cash obtained from selling B's bond is denoted by γ_{P_B}. Thus, the funding account, denoted by γ_F, is defined as the difference between the cost of the hedging portfolio and the price of the position in counterparty B's bond, $\gamma_F = \gamma_P - \gamma_{P_B}$.
  2. The cash needed to buy a position in C's bond, or the cash received from selling a C's bond, is denoted by γ_{P_C}.
  3. The repo account contains the amount of cash invested or borrowed in order to fund the stock position $\Delta(t)S_t$ through a repurchase agreement, and is denoted by γ_R.
  4. Although γ_P, γ_{P_B} and γ_F depend on t, for simplicity we do not explicit this dependence in the forthcoming expressions.

The values of the different bonds, in which the cash amount is invested or borrowed, satisfy the following relations for $s \geq t$:

$$dB_F(t,s) = \begin{cases} r_F(s)B_F(s)ds & \text{if} \quad \gamma_F \leq 0 \\ r(s)B_F(s)ds & \text{if} \quad \gamma_F > 0\,, \end{cases}$$

and

$$dB_{P_C}(t,s) = r(s)B_{P_C}(s)\,ds\,,$$
$$dB_R(t,s) = r_R(s)B_R(s)\,ds\,,$$

jointly with $B_F(t,t) = B_{P_C}(t,t) = B_R(t,t) = 1$. B_F and B_R are two bonds with different interest rates. Moreover, r denotes the risk–free interest rate, r_F represents the funding rate from the issuer and r_R is the rate paid for the underlying asset in a repurchase agreement. Thus, the portfolio value is equal to:

$$\Pi_t = \Delta(t)S_t + \alpha_B(t)P_{B_t} + \alpha_C(t)P_{C_t} + \gamma(t)\,,$$

and, according to the self–financing condition, $d\Pi_t = -d\widehat{V}_t$.

Next, since P_B and P_C are zero recovery bonds, their spreads are equal to the default intensities λ_B and λ_C, respectively:

$$\lambda_B = r_{P_B} - r\,, \qquad \lambda_C = r_{P_C} - r\,. \tag{1.4}$$

Now, imposing the self–financing feature of the portfolio, we deduce:

$$d\Pi_t = \Delta(t)dS_t + \alpha_B(t)dP_{B_t} + \alpha_C(t)dP_{C_t} + (r\gamma_F^+ + r_F\gamma_F^- - r\gamma_{P_C} - r_R\gamma_R)(t)dt\,. \tag{1.5}$$

In order to compute the change in the derivative price we use Itô's lemma for jump–diffusion processes (see [43], for example), thus leading to:

$$\begin{aligned} d\widehat{V}_t &= \frac{\partial \widehat{V}}{\partial t}dt + \frac{\partial \widehat{V}}{\partial S}dS_t + \frac{1}{2}\sigma^2 S_t^2\frac{\partial^2 \widehat{V}}{\partial S^2}dt + \Delta\widehat{V}_{B_t}dJ_t^B + \Delta\widehat{V}_{C_t}dJ_t^C \\ &= \left(\frac{\partial \widehat{V}}{\partial t} + r_R\frac{\partial \widehat{V}}{\partial S} + \frac{1}{2}\sigma^2 S_t^2\frac{\partial^2 \widehat{V}}{\partial S^2}\right)dt + \sigma S_t\frac{\partial \widehat{V}}{\partial S}dW_t + \Delta\widehat{V}_{B_t}dJ_t^B + \Delta\widehat{V}_{C_t}dJ_t^C\,, \end{aligned} \tag{1.6}$$

where $\widehat{V}$ and all partial derivatives of $\widehat{V}$ are evaluated at (t, S_t, J_t^B, J_t^C). Moreover, we use the notations

$$\begin{aligned} \Delta\widehat{V}_{B_t} &= \widehat{V}(t,S_t,1,0) - \widehat{V}(t,S_t,0,0)\,, \\ \Delta\widehat{V}_{C_t} &= \widehat{V}(t,S_t,0,1) - \widehat{V}(t,S_t,0,0)\,, \end{aligned} \tag{1.7}$$

which can be computed using the default conditions (1.2)–(1.3).

Keeping in mind expressions (1.5) and (1.6) we deduce the following equation:

$$\begin{aligned}&\Delta(t)dS_t + \alpha_B(t)dP_{B_t} + \alpha_C(t)dP_{C_t} + (r\gamma_F^+ + r_F\gamma_F^- - r\gamma_{P_C} - r_R\gamma_R)dt \\ &\quad = -\left(\frac{\partial \widehat{V}}{\partial t}dt + \frac{\partial \widehat{V}}{\partial S}dS_t + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 \widehat{V}}{\partial S^2}dt + \Delta\widehat{V}_{B_t}dJ_t^B + \Delta\widehat{V}_{C_t}dJ_t^C\right).\end{aligned} \tag{1.8}$$

According to the SDEs in (1.1) we obtain:

$$\begin{aligned}&\Delta(t)dS_t + \alpha_B(t)(r_{P_B}P_{B_t}dt - P_{B_t}dJ_t^B) + \alpha_C(t)(r_{P_C}P_{C_t}dt - P_{C_t}dJ_t^C) \\ &\quad + (r\gamma_F^+ + r_F\gamma_F^- - r\gamma_{P_C} - r_R\gamma_R)dt \\ &\quad = -\left(\frac{\partial \widehat{V}}{\partial t}dt + \frac{\partial \widehat{V}}{\partial S}dS_t + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 \widehat{V}}{\partial S^2}dt + \Delta\widehat{V}_{B}dJ_t^B + \Delta\widehat{V}_{C}dJ_t^C\right).\end{aligned} \tag{1.9}$$

Moreover, we choose the following weights:

$$\begin{aligned}\Delta(t) &= -\frac{\partial \widehat{V}}{\partial S}, \\ \alpha_B(t) &= \frac{\Delta\widehat{V}_{B_t}}{P_{B_t}} = -\frac{\widehat{V}_t - (M_t^+ + R_B M_t^-)}{P_{B_t}}, \\ \alpha_C(t) &= \frac{\Delta\widehat{V}_{C_t}}{P_{C_t}} = -\frac{\widehat{V}_t - (M_t^- + R_C M_t^+)}{P_{C_t}}\end{aligned} \tag{1.10}$$

in order to remove all risks in the portfolio Π_t. Thus, equation (1.9) leads to

$$\alpha_B r_{P_B} P_B + \alpha_C r_{P_C} P_C + (r\gamma_F^+ + r_F\gamma_F^- - r\gamma_{P_C} - r_R\gamma_R) + \frac{\partial \widehat{V}}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 \widehat{V}}{\partial S^2} = 0\,. \tag{1.11}$$

In order to obtain the PDE that models the derivative value, we consider the equivalences: $\gamma_{P_B} = \alpha_B P_{B_t}$, $\gamma_{P_C} = \alpha_C P_{C_t}$, $r_F = r + s_F$ and $\gamma_F = \gamma_P - \gamma_{P_B}$, so that

$$\begin{aligned}&\alpha_B r_{P_B} P_B + \alpha_C r_{P_C} P_C + r\gamma_F^+ + r_F\gamma_F^- - r\gamma_{P_C} - r_R\gamma_R \\ &\quad = \alpha_B r_{P_B} P_B + \alpha_C r_{P_C} P_C + r(\gamma_P - \gamma_{P_B})^+ + r_F(\gamma_P - \gamma_{P_B})^- - r\alpha_C P_C - r_R\gamma_R \\ &\quad = \alpha_B r_{P_B} P_B + \alpha_C r_{P_C} P_C + r(\gamma_P - \alpha_B P_B) + s_F(\gamma_P - \alpha_B P_B)^- - r\alpha_C P_C - r_R\gamma_R\,.\end{aligned}$$

According to the repo account we have $\gamma_R = \Delta S$, so that the previous identity becomes:

$$\alpha_B r_{P_B} P_B + \alpha_C r_{P_C} P_C + r\gamma_F^+ + r_F\gamma_F^- - r\gamma_{P_C} - r_R\gamma_R$$
$$= r\gamma_P + s_F\gamma_F^- - r_R\Delta S + (r_{P_C} - r)\alpha_C P_C + (r_{P_B} - r)\alpha_B P_B \,.$$

In order to avoid arbitrage opportunities, the hedging portfolio value has to be equal to the derivative value, so that $\gamma_P = -\widehat{V}$. Moreover, by considering the expressions in (1.4) the previous equation can be further reduced to

$$\alpha_B r_{P_B} P_B + \alpha_C r_{P_C} P_C + r\gamma_F^+ + r_F\gamma_F^- - r\gamma_{P_C} - r_R\gamma_R$$
$$= -r\widehat{V} + s_F\gamma_F^- - r_R\Delta S + \lambda_C\alpha_C P_C + \lambda_B\alpha_B P_B \,.$$

Finally, considering the addends in which $\alpha_B P_{B_t}$ and $\alpha_C P_{C_t}$ take place and expressing them in terms of the mark–to–market value we get

$$\alpha_B r_{P_B} P_B + \alpha_C r_{P_C} P_C + r\gamma_F^+ + r_F\gamma_F^- - r\gamma_{P_C} - r_R\gamma_R$$
$$= -(r + \lambda_B + \lambda_C)\widehat{V} + s_F\gamma_F^- - r_R\Delta S + \lambda_B(M^+ + R_B M^-) + \lambda_C(M^- + R_C M^+) \,.$$

Thus, we introduce the previous expression in (1.11) to obtain the PDE that models the value of the derivative including the counterparty risk:

$$\begin{cases} \partial_t\widehat{V} + \mathcal{A}\widehat{V} - r\widehat{V} = (\lambda_B + \lambda_C)\widehat{V} + s_F M^+ \\ \qquad\qquad -\lambda_B(R_B M^- + M^+) - \lambda_C(R_C M^+ + M^-) \\ \widehat{V}(T,S) = H(S)\,, \end{cases} \tag{1.12}$$

where s_F is the funding cost of the entity, M represents the mark–to–market and the differential operator $\mathcal{A}$ is given by

$$\mathcal{A}V \equiv \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + r_R S\frac{\partial V}{\partial S}\,. \tag{1.13}$$

According to the two scenarios usually considered for the choice of the derivative mark–to–market value at default, M, two different PDE problems are obtained:

- If $M = \widehat{V}$,

$$\begin{cases} \partial_t \widehat{V} + \mathcal{A}\widehat{V} - r\widehat{V} = (1-R_B)\lambda_B \widehat{V}^- + (1-R_C)\lambda_C \widehat{V}^+ + s_F \widehat{V}^+ \\ \widehat{V}(T,S) = H(S)\,. \end{cases}$$

- If $M = V$,

$$\begin{cases} \partial_t \widehat{V} + \mathcal{A}\widehat{V} - (r+\lambda_B+\lambda_C)\widehat{V} \\ \qquad\qquad = -(R_B\lambda_B+\lambda_C)V^- - (R_C\lambda_C+\lambda_B)V^+ + s_F V^+ \\ \widehat{V}(T,S) = H(S)\,, \end{cases}$$

where $H(S)$ represents the pay–off of the derivative. In this chapter, European vanilla call and put options and forwards will be considered.

The derivative value with counterparty risk can be written as:

$$\widehat{V} = V + U\,,$$

where U is the total value adjustment (XVA) and the counterparty risk–free value of the derivatives, V, satisfies the classical linear Black–Scholes equation:

$$\begin{cases} \partial_t V + \mathcal{A}V - rV = 0\,, \\ V(T,S) = H(S)\,. \end{cases} \tag{1.14}$$

Thus, the PDE problems satisfied by U are the following:

- If $M = \widehat{V}$, we get a final value nonlinear problem:

$$\begin{cases} \partial_t U + \mathcal{A}U - rU = (1-R_B)\lambda_B(V+U)^- \\ \qquad\qquad +(1-R_C)\lambda_C(V+U)^+ + s_F(V+U)^+ \\ U(T,S) = 0\,. \end{cases} \tag{1.15}$$

- If $M = V$, an analogous linear problem is deduced:

$$\begin{cases} \partial_t U + \mathcal{A}U - (r+\lambda_B+\lambda_C)U = (1-R_B)\lambda_B V^- \\ \qquad\qquad +(1-R_C)\lambda_C V^+ + s_F V^+ \\ U(T,S) = 0\,. \end{cases} \tag{1.16}$$

In both cases, variable S lies in the unbounded domain $[0, +\infty)$ while $t \in [0, T]$.

1.2.2 Pricing with counterparty credit risk, funding costs and collateral

Many contracts include the collateralization of an asset. Collateral is a property or other assets that a borrower offers a lender to secure a loan. If the borrower stops making the promised loan payments, the lender can seize the collateral to fully or partly recover its losses.

In this section, mainly following [16], a credit risky collateralised derivative value is modelled in terms of PDEs, thus a more generalized framework is studied. For this purpose, we assume an agreement between two risky counterparties B and C, where B is the issuer. As in the previous section, a self–financing hedging portfolio is used. The main difference with respect to the former setting is that in the present one the hedging portfolio only hedges out the derivative when counterparty does not default, whereas in the previous section the hedging portfolio perfectly hedges the derivative.

When the counterparty B defaults, there is a difference between the hedging portfolio and the short derivative value, which is known as hedge error.

In a similar way to the previous section, we want to deduce the PDE model for a collateralised derivative. Thus, we need to describe all the items taking part in this new setting. For this purpose, in [16] the authors consider the general case in which B has a portfolio made up of two bonds, P_1 and P_2, with different seniorities and different recoveries, R_1 and R_2, respectively. More precisely,

- P_1 is an issued junior bond with recovery $R_1 \geq 0$ and yield r_1

- P_2 is an issued senior bond with recovery $R_2 > 0$ and yield r_2

and $R_2 > R_1$. Thus, we assume the price processes satisfy the following SDEs:

$$dS_t = r_R(t)S_t dt + \sigma(t)S_t dW_t \tag{1.17}$$

$$dP_{C_t} = r_{P_C}(t)P_{C_t}dt - P_C dJ_t^C \tag{1.18}$$

$$dP_{1_t} = r_1(t)P_{1_t}dt - (1 - R_1)P_{1_t}dJ_t^B \tag{1.19}$$

$$dP_{2_t} = r_2(t)P_{2_t}dt - (1 - R_2)P_{2_t}dJ_t^B \, . \tag{1.20}$$

The total position, at time t, in the B issued bond is given by

$$P_{B_t} = \alpha_1(t)P_{1_t} + \alpha_2(t)P_{2_t} \tag{1.21}$$

and the value of P_B in the issuer's default instant is defined as

$$P_{D_t} = \alpha_1(t)R_1P_{1_t} + \alpha_2(t)R_2P_{2_t} \, . \tag{1.22}$$

The conditions of the collateral derivative value upon default of both counterparties are:

- if B defaults first, then

$$\widehat{V}(t, S_t, 1, 0) = g_B(M_t, X_t) = X_t + (M_t - X_t)^+ + R_B(M_t - X_t)^- \tag{1.23}$$

- if C defaults first, then

$$\widehat{V}(t, S_t, 0, 1) = g_C(M_t, X_t) = X_t + (M_t - X_t)^- + R_C(M_t - X_t)^+ \, , \tag{1.24}$$

where X_t represents the collateral and M_t is the mark–to–market value. These conditions represent an extension of the ones given in (1.2)–(1.3), which are clearly recovered for $X_t = 0$.

The hedging portfolio built up in this model only hedges out the derivative when the counterparty B does not default, so that, in this case

$$\Pi_t + \widehat{V}_t = 0 \, . \tag{1.25}$$

Moreover, the self–financing hedging portfolio is made up of

- $\Delta(t)$ units of the underlying asset S_t.
- One unit of counterparty B bonds, P_{B_t}.
- $\alpha_C(t)$ units of counterparty C bonds, P_{C_t}.
- $\gamma(t)$ units of cash, which consists on an amount of stock position in a repurchase agreement, $\gamma_R(t)$, and the cash amount necessary to purchase $\alpha_C(t)$ bonds of C, γ_{P_C}.
- An amount of collateral, X_t.

Thus, the total value of the portfolio at time t is given by:

$$\Pi_t = \Delta(t)S_t + P_{B_t} + \alpha_C(t)P_{C_t} + \gamma(t) - X_t\,. \tag{1.26}$$

As the portfolio is self–financing, the change in the hedging portfolio is

$$d\Pi_t = \Delta(t)dS_t + dP_{B_t} + \alpha_C(t)dP_{C_t} - (r\gamma_{P_C} + r_R\gamma_R)(t)dt - r_X X_t dt\,. \tag{1.27}$$

We now consider the expressions given in (1.17)–(1.20), and by replacing them in the hedging equation (1.27) we obtain:

$$\begin{aligned} d\Pi_t &= \Delta dS_t + \alpha_1(t)(r_1(t)P_{1_t}dt - (1-R_1)P_{1_t}dJ_t^B) \\ &\quad + \alpha_2(t)(r_2(t)P_{2_t}dt - (1-R_2)P_{2_t}dJ_t^B) \\ &\quad + \alpha_C(t)(r_{P_C}(t)P_{C_t}dt - P_C dJ_t^C) - r\alpha_C(t)P_C dt - r_R\Delta(t)S_t dt - r_X X_t dt\,, \end{aligned}$$

and reordering terms we get:

$$\begin{aligned} d\Pi_t &= \Delta(t)dS_t + \alpha_1(t)r_1P_{1_t}dt + \alpha_2(t)r_2P_{2_t}dt + \alpha_C(t)\lambda_C P_{C_t}dt \\ &\quad - r_R\Delta S_t dt - r_X X_t dt - (\alpha_1(t)P_{1_t} + \alpha_2(t)P_{2_t})dJ_t^B \\ &\quad + (\alpha_1(t)R_1P_{1_t} + \alpha_2(t)R_2P_{2_t})dJ_t^B - \alpha_C(t)P_{C_t}dJ_t^C\,. \end{aligned} \tag{1.28}$$

Taking into account equations (1.21) and (1.22), the hedging equation (1.28) reads:

$$\begin{aligned} d\Pi_t &= (\alpha_1(t)r_1P_{1_t} + \alpha_2(t)r_2P_{2_t} + \alpha_C(t)\lambda_C P_{C_t} - r_R\Delta(t)S_t - r_X X_t)dt \\ &\quad + \Delta(t)dS_t + (P_{D_t} - P_{B_t})dJ_t^B - \alpha_C(t)P_C dJ_t^C\,. \end{aligned}$$

Moreover, applying again Itô's lemma for jump–diffusion processes, the dynamics of the risky derivative value is obtained:

$$\begin{aligned} d\widehat{V}_t &= \frac{\partial \widehat{V}}{\partial t} dt + \frac{\partial \widehat{V}}{\partial S} dS_t + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 \widehat{V}}{\partial S^2} dt + \Delta \widehat{V}_{B_t} dJ_t^B + \Delta \widehat{V}_{C_t} dJ_t^C \\ &= \left(\frac{\partial \widehat{V}}{\partial t} + r_R \frac{\partial \widehat{V}}{\partial S} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 \widehat{V}}{\partial S^2} \right) dt + \sigma S_t \frac{\partial \widehat{V}}{\partial S} dW_t + \Delta \widehat{V}_{B_t} dJ_t^B + \Delta \widehat{V}_{C_t} dJ_t^C , \end{aligned} \tag{1.29}$$

where:

$$\begin{aligned} \Delta \widehat{V}_{B_t} &= \widehat{V}(t, S_t, 1, 0) - \widehat{V}(t, S_t, 0, 0) = g_B(M_t, X_t) - \widehat{V}(t, S_t, 0, 0) \\ \Delta \widehat{V}_{C_t} &= \widehat{V}(t, S_t, 0, 1) - \widehat{V}(t, S_t, 0, 0) = g_C(M_t, X_t) - \widehat{V}(t, S_t, 0, 0) . \end{aligned}$$

By combining the change in the hedging portfolio and the change in the derivative value, we obtain

$$\begin{aligned} d\Pi_t + d\widehat{V}_t = \Bigg(& \alpha_1(t) r_1 P_{1_t} + \alpha_2(t) r_2 P_{2_t} + \alpha_C(t) \lambda_C P_{C_t} - r_R \Delta(t) S_t - r_X X_t + \frac{\partial \widehat{V}}{\partial t} \\ & + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 \widehat{V}}{\partial S^2} \Bigg) dt + \left(\Delta(t) + \frac{\partial \widehat{V}}{\partial S} \right) dS_t \\ & + (P_{D_t} - P_{B_t} + \Delta \widehat{V}_{B_t}) dJ_t^B + (\Delta \widehat{V}_{C_t} - \alpha_C(t) P_C) dJ_t^C . \end{aligned} \tag{1.30}$$

Next, we can remove the counterparty C's credit risk and the market risk by choosing

$$\alpha_C(t) = \frac{\Delta V_{C_t}}{P_C} , \qquad \Delta(t) = -\frac{\partial \widehat{V}}{\partial S} , \tag{1.31}$$

so that equation (1.30) is reduced to:

$$\begin{aligned} d\Pi_t + d\widehat{V}_t = & \left(\frac{\partial \widehat{V}}{\partial t} + \mathcal{A}\widehat{V} + \alpha_1(t) r_1 P_{1_t} + \alpha_2(t) r_2 P_{2_t} + \alpha_C(t) \lambda_C P_{C_t} - r_X X_t \right) dt \\ & + \left(\Delta \widehat{V}_{B_t} - P_{B_t} + P_{D_t} \right) dJ_t^B \end{aligned} \tag{1.32}$$

where the differential operator $\mathcal{A}$ is defined as in (1.13).

Furthermore, when the counterparty B does not default, the difference between the hedging portfolio and the short derivative is given by

$$\begin{aligned}\Pi_t - (-\widehat{V}_t) = \Pi_t + \widehat{V}_t &= \Delta(t)S_t + P_{B_t} + \alpha_C(t)P_{C_t} + \gamma - X_t + \widehat{V}_t \\ &= \Delta(t)S_t + P_{B_t} + \alpha_C(t)P_{C_t} - (\gamma_R + \gamma_{P_C}) - X_t + \widehat{V}_t \\ &= P_{B_t} + \widehat{V}_t - X_t\,. \end{aligned} \tag{1.33}$$

In this case, while the counterparty B is alive, we have a perfectly hedged portfolio, so that the following funding constraint is obtained:

$$\widehat{V}_t + P_{B_t} - X_t = 0\,. \tag{1.34}$$

We can interpret this equation in the following way: if $\widehat{V}_t - X_t < 0$, then B bonds are used to fund the difference between the derivative value and the collateral. Conversely, if that difference is positive then they are used to repurchase B issued bonds. Finally, if the risky value is fully hedged by the collateral then the bond position will be reduced to zero. If the collateral is zero, the trade will be financed by B's bonds.

Therefore, we have assumed that the issuer wants a self–financing portfolio while he/she is alive. In this case, the jump indicator is zero, because B does not default. Thus, the drift term has to be equal to zero to obtain a self–financing hedging portfolio according to (1.25). So, the PDE for the collateralized risky value is given by:

$$\frac{\partial \widehat{V}}{\partial t} + \mathcal{A}\widehat{V} + \alpha_1 r_1 P_1 + \alpha_2 r_2 P_2 + \alpha_C \lambda_C P_C - r_X X = 0\,. \tag{1.35}$$

Next, let us consider the case when the counterparty B defaults. In this situation the derivative value is $g_B(M_t, X_t)$ and P_D is the total B bond position, so that the previous difference (1.33), in the case that B defaults, turns into the hedge error given by:

$$h_e = P_{D_t} + g_B(M_t, X_t) - X_t\,, \tag{1.36}$$

which depends on the mark–to–market value.

Taking into account equations (1.31), (1.34) and (1.36), the PDE (1.35) can be reduced to:

$$\frac{\partial \widehat{V}}{\partial t} + \mathcal{A}\widehat{V} - (r + \lambda_B + \lambda_C)\widehat{V} + \lambda_C g_C(M,X) + \lambda_B g_B(M,X) - \lambda_B h_e - s_X X = 0\,,$$

so that the final value problem consists in finding the function $\widehat{V}$ as the solution of:

$$\begin{cases} \dfrac{\partial \widehat{V}}{\partial t} + \mathcal{A}\widehat{V} - (r + \lambda_B + \lambda_C)\widehat{V} \\ \qquad\qquad\qquad = \lambda_B h_e - \lambda_B g_B(M, X) - \lambda_C g_C(M, X) + s_X X \\ \widehat{V}(T, S) = H(S)\,. \end{cases} \tag{1.37}$$

If we compare (1.37) with the PDE problem (1.12) obtained in the case without collateral, the two additional terms $\lambda_B h_e$ and $s_X X$ appear. Furthermore, the terms g_B and g_C are now more general.

Moreover, in case of counterparty B default a hedge error arises. Nevertheless, while the issuer B is alive, it will incur a cost or gain of size $\lambda_B h_e$ per time unit. We can prove that this gain is equal to the hedge error. The gain is defined as the coefficient of J^B in (1.32):

$$\Delta \widehat{V}_B + P_D - P_B = g_B - \widehat{V} + P_D - P_B = g_B + P_D - X = h_e.$$

As in the case without collateral (described in the previous section), our goal is the computation of the total value adjustment. For this purpose, we write the risky value as the sum of the risk–free value V and the total value adjustment, U, i.e.:

$$\widehat{V} = V + U$$

where V is solution of (1.14). Thus, the total value adjustment satisfies the following PDE problem:

$$\begin{cases} \dfrac{\partial U}{\partial t} + \mathcal{A}U - rU = \lambda_B h_e + \lambda_B(\widehat{V} - g_B(M, X)) \\ \qquad\qquad\qquad + \lambda_C(\widehat{V} - g_C(M, X)) + s_X X \\ U(T, S) = 0\,. \end{cases} \tag{1.38}$$

If we analyze the terms involved in the right hand side of the equation, the following adjustments are taken into account: the first term is related to the amount of gain or cost, and takes part of the FCA; the second and third terms are related to the DVA and CVA, respectively; and the last term is related to collateral value adjustment. Depending on the mark–to–market value, we obtain two different equations:

- If $M = \widehat{V}$, we get a final value problem governed by a nonlinear PDE:

$$\begin{cases} \dfrac{\partial U}{\partial t} + \mathcal{A}U - rU = \lambda_B h_e + \lambda_B(1-R_B)(V+U-X)^- \\ \qquad\qquad\qquad\qquad +\lambda_C(1-R_C)(V+U-X)^+ + s_X X \\ U(T,S) = 0\,. \end{cases} \tag{1.39}$$

- If $M = V$, an analogous linear problem is deduced:

$$\begin{cases} \dfrac{\partial U}{\partial t} + \mathcal{A}U - (r+\lambda_B+\lambda_C)U = \lambda_B h_e + \lambda_B(1-R_B)(V-X)^- \\ \qquad\qquad\qquad\qquad\qquad\qquad +\lambda_C(1-R_C)(V-X)^+ + s_X X \\ U(T,S) = 0\,. \end{cases} \tag{1.40}$$

As in the non–collateralized problems, variable S lies in $[0,+\infty)$ while $t \in [0,T]$.

Finally, different assumptions are made on counterparty B bond. As a result, three particular different models can be proposed. Note that the linear versions corresponding to (1.40) have been proposed in [14].

Collateral model 1: Perfect hedging

If all risks are perfectly hedged, then h_e is reduced to zero; thus we get:

$$\begin{aligned} h_e &= g_B(M_t, X_t) + P_{D_t} - X_t \\ &= g_B(M_t, X_t) + \alpha_1(t)R_1P_{1_t} + \alpha_2(t)R_2P_{2_t} - X_t = 0\,. \end{aligned} \tag{1.41}$$

Moreover, according to the funding constraint (1.34), we have:

$$\widehat{V}_t + \alpha_1(t)P_{1_t} + \alpha_2(t)P_{2_t} - X_t = 0\,,$$

so that we get the identity:

$$\alpha_1(t) = \frac{X_t - \widehat{V}_t - \alpha_2 P_{2_t}}{P_{1_t}}. \tag{1.42}$$

Replacing this value in (1.41), we obtain the number of senior bonds

$$\alpha_2(t) = \frac{-g_B(M_t, X_t) + (1-R_1)X_t + R_1\widehat{V}_t}{P_{2_t}(R_2 - R_1)}. \tag{1.43}$$

Next, replacing (1.43) in (1.42), we obtain the number of junior bonds:

$$\alpha_1(t) = \frac{(R_2 - 1)X_t - R_2\widehat{V}_t + g_B(M_t, X_t)}{P_{1_t}(R_2 - R_1)},$$

with $R_2 > R_1$, $P_1 \neq 0$ and $P_2 \neq 0$. With that position of the counterparty B's bonds, a perfect hedging portfolio is obtained, eventhough B defaults. In this case the PDE which models the risky derivative value is reduced to

$$\begin{cases} \dfrac{\partial \widehat{V}}{\partial t} + \mathcal{A}\widehat{V} - (r + \lambda_B + \lambda_C)\widehat{V} = -\lambda_C g_C(M, X) - \lambda_B g_B(M, X) + s_X X \\ \widehat{V}(T, S) = H(S), \end{cases}$$

and the PDEs for the total value adjustment, U, are:

- If $M = \widehat{V}$,

$$\begin{cases} \dfrac{\partial U}{\partial t} + \mathcal{A}U - rU = \lambda_B(1 - R_B)(V + U - X)^- \\ \qquad\qquad\qquad +\lambda_C(1 - R_C)(V + U - X)^+ + s_X X \\ U(T, S) = 0\,. \end{cases}$$

- If $M = V$,

$$\begin{cases} \dfrac{\partial U}{\partial t} + \mathcal{A}U - (r + \lambda_B + \lambda_C)U = \lambda_B(1 - R_B)(V - X)^- \\ \qquad\qquad\qquad +\lambda_C(1 - R_C)(V - X)^+ + s_X X \\ U(T, S) = 0\,. \end{cases}$$

Notice that funding cost adjustment vanishes because the hedge error is null, so that only CVA, DVA and CollVA are taken into account in the XVA.

Collateral model 2: Two bonds model

In this model, we assume that counterparty B has two bonds. More precisely, a zero recovery bond P_1 and a bond P_2 with recovery R_2. This recovery is equal to the recovery rate of counterparty B on a derivative trade, i.e. $R_2 = R_B$. Under this assumption, the corresponding PDE is deduced.

Assuming the funding constraint (1.34), we write:

$$P_{B_t} = \alpha_1(t)P_{1_t} + \alpha_2(t)P_{2_t} = -(\widehat{V}_t - X_t)\,. \tag{1.44}$$

According to the model without collateral and taking into account (1.10), the zero recovery B bond position value is equal to the difference between the risky value and the mark–to–market value, thus we have:

$$\alpha_1(t)P_{1_t} = -(\widehat{V}_t - M_t)\,. \tag{1.45}$$

Note that if the mark–to–market value is equal to the risk–free value, then the first B bond position is bought or issued to invest or fund the XVA amount. Otherwise, when the mark–to–market value is equal to the risky derivative value the situation becomes equivalent to a one bond case, which will be later explained.

Including the first B bond position in (1.44) we obtain the second B bond position:

$$\alpha_2(t)P_{2_t} = -(\widehat{V}_t - X_t) + (\widehat{V}_t - M_t) = X_t - M_t\,. \tag{1.46}$$

Next, by considering expressions (1.23), (1.24), (1.44), (1.45) and (1.46), the hedge error becomes

$$h_e = (1 - R_B)(M_t - X_t)^+\,.$$

Now, the hedge error is replaced into the general PDE (1.37), thus obtaining:

$$\begin{cases} \dfrac{\partial \widehat{V}}{\partial t} + \mathcal{A}\widehat{V} - (r + \lambda_B + \lambda_C)\widehat{V} = \lambda_B(1 - R_B)(M - X)^+ \\ \qquad\qquad\qquad\qquad\qquad -\lambda_B g_B(M, X) - \lambda_C g_C(M, X) + s_X X \\ \widehat{V}(T, S) = H(S)\,, \end{cases}$$

and the PDE models satisfied by XVA are given by:

- If $M = \widehat{V}$,

$$\begin{cases} \dfrac{\partial U}{\partial t} + \mathcal{A}U - (r + \lambda_B(1 - R_B))U = \lambda_B(1 - R_B)(V - X) \\ \qquad\qquad\qquad\qquad\qquad +\lambda_C(1 - R_C)(V + U - X)^+ + s_X X \\ U(T, S) = 0\,. \end{cases}$$

- If $M = V$,

$$\begin{cases} \dfrac{\partial U}{\partial t} + \mathcal{A}U - (r + \lambda_B + \lambda_C)U = \lambda_B(1 - R_B)(V - X) \\ \qquad\qquad\qquad\qquad\qquad\qquad + \lambda_C(1 - R_C)(V - X)^+ + s_X X \\ U(T, S) = 0\,. \end{cases}$$

Collateral model 3: One bond model

Finally, only one bond from B, with recovery rate R_B, is considered so that:

$$\alpha_1(t)P_{1_t} = 0 \quad \Longrightarrow \quad P_{B_t} = \alpha_2(t)P_{2_t}\,.$$

According to the funding constraint (1.34), we get:

$$\alpha_2(t)P_{2_t} = -(\widehat{V}_t - X_t)\,,$$

and the hedge error is given by

$$h_e = g_B(M_t, X_t) - R_B(\widehat{V}_t - X_t) - X_t = g_B(M_t, X_t) + (R_B - 1)X_t - R_B\widehat{V}_t\,.$$

After replacing the hedge error expression in (1.37), the following PDE modelling the risky value is obtained:

$$\begin{cases} \dfrac{\partial \widehat{V}}{\partial t} + \mathcal{A}\widehat{V} - (r + \lambda_B(1 - R_B) + \lambda_C)\widehat{V} = \lambda_B(R_B - 1)X - \lambda_C g_C(M, X) + s_X X \\ \widehat{V}(T, S) = H(S)\,, \end{cases}$$

and the models for the XVA are:

- If $M = \widehat{V}$,

$$\begin{cases} \dfrac{\partial U}{\partial t} + \mathcal{A}U - (r + \lambda_B(1 - R_B))U = \lambda_B(1 - R_B)(V - X) \\ \qquad\qquad\qquad\qquad + \lambda_C(1 - R_C)(V + U - X)^+ + s_X X \\ U(T, S) = 0\,. \end{cases} \tag{1.47}$$

- If $M = V$,

$$\begin{cases} \dfrac{\partial U}{\partial t} + \mathcal{A}U - (r + \lambda_B(1-R_B) + \lambda_C)U = \lambda_B(1-R_B)(V-X) \\ \qquad\qquad +\lambda_C(1-R_C)(V-X)^+ + s_X X \\ U(T,S) = 0\,. \end{cases} \tag{1.48}$$

We can observe that in the linear problem (1.48), when $M = V$, if a fully collateralised derivative is considered then only CollVA exists in the adjustment upon risk–neutral value, i.e. CVA, DVA and FCA vanish.

If we analyze the current situation, in which only funding desk can issue bonds in the bank, the present model results the most realistic one because the trader cannot issue bonds in order to raise cash for trade, so that only one bond from B has to be considered.

1.3 Mathematical analysis

In the previous section, the mathematical model for XVA associated to European options depending on one stochastic factor has been posed as a nonlinear or a linear final value problem. In this section we study the existence and uniqueness of solution for the nonlinear final value problem (1.15), as the linear case can be studied as a particular one. Using a similar procedure, the mathematical analysis of the problem considering collateral can be addressed.

For this purpose, we transform the associated partial differential equation (1.15) into an equivalent one, governed by a sectorial operator. Thus, the following changes of variables and unknown

$$x = \ln\left(\frac{S}{K}\right), \quad \tau = \frac{\sigma^2}{2}(T-t), \quad \omega(\tau,x) = \frac{1}{K}U(t,S)$$

are introduced in problem (1.15). Note that $x \in \mathbb{R}$ and $\tau \in \left[0, \frac{\sigma^2 T}{2}\right]$. Thus, the equivalent initial value problem becomes

$$\begin{cases} \dfrac{\partial \omega}{\partial \tau} = \dfrac{\partial \omega}{\partial x^2} - \dfrac{2}{\sigma^2 K} g(\tau, \omega) + \left(\dfrac{2r_R}{\sigma^2} - 1\right) \dfrac{\partial \omega}{\partial x} - \dfrac{2r}{\sigma^2}\omega, \quad x \in \mathbb{R}, \ 0 < \tau \leq \dfrac{\sigma^2 T}{2} \\ \omega(0, x) = 0\,, \end{cases} \tag{1.49}$$

where function g is defined as:

$$g(\tau, \omega) = (1 - R_B)\lambda_B(V + K\omega)^- + (1 - R_C)\lambda_C(V + K\omega)^+ + s_F(V + K\omega)^+\,.$$

Next, we introduce a new change of variable in order to remove the last two terms in the right hand side of the first equation in (1.49):

$$v(\tau, x) = \exp(\alpha x + \beta\tau)\,\omega(\tau, x)\,,$$

with

$$\alpha = -\frac{1}{2}\left(1 - \frac{2r_R}{\sigma^2}\right), \qquad \beta = \left(1 - \frac{2r_R}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}\,.$$

As a consequence, the following equivalent problem is posed:

$$\begin{cases} \dfrac{\partial v}{\partial \tau} - \dfrac{\partial^2 v}{\partial x^2} = h(\tau, v), \qquad x \in \mathbb{R},\ \tau \in \left(0, \dfrac{\sigma^2 T}{2}\right] \\ v(0, x) = 0\,. \end{cases}$$

The function h is given by

$$\begin{aligned} h(\tau, \varphi)(x) = &- \exp\left(-\frac{1}{2}\left(1 - \frac{2r_R}{\sigma^2}\right)x + \left[\left(1 - \frac{2r_R}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}\right]\tau \right) \\ &\times \frac{2}{K\sigma^2}\left[(1 - R_B)\lambda_B\big(\mathcal{G}(\tau, \varphi)(x)\big)^- + [(1 - R_C)\lambda_C + s_F]\,\big(\mathcal{G}(\tau, \varphi)(x)\big)^+\right] \end{aligned}$$

where $\mathcal{G}(\tau, \varphi)(x) = V(\tau, Ke^x) + K\varphi(x)\exp\left(\frac{1}{2}\left(1 - \frac{2r_R}{\sigma^2}\right)x - \left[\left(1 - \frac{2r_R}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}\right]\tau\right)$.

Finally, we apply the last change of variable in order to obtain a well defined function in the second term of the equation:

$$u(\tau, x) = e^{\gamma x} v(\tau, x)$$

where γ will be later deduced. Thus, the following problem is obtained:

$$\begin{cases} \dfrac{\partial u}{\partial \tau} - \dfrac{\partial^2 u}{\partial x^2} = J(\tau, u), & x \in \mathbb{R},\, \tau \in \left(0, \dfrac{\sigma^2 T}{2}\right] \\ v(0,x) = 0\,, \end{cases} \tag{1.50}$$

where function $J : \left[0, \frac{\sigma^2}{2}T\right] \times H^1(\mathbb{R}) \to L^2(\mathbb{R})$ is defined as follows:

$$J(\tau, \varphi)(x) = \gamma^2 \varphi(x) - 2\gamma \frac{\partial \varphi}{\partial x}(x) + e^{\gamma x} h(\tau, e^{-\gamma x}\varphi(x)) \tag{1.51}$$

for all $\tau \in \left[0, \frac{\sigma^2 T}{2}\right]$, $\varphi \in H^1(\mathbb{R})$.

Next, we recall the definition of a sectorial operator, and a theorem that establishes the conditions for the existence and uniqueness of solution for a nonlinear PDE problem associated to a sectorial operator (see [30]).

Definition 1.3.1. *A linear operator $\mathcal{B}$ in a Banach space X is a sectorial operator if it is a closed densely defined operator such that, for some $\phi \in (0, \pi/2)$, $M_0 \geq 1$ and a real a, the sector $S_{a,\phi} = \{\lambda \mid \phi \leq |arg(\lambda - a)| \ \leq \pi, \lambda \neq a\}$ is in the resolvent set of $\mathcal{B}$, and*

$$\left\|(\lambda - \mathcal{B})^{-1}\right\| \leq \frac{M_0}{|\lambda - a|}, \quad \textit{for any } \lambda \in S_{a,\phi}.$$

Recall that for a sectorial operator $\mathcal{B}$ one can introduce a scale of fractional power spaces $X^\alpha = Range(\mathcal{B}^{-\alpha})$, such that $X = X^0$ and $X^1 = Dom(\mathcal{B})$, equipped with the norm $\|y\| = \|\mathcal{B}^\alpha y\|$, where $\mathcal{B}^\alpha$ is a fractional power of $\mathcal{B}$ $(\alpha > 0)$.

Theorem 1.3.2 (Henry, [30])**.** *Assume that $\mathcal{B}$ is a sectorial operator in a Hilbert space X, $0 \leq \alpha < 1$ and $f : \mathcal{U} \to X$, with $\mathcal{U}$ an open subset of $\mathbb{R} \times X^\alpha$ and $f(\tau, y)$ a locally Hölder continuous function in τ and locally Lipschitzian in y. Then, for any $(\tau_0, y_0) \in \mathcal{U}$ there exists $T_0 = T_0(\tau_0, y_0) > 0$, such that the initial value nonlinear PDE problem:*

$$\begin{cases} \dfrac{dy}{d\tau} + \mathcal{B}y = f(\tau, y)\,, & \tau > \tau_0, \\ y(\tau_0) = y_0\,, \end{cases} \tag{1.52}$$

has a unique solution y on $(\tau_0, \tau_0 + T_0)$.

In order to apply the theorem, we will consider $X = L^2(\mathbb{R})$, $X^\alpha = H^1(\mathbb{R})$ with $\alpha = 1/2$ and $\mathcal{U} = \left(0, \frac{\sigma^2 T}{2}\right) \times H^1(\mathbb{R})$. Next, we will prove that the operator $-\frac{\partial^2}{\partial x^2}$ is a sectorial operator and that the function h satisfies the conditions assumed by f in the previous theorem. For the first purpose, we first recall a lemma from [30].

Let Δ_D denote the closure of the Laplacian operator.

Lemma 1.3.3 (Henry, [30])**.** *The operator -Δ_D is a sectorial operator in $L^2(\mathbb{R}^n)$.*

Therefore, by Lemma 1.3.3 the operator $-\dfrac{\partial^2}{\partial^2 x}$ is a sectorial operator in $L^2(\mathbb{R})$.

Proposition 1.3.4. *For $\gamma < -\dfrac{1}{2} - \dfrac{r_R}{\sigma^2}$ in the case of a call option and $\gamma > \dfrac{1}{2} - \dfrac{r_R}{\sigma^2}$ in the case of a put option, the function $J : \mathcal{U} \to X$ given by (1.51) is well defined, is locally Hölder continuous in τ and locally Lipschitzian in φ.*

Proof. Note that function V is given by the classical Black–Scholes formula for European call or put options. Thus, depending on the kind of option we have:

- for a call option:

$$V(\tau, x) = K \exp(x) \exp\left(-D_0 \frac{2}{\sigma^2}\tau\right) N(d_1^*) - K \exp\left(-r\frac{2}{\sigma^2}\tau\right) N(d_2^*)\,,$$

- for a put option:

$$V(\tau, x) = K \exp\left(-r\frac{2}{\sigma^2}\tau\right) N(-d_2^*) - K \exp(x) \exp\left(-D_0 \frac{2}{\sigma^2}\tau\right) N(-d_1^*)\,,$$

where

$$d_1^* = \frac{x + (r - D_0 + \sigma^2/2)\frac{2}{\sigma^2}\tau}{\sqrt{2\tau}}\,, \qquad d_2^* = \frac{x + (r - D_0 - \sigma^2/2)\frac{2}{\sigma^2}\tau}{\sqrt{2\tau}}\,, \tag{1.53}$$

with $D_0 = r - r_R$ and $N(x)$ represents the distribution function of the standard $\mathcal{N}(0,1)$ random variable.

In order to prove that $J(\tau, \varphi) \in L^2(\mathbb{R})$, we need to study the behaviour of function $J(\tau, \varphi)$ in the whole domain. For this purpose, we rewrite function $J(\tau, \varphi)$ as follows

$$J(\tau, \varphi) = J_1(\tau, \varphi) + J_2(\tau, \varphi)\,,$$

where

$$J_1(\tau,\varphi)(x) = \gamma^2\varphi(x) - 2\gamma\frac{\partial\varphi}{\partial x}(x) \quad \text{and} \quad J_2(\tau,\varphi)(x) = e^{\gamma x}h(\tau, e^{-\gamma x}\varphi(x))\,.$$

Due to $\varphi \in H^1(\mathbb{R})$, then $J_1(\tau,\varphi) \in L^2(\mathbb{R})$. Next, we need to prove that function $J_2(\tau,\varphi) \in L^2(\mathbb{R})$.

$$\begin{aligned}
J_2(\tau,\varphi)(x) &= e^{\gamma x}h(\tau, e^{-\gamma x}\varphi(x)) \\
&= -\exp(\gamma x)\exp\left(-\frac{1}{2}\left(1-\frac{2r_R}{\sigma^2}\right)x + \left[\left(1-\frac{2r_R}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}\right]\tau\right) \\
&\quad \times \frac{2}{K\sigma^2}\Big[(1-R_B)\lambda_B\mathcal{G}(\tau, e^{-\gamma x}\varphi)^- + [(1-R_C)\lambda_C + s_F]\,\mathcal{G}(\tau, e^{-\gamma x}\varphi)^+\Big] \\
&= -\frac{2}{K\sigma^2}\Bigg[(1-R_B)\lambda_B\left(\exp\left(\Theta_1 x + \Theta_2\tau\right)V(\tau, Ke^x) + K\varphi(x)\right)^- \\
&\quad + [(1-R_C)+s_F]\left(\exp\left(\Theta_1 x + \Theta_2\tau\right)V(\tau, Ke^x) + K\varphi(x)\right)^+\Bigg] \qquad (1.54)
\end{aligned}$$

with

$$\Theta_1 = \gamma - \frac{1}{2} + \frac{r_R}{\sigma^2}\,, \qquad \Theta_2 = \left(1-\frac{2r_R}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}\,.$$

So, a choice to prove that $J_2(\tau,\varphi) \in L^2(\mathbb{R})$ consists in proving that $J_3(\tau,\varphi) \in L^2(\mathbb{R})$, with

$$J_3(\tau,\varphi)(x) = \exp\left(\Theta_1 x + \Theta_2\tau\right)V(\tau, Ke^x)\,.$$

Thus, we will study the limits of $J_3(\tau,\varphi)$ when $x \to \pm\infty$. First note that depending on the option type and taking into account the behaviour of the terms in the Black–Scholes solution, we deduce:

- When $x \to \infty$,

$$\begin{aligned}
d_1^* \to \infty \quad &\Longrightarrow \quad N(d_1^*) \to 1, \quad N(-d_1^*) \to 0 \\
d_2^* \to \infty \quad &\Longrightarrow \quad N(d_2^*) \to 1, \quad N(-d_2^*) \to 0\,.
\end{aligned}$$

  and we obtain:

– For a call option,

$$J_3(\tau,\varphi)(x) = K\exp\left((\Theta_1+1)x + \left(\Theta_2 - D_0\frac{2}{\sigma^2}\right)\tau\right)N(d_1^*)$$
$$- K\exp\left(\Theta_1 x + \left(\Theta_2 - r\frac{2}{\sigma^2}\right)\tau\right)N(d_2^*)\,.$$

Thus, if we impose $\Theta_1 + 1 < 0$ and $\Theta_1 < 0$, then $e^{(\Theta_1+1)x}$ and $e^{\Theta_1 x}$ tend to zero. Thus, for $\gamma < -\frac{1}{2} - \frac{r_R}{\sigma^2}$, we deduce that function $J_3(\tau,\varphi) \in L^2(\mathbb{R})$.

– For a put option,

$$J_3(\tau,\varphi)(x) = K\exp\left(\Theta_1 x + \left(\Theta_2 - r\frac{2}{\sigma^2}\right)\tau\right)N(-d_2^*)$$
$$- K\exp\left((\Theta_1+1)x + \left(\Theta_2 - D_0\frac{2}{\sigma^2}\right)\tau\right)N(-d_1^*)\,.$$

In this case, $J_3(\tau,\varphi)(x) \to 0$ for all $\gamma \in \mathbb{R}$. Thus, $J_3(\tau,\varphi) \in L^2(\mathbb{R})$ for all $\gamma \in \mathbb{R}$.

- When $x \to -\infty$,

$$d_1^* \to -\infty \quad \Longrightarrow \quad N(d_1^*) \to 0, \quad N(-d_1^*) \to 1$$
$$d_2^* \to -\infty \quad \Longrightarrow \quad N(d_2^*) \to 0, \quad N(-d_2^*) \to 1\,.$$

– For a call option,

$$J_3(\tau,\varphi)(x) = K\exp\left((\Theta_1+1)x + \left(\Theta_2 - D_0\frac{2}{\sigma^2}\right)\tau\right)N(d_1^*)$$
$$- K\exp\left(\Theta_1 x + \left(\Theta_2 - r\frac{2}{\sigma^2}\right)\tau\right)N(d_2^*)\,.$$

In this case, $J_3(\tau,\varphi)(x) \to 0$ thus $J_3(\tau,\varphi) \in L^2(\mathbb{R})$ for all $\gamma \in \mathbb{R}$.

– For a put option,

$$J_3(\tau,\varphi)(x) = K\exp\left(\Theta_1 x + \left(\Theta_2 - r\frac{2}{\sigma^2}\right)\tau\right)N(-d_2^*)$$
$$- K\exp\left((\Theta_1+1)x + \left(\Theta_2 - D_0\frac{2}{\sigma^2}\right)\tau\right)N(-d_1^*)\,.$$

Choosing Θ_1 such that $\Theta_1 + 1 > 0$ and $\Theta_1 > 0$, we get $e^{(\Theta_1+1)x} \to 0$ and $e^{\Theta_1 x} \to 0$. Thus, $J_3(\tau,\varphi)(x) \to 0$ which means that $J_3(\tau,\varphi) \in L^2(\mathbb{R})$ for $\gamma > \dfrac{1}{2} - \dfrac{r_R}{\sigma^2}$.

In the previous results we have used that $N(d) \to 0$ faster than $e^x \to \infty$ when $x \to \infty$ or $x \to -\infty$.

Hence, $J_2(\tau,\varphi) \in L^2(\mathbb{R})$ if $\gamma < -\dfrac{1}{2} - \dfrac{r_R}{\sigma^2}$ for an European call option and $\gamma > \dfrac{1}{2} - \dfrac{r_R}{\sigma^2}$ for an European put option. Therefore, under this assumptions on γ, $J(\tau,\cdot): H^1(\mathbb{R}) \to L^2(\mathbb{R})$ is well defined.

Next, we will prove that J is locally Lipschitz in φ, i.e.

$$\|J(\tau,\varphi_1) - J(\tau,\varphi_2)\|_{L^2(\mathbb{R})} \le L_J \|\varphi_1 - \varphi_2\|_{H^1(\mathbb{R})}, \quad \text{for all } \varphi_1, \varphi_2 \in H^1(\mathbb{R}).$$

For this purpose, we estimate the difference

$$\begin{aligned}
\Big|J(\tau,\varphi_1)(x) - J(\tau,\varphi_2)(x)\Big| &= \Big|\gamma^2\varphi_1(x) - 2\gamma\frac{\partial\varphi_1}{\partial x}(x) + e^{\gamma x}h(\tau, e^{-\gamma x}\varphi_1)(x) \\
&\quad -\Big(\gamma^2\varphi_2(x) - 2\gamma\frac{\partial\varphi_2}{\partial x}(x) + e^{\gamma x}h(\tau, e^{-\gamma x}\varphi_2)(x)\Big)\Big| \\
&\le \gamma^2 |\varphi_1(x) - \varphi_2(x)| + 2\gamma\left|\frac{\partial\varphi_1}{\partial x}(x) - \frac{\partial\varphi_2}{\partial x}(x)\right| \\
&\quad + e^{\gamma x}L_h\left|e^{-\gamma x}\varphi_1(x) - e^{-\gamma x}\varphi_2(x)\right| \\
&\le (\gamma^2 + L_h)|\varphi_1(x) - \varphi_2(x)| + 2\gamma\left|\frac{\partial\varphi_1(x)}{\partial x} - \frac{\partial\varphi_2(x)}{\partial x}\right|,
\end{aligned}$$

where we have used the fact that $\left|\chi_1^+ - \chi_2^+\right| \le |\chi_1 - \chi_2|$ and $\left|\chi_1^- - \chi_2^-\right| \le |\chi_1 - \chi_2|$, with $\chi_i = V(\tau,\cdot) + Ke^{\left(\frac{1}{2} - \frac{r_R}{\sigma^2} - \gamma\right)x - \left[\left(1 - \frac{2r_R}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}\right]\tau}\varphi_i$. Moreover, we introduced the constant

$$L_h = \frac{2}{\sigma^2}\Big(|(1-R_B)\lambda_B| + |(1-R_C)\lambda_C + s_F|\Big).$$

Then, by integration we get

$$\begin{aligned}
\int_{\mathbb{R}} |J(\tau,\varphi_1)(x) - J(\tau,\varphi_2)(x)|^2\,dx &\le (\gamma^2 + L_h)^2\int_{\mathbb{R}} |\varphi_1(x) - \varphi_2(x)|^2\,dx \\
&\quad + (2\gamma)^2\int_{\mathbb{R}}\left|\frac{\partial\varphi_1}{\partial x}(x) - \frac{\partial\varphi_2}{\partial x}(x)\right|^2 dx
\end{aligned}$$

which is equivalent to

$$\|J(\tau,\varphi_1)-J(\tau,\varphi_2)\|^2_{L^2(\mathbb{R})} \le L_J^2 \|\varphi_1-\varphi_2\|^2_{H^1(\mathbb{R})} \ ,$$

where $L_J = \max\{\gamma^2 + L_h, 2\gamma\}$. Therefore, J is locally Lipschitz in the second variable φ.

Next, we prove that J is locally Lipschitz continuous in τ. Thus, for $\tau_1, \tau_2 \in \left[0, \frac{\sigma^2}{2}T\right]$, we obtain

$$\begin{aligned}
\Big|J(\tau_1,\varphi)(x) - J(\tau_2,\varphi)(x)\Big| &= \left|e^{\gamma x}(h(\tau_1, e^{-\gamma x}\varphi)(x) - h(\tau_2, e^{-\gamma x}\varphi)(x))\right| \\
&= \Bigg| -e^{(\gamma+\alpha)x}\frac{2}{K\sigma^2}\Big[(1-R_B)\lambda_B\left(\tilde{\mathcal{G}}(\tau_1,x)^- - \tilde{\mathcal{G}}(\tau_2,x)^-\right) \\
&\quad + [(1-R_C)\lambda_C + s_F]\left(\tilde{\mathcal{G}}(\tau_1,x)^+ - \tilde{\mathcal{G}}(\tau_2,x)^+\right)\Big]\Bigg| \\
&\le \left|-e^{(\gamma+\alpha)x}\frac{2}{K\sigma^2}\right|\Big(|(1-R_B)\lambda_B|\left|V(\tau_1,\cdot)e^{-\beta\tau_1} - V(\tau_2,\cdot)e^{-\beta\tau_2}\right| \\
&\quad + |(1-R_C)\lambda_C + s_F|\left|V(\tau_1,\cdot)e^{-\beta\tau_1} - V(\tau_2,\cdot)e^{-\beta\tau_2}\right|\Big) \\
&= \mathcal{M}\left|V(\tau_1,\cdot)e^{-\beta\tau_1} - V(\tau_2,\cdot)e^{-\beta\tau_2}\right|
\end{aligned}$$

where $\tilde{\mathcal{G}}(\tau,\varphi) = V(\tau,\cdot)e^{-\beta\tau} + Ke^{(-\alpha-\gamma)x}\varphi$ and

$$\mathcal{M} = \left|-e^{(\gamma+\alpha)x}\frac{2}{K\sigma^2}\right|\Big(|(1-R_B)\lambda_B| + |(1-R_C)\lambda_C + s_F|\Big).$$

Moreover, function $e^{-\beta\tau}$ is Lipschitz continuous in τ in the interval $\left[0, \dfrac{\sigma^2}{2}T\right]$. Then, using that $V \in \mathcal{C}((0, \frac{\sigma^2}{2}T), X)$ we can apply that V is also Lipschitz continuous in τ. Therefore, in terms of the norm, we get

$$\|J(\tau_1,\varphi) - J(\tau_2,\varphi)\|^2 = \int_0^T |J(\tau_1,\varphi) - J(\tau_2,\varphi)|^2\, d\tau \le C\,\|\tau_1-\tau_2\|_{L^2\left(0,\frac{\sigma^2}{2}T\right)} \ ,$$

where $C = \mathcal{M}L_V$ and L_V is the Lipschitz constant associated to function $V(\tau,x)e^{-\beta\tau}$. As $J(\tau,\varphi)$ is Lipschitz continuous in τ, in particular it is Hölder continuous in τ.

□

Corollary 1.3.5. *For any initial condition* $u_0 \in H^1(\mathbb{R})$ *there exists* $T_0 = T_0(0, u_0) > 0$ *so that the initial value problem (1.50) has a unique solution in* $(0, T_0)$.

Corollary 1.3.5 follows from Proposition 1.3.4 and Theorem 1.3.2, and provides the existence and uniqueness of a local solution, as $T_0 = T_0(0, u_0)$ is a local time. In order to extend it to any interval $(0, T)$ for a given $T > 0$ we need to apply Corollary 3.3.5 in [30].

Proposition 1.3.6. *The following inequality holds:*

$$\|J(\tau, \varphi)\|_{L^2(\mathbb{R})} \leq \mathcal{K}(\tau)\Big(1 + \|\varphi\|_{H^1(\mathbb{R})}\Big), \text{ for all } (\tau, \varphi) \in (0, \infty) \times H^1(\mathbb{R}), \qquad (1.55)$$

where $\mathcal{K}$ *is continuous in* $(0, \infty)$. *Therefore, there exists a unique solution of problem (1.50) defined on the entire time interval* $\Big(0, \frac{\sigma^2}{2} T\Big]$.

Proof. First, we note that the Lipschitz continuity properties also hold for $\tau \in (0, \infty)$ and prove the inequality (1.55). Thus, for any $(\tau, \varphi) \in (0, \infty) \times H^1(\mathbb{R})$ we have

$$\begin{aligned}
\|J(\tau, \varphi)\|_{L^2(\mathbb{R})} &\leq \|J(\tau, \varphi) - J(\tau, 0)\|_{L^2(\mathbb{R})} + \|J(\tau, 0)\|_{L^2(\mathbb{R})} \\
&\leq L_J \|\varphi - 0\|_{H^1(\mathbb{R})} + \|J(\tau, 0)\|_{L^2(\mathbb{R})} \\
&= \Big(L_J + \|J(\tau, 0)\|_{L^2(\mathbb{R})}\Big)\Big(\|\varphi\|_{H^1(\mathbb{R})} + 1\Big),
\end{aligned}$$

where L_J is the Lipschitz constant for J, so that we can take

$$K(\tau) = L_J + \|J(\tau, 0)\|_{L^2(\mathbb{R})}$$

which is continuous in τ on $(0, \infty)$.

Next, we can apply Corollary 3.3.5 in [30]. Thus, we consider $u(\tau, \cdot)$ as the unique solution of (1.50) at time $\tau_0 = T_0/2$ obtained from Corollary 1.3.5, so that from Corollary 3.3.5 in [30], the unique solution of (1.50) through $(\tau_0, u(\tau_0, \cdot))$ exists for all $\tau \geq \tau_0$. Therefore, we obtain existence and uniqueness of solution of (1.50) in $\Big(0, \frac{\sigma^2}{2} T\Big]$.

□

Corollary 1.3.7. *There exists a unique solution of Problem (1.15)*

Proof. It follows from the existence and uniqueness of solution of the equivalent problem (1.50). □

1.4 Numerical methods

In order to solve the previous models, in this section different numerical methods are proposed. We will mainly focus on nonlinear problems, similar methods being used in the corresponding linear ones. Moreover, we only develop the problem with collateral, as we can consider the model without collateral as a particular case.

On one hand, as the initial domain of the problem is unbounded in variable S, a localization procedure to define a suitable finite domain is required and adequate boundary conditions are deduced and implemented. On the other hand, the time discretization is made using a semi Lagrangian method combined with a piecewise linear finite element spatial discretization.

The previous set of numerical methods is proposed to solve problem (1.39), the solution of which is the adjustment value considering CVA, DVA, FCA and CollVA.

In order to state the problem (1.39) as an equivalent initial value problem, the change of time variable $\tau = T - t$ is considered, then (1.39) is transformed into the following forward in time problem:

$$\begin{cases} \dfrac{\partial U}{\partial \tau} - \dfrac{\sigma^2}{2} S^2 \dfrac{\partial^2 U}{\partial S^2} - r_R S \dfrac{\partial U}{\partial S} + rU = -\lambda_B h_e - s_X X \\ \qquad\qquad -(1-R_B)\lambda_B (V+U-X)^- - (1-R_C)\lambda_C (V+U-X)^+ \\ U(0,S) = 0\,. \end{cases} \tag{1.56}$$

Moreover, as we propose to solve (1.56) by a finite element method, we write it in divergencial form:

$$\begin{aligned} &\frac{\partial U}{\partial \tau} - \frac{\partial}{\partial S}\left(\frac{\sigma^2}{2} S^2 \frac{\partial U}{\partial S}\right) + (\sigma^2 - r_R) S \frac{\partial U}{\partial S} + rU = -\lambda_B h_e \\ &\qquad - (1-R_B)\lambda_B (V+U-X)^- - (1-R_C)\lambda_C (V+U-X)^+ - s_X X\,. \end{aligned} \tag{1.57}$$

1.4.1 Method of characteristics

Analogously to other advection–diffusion equations, we propose a semi–Lagrangian discretization combined with finite elements. More precisely, for time discretization we use a characteristics method first proposed in financial setting in [50]. It is based on a finite difference scheme for the discretization of the material derivative, i.e., the time derivative along the characteristic lines. For this purpose, we consider the material derivative of function U:

$$\frac{DU}{D\tau} = \frac{\partial U}{\partial \tau} + \frac{\partial U}{\partial S}\frac{dS}{d\tau}$$

for a given function $S = S(\tau)$. Thus, we can write equation (1.57) as:

$$\begin{aligned}\frac{DU}{D\tau} - \frac{\sigma^2}{2}\frac{\partial}{\partial S}\left(S^2\frac{\partial U}{\partial S}\right) + rU &= -\lambda_B h_e - (1-R_B)\lambda_B(V+U-X)^- \\ &\quad - (1-R_C)\lambda_C(V+U-X)^+ - s_X X\,. \end{aligned} \tag{1.58}$$

We will call velocity the coefficient of the advective term in (1.57), i.e. $(\sigma^2 - r_R)S$. Then, we introduce $N_T > 0$, a time step $\Delta\tau = T/N_T$, the time discretization given by $\tau^n = n\Delta\tau$ for $n = 0, 1, 2, \dots, N_T$ and the final value ODE problem:

$$\begin{cases} \dfrac{\partial \chi}{\partial \tau} = (\sigma^2 - r_R)\chi(\tau) \\ \chi(\tau^{n+1}) = S\,, \end{cases} \tag{1.59}$$

the analytical solution of which is:

$$\chi(S, \tau^{n+1}; \tau^n) = S\exp((r_R - \sigma^2)\Delta\tau)$$

for $n = 0, 1, \dots, N_T - 1$. Note that function χ represents the characteristic curve associated to the velocity passing through point S at time τ^{n+1}.

We approximate the material derivative in (1.58) by a first order quotient, so that equation (1.58) is approximated by:

$$\begin{aligned}\frac{U^{n+1} - U^n \circ \chi^n}{\Delta\tau} &- \frac{\sigma^2}{2}\frac{\partial}{\partial S}\left(S^2\frac{\partial U^{n+1}}{\partial S}\right) + rU^{n+1} \\ &= -\lambda_B h_e - (1-R_B)\lambda_B(V+U^{n+1}-X)^- \\ &\quad - (1-R_C)\lambda_C(V+U^{n+1}-X)^+ - s_X X. \end{aligned} \tag{1.60}$$

We can evaluate $U^n \circ \chi^n$ at each step of (1.60) in the mesh points by piecewise linear interpolation.

1.4.2 Fixed point scheme

In order to solve the nonlinear equation (1.60) at each iteration of the characteristics method, we propose a fixed point algorithm. Thus, the global scheme can be written in the following way:

Algorithm 1.1

---

1. Let $N_T > 1$, $\varepsilon > 0$, U^0 given.
2. For $n = 0, 1, 2, \ldots, N_T - 1$
   - Let $U^{n+1,0} = U^n$
   - For $k = 0, 1, 2, \ldots,$ we compute $U^{n+1,k+1}$ satisfying:

$$\begin{aligned}(1 + r\Delta\tau)\, U^{n+1,k+1} - \frac{\sigma^2\Delta\tau}{2}\frac{\partial}{\partial S}\left(S^2\frac{\partial U^{n+1,k+1}}{\partial S}\right) \\ = U^n \circ \chi^n - \Delta\tau\left[\lambda_B h_e + (1 - R_B)\lambda_B(V^{n+1} + U^{n+1,k} - X)^-\right. \\ \left. +(1 - R_C)\lambda_C(V^{n+1} + U^{n+1,k} - X)^+ + s_X X\right] \qquad (1.61)\end{aligned}$$

   until $\dfrac{\|U^{n+1,k+1} - U^{n+1,k}\|}{\|U^{n+1,k+1}\|} < \varepsilon.$

1.4.3 Boundary conditions

As previously indicated, we will use a finite element method to discretize the previous equations and approximate the solution. Thus, we need to truncate the unbounded domain $[0, +\infty)$ into a bounded one, so that the solution is not affected by the truncation in the region of financial interest. We will assume $S \in [0, S_\infty]$, where $S_\infty > 0$ is a large enough value; a typical choice in financial problems is $S_\infty = 4K$ where K represents the strike of the option.

Next, we deduce the boundary conditions from the partial differential equation. More precisely, let us introduce function f, defined by:

$$f(\widehat{V}) = \lambda_B h_e + (1-R_B)\lambda_B(\widehat{V}-X)^- + (1-R_C)\lambda_C(\widehat{V}-X)^+ + s_X X\,, \qquad (1.62)$$

representing the right hand side of (1.60).

The boundary condition at $S = 0$ is obtained just by replacing $S = 0$ in (1.56). Thus, we obtain the nonlinear ODE:

$$\partial_\tau U + rU = -f(U+V)\,.$$

This equation is discretized by a characteristics (in this case, equivalent to an implicit Euler) method combined with a fixed point scheme:

$$U^{n+1,k+1}(0) - U^n(0) + r\Delta\tau\, U^{n+1,k+1}(0) = -\Delta\tau\, f(U^{n+1,k}(0) + V^{n+1}(0))\,,$$

for $k \geq 0$ and $n \geq 0$, so that a nonhomogeneous Dirichlet boundary condition is obtained at each step of the global algorithm:

$$\begin{aligned} U^{n+1,k+1}(0) = \frac{1}{1+r\Delta\tau}\Big(U^n(0) - \Delta\tau\Big[&\lambda_B h_e \\ &+ (1-R_B)\lambda_B(V^{n+1}(0) + U^{n+1,k}(0) - X)^- \\ &+ (1-R_C)\lambda_C(V^{n+1}(0) + U^{n+1,k}(0) - X)^+ + s_X X\Big]\Big)\,. \end{aligned} \qquad (1.63)$$

In order to deduce the boundary condition at $S = S_\infty$, we first multiply equation (1.56) by S^{-2}. Next, by taking the limit when S tends to infinity the following property is obtained:

$$\lim_{S\to\infty} \frac{\partial^2 U}{\partial S^2} = 0\,. \qquad (1.64)$$

Then, following [19], when $S \to \infty$ we consider a solution of the form:

$$U = H_0(\tau) + H_1(\tau)S\,, \qquad (1.65)$$

where $H_0(\tau)$ and $H_1(\tau)$ are coefficients not depending on S. Next, by assuming $S^2\dfrac{\partial^2 U}{\partial S^2} \to 0$ when $S \to \infty$ in (1.56) we have

$$\frac{\partial U}{\partial \tau} - r_R S\frac{\partial U}{\partial S} + rU = -f(U+V) \qquad (1.66)$$

when $S \to \infty$.

Discretizing (1.66) by the characteristic curve, we have:

$$(1 + r\Delta\tau)\, U^{n+1,k+1} = U^n \circ \chi^n - \Delta\tau\, f(U^{n+1,k} + V^{n+1}) \tag{1.67}$$

where $\chi^n \equiv \chi(S, \tau^{n+1}; \tau^n)$ is solution of the final value problem

$$\begin{cases} \dfrac{d\chi}{d\tau} = -r_R \chi(\tau) \\ \chi(\tau^{n+1}) = S\,. \end{cases} \tag{1.68}$$

Thus, the characteristic curve is given by $\chi(S, \tau^{n+1}; \tau^n) = S \exp(r_R \Delta\tau)$.

Introducing the expression (1.65) into each fixed point iteration (1.67), we obtain:

$$\begin{aligned} &(1 + r\Delta\tau)\,(H_0^{n+1,k+1} + H_1^{n+1,k+1} S_\infty) \\ &\quad = U^n \circ \chi^n - \Delta\tau \left[\lambda_B h_e + (1 - R_B)\lambda_B (V^{n+1} + U^{n+1,k} - X)^- \right. \\ &\qquad \left. + (1 - R_C)\lambda_C (V^{n+1} + U^{n+1,k} - X)^+ + s_X X\right]\,. \end{aligned} \tag{1.69}$$

If we choose $H_0^{n+1,k+1} = 0$, a nonhomogeneous Dirichlet boundary condition is deduced:

$$\begin{aligned} U^{n+1,k+1}(S_\infty) &= H_1^{n+1,k+1} S_\infty \\ &= \frac{1}{(1 + r\Delta\tau)} \Big((U^n \circ \chi^n)(S_\infty) \\ &\quad - \Delta\tau \left[\lambda_B h_e + (1 - R_B)\lambda_B (V^{n+1}(S_\infty) + U^{n+1,k}(S_\infty) - X)^- \right. \\ &\quad \left. + (1 - R_C)\lambda_C (V^{n+1}(S_\infty) + U^{n+1,k}(S_\infty) - X)^+ + s_X X\right] \Big)\,. \end{aligned} \tag{1.70}$$

Thus, (1.63) and (1.70) are evaluated at each iteration of the fixed point algorithm as a previous step to the stating of the linear system of equations issued from the finite element method.

1.4.4 Finite element method

As we mention at the beginning of the section, we use the semi–Lagrangian method for the time discretization jointly with finite elements for the spatial discretization.

Therefore, at each time step, $n = 0, 1, 2, \dots, N_T - 1$, and each fixed point iteration, $k = 0, 1, 2, \dots$, a variational formulation for (1.61) is posed: find $U^{n+1,k+1} \in H^1(0, S_\infty)$ such that:

$$
\begin{aligned}
&(1 + r\Delta\tau) \int_0^{S_\infty} U^{n+1,k+1} \varphi dS - \Delta\tau \int_0^{S_\infty} \frac{\partial}{\partial S} \left(\frac{\sigma^2}{2} S^2 \frac{\partial U^{n+1,k+1}}{\partial S} \right) \varphi dS \\
&= \int_0^{S_\infty} (U^n \circ \chi^n)(S) \varphi dS - \Delta\tau \int_0^{S_\infty} f(U^{n+1,k} + V^{n+1}) \varphi dS \,, \;\; \forall \varphi \in H_0^1(0, S_\infty) \,,
\end{aligned}
$$

or, after applying Green's theorem,

$$
\begin{aligned}
&(1 + r\Delta\tau) \int_0^{S_\infty} U^{n+1,k+1} \varphi dS + \Delta\tau \frac{\sigma^2}{2} \int_0^{S_\infty} S^2 \frac{\partial U^{n+1,k+1}}{\partial S} \frac{\partial \varphi}{\partial S} dS \\
&= \int_0^{S_\infty} (U^n \circ \chi^n)(S) \varphi dS - \Delta\tau \int_0^{S_\infty} f(U^{n+1,k} + V^{n+1}) \varphi dS \,, \;\; \forall \varphi \in H_0^1(0, S_\infty) \,.
\end{aligned}
$$

For a fixed natural number $N_S > 0$, we consider a uniform mesh of the computational domain $\Omega = [0, S_\infty]$, the nodes of which are $S_j = j\Delta S, \; j = 0, \dots N_S + 1$, where $\Delta S = S_\infty/(N_S + 1)$ denotes the constant mesh step. Associated to this uniform mesh a piecewise linear Lagrange finite element discretization is considered.

More precisely, we search $U_h^{n+1,k+1} \in W_h$ such that, for all $\varphi_h \in W_{h,0}$,

$$
\begin{aligned}
&(1 + r\Delta\tau) \int_0^{S_\infty} U_h^{n+1,k+1} \varphi_h dS + \Delta\tau \frac{\sigma^2}{2} \int_0^{S_\infty} S^2 \frac{\partial U_h^{n+1,k+1}}{\partial S} \frac{\partial \varphi_h}{\partial S} dS \\
&= \int_0^{S_\infty} (U_h^n \circ \chi^n)(S) \varphi_h dS - \Delta\tau \int_0^{S_\infty} f(U_h^{n+1,k} + V^{n+1}) \varphi_h dS \,, \qquad (1.71)
\end{aligned}
$$

where the finite element spaces are

$$
\begin{aligned}
W_h &= \{\varphi_h : (0, S_\infty) \to \mathbb{R} / \varphi_h \in \mathcal{C}(0, S_\infty), \varphi_h|_{[S_j, S_{j+1}]} \in \mathcal{P}_1\} \,, \\
W_{h,0} &= \{\varphi_h \in W_h / \varphi_h(0) = 0, \varphi_h(S_\infty) = 0\} \,,
\end{aligned}
$$

$\mathcal{P}_1$ being the space of polynomials of degree less or equal than one.

The coefficients of the matrix and right hand side vector defining the linear system associated to the fully discretized problem are approximated by adequate quadrature formulae. In particular, Simpson, three nodes Gaussian, midpoint and trapezoidal formulae have been used for the different terms, depending on the degree of the resulting polynomials to be integrated in each term. Finally, the system of linear equations is solved by a partial pivoting LU factorization method. The implementation has taken into account the sparse structure of the global matrices.

Remark 1.4.1. *The value of the derivative without counterparty risk, V^{n+1}, can be obtained at each time step as the solution of the Black–Scholes equation for options with dividends:*

$$\begin{cases} \dfrac{\partial V}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 V}{\partial S^2} + (r - D_0) S \dfrac{\partial V}{\partial S} - rV = 0 & \textit{in} \quad [0,T) \times [0,\infty) \\ V(T,S) = H(S) & S > 0\,, \end{cases} \tag{1.72}$$

where $D_0 \equiv r - r_R$. Thus, depending on the type of financial derivative we have different payoff functions. In some cases, the value of the derivative admits an analytical expression. For example, in the three cases here treated these expressions come from the well–known formulae:

- *Call option:*

$$V(t,S) = S\exp(-D_0(T-t))N(d_1) - K\exp(-r(T-t))N(d_2)$$

- *Put option:*

$$V(t,S) = K\exp(-r(T-t))N(-d_2) - S\exp(-D_0(T-t))N(-d_1)$$

- *Forward:*

$$\begin{aligned} V(t,S) = S\exp&\left(\left(\frac{\sigma^2}{4} + \frac{r_R^2}{\sigma^2} - r\right)(T-t)\right) \\ &- K\exp\left(\left(\sigma^2\left(\frac{r_R}{\sigma^2} - \frac{1}{2}\right)^2 - r\right)(T-t)\right) \end{aligned}$$

where:

$$d_1 = \frac{\log(S/K) + (r - D_0 + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$
$$d_2 = \frac{\log(S/K) + (r - D_0 - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$

and $N(x)$ represents the distribution function of the standard $\mathcal{N}(0,1)$ random variable. Equivalent expressions to the first two formulae had been introduced in Proposition 1.3.4 in terms of variables x and τ.

1.5 A Monte Carlo method

In the previous sections, the problems which model the total value adjustment associated with the European options have been posed. Moreover, some numerical methods based on finite element method for spatial discretization and semi–Lagrangian method for time discretization have been proposed to solve the PDEs in a numerical way.

Nevertheless, it is usual to obtain the total value adjustment in terms of expectations [12, 42]. In this section, we apply Monte Carlo simulation technique to compute the total value adjustment for European options depending on one stochastic factor.

We assume that the price, S_t, follows a general geometric Brownian motion, thus satisfying:

$$dS_t = r_R S_t dt + \sigma S_t dW_t, \tag{1.73}$$

where r_R and σ have been described in Section 1.2.1 as the rate paid for the underlying asset in a repurchase agreement and the volatility of the price, respectively, and W_t denotes a Wiener process.

Next, we focus on the problem without collateral; the case with collateral can be computed by a similar procedure. As in Section 1.2 we distinguish two cases depending on the mark–to–market value at default, M: the risky derivative value leading to problem (1.15), or the risk–free value leading to problem (1.16).

Using Feynman–Kac theorem, we can obtain the expected value of the XVA, U, from the partial differential equations which model the adjustments. Then, the total value adjustment at the time instant t is given by the following expressions:

- If $M = \widehat{V}$,

$$U(t,S_t) = \mathbb{E}_t\Bigg[-\int_t^T e^{-\int_t^u r d\tau}\big[(1-R_B)\lambda_B(V(u,S(u))+U(u,S(u)))^-$$
$$+(1-R_C)\lambda_C(V(u,S(u))+U(u,S(u)))^+$$
$$+s_f(V(u,S(u))+U(u,S(u)))^+\big]\,du \mid S_t = s\Bigg]$$
$$= -\mathbb{E}_t\left[(1-R_B)\lambda_B\int_t^T e^{-r(u-t)}(V(u,S(u))+U(u,S(u)))^- du \mid S_t = s\right]$$
$$-\mathbb{E}_t\left[(1-R_C)\lambda_C\int_t^T e^{-r(u-t)}(V(u,S(u))+U(u,S(u)))^+ du \mid S_t = s\right]$$
$$-\mathbb{E}_t\left[s_F\int_t^T e^{-r(u-t)}(V(u,S(u))+U(u,S(u)))^+ du \mid S_t = s\right].$$

We are interested in finding the value of the adjustment at the initial time, when the derivative is priced. Then, the XVA value at current time, $t = 0$, is given by:

$$U(0,S_0) = \mathbb{E}_0\Bigg[-\int_0^T e^{-\int_0^u r d\tau}\big[(1-R_B)\lambda_B(V(u,S(u))+U(u,S(u)))^-$$
$$+(1-R_C)\lambda_C(V(u,S(u))+U(u,S(u)))^+$$
$$+s_f(V(u,S(u))+U(u,S(u)))^+\big]\,du \mid S_0 = s\Bigg]$$
$$= -\mathbb{E}_0\left[(1-R_B)\lambda_B\int_0^T e^{-ru}(V(u,S(u))+U(u,S(u)))^- du \mid S_0 = s\right]$$
$$-\mathbb{E}_0\left[(1-R_C)\lambda_C\int_0^T e^{-ru}(V(u,S(u))+U(u,S(u)))^+ du \mid S_0 = s\right]$$
$$-\mathbb{E}_0\left[s_F\int_0^T e^{-ru}(V(u,S(u))+U(u,S(u)))^+ du \mid S_0 = s\right].$$

- If $M = V$,

$$U(t,S_t) = \mathbb{E}_t\Bigg[-\int_t^T e^{-\int_t^u (r+\lambda_B+\lambda_C)d\tau}\big[(1-R_B)\lambda_B V(u,S(u))^-$$

$$
\begin{aligned}
&+(1-R_C)\lambda_C V(u,S(u))^+ + s_f V(u,S(u))^+\big]\, du \mid S_t = s\Bigg] \\
=& -\mathbb{E}_t\left[\int_t^T e^{-(r+\lambda_B+\lambda_C)(u-t)}(1-R_B)\lambda_B V(u,S(u))^- du \mid S_t = s\right] \\
& -\mathbb{E}_t\left[\int_t^T e^{-(r+\lambda_B+\lambda_C)(u-t)}(1-R_C)\lambda_C V(u,S(u))^+ du \mid S_t = s\right] \\
& -\mathbb{E}_t\left[\int_t^T e^{-(r+\lambda_B+\lambda_C)(u-t)} s_F V(u,S(u))^+ du \mid S_t = s\right].
\end{aligned}
$$

Thus, the value at current time $t = 0$ is given by:

$$
\begin{aligned}
U(0,S) =& \mathbb{E}_0\Bigg[-\int_0^T e^{-\int_0^u (r+\lambda_B+\lambda_C)d\tau}\big[(1-R_B)\lambda_B V(u,S(u))^- \\
&+(1-R_C)\lambda_C V(u,S(u))^+ + s_f V(u,S(u))^+\big]\, du \mid S_0 = s\Bigg] \\
=& -\mathbb{E}_0\left[\int_0^T e^{-(r+\lambda_B+\lambda_C)u}(1-R_B)\lambda_B V(u,S(u))^- du \mid S_0 = s\right] \\
& -\mathbb{E}_0\left[\int_0^T e^{-(r+\lambda_B+\lambda_C)u}(1-R_C)\lambda_C V(u,S(u))^+ du \mid S_0 = s\right] \\
& -\mathbb{E}_0\left[\int_0^T e^{-(r+\lambda_B+\lambda_C)u} s_F V(u,S(u))^+ du \mid S_0 = s\right].
\end{aligned}
$$

For both values of the mark–to–market, the risky derivative value or the risk–free value, the previous expression of the XVA has been split up into three terms, each one of which represents a kind of adjustment: credit value adjustment (CVA), debit value adjustment (DVA) or funding value adjustment (FVA), respectively.

In order to obtain the numerical value, a discrete approximation of the integrals which appear in the expression of the expected value has to be used. For this purpose, we consider a set of fixed points $0 = t_0 < t_1 < \ldots < t_{N_T} = T$, with T the maturity time, when the payoff is received. Taking into account the fixed instant times, we denote by $S_i = S(t_i), i = 1, 2, \ldots, N_T$, the asset price at the i–th instant of time. We approximate those values, solution of the stochastic differential equation (1.73), by the Euler–Maruyama scheme:

$$
S_i = S_{i-1} + r_R S_{i-1}\Delta t + \sigma S_{i-1}\Delta W_i\,, \quad i = 1, 2, \ldots, N_T\,,
$$

where $\Delta t = t_i - t_{i-1}$ is the size of the time interval and $\Delta W_i = W_i - W_{i-1}$ is the independent Brownian increment, which follows a normal distribution $\mathcal{N}(0, \sqrt{\Delta t})$.

Finally, in order to compute the XVA when $M = \widehat{V}$, a fixed point implementation is carried out at each time step.

1.6 Numerical results

Following the numerical methods introduced in Section 1.4, in the present one we give some numerical results which show the behaviour of the adjustment according to the asset price value. In order to illustrate the good behaviour of the proposed numerical strategy, we have first compared the results obtained in specific cases for which an analytical solution is known. Moreover, other examples in which we compute the XVA in different situations are also presented.

In the following tests we have used some common parameters, which are gathered in Tables 1.1 and 1.2.

Table 1.1: Financial data for numerical tests

$\sigma = 0.25$	$K = 15$	$T = 0.5$	$S_\infty = 4K$
$r = 0.03$	$r_R = 0.015$	$R_B = 0.4$	$R_C = 0.4$

Table 1.2: Financial data for numerical tests

$\sigma = 0.25$	$r = 0.04$	$r_R = 0.06$	$S_\infty = 20$
$T = 0.5$	$K = 10e^{r_R T}$	$\lambda_B = 0.04$	$\lambda_C = 0.04$
$R_B = 0.3$	$R_C = 0.3$	$r_{P_B} = 0.08$	$r_{P_C} = 0.08$

1.6.1 Test 1: Convergence

We first study the error and the order of convergence of the applied numerical methods, for which we take advantage of the analytical solution of the XVA problem in particular cases [15]. For example, we consider a not collateralized call option bought by B, with $M = \widehat{V}$ and funding costs. Note that as we consider $s_F = (1 - R_B)\lambda_B$, the analytical expression of the XVA is:

$$U(t, S) = -(1 - \exp(-((1 - R_B)\lambda_B + (1 - R_C)\lambda_C)(T - t)))V(t, S)\,.$$

Table 1.3: Relative errors in norm $L^\infty((0, T) \times L^2([0, S_\infty]))$, convergence ratios and order. Example with finite element scheme (Test 1). The input parameters used are from Table 1.1 and $\lambda_B = 0.02$, $\lambda_C = 0.05$

Time steps	Space steps	Error	R	Order
400	50	0.02232872	-	-
800	100	0.01192059	1.87312280	0.90544548
1600	200	0.00617545	1.93031711	0.94883787
3200	400	0.00315299	1.95860211	0.96982435
6400	800	0.00160323	1.96665313	0.97574253

As we can observe in Table 1.3, the experimental order of convergence obtained with the discrete norm $L^\infty((0, T) \times L^2([0, S_\infty]))$ is one.

In Figures 1.1, 1.2 and 1.3 we show the XVA value as a percentage of the risk–free value, V. We can observe the relevance of the choice of the mark–to–market value at default (either V or $\widehat{V}$), as well as the funding costs. These results correspond to time $t = 0$ and the set of financial parameters are taken from Table 1.1.

Notice that in the four considered cases, with and without funding costs and both possibilities of the mark–to–market value, the value of XVA grows as the default intensity of C increases. Moreover, in the cases which do not consider funding cost the XVA remains constant, independently of the changes of the default intensity of B, λ_B. Nevertheless, when funding costs are considered, the XVA increases with λ_B.

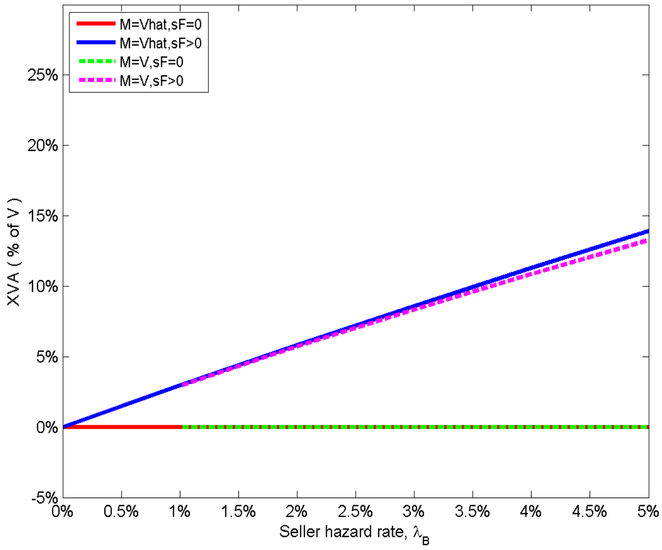


Figure 1.1: XVA in the cases $M = \widehat{V}$ and $M = V$ for counterparty hazard rate, $\lambda_C = 0\%$ (Test 1)

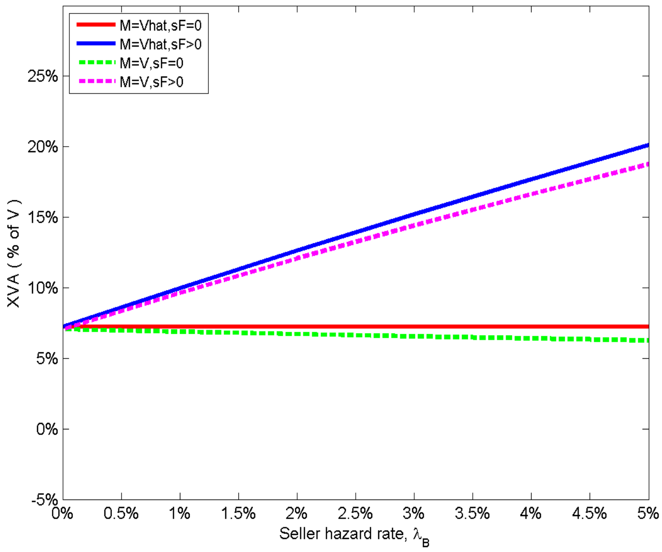


Figure 1.2: XVA in the cases $M = \widehat{V}$ and $M = V$ for counterparty hazard rate, $\lambda_C = 2.5\%$ (Test 1)

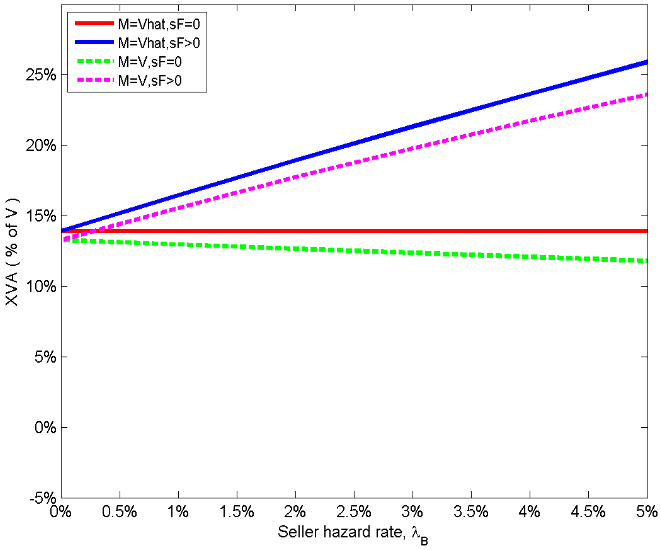


Figure 1.3: XVA in the cases $M = \widehat{V}$ and $M = V$ for counterparty hazard rate, $\lambda_C = 5\%$ (Test 1)

Concerning the fixed–point algorithm (1.61) introduced in Section 1.4.2, we have not proved its theoretical convergence. However, convergence is attained in a reduced number of iterations (less than five) in all the experiments for European options. We have used $\epsilon = 10^{-11}$ as the tolerance for the relative quadratic error between two iterations.

1.6.2 Test 2: European put option

In this example we analyze the time evolution of the CVA and FVA, in terms of the spot value. We have considered the case in which no collateral is posted in the trade.

We assume counterparty B buys a put option from C, the strike depending on the repo rate ($K = 10e^{r_R T}$), and a maturity period of 0.5 years. The rest of financial parameters are given in Table 1.2.

We have used $N_S = 600$ nodes and $N_T = 1000$ time steps. The same discretization parameters have also been used in the subsequent tests.

Figure 1.4 shows the total value adjustment for the European put option. The XVA value is negative because it represents the decrease in the risk–free put value due to the probability of default from both counterparties.

Figure 1.5 shows the credit value adjustment surface for the put option. The function takes negative values, since it represents the amount that B has to charge to C due to C's probability of default. The value is null when the option expires, because at maturity date the exposure at the counterparty default disappears. Furthermore, the absolute value is larger when the put option is in the money. In this case, B will be interested in exercising and will be (more) exposed to C's default.

Figure 1.6 represents the funding cost adjustment surface for the same European put option. The value is negative because it represents the funding costs that B charges to C; i.e., B will pay less money to C due to B's incurring in funding cost associated to the financing agreements. Thus, the FCA increases when the option is in the money, as the funding needed to pay the prime in the money is larger than if the option is out of the money.

1.6.3 Test 3: European call option and forward including funding costs

Now, according to the counterparties which take part in the agreement, we compare the risk–free value and the risky value considering and not considering funding costs.

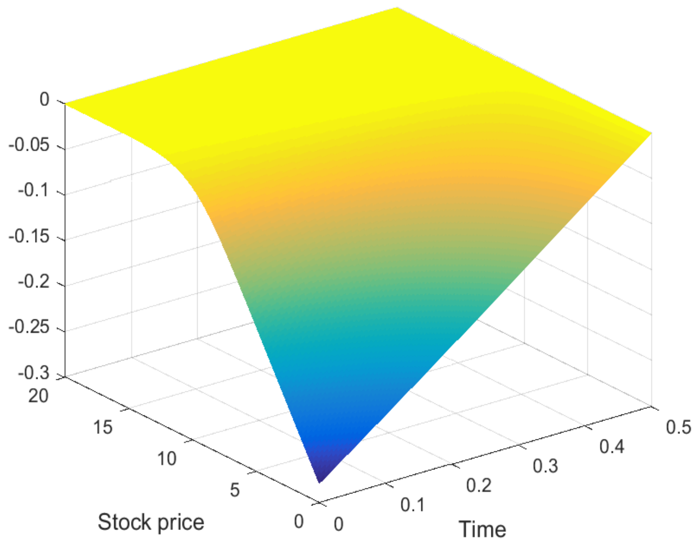


Figure 1.4: XVA surface for European put option (Test 2). Input arguments are given in Table 1.2

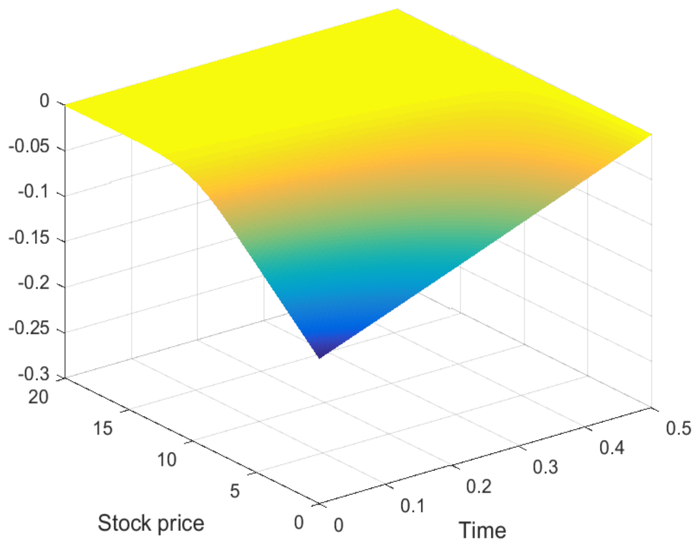


Figure 1.5: CVA+DVA surface for European put option (Test 2). Input arguments are given in Table 1.2

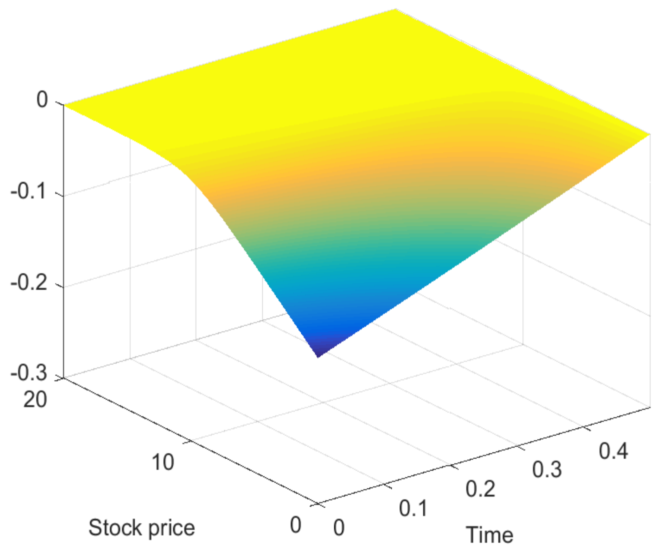


Figure 1.6: FCA surface for European put option (Test 2). Input arguments are given in Table 1.2 and $s_F = (1 - R_B)\lambda_B$

We have studied the value for an European call option with strike $K = 10e^{r_R T}$ and a maturity time of 3 years; the rest of the input parameters are taken from Table 1.2.

On one hand, if we assume the trade takes place between banks before the crisis, these counterparties are considered to be risk–free. Therefore, no CVA is taken into account and the FCA is negligible; thus the price is equal to the derivative value without counterparty risk.

Let us now assume that counterparty B is a bank, and C is a risky client. Thus, the bank will charge C a credit value adjustment on the trade, i.e., the price B charges to C is equal to the risk–free price plus CVA.

On the other hand, if the trading takes place after the financial crisis, the banks are no more considered parts without counterparty risk (risk–free). Moreover, they charge a prime due to funds lending in the capital market and counterparty B will not be able to fund the premium of the trade at the risk–free rate anymore. This means that B will incur in a funding cost in the agreement. Hence, the price that B

will offer to counterparty C is the risk–free value plus CVA and FCA. These three situations are represented in Figure 1.7.

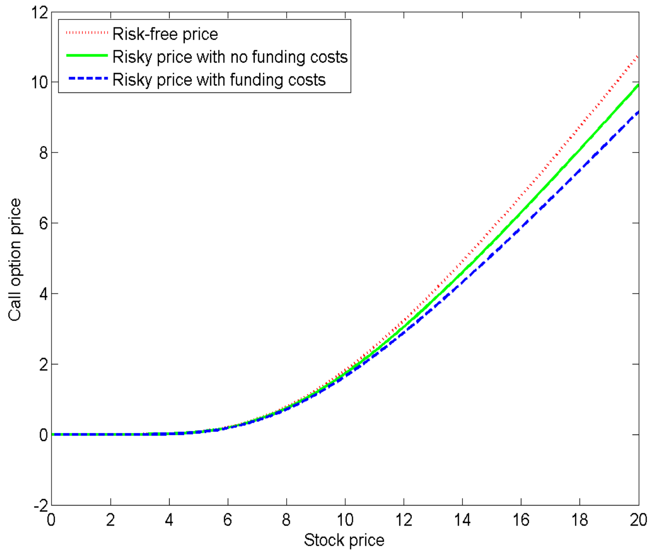
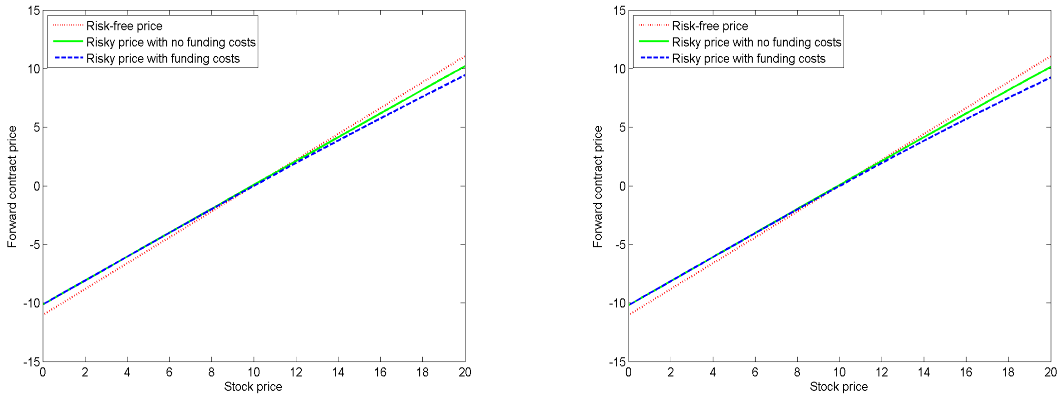


Figure 1.7: European call option values with CVA and FCA (Test 3)

A similar test concerning a forward contract has been done. The risk–free value and the risky values (with and without funding costs) are presented in Figure 1.8(a) for the mark–to–market equal to the risky derivative (nonlinear model) and in Figure 1.8(b) for the mark–to–market equal to the risk–free derivative (linear model). We can appreciate that when the forward has a positive value, B has the choice of exercising the contract thus being exposed to C default. On the other hand, if the forward has a negative value, then B may not be interested in exercising the contract, so that all the counterparty risk (from the point of view of B) is included in DVA. As we can observe, the computed results are similar in both cases. So, there is not a big difference in the choice of the mark–to–market close out.



(a) Case with $M = \widehat{V}$ (b) Case with $M = V$

Figure 1.8: Forward values with CVA and FCA (Test 3)

1.6.4 Test 4: Collateralized European options

In this example we study again a European put option bought by B. However, in this example the trading is now on a collateralized derivative and we use model 3 of Section 1.2.2. The strike is $K = 10e^{r_R T}$ and the maturity time is equal to 0.5 years. The rest of the parameters are in Table 1.2 and the collateral rate is $r_C = 0.05$. Thus, we show in Figure 1.9 the difference between the fully collateralized and a partially collateralized derivative prices. The difference is positive, because it represents the additional amount that has to be paid by B if the derivative is collateralized. So, this price increases as the collateral is larger, thus the exposure facing C's default is lower. Therefore, the price of a collateralized European put option is larger than the not collateralized one. This difference between both of them is the CollVA.

In Figure 1.10, the XVA surface is represented when the trading takes place with a collateralized derivative. We show the variation in the XVA value for different collateral values, which are in all cases a percentage of the derivative risk–free value. As expected, if the derivative is not collateralized, $X = 0$ and the XVA value corresponds with the results obtained in Figure 1.4. Nevertheless, the XVA values decrease when the derivative approaches to the fully collateralized case.

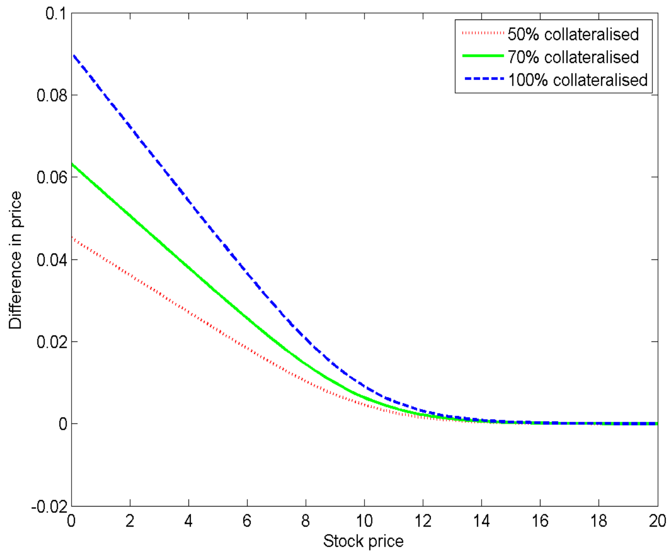


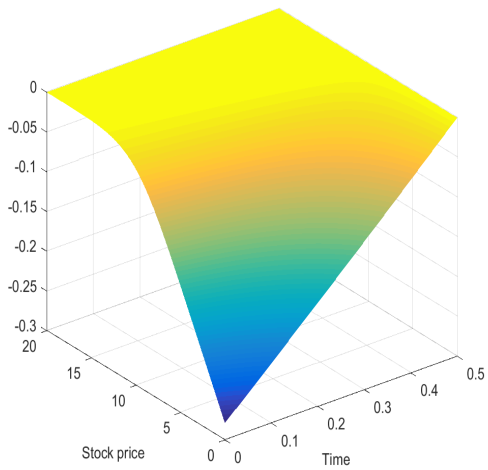
Figure 1.9: Collateral Value Adjustment for different amount of collateral (Test 4)

Moreover, we compare the three particular models explained in Section 1.2.2. Figure 1.11 represents the computed XVA value according to the different assumptions made about counterparty B's bond. We can observe that for a stock price in the money area, the results obtained using model 2 and model 3 are similar, whereas the XVA is higher in absolute terms if model 1 is employed. In any case, the differences between the models are negligible.

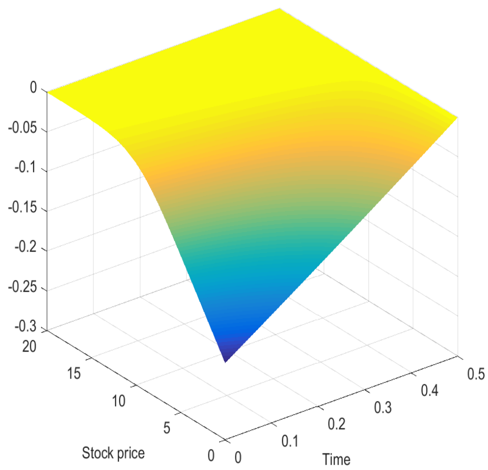
In all cases, tests have been performed by using MATLAB on an Intel(R) Xeon(R) CPU E3-1241 3.50GHz computer. In all examples, the elapsed computational time is less than 25 seconds.

1.6.5 Test 5: Monte Carlo simulation

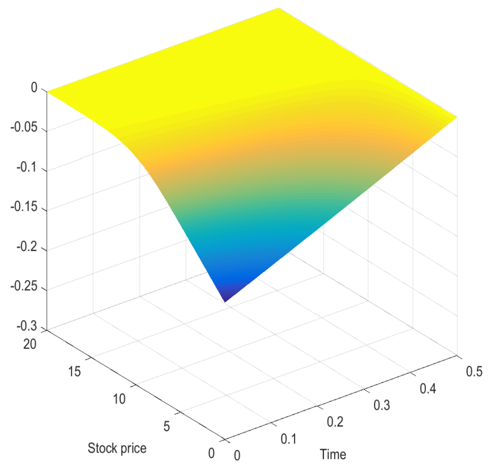
In this test, we estimate the XVA of an European put option by Monte Carlo techniques. In Table 1.4 we show the value for the nonlinear problem (1.15) and in Table 1.5 the solution for the linear one (1.16), both given in Section 1.5. The parameters are $K = 10$, $r = 0.03$, $r_R = 0.06$, $\sigma = 0.3$, $t \in [0, 0.5]$, $\lambda_B = 0.04$, $\lambda_C = 0.04$,



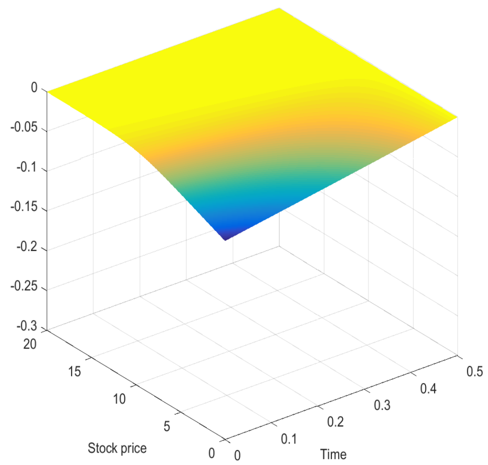
(a) Collateral = 0



(b) Collateral = 0.33V



(c) Collateral = 0.66V



(d) Collateral = V

Figure 1.10: XVA surfaces for different collateral values (Test 4)

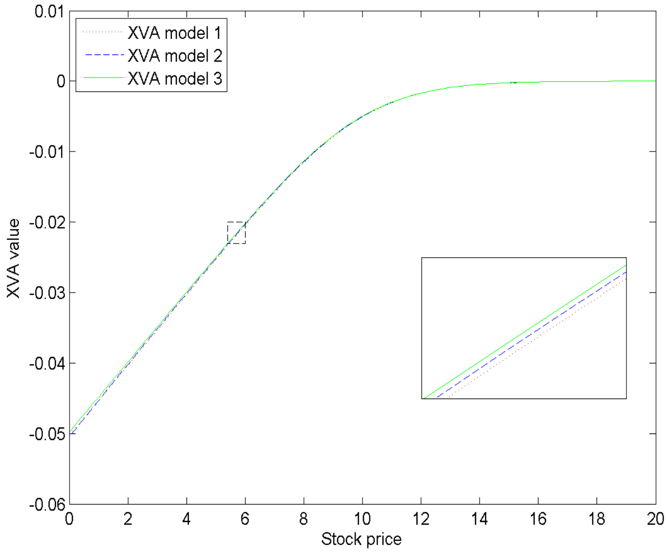


Figure 1.11: XVA according to the different collateral models (Test 4)

$s_F = \lambda_B(1-R_B)$, $R_B = 0.3$ and $R_C = 0.3$. For each problem, we show the asset value S, the XVA value obtained by the finite element method, the XVA value obtained by Monte Carlo techniques and the 99% confident intervals with Monte Carlo simulation in $t = 0$. As expected, the XVA value computed from the PDE model belongs to the 99% confidence interval obtained by classical Monte Carlo techniques, which in the case with $M = \widehat{V}$ have been combined with a fixed point iteration algorithm. The elapsed computational time needed to compute the value in one only node using Monte Carlo techniques with $N_P = 10000$ paths and $N_T = 1000$ time steps is 284 seconds when $M = V$ and 319 seconds when $M = \widehat{V}$. The PDE is solved with $N_S = 401$ for $S \in [0, 5K]$ and $N_T = 400$. The elapsed computational time in that case is 16 seconds for the mesh when $M = \widehat{V}$ and 10 seconds when $M = V$.

Table 1.4: Total Value Adjustment for European option with $M = \widehat{V}$

S	Partial differential equation	Monte Carlo	Confidence interval
0.0	-0.27060363	-0.27417357	(-0.27417357 , -0.27417357)
2.5	-0.20087696	-0.20349478	(-0.20384583 , -0.20340337)
5.0	-0.13116267	-0.13296892	(-0.13336176 , -0.13246800)
7.5	-0.06425927	-0.06512770	(-0.06535973 , -0.06420641)
10.0	-0.01944521	-0.01949560	(-0.01968523 , -0.01892315)
12.5	-0.00375659	-0.00379364	(-0.00382113 , -0.00353373)
15.0	-0.00054124	-0.00054131	(-0.00058505 , -0.00049654)
17.5	-0.00006629	-0.00006448	(-0.00007087 , -0.00004932)
20.0	-0.00000752	-0.00000586	(-0.00001422 , 0.00000304)
22.5	-0.00000083	-0.00000064	(-0.00000084 , -0.00000034)
25.0	-0.00000009	-0.00000007	(-0.00000023 , -0.00000002)
27.5	-0.00000001	-0.00000001	(-0.00000001 , 0.00000000)
30.0	-0.00000000	-0.00000000	(-0.00000000 , -0.00000000)

Table 1.5: Total Value Adjustment for European option with $M = V$

S	Partial differential equation	Monte Carlo	Confidence interval
0.0	-0.26898638	-0.26878065	(-0.26878065 , -0.26878065)
2.5	-0.19967652	-0.19964729	(-0.19986386 , -0.19943072)
5.0	-0.13037896	-0.13082446	(-0.13126011 , -0.13038882)
7.5	-0.06387518	-0.06389601	(-0.06445993 , -0.06333210)
10.0	-0.01932858	-0.01922398	(-0.01959828 , -0.01884968)
12.5	-0.00373389	-0.00374918	(-0.00389034 , -0.00360801)
15.0	-0.00053793	-0.00053692	(-0.00057657 , -0.00049727)
17.5	-0.00006588	-0.00006122	(-0.00006932 , -0.00005311)
20.0	-0.00000747	-0.00000539	(-0.00000629 , -0.00000450)
22.5	-0.00000083	-0.00000102	(-0.00000173 , -0.00000032)
25.0	-0.00000009	-0.00000005	(-0.00000008 , -0.00000003)
27.5	-0.00000001	-0.00000001	(-0.00000001 , -0.00000000)
30.0	-0.00000000	-0.00000000	(-0.00000000 , -0.00000000)

Chapter 2

One stochastic factor model for American options with XVA

2.1 Introduction

In the previous chapter we have modelled the total value adjustment associated with European options. With this purpose, a self–financing portfolio was built. Moreover, we assumed a constant behaviour of intensity of default from each counterparty. In the present chapter, we study the total value adjustment in the case of American options. A similar framework than for European–style options is considered.

As we mentioned in the Introduction, the hedging strategy is imposed by taking into account the period of time where it can be exercised. Then, we built the portfolio following [15], where non arbitrage opportunities are also imposed. As a result, analogous models to European options are posed in terms of linear and nonlinear complementarity problems, depending on the mark–to–market value. As we did with European options, several models are also obtained if different risks are taken into account and the appropriate adjustments are applied when pricing the derivative.

Similar numerical methods to the European case are suggested. Additionally, the augmented Lagrangian active set algorithm is introduced to solve the discretized system. Moreover, after modelling by linear and nonlinear complementarity problems, the solution of the risky derivative value is written in terms of expectations. Next, we extend the works by Longstaff and Schwartz [38] and Glasserman [28] for the approximation of the riskless American option value in order to obtain the approximation including counterparty risk. In this way, a dynamic programming technique is implemented: at each time step an optimal stopping problem is solved, an optimal exercise criterion is stated and the expected discounted payoff of the option price under this criterion is computed. Finally, both methods, finite element discretization and Monte Carlo techniques, are used to compute the total value adjustment as the difference between the risky and the risk–free values.

The scheme of the chapter is the following. In Section 2.2 we introduce the model of the American options considering counterparty risk. In Section 2.3 the numerical methods to solve the complementarity problems are proposed. Section 2.4 introduces an alternative way to obtain the risky derivative value by means of Monte Carlo techniques. Finally in Section 2.5, different results obtained with the numerical methods introduced along the chapter are shown.

Most of the results in this chapter are included in [2] and [4].

2.2 Mathematical model

In this section, as we did for European options in Chapter 1, we deduce several models which represent the American options value including different adjustments when counterparty risk is considered. As a result, linear and nonlinear complementarity problems are obtained. Unlike the European options, in this chapter we do not deduce a problem which directly models the XVA. On the opposite, we obtain the XVA value as the difference between the risky derivative value, and the risk–free value, i.e. $U = \widehat{V} - V$.

2.2.1 Pricing with counterparty credit risk and funding costs

We consider a similar scenario to that of European options: two bonds of counterparties B and C and the underlying asset with no default risk, the processes of which will be modelled by the SDEs given in (1.1).

Thus, we consider a derivative trade between two default counterparties, the issuer B and the buyer C. From the point of view of the seller the risky derivative value at time t is denoted by $\widehat{V}(t, S_t, J_t^B, J_t^C)$, where J^B and J^C are the same jump processes defined in the case of European options. The counterparty risk–free American option price is denoted by $V(t, S_t)$, which can be computed using the Black–Scholes complementarity problem for American options (see [51, 52], for example).

Conditions of the defaultable American option price upon the default of different counterparties are given by (1.2)–(1.3). In order to derive the value of the American option with counterparty risk, we consider the self–financing portfolio Π_t, used in the European option case (see Section 1.2.1), which at time t is given by:

$$\Pi_t = \Delta(t)S_t + \alpha_B(t)P_{B_t} + \alpha_C(t)P_{C_t} + \gamma_t\,. \tag{2.1}$$

As the portfolio is self–financing, its change is given by

$$d\Pi_t = \Delta(t)dS_t + \alpha_B(t)dP_{B_t} + \alpha_C(t)dP_{C_t} + (r\gamma_F^+ + r_F\gamma_F^- - r\gamma_{P_C} - r_R\gamma_R)(t)dt\,. \tag{2.2}$$

In addition, to avoid arbitrage opportunities we introduce the hedging inequality:

$$d\Pi_t + d\widehat{V}_t \leq 0\,. \tag{2.3}$$

The change in the derivative value is obtained by applying Itô's lemma for jump diffusion, and is given by (1.6):

$$\begin{aligned} d\widehat{V}_t &= \frac{\partial\widehat{V}}{\partial t}dt + \frac{\partial\widehat{V}}{\partial S}dS_t + \frac{1}{2}\sigma^2S_t^2\frac{\partial^2\widehat{V}}{\partial S^2}dt + \Delta\widehat{V}_{B_t}dJ_t^B + \Delta\widehat{V}_{C_t}dJ_t^C \\ &= \left(\frac{\partial\widehat{V}}{\partial t} + r_R\frac{\partial\widehat{V}}{\partial S} + \frac{1}{2}\sigma^2S_t^2\frac{\partial^2\widehat{V}}{\partial S^2}\right)dt + \sigma S_t\frac{\partial\widehat{V}}{\partial S}dW_t + \Delta\widehat{V}_{B_t}dJ_t^B + \Delta\widehat{V}_{C_t}dJ_t^C\,, \end{aligned} \tag{2.4}$$

where $\widehat{V}$ and all partial derivatives of $\widehat{V}$ are evaluated at point (t, S_t, J_t^B, J_t^C). Moreover, we use the notation introduced in (1.7)

$$\Delta\widehat{V}_{B_t} = \widehat{V}(t, S_t, 1, 0) - \widehat{V}(t, S_t, 0, 0)\,,$$
$$\Delta\widehat{V}_{C_t} = \widehat{V}(t, S_t, 0, 1) - \widehat{V}(t, S_t, 0, 0)\,,$$

which can be computed using the default conditions (1.2) and (1.3).

Keeping in mind expressions (2.2) and (2.4) we deduce the following inequality:

$$\begin{aligned}\Delta(t)dS_t + \alpha_B(t)dP_{B_t} + \alpha_C(t)dP_{C_t} + (r\gamma_F^+ + r_F\gamma_F^- - r\gamma_{P_C} - r_R\gamma_R)dt \\ \leq -\left(\frac{\partial\widehat{V}}{\partial t}dt + \frac{\partial\widehat{V}}{\partial S}dS_t + \frac{1}{2}\sigma^2 S_t^2\frac{\partial^2\widehat{V}}{\partial S^2}dt + \Delta\widehat{V}_{B_t}dJ_t^B + \Delta\widehat{V}_{C_t}dJ_t^C\right)\,,\end{aligned} \tag{2.5}$$

analogous to (1.8). According to the SDEs in (1.1) we obtain:

$$\begin{aligned}\Delta(t)dS_t + \alpha_B(t)(r_{P_B}P_{B_t}dt - P_{B_t}dJ_t^B) + \alpha_C(t)(r_{P_C}P_{C_t}dt - P_{C_t}dJ_t^C) \\ + (r\gamma_F^+ + r_F\gamma_F^- - r\gamma_{P_C} - r_R\gamma_R)dt \\ \leq -\left(\frac{\partial\widehat{V}}{\partial t}dt + \frac{\partial\widehat{V}}{\partial S}dS_t + \frac{1}{2}\sigma^2 S_t^2\frac{\partial^2\widehat{V}}{\partial S^2}dt + \Delta\widehat{V}_B dJ_t^B + \Delta\widehat{V}_C dJ_t^C\right)\,.\end{aligned} \tag{2.6}$$

Choosing, as in (1.10), the following weights,

$$\begin{aligned}\Delta(t) &= -\frac{\partial\widehat{V}}{\partial S}, \\ \alpha_B(t) &= \frac{\Delta\widehat{V}_{B_t}}{P_{B_t}} = -\frac{\widehat{V}_t - (M_t^+ + R_B M_t^-)}{P_{B_t}}, \\ \alpha_C(t) &= \frac{\Delta\widehat{V}_{C_t}}{P_{C_t}} = -\frac{\widehat{V}_t - (M_t^- + R_C M_t^+)}{P_{C_t}}\end{aligned} \tag{2.7}$$

we remove all risks in the portfolio Π_t. Thus, equation (2.6) leads to

$$\begin{aligned}\alpha_B r_{P_B} P_B + \alpha_C r_{P_C} P_C + (r\gamma_F^+ + r_F\gamma_F^- - r\gamma_{P_C} - r_R\gamma_R) + \\ + \frac{\partial\widehat{V}}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2\widehat{V}}{\partial S^2} \leq 0\,.\end{aligned} \tag{2.8}$$

As we did for European options (cf. page 13), we consider the equivalences $\gamma_{P_B} = \alpha_B P_{B_t}$, $\gamma_{P_C} = \alpha_C P_{C_t}$, $r_F = r + s_F$ and $\gamma_F = \gamma_P - \gamma_{P_B}$, and write $\alpha_B P_{B_t}$ and $\alpha_C P_{C_t}$ in terms of the mark–to–market value to deduce:

$$\begin{aligned}\alpha_B r_{P_B} P_B + \alpha_C r_{P_C} P_C + r\gamma_F^+ + r_F\gamma_F^- - r\gamma_{P_C} - r_R\gamma_R \\ = -(r + \lambda_B + \lambda_C)\widehat{V} + s_F\gamma_F^- - r_R\Delta S \\ + \lambda_B(R_B M^- + M^+) + \lambda_C(R_C M^+ + M^-)\,.\end{aligned}$$

Thus, we introduce the previous expression in (2.8) to obtain the inequality that models the value of the derivative including the counterparty risk:

$$\begin{aligned}\partial_t\widehat{V} + \mathcal{A}\widehat{V} - r\widehat{V} \leq (\lambda_B + \lambda_C)\widehat{V} + s_F M^+ \\ - \lambda_B(R_B M^- + M^+) - \lambda_C(R_C M^+ + M^-)\,,\end{aligned} \tag{2.9}$$

where the operator $\mathcal{A}$ is defined in (1.13). Thereafter, the complementarity problem which models the American options price in the presence of counterparty risk reads:

$$\begin{cases}\mathcal{L}(\widehat{V}) = \partial_t\widehat{V} + \mathcal{A}\widehat{V} - (r + \lambda_B + \lambda_C)\widehat{V} \\ \qquad -s_F M^+ + \lambda_B(R_B M^- + M^+) + \lambda_C(R_C M^+ + M^-) \leq 0 \\ \widehat{V}(t,S) \geq H(S) \\ \mathcal{L}(\widehat{V})\,(\widehat{V} - H) = 0 \\ \widehat{V}(T,S) = H(S)\end{cases} \tag{2.10}$$

where H denotes the payoff function.

According to the choice of the mark–to–market value, two different complementarity problems are obtained:

- If $M = \widehat{V}$,

$$
\begin{cases}
\mathcal{L}_1(\widehat{V}) = \partial_t \widehat{V} + \mathcal{A}\widehat{V} - r\widehat{V} \\
\qquad\qquad -(1-R_B)\lambda_B \widehat{V}^- - (1-R_C)\lambda_C \widehat{V}^+ - s_F \widehat{V}^+ \leq 0 \\
\widehat{V}(t,S) \geq H(S) \\
\mathcal{L}_1(\widehat{V})\,(\widehat{V} - H) = 0 \\
\widehat{V}(T,S) = H(S)\,.
\end{cases}
\tag{2.11}
$$

- If $M = V$,

$$
\begin{cases}
\mathcal{L}_2(\widehat{V}) = \partial_t \widehat{V} + \mathcal{A}\widehat{V} - (r + \lambda_B + \lambda_C)\widehat{V} \\
\qquad\qquad +(R_B\lambda_B + \lambda_C)V^- + (R_C\lambda_C + \lambda_B)V^+ - s_F V^+ \leq 0 \\
\widehat{V}(t,S) \geq H(S) \\
\mathcal{L}_2(\widehat{V})\,(\widehat{V} - H) = 0 \\
\widehat{V}(T,S) = H(S)\,.
\end{cases}
\tag{2.12}
$$

Thus, the price of an American option including counterparty risk is the solution of either a nonlinear or a linear complementarity problem.

Remark 2.2.1. *In the particular case of American options, where the payoff is always positive, problem (2.11) becomes linear. We prefer to study a more general obstacle problem (not only restricted to American options) in which function H can be negative.*

In order to compute the XVA value, the Black–Scholes equation for American options without counterparty risk has to be previously solved. More precisely, the risk–free price, V, is solution of the classical problem:

$$
\begin{cases}
\widetilde{\mathcal{L}}(V) = \partial_t V + \mathcal{A}V - rV \leq 0 \\
V(t,S) \geq H(S) \\
\widetilde{\mathcal{L}}(V)\,(V - H) = 0 \\
V(T,S) = H(S)\,.
\end{cases}
\tag{2.13}
$$

Finally, the XVA value is obtained after solving the two obstacle problems and is given by $U = \widehat{V} - V$.

2.2.2 Pricing with counterparty credit risk, funding costs and collateral

As we have done for European options, we deduce the American option value when a collateral is included in the contract between both counterparties. Then, due to the presence of collateral, the risk of the contract is reduced.

Considering a similar scenario, we assume an agreement between counterparties B and C. Moreover, a self–financing portfolio is built, the main difference with the case without collateral is that now the portfolio only hedges the derivative when the counterparty does not default; in other case, the difference between the hedge portfolio and the derivative is the hedge error.

We make the same assumptions that in Section 1.2.2 for collateralized European options. Then, B has a portfolio made up of two bonds, P_1 and P_2. The different bonds and the asset price that take part in the contract satisfy the SDEs given by (1.18)–(1.22).

When one of the counterparties defaults, the risky derivative value is given by the conditions (1.23) and (1.24). The hedging inequality is given by (2.3), where now the portfolio is made up of

$$\Pi_t = \Delta(t)S_t + P_{B_t} + \alpha_C(t)P_{C_t} + \gamma(t) - X_t\,, \tag{2.14}$$

and the financial instruments are the same than in the European case (cf. Section 1.2.2).

Then, replacing the expressions given in Section 2.2.1 in the hedging equation and removing the risky terms as we did for European options in Section 1.2.2, we obtain the following inequality

$$\frac{\partial \widehat{V}}{\partial t} + \mathcal{A}\widehat{V} + \alpha_1 r_1 P_1 + \alpha_2 r_2 P_2 + \alpha_C \lambda_C P_C - r_X X \leq 0\,, \tag{2.15}$$

analogous to (1.35).

Next, let us consider the case when the counterparty B defaults. In this situation the derivative value is the solution of the complementarity problem

$$\begin{cases} \mathcal{L}_X(\widehat{V}) = \dfrac{\partial \widehat{V}}{\partial t} + \mathcal{A}\widehat{V} - (r + \lambda_B + \lambda_C)\widehat{V} - \lambda_B h_e \\ \qquad\qquad +\lambda_B g_B(M, X) + \lambda_C g_C(M, X) - s_X X \leq 0 \\ \widehat{V}(t, S) \geq H(S) \\ \mathcal{L}_X(\widehat{V})(\widehat{V} - H) = 0 \\ \widehat{V}(T, S) = H(S)\,. \end{cases} \tag{2.16}$$

The difference between (2.16) with the complementarity problem (2.10) obtained in the case without collateral is the presence of the terms $\lambda_B h_e$ and $s_X X$. Furthermore, the terms g_B and g_C are now more general than in the non collateralized case.

As in the European case, when counterparty B defaults a hedge error arises. Nevertheless, while the issuer B is alive, B will incur a cost or gain of size $\lambda_B h_e$ per time unit.

Once again, depending on the chosen of the mark–to–market two different complementarity problems are obtained:

- If $M = \widehat{V}$,

$$\begin{cases} \mathcal{L}_3(\widehat{V}) = \partial_t \widehat{V} + \mathcal{A}\widehat{V} - r\widehat{V} - \lambda_B h_e + (R_B - 1)\lambda_B(\widehat{V} - X)^- \\ \qquad +(R_C - 1)\lambda_C(\widehat{V} - X)^+ - s_X X \leq 0 \\ \widehat{V}(t, S) \geq H(S) \\ \mathcal{L}_3(\widehat{V})\,(\widehat{V} - H) = 0 \\ \widehat{V}(T, S) = H(S)\,. \end{cases} \tag{2.17}$$

- If $M = V$,

$$\begin{cases} \mathcal{L}_4(\widehat{V}) = \partial_t \widehat{V} + \mathcal{A}\widehat{V} - (r + \lambda_B + \lambda_C)\widehat{V} - \lambda_B h_e \\ \qquad +(\lambda_B + \lambda_C - s_X)X + (\lambda_B + \lambda_C R_C)(V - X)^+ \\ \qquad +(\lambda_C + \lambda_B R_B)(V - X)^- \leq 0 \\ \widehat{V}(t, S) \geq H(S) \\ \mathcal{L}_4(\widehat{V})\,(\widehat{V} - H) = 0 \\ \widehat{V}(T, S) = H(S)\,. \end{cases} \tag{2.18}$$

According to the different assumptions made on counterparty B bonds (presented in Section 1.2.2), three particular different models are posed.

Collateral model 1: Perfect hedging

We consider that all risks are perfectly hedged, so $h_e = 0$. Thus, we get

$$h_e = g_B(M_t, X_t) + P_{D_t} - X_t = g_B(M_t, X_t) + \alpha_1(t) R_1 P_{1_t} + \alpha_2(t) R_2 P_{2_t} - X_t = 0\,. \tag{2.19}$$

Then, the complementarity problem that models the American option price (2.16) is reduced to

$$\begin{cases} \mathcal{L}_X(\widehat{V}) = \dfrac{\partial \widehat{V}}{\partial t} + \mathcal{A}\widehat{V} - (r + \lambda_B + \lambda_C)\widehat{V} \\ \qquad\qquad +\lambda_C g_C(M, X) + \lambda_B g_B(M, X) - s_X X \leq 0 \\ \widehat{V}(t, S) \leq H(S) \\ \mathcal{L}_X(\widehat{V})\,(\widehat{V} - H(S)) = 0 \\ \widehat{V}(T, S) = H(S)\,, \end{cases}$$

and depending on the mark–to–market value, we obtain

- If $M = \widehat{V}$,

$$\begin{cases} \mathcal{L}_3(\widehat{V}) = \partial_t \widehat{V} + \mathcal{A}\widehat{V} - r\widehat{V} \\ \qquad +\lambda_B(R_B - 1)(\widehat{V} - X)^- + \lambda_C(R_C - 1)(\widehat{V} - X)^+ - s_X X \leq 0 \\ \widehat{V}(t,S) \geq H(S) \\ \mathcal{L}_3(\widehat{V})\,(\widehat{V} - H) = 0 \\ \widehat{V}(T,S) = H(S)\,. \end{cases}$$

- If $M = V$,

$$\begin{cases} \mathcal{L}_4(\widehat{V}) = \partial_t \widehat{V} + \mathcal{A}\widehat{V} - (r + \lambda_B + \lambda_C)\widehat{V} + (\lambda_B + \lambda_C - s_X)X \\ \qquad +(\lambda_B + \lambda_C R_C)(V - X)^+ + (\lambda_C + \lambda_B R_B)(V - X)^- \leq 0 \\ \widehat{V}(t,S) \geq H(S) \\ \mathcal{L}_4(\widehat{V})\,(\widehat{V} - H) = 0 \\ \widehat{V}(T,S) = H(S)\,. \end{cases}$$

Due to the nullity of the hedge error, funding cost vanishes and only CVA, DVA and CollVA are included in the XVA.

Collateral model 2: Two bonds model

In this model, we assume that counterparty B has two bonds. More precisely, a zero recovery bond P_1 and a bond P_2 with recovery R_2 which is equivalent to the recovery rate of counterparty B on a derivative trade, i.e. $R_2 = R_B$.

Assuming the funding constraint introduced in (1.34)

$$\widehat{V}_t + P_{B_t} - X_t = 0\,, \tag{2.20}$$

we write

$$P_{B_t} = \alpha_1(t)P_{1_t} + \alpha_2(t)P_{2_t} = -(\widehat{V}_t - X_t)\,.$$

Now, taking into account this assumption, the general complementarity problem (2.16) turns into:

$$
\begin{cases}
\mathcal{L}_X(\widehat{V}) = \dfrac{\partial \widehat{V}}{\partial t} + \mathcal{A}\widehat{V} - (r + \lambda_B + \lambda_C)\widehat{V} - \lambda_B(1 - R_B)(M - X)^+ \\
\qquad\qquad\qquad + \lambda_B g_B(M, X) + \lambda_C g_C(M, X) - s_X X \leq 0 \\
\widehat{V}(t, S) \geq H(S) \\
\mathcal{L}_X(\widehat{V})(\widehat{V} - H) = 0 \\
\widehat{V}(T, S) = H(S)\,.
\end{cases}
\tag{2.21}
$$

and, depending on the mark–to–market value,

- If $M = \widehat{V}$,

$$
\begin{cases}
\mathcal{L}_3(\widehat{V}) = \partial_t \widehat{V} + \mathcal{A}\widehat{V} - r\widehat{V} \\
\qquad\qquad + \lambda_B(R_B - 1)(\widehat{V} - X) + \lambda_C(R_C - 1)(\widehat{V} - X)^+ - s_X X \leq 0 \\
\widehat{V}(t, S) \geq H(S) \\
\mathcal{L}_3(\widehat{V})\,(\widehat{V} - H) = 0 \\
\widehat{V}(T, S) = H(S)\,.
\end{cases}
$$

- If $M = V$,

$$
\begin{cases}
\mathcal{L}_4(\widehat{V}) = \partial_t \widehat{V} + \mathcal{A}\widehat{V} - (r + \lambda_B + \lambda_C)\widehat{V} + (\lambda_B + \lambda_C - s_X)X \\
\qquad\qquad + (\lambda_B R_B + \lambda_C R_C)(V - X)^+ + (\lambda_C + \lambda_B R_B)(V - X)^- \leq 0 \\
\widehat{V}(t, S) \geq H(S) \\
\mathcal{L}_4(\widehat{V})\,(\widehat{V} - H) = 0 \\
\widehat{V}(T, S) = H(S)\,.
\end{cases}
$$

Collateral model 3: One bond model

Finally, only one bond from B, with recovery rate R_B, is considered. Taking $\alpha_1(t) = 0$ in (2.20) we set $P_{B_t} = \alpha_2(t)P_{2_t}$. Under this assumption, the following complementarity problem modelling the risky derivative value is obtained:

$$\begin{cases} \mathcal{L}_X(\widehat{V}) = \dfrac{\partial \widehat{V}}{\partial t} + \mathcal{A}\widehat{V} - (r + \lambda_B(1-R_B) + \lambda_C)\widehat{V} \\ \qquad\qquad -\lambda_B(R_B - 1)X + \lambda_C g_C(M,X) - s_X X \leq 0 \\ \widehat{V}(t,S) \geq H(S) \\ \mathcal{L}_X(\widehat{V})\,(\widehat{V} - H) = 0 \\ \widehat{V}(T,S) = H(S) \end{cases}$$

and the complementarity problems related to the possible choices of the mark–to–market value are:

- If $M = \widehat{V}$,

$$\begin{cases} \mathcal{L}_3(\widehat{V}) = \partial_t \widehat{V} + \mathcal{A}\widehat{V} - (r + \lambda_B(1-R_B))\widehat{V} \\ \qquad +\lambda_C(R_C - 1)(\widehat{V} - X)^+ - (s_X + \lambda_B(R_B - 1))X \leq 0 \\ \widehat{V}(t,S) \geq H(S) \\ \mathcal{L}_3(\widehat{V})\,(\widehat{V} - H) = 0 \\ \widehat{V}(T,S) = H(S)\,. \end{cases}$$

- If $M = V$,

$$\begin{cases} \mathcal{L}_4(\widehat{V}) = \partial_t \widehat{V} + \mathcal{A}\widehat{V} - (r + \lambda_B(1-R_B) + \lambda_C)\widehat{V} + \lambda_C R_C (V - X)^+ \\ \qquad +\lambda_C(V - X)^- + (\lambda_C - \lambda_B(R_B - 1) - s_X)X \leq 0 \\ \widehat{V}(t,S) \geq H(S) \\ \mathcal{L}_4(\widehat{V})\,(\widehat{V} - H) = 0 \\ \widehat{V}(T,S) = H(S)\,. \end{cases}$$

2.2.3 Mathematical analysis

We have not done a detailed study of the existence and uniqueness of solution of the one–dimensional problem (2.11). However, we will analyze the two–dimensional problem in the forthcoming Chapter 4, proving the existence and uniqueness of solution.

The one–dimensional problem can be faced in a similar way: with the adequate changes of variable $\tau = T - t$, $x = \ln(S/K)$, $u(\tau, x) = U(t, S)$ and $v(\tau, x) = V(t, S)$, we can write (2.11) on the XVA variable as:

$$\begin{cases} \mathcal{L}(u) = -\dfrac{\partial u}{\partial \tau} - \mathcal{A}u + \Phi(\tau, u) + \ell(\tau) \leq 0 \\ u(t, S) \geq \psi(\tau, x) \\ \mathcal{L}(u)\,(u - \psi) = 0 \\ u(0, S) = 0\,, \end{cases}$$

so that the application of Theorem 4.3.3 is straightforward. Further details are given in Chapter 4.

2.3 Numerical methods

In order to solve the previous models, we propose in this section some numerical methods. We develop the problem with collateral, as the problem without collateral can be considered as a particular case. Moreover, as we made in Section 1.4 for the European options case, we focus on the nonlinear problems, similar methods being used in the linear ones.

We have developed an approach based on the method of characteristics for time discretization jointly with a finite element method for spatial discretization. Due to the fact that the domain is unbounded in variable S, a localization procedure is required. Once again, reasonable boundary conditions are deduced and implemented.

Unlike European options, where a XVA problem was directly solved, for American options we compute the derivative value considering counterparty risk and the risk–free derivative value, and then we obtain the XVA as the difference of both. Then, we propose the numerical methods to solve the risky derivative problem; since there is not an analytical solution for the classical Black–Scholes inequality, similar methods are applied to obtain the risk–free American option value.

Thus, we solve problem (2.17), the solution of which is the risky value considering CVA, DVA, FCA and CollVA. Problems that do not consider a collateral can be assumed as a particular case, and we will use the same set of numerical methods.

Once again, in order to write the problem (2.17) forward in time, the change of variable $\tau = T - t$ is applied. Then, the following non linear complementarity problem is obtained:

$$\begin{cases} \mathcal{L}_3(\widehat{V}) = \dfrac{\partial \widehat{V}}{\partial \tau} - \dfrac{\sigma^2}{2} S^2 \dfrac{\partial^2 \widehat{V}}{\partial S^2} - r_R S \dfrac{\partial \widehat{V}}{\partial S} + r\widehat{V} + \lambda_B h_e \\ \qquad\quad -(R_B - 1)\lambda_B(\widehat{V} - X)^- - (R_C - 1)\lambda_C(\widehat{V} - X)^+ + s_X X \geq 0 \\ \widehat{V}(\tau, S) \geq H(S) \\ \mathcal{L}_3(\widehat{V})\,(\widehat{V} - H) = 0 \\ \widehat{V}(T, S) = H(S)\,. \end{cases} \tag{2.22}$$

Moreover, we rewrite the equation in divergencial form, in order to be solved by a finite element method:

$$\begin{cases} \mathcal{L}_3(\widehat{V}) = \dfrac{\partial \widehat{V}}{\partial \tau} - \dfrac{\partial}{\partial S}\left(\dfrac{\sigma^2}{2} S^2 \dfrac{\partial \widehat{V}}{\partial S}\right) + (\sigma^2 - r_R) S \dfrac{\partial \widehat{V}}{\partial S} + r\widehat{V} + \lambda_B h_e \\ \qquad\quad -(R_B - 1)\lambda_B(\widehat{V} - X)^- - (R_C - 1)\lambda_C(\widehat{V} - X)^+ + s_X X \geq 0 \\ \widehat{V}(\tau, S) \geq H(S) \\ \mathcal{L}_3(\widehat{V})\,(\widehat{V} - H) = 0 \\ \widehat{V}(T, S) = H(S)\,. \end{cases} \tag{2.23}$$

2.3.1 Method of characteristics

In order to solve the problem, we propose a semi–Lagrangian discretization combined with finite elements.

With this purpose, as we made for European options, we rewrite the inequality in terms of the material derivative. Applying the time discretization explained in Section 1.4.1, the first inequality in (2.23) is approximated by:

$$\mathcal{L}_3^n(\widehat{V}^{n+1}) = \frac{\widehat{V}^{n+1} - \widehat{V}^n \circ \chi^n}{\Delta\tau} - \frac{\sigma^2}{2}\frac{\partial}{\partial S}\left(S^2\frac{\partial \widehat{V}^{n+1}}{\partial S}\right) + r\widehat{V}^{n+1} + \lambda_B h_e$$
$$- (R_B - 1)\lambda_B(\widehat{V}^{n+1} - X)^- - (R_C - 1)\lambda_C(\widehat{V}^{n+1} - X)^+ + s_X X \geq 0\,, \quad (2.24)$$

for $n = 0, 1, 2, \ldots, N_T - 1$ (N_T being the number of time steps), where $\widehat{V}^n(\cdot) = \widehat{V}(\tau^n, \cdot)$ and $\chi^n \equiv \chi(S, \tau^{n+1}; \tau^n)$ represents the characteristic curve passing through point S at time τ^{n+1}, so that function χ satisfies the final value ODE problem (1.59).

2.3.2 Fixed point scheme

As we have proceeded for European options, in this section we introduce a fixed point algorithm at each iteration of the method of characteristics, in order to linearize the nonlinear inequality (2.24). The global scheme is shown in Algorithm 2.1.

2.3.3 Boundary conditions

We follow a similar reasoning as in European options: we truncate the unbounded domain $[0, \infty)$ into a bounded one, $[0, S_\infty]$ (with S_∞ large enough), so that the solution is not affected by the truncation in the interest region from the financial point of view.

In this section, we propose adequate boundary conditions for problem (2.25). We recall the function (1.62) introduced for European options

$$f(\widehat{V}) = \lambda_B h_e - (R_B - 1)\lambda_B(\widehat{V} - X)^- - (R_C - 1)\lambda_C(\widehat{V} - X)^+ + s_X X$$

in order to simplify the right hand side of (2.25), that also takes part in (2.23). The boundary condition at $S = 0$ is obtained by replacing $S = 0$ in the first inequality of

Algorithm 2.1

1. Let $N_T > 1$, $\varepsilon > 0$, $\widehat{V}^0$ given.
2. For $n = 0, 1, 2, \ldots, N_T - 1$
   - Let $\widehat{V}^{n+1,0} = \widehat{V}^n$
   - For $k = 0, 1, 2, \ldots,$ we compute $\widehat{V}^{n+1,k+1}$ satisfying:

$$
\begin{aligned}
(1 + r\Delta\tau)\,\widehat{V}^{n+1,k+1} &- \frac{\sigma^2\Delta\tau}{2}\frac{\partial}{\partial S}\left(S^2\frac{\partial\widehat{V}^{n+1,k+1}}{\partial S}\right) \\
&\geq \widehat{V}^n \circ \chi^n - \Delta\tau\left[\lambda_B h_e - (R_B - 1)\lambda_B(\widehat{V}^{n+1,k} - X)^-\right. \\
&\qquad \left. -(R_C - 1)\lambda_C(\widehat{V}^{n+1,k} - X)^+ + s_X X\right] \qquad (2.25) \\
&\widehat{V}^{n+1,k+1}(S) \geq H(S) \\
&\mathcal{L}_3^n(\widehat{V}^{n+1,k+1})\,(\widehat{V}^{n+1,k+1} - H) = 0
\end{aligned}
$$

   until $\dfrac{\|\widehat{V}^{n+1,k+1} - \widehat{V}^{n+1,k}\|}{\|\widehat{V}^{n+1,k+1}\|} < \varepsilon.$

(2.23). Thus we deduce the nonlinear inequality

$$\partial_\tau\widehat{V} + r\widehat{V} \geq -f(\widehat{V})\,.$$

This inequality is discretized by the method of characteristics (in this case equivalent to the implicit Euler method), combined with a fixed point scheme

$$\widehat{V}^{n+1,k+1}(0) - \widehat{V}^n(0) + r\,\Delta\tau\,\widehat{V}^{n+1,k+1}(0) \geq -\Delta\tau\,f(\widehat{V}^{n+1,k}(0))\,,$$

for $k \geq 0$ and $n \geq 0$, so that a nonhomogeneous Dirichlet boundary condition is obtained at each step of the global algorithm:

$$
\begin{aligned}
\widehat{V}^{n+1,k+1}(0) \geq \frac{1}{1 + r\Delta\tau}\Big(\widehat{V}^n(0) - \Delta\tau\Big[&\lambda_B h_e - (R_B - 1)\lambda_B(\widehat{V}^{n+1,k}(0) - X)^- \\
&-(R_C - 1)\lambda_C(\widehat{V}^{n+1,k}(0) - X)^+ + s_X X\Big]\Big)\,.
\end{aligned}
$$

In order to simplify the notations, let

$$\widehat{f} = \frac{1}{1+r\Delta\tau}\left(\widehat{V}^n(0) - \Delta\tau\left[\lambda_B h_e - (R_B-1)\lambda_B(\widehat{V}^{n+1,k}(0)-X)^- \right.\right.$$
$$\left.\left. -(R_C-1)\lambda_C(\widehat{V}^{n+1,k}(0)-X)^+ + s_X X\right]\right).$$

Moreover, the value on the boundary has to satisfy the obstacle condition; thus, the following boundary condition is proposed:

$$\widehat{V}^{n+1,k+1}(0) = \max\left(\widehat{f}, H(0)\right).$$

In order to deduce the boundary condition for problem (2.24) at $S = S_\infty$ we compute the boundary condition for the associated European option problem, as a particular solution of the American option problem, for which we follow the procedure in Section 1.4.3. Thus, if $\widehat{V}_\mathcal{E}$ denotes the value of the associated European option, taking the limit when S tends to infinity the following condition is obtained

$$\lim_{S\to\infty} \frac{\partial^2 \widehat{V}_\mathcal{E}}{\partial S^2} = 0\,. \tag{2.26}$$

Then, following [19], when $S \to \infty$ we consider a solution of the form:

$$\widehat{V}_\mathcal{E} = H_0(\tau) + H_1(\tau)S\,, \tag{2.27}$$

where $H_0(\tau)$ and $H_1(\tau)$ are constant coefficients with respect to variable S.

Discretizing the associated equation in S_∞

$$\frac{\partial \widehat{V}_\mathcal{E}}{\partial \tau} - r_R S\frac{\partial \widehat{V}_\mathcal{E}}{\partial S} + r\widehat{V}_\mathcal{E} = -f(\widehat{V}_\mathcal{E})\,, \tag{2.28}$$

on the characteristic curve we have:

$$\frac{\widehat{V}_\mathcal{E}^{n+1} - \widehat{V}_\mathcal{E}^{n}\circ\chi^n}{\Delta\tau} + r\widehat{V}_\mathcal{E}^{n+1} = -f(\widehat{V}_\mathcal{E})\,, \tag{2.29}$$

where $\chi^n \equiv \chi(S, \tau^{n+1}; \tau^n)$ is the solution of the final value problem

$$\begin{cases} \dfrac{d\chi}{d\tau} = -r_R\chi(\tau) \\ \chi(\tau^{n+1}) = S\,. \end{cases} \tag{2.30}$$

Thus, the characteristic curve is given by $\chi(S, \tau^{n+1}; \tau^n) = S \exp(r_R \Delta\tau)$. Introducing (2.27) into each fixed point iteration of equation (2.29), we obtain two more simple equations:

$$\begin{cases} (1 + r\Delta\tau) H_0^{n+1,k+1} = 0 \\ (1 + r\Delta\tau) H_1^{n+1,k+1} S_\infty = (\widehat{V}_\mathcal{E}^n \circ \chi^n)(S_\infty) - \Delta\tau \Big[\lambda_B h_e + s_X X \\ \qquad -(R_B - 1)\lambda_B (\widehat{V}_\mathcal{E}^{n+1,k}(S_\infty) - X)^- - (R_C - 1)\lambda_C (\widehat{V}_\mathcal{E}^{n+1,k}(S_\infty) - X)^+ \Big], \end{cases}$$

so that $H_0^{n+1,k+1} = 0$ and the following expression of $\widehat{V}_\mathcal{E}$ is deduced:

$$\begin{aligned} \widehat{V}_\mathcal{E}^{n+1,k+1}(S_\infty) &= H_1^{n+1,k+1} S_\infty \\ &= \frac{1}{(1 + r\Delta\tau)} \Big((\widehat{V}_\mathcal{E}^n \circ \chi^n)(S_\infty) - \Delta\tau \Big[\lambda_B h_e \\ &\quad - (R_B - 1)\lambda_B (\widehat{V}_\mathcal{E}^{n+1,k}(S_\infty) - X)^- \\ &\quad - (R_C - 1)\lambda_C (\widehat{V}_\mathcal{E}^{n+1,k}(S_\infty) - X)^+ + s_X X \Big] \Big) . \end{aligned} \tag{2.31}$$

Moreover, as we did at $S = 0$, the derivative value has to satisfy the obstacle condition. Then we impose the following boundary condition at $S = S_\infty$ for each fixed point iteration:

$$\widehat{V}^{n+1,k+1}(S_\infty) = \max \Big(\widehat{V}_\mathcal{E}^{n+1,k+1}(S_\infty), H(S_\infty) \Big) . \tag{2.32}$$

As a result, nonhomogeneous Dirichlet conditions are obtained for both boundaries of the domain.

Remark 2.3.1. *Note that in the particular case of American options, at each step of the fixed point iteration the boundary condition considered in (2.32) is always equivalent to the payoff, $H(S_\infty)$. The previous calculation is more interesting for a general derivative product, where the involved obstacle is different. Then, the maximum in (2.32) does not always take the same value.*

2.3.4 Finite element method

Next, we proceed with the spatial discretization. As we previously mentioned, a finite element method is applied. First, we introduce a convex closed subset

$$\widehat{\mathcal{K}} = \left\{ \varphi \in H^1(0, S_\infty) \,/\, \varphi(0) = \widehat{V}(0),\, \varphi(S_\infty) = \widehat{V}(S_\infty) \text{ and } \varphi \geq H(S) \right\},$$

and a spatial discretization of nodes S_j for $j = 1, 2, \ldots, N_S$, similarly to what we did in Chapter 1. At each time step, $n = 0, 1, 2, \ldots, N_T - 1$, and each fixed point iteration, $k = 0, 1, 2, \ldots$, a variational formulation for (2.25) is posed after applying Green's theorem: find $\widehat{V}^{n+1,k+1} \in \widehat{\mathcal{K}}$ such that:

$$\begin{aligned}
(1 + r\Delta\tau) \int_0^{S_\infty} & \widehat{V}^{n+1,k+1}(\varphi - \widehat{V}^{n+1,k+1}) dS \\
& + \Delta\tau \frac{\sigma^2}{2} \int_0^{S_\infty} S^2 \frac{\partial \widehat{V}^{n+1,k+1}}{\partial S} \frac{\partial(\varphi - \widehat{V}^{n+1,k+1})}{\partial S} dS \\
& \geq \int_0^{S_\infty} (\widehat{V}^n \circ \chi^n)(S)(\varphi - \widehat{V}^{n+1,k+1}) dS \\
& - \Delta\tau \int_0^{S_\infty} f(\widehat{V}^{n+1,k})(\varphi - \widehat{V}^{n+1,k+1}) dS\,, \quad \forall \varphi \in \widehat{\mathcal{K}}\,.
\end{aligned}$$

Associated to this uniform mesh a piecewise linear Lagrange finite element discretization is considered.

More precisely, we search $\widehat{V}_h^{n+1,k+1} \in \mathcal{K}_h$ such that:

$$\begin{aligned}
(1 + r\Delta\tau) \int_0^{S_\infty} & \widehat{V}_h^{n+1,k+1}(\varphi_h - \widehat{V}_h^{n+1,k+1}) dS \\
& + \Delta\tau \frac{\sigma^2}{2} \int_0^{S_\infty} S^2 \frac{\partial \widehat{V}_h^{n+1,k+1}}{\partial S} \frac{\partial(\varphi_h - \widehat{V}_h^{n+1,k+1})}{\partial S} dS \\
& \geq \int_0^{S_\infty} (U_h^n \circ \chi^n)(S)(\varphi_h - \widehat{V}_h^{n+1,k+1}) dS \\
& - \Delta\tau \int_0^{S_\infty} f(\widehat{V}_h^{n+1,k})(\varphi_h - \widehat{V}_h^{n+1,k+1}) dS\,, \quad \forall \varphi_h \in \mathcal{K}_h\,,
\end{aligned} \tag{2.33}$$

where the finite element space $\mathcal{K}_h$ is given by:

$$\mathcal{K}_h = \{\varphi_h : (0, S_\infty) \to \mathbb{R} : \varphi_h|_{[S_j, S_{j+1}]} \in \mathcal{P}_1 \text{ for } j = 1, 2, \ldots, N_S - 1\,,\ \varphi_h \in \widehat{\mathcal{K}}\}\,.$$

The coefficients of the matrix and right hand side vector defining the linear system associated to the fully discretized problem are approximated by adequate quadrature formulae. Once again, Simpson, three nodes Gaussian, midpoint and trapezoidal formulae have been used for the different terms. Finally, the system of linear equations is solved by the augmented Lagrangian active set algorithm, which is introduced in the next subsection.

2.3.5 An Augmented Lagrangian Active Set method

In this section we introduce the Augmented Lagrangian Active Set (ALAS) algorithm [35] to solve the discretized obstacle problem obtained after applying the numerical techniques previously described.

For the pricing of American options, the unknowns V^{n+1} and $\widehat{V}^{n+1,k+1}$ satisfy complementarity problems associated to linear and nonlinear partial differential equations (2.13) and (2.17), respectively. In order to explain their numerical solution, let us first focus on the nonlinear problem for $\widehat{V}^{n+1,k+1}$. After a time discretization by the method of characteristics and a spatial discretization with finite elements, the fully discretized problem can be written in the form:

$$\begin{cases} A_h \widehat{V}_h^{n+1,k+1} \geq b_h^{n+1,k+1} \\ \widehat{V}_h^{n+1,k+1} \geq \Psi_h \\ \left(A_h \widehat{V}_h^{n+1,k+1} - b_h^{n+1,k+1} \right) \left(\widehat{V}_h^{n+1,k+1} - \Psi_h \right) = 0 \end{cases} \tag{2.34}$$

for $n = 0, 1, \ldots, N_T - 1$ and $k = 0, 1, \ldots$, where Ψ_h denotes the discretized exercise value, $H(S)$, which also coincides with the value at maturity.

Following [6], the ALAS algorithm proposed by [35] has been implemented to solve (2.34). For this purpose, we introduce a multiplier P_h in order to write (2.34) in the equivalent form:

$$\begin{cases} A_h \widehat{V}_h^{n+1,k+1} + P_h^{n+1,k+1} = b_h^{n+1,k+1} \\ \widehat{V}_h^{n+1,k+1} \geq \Psi_h \\ P_h^{n+1,k+1} \leq 0 \\ (\widehat{V}_h^{n+1,k+1} - \Psi_h)\, P_h^{n+1,k+1} = 0\,. \end{cases} \tag{2.35}$$

Note that the last equation in (2.34) and (2.35) should be understood as componentwise.

ALAS algorithm consists of two steps. The first step decomposes the domain into active (that is, nodes where $P_h^{n+1,k+1} < 0$) and inactive (nodes where $P_h^{n+1,k+1} = 0$) regions. In the second step, a reduced linear system associated to the inactive part is solved.

First, let $\mathcal{N} := \{1, 2, \ldots, N_{dof}\}$ be the set of degrees of freedom. For any decomposition $\mathcal{N} = \mathcal{I} \cup \mathcal{J}$, the principal minor of matrix A_h is denoted by $[A_h]_{\mathcal{I},\mathcal{I}}$, while $[A_h]_{\mathcal{I},\mathcal{J}}$ is the codiagonal block indexed by $\mathcal{I}$ and $\mathcal{J}$. Therefore, for each time step $n+1$ and each fixed point iteration $k+1$, ALAS algorithm computes the decomposition $\mathcal{N} = \mathcal{I}^{n+1,k+1} \cup \mathcal{J}^{n+1,k+1}$ such that $\widehat{V}_h^{n+1,k+1}$ and $P_h^{n+1,k+1}$ are the solution of the following system:

$$\begin{aligned} A_h \widehat{V}_h^{n+1,k+1} + P_h^{n+1,k+1} &= b_h^{n+1,k+1} \\ [P_h^{n+1,k+1}]_j + \beta[\widehat{V}_h^{n+1,k+1} - \Psi_h]_j &\leq 0\,, \qquad && \forall j \in \mathcal{J}^{n+1,k+1} \\ [P_h^{n+1,k+1}]_i &= 0\,, && \forall i \in \mathcal{I}^{n+1,k+1} \end{aligned}$$

for a given positive parameter β. In the previous equations, $\mathcal{I}^{n+1,k+1}$ and $\mathcal{J}^{n+1,k+1}$ represent the inactive and the active sets, respectively. Namely, the iterative algorithm builds sequences $\{\widehat{V}_{h,m}^{n+1,k+1}\}_m$, $\{P_{h,m}^{n+1,k+1}\}_m$, $\{\mathcal{I}_m^{n+1,k+1}\}_m$ and $\{\mathcal{J}_m^{n+1,k+1}\}_m$ converging to $\widehat{V}_h^{n+1,k+1}$, $P_h^{n+1,k+1}$, $\mathcal{I}^{n+1,k+1}$ and $\mathcal{J}^{n+1,k+1}$, respectively, through the following steps:

1. Let be $\widehat{V}_{h,0}^{n+1,k+1} = \Psi_h$ and $P_{h,0}^{n+1,k+1} = \min\{b_h^{n+1,k+1} - A_h \widehat{V}_{h,0}^{n+1,k+1}, 0\} \leq 0$. Choose $\beta > 0$. Set $m = 0$.

2. Compute

$$
\begin{aligned}
Q_{h,m}^{n+1,k+1} &= \min\{0, P_{h,m}^{n+1,k+1} + \beta(\widehat{V}_{h,m}^{n+1,k+1} - \Psi_h)\} \\
\mathcal{J}_m^{n+1,k+1} &= \{j \in \mathcal{N}, [Q_{h,m}^{n+1,k+1}]_j < 0\} \\
\mathcal{I}_m^{n+1,k+1} &= \{i \in \mathcal{N}, [Q_{h,m}^{n+1,k+1}]_i = 0\}
\end{aligned}
$$

3. If $m \geq 1$ and $\mathcal{J}_m^{n+1,k+1} = \mathcal{J}_{m-1}^{n+1,k+1}$, then convergence is achieved.

4. Let $\bar{V}$ and $\bar{P}$ be the solution of the linear system:

$$
\begin{aligned}
&A_h\bar{V} + \bar{P} = b_h \\
&\bar{P} = 0 \quad \text{on} \quad \mathcal{I}_m^{n+1,k+1} \quad \text{and} \quad \bar{V} = \Psi_h \quad \text{on} \quad \mathcal{J}_m^{n+1,k+1}\,.
\end{aligned} \tag{2.36}
$$

   Set $\widehat{V}_{h,m+1}^{n+1,k+1} = \bar{V}$, $P_{h,m+1}^{n+1,k+1} = \min\{0, \bar{P}\}$, $m = m+1$ and go to step 2.

It is important to notice that, instead of solving the full linear system in (2.36), the following reduced system on the inactive set is solved:

$$
\begin{aligned}
[A_h]_{\mathcal{I},\mathcal{I}}[\bar{V}]_{\mathcal{I}} &= [b_h]_{\mathcal{I}} - [A_h]_{\mathcal{I},\mathcal{J}}[\Psi]_{\mathcal{J}} \\
[\bar{V}]_{\mathcal{J}} &= [\Psi]_{\mathcal{J}} \\
\bar{P} &= b_h - A_h\bar{V}\,,
\end{aligned}
$$

where we have denoted $\mathcal{I} = \mathcal{I}_m^{n+1,k+1}$ and $\mathcal{J} = \mathcal{J}_m^{n+1,k+1}$. Therefore, after applying the ALAS method to problems (2.17) and (2.13) or to problems (2.18) and (2.13), we can compute the XVA value as $U_h = \widehat{V}_h - V_h$. Analogously, the XVA is computed when collateral is not included in the contract.

2.4 A Monte Carlo approach

In this section, we introduce the most used methodology to price derivative products with counterparty risk. The derivative value is expressed in terms of expectations, then Monte Carlo methods are involved.

We mainly follow Longstaff and Schwartz [38] and Glasserman [28] in order to obtain the approximation of the risk–free option price and the risky option price. In this way, we finally compute the total value adjustment as the difference between both prices. A dynamic programming technique is implemented: at each time step an optimal stopping problem is solved, an optimal exercise criterion is stated and the expected discounted payoff of the option price under this criterion is computed.

We focus on problem (2.11) and (2.12), as the problem considering collateral can be solved using a similar procedure.

First, we introduce the description of the numerical algorithms implemented to compute the value of the risky option in the linear case, and in a second part, we present their adaption to numerically solve the analogous nonlinear complementarity problem.

2.4.1 The linear problem ($M = V$)

As we have introduced in Section 1.5, we assume that S_t follows a general geometric Brownian motion, thus satisfying:

$$dS_t = r_R S_t \, dt + \sigma S_t \, dW_t \,, \tag{2.37}$$

where r_R is the rate paid for the underlying asset in a repurchase agreement, σ is its volatility and W_t is a Wiener process.

Unlike the European option, which can only be exercised at maturity time T, an American option can be exercised at any time $t \in (0, T]$. We denote its exercise value at any time $t \in (0, T]$ as

$$h^*(t, S_t) = H(S_t) \,, \tag{2.38}$$

where $H(S_t)$ represents the payoff of the option. Note that the price process S_t is Markovian.

In our numerical approach, the value of V solving (2.13) will be estimated by a classical Monte Carlo technique for American options without counterparty risk.

In a first step, we consider problem (2.12). Let g be the function defined by:

$$g(V) = (R_B\lambda_B + \lambda_C)V^- + (R_C\lambda_C + \lambda_B)V^+ - s_F V^+ .$$

Following [43] we can deduce that, in terms of expectations, the risky derivative value at time $t = 0$ for the underlying value S_0 is given by:

$$\widehat{V}_0(S_0) = \sup_{\tau\in\mathcal{T}_0} \mathbb{E}_0 \left[e^{-r_0\tau} h^*(\tau, S_\tau) + \int_0^\tau e^{-r_0 u} g(V(u, S(u)))\, du \right],$$

where $r_0 = r + \lambda_B + \lambda_C$ and $\mathcal{T}_t$ is the set of admissible stopping instants in $[t, T]$.

In order to price the option, we first discretize the time interval by introducing a finite and increasing set of instants, $0 = t_0 < t_1 < t_2 < \ldots < t_{N_T} = T \subset [0, T]$.

We will assume that the option can only be exercised in t_i ($i = 0, 1, \ldots, N_T$). Therefore, we are approaching the American option by a Bermudan one. Taking into account the fixed instant times, we denote by $S_i = S(t_i)$, $i = 1, 2, \ldots, N_T$, the asset price at the i–th exercise opportunity. We approximate those values, solution of the stochastic differential equation (2.37), by the Euler–Maruyama scheme:

$$S_i = S_{i-1} + r_R S_{i-1}\Delta t + \sigma S_{i-1}\Delta W_i, \qquad i = 1, 2, \ldots, N_T\,, \tag{2.39}$$

where $\Delta t = t_i - t_{i-1}$ is the size of the time interval and $\Delta W_i = W_i - W_{i-1}$ is the independent Brownian increment, which follows a normal distribution $\mathcal{N}(0, \sqrt{\Delta t})$.

A dynamic programming formulation

Considering the previous time discretization for the asset price evolution, the American option with counterparty risk can be priced through a dynamic programming approach. Thus, in a particular time instant $t = t_i$, the risky derivative value is given by

$$\widehat{V}_i^*(s) = \sup_{\tau\in\mathcal{T}_{t_i}} \mathbb{E}_{t_i} \left[e^{-r_0(\tau - t_i)} h^*(\tau, S_\tau) + \int_{t_i}^\tau e^{-r_0(u-t_i)} g(V(u, S(u)))\, du \;\Big|\; S_i = s \right].$$

If we compute $\widehat{V}_i^*(s)$ for $i = N_T, \ldots, 1, 0$ (thus, from $t = T$ to $t = 0$), we define a strategy for pricing American options.

We know the option value at maturity ($t_{N_T} = T$):

$$\widehat{V}^*_{N_T}(s) = h^*(T, s)$$

for a given underlying value s. At time $t = t_{N_T-1}$, an investor will choose to exercise the option if and only if the payoff at this instant is greater than the discounted expected value to be received if the investor decides not to exercise. From this consideration, we have:

$$\widehat{V}^*_{N_T-1}(s) = \max\Bigg\{ h^*(t_{N_T-1}, s),$$
$$\mathbb{E}_{t_{N_T-1}}\left[D_{N_T-1,N_T}\widehat{V}^*_{N_T}(S_{N_T}) + \int_{t_{N_T-1}}^{t_{N_T}} e^{-r_0(u-t_{N_T-1})} g(V(u, S(u)))\, du \;\Big|\; S_{N_T-1} = s \right] \Bigg\},$$

where the discounting factor is defined by $D_{i-1,i} = e^{-r_0(t_i - t_{i-1})}$. Thus, the recursive formula is given by:

$$\widehat{V}^*_{N_T}(s) = h^*(T, s)\,, \quad S_{N_T} = s\,,$$
$$\widehat{V}^*_{i-1}(s) = \max\Bigg\{ h^*(t_{i-1}, s),$$
$$\mathbb{E}_{t_{i-1}}\left[D_{i-1,i}\widehat{V}^*_i(S_i) + \int_{t_{i-1}}^{t_i} e^{-r_0(u-t_{i-1})} g(V(u, S(u)))\, du \;\Big|\; S_{i-1} = s \right] \Bigg\}, \quad (2.40)$$

for $i = N_T, N_T - 1, \ldots, 1$.

Note that we are interested in obtaining the discounted values at $t_0 = 0$, so we consider

$$h_i(s) = D_{0,i}h^*(t_i, s)\,, \qquad \widehat{V}_i(s) = D_{0,i}\widehat{V}^*_i(s) \quad (i = 0, \ldots, N_T)\,.$$

Taking into account that $\widehat{V}_0(s) = \widehat{V}^*_0(s)$ and the recursive expression given in (2.40), we obtain:

$$\widehat{V}_{N_T}(s) = h_{N_T}(s)$$
$$\widehat{V}_{i-1}(s) = D_{0,i-1}\widehat{V}^*_{i-1}(s)$$

$$
\begin{aligned}
&= D_{0,i-1} \max\Bigg\{ h^*(t_{i-1}, s), \quad \mathbb{E}_{t_{i-1}}\Bigg[D_{i-1,i}\widehat{V}_i^*(S_i) \\
&\qquad\qquad + \int_{t_{i-1}}^{t_i} e^{-r_0(u-t_{i-1})} g(V(u,S(u)))\,du \;\Big|\; S_{i-1} = s \Bigg]\Bigg\} \\
&= \max\Bigg\{ h_{i-1}(s), \quad \mathbb{E}_{t_{i-1}}\Bigg[D_{0,i-1}D_{i-1,i}\widehat{V}_i^*(S_i) \\
&\qquad\qquad + \int_{t_{i-1}}^{t_i} D_{0,i-1}e^{-r_0(u-t_{i-1})} g(V(u,S(u)))\,du \;\Big|\; S_{i-1} = s \Bigg]\Bigg\} \\
&= \max\Bigg\{ h_{i-1}(s),\ \mathbb{E}_{t_{i-1}}\Bigg[D_{0,i}\widehat{V}_i^*(S_i) + \int_{t_{i-1}}^{t_i} e^{-r_0 u} g(V(u,S(u)))\,du \;\Big|\; S_{i-1} = s \Bigg]\Bigg\},
\end{aligned}
$$

for $i = N_T, N_T - 1, \ldots, 1$. Introducing the discounting factor in the payoff and in the functions, the previous expressions can be simplified:

$$
\begin{aligned}
&\widehat{V}_{N_T}(s) = h(T,s)\,, \quad S_{N_T} = s \\
&\widehat{V}_{i-1}(s) = \max\Bigg\{ h_{i-1}(s), \mathbb{E}_{t_{i-1}}\Bigg[\widehat{V}_i(S_i) \\
&\qquad\qquad + \int_{t_{i-1}}^{t_i} e^{-r_0 u} g(V(u,S(u)))\,du \;\Big|\; S_{i-1} = s \Bigg]\Bigg\}, \qquad (2.41)
\end{aligned}
$$

for $i = N_T, N_T - 1, \ldots, 1$.

Optimal stopping rule and continuation value

In the previous section we have approximated the option value in a recursive way. However, it is also important to price the option through stopping rules and exercise region. In that sense, any stopping time τ determines the sub–optimal value

$$
\widehat{V}_0^{\tau}(S_0) = \mathbb{E}_0\left[h_\tau(S_\tau) + \int_0^\tau e^{-r_0 u} g(V(u,S(u)))\,du \right].
$$

Our aim is to choose the optimal stopping time, which will be determined by

$$
\tau^* = \min\left\{ \tau_i \in \{t_1, \ldots, t_{N_T}\} : h_i(S_i) \geq \widehat{V}_i(S_i) \right\}, \qquad (2.42)
$$

so that the exercise region associated to $\widehat{V}_i$ at the i–th exercise date is the set

$$\left\{ s : \, h_i(s) = \widehat{V}_i(s) \right\}.$$

After defining the optimal stopping rule we introduce the continuation value, which is the value of holding instead of exercising the option. This continuation value can be computed in a recursive way as:

$$\begin{aligned} &C_{N_T}(s) = 0, \\ &C_i(s) = \mathbb{E}_{t_i}\left[\widehat{V}_{i+1}(S_{i+1}) + \int_{t_i}^{t_{i+1}} e^{-r_0 u} g(V(u, S(u)))\, du \;\Big|\; S_i = s \right], \end{aligned}$$

for $i = N_T - 1, \ldots, 0$, where $\widehat{V}_i$ is obtained as the solution of the recursive dynamic programming problem. Moreover, according to (2.41) the option value is given in terms of the continuation and exercise values as follows:

$$\widehat{V}_i(s) = \max\{h_i, C_i\}, \qquad i = 1, \ldots, N_T.$$

Thus, the optimal stopping rule can be rewritten as

$$\tau^* = \min \left\{ \tau_i \in \{t_1, \ldots, t_{N_T}\} : \, h_i(S_i) \geq C_i(S_i) \right\}. \tag{2.43}$$

In terms of the optimal stopping time, the option value is determined by

$$\widehat{V}_0^{\tau^*}(S_0) = \mathbb{E}_0\left[h_{\tau^*}(S_{\tau^*}) + \int_0^{\tau^*} e^{-r_0 u} g(V(u, S(u)))\, du \right].$$

Lower bounds estimator using least-squares regressions

We now introduce the approximations, $\kappa_i(s)$, of the continuation values, $C_i(s)$. Several authors, cf. Longstaff and Schwartz [38] for example, have proposed a least–squares regression to estimate these values from the simulated paths. In this way, the value $C_i(s)$ can be obtained as the regression of

$$\widehat{V}_{i+1}(S_{i+1}) + \int_{t_i}^{t_{i+1}} e^{-r_0 u} g(V(u, S(u)))\, du$$

on the current state of the asset price s. Thus, C_i is approximated by a linear combination of known functions of the current state using a least–squares regression that leads to coefficients κ_i.

Following this idea, we introduce how to approximate the continuation values considering counterparty risk. We will write the continuation value as a linear combination of basis functions as follows:

$$\begin{aligned} C_i(s) &= \mathbb{E}_{t_i}\left[\widehat{V}_{i+1}(S_{i+1}) + \int_{t_i}^{t_{i+1}} e^{-r_0 u} g(V(u, S(u)))\, du \,\Big|\, S_i = s\right] \\ &= \sum_{j=1}^{J} b_{ij}\psi_j(s) = b_i^T \psi(s)\,, \end{aligned} \tag{2.44}$$

where $b_i = (b_{i1}, \ldots, b_{iJ})^T$ are the regression coefficients at time t_i and

$$\psi(s) = (\psi_1(s), \ldots, \psi_J(s))^T$$

is the vector of basis functions.

Different bases can be used to approximate the continuation value. We focus on the weighted Laguerre polynomials:

$$\psi_j(x) = e^{-x/2} L_{j-1}(x)\,, \quad j = 1, 2, \ldots$$

where L_j is the j–th Laguerre polynomial.

Next, we determine the expression of the regression coefficients b_i using a least–squares optimization technique. Let φ the function to minimize:

$$\varphi(b_i) = \mathbb{E}_{t_i}\left[\left(\psi(S_i)^T b_i - \mathbb{E}_{t_i}\left[\widehat{V}_{i+1}(S_{i+1}) + \int_{t_i}^{t_{i+1}} e^{-r_0 u} g(V(u, S(u)))\, du \,\Big|\, S_i = s\right]\right)^2\right].$$

In order to minimize, we vanish the derivatives with respect to b_i, so that we get:

$$\mathbb{E}_{t_i}\left[\psi(S_i)\left(\psi(S_i)^T b_i - \mathbb{E}_{t_i}\left[\widehat{V}_{i+1}(S_{i+1}) + \int_{t_i}^{t_{i+1}} e^{-r_0 u} g(V(u, S(u)))\, du \,\Big|\, S_i = s\right]\right)\right] = 0$$

or, equivalently,

$$\mathbb{E}_{t_i}[\psi(S_i)\psi(S_i)^T]\, b_i = \mathbb{E}_{t_i}\left[\psi(S_i)\,\mathbb{E}_{t_i}\left[\widehat{V}_{i+1}(S_{i+1}) \;\Big|\; S_i\right]\right]$$

$$+\, \mathbb{E}_{t_i}\left[\psi(S_i)\,\mathbb{E}_{t_i}\left[\int_{t_i}^{t_{i+1}} e^{-r_0 u} g(V(u,S(u)))\,du \;\Big|\; S_i = s\right]\right]$$

$$= \mathbb{E}_{t_i}\left[\psi(S_i)\,\widehat{V}_{i+1}(S_{i+1})\right] + \mathbb{E}_{t_i}\left[\psi(S_i)\,\mathbb{E}_{t_i}\left[\int_{t_i}^{t_{i+1}} e^{-r_0 u} g(V(u,S(u)))\,du \;\Big|\; S_i = s\right]\right].$$

Thus, the expression of b_i is approximated by β_i, which satisfies the linear system:

$$A_i^{\psi}\,\beta_i = d_i^{\psi}\,,$$

where A_i^{ψ} and d_i^{ψ} can be easily estimated by Monte Carlo simulations. For this purpose, let us consider independent paths $(S_{j,1}, S_{j,2}, \ldots, S_{j,N_T})$ $(j = 1, 2, \ldots, N_P)$, that can be deduced by (2.39), and assume that the value $V_{i+1}(S_{j,i+1})$ is known at time t_i. Then, A_i^{ψ} is a $N_T \times N_T$ matrix with coefficients:

$$(A_i^{\psi})_{l,k} = \frac{1}{N_P}\sum_{j=1}^{N_P} \psi_l(S_{j,i})\,\psi_k(S_{j,i})$$

and d_i^{ψ} is the N_T–array with the k–th element given by

$$(d_i^{\psi})_k = \frac{1}{N_P}\sum_{j=1}^{N_P} \psi_k(S_{j,i})\,\widehat{W}_{i+1}(S_{j,i+1}) + \frac{1}{N_P}\sum_{j=1}^{N_P} \psi_k(S_{j,i}) \int_{t_i}^{t_{i+1}} e^{-r_0 u} g(W(u,S(u)))\,du\,,$$

where $S_{j,i}$ and $S_{j,i+1}$ correspond to the same trajectory. Moreover, W denotes the risk–free value estimated by the classical Longstaff–Schwartz algorithm while $\widehat{W}_{i+1}$ is the estimation of the risky value in the previous time step.

Thus, the continuation value C_i can be approximated by:

$$\kappa_i = \beta_i^T \psi(S_i) \tag{2.45}$$

and the risky derivative value can be replaced by its estimated value

$$\widehat{W}_{i+1} = \max\left\{h_{i+1}(S_{i+1}), \kappa_{i+1}\right\}.$$

All these steps are merged in Algorithm 2.2.

Let us remark that in Algorithm 2.2 we have to apply an *inner* Monte Carlo method at each step of time and for each asset price path, what makes this solution very expensive from the computational point of view.

With the aim of reducing this computational cost, we introduce a second alternative to solve the same problem (Algorithm 2.3). In this alternative, we propose to compute the risk–free derivative value, W, for a set of asset prices at each instant time of the discretization used to obtain the risky derivative value. The classical Longstaff–Schwartz algorithm is employed. Then, in each integral, the risk–free derivative value has to be evaluated in the state of the asset price at instant t_i. Instead of the exact value, we propose the use of the interpolated value computed from the set of fixed values previously obtained for different asset prices.

Low–biased estimator using optimal stopping rule

After obtaining the regression coefficients, we compute the value of the American option with counterparty risk, by simulating a new set of paths independent from the previously used prices. Then, the optimal stopping strategy is determined with the previous algorithm, given the state of the asset price S_i. Thus,

$$\widehat{\tau} = \min \Big\{ \tau_i \in \{t_1, \ldots, t_{N_T}\} : h_i(S_i) \geq \kappa_i(S_i) \Big\}.$$

By using this stopping strategy, with the second set of paths, the risky American option value is estimated as

$$\widehat{W}_0(S_0) = \mathbb{E}_0 \Bigg[h_{\widehat{\tau}}(S_{\widehat{\tau}}) + \int_0^{\widehat{\tau}} e^{-r_0 u} \left[(R_B \lambda_B + \lambda_C) W(u, S(u))^- \right. \\ \left. + (R_C \lambda_C + \lambda_B) W(u, S(u))^+ - s_F W(u, S(u))^+ \right] du \Bigg]. \qquad (2.46)$$

Algorithm 2.2 Regression coefficients β_i (without interpolation)

1. Simulate N_P independent paths $\{S_{j,1}, S_{j,2}, \ldots, S_{j,N_T}\}$ (for $j = 1, \ldots, N_P$) of the asset prices process.
2. At maturity time t_{N_T}, $\widehat{W}_{N_T}(S_{j,N_T}) = h_{N_T}(S_{j,N_T})$.
3. Apply backward induction for $i = N_T - 1, \ldots, 1$.
   - Compute the classical Longstaff–Schwartz approximation with $S_0 = S_{j,i}$ for the time interval $[t_i, T]$ to obtain $W_{j,i}$.
   - Given the estimated value $\widehat{W}_{j,i+1}$ and $W_{j,i}$ ($j = 1, \ldots, N_P$), compute β_i as the solution of the linear system $A_i^{\psi}\beta_i = d_i^{\psi}$.
   - Estimate the continuation value $\kappa_i(S_{j,i}) = \beta_i^T \psi(S_{j,i})$ ($j = 1, \ldots, N_P$).
   - Compute $\widehat{W}_{j,i}^{k+1} = \max\{h_i(S_{j,i}), \kappa_i(S_{j,i})\}$.
4. Save the regression coefficients β_i to compute the risky derivative value.

Algorithm 2.3 Regression coefficients β_i (with interpolation)

1. Simulate N_P independent paths $\{S_{j,1}, S_{j,2}, \ldots, S_{j,N_T}\}$ (for $j = 1, \ldots, N_P$) of the asset prices process.
2. Apply forward induction for $i = 0, 1, \ldots, N_T - 1$. Compute the risk–free derivative value for different asset values in the time interval $[t_i, T]$.
3. At maturity time t_{N_T}, $\widehat{W}_{N_T}(S_{j,N_T}) = h_{N_T}(S_{j,N_T})$.
4. Apply backward induction for $i = N_T - 1, \ldots, 1$.
   - Interpolate the risk–free derivative value for the asset price $S_{j,i}$ at time t_i.
   - Given the estimated values $\widehat{W}_{j,i+1}$ and $W_{j,i}$ ($j = 1, \ldots, N_P$), compute β_i as the solution of the linear system $A_i^{\psi}\beta_i = d_i^{\psi}$.
   - Estimate the continuation value $\kappa_i(S_{j,i}) = \beta_i^T \psi(S_{j,i})$ ($j = 1, \ldots, N_P$).
   - Compute $\widehat{W}_{j,i}^{k+1} = \max\{h_i(S_{j,i}), \kappa_i(S_{j,i})\}$.
5. Save the regression coefficients β_i to compute the risky derivative value.

Taking into account the expression of the risky derivative value $\widehat{V}_0(S_0)$, given by

$$\widehat{V}_0(S_0) = \sup_{\tau \in \mathcal{T}_0} \mathbb{E}_0 \Bigg[h(\tau, S_\tau) + \int_0^\tau e^{-r_0 u} \left[(R_B\lambda_B + \lambda_C) V(u, S(u))^- \right.$$

$$\left. + (R_C\lambda_C + \lambda_B) V(u, S(u))^+ - s_F V(u, S(u))^+ \right] du \Bigg]$$

$$\geq \mathbb{E}_0 \Bigg[h_{\widehat{\tau}}(S_{\widehat{\tau}}) + \int_0^{\widehat{\tau}} e^{-r_0 u} \left[(R_B\lambda_B + \lambda_C) W(u, S(u))^- \right.$$

$$\left. + (R_C\lambda_C + \lambda_B) W(u, S(u))^+ - s_F W(u, S(u))^+ \right] du \Bigg] = \widehat{W}_0(S_0)\,,$$

we deduce that the estimator defined in (2.46) is a low–biased estimator which provides a lower bound of the theoretical value. The algorithm that provides the low estimator is shown as Algorithm 2.4.

Algorithm 2.4 Derivative value estimation

1. Load regression coefficients β_i $(i = 1, \ldots, N_T)$.
2. Simulate N_P independent paths $\{S_{j,1}, S_{j,2}, \ldots, S_{j,N_T}\}$ (for $j = 1, \ldots, N_P$) of the asset prices process from the first one used.
3. Apply forward induction for $i = 1, \ldots, N_T - 1$ and $j = 1, \ldots, N_P$.
   - Compute the continuation value $\kappa_i(S_{j,i}) = \beta_i^T \psi(S_{j,i})$ $(j = 1, \ldots, N_P)$.
   - Compute the payoff functions $h_i(S_{j,i})$.
4. At maturity time t_{N_T}, $\widehat{W}_{N_T}(S_{j,N_T}) = h_{N_T}(S_{j,N_T})$ and $C_{N_T}(S_{j,N_T}) = 0$.
5. Compute $\widehat{W}_{j,0}(S_0) = h_{i^*}(S_{j,i^*})$ $(i^* = \min\{i \in \{1, \ldots, N_T\} : h_i(S_{j,i}) \geq \kappa_i(S_{j,i})\})$.
6. Calculate the estimated value of the option: $\widehat{W}_0(S_0) = \frac{1}{N_P} \sum_{j=1}^{N_P} \widehat{W}_{j,0}$.

Duality. Upper bounds estimator using martingales

As we have seen in the previous paragraph, the estimator of the American option, obtained by using least square regression, was a lower estimator on the real American option value. In this section an upper estimator using martingales is considered. For this purpose, we follow the works of Haugh and Kogan [29] and Rogers [46]. Both have established dual formulations which represent the price of an American option through a suitable minimization problem. The duality technique minimizes over a class of supermartingales or martingales and leads to a high–biased approximation, therefore obtaining upper bounds on prices.

As we have seen in (2.41), the discounted value $\widehat{V}_i(S_i)$ satisfies the recursive formulation

$$
\begin{aligned}
\widehat{V}_{N_T}(s) &= h(T,s)\,, \quad S_{N_T} = s \\
\widehat{V}_{i-1}(s) &= \max\left\{ h_{i-1}(s), \mathbb{E}_{t_{i-1}}\left[\widehat{V}_i(S_i) + \int_{t_{i-1}}^{t_i} e^{-r_0 u}\left[(R_B\lambda_B + \lambda_C)V(u,S(u))^- \right.\right.\right. \\
&\qquad \left.\left.\left. +(R_C\lambda_C + \lambda_B)V(u,S(u))^+ - s_F V(u,S(u))^+ \right] du \;\middle|\; S_{i-1} = s \right]\right\},
\end{aligned}
$$

for $i = N_T, N_T - 1, \ldots, 1$. From the previous recursive formula, the following inequality is obtained:

$$
\begin{aligned}
\widehat{V}_i(S_i) &\geq \mathbb{E}_{t_i}\left[\widehat{V}_{i+1}(S_{i+1}) + \int_{t_i}^{t_{i+1}} e^{-r_0 u}\left[(R_B\lambda_B + \lambda_C)V(u,S(u))^- \right.\right. \\
&\qquad \left.\left. +(R_C\lambda_C + \lambda_B)V(u,S(u))^+ - s_F V(u,S(u))^+ \right] du \;\middle|\; S_i \right] \\
&\geq \mathbb{E}_{t_i}\left[\widehat{V}_{i+1}(S_{i+1}) \;\middle|\; S_i \right],
\end{aligned}
$$

for $i = 0, \ldots, N_T - 1$. Thus, we can conclude that $\widehat{V}_i$ is a supermartingale [43].

On the other hand, the American option price satisfies:

$$
\widehat{V}_i(S_i) \geq h_i(S_i), \quad i = 0, \ldots, N_T\,.
$$

Thus, the value function process $\widehat{V}_i(S_i)$ $(i = 0, \ldots N_T)$ is the minimal supermartingale dominating $h_i(S_i)$ at each exercise time t_i.

Let $\mathcal{M} = \{\mathcal{M}_i, i = 0, \ldots, N_T\}$ be a martingale, with $\mathcal{M}_0 = 0$. By the optimal stopping theorem of martingales, the expected value of a martingale at a stopping time is equal to the expected value of its initial value. Then, for any stopping time $\tau \in \{t_1, t_2, \ldots, t_{N_T}\}$, we have $\mathbb{E}[\mathcal{M}_\tau] = \mathcal{M}_0 = 0$ and we can deduce:

$$\begin{aligned}
&\mathbb{E}_0\left[h_\tau(S_\tau) + \int_0^\tau e^{-r_0 u} g(V(u, S(u)))\, du\right] \\
&\quad = \mathbb{E}_0\left[h_\tau(S_\tau) + \int_0^\tau e^{-r_0 u} g(V(u, S(u)))\, du - \mathcal{M}_\tau\right] \\
&\quad \leq \mathbb{E}_0\left[\max_{i=1,\ldots,N_T}\left(h_i(S_i) + \int_0^{t_i} e^{-r_0 u} g(V(u, S(u)))\, du - \mathcal{M}_i\right)\right]. \qquad (2.47)
\end{aligned}$$

Moreover, in terms of the infimum over martingales $\mathcal{M}$ with initial value $\mathcal{M}_0 = 0$, we obtain

$$\begin{aligned}
&\mathbb{E}_0\left[h_\tau(S_\tau) + \int_0^\tau e^{-r_0 u} g(V(u, S(u)))\, du\right] \\
&\quad \leq \inf_{\mathcal{M}} \mathbb{E}_0\left[\max_{i=1,\ldots,N_T}\left(h_i(S_i) + \int_0^{t_i} e^{-r_0 u} g(V(u, S(u)))\, du - \mathcal{M}_i\right)\right], \qquad (2.48)
\end{aligned}$$

which holds for any stopping time τ. Thus, the American option price written in terms of the supremum over τ leads to the following inequality:

$$\begin{aligned}
\widehat{V}_0(S_0) &= \sup_\tau \mathbb{E}_0\left[h_\tau(S_\tau) + \int_0^\tau e^{-r_0 u} g(V(u, S(u)))\, du\right] \\
&\leq \inf_{\mathcal{M}} \mathbb{E}_0\left[\max_{i=1,\ldots,N_T}\left(h_i(S_i) + \int_0^{t_i} e^{-r_0 u} g(V(u, S(u)))\, du - \mathcal{M}_i\right)\right] \qquad (2.49)
\end{aligned}$$

for every martingale $\mathcal{M}$. The minimization problem on the right hand side is known as dual problem.

Next, let us consider the stochastic process defined by:

$$\mathcal{M}_0 = 0\,, \qquad \mathcal{M}_i = \sum_{k=1}^{i} \Delta_k\,, \quad i = 1, \ldots, N_T\,, \qquad (2.50)$$

where $\Delta_k = \widehat{V}_k(S_k) - \mathbb{E}_{t_{k-1}}[\widehat{V}_k(S_k) \mid S_{k-1}]$. We can easily prove that this process is a martingale, so that it satisfies (2.49).

Taking into account the definition of Δ_k, we have

$$\mathbb{E}_{t_{i-1}}[\Delta_i \mid S_{i-1}] = \mathbb{E}_{t_{i-1}}\Big[\widehat{V}_i(S_i) - \mathbb{E}_{t_{i-1}}[\widehat{V}_i(S_i) \mid S_{i-1}] \mid S_{i-1}\Big] = 0.$$

For this purpose, first we have

$$\mathbb{E}_{t_{i-1}}[\mathcal{M}_i \mid S_{i-1}] = \mathbb{E}_{t_{i-1}}\Big[\sum_{k=1}^{i} \Delta_k \mid S_{i-1}\Big] = \sum_{k=1}^{i-1} \Delta_k = \mathcal{M}_{i-1}, \tag{2.51}$$

which shows that $\mathcal{M}$ satisfies the martingale property.

Furthermore, we can also prove [28]:

$$\widehat{V}_0(S_0) = \mathbb{E}_0\left[\max_{i=1,\dots,N_T}\left\{h_i(S_i) + \int_0^{t_i} e^{-r_0 u} g(V(u, S(u)))\, du - \mathcal{M}_i\right\}\right]. \tag{2.52}$$

Thus, inequality (2.49) holds for our particular choice of martingale.

Next, we use backward induction to prove that

$$\begin{aligned}\widehat{V}_i(S_i) = \mathbb{E}_{t_i}\Bigg[\max\Big\{&h_i(S_i) + \int_{t_i}^{t_i} e^{-r_0 u} g(V(u, S(u)))du,\\ &h_{i+1}(S_{i+1}) + \int_{t_i}^{t_{i+1}} e^{-r_0 u} g(V(u, S(u)))du - \Delta_{i+1},\\ &h_{i+2}(S_{i+2}) + \int_{t_i}^{t_{i+2}} e^{-r_0 u} g(V(u, S(u)))du - \Delta_{i+2} - \Delta_{i+1}, \dots,\\ &h_{N_T} + \int_{t_i}^{t_{N_T}} e^{-r_0 u} g(V(u, S(u)))du - \Delta_{N_T} - \dots - \Delta_{i+1}\Big\} \;\Big|\; S_i\Bigg].\end{aligned} \tag{2.53}$$

For the maturity time t_{N_T}, we have $\widehat{V}_{N_T}(S_{N_T}) = h_{N_T}(S_{N_T}) = \mathbb{E}[h_{N_T}(S_{N_T}) \mid S_{N_T}]$. So, equality (2.53) is satisfied.

Next, we assume that (2.53) is satisfied at time t_i. We obtain

$$\begin{aligned}\widehat{V}_{i-1}(S_{i-1}) &= \max\left\{h_{i-1}(S_{i-1}), \mathbb{E}_{t_i}\left[\widehat{V}_i(S_i) + \int_{t_{i-1}}^{t_i} e^{-r_0 u} g(V(u, S(u)))du \;\Big|\; S_{i-1}\right]\right\}\\ &= \mathbb{E}_{t_{i-1}}\Bigg[\max\Bigg\{h_{i-1}(S_{i-1}), \mathbb{E}_{t_i}\Bigg[\widehat{V}_i(S_i)\end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad + \int_{t_{i-1}}^{t_i} e^{-r_0 u} g(V(u,S(u)))du \;\Big|\; S_{i-1}\Big]\Big\} \;\Big|\; S_{i-1}\Big] \\
&= \mathbb{E}_{t_{i-1}}\Big[\max\Big\{h_{i-1}(S_{i-1}), \widehat{V}_i(S_i) \\
&\qquad\qquad + \int_{t_{i-1}}^{t_i} e^{-r_0 u} g(V(u,S(u)))du - \Delta_i\Big\} \;\Big|\; S_{i-1}\Big] \\
&= \mathbb{E}_{t_{i-1}}\Big[\max\Big\{h_{i-1}(S_{i-1}), h_i(S_i) + \int_{t_{i-1}}^{t_i} e^{-r_0 u} g(V(u,S(u)))du - \Delta_i, \\
&\quad h_{i+1}(S_{i+1}) + \int_{t_{i-1}}^{t_{i+1}} e^{-r_0 u} g(V(u,S(u)))du - \Delta_{i+1} - \Delta_i, \ldots, \\
&\quad h_{N_T}(S_{N_T}) + \int_{t_{i-1}}^{t_{N_T}} e^{-r_0 u} g(V(u,S(u)))du - \Delta_{N_T} - \ldots - \Delta_i\Big\} \;\Big|\; S_{i-1}\Big],
\end{aligned}
$$

so that (2.53) also holds for t_{i-1}. Finally, at $t = t_0$ the American option value is given by

$$
\begin{aligned}
\widehat{V}_0(S_0) &= \mathbb{E}_0\Big[\widehat{V}_1(S_1) + \int_0^{t_1} e^{-r_0 u} g(V(u,S(u)))\,du \;\Big|\; S_0\Big] \\
&= \widehat{V}_1(S_1) + \int_0^{t_1} e^{-r_0 u} g(V(u,S(u)))du - \Delta_1. \qquad (2.54)
\end{aligned}
$$

Moreover, according to (2.53)

$$
\begin{aligned}
\widehat{V}_1(S_1) = \mathbb{E}_{t_1}\Big[\max\Big\{&h_1(S_1) + \int_{t_1}^{t_1} e^{-r_0 u} g(V(u,S(u)))\,du, \\
&h_2(S_2) + \int_{t_1}^{t_2} e^{-r_0 u} g(V(u,S(u)))\,du - \Delta_2, \\
&h_3(S_3) + \int_{t_1}^{t_3} e^{-r_0 u} g(V(u,S(u)))\,du - \Delta_3 - \Delta_2, \ldots, \\
&h_{N_T}(S_{N_T}) + \int_{t_1}^{t_{N_T}} e^{-r_0 u} g(V(u,S(u)))\,du - \Delta_{N_T} - \ldots - \Delta_2\Big\} \;\Big|\; S_1\Big]. \qquad (2.55)
\end{aligned}
$$

Then, we have

$$\widehat{V}_0(S_0) = \mathbb{E}_{t_1}\Bigg[\max\Bigg\{h_1(S_1) + \int_0^{t_1} e^{-r_0 u} g(V(u, S(u)))\,du - \Delta_1,$$
$$h_2(S_2) + \int_0^{t_2} e^{-r_0 u} g(V(u, S(u)))\,du - \Delta_2 - \Delta_1,$$
$$h_3(S_3) + \int_0^{t_3} e^{-r_0 u} g(V(u, S(u)))\,du - \Delta_3 - \Delta_2 - \Delta_1, \dots,$$
$$h_{N_T}(S_{N_T}) + \int_0^{t_{N_T}} e^{-r_0 u} g(V(u, S(u)))\,du - \Delta_{N_T} - \dots - \Delta_1\Bigg\} \;\Bigg|\; S_1\Bigg]. \tag{2.56}$$

In consequence, we get

$$\widehat{V}_0(S_0) = \mathbb{E}_0\left[\max_{i=1,\dots,N_T}\left\{h_i(S_i) + \int_0^{t_i} e^{-r_0 u} g(V(u, S(u)))\,du - \mathcal{M}_i\right\}\right], \tag{2.57}$$

which proves inequality (2.49) for the martingale defined by (2.50). Moreover, by (2.57) we have obtained an upper estimator for the American options price with counterparty risk.

Our next goal is, for practical purpose, to find a computable estimated martingale $\widehat{\mathcal{M}}$ close to the optimal one, $\mathcal{M}$, in order to obtain the following estimated value of $\widehat{V}_0$:

$$\widehat{W}_0(S_0) = \mathbb{E}_0\left[\max_{i=1,\dots,N_T}\left\{h_i(S_i) + \int_0^{t_i} e^{-r_0 u} g(W(u, S(u)))du - \widehat{\mathcal{M}}_i\right\}\right], \tag{2.58}$$

which is the so called duality estimator.

Next, the computation of such martingale is detailed. We construct the martingale $\widehat{\mathcal{M}}_i$. Thus, we follow the definition given in (2.50) to find the suitable martingale.

$$\widehat{\mathcal{M}}_0 = 0\,, \qquad\qquad \widehat{\mathcal{M}}_i = \sum_{k=1}^{i} \widehat{\Delta}_k, \quad i = 1, \dots, N_T, \tag{2.59}$$

where $\widehat{\Delta}_k$ is given by $\widehat{\Delta}_i = \widehat{W}_i(S_i) - \mathbb{E}_{t_{i-1}}[\widehat{W}_i(S_i) \mid S_{i-1}]$. Then, $\widehat{\mathcal{M}}$ satisfies the general martingale property.

Note that $\widehat{\Delta}_k$ is now expressed in terms of the estimated value of the American options, which was given by

$$\widehat{W}_i = \max\{h_i(S_i), \kappa_i(S_i)\}\,, \tag{2.60}$$

where κ_i was defined in (2.45). In (2.45) the vector β_i and the function bases ψ are the same as for the least square method.

Next, we explain how to estimate the martingale value. For this purpose, we assume that we have simulated the main Monte Carlo paths $\{S_{j,i}, j = 1, \ldots, N_P\}$. Then, for each S_{i-1} we simulate N_I successors $\{\tilde{S}_{k,i}, k = 1, \ldots, N_I\}$, and estimate the conditional expectation $\mathbb{E}_{t_{i-1}}[\widehat{W}_i(S_i) \mid S_{i-1}]$ by

$$\mathbb{E}_{t_{i-1}}[\widehat{W}_i(S_i) \mid S_{i-1}] = \frac{1}{N_I}\sum_{k=1}^{N_I} \widehat{W}_i(\tilde{S}_{k,i}), \tag{2.61}$$

where $\widehat{W}_i(\tilde{S}_{k,i})$ is calculated as in (2.60). Then, the estimated value $\widehat{\Delta}_i$ is given by

$$\widehat{\Delta}_i = \widehat{W}_i(S_i) - \frac{1}{N_T}\sum_{k=1}^{N_T} \widehat{W}_i(\tilde{S}_{k,i}) \tag{2.62}$$

which gives the upper–biased estimator.

Finally, Algorithm 2.5 sketches the computation of this dual estimator.

Confidence intervals

We take into account the lower and upper estimators developed in the previous sections to propose confidence intervals that contain the American option price.

We denote by $\underline{V}$ and $\overline{V}$ the lower and upper estimators, respectively, both computed with N_P paths. Then, the $(1-\alpha)$ confidence interval is given by

$$\left(\underline{V} - z_{\alpha/2}\frac{s_{\underline{V}}(N_P)}{\sqrt{N_P}}\,,\ \overline{V} + z_{\alpha/2}\frac{s_{\overline{V}}(N_P)}{\sqrt{N_P}}\right),$$

where $s_{\underline{V}}(N_P)$ and $s_{\overline{V}}(N_P)$ denote the respective sample standard deviations and $z_{\alpha/2}$ represents the $(1-\alpha/2)$ quantile of the normal distribution.

Algorithm 2.5 Dual estimator using martingales

1. Load regression coefficients β_i, $i = 1, \ldots, N_T$ given by Algorithms 2.2 or 2.3
2. Simulate N_P independent paths $\{S_{j,1}, S_{j,2}, \ldots, S_{j,N_T}\}$ (for $j = 1, \ldots, N_P$) of the asset prices process.
3. Set the initial martingale $\widehat{\mathcal{M}}_0 = 0$
4. For each $j = 1, \ldots, N_P$, apply forward induction for $i = 1, \ldots, N_T$.
   - Compute the continuation values κ_i.
   - Estimate the American option price, $\widehat{W}_i(S_{j,i}) = \max\{h_i(S_{j,i}), \kappa_i(S_{j,i})\}$.
   - Simulate N_I subpaths $\{\tilde{S}_{1,i}, \tilde{S}_{2,i}, \ldots, \tilde{S}_{N_I,i}\}$ starting from $S_{j,i-1}$.
   - Compute the estimation of the martingale differential $\widehat{\Delta}_i$
   - Obtain the martingales $\widehat{\mathcal{M}}_i = \widehat{\mathcal{M}}_{i-1} + \widehat{\Delta}_i$
5. Set $\widehat{W}_{0,j}(S_0) = \max\limits_{i=1,\ldots,N_T} \left(h_i(S_{j,i}) + \int_0^{t_i} e^{-r_0 u} g(V(u, S(u)))du - \widehat{\mathcal{M}}_{j,i} \right)$.
6. Compute the dual estimated value as $\widehat{W}_0(S_0) = \frac{1}{N_P} \sum_{j=1}^{N_P} \widehat{W}_{0,j}(S_0)$.

2.4.2 The nonlinear problem ($M = \widehat{V}$)

In the previous section we have deduced how to price the American option value considering counterparty risk, when the mark–to–market is equal to the risk–free derivative value. Two alternative algorithms have been proposed, transforming the classical Longstaff–Schwartz scheme. More precisely, Algorithm 2.2 consists of two nested Monte Carlo methods while Algorithm 2.3 combines a Monte Carlo method with an interpolation technique.

Now, when the mark–to–market value is equal to the price of the derivative with counterparty risk ($M = \widehat{V}$), in the corresponding complementarity problem (2.11) we identify a nonlinear dependence on the solution $\widehat{V}$. In this case, Feynman–Kac theorem [43] provides the risky American option value at time $t = 0$, which satisfies:

$$\widehat{V}_0(S_0) = \sup_{\tau \in \mathcal{T}_0} \mathbb{E}_0 \left[e^{-r\tau} h^*(\tau, S_\tau) + \int_0^\tau e^{-ru} \widehat{g}(\widehat{V}(u, S(u)))\, du \right],$$

where function $\widehat{g}$ is defined by:

$$\widehat{g}(\widehat{V}) = -(1 - R_B)\lambda_B \widehat{V}^- - (1 - R_C)\lambda_C \widehat{V}^+ - s_F \widehat{V}^+ .$$

Recall that the asset prices follow the geometric Brownian motion process defined in (2.37). Once again, to simulate a continuously exercisable American option the period of time is discretized in $N_T + 1$ time steps. Thus, the asset price value at each time step is approximated by Euler–Maruyama scheme like in (2.39).

Now, using a dynamic programming formulation the American option value can be written in a recursive formula

$$\begin{aligned} &\widehat{V}_{N_T}(s) = h(T, s)\,, \quad S_{N_T} = s \\ &\widehat{V}_{i-1}(s) = \max \left\{ h_{i-1}(s), \mathbb{E}_{t_{i-1}} \left[D_{0,i} \widehat{V}_i^*(S_i) + \int_{t_{i-1}}^{t_i} e^{-ru} \widehat{g}(\widehat{V}(u, S(u)))\, du \;\middle|\; S_{i-1} = s \right] \right\}, \end{aligned}$$

for $i = N_T, N_T - 1, \ldots, 1$, the discounting factor being defined as

$$D_{i-1,i} = e^{-r(t_i - t_{i-1})}.$$

Introducing the discounting factor in each term, the recursive formula becomes:

$$\begin{aligned} &\widehat{V}_{N_T}(s) = h(T, s)\,, \quad S_{N_T} = s \\ &\widehat{V}_{i-1}(s) = \max \left\{ h_{i-1}(s), \mathbb{E}_{t_{i-1}} \left[\widehat{V}_i(S_i) + \int_{t_{i-1}}^{t_i} e^{-ru} \widehat{g}(\widehat{V}(u, S(u)))\, du \;\middle|\; S_{i-1} = s \right] \right\}, \end{aligned}$$

for $i = 1, \ldots, N_T$.

Next, we write the continuation value, which is also approximated by a regression function, as follows:

$$\begin{aligned} C_i(s) &= \mathbb{E}_{t_i} \left[\widehat{V}_{i+1}(S_{i+1}) + \int_{t_i}^{t_{i+1}} e^{-ru} \widehat{g}(\widehat{V}(u, S(u)))\, du \;\middle|\; S_i = s \right] \\ &= \sum_{j=1}^{J} b_{ij} \psi_j(s) = b_i^T \psi(s)\,. \end{aligned} \tag{2.63}$$

Let us remark that the main difference with respect to the case where the mark–to–market is equal to the risk–free derivative value arises in the continuation value, which leads to a different expression of d_i^{ψ}. Furthermore, the continuation value at time t_i is defined in terms of the risky derivative value in the previous time step, which has been previously computed, and the risky derivative value at the same instant of time.

In order to deal with the nonlinear feature of this problem, we propose a fixed point algorithm to compute coefficients β_i as the estimators of b_i (Algorithm 2.6).

Algorithm 2.6 Regression coefficients β_i with fixed point iteration

---

1. Simulate N_P independent paths $\{S_{j,1}, S_{j,2}, \ldots, S_{j,N_T}\}$ (for $j = 1, \ldots, N_P$) of the asset prices process.
2. At maturity time t_{N_T}, $\widehat{W}_{N_T}(S_{j,N_T}) = h_{N_T}(S_{j,N_T})$.
3. Set the tolerance ϵ.
4. For $i = N_T - 1, \ldots, 1$, perform a fixed point algorithm:
   - Initialize $\ell = 0$ and set $\widehat{W}_{j,i}^{0} = \widehat{W}_{j,i+1}$.
   - Given the estimated value $\widehat{W}_{j,i+1}$ ($j = 1, \ldots, N_P$), compute A^{ψ}.
   - Iterate the following steps while $e \geq \epsilon$
     - Compute $d_i^{\psi,\ell}$ in terms of $\widehat{W}_{j,i}^{\ell}$.
     - Compute β_i as the solution of the linear system $A_i^{\psi}\beta_i^{\ell} = d_i^{\psi,\ell}$.
     - Estimate the continuation value $\kappa_i(S_{j,i}) = \beta_i^T\psi(S_{j,i})$ for $j = 1, \ldots, N_P$.
     - Compute $\widehat{W}_{j,i}^{\ell+1} = \max\{h_i(S_{j,i}), \kappa_i(S_{j,i})\}$.
     - $e = \dfrac{\|\widehat{W}_{j,i}^{\ell+1} - \widehat{W}_{j,i}^{\ell}\|}{\|\widehat{W}_{j,i}^{\ell+1}\|}$ and set $\ell = \ell + 1$
5. Save the regression coefficients β_i to compute the risky derivative value.

Therefore, to obtain the lower estimator of the risky derivative value at time $t = 0$ we apply Algorithm 2.4, using the β_i coefficients obtained with Algorithm 2.6.

Using a similar procedure to the one followed in the linear complementarity problem (when $M = V$), an upper estimator of the derivative value can be obtained. In this case, after computing the regression coefficients β_i by Algorithm 2.6, we apply Algorithm 2.5 to obtain the estimator of the American option value. Remark that function $g(V)$ in Algorithm 2.5 is replaced by function $\widehat{g}(\widehat{V})$. Again the confidence intervals are obtained like in Section 2.4.1.

2.5 Numerical results

In this section, we show the results obtained for American options bought by counterparty B, the value of the parameters being the same than in the analogous example for European options in Chapter 1. For the ALAS algorithm, we consider $\beta = 10^5$ and the stopping test parameter equal to 10^{-5}, thus obtaining the convergence in two or three iterations.

2.5.1 Test 1: American call option

In Figure 2.1 we compare the American call option value considering different adjustments upon risk free value. The maturity time is $T = 3$, and the rest of the input parameters are given in Table 1.2. As in the European call option case, when counterparty B buys a call option, the price that B has to pay by the risk–free derivative is higher than the amount that has to be paid for an option if default risk and funding costs are considered. Moreover, as expected in an option that pays no dividends, risk–free value is the same for both options; in other case, when risky values are considered the American option value is larger than the European one, due to the fact that the American option can be exercised before the maturity date.

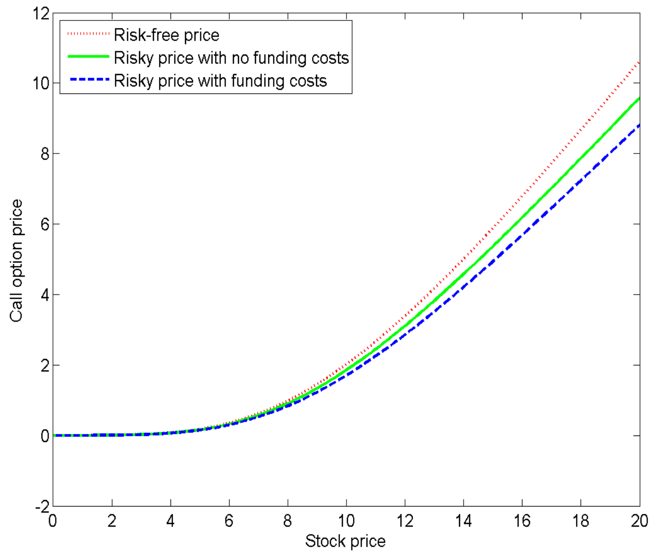
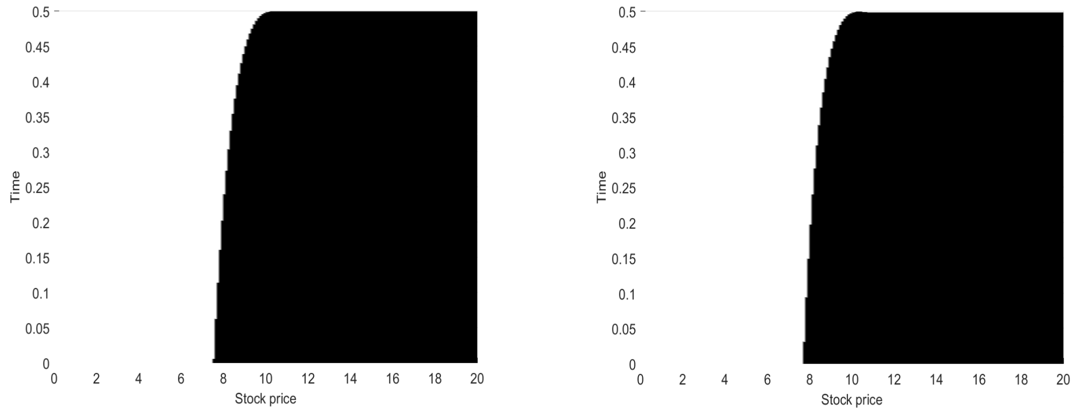


Figure 2.1: American call option value (Test 1)

2.5.2 Test 2: American put option

In Figures 2.2(a) and 2.2(b), the exercise region for an American put option is represented in white. The example corresponds with an American put option bought by B with the same data than the call option in Test 1, except the maturity date which is $T = 0.5$ years. We can see that in the case with counterparty risk this region is larger than the same area in the case of an American put option without counterparty risk. According to these regions, we can interpret Figure 2.3, which represents the XVA surface for an American put option. We can observe that the XVA is negative because it represents the discounted value upon the risk–free value, due to the risk exposure of counterparty B. Moreover, in terms of absolute value this is larger when the asset value approaches the exercise area because the buyer B is more interested in exercising the option. Moreover, when the spot price is in the exercise region, the XVA surface tends to zero. This is due to the fact that the risky value and the risk–free value reach the exercise price, so that XVA $= \widehat{V} - V = 0$. Finally, the XVA value is zero at maturity, because the counterparty is no more exposed.



(a) Risk–free American put option

(b) Risky American put option

Figure 2.2: Exercise region (white) for an American put option (Test 2)

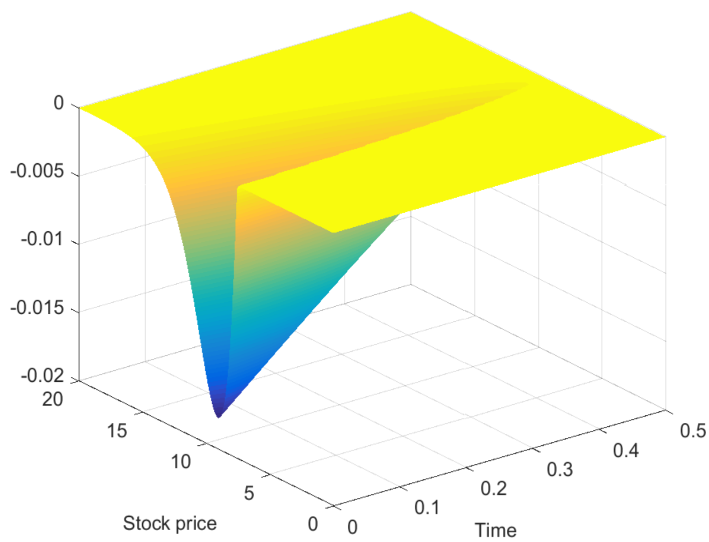


Figure 2.3: Total Value Adjustment surface for American put option (Test 2)

2.5.3 Test 3: Collateralized American option

In this example, we study again an American put option. However, the derivative in the contract is collateralized. As we did for European options, we show the results of model 3 of Section 2.2.2. We have used the same parameters than for the collateralized European option model (cf. Test 4 in Chapter 1), where the collateral rate is $r_C = 0.05$ and the rest of the parameters are given in Table 1.2.

Thus, we show in Figure 2.4 the difference between the fully, partially and non collateralized derivative prices. As for European options, the difference is positive, because it represents the additional amount that has to be paid by B if the derivative is collateralized. This price increases as the collateral is larger, thus the exposure facing C's default is lower. Therefore, the price of a collateralized American put option, out of the exercise region, is larger than the not collateralized one. We can also appreciate how this difference is null for asset prices in the exercise region, which is almost equal for all collateral amounts. This difference between both of them is the CollVA.

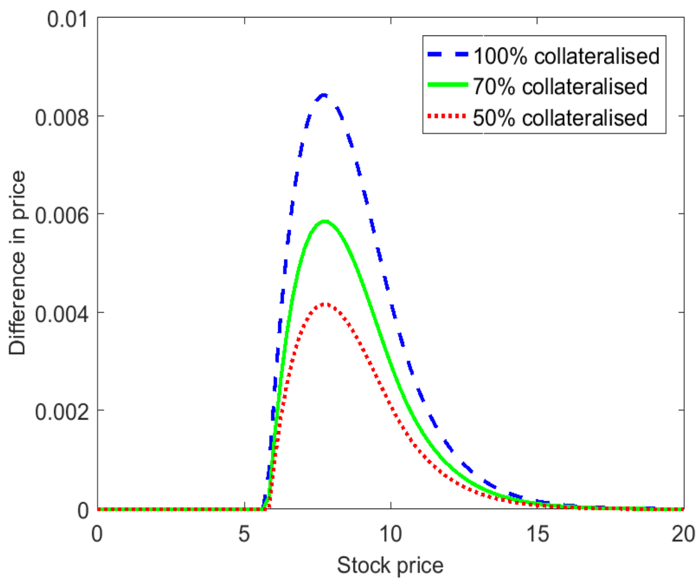
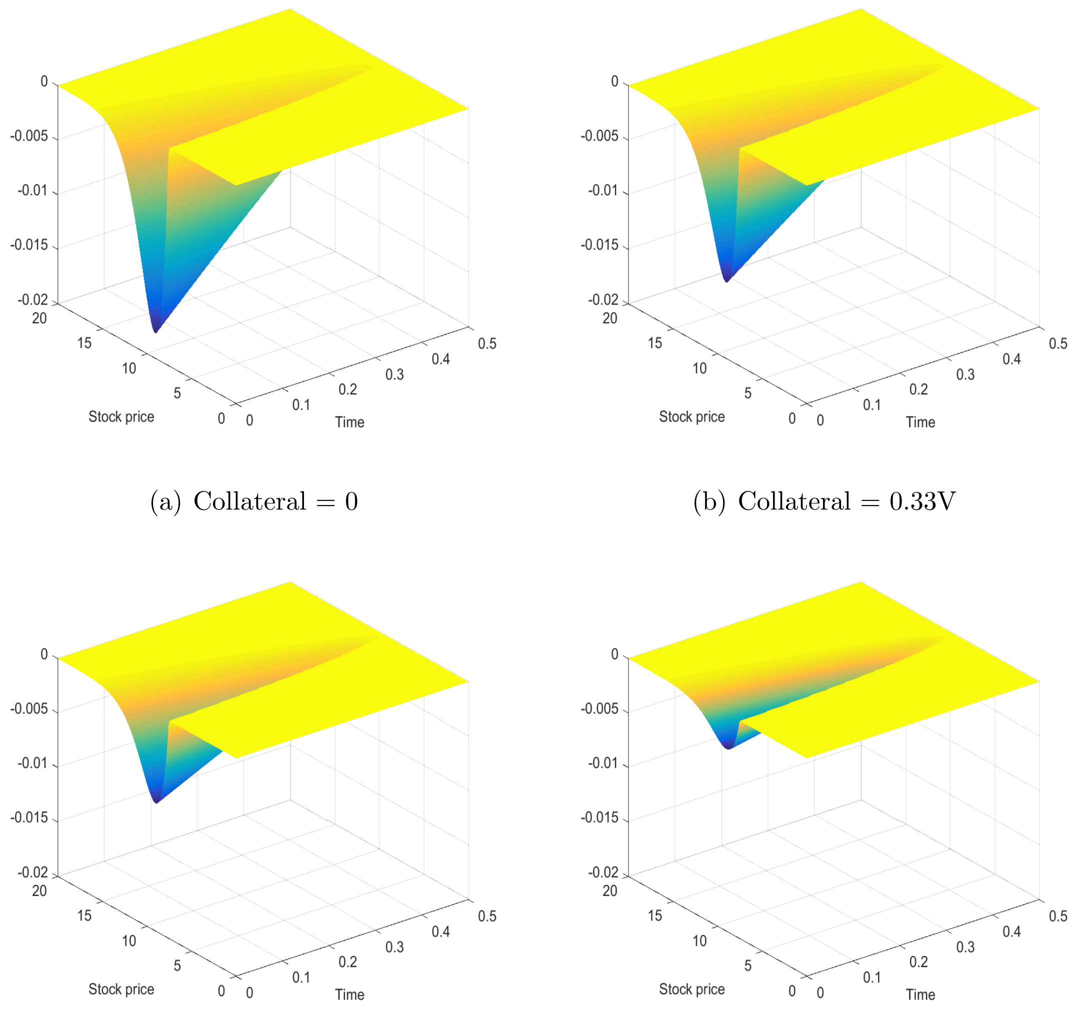


Figure 2.4: Collateral Value adjustment for different amount of collateral (Test 3)

In Figure 2.5, we show the XVA for a contract with a collateralized derivative. The variation of the XVA is represented according to the percentage of risk–free derivative which has been collateralized. As expected, when the derivative is totally collateralized, the total value adjustment in absolute terms is lower, because the exposure facing C decreases. Moreover, when the derivative is not collateralized, $(X = 0)$ the XVA value corresponds with the results shown in Figure 2.3



(a) Collateral = 0

(b) Collateral = 0.33V

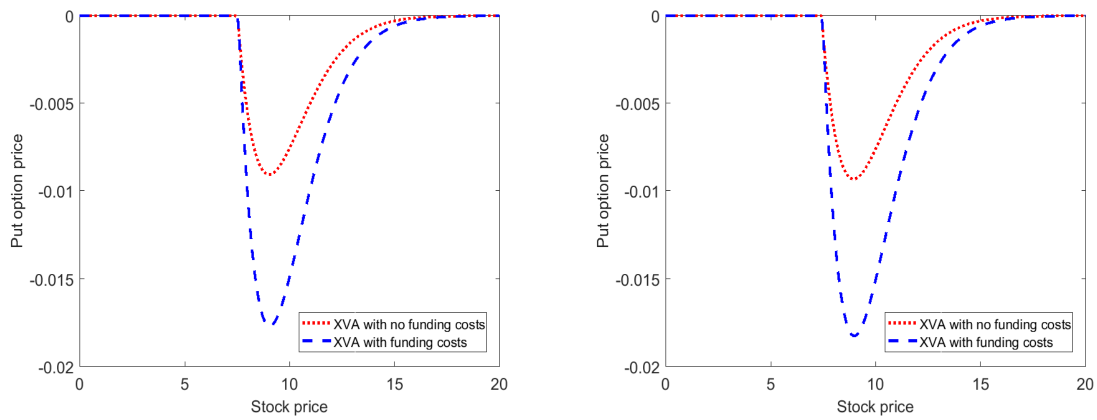
(c) Collateral = 0.66V

(d) Collateral = V

Figure 2.5: XVA surfaces for different collateral values (Test 3)

2.5.4 Test 4: Some results on the linear problem ($M = V$)

In the previous examples, the mark–to–market value was the derivative value considering counterparty risk. In the present test we show the values obtained for an American put option when the mark–to–market is the risk–free value. We consider the same value of parameters than in the previous test. In Figure 2.6 we compare the value adjustments at current time, for both linear and nonlinear problems. Moreover, for each case, the XVA including or not including FCA are plotted. Finally we can conclude there is not a big difference in the choice of the mark–to–market close out, being the total value adjustment more negative when M is the risk–free derivative value.



(a) Case with $M = \widehat{V}$ (b) Case with $M = V$

Figure 2.6: Forward values with CVA and FCA (Test 4)

2.5.5 Test 5: The influence of parameters in the model

Next, we show how the different parameters of the model affect the value of a put American option. The results correspond with the nonlinear problem, where the mark–to–market is the risky derivative value. In the previous example, we have proved that a similar behaviour is obtained for the linear one. In Table 2.1 we can appreciate how the intensity of default from B and the recovery rate from the same counterparty have an effect on the derivative value. Nevertheless, in Table 2.2 the derivative value changes according to the probability of default and recovery rate of counterparty C. As expected, when the intensity of default from each counterparty increases, the derivative value decreases. A similar behaviour, on the opposite sense, is found when the recovery rate grows. Moreover, we can appreciate that the derivative value is larger for a long maturity term, as a large maturity period increases the uncertainty.

The common parameters for both tables are $K = 15$, $\sigma = 0.3$, $r_R = 0.015$, $r = 0.05$ and $s_F = (1 - R_B)\lambda_B$.

Table 2.1: American put option, with $\lambda_C = 0.08$ and $R_C = 0.3$ (Test 5)

T	λ_B	R_B	$S = 5$	$S = 12$	$S = 20$
0.5	0.04	0.1	10.00000000	3.13308090	0.16323770
		0.5	10.00000000	3.14166452	0.16437717
	0.2	0.1	10.00000000	3.07218167	0.15345619
		0.5	10.00000000	3.10268759	0.15878868
	0.6	0.1	10.00000000	3.00469239	0.13213340
		0.5	10.00000000	3.03875867	0.14589116
3	0.04	0.1	10.00000000	3.77230534	1.20695426
		0.5	10.00000000	3.82986828	1.24530828
	0.2	0.1	10.00000000	3.40960804	0.93051979
		0.5	10.00000000	3.58192861	1.07095180
	0.6	0.1	10.00000000	3.06610652	0.54715149
		0.5	10.00000000	3.23836762	0.76713906

We have not theoretically proved the convergence of the fixed point iteration. Nevertheless, all tests have converged in a reduced number of iterations.

Table 2.2: American put option, with $\lambda_B = 0.06$ and $R_B = 0.4$ (Test 5)

T	λ_C	R_C	$S = 5$	$S = 12$	$S = 20$
0.5	0.04	0.1	10.00000000	3.14387856	0.16466381
		0.5	10.00000000	3.15302397	0.16581763
	0.2	0.1	10.00000000	3.07911138	0.15476659
		0.5	10.00000000	3.11155725	0.16016080
	0.6	0.1	10.00000000	3.00621647	0.13321601
		0.5	10.00000000	3.04341163	0.14711791
3	0.04	0.1	10.00000000	3.84500310	1.25519575
		0.5	10.00000000	3.90880999	1.29601308
	0.2	0.1	10.00000000	3.44731946	0.96278708
		0.5	10.00000000	3.63539725	1.11044543
	0.6	0.1	10.00000000	3.07521571	0.56160892
		0.5	10.00000000	3.26143931	0.79122318

2.5.6 Test 6: A Monte Carlo simulation

We now present some numerical results obtained with the previously described Monte Carlo algorithms. Our aim is to compare the efficiency of these methods with the algorithms proposed to solve the analogous complementarity problem formulations.

In all examples, the initially chosen financial parameters are: $K = 15$, $r = 0.04$, $r_R = 0.06$, $\sigma = 0.25$, $R_B = R_C = 0.3$, $\lambda_B = \lambda_C = 0.04$, $s_F = (1 - R_B)\lambda_B$ and $T = 0.5$. We will also show the sensitivity of the option price with respect to parameters λ_B, λ_C, R_B and R_C by shifting these initial values.

For the numerical simulation with Monte Carlo techniques, we have used $N_P = 500$ paths and $N_T = 1000$ time steps. In particular, for Algorithm 2.2 we have additionally considered 8 inner paths, while for Algorithm 2.5 we use $N_I = 50$. Moreover, we consider a basis consisting of three Laguerre polynomials in the regression formula (2.44).

In Tables 2.3 to 2.8 we include results issued from the solution of the complementarity problems, for which we have discretized the spatial domain with $N_S = 601$ nodes and we have used $N_T = 200$ time steps.

Example with mark–to–market $M = V$

In this test we focus on the linear problem, posed when $M = V$, on an American put option.

Table 2.3 presents some numerical results obtained when the mark–to–market is $M = V$. More precisely, for different underlying prices, the numerical solution of the linear complementarity problem (2.12), the lower (2.46) and upper (2.58) estimators and the 99% confidence interval are shown jointly with the excercise value.

The numerical solution of (2.12) is computed with the numerical techniques described in Section 2.3 and [4]. We can appreciate that it lies in the confidence interval, except in the first critical case for $S = 0$ where Monte Carlo approximation is very close to the exercise value. For the larger underlying prices ($S \geq 25$), all values become naturally close to zero, as expected.

A similar behaviour is observed with Algorithm 2.3, where the risk–free price V is interpolated from the values previously obtained in a thin mesh for the asset, instead of being computed by an inner Monte Carlo algorithm (see Table 2.4).

Table 2.3: American option value considering counterparty risk and $M = V$ (Algorithms 2.2 and 2.5).

S	Pay-off	Complementarity problem approximation	Lower estimator	Upper estimator	Confidence interval
0.0	15.0	15.00000000	14.99910003	14.99919002	(14.99910003 , 14.99919002)
2.5	12.5	12.50000000	12.49809274	12.50113807	(12.49633159 , 12.50151772)
5.0	10.0	10.00000000	9.99706459	10.00999906	(9.98974544 , 10.01151316)
7.5	7.5	7.50000000	7.49637060	7.52156681	(7.48397263 , 7.52788763)
10.0	5.0	5.00000000	4.99038366	5.05447309	(4.96725963 , 5.06112388)
12.5	2.5	2.51600182	2.31297960	2.67428026	(2.14455680 , 2.68999472)
15.0	0.0	0.87745370	0.81300011	1.04625567	(0.67303811 , 1.06865764)
17.5	0.0	0.22907711	0.20395622	0.50740824	(0.13398242 , 0.53206161)
20.0	0.0	0.04701583	0.06316155	0.32741755	(0.02434924 , 0.34431668)
22.5	0.0	0.00810599	0.01550121	0.17101847	(0.00048283 , 0.17990265)
25.0	0.0	0.00124366	0.00070868	0.02497713	(-0.00071642 , 0.02650098)
27.5	0.0	0.00017746	0.00000000	0.00037909	(0.00000000 , 0.00039856)
30.0	0.0	0.00002432	0.00000000	0.00003384	(0.00000000 , 0.00009372)

Table 2.4: American option value considering counterparty risk and $M = V$ (Algorithms 2.3 and 2.5).

S	Pay-off	Complementarity problem approximation	Lower estimator	Upper estimator	Confidence interval
0.0	15.0	15.00000000	14.99910003	14.99919002	(14.99910003 , 14.99919002)
2.5	12.5	12.50000000	12.49878788	12.50074403	(12.49701990 , 12.50109953)
5.0	10.0	10.00000000	9.99825352	10.00895521	(9.99516292 , 10.01031047)
7.5	7.5	7.50000000	7.49814808	7.52543188	(7.48526493 , 7.52874533)
10.0	5.0	5.00000000	5.00220584	5.05834509	(4.98403716 , 5.06774399)
12.5	2.5	2.51600182	2.46371870	2.67394114	(2.30502657 , 2.68898177)
15.0	0.0	0.87745370	0.84859226	1.06127342	(0.70565317 , 1.08186773)
17.5	0.0	0.22907711	0.20054059	0.49115613	(0.13612061 , 0.51495675)
20.0	0.0	0.04701583	0.03959987	0.31201452	(0.01426014 , 0.32890688)
22.5	0.0	0.00810599	0.00823601	0.16045927	(-0.00358441 , 0.16930795)
25.0	0.0	0.00124366	0.00052809	0.01281424	(-0.00048394 , 0.01380686)
27.5	0.0	0.00017746	0.00000000	0.00003246	(0.00000000 , 0.00003439)
30.0	0.0	0.00002432	0.00000000	0.00000442	(0.00000000 , 0.00000468)

Table 2.5: American option value considering counterparty risk and $M = \widehat{V}$ (Algorithms 2.6 and 2.5).

S	Pay-off	Complementarity problem approximation	Lower estimator	Upper estimator	Confidence interval
0.0	15.0	15.00000000	14.99970000	14.99979001	(14.99970000 , 14.99979001)
2.5	12.5	12.50000000	12.49987807	12.50229450	(12.49825098 , 12.50281445)
5.0	10.0	10.00000000	9.99848975	10.01481249	(9.99246462 , 10.01684125)
7.5	7.5	7.50000000	7.50417414	7.53459612	(7.49600267 , 7.53920122)
10.0	5.0	5.00000000	4.96081214	5.07280533	(4.90694567 , 5.08156060)
12.5	2.5	2.52410327	2.32223198	2.69563668	(2.15396736 , 2.71232058)
15.0	0.0	0.89012915	0.81039447	1.08655935	(0.66794198 , 1.10885981)
17.5	0.0	0.23345859	0.23289102	0.50408124	(0.15736865 , 0.52928497)
20.0	0.0	0.04802108	0.04619341	0.31320212	(0.01305985 , 0.32997245)
22.5	0.0	0.00828968	0.01042668	0.14064556	(-0.00236734 , 0.14839822)
25.0	0.0	0.00127290	0.00036507	0.02547380	(-0.00037522 , 0.02682979)
27.5	0.0	0.00018174	0.00000000	0.00000000	(0.00000000 , 0.00000000)
30.0	0.0	0.00002492	0.00000000	0.00000000	(0.00000000 , 0.00000000)

In order to compare the efficiency of algorithms 2.2 and 2.3, we have measured the elapsed CPU time in both cases. In all examples, tests have been performed with Matlab on an Intel(R) Xeon(R) CPU E3-1241 3.50 GHz computer. Algorithm 2.2 takes 55134 seconds for computing the lower estimator and 37390 seconds for the upper estimator. However, Algorithm 2.3 only needs 5.4863 seconds to obtain the regression coefficients. Nevertheless, note that Algorithm 2.3 needs a large computational time to previously obtain the risk–free derivative value on the thin mesh used to interpolate. More precisely, it takes 122960 seconds to obtain the lower and upper estimators of the risk–free derivative price for the whole set of asset nodes. Furthermore, Algorithms 2.4 and 2.5 take 0.0759 and 2.1875 seconds, respectively, for the computation of the risky American option price.

All these computational times correspond to the approximation of the option price for just one asset price. We can observe that the interpolation of the risk–free option values implies a larger time in obtaining the lower and upper estimators for a unique initial asset price. Nevertheless, once the values of the risk–free derivative on the fine mesh are available, the computation of the option price for several asset prices by Algorithm 2.3 (interpolation) is much more efficient than by Algorithm 2.2 (inner Longstaff–Schwartz scheme). Indeed, only six additional seconds per asset price are required in Algorithm 2.3.

Alternatively, the numerical solution of the complementarity problem (2.12) is clearly more efficient, as only 6.89 seconds are needed to approximate the solution on a mesh of 601 nodes (each node represents an initial asset price) and 200 time steps.

Example with mark–to–market $M = \widehat{V}$

Table 2.5 shows the results obtained in the example with mark–to–market $M = \widehat{V}$, which corresponds to problem (2.11). The associated Monte Carlo technique has been described in Section 2.4.2. In this example, Algorithm 2.6 takes 6.2608 seconds, while the numerical methods [4] employed to approximate the solution of the nonlinear complementarity problem take 270 seconds with a 601 nodes mesh and 200 time steps. We point out the good agreement between the values computed from the PDE formulation and the confidence intervals obtained with the proposed Monte Carlo technique.

As we have done in Test 5 for the finite element case, we show how Monte Carlo techniques also reflect the influence of different parameters on the option value. Table 2.6 shows, for an initial price $S_0 = 20$, the numerical solution of the complementarity problem, the Monte Carlo lower and upper estimators, and the confidence intervals computed for different values of the intensity of default λ_B. As expected, we appreciate that for increasing values of this parameter both estimators decrease. We have observed the same effect when we have fixed λ_B and taken different increasing values for the intensity of default λ_C.

Table 2.6: American put option value considering counterparty risk and $M = \widehat{V}$ (Algorithms 2.6 and 2.5). Effect of the intensity of default. $S_0 = 20$, $\lambda_C = 0.04$, $R_B = R_C = 0.30$.

λ_B	Complementarity problem approximation	Lower estimator	Upper estimator	Confidence interval
0.04	0.04802108	0.04942329	0.31842569	(0.01289458 , 0.33546431)
0.10	0.04715281	0.04930002	0.31715650	(0.01287656 , 0.33380921)
0.30	0.04439205	0.04895565	0.30504576	(0.01282437 , 0.32139431)

A similar behaviour, in the opposite sense, is observed when we increase the recovery rates R_B or R_C. Tables 2.7 and 2.8 show the obtained results for $S_0 = 20$.

Table 2.7: American put option value considering counterparty risk and $M = \widehat{V}$ (Algorithms 2.6 and 2.5). Effect of the recovery rate. $S_0 = 20$, $\lambda_B = \lambda_C = 0.30$, $R_C = 0.30$.

R_B	Complementarity problem approximation	Lower estimator	Upper estimator	Confidence interval
0.10	0.04005223	0.04732412	0.29536326	(0.01512059 , 0.31200451)
0.30	0.04107955	0.04766287	0.30351023	(0.01513649 , 0.32043366)
0.90	0.04435412	0.04790897	0.31169431	(0.01514366 , 0.32841816)

Table 2.8: American put option value considering counterparty risk and $M = \widehat{V}$ (Algorithms 2.6 and 2.5). Effect of the recovery rate. $S_0 = 20$, $\lambda_B = \lambda_C = 0.30$, $R_B = 0.30$.

R_C	Complementarity problem approximation	Lower estimator	Upper estimator	Confidence interval
0.10	0.04005223	0.04732412	0.29615655	(0.01512059 , 0.31268850)
0.30	0.04107955	0.04766287	0.30497545	(0.01513649 , 0.32197302)
0.90	0.04435412	0.04790897	0.31058947	(0.01514366 , 0.32782823)

Chapter 3

Two stochastic factors model for European options with XVA

3.1 Introduction

In the previous chapters, and also in [4], a one factor model to price the adjustments associated to European and American options with counterparty risk has been analyzed and numerically solved. In particular, funding value adjustment (FVA), debit value adjustment (DVA) and credit value adjustment (CVA) have been considered. Furthermore, the model in [4] is extended to incorporate the collateral value adjustment (CollVA), in case that a collateral is used to guarantee the obligations related to the options contract. In this model, constant default intensities for both counterparties have been considered, so that a model depending on just one underlying stochastic factor (the underlying asset) is deduced and numerically solved.

However, counterparties default intensities do not always exhibit constant behaviours. In a general framework, intensities might follow a stochastic process [27]. In the present chapter we focus on the European options pricing and the corresponding XVA adjustments when stochastic intensities are assumed. More precisely, we state PDE models for the derivative value, from the point of view of an investor, when the trade takes place between two counterparties: an investor and a hedger. If we consider stochastic intensities of default for both counterparties then a model with three stochastic factors is obtained [27]. Our approach is based on the same framework and assumptions as in [27], although with the additional hypothesis of a zero default intensity for the hedger, thus leading to a two stochastic factors model. The three factors model could be approached by the theoretical analysis and numerical methods that we present in this chapter.

As in [27], we include all the components in the pricing of uncollateralized derivatives with counterparty risk, with the following assumptions:

- The price of a derivative should reflect all of its hedging costs.
- Since in a high percentage of uncollateralized transactions the presence of an investor (risk taker) and a hedger (risk hedger) is implied, the price of the derivative should just reflect the hedging costs transmitted by the hedger.
- The hedger will only be willing to hedge the fluctuations in the price of the derivative that he will experience while not having defaulted.
- There is neither CVA nor FVA to be applied to fully collateralized derivatives (with continuous collateral margining in cash, symmetrical collateral mechanism and no threshold, minimun transfer amount, etc).

Moreover, we will consider the following market assumptions:

- There is a liquid credit default swap (CDS) curve for the investor.
- There is a liquid curve of bonds issued by the hedger.
- Continuous hedging, unlimited liquidity, no bid–offer spreads, no trading costs.
- Recovery rates are either deterministic or there are recovery locks available so that recovery risk is not a concern,

as well as the following model assumptions:

- Only the investor is defaultable.

- The underlying asset follows a diffusion process under the real world measure.

- The underlying asset of a derivative is unaffected by a default event of the investor.

- The investor credit spread is stochastic and follows a diffusion process correlated with the asset price under the real world measure.

Keeping in mind these assumptions, in the present chapter we state a PDE formulation by means of suitable hedging arguments and the use of Itô's Lemma for jump–diffusion processes [43]. After arguing the hedging strategy, as we did in Chapter 1, different linear or nonlinear PDEs arise depending on the choice of the mark–to–market value at default. For the nonlinear PDE formulation we develop the mathematical analysis of the model to obtain existence and uniqueness of a solution in the appropriate functional space on a bounded domain. For this purpose, we use the tools of nonlinear parabolic PDEs involving sectorial operators [30].

In addition, we propose a set of numerical methods to solve the PDEs for both choices of the mark–to–market value. First, we truncate the unbounded domain and formulate suitable boundary conditions at the boundaries of the localized domain, following some ideas in [23]. Next, we propose a time discretization based on the method of characteristics combined with a finite element discretization in the asset and spread variables. The method of characteristics has been proposed in [44] in the context of fluid mechanics problems and used in finance in [50] for vanilla options, in [23, 7] for Asian options or in [17] for pension plans. For the nonlinear PDE a fixed point iteration algorithm is additionally proposed.

This chapter is organized as follows. In Section 3.2 we propose the mathematical model. Section 3.3 is devoted to the mathematical analysis of the nonlinear PDE problem that models the price of the XVA. Furthermore, we prove the existence and uniqueness of solution. In Section 3.4 we describe the numerical methods we propose to compute a solution of our models. In Section 3.5, we show and discuss the numerical results for some illustrative examples. Most of the results in this chapter are included in [5].

3.2 Mathematical model

In this section, we deduce the models for European options and their associated XVA pricing when the counterparty risk and funding costs are taken into account. The main difference with the one factor model presented in Chapter 1 comes from the consideration of stochastic default intensities instead of constant ones. As previously indicated, we assume an investor as a risky counterparty and consider that the issuer's intensity of default is null. Thus, the underlying asset price S, and the short term CDS spread of the investor h, are modelled by means of stochastic processes satisfying the following stochastic differential equations (SDEs):

$$dS_t = (r(t) - q(t))\, S_t\, dt + \sigma^S(t)\, S_t\, dW_t^S\,, \tag{3.1}$$

$$dh_t = (\mu^h(t) - M^h(t)\sigma^h(t))\, dt + \sigma^h(t)\, dW_t^h\,, \tag{3.2}$$

where $(r(t)-q(t))$ and $(\mu^h(t)-M^h(t)\sigma^h(t))$ are the (respective) drifts of the processes. Moreover, $r(t)$ denotes the risk–free interest rate, $q(t)$ is the asset dividend yield rate, $M^h(t)$ is the market price of investor's credit risk, $\sigma^S(t,S)$ and $\sigma^h(t,h)$ are the volatility functions, and W_t^S and W_t^h are two correlated Wiener processes

$$\rho\, dt = dW_t^S\, dW_t^h$$

such that ρ is the instantaneous correlation between S_t and h_t.

In terms of the spread, the default intensity of the investor, λ_t, is defined as:

$$\lambda_t = \frac{h_t}{1-R}\,, \tag{3.3}$$

where $0 \leq R < 1$ denotes the investor recovery rate.

We consider a derivative trade between a hedger and an investor, where only the last one is defaultable. The main risk factors in the trading are the market risk due to changes produced in the asset value, investor spread risk and investor default instant. Thus, from the point of view of the investor, the derivative value at time t is denoted by $\widehat{V}_t = \widehat{V}(t, S_t, h_t, J_t^I)$ and depends on the spot value of the asset (S_t), on the spread of the investor (h_t) and on the investor default state at time t (J_t^I). Note that $J_t^I = 1$ in case of default before or at time t, otherwise $J_t^I = 0$. The price of the same derivative between two default–free counterparties (risk–free derivative) is denoted by $V_t = V(t, S_t)$. The risky derivative price $\widehat{V}_t$ includes adjustments (such as DVA, FCA and/or CollVA) into valuation, whereas the risk–free derivative price V_t does not include any counterparty risk adjustment.

The price of the risky derivative upon default of the investor is given by:

$$\widehat{V}(t, S_t, h_t, 1) = RM^+(t, S_t, h_t) + M^-(t, S_t, h_t), \tag{3.4}$$

where $M(t, S_t, h_t)$ denotes the mark–to–market price. Moreover, $Z^+ = \max(Z, 0)$ and $Z^- = \min(0, Z)$.

In terms of the mark–to–market condition (3.4), we introduce $\Delta\widehat{V}$ as the variation of $\widehat{V}$ at default, which is given by:

$$\Delta\widehat{V}_t = RM_t^+ + M_t^- - \widehat{V}_t\,, \tag{3.5}$$

where $M_t = M(t, S_t, h_t)$. Note that, this expresion corresponds with (1.7) in Chapter 1, i.e. the variation of the risky derivative value when counterparty C makes default. As we have considered in the model for constant intensities of default and following the literature [15], we only consider two possible choices for M_t: either the risk–free either the risky derivative value. In order to state the pricing model of the risky derivative, this one is hedged by a self–financing portfolio, Π_t, which is designed to hedge all underlying risk factors.

With this aim, the hedger will trade with different financial instruments in order to hedge the following risk factors:

- Market risk: a fully collateralized derivative is employed to hedge this kind of risk. We denote by H_t the net present value associated to that derivative, from the point of view of the hedger.

- Spread risk and default risk of the investor: the hedger will trade with two credit default swaps with different maturity times. The first one, $\text{CDS}(t, t+dt)$, for which the buyer pays a premium $h(t)dt$ at time $t + dt$, presents a short maturity date. If the default time takes place before the maturity time $t + dt$, the buyer of the protection receives $(1 - R)$, where R denotes the recovery rate at time $t + dt$. Moreover, the premium $h(t)dt$ is such that $\text{CDS}(t, t + dt) = 0$. The second credit default swap, $\text{CDS}(t, T)$, represents the amount of money guaranteed until a longer maturity time, $T > t$.

Thus, from no arbitrage arguments we have $\widehat{V}_t = \Pi_t$. Let us assume that the portfolio at time t, Π_t, is made up of:

- $\alpha(t)$ units of the net present value of a fully collateralized derivative H_t,
- $\beta(t)$ units of cash in collateral accounts,
- $\gamma(t)$ units of a long term credit default swap,
- $\varepsilon(t)$ units of a short term credit default swap,
- $\Omega(t)$ units of a short term bond,

such that:

$$\Pi_t = \alpha(t)H(t) + \beta(t) + \gamma(t)\text{CDS}(t, T) + \varepsilon(t)\text{CDS}(t, t + dt) + \Omega(t)B(t, t + dt)\,. \quad (3.6)$$

The hedger trades on bonds that mature on $t + dt$ to match the spread duration of the uncollateralized derivative, imposing that the net buyback is equal to $\widehat{V}_t$. This is known as a self–financing condition of the replication strategy, so that

$$\widehat{V}_t = \Omega(t)B(t, t + dt)\,, \quad (3.7)$$

which implies that the number of units of $B(t, t + dt)$ is given by:

$$\Omega(t) = \frac{\widehat{V}_t}{B(t, t + dt)}\,.$$

Therefore, as a consequence of the self–financing condition, the portfolio evolution comes from the changes in each component:

$$
\begin{aligned}
d\widehat{V}_t &= \alpha(t)dH(t) + d\beta(t) + \gamma(t)d\mathrm{CDS}(t,T) \\
&\quad + \varepsilon(t)d\mathrm{CDS}(t,t+dt) + \frac{\widehat{V}_t}{B(t,t+dt)}\, dB(t,t+dt)\,.
\end{aligned} \tag{3.8}
$$

Applying Itô's lemma for jump diffusion processes [43], the change $d\widehat{V}_t$ of $\widehat{V}_t$ from t to $t+dt$ is given by:

$$
\begin{aligned}
d\widehat{V}_t &= \frac{\partial \widehat{V}}{\partial t}(t,S_t,h_t)\,dt + \frac{\partial \widehat{V}}{\partial S}(t,S_t,h_t)\,dS_t + \frac{\partial \widehat{V}}{\partial h}(t,S_t,h_t)\,dh_t \\
&\quad + \left(\frac{(\sigma^S)^2 S^2}{2}\frac{\partial^2 \widehat{V}}{\partial S^2} + \frac{(\sigma^h)^2}{2}\frac{\partial^2 \widehat{V}}{\partial h^2} + \rho\sigma^S\sigma^h S \frac{\partial^2 \widehat{V}}{\partial S \partial h} \right)(t,S_t,h_t)\,dt \\
&\quad + \Delta \widehat{V}(t,S_t,h_t)\,dJ_t^I \\
&= \left(\frac{\partial \widehat{V}}{\partial t} + (r-q)S\frac{\partial \widehat{V}}{\partial S} + (\mu^h - M^h\sigma^h)\frac{\partial \widehat{V}}{\partial h} + \frac{(\sigma^S)^2 S^2}{2}\frac{\partial^2 \widehat{V}}{\partial S^2} + \frac{(\sigma^h)^2}{2}\frac{\partial^2 \widehat{V}}{\partial h^2} \right. \\
&\quad \left. + \rho\sigma^S\sigma^h S\frac{\partial^2 \widehat{V}}{\partial S\partial h} \right)(t,S_t,h_t)\,dt + \sigma^S S\frac{\partial \widehat{V}}{\partial S}(t,S_t,h_t)\,dW_t^S \\
&\quad + \sigma^h \frac{\partial \widehat{V}}{\partial h}(t,S_t,h_t)\,dW_t^h + \Delta\widehat{V}(t,S_t,h_t)\,dJ_t^I\,.
\end{aligned} \tag{3.9}
$$

Then, let us show the evolution of the rest of financial instruments in the portfolio. The cash amount $\beta(t)$ is a sum of $-\alpha(t)H(t)$ and $-\gamma(t)\mathrm{CDS}(t,T)$ that has been posted to the hedger. Thus, the change in $\beta(t)$ is given by

$$
d\beta(t) = \Big(-\alpha(t)H_t - \gamma(t)\mathrm{CDS}(t,T) \Big)\, c(t)\,dt\,,
$$

where $c(t)$ represents the accrual rate, that is the rate of interest that is added to the principal of a financial instrument between cash payments of that interest.

Applying Itô's lemma [40] to the fully collateralized product

$$
\begin{aligned}
dH_t &= \left(\frac{\partial H}{\partial t} + (r-q)S\frac{\partial H}{\partial S} + \frac{(\sigma^S)^2 S^2}{2}\frac{\partial^2 H}{\partial S^2} \right)(t,S_t,h_t)\,dt \\
&\quad + \sigma^S S_t \frac{\partial H}{\partial t}(t,S_t,h_t)\,dW_t^S\,.
\end{aligned} \tag{3.10}
$$

The differential change in the short term CDS and bond are respectively given by:

$$d\mathrm{CDS}(t, t+dt) = h(t)\,dt - (1-R)\,dJ_t^I\,, \tag{3.11}$$

$$dB(t, t+dt) = f(t)B(t, t+dt)\,dt\,, \tag{3.12}$$

where $f(t)$ represents the EONIA rate, i.e. the weighted average of overnight euro interbank offer rates (EURIBOR) for inter–bank loans.

Applying Itô's lemma for jump–diffusion, the change of the long term CDS is

$$\begin{aligned} d\mathrm{CDS}(t,T) = &\left(\frac{\partial CDS(t,T)}{\partial t} + (\mu^h - M^h\sigma^h)\frac{\partial \mathrm{CDS}(t,T)}{\partial h}\right. \\ &+ \left.\frac{(\sigma^h)^2}{2}\frac{\partial^2 \mathrm{CDS}(t,T)}{\partial h^2}\right)dt + \sigma^h\frac{\partial \mathrm{CDS}(t,T)}{\partial h}\,dW_t^h + \Delta\mathrm{CDS}(t,T)\,dJ_t^I\,, \end{aligned} \tag{3.13}$$

where ΔCDS represents the variation of the CDS price at default.

Next, replacing (3.9)–(3.13) into equation (3.8), the latter can be written as:

$$\begin{aligned} &\left(\frac{\partial \widehat{V}}{\partial t} + (r-q)S\frac{\partial \widehat{V}}{\partial S} + (\mu^h - M^h\sigma^h)\frac{\partial \widehat{V}}{\partial h} + \frac{(\sigma^S)^2S^2}{2}\frac{\partial^2 \widehat{V}}{\partial S^2} + \frac{(\sigma^h)^2}{2}\frac{\partial^2 \widehat{V}}{\partial h^2}\right. \\ &\quad + \left.\rho\sigma^S\sigma^h S\frac{\partial^2 \widehat{V}}{\partial S\partial h}\right)dt + \sigma^S S\frac{\partial \widehat{V}}{\partial S}\,dW_t^S + \sigma^h\frac{\partial \widehat{V}}{\partial h}\,dW_t^h + \Delta\widehat{V}dJ_t^I \\ &= \alpha(t)\left(\left(\frac{\partial H}{\partial t} + (r-q)S\frac{\partial H}{\partial S} + \frac{(\sigma^S)^2S^2}{2}\frac{\partial^2 H}{\partial S^2}\right)dt + \sigma^S S\frac{\partial H}{\partial S}\,dW_t^S\right) \\ &\quad - \Big(\alpha(t)H(t) + \gamma(t)CDS(t,T)\Big)c(t)\,dt + \gamma(t)\left[\left(\frac{\partial \mathrm{CDS}(t,T)}{\partial t}\right.\right. \\ &\quad + \left.(\mu^h - M^h\sigma^h)\frac{\partial \mathrm{CDS}(t,T)}{\partial h} + \frac{(\sigma^h)^2}{2}\frac{\partial^2 \mathrm{CDS}(t,T)}{\partial h^2}\right)dt \\ &\qquad + \left.\sigma^h\frac{\partial \mathrm{CDS}(t,T)}{\partial h}\,dW_t^h + \Delta\mathrm{CDS}(t,T)\,dJ_t^I\right] \\ &\quad + \varepsilon(t)\Big(h(t)dt - (1-R)dJ_t^I\Big) + \frac{\widehat{V}_t}{B(t,t+dt)}f(t)B(t,t+dt)\,dt\,. \end{aligned} \tag{3.14}$$

In order to obtain a risk–free portfolio, we remove the risky terms in (3.14) with the following choices of coefficients:

$$\alpha(t) = \frac{\partial \widehat{V}/\partial S}{\partial H/\partial S}\,,\quad \gamma(t) = \frac{\partial \widehat{V}/\partial h}{\partial \mathrm{CDS}(t,T)/\partial h}\,,\quad \varepsilon(t) = \gamma(t)\frac{\Delta\mathrm{CDS}(t,T)}{1-R} - \frac{\Delta\widehat{V}}{1-R}\,. \tag{3.15}$$

Moreover, as in [4] we consider the Black–Scholes equations modelling $H(t)$ and $\mathrm{CDS}(t,T)$:

$$\frac{\partial H}{\partial t} + \frac{(\sigma^S)^2 S^2}{2}\frac{\partial^2 H}{\partial S^2} + (r-q)S\frac{\partial H}{\partial S} - cH = 0, \tag{3.16}$$

$$\begin{aligned}\frac{\partial \mathrm{CDS}(t,T)}{\partial t} &+ \frac{(\sigma^h)^2}{2}\frac{\partial^2 \mathrm{CDS}(t,T)}{\partial h^2} + \left(\mu^h - M^h\sigma^h\right)\frac{\partial \mathrm{CDS}(t,T)}{\partial h} \\ &+ \frac{h_t}{1-R}\Delta \mathrm{CDS}(t,T) - c\,\mathrm{CDS}(t,T) = 0\,.\end{aligned} \tag{3.17}$$

Next, by using (3.16)–(3.17), the hedging equation (3.14) is simplified to:

$$\begin{aligned}&\frac{\partial \widehat{V}}{\partial t} + \frac{(\sigma^S)^2 S^2}{2}\frac{\partial^2 \widehat{V}}{\partial S^2} + \frac{(\sigma^h)^2}{2}\frac{\partial^2 \widehat{V}}{\partial h^2} + \rho\sigma^S\sigma^h S\frac{\partial^2 \widehat{V}}{\partial S\partial h} \\ &\quad = \alpha\left(cH - (r-q)S\frac{\partial H}{\partial S}\right) - \alpha c H \\ &\quad + \gamma\left(-\frac{h}{1-R}\Delta \mathrm{CDS}(t,T) - \left(\mu^h - M^h\sigma^h\right)\frac{\partial \mathrm{CDS}(t,T)}{\partial h}\right) + \varepsilon h + f\widehat{V}\end{aligned} \tag{3.18}$$

in $[0,T)\times(0,\infty)\times(0,\infty)$ where α, γ and ε are given by (3.15).

Thus, the derivative price is modelled by the following final value PDE problem:

$$\begin{cases}\dfrac{\partial \widehat{V}}{\partial t} + \widetilde{\mathcal{L}}_{Sh}\widehat{V} + \dfrac{\Delta \widehat{V}}{1-R}h - f\widehat{V} = 0, & \text{in } [0,T)\times(0,\infty)\times(0,\infty), \\ \widehat{V}(T,S,h) = G(S)\,, & \end{cases} \tag{3.19}$$

where $G(S)$ represents the option payoff and the differential operator $\widetilde{\mathcal{L}}_{Sh}$ is given by

$$\begin{aligned}\widetilde{\mathcal{L}}_{Sh}V \equiv &\frac{(\sigma^S)^2 S^2}{2}\frac{\partial^2 V}{\partial S^2} + \frac{(\sigma^h)^2}{2}\frac{\partial^2 V}{\partial h^2} + \rho\sigma^S\sigma^h S\frac{\partial^2 V}{\partial h\partial S} \\ &+ (r-q)S\frac{\partial V}{\partial S} + \left(\mu^h - M^h\sigma^h\right)\frac{\partial V}{\partial h}\,.\end{aligned} \tag{3.20}$$

In order to write $\widetilde{\mathcal{L}}_{Sh}$ in terms of the spread h, we use the relationship between the drift of the spread $(\mu^h - M^h\sigma^h)$ and the investor's intensity of default λ:

$$\mu^h - M^h\sigma^h = -\kappa\lambda\,. \tag{3.21}$$

Thus, using the relationship (3.3) between h_t and λ_t in (3.21), we get

$$\mu^h - M^h\sigma^h = -\frac{\kappa}{1-R}h\,.$$

Therefore, the differential operator (3.20) turns into:

$$\mathcal{L}_{Sh}V \equiv \frac{(\sigma^S)^2 S^2}{2}\frac{\partial^2 V}{\partial S^2}+\frac{(\sigma^h)^2}{2}\frac{\partial^2 V}{\partial h^2}+\rho\sigma^S\sigma^h S\frac{\partial^2 V}{\partial h\partial S}+(r-q)S\frac{\partial V}{\partial S}-\kappa\frac{h}{1-R}\frac{\partial V}{\partial h}\,. \quad (3.22)$$

According to expression (3.5) and the possible choices for the mark–to–market value at default, different kinds of PDEs arise: the risk–free derivative value leads to a linear PDE, while the risky one gives rise to a nonlinear PDE. Therefore, two alternative problems are posed:

- If $M = \widehat{V}$, a nonlinear PDE model for the risky derivative value is obtained:

$$\begin{cases} \dfrac{\partial \widehat{V}}{\partial t} + \mathcal{L}_{Sh}\widehat{V} - f\widehat{V} = h\,\widehat{V}^+, & \text{in } [0,T)\times(0,\infty)\times(0,\infty), \\ \widehat{V}(T,S,h) = G(S)\,. \end{cases} \quad (3.23)$$

- If $M = V$, we obtain a problem governed by a linear PDE:

$$\begin{cases} \dfrac{\partial \widehat{V}}{\partial t} + \mathcal{L}_{Sh}\widehat{V} - \left(\dfrac{h}{1-R}+f\right)\widehat{V} = hV^+ - \dfrac{h}{1-R}V, \\ \qquad\qquad\qquad\qquad \text{in } [0,T)\times(0,\infty)\times(0,\infty), \\ \widehat{V}(T,S,h) = G(S)\,. \end{cases} \quad (3.24)$$

Next, in order to pose the PDEs modelling the XVA, the risky derivative value is split up into $\widehat{V} = V + U$, where V is the value of the risk–free derivative and U represents the XVA. Thus V satisfies:

$$\begin{cases} \dfrac{\partial V}{\partial t} + \mathcal{L}_S V - fV = 0, & \text{in } [0,T)\times(0,\infty), \\ V(T,S) = G(S)\,, \end{cases} \quad (3.25)$$

where the classical linear Black–Scholes operator $\mathcal{L}_S$ is given by

$$\mathcal{L}_S V \equiv \frac{(\sigma^S)^2 S^2}{2}\frac{\partial^2 V}{\partial S^2} + (r-q)S\frac{\partial V}{\partial S}\,.$$

Therefore, the XVA price U satisfies either a linear, either a nonlinear problem depending on the choice of the mark–to–market:

- If $M = \widehat{V}$, we obtain the nonlinear problem:

$$\begin{cases} \dfrac{\partial U}{\partial t} + \mathcal{L}_{Sh} U - fU = h(V+U)^+, & \text{in } [0,T) \times (0,\infty) \times (0,\infty), \\ U(T,S,h) = 0\,. \end{cases} \tag{3.26}$$

- If $M = V$, we obtain the linear problem:

$$\begin{cases} \dfrac{\partial U}{\partial t} + \mathcal{L}_{Sh} U - \left(\dfrac{h}{1-R} + f\right) U = hV^+, \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{in } [0,T) \times (0,\infty) \times (0,\infty), \\ U(T,S,h) = 0\,. \end{cases} \tag{3.27}$$

As our goal is to solve numerically problems (3.26) and (3.27) by a finite element method, we first proceed to localize the problems on a bounded domain. For this purpose, let us consider $\Omega = (0, S_\infty) \times (0, h_\infty)$ for large enough values of S_∞ and h_∞, so that the choice of these values does not affect the solution in the domain of financial interest. In the bounded domain we need to impose appropriate boundary conditions to be satisfied by U. For this purpose, we first consider the conditions satisfied by the risky value $\widehat{V}$ and the risk–free value V at $S = 0$ and $S = S_\infty$, that is

$$\begin{cases} \widehat{V}(t, S_\infty, h) = V(t, S_\infty) = V_\infty(t), \\ \widehat{V}(t, 0, h) = V(t, 0) = V_0(t), \end{cases} \tag{3.28}$$

where the values of $V_\infty(t)$ and $V_0(t)$ are respectively given by

$$V_\infty(t) = \begin{cases} S_\infty - K\,, & \text{for a call option,} \\ 0\,, & \text{for a put option,} \end{cases} \tag{3.29}$$

$$V_0(t) = \begin{cases} 0\,, & \text{for a call option,} \\ K \exp(-f(T-t))\,, & \text{for a put option.} \end{cases} \tag{3.30}$$

Moreover, the boundary $h = 0$ corresponds to zero spread, which is equivalent to a null intensity of default. Therefore, when $h = 0$ the derivative has no counterparty risk and behaves like the risk–free derivative, so that we impose the reasonable condition $\widehat{V}(t,S,0) = V(t,S)$.

In order to impose the boundary condition on U at $h = h_\infty$, we introduce the matrix

$$A = \frac{1}{2}\begin{pmatrix} (\sigma^S)^2 S^2 & \rho\sigma^S\sigma^h S \\ \rho\sigma^S\sigma^h S & (\sigma^h)^2 \end{pmatrix} \tag{3.31}$$

and we assume that U satisfies the Neumann condition $(A\nabla U \cdot \vec{n}) = 0$ for $h = h_\infty$, where $\vec{n}$ represents the unit outer normal vector on $\partial\Omega$.

Next, we introduce the new time variable $\tau = T - t$ to write problem (3.26) forward in time. We also rewrite the boundary conditions, previously formulated for $\widehat{V}$, in terms of U. Thus, the problem reads:

$$\begin{cases} \dfrac{\partial U}{\partial \tau} - \mathcal{L}_{Sh} U + fU = -h\,(V+U)^+\,, & (S,h) \in \Omega, \quad \tau \in (0,T] \\ U(\tau, S_\infty, h) = 0, & \\ U(\tau, 0, h) = 0, & \\ U(\tau, S, 0) = 0, & \\ (A\nabla U \cdot \vec{n})(\tau, S, h_\infty) = 0, & \\ U(0,S,h) = 0\,. & \end{cases} \tag{3.32}$$

For the linear equation in (3.27), we consider the same boundary conditions.

In Section 3.4 (Numerical methods) we consider a bounded computational domain and, using the properties of the differential operator, we show that prescribing a boundary condition at the boundary $S = 0$ is neither necessary for the analytical nor the numerical solution.

3.3 Mathematical analysis

As we have done in Chapter 1 for the model depending on one stochastic factor, in this section we study the existence and uniqueness of solution of problem (3.32). The mathematical analysis for the linear problem (3.27) can be studied as a particular case of the previous one. The work presented by Henry [30] is also followed to prove the existence and uniqueness of solution for bi–dimensional problems.

For the mathematical analysis of the model (3.32), we transform the associated PDE into an equivalent one governed by a sectorial operator. Thus, we introduce in (3.32) the following change of variables:

$$x = \ln\left(\frac{S}{K}\right)\,, \quad u(\tau, x, h) = U(\tau, S, h)\,.$$

Note that $x \in (-\infty, x_\infty)$. Therefore, we introduce a new truncation by considering the bounded domain $\widehat{\Omega} = (x_0, x_\infty) \times (0, h_\infty)$ and the following problem is posed:

$$\begin{cases} \dfrac{\partial u}{\partial \tau} + \mathcal{A}u = H(\tau, u), \qquad \text{in } (0, T] \times \widehat{\Omega} \\ u(\tau, x_\infty, h) = 0, \\ u(\tau, x_0, h) = 0, \\ u(\tau, x, 0) = 0, \\ (\widehat{A}\nabla u \cdot \vec{n})(\tau, x, h_\infty) = 0, \\ u(0, x, h) = 0\,, \end{cases} \tag{3.33}$$

where $\mathcal{A}u = -\text{div}\,(\widehat{A}\nabla u)$, with the constant matrix $\widehat{A}$ given by:

$$\widehat{A} = \frac{1}{2}\begin{pmatrix} (\sigma^S)^2 & \rho\sigma^S\sigma^h \\ \rho\sigma^S\sigma^h & (\sigma^h)^2 \end{pmatrix}.$$

The matrix $\widehat{A}$ is positive definite if and only if $|\rho| < 1$. Moreover, H is given by:

$$\begin{aligned} H(\tau, \varphi)(x, h) = &-h(V(\tau, Ke^x) + \varphi(x, h))^+ - c_0\,\varphi(x, h) - c_1\,\frac{\partial \varphi}{\partial x}(x, h) \\ &- c_2(h)\,\frac{\partial \varphi}{\partial h}(x, h), \qquad \forall \tau \in [0, T],\ \varphi \in H^1_\Gamma(\widehat{\Omega}), \end{aligned} \tag{3.34}$$

with

$$c_0 = f\,, \qquad c_1 = \frac{(\sigma^S)^2}{2} - (r - q)\,, \qquad c_2(h) = \frac{\kappa}{1 - R}h\,.$$

In the definition of H we use the notation $\Gamma = \{(x,h) \in \partial\widehat{\Omega} \,/\, h \neq h_\infty\}$ and the space $H^1_\Gamma(\widehat{\Omega}) = \{v \in H^1(\widehat{\Omega})/v = 0 \text{ on } \Gamma\}$ with the norm:

$$\|v\|^2_{H^1_\Gamma(\widehat{\Omega})} = \int_{\widehat{\Omega}} |\nabla v|^2 \, dx\, dh, \tag{3.35}$$

which is equivalent to the usual norm in $H^1(\widehat{\Omega})$ (see [1], for example).

In Section 1.3 we have introduced the definition of sectorial operator (Definition 1.3.1). Recall that for a sectorial operator $\mathcal{B}$ one can introduce a scale of fractional power spaces $X^\alpha = Range(\mathcal{B}^{-\alpha})$, such that $X = X^0$ and $X^1 = Dom(\mathcal{B})$, equipped with the norm $\|y\| = \|\mathcal{B}^\alpha y\|$, where $\mathcal{B}^\alpha$ for $\alpha > 0$ is a fractional power of $\mathcal{B}$.

Moreover, Theorem 1.3.2 introduced in Chapter 1 stablishes the hypotheses required to prove the existence of a unique solution of non linear problem (3.32). In order to apply Theorem 1.3.2, we will consider $X = L^2(\widehat{\Omega})$, $X^\alpha = H^1_\Gamma(\widehat{\Omega})$ with $\alpha = 1/2$, and $\mathcal{U} = (0,T) \times H^1_\Gamma(\widehat{\Omega})$. We will prove that operator $\mathcal{A}$ in (3.33) is a sectorial operator and that function H satisfies the conditions assumed for f in Theorem 1.3.2. For the first purpose, we first recall a lemma by Henry.

Lemma 3.3.1 (Section 1.3, Henry [30])**.** *If $\mathcal{B}$ is a bounded below, self–adjoint densely defined closed operator in a Hilbert space X, then $\mathcal{B}$ is sectorial.*

Proposition 3.3.2. *The operator $\mathcal{A}$ in (3.33) is a self–adjoint closed operator bounded from below. Therefore, $\mathcal{A}$ is sectorial.*

Proof: In order to prove that $\mathcal{A}$ is self–adjoint, for all φ, $\chi \in H^1_\Gamma(\widehat{\Omega})$ we compute

$$\begin{aligned}\langle \mathcal{A}\varphi, \chi\rangle &= \int_{\widehat{\Omega}} (\mathcal{A}\varphi, \chi)\, dx\, dh = -\int_{\widehat{\Omega}} \operatorname{div}(\widehat{A}\nabla\varphi)\chi \, dx\, dh \\ &= \int_{\widehat{\Omega}} \widehat{A}\nabla\varphi \cdot \nabla\chi \, dx\, dh - \int_{\partial\widehat{\Omega}} (\widehat{A}\nabla\varphi \cdot n)\chi \, d\gamma = \int_{\widehat{\Omega}} \widehat{A}\nabla\varphi \cdot \nabla\chi \, dx\, dh \,,\end{aligned}$$

where the last equality holds thanks to the boundary conditions. Moreover, we have

$$\begin{aligned}\langle \varphi, \mathcal{A}\chi\rangle &= \int_{\widehat{\Omega}} (\varphi, \mathcal{A}\chi)\, dx\, dh = -\int_{\widehat{\Omega}} \varphi \operatorname{div}(\widehat{A}\nabla\chi) \, dx\, dh \\ &= \int_{\widehat{\Omega}} \nabla\varphi \cdot \widehat{A}\nabla\chi \, dx\, dh - \int_{\partial\widehat{\Omega}} \varphi(\widehat{A}\nabla\chi \cdot n) \, d\gamma = \int_{\widehat{\Omega}} \nabla\varphi \cdot \widehat{A}\nabla\chi \, dx\, dh \,.\end{aligned}$$

The matrix $\widehat{A}$ is symmetric, hence we prove that $\langle \mathcal{A}\varphi, \chi\rangle = \langle \varphi, \mathcal{A}\chi\rangle$. Therefore, $\mathcal{A}$ is a self–adjoint operator in $H^1_\Gamma(\widehat{\Omega})$.

Next, we prove that the operator $\mathcal{A}$ is bounded from below.

$$\begin{aligned}
\langle \varphi, \mathcal{A}\varphi\rangle &= -\int_{\widehat{\Omega}} \varphi \operatorname{div}(\widehat{A}\nabla\varphi)\, dx\, dh = -\int_{\widehat{\Omega}} \operatorname{div}(\widehat{A}\nabla\varphi)\varphi\, dx\, dh \\
&= \int_{\widehat{\Omega}} \widehat{A}\nabla\varphi \cdot \nabla\varphi\, dx\, dh - \int_{\partial\widehat{\Omega}} \varphi(\widehat{A}\nabla\varphi)\, d\gamma = \int_{\widehat{\Omega}} \widehat{A}\nabla\varphi \cdot \nabla\varphi\, dx\, dh \\
&= \int_{\widehat{\Omega}} (\widehat{A}\nabla\varphi, \nabla\varphi)\, dx\, dh \geq \lambda_{min} \int_{\widehat{\Omega}} |\nabla\varphi|^2\, dx\, dh = \lambda_{min} \|\varphi\|^2_{H^1_\Gamma(\widehat{\Omega})},
\end{aligned}$$

where we have used that $\widehat{A}$ is a positive definite matrix and $\lambda_{min} = \min(\sigma(\widehat{A})) > 0$ is the minimum of the eigenvalues of $\widehat{A}$.

Thus, from the previous lemma we have shown that $\mathcal{A}$ is a sectorial operator. □

Proposition 3.3.3. *The function $H : \mathcal{U} \to X$ given by (3.34) is well defined, locally Lipschitz continuous in τ and locally Lipschitzian in φ.*

Proof: First note that function c_2 belongs to $L^\infty(\widehat{\Omega})$. Moreover, function V is given by the classical Black–Scholes formula for European call or put options, so that $x \mapsto V(\tau, K\, e^x) \in L^2(\widehat{\Omega})$. Therefore, $(V(\tau, \cdot) + \varphi)^+ \in L^2(\widehat{\Omega})$ for any function $\varphi \in H^1_\Gamma(\widehat{\Omega})$, thus implying $H(\tau, \varphi) \in L^2(\widehat{\Omega})$ so that $H(\tau, .) : L^2(\widehat{\Omega}) \to L^2(\widehat{\Omega})$ is well defined.

Next, we prove that H is locally Lipschitzian in φ, i.e.

$$\|H(\tau, \varphi_1) - H(\tau, \varphi_2)\|_{L^2(\widehat{\Omega})} \leq L_H\, \|\varphi_1 - \varphi_2\|_{H^1_\Gamma(\widehat{\Omega})}\,, \qquad \text{for all } \varphi_1,\, \varphi_2 \in H^1_\Gamma(\widehat{\Omega})\,.$$

For this purpose, let us estimate the difference

$$\begin{aligned}
|H(\tau, \varphi_1) - H(\tau, \varphi_2)| \;\leq\; & |c_1| \left|\frac{\partial\varphi_1}{\partial x} - \frac{\partial\varphi_2}{\partial x}\right| + |c_2(h)| \left|\frac{\partial\varphi_1}{\partial h} - \frac{\partial\varphi_2}{\partial h}\right| + |c_0|\; |\varphi_1 - \varphi_2| \\
& + |h|\; \left|(V(\tau, \cdot) + \varphi_1)^+ - (V(\tau, \cdot) + \varphi_2)^+\right| \\
\leq\; & |c_1| \left|\frac{\partial\varphi_1}{\partial x} - \frac{\partial\varphi_2}{\partial x}\right| + |c_2(h)| \left|\frac{\partial\varphi_1}{\partial h} - \frac{\partial\varphi_2}{\partial h}\right| \\
& + |c_0 + h|\; |\varphi_1 - \varphi_2|\,,
\end{aligned}$$

where we have used the fact that $\left|\chi_1^+ - \chi_2^+\right| \leq |\chi_1 - \chi_2|$, with $\chi_i = V(\tau,\cdot)+\varphi_i$. Then, by integration we get

$$\int_{\widehat{\Omega}} |H(\tau,\varphi_1) - H(\tau,\varphi_2)|^2 \, dx\, dh \leq |c_1|^2 \int_{\widehat{\Omega}} \left|\frac{\partial \varphi_1}{\partial x} - \frac{\partial \varphi_2}{\partial x}\right|^2 dx\, dh$$
$$+ \bar{c}_2^2 \int_{\widehat{\Omega}} \left|\frac{\partial \varphi_1}{\partial h} - \frac{\partial \varphi_2}{\partial h}\right|^2 dx\, dh + \bar{c}_0^2 \int_{\widehat{\Omega}} |\varphi_1 - \varphi_2|^2 \, dx\, dh$$

and, in terms of the norm,

$$\|H(\tau,\varphi_1) - H(\tau,\varphi_2)\|_{L^2(\widehat{\Omega})} \leq L_H \|\varphi_1 - \varphi_2\|_{H^1_\Gamma(\widehat{\Omega})}, \tag{3.36}$$

with $L_H = \max\{|c_1|, \bar{c}_2, C_0\, \bar{c}_0\}$, where the new constants are $\bar{c}_2 = \max\{|c_2(h)| \,/ h \in [0,h_\infty]\}$, $\bar{c}_0 = \max\{|c_0 + h| \,/ h \in [0,h_\infty]\}$ and $C_0 > 0$ is the constant associated to the Poincaré–Friedrichs inequality.

Next, we prove that H is locally Lipchitz continuous in τ. Thus, for τ_1, $\tau_2 \in [0,T]$ we compute

$$\begin{aligned} |H(\tau_1,\varphi) - H(\tau_2,\varphi)| &\leq |h| \left|(V(\tau_1,\cdot)+\varphi)^+ - (V(\tau_2,\cdot)+\varphi)^+\right| \\ &\leq h_\infty |V(\tau_1,\cdot) - V(\tau_2,\cdot)|, \end{aligned}$$

where we have used the inequality $\left|\chi_1^+ - \chi_2^+\right| \leq |\chi_1 - \chi_2|$, with $\chi_i = V(\tau_i,\cdot) + \varphi$. Therefore, in terms of norms we have

$$\|H(\tau_1,\varphi) - H(\tau_2,\varphi)\|^2_{L^2(\widehat{\Omega})} \leq h_\infty^2 \|V(\tau_1,\cdot) - V(\tau_2,\cdot)\|^2_{L^2(\widehat{\Omega})}. \tag{3.37}$$

Next, using that $V \in \mathcal{C}^1((0,T),X)$, and V is Lipschitz continuous in τ, we obtain that $H(\tau,u)$ is a Lipschitz function in τ. □

We introduce a corollary similar to Corollary 1.3.5 to prove the existence of a unique local solution of problem (3.33).

Corollary 3.3.4. *For any initial condition* $u_0 \in H^1_\Gamma(\widehat{\Omega})$ *there exists* $T_0 = T_0(0,u_0) > 0$ *such that the initial value problem (3.33) has a unique solution in* $(0,T_0)$.

The previous corollary follows from Theorem 1.3.2 and provides the existence and uniqueness of a local solution, as $T_0 = T_0(0,u_0)$ is a local time. Finally, in order to extend it to any interval $(0,T)$ for a given $T>0$, as we did in Section 1.3 for the one dimensional model, we need to apply Corollary 3.3.5 in [30].

Proposition 3.3.5. *The following inequality holds:*

$$\|H(\tau,\varphi)\|_{L^2(\widehat{\Omega})} \leq K(\tau)\big(1+\|\varphi\|_{H^1_\Gamma(\widehat{\Omega})}\big)\,,\ \textit{for all}\ \ (\tau,\varphi)\in(0,\infty)\times H^1_\Gamma(\widehat{\Omega})\,,$$

where K is continuous in $(0,\infty)$. Therefore, there exists a unique solution of problem (3.33) defined on the entire time interval $(0,T]$.

Proof: First, we note that the Lipschitz continuity properties also hold for $\tau\in(0,\infty)$ and prove the stated inequality. Thus, for any $(\tau,\varphi)\in(0,\infty)\times H^1_\Gamma(\widehat{\Omega})$ we have

$$\begin{aligned}
\|H(\tau,\varphi)\|_{L^2(\widehat{\Omega})} &\leq \|H(\tau,\varphi)-H(\tau,0)\|_{L^2(\widehat{\Omega})}+\|H(\tau,0)\|_{L^2(\widehat{\Omega})}\\
&\leq L_H\,\|\varphi-0\|_{H^1_\Gamma(\widehat{\Omega})}+\|H(\tau,0)\|_{L^2(\widehat{\Omega})}\\
&\leq \Big(L_H+\|H(\tau,0)\|_{L^2(\widehat{\Omega})}\Big)\Big(\|\varphi\|_{H^1_\Gamma(\widehat{\Omega})}+1\Big)\,,
\end{aligned}$$

where L_H is the Lipschitz constant for H, so that we can take

$$K(\tau)=L_H+\|H(\tau,0)\|_{L^2(\widehat{\Omega})}\ ,$$

which is continuous in τ in the interval $(0,\infty)$.

Next, we can apply Corollary 3.3.5 in [30]. Thus, we consider $u(\tau_0,\cdot)$ as the unique solution of (3.33) at time $\tau_0=T_0/2$ obtained from Corollary 3.3.4, so that from the Corollary 3.3.5 in [30] the unique solution of (3.33) through $(\tau_0,u(\tau_0,\cdot))$ exists for all $\tau\geq\tau_0$. Therefore, we obtain existence and uniqueness of solution of (3.33) in $(0,T]$. $\square$

3.4 Numerical methods

In this section we describe the numerical techniques we propose to solve the nonlinear problem (3.32). The corresponding linear problem can be considered as a particular case and is solved by similar methods.

The numerical approximation is mainly based on finite elements for spatial discretization. As usually in European options, we choose the maximum for the asset price coordinate, S_∞, equal to four times the strike price. Concerning the spread coordinate we consider the interval $[0, h_\infty]$, with $h_\infty = 0.2 = 20\%$ as a large enough value to not affect the numerical solution in the region of financial interest.

In order to solve it with a finite element method, we rewrite the PDE in (3.32) in a divergence form. Thus, we use matrix A from (3.31) and the vector

$$b = \begin{pmatrix} \left((\sigma^S)^2 - (r-q)\right) S \\ \dfrac{\rho\sigma^S\sigma^h}{2} + \dfrac{\kappa}{1-R}h \end{pmatrix}, \tag{3.38}$$

so that the PDE in (3.32) becomes:

$$\frac{\partial U}{\partial \tau} - \operatorname{div}(A\nabla U) + b \cdot \nabla U + fU = -h(V+U)^+, \quad (S,h) \in \Omega. \tag{3.39}$$

3.4.1 Time discretization and the method of characteristics

For the time discretization we use a semi–Lagrangian method, also known as the method of characteristics, first used in finance in [50]. As in the one factor model [4], we introduce the material derivative of U, i.e.

$$\frac{DU}{D\tau} = \frac{\partial U}{\partial \tau} + \frac{\partial U}{\partial S}\frac{\partial S}{\partial \tau} + \frac{\partial U}{\partial h}\frac{\partial h}{\partial \tau}$$

for given functions $S = S(\tau)$ and $h = h(\tau)$. Thus, in our problem the material derivative term is given by:

$$\frac{DU}{D\tau} = \frac{\partial U}{\partial \tau} + \left((\sigma^S)^2 - (r-q)\right) S\frac{\partial U}{\partial S} + \left(\frac{\rho\sigma^S\sigma^h}{2} + \frac{\kappa}{1-R}h\right)\frac{\partial U}{\partial h} \tag{3.40}$$

and equation (3.39) becomes:

$$\frac{DU}{D\tau} - \operatorname{div}(A\nabla U) + fU = -h(V+U)^+. \tag{3.41}$$

Taking into account the advective term in (3.40), we introduce $N_T > 0$, a constant time step $\Delta\tau = T/N_T > 0$, the time instants $\tau^n = n\Delta\tau$ ($n = 0, 1, \ldots, N_T$) and the

ODE problems associated to the computation of the characteristic curves:

$$\begin{cases} \dfrac{d\chi_1}{d\tau} = \Big(\left(\sigma^S\right)^2 - (r-q) \Big)\, \chi_1, \\ \chi_1(\tau^{n+1}) = S, \end{cases} \qquad \begin{cases} \dfrac{d\chi_2}{d\tau} = \dfrac{\rho\sigma^S\sigma^h}{2} + \dfrac{\kappa}{1-R}\chi_2, \\ \chi_2(\tau^{n+1}) = h, \end{cases} \tag{3.42}$$

the solution of which, $\chi(\tau) = \chi((S,h),\tau^{n+1};\tau)$ represents the characteristic curve associated to the vector field b passing through the point (S,h) at time instant τ^{n+1}. This characteristic curve, given by a two components expression, is a generalization of the characteristic curve introduced in Section 1.4, in particular:

$$\begin{aligned} \chi_1(\tau) &= S \exp\left(-((\sigma^S)^2 - r + q)(\tau^{n+1} - \tau)\right) , \\ \chi_2(\tau) &= -\frac{(1-R)\sigma^S\sigma^h\rho}{2\kappa} + \left(h + \frac{(1-R)\sigma^S\sigma^h\rho}{2\kappa}\right) \exp\left(\frac{-\kappa}{1-R}(\tau^{n+1} - \tau)\right) . \end{aligned}$$

Next, using the method of characteristics we approximate the material derivative in (3.41) and pose the semi–discrete problem:

$$\begin{cases} \dfrac{U^{n+1} - U^n \circ \chi^n}{\Delta\tau} - \operatorname{div}(A\nabla U^{n+1}) + fU^{n+1} = -h(V^{n+1} + U^{n+1})^+ , \\ U^0(S,h) = 0 , \end{cases} \tag{3.43}$$

where $\chi^n = \chi(\tau^n) = \chi((S,h),\tau^{n+1};\tau^n)$ and $U^n(\cdot) \approx U(\tau^n,\cdot)$. A piecewise bilinear interpolation method will be applied to evaluate $U^n \circ \chi^n$ in (3.43) at the nodes of the finite element mesh.

Remark 3.4.1. *When applying the method of characteristics, the displaced points on the characteristic line can be outside the domain. In that case, we consider the intersection of the characteristic curve* $\chi((S,h),\tau^{n+1},\tau^n)$ *with the boundary of the domain and interpolate the function on that new point.*

3.4.2 Fixed point scheme

Due to the nonlinearity of the problem (3.43), a fixed point scheme is proposed in each iteration of the method of characteristics. As a result, the global scheme, sketched in Algorithm 3.1 is implemented.

Algorithm 3.1

Let $N_T > 1$, $n = 0$, $\varepsilon > 0$ and U^0 given
For $n = 0, 1, 2, \dots, N_T - 1$

1. Let $U^{n+1,0} = U^n$, $k = 0$, $e = \varepsilon + 1$
2. For $k = 0, 1, \dots,$
   - Search $U^{n+1,k+1}$ solution of:

     $(1 + \Delta\tau f)\, U^{n+1,k+1} - \Delta\tau\, \mathrm{div}(A\nabla U^{n+1,k+1}) = U^n \circ \chi^n - \Delta\tau\, h\, (V^{n+1} + U^{n+1,k})^+$
   - Compute the relative error: $e = \dfrac{\|U^{n+1,k+1} - U^{n+1,k}\|}{\|U^{n+1,k+1}\|}$

   until $e < \varepsilon$

3.4.3 Boundary conditions

In Section 3.3 we have considered appropriate boundary conditions in order to prove the existence and uniqueness of a solution of (3.33). We will now adapt them for the numerical solution of the equivalent problem (3.32). First, we introduce the notation $x_0 = \tau$, $x_1 = S$ and $x_2 = h$, and the domain $\Omega^* = (0, x_0^\infty) \times (0, x_1^\infty) \times (0, x_2^\infty)$, where $x_0^\infty = T$, $x_1^\infty = S_\infty$ and $x_2^\infty = h_\infty$. The boundary of Ω^* is

$$\partial\Omega^* = \bigcup_{i=0}^{2} (\Gamma_i^{*,-} \cup \Gamma_i^{*,+}),$$

where we use the notation

$$\Gamma_i^{*,-} = \{(x_0, x_1, x_2) \in \partial\Omega^* \,/\, x_i = 0\}, \tag{3.44}$$

$$\Gamma_i^{*,+} = \{(x_0, x_1, x_2) \in \partial\Omega^* \,/\, x_i = x_i^\infty\}. \tag{3.45}$$

Then, the PDE in problem (3.32) can be written in the form:

$$\sum_{i,j=0}^{2} b_{ij} \frac{\partial^2 V}{\partial x_i \partial x_j} + \sum_{j=0}^{2} p_j \frac{\partial V}{\partial x_j} + c_0 V = g_0,$$

where the involved data are defined as follows:

$$B(x_0,x_1,x_2) = (b_{ij}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \dfrac{(\sigma^S)^2}{2}x_1^2 & \dfrac{\rho\sigma^S\sigma^h}{2}x_1 \\ 0 & \dfrac{\rho\sigma^S\sigma^h}{2}x_1 & \dfrac{(\sigma^h)^2}{2} \end{pmatrix}, \quad c_0(x_0,x_1,x_2) = -f\,,$$

$$p(x_0,x_1,x_2) = (p_j) = \begin{pmatrix} -1 \\ (r-q)x_1 \\ -\kappa\dfrac{x_2}{1-R} \end{pmatrix}, \qquad g_0(x_0,x_1,x_2) = (V+U)^+ x_2\,.$$

Following [41], that includes the theory of Fichera [25], we introduce the following subsets of Γ^* in terms of the normal vector to the boundary pointing inwards Ω^*, $\vec{m} = (m_0, m_1, m_2)$:

$$\Sigma^0 = \left\{ x \in \partial\Omega^* \,/\, \sum_{i,j=0}^{2} b_{ij} m_i m_j = 0 \right\}, \qquad \Sigma^1 = \partial\Omega^* - \Sigma^0\,,$$

$$\Sigma^2 = \left\{ x \in \Sigma^0 \,/\, \sum_{i=0}^{2} \left(p_i - \sum_{j=0}^{2} \frac{\partial b_{ij}}{\partial x_j} \right) m_i < 0 \right\}.$$

In our particular case, we have

$$\Sigma^0 = \Gamma_0^{*,-} \cup \Gamma_0^{*,+} \cup \Gamma_1^{*,-}\,, \quad \Sigma^1 = \Gamma_1^{*,+} \cup \Gamma_2^{*,-} \cup \Gamma_2^{*,+} \quad \text{and} \quad \Sigma^2 = \Gamma_0^{*,-}\,.$$

Thus, the boundary conditions must be imposed over the subset $\Sigma^1 \cup \Sigma^2$ [41], which matches with the set $\Gamma_0^{*,-} \cup \Gamma_1^{*,+} \cup \Gamma_2^{*,-} \cup \Gamma_2^{*,+}$.

After studying the boundaries which need a boundary condition to be imposed in order to solve the problem, we proceed to their effective deduction. Let us remark that the condition imposed on the boundary $\Gamma_0^{*,-}$ corresponds with the initial condition. On the boundary $\Gamma_1^{*,+}$, corresponding with the nodes (S_∞, h), a similar reasoning to the one in Section 1.4.3 is applied. We divide equation (3.32) by S^2 and pass to the limit, so that the following condition is obtained [23, 19]:

$$\lim_{S\to\infty} \frac{\partial^2 U}{\partial S^2} = 0\,. \tag{3.46}$$

Analogously to [19], we search a solution of the form

$$U(\tau, S, h) = H_1(\tau)S + H_2(\tau)h^2 + H_3(\tau)Sh + H_4(\tau)h + H_5(\tau)\,, \qquad (3.47)$$

where $H_1(\tau)$, $H_2(\tau)$, $H_3(\tau)$, $H_4(\tau)$ and $H_5(\tau)$ are independent of S and h.

More precisely, assuming $S^2\dfrac{\partial^2 U}{\partial S^2} \to 0$ when $S \to \infty$ in (3.32), we have:

$$\frac{\partial U}{\partial \tau} - \frac{1}{2}(\sigma^h)^2\frac{\partial^2 U}{\partial h^2} - \rho\sigma^S\sigma^h S\frac{\partial^2 U}{\partial h \partial S} - (r-q)S\frac{\partial U}{\partial S} + \kappa\frac{h}{1-R}\frac{\partial U}{\partial h} + fU = -h(V+U)^+\,.$$

This equation can be equivalently written as:

$$\frac{\partial U}{\partial \tau} - \mathrm{div}(\widetilde{A}\nabla U) + \widetilde{b}\cdot\nabla U + fU = -h(U+V)^+, \qquad (3.48)$$

where the matrix $\widetilde{A}$ and vector $\widetilde{b}$ are defined as follows:

$$\widetilde{A} = \begin{pmatrix} 0 & \dfrac{\rho\sigma^S\sigma^h}{2}S \\ \dfrac{\rho\sigma^S\sigma^h}{2}S & \dfrac{(\sigma^h)^2}{2} \end{pmatrix}, \qquad \widetilde{b} = \begin{pmatrix} -(r-q)S \\ \dfrac{\rho\sigma^S\sigma^h}{2} + \dfrac{\kappa}{1-R}h \end{pmatrix}. \qquad (3.49)$$

By using the method of characteristics in (3.48), we pose:

$$\frac{U^{n+1} - U^n\circ\chi^n}{\Delta\tau} - \mathrm{div}(\widetilde{A}\nabla U^{n+1}) + fU^{n+1} = -h(U+V)^+, \qquad (3.50)$$

where $\chi^n \equiv \chi((S,h),\tau^{n+1};\tau^n)$ is obtained from the solution of the problems:

$$\begin{cases} \dfrac{d\chi_1}{d\tau} = -(r-q)\chi_1, \\ \chi_1(\tau^{n+1}) = S, \end{cases} \qquad \begin{cases} \dfrac{d\chi_2}{d\tau} = \dfrac{\rho\sigma^S\sigma^h}{2} + \dfrac{\kappa}{1-R}\chi_2, \\ \chi_2(\tau^{n+1}) = h, \end{cases} \qquad (3.51)$$

and its components are given by

$$\chi_1^n = S\exp((r-q)\Delta\tau)\,,$$
$$\chi_2^n = -\frac{(1-R)\sigma^S\sigma^h\rho}{2\kappa} + \left(h + \frac{(1-R)\sigma^S\sigma^h\rho}{2\kappa}\right)\exp\left(\frac{-\kappa}{1-R}\Delta\tau\right)\,.$$

Now, replacing the solution (3.47) in each fixed point step of the discretized equation in (3.50), we obtain the following equation:

$$
\begin{aligned}
&(1+\Delta\tau f)H_1^{n+1,k+1}S + (1+\Delta\tau f)H_2^{n+1,k+1}h^2 \\
&\quad - \Delta\tau\left((\sigma^h)^2 H_2^{n+1,k+1} + \rho\sigma^S\sigma^h H_2^{n+1,k+1}h\right) \\
&\quad + (1+\Delta\tau f)\,H_3^{n+1,k+1}Sh - \Delta\tau\frac{3\rho\sigma^S\sigma^h}{2}SH_3^{n+1,k+1} + (1+\Delta\tau f)\,H_4^{n+1,k+1}h \\
&\quad - \Delta\tau\frac{\rho\sigma^S\sigma^h}{2}H_4^{n+1,k+1} + (1+\Delta\tau f)\,H_5^{n+1,k+1} \\
&\quad = -\Delta\tau h(V^{n+1}+U^{n+1,k})^+ + U^n \circ \chi^n\,.
\end{aligned}
$$

If we choose $H_1^{n+1,k+1} = H_2^{n+1,k+1} = H_3^{n+1,k+1} = H_4^{n+1,k+1} = 0$, the following nonhomogeneous Dirichlet boundary condition is deduced:

$$
U^{n+1,k+1}(S_\infty, h) = H_5^{n+1,k+1} = \frac{-\Delta\tau h(V^{n+1}+U^{n+1,k})^+ + U^n\circ\chi^n}{1+\Delta\tau f}\,. \tag{3.52}
$$

Note that this Dirichlet condition on $\Gamma_1^{*,+}$ tends to the boundary condition proposed in (3.32) for $S = S_\infty$ when $\Delta\tau$ tends to zero.

Next, we analyze the boundary conditions on $\Gamma_2^{*,+}$ and $\Gamma_2^{*,-}$. First, note that on $\Gamma_2^{*,-}$ we have $h = 0$, which means that the probability of default is zero. Thus, the value with counterparty risk is equal to the risk–free value and then $U(\tau, S, 0) = 0$. Thus, we will impose

$$
U^{n,k}(S,0) = 0\,, \qquad \text{for } n = 0,1,\ldots, \quad k = 0,1,\ldots\,.
$$

Following (3.32), for $h = h_\infty$ we impose $(A\nabla U^{n,k}\cdot\vec{n}) = 0\,.$

3.4.4 Finite element method

As we have already mentioned, we use the finite element method for spatial discretization. For this purpose, a triangular mesh of Ω and the associated finite element space of piecewise linear Lagrange polynomials are considered. First, at each time step $n = 0, 1, 2, \ldots, N_T - 1$ and each fixed point iteration $k = 0, 1, \ldots$, by using Green's formula the following variational formulation is posed:

Find $U^{n+1,k+1} \in \{\varphi \in H^1(\Omega)/\varphi = 0 \text{ on } \Gamma_2^-,\, \varphi = H_5^{n+1,k+1} \text{ on } \Gamma_1^+\}$, such that:

$$\int_\Omega (1+\Delta\tau f) U^{n+1,k+1} \varphi \, dS\, dh + \Delta\tau \int_\Omega A \nabla U^{n+1,k+1} \nabla\varphi \, dS\, dh$$
$$= \int_\Omega (U^n \circ \chi^n)\varphi \, dS\, dh - \Delta\tau \int_\Omega h(V^{n+1} + U^{n+1,k})^+ \varphi \, dS\, dh\,, \quad \forall \varphi \in H_*^1(\Omega)\,,$$

where $H_*^1(\Omega) = \{\varphi \in H^1(\Omega) \,/\, \varphi = 0 \text{ on } \Gamma_1^{*,+} \cup \Gamma_2^{*,-}\}$.

Next, for fixed natural numbers $N_S > 0$ and $N_h > 0$, we consider a uniform mesh of the computational domain Ω, the nodes of which are (S_i, h_j), with $S_i = i\Delta S$ $(i = 0, \ldots, N_S + 1)$ and $h_j = j\Delta h$ $(j = 0, \ldots, N_h + 1)$, where $\Delta S = S_\infty/(N_S + 1)$ and $\Delta h = h_\infty/(N_h + 1)$ denote the constant mesh steps in each coordinate. Associated to this uniform mesh, a piecewise linear Lagrange finite element discretization is considered. More precisely, we introduce the finite element spaces

$$W_h = \{\varphi_h \in \mathcal{C}(\overline{\Omega}) \,/\, \varphi_h|_{T_j} \in \mathcal{P}_1\,,\ \forall T_j \in \mathcal{T}\},$$
$$W_{h,*} = \{\varphi_h \in W_h \,/\, \varphi_h = 0 \text{ on } \Gamma_1^{*,+} \cup \Gamma_2^{*,-}\}\,,$$

in order to find $U_h^{n+1,k+1} \in W_h$, satisfying the boundary conditions and such that:

$$\int_\Omega (1+\Delta\tau f) U_h^{n+1,k+1} \varphi_h \, dS\, dh + \Delta\tau \int_\Omega A \nabla U_h^{n+1,k+1} \nabla\varphi_h \, dS\, dh$$
$$= \int_\Omega (U_h^n \circ \chi^n)\varphi_h \, dS\, dh - \Delta\tau \int_\Omega h(V^{n+1} + U_h^{n+1,k})^+ \varphi_h \, dS\, dh\,, \quad \forall \varphi_h \in W_{h,*}\,.$$

Quadrature formula based on the midpoints of the edges of the triangles has been used to obtain the coefficients of the matrix and the right hand side vector which define the linear system associated to the discretized problem. Moreover, the system has been solved by a partial pivoting LU factorization method [20].

The risk–free derivative value V is analytically given by the Black–Scholes formula for European options with a dividend yield [52]. We proceed as in Chapter 1 to transform the problem (3.25) in a model for an option which pays dividends.

3.4.5 Monte Carlo method

As we have made for the one dimensional model in Chapter 1, we also compute the XVA value in terms of expectations. With this purpose, the multi–dimensional Feynman–Kac theorem is applied on the nonlinear (3.26) and linear (3.27) PDEs.

We will assume that the evolution of the asset price and the evolution of the spread under the risk neutral measure are described by the following stochastic differential equations:

$$\begin{aligned} dS_t &= (r(t) - q(t))\, S_t\, dt + \sigma^S(t)\, S_t\, dW_t^{S,Q}\,, \\ dh_t &= (\mu^h(t) - M^h(t)\sigma^h(t))\, dt + \sigma^h(t)\, dW_t^{h,Q}\,, \end{aligned}$$

where $dW^{S,Q}$ and $dW^{h,Q}$ denote two correlated Wiener processes under measure Q, such that $\rho dt = dW^{S,Q} dW^{h,Q}$. The parameters which take part in the SDEs have been described in Section 3.2.

Next, applying Feynman–Kac theorem, the following expressions on the XVA at time instant t are deduced:

- If $M = \widehat{V}$,

$$\begin{aligned} U(t,S,h) = \mathbb{E}_t^Q \Bigg[- \int_t^T e^{-\int_t^u f d\tau} h \Big(& V(u,S(u),h(u)) \\ & + U(u,S(u),h(u)) \Big)^+ du \mid S_t = S, h_t = h \Bigg]. \end{aligned}$$

- If $M = V$,

$$U(t,S,h) = \mathbb{E}_t^Q \Bigg[- \int_t^T e^{-\int_t^u \left(\frac{h}{1-R} + f\right) d\tau} h \Big(V(u,S(u),h(u)) \Big)^+ du \mid S_t = S, h_t = h \Bigg].$$

Then, the XVA value at the current time is given by the following expressions:

- If $M = \widehat{V}$,

$$U(0,S,h) = \mathbb{E}_0^Q\Bigg[-\int_0^T e^{-\int_0^u f d\tau} h\Big(V(u,S(u),h(u)) + U(u,S(u),h(u))\Big)^+ du \mid S_0 = S, h_0 = h\Bigg]$$

$$= \mathbb{E}_0^Q\Bigg[-\int_0^T e^{-fu} h\Big(V(u,S(u),h(u)) + U(u,S(u),h(u))\Big)^+ du \mid S_0 = S, h_0 = h\Bigg].$$

- If $M = V$,

$$U(0,S,h) = \mathbb{E}_0^Q\Bigg[-\int_0^T e^{-\int_0^u \left(\frac{h}{1-R}+f\right) d\tau} h\Big(V(u,S(u),h(u))\Big)^+ du \mid S_0 = S, h_0 = h\Bigg]$$

$$= \mathbb{E}_0^Q\Bigg[-\int_0^T e^{-\left(\frac{h}{1-R}+f\right) u} h\Big(V(u,S(u),h(u))\Big)^+ du \mid S_0 = S, h_0 = h\Bigg].$$

The expressions in the previous integrals are discretized on a time mesh and approximated by numerical formulae. For this purpose, we consider $N_T > 0$ and a set of fixed instant times $t = 0 < t_1 < \ldots < t_{N_T} = T$, being T the maturity time. Thus, denoting $S_j = S(t_j)$ and $h_j = h(t_j)$ and using Euler–Maruyama scheme, the simulated asset price $S(t_j)$ and the simulated spread $h(t_j)$ are derived as follows:

$$S_j = S_{j-1}\Big(1 + (r-q)\Delta t\Big) + \sigma^S S_{j-1} \Delta W_j^S$$
$$h_j = h_{j-1} + (\mu - M\sigma^h)\Delta t + \sigma^h \Delta W_j^h$$

for $j = 1, \ldots, N_T$, where Δt is the size of the time interval and ΔW_j^i $(i = S, h)$ are independent Brownian increments which follow a normal distribution $\mathcal{N}(0, \sqrt{\Delta t})$. In order to build correlated Brownian processes, the Cholesky factorization is applied. Moreover, to reduce the discretization error the number of time steps N_T must be large enough.

As in Chapter 1, a fixed point iteration is implemented to compute the XVA when $M = \widehat{V}$.

3.5 Numerical results

In this section we present some examples to illustrate the performance of the models and the numerical methods in order to reproduce the expected behaviour of the risk–free value V, the risky value $\widehat{V}$, and the associated total value adjustment U, for different European options.

In all the tests we have used the same financial data, which are given in Table 3.1.

Table 3.1: Financial data for numerical tests

$\sigma^S = 0.3$	$\sigma^h = 0.2$	$\rho = 0.2$	$K = 15$	$T = 0.5$
$r = 0.3$	$q = 0.24$	$R = 0.3$	$\kappa = 0.01$	$f = 0.04$

The XVA represents the amount that has to be discounted from the risk–free derivative value due to the investor probability of default. We have developed the model from the point of view of the investor, thus we expect the XVA to be negative, as we can observe in the following examples. Moreover, we have considered both values for the mark–to–market, $M = \widehat{V}$ and $M = V$, so that a nonlinear problem and a linear one are formulated and numerically solved.

In practice, due to the great difference in S and h ranges of values, we have scaled the equations and solved the problem in the computational dimensionless domain $\widetilde{\Omega} = [0,1] \times [0,1]$, with step sizes $\Delta\widetilde{S}$ and $\Delta\widetilde{h}$ in the respective directions.

3.5.1 Test 1: Convergence

Table 3.2 shows the order of convergence of the proposed algorithm when the XVA of a call option is computed. Following [23], we use the convergence ratio CR

$$CR = \frac{\left\|U_{h/2} - U_{h/4}\right\|_\infty}{\left\|U_h - U_{h/2}\right\|_\infty},$$

from which we compute the experimental order of convergence $p = \log_2(CR)$. In Table 3.2 we can see how the computed values of p tend to one, which is the expected order of convergence taking into account that we use the piecewise linear finite elements and a first order time discretization.

Table 3.2: Empirical illustration of the order of convergence (p) for Test 1

$\Delta\widetilde{S} = \Delta\widetilde{h}$	$\Delta\tau$	CR	p
2^{-3}	1/10		
2^{-4}	1/20		
2^{-5}	1/40	2.02925134	1.02094757
2^{-6}	1/80	2.01447211	1.01040183
2^{-7}	1/160	2.00729238	1.00525078
2^{-8}	1/320	2.00367719	1.00265010
2^{-9}	1/640	2.00185420	1.00133690

3.5.2 Test 2: European call options

In this example, we study a European call option sold by the investor. Figure 3.1 shows the total value adjustment (XVA) for the European call option at $t = 0$. In this and all forthcoming examples we consider $N_S = N_h = 200$ and $\Delta\tau = 0.001$.

We can observe that the XVA becomes more negative when the underlying asset price increases, that is, when the option is "in the money". In this framework, the buyer will be more interested in exercising the option and will be more exposed to seller's default. Moreover, when the spread is higher, the total value adjustment increases in absolute terms.

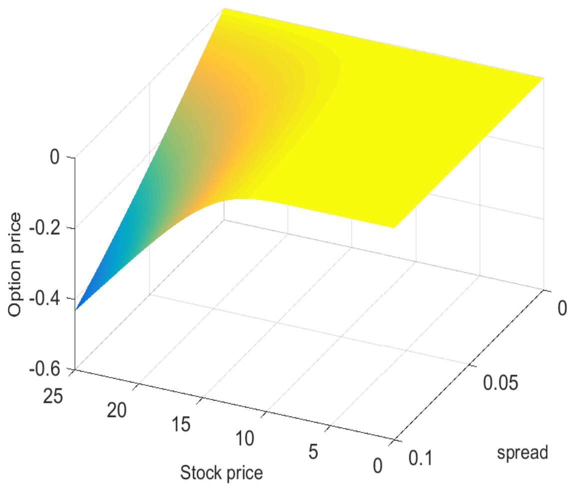


Figure 3.1: XVA for a European call option (Test 2)

3.5.3 Test 3: European put options

In this example we assume that the investor sells a European put option. In Figure 3.2 (left) the total value adjustment associated to this option is represented. In this case, the XVA is more negative when the asset price approaches to zero, that is when the put option is "in the money". Moreover, the XVA increases with the probability of default of the investor.

Next, the option value with counterparty risk is also shown in Figure 3.2 (right). Note that the difference between functions represented in both figures provides the price of the European option without counterparty risk.

3.5.4 Test 4: The linear problem ($M = V$)

In this test, we show the total value adjustment when the mark–to–market is chosen to be equal to the risk–free derivative. In Figure 3.3 we show the XVA associated to European call and put options, respectively. Thus, if these values are compared with the computed XVA when mark–to–market is equal to risky derivative (see Figures

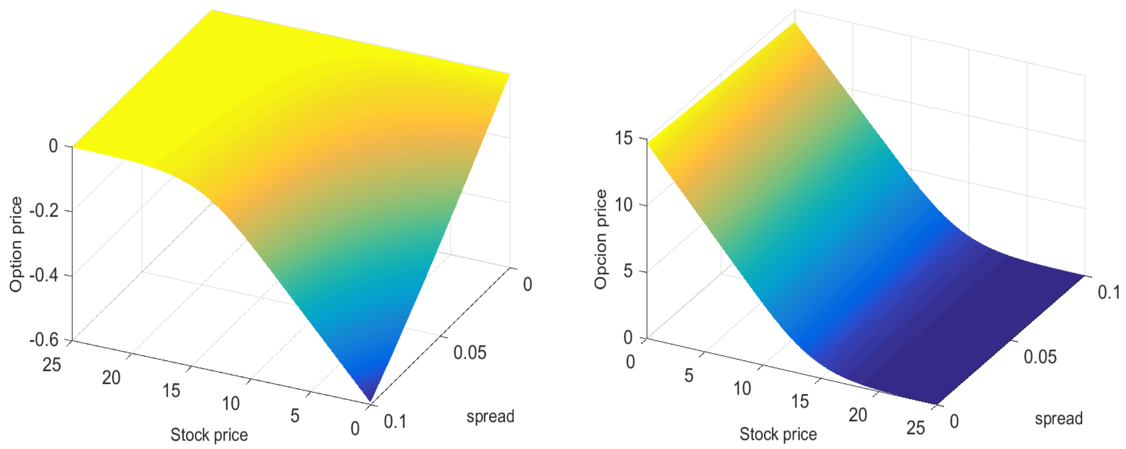


Figure 3.2: XVA (left) and price (right) of a European put option (Test 3)

3.1 and 3.2), we can conclude that there is not a significative difference between the choices of the mark–to–market close out.

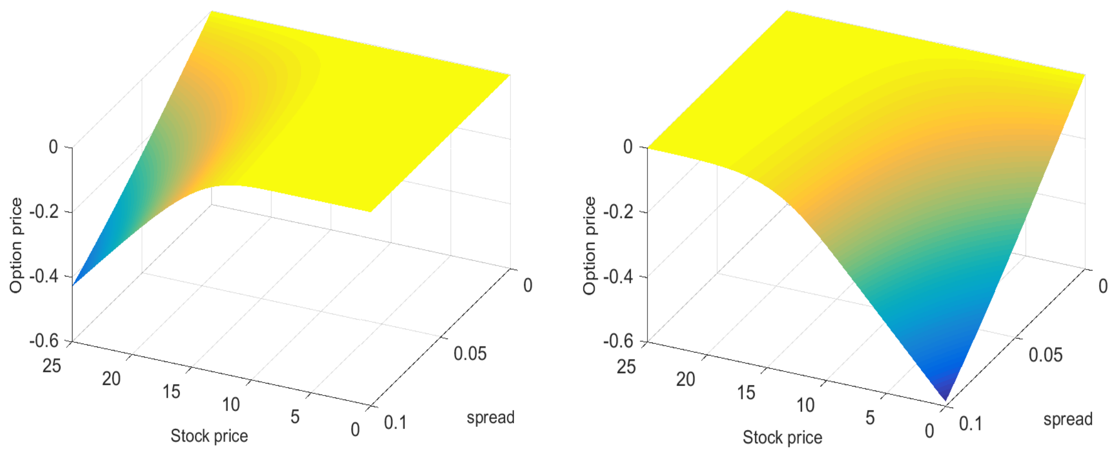


Figure 3.3: XVA for European call (left) and put (right) options (Test 4)

3.5.5 Test 5: Monte Carlo simulation

Finally, with this test we show the value obtained using the equation given in expectation terms. With this purpose, Monte Carlo techniques explained in Section 3.4.5

have been used with $N_T = 1000$ time steps and $N_P = 1000$ paths of asset price and spread. Moreover a 99% confidence interval has been built.

Table 3.3: Total value adjustment for an European put option with $M = \widehat{V}$. The parameter values of the problem are: $K = 15$, $T = 0.5$, $\sigma^S = 0.3$, $\sigma^h = 0.2$, $\rho = 0.2$, $r = 0.3$, $q = 0.24$, $\kappa = 0.01$, $R = 0.3$, $f = 0.04$.

S	h	Finite Elements	Confidence interval
0.0	0.00	-0.00000000	(-0.04871611 , 0.05091447)
27.0		-0.00000000	(-0.00007702 , 0.00029186)
3.0	0.05	-0.24748739	(-0.30204694 , -0.22295254)
9.0		-0.01164486	(-0.13335754 , -0.09750527)
15.0		-0.01994645	(-0.02350389 , -0.01447736)
24.0		-0.00001699	(-0.00023914 , 0.00017289)
9.0	0.10	-0.22398023	(-0.26166560 , -0.22317681)
12.0		-0.11223931	(-0.12786952 , -0.10594350)
15.0		-0.04017497	(-0.04563772 , -0.03546747)
18.0		-0.01065965	(-0.01236094 , -0.00746687)
27.0		-0.00007311	(-0.00010830 , -0.00002189)
18.0	0.15	-0.01575309	(-0.02016913 , -0.01561121)
21.0		-0.00346883	(-0.00491944 , -0.00318176)
24.0		-0.00066395	(-0.00083117 , -0.00048701)
21.0	0.20	-0.00441313	(-0.00612804 , -0.00433967)
27.0		-0.00015780	(-0.00045110 , -0.00004559)
30.0		-0.00002719	(-0.00045110 , -0.00004559)

We can observe that the numerical solution of the PDE model belongs, in all cases, to the confidence interval associated with the Monte Carlo simulation technique. The elapsed time to compute the XVA by the finite element method with $N_S = N_h = 200$ and $N_T = 500$ is 38314 seconds. On the other hand, the Monte Carlo resolution takes 21.9014 seconds for a only initial price. It is easy to deduce that the resolution of the PDE is more efficient that the Monte Carlo simulation for a large number of nodes.

Table 3.4: Total value adjustment for an European put option with $M = V$. The parameter values of the problem are: $K = 15$, $T = 0.5$, $\sigma^S = 0.3$, $\sigma^h = 0.2$, $\rho = 0.2$, $r = 0.3$, $q = 0.24$, $\kappa = 0.01$, $R = 0.3$, $f = 0.04$.

S	h	Finite Elements	Confidence interval
0.0	0.00	-0.00000000	(-0.01264495 , 0.08785706)
27.0		-0.00000000	(-0.00009733 , 0.00035472)
3.0	0.05	-0.24483526	(-0.27005760 , -0.19318396)
9.0		-0.11509726	(-0.11970139 , -0.08462493)
15.0		-0.01973870	(-0.02108726 , -0.01220414)
24.0		-0.00019265	(-0.00022632 , 0.00022493)
9.0	0.10	-0.22124095	(-0.24111938 , -0.20488754)
12.0		-0.11087184	(-0.11799153 , -0.09700050)
15.0		-0.03968885	(-0.04207405 , -0.03236791)
18.0		-0.01053191	(-0.01110796 , -0.00659774)
27.0		-0.00007226	(-0.00010330 , -0.00001913)
18.0	0.15	-0.01553640	(-0.01867646 , -0.01460365)
21.0		-0.00342128	(-0.00456478 , -0.00296129)
24.0		-0.00065490	(-0.00077348 , -0.00045896)
21.0	0.20	-0.00434864	(-0.00565678 , -0.00406874)
27.0		-0.00015550	(-0.00040484 , -0.00004905)
30.0		-0.00002680	(-0.00003033 , -0.00001152)

Chapter 4

Two stochastic factors model for American options with XVA

4.1 Introduction

In this chapter, we extend the model introduced in Chapter 2 to price the American options considering counterparty risk and compute the associated total value adjustment. In Chapter 2 a one dimensional model was deduced and analyzed to price the derivative value. In a first step, funding value adjustment (FVA), debit value adjustment (DVA) and credit value adjustment (CVA) were considered, and the model was later modified in order to include collateral value adjustment (CollVA). Moreover the intensities of default from both counterparties —the hedger and the investor— were considered constant. Thus, a model depending on one stochastic factor, the underlying active, was presented.

Nevertheless, default intensities from counterparties do not always exhibit constant behaviour. In particular, if both risky counterparties are considered to have stochastic intensities of default, a three underlying stochastic factors model is obtained [27].

In this chapter, as we have done in the previous one for European options, we consider that only the investor is defaultable and presents an stochastic intensity of default. Similar hypotheses as in the European options model introduced in the previous chapter are assumed (see page 116). Thus, we can analogously deduce the two–dimensional PDE models for the derivative value $\widehat{V}$ from the point of view of the investor.

We follow an approach based on complementarity problem formulation by means of suitable hedging arguments and the use of Itô's lemma for jump–diffusion processes, which extends the classical Black–Scholes inequality for American options. After imposing the hedging strategy, different kinds (linear or nonlinear, depending on the assumption of the mark–to–market value at default) of complementarity problems arise: a mark–to–market value equal to the riskless derivative leads to a linear complementarity problem involving the value of the riskless derivative, while a mark–to–market value equal to the risky derivative leads to a nonlinear complementarity problem.

In order to state the existence and uniqueness of the solution for the nonlinear complementarity problem we follow the methodology introduced by Jeong–Park [34], based in previous works by Brézis [10, 11]. Fichera [26] and Stampacchia [48, 49, 36] have also done important contributions to the analysis of variational inequalities and complementarity problems.

In addition, we propose a set of numerical methods to solve the complementarity problems for both choices of the mark–to–market value. For this purpose, we truncate the unbounded domain and pose suitable boundary conditions at the boundaries of the resulting bounded domain, following some ideas in [19]. After this truncation, we propose a time discretization based on the method of characteristics combined with a finite element discretization in the asset and spread variables. For the nonlinear complementarity problem, a fixed point iteration algorithm is proposed. Finally, the Augmented Lagrangian Active Set (ALAS) algorithm is used to solve the discretized complementarity problems.

The plan of the chapter is the following. In Section 4.2 we pose the complementarity problems deduced from the hedging arguments. In Section 4.3 we present the mathematical analysis of the previous problems. Section 4.4 presents the numerical methods and Section 4.5 shows some illustrative numerical results. In order to validate these results, some tests have also been solved by the Monte Carlo techniques described in [2].

4.2 Mathematical model

In this section, we obtain the models for American options considering counterparty risk. The main difference with the one factor model introduced in Chapter 2 comes from the consideration of stochastic intensities of default instead of constant ones. Moreover, assumptions and techniques similar to those ones of Chapter 3 for European options will be used, namely self–financing portfolio and non–arbitrage scenarios. For these reason, we will not enter into the details and make reference to the previous chapters.

As in Chapter 3, we assume an investor as a risky counterparty and consider that the issuer's intensity of default is null. Thus, the underlying asset price S, and the short term CDS spread of the investor h are modelled by the system of stochastic differential equations (3.1)–(3.2).

Thus, we consider a derivative trade between a hedger and an investor, where the latter has probability of default. The risky derivative value from the point of view of the investor, at time t, is denoted by $\widehat{V}(t, S_t, h_t, J_t^I)$, and depends on the spot value of the asset (S_t), on the spread of the investor (h_t) and on the investor's default state at time t (J_t^I). Remind that $J_t^I = 1$ in case of default before or at time t, otherwise $J_t^I = 0$. The risk–free American option value, corresponding to the same contract between two free–bankruptcy counterparties, is denoted by $\widehat{V}(t, S_t)$ and does not include any counterparty risk adjustment, whereas the risky derivative price $\widehat{V}_t$ includes adjustments such as DVA, FCA and/or CollVA into valuation.

As we introduced in Chapter 3, the intensity of default of the investor can be given in terms of the spread by (3.3). Moreover, the price of the derivative in case the investor goes bankrupt is given by (3.4), where $M(t, S_t, h_t)$ denotes the mark–to–market value, $Z^+ = \max(Z, 0)$ and $Z^- = \min(Z, 0)$. We also define $\Delta\widehat{V}$ as the variation of the derivative value, $\widehat{V}$, when the investor defaults and is given by (3.5).

The hedger will trade with different financial instruments to hedge the market risk, the spread risk and the investor's default risk, described in page 119. Thus, in order to derive the value of American options with counterparty risk, we consider the same self–financing portfolio built for European options in (3.6), Π_t, which is designed to hedge all underlying risk factors:

$$\Pi_t = \alpha(t)H(t) + \beta(t) + \gamma(t)\text{CDS}(t,T) + \varepsilon(t)\text{CDS}(t, t+dt) + \Omega(t)B(t, t+dt)\,. \quad (4.1)$$

Furthermore, as we did in Chapter 2 for one stochastic factor American options, in order to avoid arbitrage opportunities we introduce the following hedging inequality:

$$d\widehat{V}_t \le d\Pi_t\,. \quad (4.2)$$

Next, by applying Itô's Lemma for jump diffusion processes, we obtain the variation $d\widehat{V}_t$ of the derivative value $\widehat{V}_t$ introduced in (3.9). Thus, replacing the change of the portfolio and the change of the derivative in (4.2), the hedging equation is transformed into:

$$\begin{aligned}
&\Bigg(\frac{\partial \widehat{V}}{\partial t} + (r-q)S_t\frac{\partial \widehat{V}}{\partial S} + (\mu^h - M^h\sigma^h)\frac{\partial \widehat{V}}{\partial h} + \frac{1}{2}(\sigma^S)^2 S_t^2\frac{\partial^2 \widehat{V}}{\partial S^2} + \frac{1}{2}(\sigma^h)^2\frac{\partial^2 \widehat{V}}{\partial h^2} \\
&\qquad + \rho\sigma^S\sigma^h S_t\frac{\partial^2 \widehat{V}}{\partial S\partial h}\Bigg)(t, S_t, h_t)dt + \sigma^S S_t\frac{\partial \widehat{V}}{\partial S}(t, S_t, h_t)dW_t^S \\
&\qquad + \sigma^h\frac{\partial \widehat{V}}{\partial h}(t, S_t, h_t)dW_t^h + \Delta\widehat{V}(t, S_t, h_t)dJ_t^I \\
&\le \alpha(t)dH(t) + d\beta(t) + \gamma(t)d\text{CDS}(t,T) \\
&\qquad + \varepsilon(t)d\text{CDS}(t, t+dt) + \frac{\widehat{V}_t}{B(t,t+dt)}dB(t, t+dt)\,. \qquad (4.3)
\end{aligned}$$

In the previous inequality, we have taken into account the self–financing condition of the replication strategy (3.7).

Next, we proceed as in Chapter 3: we introduce the variation of the other financial instruments that take part in the portfolio, we apply Itô's lemma for jump–diffusion and we remove the risky contributions with the choice of coefficients (3.15). Finally, we consider the Black–Scholes equations (3.16)–(3.17) modelling $H(t)$ and $\mathrm{CDS}(t,T)$, so that the hedging inequality (4.3) becomes:

$$\begin{aligned}
&\frac{\partial \widehat{V}}{\partial t} + \frac{1}{2}(\sigma^S)^2 S^2 \frac{\partial^2 \widehat{V}}{\partial S^2} + \frac{1}{2}(\sigma^h)^2 \frac{\partial^2 \widehat{V}}{\partial h^2} + \rho \sigma^S \sigma^h S \frac{\partial^2 \widehat{V}}{\partial S \partial h} \\
&\quad \leq \frac{\partial \widehat{V}/\partial S}{\partial H/\partial S}\left(cH - (r-q)S\frac{\partial H}{\partial S}\right) + \frac{\partial \widehat{V}/\partial S}{\partial H/\partial S}(-fH) \\
&\quad + \frac{\partial \widehat{V}/\partial h}{\partial \mathrm{CDS}(t,T)/\partial h}\left(-\frac{h}{1-R}\Delta \mathrm{CDS}(t,T) - \left(\mu^h - M\sigma^h\right)\frac{\partial \mathrm{CDS}(t,T)}{\partial h}\right) \\
&\quad + \left(\frac{\partial \widehat{V}/\partial h}{\partial \mathrm{CDS}(t,T)/\partial h}\frac{\Delta \mathrm{CDS}(t,T)}{1-R} - \frac{\Delta \widehat{V}}{1-R}\right) h + f\widehat{V}\,, \qquad (4.4)
\end{aligned}$$

in $[0,T) \times (0,\infty) \times (0,\infty)$. Then, the American option value when considering counterparty risk is modelled by the following complementarity problem:

$$\begin{cases}
\mathcal{L}(\widehat{V}) = \dfrac{\partial \widehat{V}}{\partial t} + \tilde{\mathcal{L}}_{Sh}\widehat{V} + \dfrac{\Delta \widehat{V}}{1-R}h - f\widehat{V} \leq 0 \\
\widehat{V}(t,S,h) \geq G(S) \\
\mathcal{L}(\widehat{V})(\widehat{V} - G) = 0 \\
\widehat{V}(T,S,h) = G(S)\,,
\end{cases} \qquad (4.5)$$

where $G(S)$ represents the option payoff and the differential operator $\widetilde{\mathcal{L}}_{Sh}$ is

$$\begin{aligned}
\widetilde{\mathcal{L}}_{Sh}V \equiv{}& \frac{1}{2}(\sigma^S)^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{1}{2}(\sigma^h)^2 \frac{\partial^2 V}{\partial h^2} + \rho\sigma^S\sigma^h S \frac{\partial^2 V}{\partial h \partial S} \\
&+ (r-q)S\frac{\partial V}{\partial S} + \left(\mu^h - M^h\sigma^h\right)\frac{\partial V}{\partial h}\,. \qquad (4.6)
\end{aligned}$$

Thus, considering the relationship between h_t and λ_t given in (3.3), we get

$$\mu^h - M^h\sigma^h = -\frac{\kappa}{1-R}h\,,$$

and, as a consequence, the differential operator (4.6) turns into:

$$\mathcal{L}_{Sh}V \equiv \frac{1}{2}(\sigma^S)^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{1}{2}(\sigma^h)^2 \frac{\partial^2 V}{\partial h^2} + \rho\sigma^S\sigma^h S\frac{\partial^2 V}{\partial h \partial S} + (r-q)S\frac{\partial V}{\partial S} - \frac{\kappa}{1-R}h\frac{\partial V}{\partial h}\,.$$

According to expression (3.5) and the possible choices of the mark–to–market value at default, two alternative complementarity problems are obtained:

- If $M = \widehat{V}$, we deduce the nonlinear complementarity problem:

$$\begin{cases} \mathcal{L}_1(\widehat{V}) = \dfrac{\partial \widehat{V}}{\partial t} + \mathcal{L}_{Sh}\widehat{V} - f\widehat{V} - h\widehat{V}^+ \leq 0, \\ \qquad\qquad\qquad \text{in } [0,T) \times (0,\infty) \times (0,\infty) \\ \widehat{V}(t,S,h) \geq G(S) \\ \mathcal{L}_1(\widehat{V})(\widehat{V} - G) = 0 \\ \widehat{V}(T,S,h) = G(S)\,. \end{cases} \tag{4.7}$$

- If $M = V$, the following linear complementarity problem is derived:

$$\begin{cases} \mathcal{L}_2(\widehat{V}) = \dfrac{\partial \widehat{V}}{\partial t} + \mathcal{L}_{Sh}\widehat{V} - \left(\dfrac{h}{1-R} + f\right)\widehat{V} \\ \qquad -((1-R)V^+ - V)\dfrac{h}{1-R} \leq 0, \quad \text{in } [0,T) \times (0,\infty) \times (0,\infty) \\ \widehat{V}(t,S,h) \geq G(S) \\ \mathcal{L}_2(\widehat{V})(\widehat{V} - G) = 0 \\ \widehat{V}(T,S,h) = G(S)\,. \end{cases} \tag{4.8}$$

Moreover, the calculus of the XVA value, $U = \widehat{V} - V$, requires the previous computing of the counterparty risk–free American option value. Thus, the following linear complementarity problem which models the derivative value V has to be solved:

$$\begin{cases} \mathcal{L}_3(V) = \dfrac{\partial V}{\partial t} + \mathcal{L}_S V - fV \leq 0\,, \quad \text{in } [0,T) \times (0,\infty) \\ V(t,S) \geq G(S) \\ \mathcal{L}_3(V)(V - G) = 0 \\ V(T,S) = G(S)\,, \end{cases} \tag{4.9}$$

where the operator $\mathcal{L}_S$ is given by

$$\mathcal{L}_S V \equiv \frac{(\sigma^S)^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + (r-q) S \frac{\partial V}{\partial S}\,.$$

Remark 4.2.1. *We will solve the previous problems considering $G(S)$ as a general function. However, notice that in the particular case of the American options, where $G(S)$ is a positive function, $\widehat{V}^+ = \widehat{V}$, and the nonlinear term disappears in (4.7).*

In order to numerically solve problems (4.7) and (4.8) by a finite element method, we proceed to localize the problems on a bounded domain. For this purpose, let us consider $\Omega = (0, S_\infty) \times (0, h_\infty)$ for large enough values of S_∞ and h_∞, so that their choice does not affect the solution in the domain of financial interest. As in Chapter 3, we need to impose appropiate boundary conditions on the risky derivative value problem in the bounded domain. For this purpose, we consider the same boundary conditions than for V and $\widehat{V}$ as in the case of European options in Chapter 3. Then, at $S = 0$ and $S = S_\infty$, the derivative value is given by:

$$\begin{cases} \widehat{V}(t, S_\infty, h) = V(t, S_\infty) = V_\infty(t)\,, \\ \widehat{V}(t, 0, h) = V(t, 0) = V_0(t)\,, \end{cases} \tag{4.10}$$

where the values of $V_\infty(t)$ and $V_0(t)$ are given by (3.29) and (3.30), respectively. When $h = 0$, the derivative has no counterparty risk, which is equivalent to a null intensity of default λ. Thus, an appropiate condition is to consider $\widehat{V}(t, S, 0) = V(t, S)$. In order to impose the boundary condition at $h = h_\infty$, we introduce the matrix (3.31) and assume a nonhomogeneous Neumann boundary condition on the risky derivative value, $(A\nabla\widehat{V} \cdot \vec{n}) = (A\nabla V \cdot \vec{n})$ for $h = h_\infty$, where $\vec{n}$ denotes the unitary normal vector to $\partial\Omega$ pointing outwards Ω.

In the next section, the existence and uniqueness of the solution of problem (4.7) are studied. For this purpose, we introduce the problem which models the XVA in order to obtain a problem with homogeneous boundary conditions. Then, we split up the risky derivative value, $\widehat{V}$, as the sum of the XVA, U, plus the total value adjustment, V, i.e. $\widehat{V} = V + U$. Introducing this breakdown in (4.7), the following nonlinear complementarity problem is deduced:

$$
\begin{cases}
\mathcal{L}_t(U) = \dfrac{\partial U}{\partial t} + \mathcal{L}_{Sh}U - fU - h(U+V)^+ \leq -\dfrac{\partial V}{\partial t} - \mathcal{L}_S V + fV\,, \\
\qquad\qquad\qquad t \in [0,T)\,, \quad (S,h) \in \Omega \\
U(t,S,h) \geq G(S) - V(t,S) \\
\Big[\mathcal{L}_t(U) - \Big(-\dfrac{\partial V}{\partial t} - \mathcal{L}_S V + fV\Big)\Big]\big[U - (G(S) - V(t,S))\big] = 0 \\
U(T,S,h) = 0 \\
U(t,0,h) = 0 \\
U(t,S_\infty,h) = 0 \\
U(t,S,0) = 0 \\
(A\nabla U \cdot \vec{n})(\tau,S,h_\infty) = 0\,.
\end{cases}
\tag{4.11}
$$

For the linear problem (4.8), we consider the same boundary conditions.

Remark 4.2.2. *In Section* 4 *(Numerical methods) we consider a bounded computational domain and, using the properties of the differential operator, we show that prescribing a boundary condition at the boundary $S = 0$ is neither necessary for the analytical nor the numerical solution.*

4.3 Mathematical analysis of the nonlinear problem

In this section, we prove the existence and uniqueness of solution for problem (4.11) for a given function V. Then, taking into account the existence and uniqueness of the solution V for the classical problem (4.9) (see [53], for example), we can state the existence and uniqueness of solution for (4.7). The mathematical analysis of the linear complementarity problem (4.8) is much simpler.

Note that problem (4.11) includes a final condition, so it is a final–boundary value problem. Moreover, matrix A defined in (3.31) contains variable coefficients and degenerates at some boundaries. So, in order to write problem (4.11) in an equivalent initial–boundary value problem with a constant matrix, we introduce the time to maturity variable, $\tau = T - t$, as well as the new variables and unknown:

$$x = \ln \frac{S}{K}, \qquad u(\tau, x, h) = U(t, S, h), \qquad v(\tau, x) = V(t, S)\,.$$

Note that $x \in (-\infty, x_\infty)$, with $x_\infty = \ln(S_\infty/K)$. Therefore, in order to get a bounded domain, we introduce a new truncation to consider the domain $\widehat{\Omega} = (x_0, x_\infty) \times (0, h_\infty)$ and pose the nonlinear complementarity problem in $(0,T) \times \widehat{\Omega}$:

$$\begin{cases} \mathcal{L}_\tau(u) = \dfrac{\partial u}{\partial \tau} + \mathcal{A}u - \Phi(\tau, u) \geq \ell\,, \quad (x,h) \in \widehat{\Omega}, \quad \tau \in (0,T] \\ u \geq \psi \\ [\mathcal{L}_\tau(u) - \ell]\ [u - \psi] = 0 \\ u(0, S, h) = 0 \\ u(\tau, x_0, h) = 0 \\ u(\tau, x_\infty, h) = 0 \\ u(\tau, x, 0) = 0 \\ (\widehat{A}\nabla u \cdot \vec{n})(\tau, x, h_\infty) = 0\,, \end{cases} \tag{4.12}$$

where

$$\mathcal{A}u = -\text{div}\,(\widehat{A}\nabla u) \tag{4.13}$$

and matrix $\widehat{A}$ is given by

$$\widehat{A} = \frac{1}{2}\begin{pmatrix} (\sigma^S)^2 & \rho\sigma^S\sigma^h \\ \rho\sigma^S\sigma^h & (\sigma^h)^2 \end{pmatrix}, \tag{4.14}$$

which is positive definite if and only if $|\rho| < 1$.

Associated to formulation (4.12) we introduce $\Gamma = \{(x,h) \in \partial\widehat{\Omega}/h \neq h_\infty\}$ and the Hilbert space

$$W = H^1_\Gamma(\widehat{\Omega}) = \{z \in H^1(\widehat{\Omega})\,/\,z = 0 \text{ on } \Gamma\}\,,$$

which is equipped with the norm:

$$\|z\|^2_{H^1_\Gamma} = \int_{\widehat{\Omega}} |\nabla z|^2 \, dx \, dh \, .$$

Moreover, we denote by W^* the dual space of W.

The operator $\Phi : [0,T] \times H^1_\Gamma(\widehat{\Omega}) \to L^2(\widehat{\Omega})$, involved in (4.12), is given by:

$$\Phi(\tau,\varphi)(x,h) = c_0\,\varphi(x,h)+c_1\,\frac{\partial\varphi}{\partial x}(x,h)+c_2(h)\,\frac{\partial\varphi}{\partial h}(x,h)+h(v(\tau,x)+\varphi(x,h))^+ \, , \quad (4.15)$$

for all $\tau \in [0,T]$ and $\varphi \in H^1_\Gamma(\widehat{\Omega})$, where:

$$c_0 = -f \, , \qquad c_1 = -\frac{(\sigma^S)^2}{2} + (r-q) \, , \qquad c_2(h) = -\frac{\kappa}{1-R} h.$$

Finally, functions ψ and ℓ involved in (4.12) do not depend on h and are given by

$$\begin{aligned} &\psi(\tau,x) = G(Ke^x) - v(\tau,x) \, , \\ &\ell(\tau,x) = -\frac{\partial v}{\partial \tau}(\tau,x) + \frac{1}{2}(\sigma^S)^2\frac{\partial^2 v}{\partial x^2}(\tau,x) + \left(r-q-\frac{1}{2}(\sigma^S)^2\right)\frac{\partial v}{\partial x}(\tau,x) - f v(\tau,x) \, . \end{aligned} \quad (4.16)$$

4.3.1 Variational formulation

In this section, we first use subdifferential calculus tools to formulate the nonlinear complementarity problem (4.12) in the framework of semilinear parabolic variational inequalities. In this way we can apply the results in [34] to obtain the existence and uniqueness of solution of problem (4.12).

For this purpose, first following [34] we introduce the functional space

$$Y = L^2(0,T;W) \cap \mathcal{C}([0,T];L^2(\widehat{\Omega})) \cap W^{1,2}(0,T;W^*)$$

and the operator $\mathcal{H} : Y \to L^2(0,T;W^*)$, defined for each $\tau \in (0,T]$ as

$$\mathcal{H}(u)(\tau,\cdot) = -\frac{\partial u}{\partial \tau}(\tau,\cdot) - \mathcal{A}u(\tau,\cdot) + \Phi(\tau,u(\tau,\cdot)) + \ell(\tau,\cdot) \, . \quad (4.17)$$

Therefore, problem (4.12) can be equivalently written as:
Find $u \in Y$ such that

$$\mathcal{H}(u) \leq 0 \, , \quad u \geq \psi \, , \quad \mathcal{H}(u)(u-\psi) = 0 \, , \quad (4.18)$$

jointly with the initial condition and the homogeneous boundary condition on $h = h_\infty$.

As the function ψ depends on τ (i.e. the obstacle function is time dependent in this obstacle problem), then we introduce for each $\tau \in [0, T]$ the closed convex set

$$K(\tau) = \{z \in W \,/\, z \geq \psi(\tau, \cdot) \text{ in } \widehat{\Omega}\}\,.$$

Associated to each convex set, we introduce the indicatrix function $\phi : W \to (-\infty, \infty]$ of the convex set $K(\tau)$ as

$$\phi(z) = \begin{cases} 0\,, & \text{if } z \in K(\tau)\,, \\ +\infty\,, & \text{if } z \notin K(\tau)\,, \end{cases}$$

which is a lower semicontinuous, proper convex function. The subdifferential of ϕ is a maximal monotone multivalued operator denoted by $\partial\phi$, which is defined by:

$$w \in \partial\phi(u) \quad \Longleftrightarrow \quad \phi(u) \leq \phi(z) + (w, u - z), \quad \forall z \in W\,,$$

where $(\cdot, \cdot)$ denotes the duality pairing between W^* and W.
In the next proposition, we reformulate the nonlinear complementarity problem (4.18) in terms of the subdifferential $\partial\phi(u)$.

Proposition 4.3.1. *For $u(\tau, \cdot) \in K(\tau)$ and τ a.e. in $(0, T)$, the following conditions are equivalent*

$$(P_1) \quad \mathcal{H}(u) \leq 0\,, \quad u \geq \psi\,, \quad \mathcal{H}(u)(u - \psi) = 0$$
$$(P_2) \quad \mathcal{H}(u) \in \partial\phi(u)\,.$$

Proof:

1. Let us assume that $u \in K(\tau)$ satisfies (P_2) .

    - Let be $\epsilon \in H^1(\widehat{\Omega})$ such that $\epsilon = 0$ on Γ and $\epsilon > 0$ in $\widehat{\Omega}$, so that $z = u + \epsilon \in K(\tau)$. As $\mathcal{H}(u) \in \partial\phi(u)$ we have

$$\phi(u) - \phi(z) \leq (\mathcal{H}(u), u - z) = (\mathcal{H}(u), -\epsilon)\,.$$

Moreover, as u, $z \in K(\tau)$ then $\phi(u) = \phi(z) = 0$, so that

$$(\mathcal{H}(u), \epsilon) \leq 0$$

for any $\epsilon > 0$. Therefore, $\mathcal{H}(u) \leq 0$ and the first condition in (P_1) is satisfied.

- As $u(\tau, \cdot) \in K(\tau)$ then $u \geq \psi$.
- Next, for a given $\tau \in (0, T)$ we consider the set

$$\widehat{\Omega}^{+}(\tau) = \{(x, h) \in \widehat{\Omega} \,/\, u(\tau, x, h) > \psi(\tau, x)\}$$

which is an open set in $\widehat{\Omega}$. Next, we take $\omega \in H^1_\Gamma(\widehat{\Omega}) \cap L^\infty(\widehat{\Omega})$ such that $\omega \neq 0$ and $\|\omega\|_{H^1_\Gamma(\widehat{\Omega})} = 1$.

For $r > 0$, we consider the functions

$$v_r^{\pm}(x, h) = \begin{cases} u(\tau, x, h), & \text{if } (x, h) \notin \widehat{\Omega}^{+}(\tau) \\ u(\tau, x, h) \pm r\omega(x, h), & \text{if } (x, h) \in \widehat{\Omega}^{+}(\tau)\,. \end{cases}$$

Let $(x_0, h_0) \in \widehat{\Omega}^{+}(\tau)$, so that $u(\tau, x_0, h_0) > \psi(\tau, x_0)$ and

$$v_r^{\pm}(x_0, h_0) = u(\tau, x_0, h_0) \pm r\omega(x_0, h_0)\,.$$

Then, $v_r^{\pm}(x_0, h_0) > \psi(\tau, x_0)$ for r sufficiently small.

Finally, using the definition of subdifferential operator and that $v_r^{\pm}$ and $u(\tau, \cdot)$ belong to $K(\tau)$, we get

$$(\mathcal{H}(u), \pm r\omega)(x_0, h_0) \geq 0\,.$$

Since $w \neq 0$ and $r > 0$, we get $\mathcal{H}(u) = 0$ in $\widehat{\Omega}^{+}(\tau)$. Therefore, the third condition in (P_1) is proved.

2. Assuming that condition (P_1) is satisfied, then we need to prove that

$$(\mathcal{H}(u), u - z) - \phi(u) + \phi(z) \geq 0\,, \quad \forall z \in H^1_\Gamma(\widehat{\Omega})\,.$$

We distinguish two cases:

- If $z \notin K(\tau)$ then $\phi(z) = +\infty$ and $\phi(u) = 0$. Moreover, we have

$$\left| \int_{\widehat{\Omega}} \mathcal{H}(u)(\psi - z) d\widehat{\Omega} \right| \leq \|\mathcal{H}(u)\|_{H^{-1}(\widehat{\Omega})} \|\psi - z\|_{H^1(\widehat{\Omega})} .$$

Then, the left hand side is finite and

$$(\mathcal{H}(u), u - z) = (\mathcal{H}(u), u - \psi) + (\mathcal{H}(u), \psi - z) ,$$

so that

$$(\mathcal{H}(u), u - z) - \phi(u) + \phi(z) = +\infty \geq 0 .$$

- If $z \in K(\tau)$ then $\phi(z) = \phi(u) = 0$, so that

$$\begin{aligned} \phi(z) - \phi(u) + (\mathcal{H}(u), u - z) &= (\mathcal{H}(u), u - z) \\ &= (\mathcal{H}(u), u - \psi) + (\mathcal{H}(u), \psi - z) = (\mathcal{H}(u), \psi - z) \geq 0 , \end{aligned}$$

where the last inequality follows from $\mathcal{H}(u) \leq 0$ and $\psi - z \leq 0$.

Therefore, we have proved that (P_2) holds. □

From Proposition 4.3.1, we obtain that problem (4.18) is equivalent to finding $u \in K(\tau)$ a.e. $\tau \in (0, T]$, such that

$$\frac{\partial u}{\partial \tau} + \mathcal{A}u + \partial\phi(u) \ni \Phi(\cdot, u) + \ell \tag{4.19}$$

jointly with the initial condition and the homogeneous Neumann boundary condition at $h = h_\infty$.

4.3.2 Existence and uniqueness of solution

In the previous section, the nonlinear complementarity problem (4.12) has been equivalently formulated in the form (4.19), which fits to the framework of [34] to obtain the existence and uniqueness of solution for semilinear parabolic variational inequalities. More precisely, we will apply the following theorem.

Theorem 4.3.2 (Jeong–Park [34]). *Let $\mathcal{A}$ be a continuous operator satisfying the Gårding's inequality and $f(t,x(t))$ be a Lipschitz continuous function in $x(t)$. Assume that $k \in L^2(0,T;V^*)$ and $x_0 \in \overline{D(\phi)}$. Then, the problem*

$$\begin{cases} \dfrac{dx(t)}{dt} + \mathcal{A}x(t) + \partial\phi(x(t)) \ni f(t,x(t)) + k(t), & 0 < t \leq T \\ x(0) = x_0\,, \end{cases} \tag{4.20}$$

has a unique solution $x \in L^2(0,T;V) \cap \mathcal{C}([0,T];H)$ and there exists a constant C_1 depending on T such that

$$\|x\|_{L^2\cap\mathcal{C}} \leq C_1\bigg(1 + \|x_0\|_H + \|k\|_{L^2(0,T;V^*)}\bigg)\,.$$

Furthermore, if $k \in L^2(0,T;H)$ then the solution x belongs to $W^{1,2}(0,T;H)$ and satisfies

$$\|x\|_{W^{1,2}(0,T;H)} \leq C_1\bigg(1 + \|x_0\|_H + \|k\|_{L^2(0,T;H)}\bigg)\,.$$

In order to apply Theorem 4.3.2, we will consider $H = L^2(\Omega)$, $V = H^1_\Gamma(\Omega)$, and the functions $f = \Phi$, $k = \ell$ and prove the following proposition.

Theorem 4.3.3. *The following statements are satisfied:*

1. *The continuous operator $\mathcal{A}$ defined in (*4.13*) satisfies Gårding's inequality, i.e.:*

$$(\mathcal{A}z, z) \geq \omega_1 \|z\|^2_{H^1_\Gamma(\widehat{\Omega})} - \omega_2 \|z\|^2_{L^2(\widehat{\Omega})}\,, \quad \forall z \in H^1_\Gamma(\widehat{\Omega})\,, \tag{4.21}$$

   with $\omega_1 > 0$ and $\omega_2 \in \mathbb{R}$.

2. *$\ell \in L^2(0,T;L^2(\widehat{\Omega})) \subset L^2(0,T;W^*)$.*

3. *Let $D(\phi) = \Big\{z \in H^1_\Gamma(\widehat{\Omega}) \,/\, \phi(z) < \infty\Big\}$ and $u_0 = u(0,x,h)$. Then, $u_0 \in \overline{D(\phi)}$.*

4. *$\Phi(\tau,\varphi)$ is Lipschitz continuous on variable φ, i.e.*

$$\|\Phi(\tau,\varphi_1) - \Phi(\tau,\varphi_2)\|_{L^2(\widehat{\Omega})} \leq L_G \|\varphi_1 - \varphi_2\|_{H^1_\Gamma(\widehat{\Omega})}\,.$$

Therefore, the nonlinear variational inequality (4.19) has a unique solution $u \in L^2(0,T;H^1_\Gamma(\widehat{\Omega})) \cap \mathcal{C}([0,T];L^2(\widehat{\Omega}))$*; in particular* $u \in W^{1,2}(0,T;L^2(\widehat{\Omega}))$ *and satisfies*

$$\|u\|_{W^{1,2}(0,T;L^2(\widehat{\Omega}))} \leq C_1 \left(1 + \|u_0\|_{L^2(\widehat{\Omega})} + \|\ell\|_{L^2(0,T;H^1_\Gamma(\widehat{\Omega}))}\right) . \tag{4.22}$$

Proof:

1. From the definition of $\mathcal{A}$ in (4.13), the operator is linear. Moreover, we have

$$\begin{aligned}(\mathcal{A}u, v) &= \int_{\widehat{\Omega}} (-\mathrm{div}\ (\widehat{A}\nabla u))v\, d\widehat{\Omega} = \int_{\widehat{\Omega}} \widehat{A}\nabla u\, \nabla v\, d\widehat{\Omega} \\ &\leq M\, \|\nabla u\|_{L^2(\widehat{\Omega})}\, \|\nabla v\|_{L^2(\widehat{\Omega})} = M\, \|u\|_{H^1_\Gamma(\widehat{\Omega})}\, \|v\|_{H^1_\Gamma(\widehat{\Omega})}\end{aligned}$$

   for $M > 0$. Therefore, $\mathcal{A}$ is continuous. In order to prove Gårding's inequality, we consider that

$$(\mathcal{A}u, u) = \int_{\widehat{\Omega}} (-\mathrm{div}\ (\widehat{A}\nabla u))u\, d\widehat{\Omega} = \int_{\widehat{\Omega}} \widehat{A}\nabla u\, \nabla u\, d\widehat{\Omega} \geq \lambda_{\min}\, \|u\|^2_{H^1_\Gamma(\widehat{\Omega})}\ ,$$

   where $\lambda_{\min} > 0$ is the minimum of the eigenvalues of $\widehat{A}$. Thus by taking $\omega_1 = \lambda_{\min}$ and $\omega_2 = 0$, we obtain (4.21).

2. From the definition of ℓ in (4.16):

$$\ell(\tau, x) = -\frac{\partial v}{\partial \tau}(\tau, x) + \frac{1}{2}(\sigma^S)^2 \frac{\partial^2 v}{\partial x^2}(\tau, x) + \left(r - q - \frac{1}{2}(\sigma^S)^2\right) \frac{\partial v}{\partial x}(\tau, x) - f v(\tau, x)\,,$$

   where v is the solution of the following complementarity problem

$$\ell \leq 0\,, \quad v \geq \widetilde{G}\,, \quad \ell(v - \widetilde{G}) = 0\,,$$

   with $\widetilde{G}(x) = G(Ke^x)$.
   If we consider a put option $\widetilde{G}(x) = K(1 - e^x)^+$. As in the region $v > \widetilde{G}$ we get $\ell = 0$, we just consider the region $v = \widetilde{G}$, so that

$$\ell(\tau, x) = -\frac{1}{2}(\sigma^S)^2 K e^x - \left(r - q - \frac{1}{2}(\sigma^S)^2\right) K e^x - fK + fKe^x = (q - r + f)Ke^x - fK\,.$$

   Therefore, in this region we have

$$|\ell(\tau, S)| \leq |q - r + f|\, Ke^{x_\infty} + fK\,,$$

so that ℓ is bounded. In particular, we have

$$|\ell|^2_{L^2(0,T;L^2(\widehat{\Omega}))} = \int_0^T \int_{\widehat{\Omega}} |\ell(\tau,x)|^2 \, d\widehat{\Omega}\, d\tau < \infty\,.$$

Analogously, we can proceed in the case of call options. Then, we have proved that $\ell \in L^2(0,T;L^2(\widehat{\Omega}))$. Moreover, since $L^2(\widehat{\Omega}) \subset W^*$, then $\ell \in L^2(0,T;W^*)$.

3. It is easy to check that $u(0,x,h) = u_0(x,h) = 0 \geq \psi(\tau,x)$, thus $u_0 \in \overline{D(\phi)}$.

4. The operator $\Phi(\tau,\varphi) : [0,T] \times H^1_\Gamma(\widehat{\Omega}) \to L^2(\widehat{\Omega})$ has been defined in (4.15). We can deduce that

$$\begin{aligned}
&\big|\Phi(\tau,\varphi_1)(x,h) - \Phi(\tau,\varphi_2)(x,h)\big| \\
&= \Bigg| h(v(\tau,x)+\varphi_1(x,h))^+ - h(v(\tau,x)+\varphi_2(x,h))^+ \\
&\quad + c_0\,\varphi_1(x,h) - c_0\varphi_2(x,h) + c_1\,\frac{\partial\varphi_1}{\partial x}(x,h) - c_1\frac{\partial\varphi_2}{\partial x}(x,h) \\
&\quad + c_2(h)\frac{\partial\varphi_1}{\partial h}(x,h) - c_2(h)\frac{\partial\varphi_2}{\partial h}(x,h)\Bigg| \\
&\leq |c_0+h|\;|\varphi_1-\varphi_2| + |c_1|\,\left|\frac{\partial\varphi_1}{\partial x}(x,h) - \frac{\partial\varphi_2}{\partial x}(x,h)\right| \\
&\quad + |c_2(h)|\,\left|\frac{\partial\varphi_1}{\partial h}(x,h) - \frac{\partial\varphi_2}{\partial h}(x,h)\right|\,.
\end{aligned}$$

Then, by integration in $\widehat{\Omega}$, we get:

$$\begin{aligned}
&\int_{\widehat{\Omega}} |\Phi(\tau,\varphi_1)(x,h) - \Phi(\tau,\varphi_2)(x,h)|^2 \, dx\,dh \\
&\quad \leq |c_0+h|^2 \int_{\widehat{\Omega}} |\varphi_1(x,h) - \varphi_2(x,h)|^2 \, dx\,dh \\
&\quad + |c_1|^2 \int_{\widehat{\Omega}} \left|\frac{\partial\varphi_1}{\partial x}(x,h) - \frac{\partial\varphi_2}{\partial x}(x,h)\right|^2 \, dx\,dh \\
&\quad + |\tilde{c}_2|^2 \int_{\widehat{\Omega}} \left|\frac{\partial\varphi_1}{\partial h}(x,h) - \frac{\partial\varphi_2}{\partial h}(x,h)\right|^2 \, dx\,dh\,.
\end{aligned}$$

In terms of the norm, we get:

$$\begin{aligned}
\|\Phi(\tau,\varphi_1) - \Phi(\tau,\varphi_2)\|_{L^2(\widehat{\Omega})} &\leq L_1\,\|\nabla\varphi_1 - \nabla\varphi_2\|_{L^2(\widehat{\Omega})} + \tilde{c}_0\,\|\varphi_1-\varphi_2\|_{L^2(\widehat{\Omega})} \\
&\leq L_G\,\|\varphi_1-\varphi_2\|_{H^1_\Gamma(\widehat{\Omega})}
\end{aligned}$$

where $\tilde{c}_0 = \max\{|c_0 + h| : h \in [0, h_\infty]\}$, $\tilde{c}_2 = \max\{c_2(h) : h \in [0, h_\infty]\}$, $L_1 = \max\{|c_1|, \bar{c}_2\}$, $L_G = \max\{C_0\tilde{c}_0, L_1\}$ and $C_0 > 0$ is the constant associated to Poincaré–Friedrichs inequality. Then, Φ is Lipschitz continuous in the second variable φ.

Thus, thanks to Theorem 4.3.2 the nonlinear differential problem (4.19) has a unique solution $u \in W^{1,2}(0, T; L^2(\widehat{\Omega}))$ that satisfies the estimation (4.22). □

Corollary 4.3.4. *There exists a unique solution $u \in Y$ of problem (4.12).*

Proof: It follows from Proposition 4.3.1. □

4.4 Numerical methods

In this section we describe the different numerical techniques proposed to compute the derivative value considering counterparty risk. The risk–free derivative value modelled by (4.9) is computed by the techniques introduced in Chapter 2 for one dimensional problems. We will describe the numerical methods for approximating the solution of the nonlinear problem (4.7), the linear case (4.8) being solved by similar methods.

The numerical approximation is mainly based on finite elements combined with the method of characteristics. As usually in vanilla options, we consider the maximum value for the asset price S_∞ as four times the strike price. Similarly, we consider the interval $[0, h_\infty]$ for the admissible spread values, where h_∞ is eight times the reference value for the spread.

In order to compute the risky derivative value using a finite element method, we rewrite the complementarity problem (4.7) forward in time and in divergence form:

$$\begin{cases} \mathcal{L}_1(\widehat{V}) = \dfrac{\partial \widehat{V}}{\partial \tau} - \text{div}\,(A\nabla\widehat{V}) + b \cdot \nabla\widehat{V} + f\widehat{V} + h\widehat{V}^+ \geq 0 & \text{in } (0, T] \times \Omega \\ \widehat{V}(\tau, S, h) \geq G(S) \\ \mathcal{L}_1(\widehat{V})(\widehat{V} - G) = 0 \\ \widehat{V}(0, S, h) = G(S)\,, \end{cases} \tag{4.23}$$

where matrix A and vector b are defined in (3.31) and (3.38) respectively.

4.4.1 Method of characteristics

As we did for European options in Chapter 3, for the time discretization a semi–Lagrangian method —also known as the method of characteristics— is applied [50].

We consider a time discretization τ^n $(n = 0, 1, \ldots, N_T)$, with $\Delta\tau^n = \tau^{n+1} - \tau^n$ not necessarily constant. Taking into account the advective term, the problem (4.23) is approximated by

$$\begin{cases} \mathcal{L}_1^n(\widehat{V}^{n+1}) = \dfrac{\widehat{V}^{n+1} - \widehat{V}^n \circ \chi^n}{\Delta\tau^n} - \operatorname{div}(A\nabla\widehat{V}^{n+1}) \\ \qquad\qquad\qquad\qquad + f\widehat{V}^{n+1} + h(\widehat{V}^{n+1})^+ \geq 0\,, \\ \widehat{V}^0(S,h) = 0\,, \\ \widehat{V}^{n+1}(S,h) \geq G(S)\,, \\ \mathcal{L}_1^n(\widehat{V}^{n+1})(\widehat{V}^{n+1} - G) = 0\,, \end{cases} \tag{4.24}$$

for $n = 0, 1, 2\ldots, N_T - 1$, where $\widehat{V}^n(\cdot) \approx \widehat{V}(\tau^n, \cdot)$ and $\chi^n = \chi(\tau^n) = \chi((S,h), \tau^{n+1}; \tau^n)$ represents the characteristic curve passing through point (S,h) at time τ^{n+1}. Then function χ is the solution of the final value ODE problem (3.42). The components of χ^n can thus be deduced and are given by:

$$\chi_1^n = S\exp\left(-((\sigma^S)^2 - r + q)(\tau^{n+1} - \tau^n)\right)\,,$$
$$\chi_2^n = -\frac{(1-R)\sigma^S\sigma^h\rho}{2\kappa} + \left(h + \frac{(1-R)\sigma^S\sigma^h\rho}{2\kappa}\right)\exp\left(\frac{-\kappa}{1-R}(\tau^{n+1} - \tau^n)\right)\,.$$

A piecewise bilinear interpolation method is applied to evaluate $\widehat{V}^n \circ \chi^n$ in (4.24) at the nodes of the finite element mesh.

4.4.2 Fixed point scheme

Due to the nonlinearity of problem (4.24), a fixed point scheme is proposed at each iteration of the characteristics method. Thus, the global scheme is shown in Algorithm 4.1.

Algorithm 4.1

Let $N_T > 1$, $n = 0$, $\varepsilon > 0$ and $\widehat{V}^0$ given
For $n = 1, 2, \ldots, N_T - 1$:

1. Let $\widehat{V}^{n+1,0} = \widehat{V}^n$, $k = 0$, $e = \varepsilon + 1$
2. For $k = 0, 1, \ldots$
   - Search $\widehat{V}^{n+1,k+1}$ solution of:

$$\begin{aligned}&(1 + \Delta\tau^n f)\,\widehat{V}^{n+1,k+1} - \Delta\tau^n \operatorname{div}(A\nabla\widehat{V}^{n+1,k+1})\\&\qquad\qquad\geq \widehat{V}^n \circ \chi^n - \Delta\tau^n\, h\,(\widehat{V}^{n+1,k})^+ \\&\widehat{V}^{n+1,k+1}(S,h) \geq G(S)\\&\mathcal{L}_1^n(\widehat{V}^{n+1,k+1})(\widehat{V}^{n+1,k+1} - G) = 0\end{aligned} \tag{4.25}$$

   - Compute the relative error $e = \dfrac{\|\widehat{V}^{n+1,k+1} - \widehat{V}^{n+1,k}\|}{\|\widehat{V}^{n+1,k+1}\|}$

   until $e < \varepsilon$.

4.4.3 Boundary conditions

In Section 4.2, we have introduced some appropriate boundary conditions for problem (4.7) in order to prove the existence of a solution of (4.11). Next, we adapt such conditions for the numerical analysis. With this aim, we follow the same reasoning made in [41] in a similar way than in Chapter 3 to obtain the boundary conditions needed to compute the derivative value considering counterparty risk.

First, we introduce the notation $x_0 = \tau$, $x_1 = S$ and $x_2 = h$, and the domain $\Omega^* = (0, x_0^\infty) \times (0, x_1^\infty) \times (0, x_2^\infty)$, where $x_0^\infty = T$, $x_1^\infty = S_\infty$ and $x_2^\infty = h_\infty$. The boundary of Γ^* is $\partial\Omega^* = \bigcup_{i=0}^{2}(\Gamma^{*,-} \bigcup \Gamma^{*,+})$ where we have used the notation

$$\begin{aligned}\Gamma_i^{*,-} &= \{(x_0, x_1, x_2) \in \partial\Omega^* / x_i = 0\}\\\Gamma_i^{*,+} &= \{(x_0, x_1, x_2) \in \partial\Omega^* / x_i = x_i^\infty\}\,.\end{aligned} \tag{4.26}$$

Then, the partial differential inequality in problem (4.23) can be written in the form:

$$\sum_{i,j=0}^{2} b_{i,j} \frac{\partial^2 \widehat{V}}{\partial x_i \partial x_j} + \sum_{j=0}^{2} p_j \frac{\partial \widehat{V}}{\partial x_j} + c_0 \widehat{V} \leq g_0$$

where the involved data are defined as follows:

$$B(x_0, x_1, x_2) = (b_{ij}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \dfrac{(\sigma^S)^2}{2} x_1^2 & \dfrac{\rho \sigma^S \sigma^h}{2} x_1 \\ & & \\ 0 & \dfrac{\rho \sigma^S \sigma^h}{2} x_1 & \dfrac{(\sigma^h)^2}{2} \end{pmatrix}, \qquad c_0(x_0, x_1, x_2) = -f\,,$$

$$\vec{p}(x_0, x_1, x_2) = (p_j) = \begin{pmatrix} -1 \\ (r-q) x_1 \\ \\ -\dfrac{\kappa}{1-R} x_2 \end{pmatrix}, \qquad g_0(x_0, x_1, x_2) = \widehat{V}^+ x_2\,.$$

Following [41], in terms of the normal vector to the boundary pointing inwards Ω^*, $\vec{m} = (m_0, m_1, m_2)$ we introduce the following subsets of Γ^*:

$$\Sigma^0 = \left\{ x \in \Gamma^* / \sum_{i,j=0}^{2} b_{i,j} m_i m_j = 0 \right\}, \quad \Sigma^1 = \Gamma^* - \Sigma^0,$$

$$\Sigma^2 = \left\{ x \in \Sigma^0 / \sum_{i=0}^{2} \left(b_i - \sum_{j=0}^{2} \frac{\partial b_{ij}}{\partial x_j} \right) m_i < 0 \right\}.$$

In our particular case, we have

$$\Sigma^0 = \Gamma_0^{*,-} \cup \Gamma_0^{*,+} \cup \Gamma_1^{*,-}, \quad \Sigma^1 = \Gamma_1^{*,+} \cup \Gamma_2^{*,-} \cup \Gamma_2^{*,+}, \quad \Sigma^2 = \Gamma_0^{*,-}.$$

As a consequence, the boundary conditions must be imposed over the subset $\Sigma^1 \bigcup \Sigma^2$ [41], which matches with the set $\Gamma_0^{*,-} \bigcup \Gamma_1^{*,+} \bigcup \Gamma_2^{*,-} \bigcup \Gamma_2^{*,+}$.

After studying the boundaries which need a boundary condition to be imposed in order to solve the problem, we proceed to obtain them. Note that the condition imposed on the boundary $\Gamma_0^{*,-}$ corresponds with the initial condition which is given by the problem.

On boundary $\Gamma_1^{*,+}$, corresponding with the nodes (S_∞, h), we proceed in a similar way to Chapter 2 following a result obtained in Section 3.4.3 for European options. In fact, we compute the boundary condition for the associated European option problem. We recall the procedure applied in Section 3.4.3. We divide the equation associated to (4.23) by S^2, so that the following condition is obtained:

$$\lim_{S\to\infty} \frac{\partial^2 \widehat{V}_{\mathcal{E}}}{\partial S^2} = 0\,, \tag{4.27}$$

where $\widehat{V}_{\mathcal{E}}$ denotes the associated value to the European option.

Reasoning in a similar way to [19], we look for a solution of the form

$$\widehat{V}_{\mathcal{E}}(\tau, S, h) = H_1(\tau)S + H_2(\tau)h^2 + H_3(\tau)Sh + H_4(\tau)h + H_5(\tau)\,, \tag{4.28}$$

where $H_1(\tau)$, $H_2(\tau)$, $H_3(\tau)$, $H_4(\tau)$ and $H_5(\tau)$ are independent of S and h.

More precisely, assuming $S^2\dfrac{\partial^2 \widehat{V}_{\mathcal{E}}}{\partial S^2} \to 0$ when $S \to \infty$ in the European option equation, we have

$$\begin{aligned}\frac{\partial \widehat{V}_{\mathcal{E}}}{\partial \tau} - \frac{1}{2}(\sigma^h)^2 \frac{\partial^2 \widehat{V}_{\mathcal{E}}}{\partial h^2} - \rho\sigma^S\sigma^h S \frac{\partial^2 \widehat{V}_{\mathcal{E}}}{\partial h \partial S} - (r-q)S\frac{\partial \widehat{V}_{\mathcal{E}}}{\partial S} \\ +\kappa \frac{h}{1-R}\frac{\partial \widehat{V}_{\mathcal{E}}}{\partial h} + f\widehat{V}_{\mathcal{E}} = -\widehat{V}_{\mathcal{E}}^{+} h\,.\end{aligned} \tag{4.29}$$

In terms of the divergence operator, equation (4.29) is written as:

$$\frac{\partial \widehat{V}_{\mathcal{E}}}{\partial \tau} - \text{div}\,(\widetilde{A}\nabla \widehat{V}_{\mathcal{E}}) + \widetilde{b}\cdot\nabla\widehat{V}_{\mathcal{E}} + f\widehat{V}_{\mathcal{E}} = -\widehat{V}_{\mathcal{E}}^{+} h \tag{4.30}$$

where the matrix $\widetilde{A}$ and vector $\widetilde{b}$ are given by (3.49). Discretizing the material derivative in (4.30) on the characteristic curve, we pose:

$$\frac{\widehat{V}_{\mathcal{E}}^{n+1} - \widehat{V}_{\mathcal{E}}^{n}\circ\chi^n}{\Delta\tau^n} - \text{ div}\,(\widetilde{A}\nabla\widehat{V}_{\mathcal{E}}^{n+1}) + f\widehat{V}_{\mathcal{E}}^{n+1} = -(\widehat{V}_{\mathcal{E}}^{n+1})^{+} h \tag{4.31}$$

where the characteristic curve $\chi^n \equiv \chi((S,h), \tau^{n+1}, \tau^n)$ is the solution of the final value problems (3.51) and its components at time τ^n are given by:

$$\begin{aligned}\chi_1^n &= S\exp((r-q)\Delta\tau^n),\\ \chi_2^n &= -\frac{(1-R)\sigma^S\sigma^h\rho}{2\kappa} + \left(h + \frac{(1-R)\sigma^S\sigma^h\rho}{2\kappa}\right)\exp\left(\frac{-\kappa}{1-R}\Delta\tau^n\right)\,.\end{aligned}$$

Now, replacing the solution (4.28) in each fixed point step of the discretized equation in (4.31), we obtain the following equation:

$$
\begin{aligned}
&(1+\Delta\tau^n f)\, H_1^{n+1,k+1} S + (1+\Delta\tau^n f)\, H_2^{n+1,k+1} h^2 \\
&\quad - \Delta\tau^n \left((\sigma^h)^2 H_2^{n+1,k+1} + \rho\sigma^S\sigma^h H_2^{n+1,k+1} h \right) \\
&\quad + (1+\Delta\tau^n f)\, H_3^{n+1,k+1} Sh - \Delta\tau^n \frac{3\rho\sigma^S\sigma^h}{2} S H_3^{n+1,k+1} \\
&\quad + (1+\Delta\tau^n f)\, H_4^{n+1,k+1} h - \Delta\tau^n \frac{\rho\sigma^S\sigma^h}{2} H_4^{n+1,k+1} \\
&\quad + (1+\Delta\tau^n f)\, H_5^{n+1,k+1} = -\Delta\tau^n (\widehat{V}_{\mathcal{E}}^{n+1,k})^+ h + \widehat{V}_{\mathcal{E}}^n \circ \chi^n \,.
\end{aligned} \tag{4.32}
$$

If we choose, similarly to the European options case,

$$
H_1^{n+1,k+1} = H_2^{n+1,k+1} = H_3^{n+1,k+1} = H_4^{n+1,k+1} = 0\,,
$$

the derivative value $\widehat{V}_{\mathcal{E}}$ satisfies the following equation:

$$
\begin{aligned}
\widehat{V}_{\mathcal{E}}^{n+1,k+1}(S_\infty, h) &= H_5^{n+1,k+1} \\
&= \frac{-\Delta\tau^n (\widehat{V}_{\mathcal{E}}^{n+1,k})^+ h + \widehat{V}_{\mathcal{E}}^n \circ \chi^n}{1+\Delta\tau^n f}\,.
\end{aligned} \tag{4.33}
$$

Note that for American options we have an obstacle problem. Taking into account the complementarity restriction, the following nonhomogeneous Dirichlet boundary condition is imposed:

$$
\begin{aligned}
\widehat{V}^{n+1,k+1}(S_\infty, h) &= \max\left(\widehat{V}_{\mathcal{E}}^{n+1,k+1}(S_\infty, h), G(S_\infty) \right) \\
&= \max\left(\frac{-\Delta\tau^n h (\widehat{V}^{n+1,k})^+ + \widehat{V}^n \circ \chi^n}{1+\Delta\tau^n f}, G(S_\infty) \right)\,.
\end{aligned}
$$

Let us remark that this Dirichlet condition on $\Gamma_1^{*,+}$ tends to the continuous boundary condition proposed in (4.10) for $S = S_\infty$ when $\Delta\tau^n$ tends to zero.

Finally, we analyze the boundary conditions on $\Gamma_2^{*,+}$ and $\Gamma_2^{*,-}$. We make the same reasoning used in the mathematical analysis to impose continuous boundary conditions. First, note that on $\Gamma_2^{*,-}$ the spread value is null, i.e. $h = 0$, which means that the probability of default from the investor is null. Thus, the derivative value considering counterparty risk is equal to the risk–free value and then $\widehat{V}(\tau, S, 0) = V(\tau, S)$. Thus, we impose

$$\widehat{V}^{n,k}(S,0) = V^{n,k}(S), \qquad \text{for } n = 0, 1, \ldots, N_T, \quad k = 0, 1, \ldots .$$

On boundary $\Gamma_2^{*,+}$, a non homogeneous Neumann boundary condition is considered:

$$(A\nabla\widehat{V}) \cdot \vec{n} = (A\nabla V) \cdot \vec{n}\,,$$

where matrix A is given by (3.31).

4.4.4 Finite element method

For the spatial discretization of (4.25) a triangular mesh of Ω and the associated finite element space of piecewise linear Lagrange polynomials are considered. First, we introduce the convex closed subset

$$\tilde{\mathcal{K}} = \{\varphi \in H^1(\Omega) \,/\, \varphi = \widehat{V}^{n+1,k+1} \text{ on } \Gamma_2^-\,,\ \varphi = \widehat{V}^{n+1,k+1} \text{ on } \Gamma_1^+ \text{ and } \varphi \geq G(S)\}\,.$$

Thus, at each time step $n = 0, 1, 2, \ldots, N_T$ and each fixed point iteration $k = 0, 1, \ldots$, the following variational formulation is posed:

Find $\widehat{V}^{n+1,k+1} \in \tilde{\mathcal{K}}$ such that:

$$\begin{aligned}
&\int_\Omega (1 + \Delta\tau^n f)\widehat{V}^{n+1,k+1}(\varphi - \widehat{V}^{n+1,k+1})\, dS\, dh + \Delta\tau^n \int_\Omega A\nabla\widehat{V}^{n+1,k+1}\,\nabla(\varphi - \widehat{V}^{n+1,k+1})\, dS\, dh\\
&\quad - \Delta\tau^n \int_{\Gamma_2^{*,+}} (A\nabla V^{n+1,k+1}, n)(\varphi - \widehat{V}^{n+1,k+1})\partial\gamma \geq \int_\Omega (\widehat{V}^n \circ \chi^n)(\varphi - \widehat{V}^{n+1,k+1})\, dS\, dh\\
&\quad - \Delta\tau^n \int_\Omega h(\widehat{V}^{n+1,k})^+(\varphi - \widehat{V}^{n+1,k+1})\, dS\, dh\,, \quad \forall \varphi \in \tilde{\mathcal{K}}\,.
\end{aligned}$$

Next, for fixed natural numbers $N_S > 0$ and $N_h > 0$, we consider a uniform mesh of the computational domain Ω, the nodes of which are (S_i, h_j), with $S_i = i\Delta S$ $(i = 0, \ldots, N_S{+}1)$ and $h_j = j\Delta h$ $(j = 0, \ldots, N_h{+}1)$, where $\Delta S = S_\infty/(N_S{+}1)$ and $\Delta h = h_\infty/(N_h{+}1)$ denote the constant mesh steps in each coordinate. Associated to this uniform mesh, a piecewise linear Lagrange finite element discretization is considered. More precisely, we introduce the finite element spaces

$$\begin{aligned}
W_h &= \{\varphi_h \in \mathcal{C}(\Omega) \,/\, \widetilde{\varphi}|_{T_j} \in \mathcal{P}_1\,,\ \forall T_j \in \mathcal{T}\}\,,\\
\mathcal{K}_h &= \{\varphi_h \in W_h \,/\, \varphi_h = \widehat{V} \text{ on } \Gamma_1^{*,+} \cup \Gamma_2^{*,-} \text{and } \varphi_h \geq G(S)\}\,,
\end{aligned}$$

in order to find $\widehat{V}_h^{n+1,k+1} \in \mathcal{K}_h$ satisfying the boundary conditions and such that:

$$
\begin{aligned}
&\int_\Omega (1+\Delta\tau^n f)\widehat{V}_h^{n+1,k+1}\big(\varphi_h - \widehat{V}_h^{n+1,k+1}\big)\, dS\, dh \\
&\quad + \Delta\tau^n \int_\Omega A\nabla\widehat{V}_h^{n+1,k+1}\,\nabla\big(\varphi_h - \widehat{V}_h^{n+1,k+1}\big)\, dS\, dh \\
&\quad - \Delta\tau^n \int_{\Gamma_2^{*,+}} (A\nabla V_h^{n+1,k+1}, n)(\varphi_h - \widehat{V}_h^{n+1,k+1})\partial\gamma \\
&\quad \geq \int_\Omega \big(\widehat{V}_h^n \circ \chi^n\big)\big(\varphi_h - \widehat{V}_h^{n+1,k+1}\big)\, dS\, dh - \Delta\tau^n \int_\Omega h\big(\widehat{V}_h^{n+1,k}\big)^+\big(\varphi_h - \widehat{V}_h^{n+1,k+1}\big)\, dS\, dh\,,
\end{aligned}
$$

for all $\varphi_h \in \mathcal{K}_h$. Quadrature formula based on the midpoints of the edges of the triangles has been used to obtain the coefficients of the matrix and the right hand side vector which define the linear system associated to the discretized problem.

After the time discretization with the method of characteristics and the spatial discretization with finite elements, the fully discretized problem can be written in the form:

$$
\begin{cases}
A_h\widehat{V}_h^{n+1,k+1} \geq b_h^{n+1,k+1}\,, \\
\widehat{V}_h^{n+1,k+1} \geq \Psi_h\,, \\
(A_h\widehat{V}_h^{n+1,k+1} - b_h^{n+1,k+1})(\widehat{V}_h^{n+1,k+1} - \Psi_h) = 0\,,
\end{cases}
\tag{4.34}
$$

where Ψ_h denotes the discretized exercise value, $G(S)$, which also coincides with the value at maturity.

In order to solve problem (4.34), the augmented Lagrangian active set (ALAS) algorithm proposed by Kärkkäinen et al. [35] and applied in the one dimensional American options problem is also employed. The details of the method can be found in Chapter 2 and in [4].

4.4.5 Monte Carlo method

In this section, as we have made for the one dimensional model in Chapter 2, we introduce and compute the total value adjustment in terms of expectations. With this aim, we combine the multi–dimensional Feynman–Kac theorem with the techniques introduced in Section 2.4, following Longstaff and Schwartz [38] and Glasserman [28].

As we have introduced in Section 4.2, we assume —under the risk neutral measure— the following evolution of the asset price and of the spread:

$$dS_t = (r(t) - q(t))\, S_t\, dt + \sigma^S(t)\, S_t\, dW_t^{S,Q}\,,$$
$$dh_t = (\mu^h(t) - M^h(t)\sigma^h(t))\, dt + \sigma^h(t)\, dW_t^{h,Q}\,,$$

where $dW^{S,Q}$ and $dW^{h,Q}$ denote two correlated Wiener processes under the measure Q, such that $\rho dt = dW^{S,Q}dW^{h,Q}$. The parameters which take part in the SDEs have been described in Section 4.2.

Following the notations for one dimensional American options, the expected values of the risky derivatives are given by:

- If $M = \widehat{V}$,

$$\widehat{V}_{N_T}(s,h) = g(T,s,h), \quad S_{N_T} = s,\ h_{N_T} = h$$
$$\widehat{V}_{i-1}(s,h) = \max\left\{ g_{i-1}(s,h), \mathbb{E}_{t_{i-1}}\left[\widehat{V}_i(S_i,h_i) + \int_{t_{i-1}}^{t_i} e^{-m_1 u} f_1(\widehat{V}(u,S(u),h(u))du | S_{i-1} = s, h_{i-1} = h \right]\right\}$$

  for $i = N_T, N_T - 1, \ldots, 1$ corresponding to the time instants t_i. Moreover, $g(T,S,h) = G(S)$ represents the payoff, $g_i(S,h) = g(t_i,S,h)$, $m_1 = f$ and $f_1(\widehat{V}) = -h\widehat{V}^+$.

- If $M = V$,

$$\widehat{V}_{N_T}(s,h) = g(T,s,h), \quad S_{N_T} = s,\ h_{N_T} = h$$
$$\widehat{V}_{i-1}(s,h) = \max\left\{ g_{i-1}(s,h), \mathbb{E}_{t_{i-1}}\left[\widehat{V}_i(S_i,h_i) + \int_{t_{i-1}}^{t_i} e^{-m_2 u} f_2(V(u,S(u),h(u))du | S_{i-1} = s, h_{i-1} = h \right]\right\}$$

  for $i = N_T, N_T - 1, \ldots, 1$, with $m_2 = \left(\frac{h}{1-R} + f\right)$ and $f_2(V) = -V^+h + V\frac{h}{1-R}$.

As we have done for the European two dimensional model in Chapter 3, we have to use a discrete approximation of the integral which appears in the expression of the risky derivative value. For this purpose, we consider a set of fixed instant times $t = 0 < t_1 < \ldots < t_{N_T} = T$

with T the maturity time. Thus, denoting $S(t) = S_t$ and $h(t) = h_t$, and using Euler–Maruyama scheme, the simulated asset price $S(t_{j+1})$ from $S(t_j)$ and the simulated spread $h(t_{j+1})$ from $h(t_j)$ are derived as follows:

$$\begin{aligned} S(t_{j+1}) &= S(t_j)(1 + (r-q)(t_{j+1} - t_j)) + \sigma^S \sqrt{t_{j+1} - t_j}\, Z^S_{j+1} \\ h(t_{j+1}) &= h(t_j) + (\mu - M\sigma^h)(t_{j+1} - t_j) + \sigma^h \sqrt{t_{j+1} - t_j}\, Z^h_{j+1}\,, \end{aligned}$$

for $j = 1, \ldots, N_T$ where $Z^\ell_1, \ldots, Z^\ell_{N_P}$ (for $\ell = S, h$) are independent standard normal random variables. This relies on the fact that $W^{i,Q}(t_{j+1}) - W^{i,Q}(t_j)$ has a zero mean and standard deviation $\sqrt{t_{j+1} - t_j}$. In order to build correlated Brownian processes, the Cholesky factorization is applied. Moreover, the number of time steps N_T must be enough large in order to reduce the discretization error.

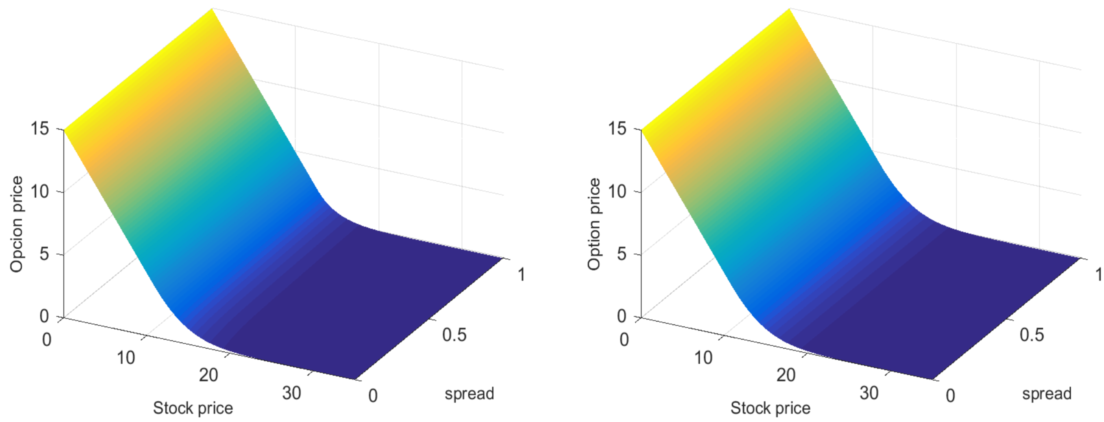
4.5 Numerical examples

In this section we show the behaviour of the risk–free value V, risky value $\widehat{V}$ and XVA value U for American options. Thus, we study the evolution of these magnitudes depending on the spot and the spread value. Following the work done for one dimensional models and European two dimensional models in the previous chapters, we compare the numerical solution obtained with the numerical techniques introduced in the previous section with the results achieved by Monte Carlo method.

In all the following tests, the financial data are taken from Table 3.1. For the first three of them, which are solved by the Lagrange–Galerkin method, the spatial mesh is uniform and consists of 160000 nodes ($N_S = N_h = 400$). On the opposite, we use a nonuniform time discretization with nodes $\tau^n = (n/N_T)^2 T$.

4.5.1 Test 1: American put options

In this first example, we study an American put option sold by the investor. The maturity time is $T = 0.5$ years and is discretized with $N_T = 700$ time steps. Figure 4.1 shows the American option value considering counterparty risk (left side) and the risk–free option value (right side). The difference between them is the XVA and is represented in Figure 4.2. We can observe that it increases, in absolute terms, when the intensity of default from the investor (spread) increases. We notice how the XVA goes down when the option goes to the "in–the–money" area and the value is null in the exercise region.



(a) Risky derivative value (b) Risk–free derivative value

Figure 4.1: American put option value (Test 1)

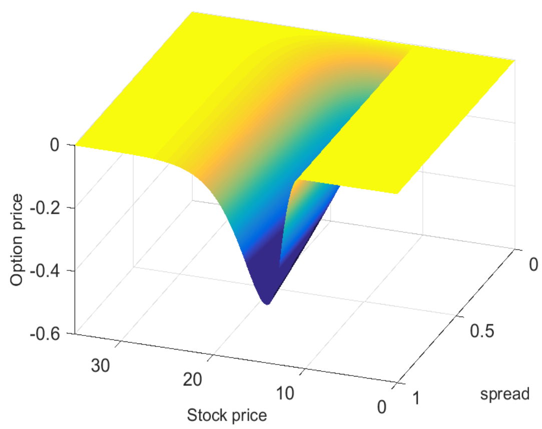
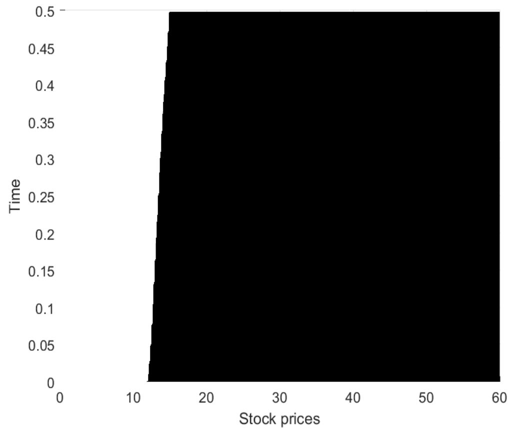
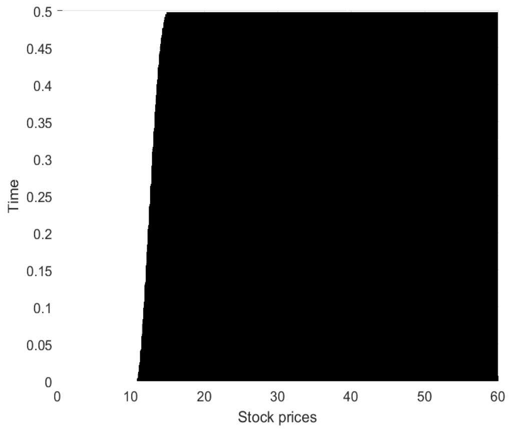


Figure 4.2: Total value adjustment (Test 1)

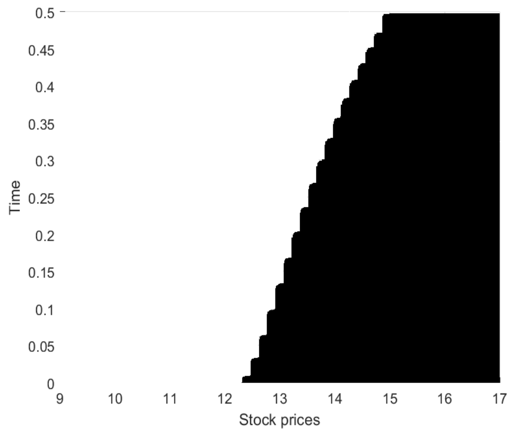
Next, Figure 4.3 shows the exercise region associated with the American option, considering counterparty risk (left) or risk–free situations (right). In the first case, the spread value is 0.25. Note that the exercise region is slightly larger when the intensity of default (spread) increases.



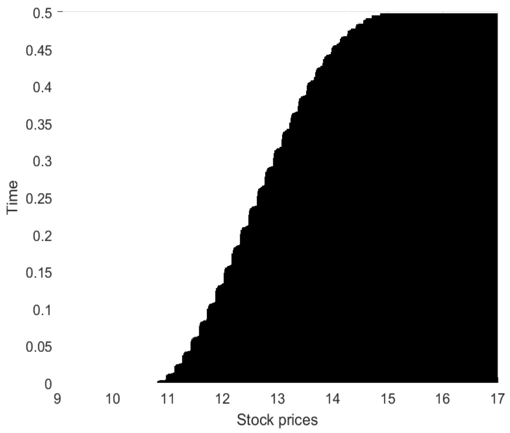
(a) Risky option



(b) Risk–free option



(c) Risky option (zoom)



(d) Risk–free option (zoom)

Figure 4.3: Exercise regions (white) of Test 1

4.5.2 Test 2: Long maturity time American put options

In this section, an American put option sold by the investor is also represented. Nevertheless the maturity time is $T = 2$ years, and we take $N_T = 1500$ time steps. As expected, comparing with Test 1, the XVA (Figure 4.4) is more negative due to a longer exposure to the risk. However, the behaviour of the total value adjustment is similar in both cases.

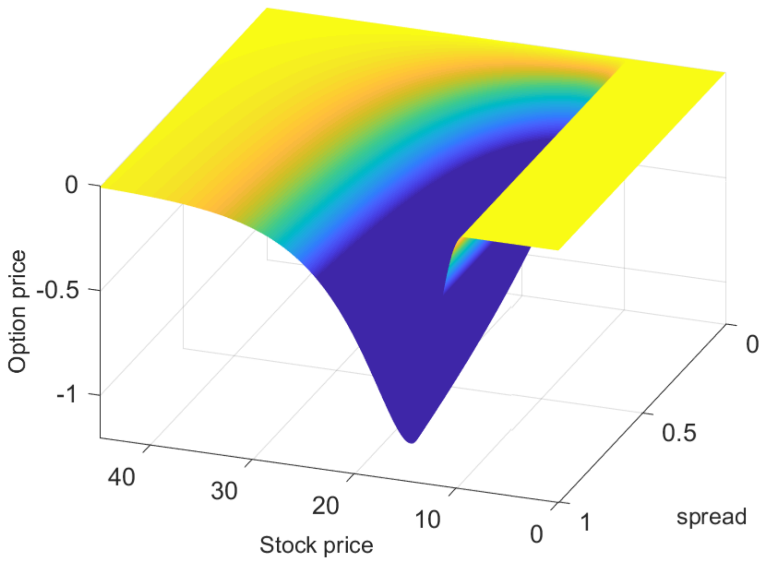
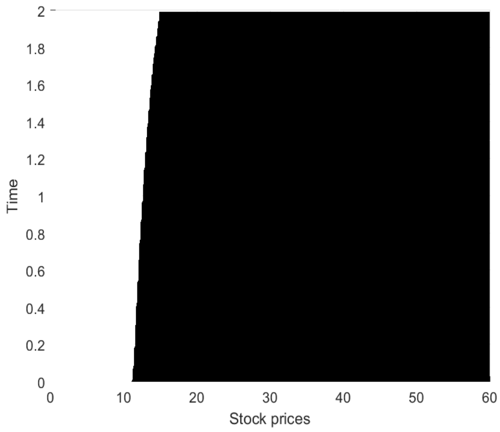


Figure 4.4: Total value adjustment (Test 2)

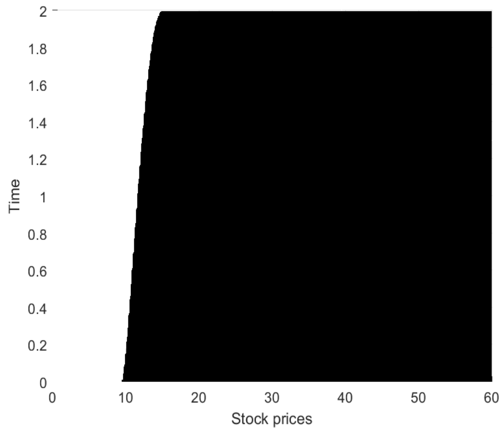
Comparing the exercise regions (Figure 4.5) with the results obtained in Test 1, we appreciate that for a long maturity time the exercise region is smaller. Nevertheless, we obtain a similar behaviour in both cases, in the sense that the exercise region for a risky option is larger than for a risk–free option.

4.5.3 Test 3: The linear problem ($M = V$)

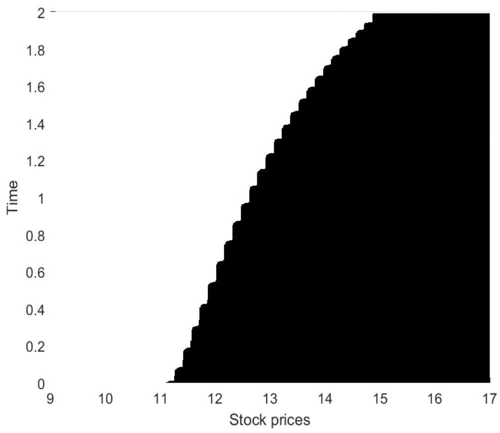
We show the behaviour of the American option value when the mark–to–market value is the risk–free derivative value. The maturity time is $T = 0.5$ years and $N_T = 700$, similarly to Test 1. We can observe a similar behaviour to the one found in the previous case. Moreover, the value of the option (Figure 4.6) and adjustment value (Figure 4.7) are also similar, being slightly more negative for $M = \widehat{V}$.



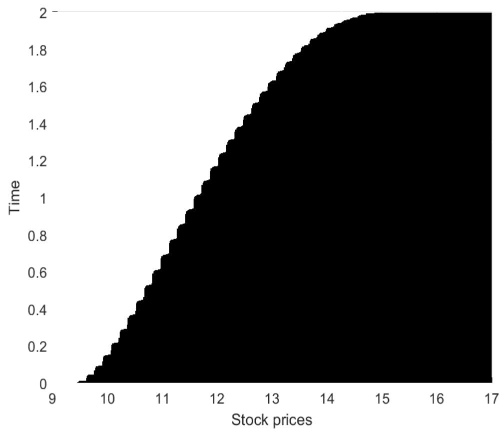
(a) Risky option



(b) Risk–free option

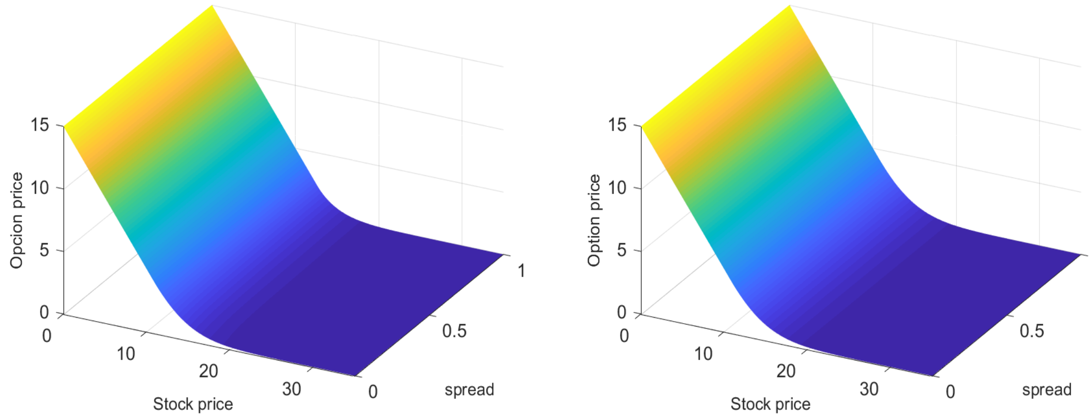


(c) Risky option (zoom)



(d) Risk–free option (zoom)

Figure 4.5: Exercise region for $T = 2$ (Test 2)



(a) Risky derivative value (b) Risk–free derivative value

Figure 4.6: American put option value, $M = V$ (Test 3)

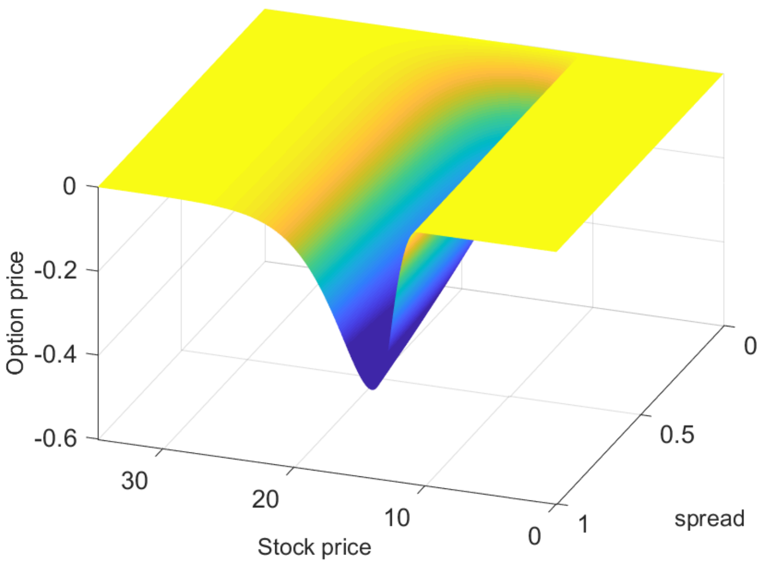


Figure 4.7: Total value adjustment (Test 3)

4.5.4 Test 4: Monte Carlo simulation

Finally, with this test we show the total value adjustment obtained by Monte Carlo techniques introduced in Section 4.4.5. As we have made for one dimensional model in Section 2.4, we have computed two estimators (lower and upper) of the XVA. Moreover, we can observe that the XVA obtained by solving the nonlinear and linear complementarity problems (Tables 4.1 and 4.2, respectively) are included in the 99% confidence interval.

In Chapter 2, we have solved the linear problem (for which the mark–to–market is $M = V$) by two different algorithms, with an inner Monte Carlo algorithm or using interpolation techniques. Both methods lead to similar results, although we have observed that the elapsed CPU time is higher for the inner iteration algorithm. Therefore, in this chapter we only compute the XVA when $M = V$ by interpolation techniques.

Table 4.1: American option value with counterparty risk $M = \widehat{V}$. The parameter values of the problem are: $K = 15$, $T = 0.5$, $\sigma^S = 0.3$, $\sigma^h = 0.2$, $\rho = 0.2$, $r = 0.3$, $q = 0.24$, $\kappa = 0.01$, $R = 0.3$, $f = 0.04$.

S	h	Finite Elements	Lower estimator	Upper estimator	Confidence interval
0.0	0.00	15.00000000	14.70298010	15.26299737	(14.70298010 , 15.28840736)
27.0		0.014580440	0.00137214	0.02527334	(-0.00022466 , 0.02730209)
3.0	0.05	12.00000000	11.99924144	12.14867741	(11.99700274 , 12.16485122)
9.0		5.999999999	5.99442104	6.33579807	(5.98367098 , 6.36138141)
15.0		1.308562527	1.00503811	1.42993109	(0.88792215 , 1.46313845)
24.0		0.039359070	0.01276533	0.14500936	(0.00370968 , 0.15321851)
9.0	0.10	5.999999999	6.00132959	6.31694827	(5.99355616 , 6.34254985)
12.0		3.064549102	2.92497307	3.49060126	(2.78070521 , 3.52303705)
15.0		1.190349256	1.09738339	1.42209030	(0.97520402 , 1.45608435)
18.0		0.381342801	0.26845411	0.68720345	(0.20919729 , 0.71412533)
27.0		0.007162736	0.00244516	0.03743066	(-0.00000067 , 0.04178265)
18.0	0.15	0.330425270	0.26696364	0.63732297	(0.20669270 , 0.66098768)
21.0		0.085487636	0.07440162	0.43350934	(0.04256772 , 0.45153917)
24.0		0.020619582	0.01324925	0.25670856	(0.00186524 , 0.26792935)
21.0	0.20	0.071796388	0.06151416	0.45182906	(0.03261546 , 0.47117011)
27.0		0.003421215	0.00099960	0.11406895	(-0.00049202 , 0.11881908)
30.0		0.000742742	0.00009576	0.00089707	(-0.00014309 , 0.00258554)

Table 4.2: American option value with counterparty risk $M = V$. The parameter values of the problem are: $K = 15$, $T = 0.5$, $\sigma^S = 0.3$, $\sigma^h = 0.2$, $\rho = 0.2$, $r = 0.3$, $q = 0.24$, $\kappa = 0.01$, $R = 0.3$, $f = 0.04$.

S	h	Finite elements	Lower estimator	Upper estimator	Confidence interval
0.0	0.00	15.00000000	14.76015652	15.31754947	(14.60199696 , 15.37917273)
27.0		0.014580440	0.00157526	0.03789128	(-0.00047226 , 0.04096308)
3.0	0.05	12.00000000	11.92717148	14.30577857	(11.80632068 , 14.60534869)
9.0		5.999999999	5.77587526	9.03605748	(5.60018068 , 9.24791036)
15.0		1.310644569	1.07192257	3.11079144	(0.94555737 , 3.23919041)
24.0		0.039554986	0.01108702	0.43742176	(0.00046278 , 0.45329951)
9.0	0.10	5.999999999	5.63823578	8.49773982	(5.47119356 , 8.68419050)
12.0		3.066098200	2.78677451	8.07659200	(2.61439440 , 8.32628955)
15.0		1.194868136	1.12406653	2.77078546	(0.99821874 , 2.89558922)
18.0		0.384468906	0.25563850	1.92442123	(0.19352922 , 2.00047634)
27.0		0.007288464	0.00040263	0.02285157	(-0.00033044 , 0.02383561)
18.0	0.15	0.335480535	0.26140404	1.70487679	(0.20130499 , 1.77073497)
21.0		0.087669807	0.06821849	1.09391591	(0.03798815 , 1.13640329)
24.0		0.021348419	0.00759706	0.52814862	(0.00026779 , 0.54853994)
21.0	0.20	0.074863949	0.04667367	0.55078911	(0.02193558 , 0.57228007)
27.0		0.003719381	0.00128050	0.01824716	(-0.00005382 , 0.02095550)
30.0		0.000821669	0.00002521	0.00040568	(-0.00003973 , 0.00089052)

The numerical solution of the complementarity problems have been computed with a mesh of 201×201 nodes and 500 time steps and the elapsed time for such simulation is 55822 seconds. For Monte Carlo simulation, we have implemented $N_P = 1000$ paths and $N_T = 1000$ time instants in the nonlinear case, and only $N_P = 500$ paths in the linear case. In both problems, we have employed three bases. The Monte Carlo computing of the risk-free option, previous to interpolation, needs 137160 seconds on a one–dimensional mesh of 100 initial prices, which states the advantage of pricing this kind of options by solving the complementarity problems.

Conclusions

When a financial contract between two parts (the hedger and the investor) takes into account the counterparty risk, different adjustments on the price of a derivative can be included and the total value adjustment (XVA) must be identified. The goal of this work is the contribution to the modelling, mathematical analysis and numerical solution of pricing problems related to vanilla options including counterparty risk.

We have considered different behaviours for the intensity of default of each counterparty of a contract. First, constant intensities of default have been assumed. Therefore, models depending on one stochastic factor —the asset price— have been deduced. In a second step, we have introduced an innovative aspect: the consideration that one of the parts —usually, the investor— is defaultable. Then, a model depending on two stochastic factors— the active price and the stochastic spread of the investor— is obtained. A further step could be achieved by considering a stochastic spread for the hedger, thus leading to a problem with three stochastic factors.

For a financial derivative without early exercise opportunity, as European vanilla options or forward contracts, different linear and nonlinear PDEs arise, depending on the choice of the mark–to–market close out. For a nonlinear partial differential equation, the existence and uniqueness of solution are obtained through the theory of sectorial differential operators. In order to solve such problem, we propose appropiate boundary conditions and numerical schemes based on the method of characteristics, finite elements and fixed point iteration techniques. The systems of linear equations at each step of the fixed point iteration are solved by a LU factorization. In the case of linear PDEs, the mathematical analysis and numerical simulation is achieved, as a particular case, by similar techniques.

In order to compute the order of convergence of the numerical methods, we have performed some one–dimensional tests for which an analytical expression of the XVA is known [15]. For the bidimensional model the analytical solution is unknown, and the convergence ratio is computed from the numerical solution obtained with different time and spatial discretizations [23]. As expected, a first order convergence is achieved in both cases. The numerical examples also illustrate the good performance of these models and methods for European vanilla options and forward contracts with and without collateral agreements, as different expected financial behaviours are recovered. In addition, these results are in agreement with the confidence intervals obtained by using Monte Carlo simulation techniques.

Furthermore, American options including counterparty risk are also modelled and analyzed. The possibility of an early exercise leads to models governed by linear and nonlinear complementarity problems. Unlike the European options, for which the XVA is the solution of the models we have proposed, for American options we obtain the total value adjustment as the difference between the risky and the risk–free derivative values, i.e. XVA=$\widehat{V} - V$.

The existence and uniqueness of solution of the nonlinear complementarity problem is studied through the theory of nonlinear functional differential problems. In order to compute the risky derivative value, the Lagrange–Galerkin method proposed for the European options is here combined with an augmented Lagrangian active set method to tackle the additional inequality constraints involved in the formulation. Numerical examples are presented to ilustrate and discuss the behaviour of the models and the proposed numerical methods.

Additionally, we express the option price in terms of expectations involving the optimal stopping times. Moreover, when the mark–to–market is equal to the option price without counterparty risk we propose two algorithms: a first one requiring two nested Monte Carlo loops and a second one considering a suitable interpolation technique for the risk–free option price. When the mark–to–market value at default is equal to the risky option price, a fixed point iteration is considered. The proposed techniques involve the computation of lower and upper estimators to build up a confidence interval for the American option price. These estimators are obtained by extending some previous results from [38] and [28]. This methodology is written in detail for constant spreads, but has been extended to compute the derivative value for stochastic intensities of default. Of course, it can be extended to other financial products with early exercise, such as callable bonds or Bermudan swaptions, for example.

We have implemented all the developed algorithms, and integrated then in a computational tool based on MATLAB. Comparing the elapsed time consumed by the different methods used to obtain the XVA, we appreciate that Monte Carlo methods require a larger computational time than the finite element techniques, specially for solving the one dimensional models. Moreover, although both families of methods need very large times for solving the two dimensional problems, the same behaviour is observed: the finite element resolution on a fine mesh is much more efficient than the Monte Carlo method for a reduced number of initial prices. The use of parallel computing techniques (like those ones related to multi–CPUs or GPUs) would allow a high speed up of the involved algorithms. These parallel computing tools result very efficient for the here considered Monte Carlo–based techniques.

As a future work, following an idea previously introduced, we could implement parallel computing techniques to improve the computational time for the American options solver. Moreover, we can also address a model depending on three stochastic factors, issued from considering stochastic spreads with two defaultable counterparties. Other types of financial derivatives —exotic options, swaptions, . . . — or new adjustments —such as capital value adjustment, KVA, or marging value adjustment, MVA— can also be incorporated by the methodologies developed in this thesis.

Resumen extenso

En este trabajo se estudian modelos para la valoración de algunos de los productos financieros derivados más usuales. En concreto, se aborda la valoración de opciones europeas y americanas, globalmente conocidas como opciones "vainilla". La principal novedad de este trabajo es la toma en consideración del riesgo de contrapartida, es decir, la posibilidad de quiebra de alguna de las partes que intervienen en el contrato.

La metodología de cobertura introducida por Black y Scholes [8] y Merton [39] para las opciones vainilla europeas no consideraba la posibilidad de que alguna de las partes del contrato pudiera caer en incumplimiento. Por otro lado, importantes instituciones financieras han asociado el estallido de la crisis financiera de 2007 a una incorrecta gestión del riesgo, además de a distintos fallos del sistema financiero. La complejidad de los nuevos derivados financieros, además de la consideración de una baja o nula probabilidad de quiebra, son dos de los factores que derivaron en la crisis.

Con objeto de realizar una valoración de los derivados financieros en un escenario más realista, diferentes ajustes —en función de las condiciones en que tiene lugar el contrato— son propuestos sobre el derivado libre de riesgo de contrapartida:

- Ajustes debido al beneficio por liquidez (Funding Benefit Adjustment, FBA).

- Ajustes debido a los costes de financiación de la entidad emisora (Funding Cost Adjustment, FCA). La diferencia de estos dos primeros, FBA y FCA, se denomina Funding Value Adjustment (FVA).

- Ajustes para compensar el riesgo de quiebra de la contrapartida (Credit Value Adjustment, CVA).

- Ajustes debido a la posibilidad de quiebra de la propia entidad emisora (Debit Value Adjustment, DVA).

- Ajustes debido a la presencia de colateral como una forma de compensar la posibilidad de quiebra de una de las partes (Collateral Value Adjustment, CollVA).

El conjunto de todos estos ajustes se conoce como Total Value Adjustment (XVA) y está dado por:

$$\mathrm{XVA} = \mathrm{DVA} - \mathrm{CVA} + (\mathrm{FBA} - \mathrm{FCA}) + \mathrm{CollVA} = \mathrm{DVA} - \mathrm{CVA} + \mathrm{FVA} + \mathrm{CollVA}\,.$$

Los objetivos de este trabajo pueden resumirse en:

- La deducción de modelos para el cálculo del XVA en opciones europeas y americanas, con el fin de obtener una valoración más adecuada de acuerdo con las exigencias actuales de los mercados financieros.

- El análisis matemático de los modelos propuestos.

- La resolución mediante un conjunto de técnicas numéricas adecuadas a las características de los modelos.

En una revisión del estado del arte encontramos principalmente tres metodologías para incluir costes de financiación, riesgo de contrapartida y ajustes por la presencia de colateral en la valoración del derivado. Una primera aproximación consiste en incluir los ajustes en términos de esperanzas; un ejemplo donde se incluye el DVA puede verse en Brigo [12] y, posteriormente, la inclusión del CollVA y costes de financiación es abordado por Pallavicini *et al.* [42]. La segunda aproximación, introducida por Crépey [21, 22], desarrolla modelos basados en ecuaciones diferenciales estocásticas hacia atrás. Más recientemente, se propone también la resolución de ecuaciones diferenciales estocásticas en [9]. Finalmente, la tercera aproximación sigue los trabajos de Piterbarg [45] y Burgard y Kjaer [15], en los que se utilizan argumentos de cobertura y el lema de Itô para deducir ecuaciones en derivadas parciales (EDPs) cuya solución nos proporciona el valor del derivado. Esta línea es también seguida por García [27] en un marco más general con spreads estocásticos, obteniéndose modelos dependientes de tres variables.

El presente trabajo sigue la tercera de las líneas explicadas previamente. Planteamos el valor de las opciones europeas como la solución de un problema de Cauchy y el valor de las opciones americanas como solución de un problema de complementariedad, ambos gobernados por ecuaciones en derivadas parciales.

Siguiendo [15], en la primera parte de la tesis se estudia la valoración de opciones europeas y americanas. En ambos casos se considera un contrato entre dos partes, un vendedor y un comprador, y se asume que ambas contrapartes tienen posibilidad de incumplimiento de las condiciones firmadas en el contrato. Con el fin de obtener un valor del derivado financiero que incluya los correspondientes ajustes debidos a los riesgos de contrapartida se emplean estrategias adecuadas de cobertura para carteras autofinanciadas y se tienen en cuenta las diferencias que presentan los dos tipos de opciones estudiadas en cuanto al periodo de ejercicio.

Debido a la posibilidad de quiebra de cada una de las partes a lo largo de la vida del contrato, es necesario, la aplicación del lema de Itô para procesos de difusión con saltos [43]. En esta primera parte se consideran intensidades de quiebra constantes, lo que conduce, para ambos tipos de opciones, a un modelo dependiente de un único factor estocástico, el activo subyacente. Además, se obtienen diferentes modelos en función de los ajustes incluidos: en primer lugar se considera un contrato sin colateral (y, por tanto, solo se modelan el CVA, DVA y FVA) para posteriormente estudiar la valoración de opciones para contratos que incluyen colateral (introduciendo el CollVA en el cálculo del XVA).

Los modelos obtenidos para ambas opciones están dados en términos del valor de mercado del derivado. Siguiendo la bibliografía, es habitual considerar dos posibles valores de mercado en el momento de quiebra: el valor libre de riesgo, que conduce a un modelo lineal, y el valor con riesgo de contrapartida, que da lugar a un modelo no lineal. Según el tipo de opción, se obtienen los siguientes problemas de EDPs, dados en términos de dicho valor de mercado.

- Opciones europeas:

$$
\begin{cases}
\dfrac{\partial \widehat{V}}{\partial t} + \mathcal{A}\widehat{V} - (r + \lambda_B + \lambda_C)\widehat{V} \\
\qquad\qquad = \lambda_B h_e - \lambda_B g_B(M,X) - \lambda_C g_C(M,X) + s_X X\,, \qquad (t,S) \in (0,T] \times (0,\infty) \\
\widehat{V}(T,S) = H(S)\,.
\end{cases}
$$

- Opciones americanas:

$$\begin{cases} \mathcal{L}_X(\widehat{V}) = \dfrac{\partial \widehat{V}}{\partial t} + \mathcal{A}\widehat{V} - (r + \lambda_B + \lambda_C)\widehat{V} - \lambda_B h_e \\ \qquad\qquad + \lambda_B g_B(M, X) + \lambda_C g_C(M, X) - s_X X \leq 0\,, \qquad (t, S) \in (0, T] \times (0, \infty) \\ \widehat{V}(t, S) \geq H(S) \\ \mathcal{L}_X(\widehat{V})(\widehat{V} - H) = 0 \\ \widehat{V}(T, S) = H(S)\,, \end{cases}$$

donde el operador $\mathcal{A}$ está dado por:

$$\mathcal{A}V \equiv \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + r_R S \frac{\partial V}{\partial S}\,.$$

Dado que el comportamiento de la intensidad de quiebra de cada una de las partes que intervienen en el contrato no es siempre constante, en una segunda parte de la tesis se consideran comportamientos estocásticos, lo cual presenta un escenario más acorde con la situación actual de los mercados financieros. Para este supuesto, seguimos el trabajo de García [27], donde la consideración de intensidades de quiebra estocásticas conduce a un modelo dependiente de tres factores: el activo subyacente y el spread de cada una de las partes que intervienen en el contrato. Con el fin de reducir la dimensión del problema, consideramos un contrato entre dos partes, el inversor y el asegurador, y suponemos que la intensidad de quiebra es estocástica pero solo una de las partes, en este caso el inversor, puede quebrar. Se obtiene así un modelo dependiente de dos factores estocásticos, el activo subyacente y el spread del inversor.

Al igual que en el caso de intensidades de quiebra constantes, el estudio se realiza sobre las opciones europeas y americanas. Nuevamente, aplicamos estrategias de cobertura en función de los distintos momentos en los que la opción puede ejercerse: solo a vencimiento (en el caso de opciones europeas) o en cualquier instante hasta el vencimiento (en el caso de opciones americanas). Se obtienen así problemas de Cauchy para ecuaciones en derivadas parciales que modelan el valor asociado a las opciones europeas y problemas de complementariedad para la valoración de opciones americanas. Al igual que sucede con los modelos unidimensionales, en función del valor que se asigne al valor de mercado se deducen problemas lineales y no lineales. De este modo, en función del tipo de opción, se obtienen los siguientes problemas en derivadas parciales:

- Opciones europeas:

$$\begin{cases} \dfrac{\partial \widehat{V}}{\partial t} + \widetilde{\mathcal{L}}_{Sh}\widehat{V} + \dfrac{\Delta \widehat{V}}{1-R}h - f\widehat{V} = 0, & \text{en } [0,T) \times (0,\infty) \times (0,\infty), \\ \widehat{V}(T,S,h) = G(S)\,. \end{cases}$$

- Opciones americanas:

$$\begin{cases} \mathcal{L}(\widehat{V}) = \dfrac{\partial \widehat{V}}{\partial t} + \tilde{\mathcal{L}}_{Sh}\widehat{V} + \dfrac{\Delta \widehat{V}}{1-R}h - f\widehat{V} \leq 0\,, & \text{en } [0,T) \times (0,\infty) \times (0,\infty), \\ \widehat{V}(t,S,h) \geq G(S) \\ \mathcal{L}(\widehat{V})(\widehat{V} - G) = 0 \\ \widehat{V}(T,S,h) = G(S)\,, \end{cases}$$

donde $\tilde{\mathcal{L}}_{Sh}$ es un operador en derivadas parciales de segundo orden.

Recordemos que el objetivo del presente trabajo es obtener el valor de los ajustes, es decir del XVA. En el caso de las opciones europeas, a partir del modelo del derivado con riesgo se deducen los problemas de Cauchy que modelan el XVA, considerando que el valor con riesgo puede descomponerse como suma del valor libre de riesgo más el valor de los ajustes. Se obtiene así el problema de EDPs que modela directamente el valor de los ajustes,

$$\begin{cases} \dfrac{\partial U}{\partial t} + \mathcal{A}U - rU = \lambda_B h_e + \lambda_B(\widehat{V} - g_B(M,X)) \\ \qquad\qquad\qquad +\lambda_C(\widehat{V} - g_C(M,X)) + s_X X\,, & (t,S) \in (0,T] \times (0,\infty) \\ U(T,S) = 0\,. \end{cases}$$

En el caso de las opciones americanas, la dificultad que acarrean las inecuaciones que intervienen en los modelos hace que el XVA tenga que calcularse después de obtener el valor con riesgo y sin riesgo del derivado, solución cada uno de ellos de los correspondientes problemas de complementariedad obtenidos mediante estrategias de cobertura. El XVA se obtiene como diferencia de ambos.

Se ha realizado el análisis matemático de los modelos no lineales (obtenidos para los distintos comportamientos de la intensidad de quiebra), estudiando la existencia y unicidad de solución. Tanto para opciones europeas como americanas, la metodología introducida se ha centrado en los problemas no lineales, considerando los lineales como un caso particular.

El análisis de las opciones europeas se ha abordado siguiendo los resultados introducidos por Henry [30]. Estos prueban la existencia de solución para problemas dados en términos de un operador sectorial y una función lipschitciana definidos en un espacio de Hilbert. En un primer paso se prueba un resultado de existencia local, para posteriormente demostrar la existencia global de solución. El problema unidimensional se estudia en el dominio no acotado $\mathbb{R}$, mientras que el estudio del modelo bidimensional se hace para un dominio acotado donde el operador es sectorial.

El análisis de las opciones americanas se ha planteado siguiendo el resultado introducido por Jeong y Park [34] para inecuaciones variacionales semilineales parabólicas. Para ello, los problemas obtenidos han sido reescritos en términos de una función lipschitciana y un operador continuo que satisface la desigualdad de Gårding en espacios de Hilbert apropiados y un operador subdiferencial definido para un espacio convexo adecuado.

Una vez probada la existencia y unicidad de solución para los modelos de EDPs obtenidos en la valoración de ambas opciones, se proponen diferentes métodos para la solución numérica.

En primer lugar, dado que el planteamiento de los problemas se hace sobre un dominio no acotado, se realiza un truncamiento del dominio y se proponen las condiciones de contorno más apropiadas desde el punto de vista financiero para resolver el problema en dicho dominio. Para la obtención de alguna de las condiciones se siguen las ideas de [19] introducidas previamente en [23]. La discretización temporal se realiza mediante un método de características que aproxima la derivada material en términos de la curva característica, y se combina con una discretización espacial basada en elementos finitos de Lagrange. Además, los problemas no lineales se resuelven mediante un método iterativo de punto fijo. En el caso de las opciones europeas, el sistema de ecuaciones lineales que se obtiene en cada paso de tiempo se ha resuelto mediante una factorización LU. Sin embargo, la resolución de las opciones americanas conduce a problemas de obstáculo para los cuales se utiliza un método de lagrangiano aumentado (ALAS) propuesto en [35].

Por otra parte, se ha utilizado también una metodología más clásica en el ámbito financiero: a partir de los modelos en términos de EDPs, se ha aplicado el teorema de Feynman–Kac para obtener el valor del XVA asociado a las opciones europeas en términos de esperanza. Una vez obtenida la expresión del mismo, se calcula su valor mediante técnicas clásicas de tipo Monte Carlo. En el caso de las opciones americanas, la expresión del XVA se ha deducido siguiendo los trabajos de Longstaff–Schwartz [38] y Glasserman [28], obteniéndose ecuaciones de valoración de las opciones americanas con riesgo de contrapartida. Esta metodología considera diferentes instantes de tiempo de ejercicio; como ocurre con las opciones de tipo Bermuda, una mayor consideración de instantes de ejercicio permitirá obtener una mejor valoración. Los resultados obtenidos mediante esta técnica se han comparado con los calculados mediante la resolución de los modelos basados en EDPs observándose que los primeros valores están incluidos en los intervalos de confianza obtenidos mediante técnicas de tipo Monte Carlo. Sin embargo, se observa cómo el tiempo computacional empleado para la resolución de las EDPs es menor que el tiempo necesario para la valoración de opciones mediante técnicas de Monte Carlo.

Finalmente, se han estudiado diferentes casos de opciones europeas y americanas, que muestran el comportamiento esperado tanto del valor de la opción como de los ajustes.

El esquema seguido en el trabajo ha sido el siguiente:

- El Capítulo 1 consta de una introducción para poner en contexto la relevancia de considerar el riesgo de contrapartida en la valoración de opciones europeas. La valoración del derivado se hace mediante técnicas de cobertura dinámica y con estrategias de no arbitraje. En este capítulo se consideran contratos entre dos contrapartes, las cuales pueden quebrar con intensidades de quiebra constantes, por lo que se obtienen modelos de EDPs lineales y no lineales dependientes de un único factor estocástico, el activo subyacente. Se estudia la existencia y unicidad de solución de los problemas no lineales, considerando el problema lineal como un caso particular. El análisis matemático de este problema se hace sobre un dominio no acotado empleando teoría de operadores sectoriales. Se proponen diferentes técnicas numéricas para la resolución de los problemas de EDPs obtenidos, el método de características combinado con elementos finitos así como un esquema de punto fijo para los problemas no lineales. Se introduce también una alternativa en la valoración del XVA en términos de esperanza mediante técnicas clásicas de Monte Carlo. El capítulo termina con varios resultados que muestran la relevancia de incorporar el riesgo de contrapartida en la valoración del derivado, comparándose los resultados obtenidos mediante la resolución de las EDPs con los obtenidos mediante técnicas de Monte Carlo.

- El Capítulo 2 comienza con una breve introducción sobre la valoración de opciones americanas incluyendo riesgo de contrapartida. Al igual que en el Capítulo 1, las intensidades de quiebra son constantes y ambas partes que intervienen en el contrato pueden quebrar. Utilizando técnicas de cobertura dinámica y estrategias de no arbitraje se deducen problemas de complementariedad lineales o no lineales, según la elección que se haga del valor de mercado en el instante de quiebra, dependientes de un único factor estocástico. Se proponen diferentes técnicas numéricas para la resolución de problemas con obstáculo. Combinado con las técnicas empleadas para el problema de opciones europeas, se implementa un algortimo de lagrangiano aumentado para resolver problemas con obstáculo. Además se propone la valoración de opciones americanas mediante técnicas de Monte Carlo, extendiendo el trabajo de Longstaff y Schwartz. Al final del capítulo se presentan resultados numéricos que muestran el comportamiento de la opción americana cuando se incluye riesgo de contrapartida en la valoración.

- En el Capítulo 3 se presenta la valoración de opciones europeas siguiendo un esquema similar al del Capítulo 1. La principal novedad respecto a éste reside en la consideración de intensidades de quiebra estocásticas. Además, se considera un contrato entre dos partes, donde solo el inversor tiene posibilidad de quebrar. Mediante técnicas de cobertura dinámica se obtiene un modelo dependiente de dos factores estocásticos, el activo subyacente y el spread del inversor. Después de la obtención de los modelos, se estudia la existencia y unicidad de solución del problema no lineal. El carácter sectorial del operador correspondiente se demuestra para dominios acotados. Técnicas numéricas similares a las introducidas en el Capítulo 1 y adaptadas a modelos de varias variables son propuestas para la resolución del problema bidimensional. Finalmente se presentan los resultados obtenidos con dichas técnicas, donde se observa el comportamiento del XVA en función del precio del activo y de la probabilidad de quiebra del inversor. El comportamiento respecto del precio del activo subyacente es similar al obtenido para los problemas unidimensionales de opciones europeas.

- El Capítulo 4 presenta un esquema similar al de los capítulos anteriores. Se estudia la valoración de opciones americanas y, al igual que en el Capítulo 3, solo el inversor puede quebrar considerandose la intensidad de quiebra estocástica. Haciendo uso de técnicas de cobertura dinámica y estrategias de ausencia de arbitraje se obtienen modelos de complementariedad asociados a ecuaciones en derivadas parciales lineales y no lineales dependientes de dos variables espaciales, el activo subyacente y el spread. Se estudia también la existencia y unicidad de solución de dichos problemas siguiendo la teoría de inecuaciones variacionales semilineales de tipo parabólico. Para la obtención de una solución numérica, se proponen métodos numéricos similares a los del Capítulo 2 para la resolución de problemas con obstáculo. Los resultados numéricos presentados muestran la variación en el valor del derivado debido a la incorporación de riesgo de contrapartida en la valoración del mismo. En este capítulo también se ha valorado la opción americana considerando riesgo de contrapartida mediante las técnicas de Monte Carlo detalladas en el Capítulo 2 adaptadas a modelos bidimenionales.

Todos los métodos y algoritmos propuestos se han implementado en un código basado en Matlab. Se dispone así de una herramienta de gran utilidad para la valoración efectiva de opciones europeas y americanas con riesgo de contrapartida. Por otra parte, los distintos tests realizados muestran la ventaja de calcular el valor de las opciones y los distintos ajustes mediante la resolución de modelos basados en EDPs, frente a los métodos de Monte Carlo más utilizados por las compañías financieras y bancos.

Resumo extenso

Neste traballo estúdanse modelos para a valoración dalgúns dos produtos financeiros derivados máis usuais. En concreto, abórdase a valoración de opcións europeas e americanas, globalmente coñecidas como opcións "vainilla". A principal novidade deste traballo é a toma en consideración do risco de contrapartida, é dicir, a posibilidade de quebra dalgunha das partes que interveñen no contrato.

A metodoloxía de cobertura introducida por Black e Scholes [8] e Merton [39] para as opcións vainilla europeas non consideraba a posibilidade de que algunha das partes do contrato puidese caer en incumprimento. Doutra banda, importantes institucións financeiras asociaron o estalido da crise financeira de 2007 a unha incorrecta xestión do risco, ademais da distintos fallos do sistema financeiro. A complexidade dos novos derivados financeiros, ademais da consideración dunha baixa ou nula probabilidade de quebra, son dous dos factores que derivaron na crise.

Con obxecto de realizar unha valoración dos derivados financeiros nun escenario máis realista, se propoñen diferentes axustes —en función das condicións en que ten lugar o contrato— sobre o derivado libre de risco de contrapartida:

- Axustes debido ao beneficio por liquidez (Funding Benefit Adjustment, FBA).
- Axustes debido aos custos de financiamento da entidade emisora (Funding Cost Adjustment, FCA). A diferenza destes dous primeiros, FBA e FCA, denomínase Funding Value Adjustment (FVA).
- Axustes para compensar o risco de quebra da contrapartida (Credit Value Adjustment, CVA).

- Axustes debido á posibilidade de quebra da propia entidade emisora (Debit Value Adjustment, DVA).

- Axustes debido á presenza de colateral como unha forma de compensar a posibilidade de quebra dunha das partes (Collateral Value Adjustment, CollVA).

O conxunto de todos estes axustes coñécese como Total Value Adjustment (XVA) e está dado por:

$$\mathrm{XVA} = \mathrm{DVA} - \mathrm{CVA} + (\mathrm{FBA} - \mathrm{FCA}) + \mathrm{CollVA} = \mathrm{DVA} - \mathrm{CVA} + \mathrm{FVA} + \mathrm{CollVA}\,.$$

Os obxectivos deste traballo poden resumirse en:

- A dedución de modelos para o cálculo do XVA en opcións europeas e americanas, co fin de obter unha valoración máis axeitada de acordo coas esixencias actuais dos mercados financeiros.

- A análise matemática dos modelos propostos.

- A resolución mediante un conxunto de técnicas numéricas adecuadas ás características dos modelos.

Nunha revisión da estado da arte atopamos principalmente tres metodoloxías para incluír custos de financiamento, risco de contrapartida e axustes pola presenza de colateral na valoración do derivado. Unha primeira aproximación consiste en incluír os axustes en termos de esperanzas; un exemplo onde se inclúe o DVA pode verse en Brigo [12] e, posteriormente, a inclusión do CollVA e custos de financiamento é abordado por Pallavicini *et al.* [42].

A segunda aproximación, introducida por Crépey [21, 22], desenvolve modelos baseados en ecuacións diferenciais estocásticas cara atrás. Máis recentemente, proponse tamén a resolución de ecuacións diferenciais estocásticas en [9]. Finalmente, a terceira aproximación segue os traballos de Piterbarg [45] e Burgard e Kjaer [15], nos que se empregan argumentos de cobertura e a lema de Itô para deducir ecuacións en derivadas parciais (EDPs) cuxa solución nos proporciona o valor do derivado. Esta liña é tamén seguida por García [27] nun marco máis xeral con spreads estocásticos, obténdose modelos dependentes de tres variables.

O presente traballo segue a terceira das liñas explicadas previamente. Obtemos o valor das opcións europeas como a solución dun problema de Cauchy e o valor das opcións americanas como solución dun problema de complementariedade, ambos os gobernados por ecuacións en derivadas parciais.

Seguindo [15], na primeira parte da tese estúdase a valoración de opcións europeas e americanas. En ambos os casos considérase un contrato entre dous partes, un vendedor e un comprador, e asúmese que ambas as contrapartes teñen posibilidade de incumprimento das condicións asinadas no contrato. Co fin de obter un valor do derivado financeiro que inclúa os correspondentes axustes debidos aos riscos de contrapartida empréganse estratexias adecuadas de cobertura para carteiras autofinanciadas e téñense en conta as diferenzas que presentan os dous tipos de opcións estudadas en canto ao período de exercicio.

Debido á posibilidade de quebra de cada unha das partes ao longo da vida do contrato, é necesario a aplicación da lema de Itô para procesos de difusión con saltos [43]. Nesta primeira parte considéranse intensidades de quebra constantes, o que conduce, para ambos os tipos de opcións, a un modelo dependente dun único factor estocástico, o activo subxacente. Ademais, obtéñense diferentes modelos en función dos axustes incluídos: en primeiro lugar considérase un contrato sen colateral (e, por tanto, só se modelan o CVA, DVA e FVA) para posteriormente estudar a valoración de opcións para contratos que inclúen colateral (introducindo o CollVA no cálculo do XVA).

Os modelos obtidos para ambas as opcións están dados en termos do valor de mercado do derivado. Seguindo a bibliografía, é habitual considerar dous posibles valores de mercado no momento de quebra: o valor libre de risco, que conduce a un modelo lineal, e o valor con risco de contrapartida, que dá lugar a un modelo non lineal. Segundo o tipo de opción, obtéñense os seguintes problemas de EDPs, dados en termos do devandito valor de mercado.

- Opcións europeas:

$$\begin{cases} \dfrac{\partial \widehat{V}}{\partial t} + \mathcal{A}\widehat{V} - (r+\lambda_B+\lambda_C)\widehat{V} \\ \qquad\qquad = \lambda_B h_e - \lambda_B g_B(M,X) - \lambda_C g_C(M,X) + s_X X\,, \qquad (t,S)\in(0,T]\times(0,\infty) \\ \widehat{V}(T,S) = H(S)\,. \end{cases}$$

- Opcións americanas:

$$
\begin{cases}
\mathcal{L}_X(\widehat{V}) = \dfrac{\partial \widehat{V}}{\partial t} + \mathcal{A}\widehat{V} - (r + \lambda_B + \lambda_C)\widehat{V} - \lambda_B h_e \\
\qquad\qquad + \lambda_B g_B(M, X) + \lambda_C g_C(M, X) - s_X X \leq 0\,, \qquad (t, S) \in (0, T] \times (0, \infty) \\
\widehat{V}(t, S) \geq H(S) \\
\mathcal{L}_X(\widehat{V})(\widehat{V} - H) = 0 \\
\widehat{V}(T, S) = H(S)\,,
\end{cases}
$$

onde o operador $\mathcal{A}$ está dado por

$$
\mathcal{A}V \equiv \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + r_R S \frac{\partial V}{\partial S}\,.
$$

Dado que o comportamento da intensidade de quebra de cada unha das partes que interveñen no contrato non é sempre constante, nunha segunda parte da tese considéranse comportamentos estocásticos, o cal presenta un escenario máis acorde coa situación actual dos mercados financeiros. Para este suposto, seguimos o traballo de García [27], onde a consideración de intensidades de quebra estocásticas conduce a un modelo dependente de tres factores: o activo subxacente e o spread de cada unha das partes que interveñen no contrato. Co fin de reducir a dimensión do problema, consideramos un contrato entre dous partes, o investidor e o asegurador, e supomos que a intensidade de quebra é estocástica pero só una das partes, neste caso o investidor, pode crebar. Obtense así un modelo dependente de dous factores estocásticos, o activo subxacente e o spread do investidor.

Do mesmo xeito que no caso de intensidades de quebra constantes, o estudo realízase sobre as opcións europeas e americanas. Novamente, aplicamos estratexias de cobertura en función dos distintos momentos nos que a opción pode exercerse: só a vencemento (no caso de opcións europeas) ou en calquera instante ata o vencemento (no caso de opcións americanas). Obtéñense así problemas de Cauchy para ecuacións en derivadas parciais que modelan o valor asociado ás opcións europeas e problemas de complementariedade para a valoración de opcións americanas. Do mesmo xeito que sucede cos modelos unidimensionais, en función do valor que se asigne ao valor de mercado dedúcense problemas lineais e non lineais. Deste xeito, en función do tipo de opción, obtéñense os seguintes problemas en derivadas parciais:

- Opcións europeas:

$$\begin{cases} \dfrac{\partial \widehat{V}}{\partial t} + \widetilde{\mathcal{L}}_{Sh}\widehat{V} + \dfrac{\Delta \widehat{V}}{1-R}h - f\widehat{V} = 0, & \text{en } [0,T) \times (0,\infty) \times (0,\infty), \\ \widehat{V}(T,S,h) = G(S)\,. \end{cases}$$

- Opcións americanas:

$$\begin{cases} \mathcal{L}(\widehat{V}) = \dfrac{\partial \widehat{V}}{\partial t} + \tilde{\mathcal{L}}_{Sh}\widehat{V} + \dfrac{\Delta \widehat{V}}{1-R}h - f\widehat{V} \leq 0 & \text{en}\,,\ [0,T) \times (0,\infty) \times (0,\infty), \\ \widehat{V}(t,S,h) \geq G(S) \\ \mathcal{L}(\widehat{V})(\widehat{V} - G) = 0 \\ \widehat{V}(T,S,h) = G(S)\,, \end{cases}$$

onde $\tilde{\mathcal{L}}_{Sh}$ é un operador en derivadas parciais de segunda orde.

Lembremos que o obxectivo do presente traballo é obter o valor dos axustes, é dicir do XVA. No caso das opcións europeas, a partir do modelo do derivado con risco dedúcense os problemas de Cauchy que modelan o XVA, considerando que o valor con risco pode descomporse como suma do valor libre de risco máis o valor dos axustes. Obtense así o problema de EDPs que modela directamente o valor dos axustes,

$$\begin{cases} \dfrac{\partial U}{\partial t} + \mathcal{A}U - rU = \lambda_B h_e + \lambda_B(\widehat{V} - g_B(M,X)) \\ \qquad\qquad\qquad\qquad +\lambda_C(\widehat{V} - g_C(M,X)) + s_X X\,, \qquad (t,S) \in (0,T] \times (0,\infty) \\ U(T,S) = 0\,. \end{cases}$$

No caso das opcións americanas, a dificultade que carrexan as inecuaciones que interveñen nos modelos fai que o XVA teña que calcularse despois de obter o valor con risco e sen risco do derivado, solución cada un deles dos correspondentes problemas de complementariedade obtidos mediante estratexias de cobertura. O XVA obtense como diferencia de ambos.

Realizouse a análise matemática dos modelos non lineais (obtidos para os distintos comportamentos da intensidade de quebra), estudando a existencia e unicidade de solución. Tanto para opcións europeas como americanas, a metodoloxía introducida centrouse nos problemas non lineais, considerando os lineais como un caso particular.

A análise das opcións europeas abordouse seguindo os resultados introducidos por Henry [30]. Estes proban a existencia de solución para problemas dados en termos dun operador sectorial e unha función lipschitciana definidos nun espazo de Hilbert. Nun primeiro paso próbase un resultado de existencia local, para posteriormente demostrar a existencia global de solución. O problema unidimensional estúdase no dominio non acoutado $\mathbb{R}$, con todo o estudo do modelo bidimensional faise para un dominio acoutado onde o operador é sectorial.

A análise das opcións americanas expúxose seguindo o resultado introducido por Jeong–Park [34] para inecuacións variacionais semilineares parabólicas. Para iso, os problemas obtidos se rescribiron en termos dunha función lipschitciana e un operador continuo que satisfai a desigualdade de Gårding en espazos de Hilbert apropiados e un operador subdiferencial definido para un espazo convexo adecuado.

Unha vez probada a existencia e unicidade de solución para os modelos de EDPs obtidos na valoración de ambas as opcións, propoñense diferentes métodos para a solución numérica.

En primeiro lugar, dado que a formulación dos problemas faise sobre un dominio non acoutado, realízase un truncamento do dominio e propóñense as condicións de contorno máis axeitadas desde o punto de vista financeiro para resolver o problema no devandito dominio. Para a obtención dalgunha das condicións séguense as ideas de [19] introducidas previamente en [23]. A discretización temporal realízase mediante un método de características que aproxima a derivada material en termos da curva característica, e combínase cunha discretización espacial baseada en elementos finitos de Lagrange. Ademais, os problemas non lineais resólvense mediante un método iterativo de punto fixo. No caso das opcións europeas, o sistema de ecuacións lineais que se obtén en cada paso de tempo resolveuse mediante unha factorización LU. Con todo, a resolución das opcións americanas conduce a problemas de obstáculo para os cales se utiliza un método de lagranxiano aumentado (ALAS) proposto en [35].

Doutra banda, utilizouse tamén unha metodoloxía máis clásica no ámbito financeiro: a partir dos modelos en termos de EDPs, aplicouse o teorema de Feynman–Kac para obter o valor do XVA asociado ás opcións europeas en termos de esperanza. Unha vez obtida a expresión do mesmo, calcúlase o seu valor mediante técnicas clásicas de tipo Monte Carlo. No caso das opcións americanas, a expresión do XVA deduciuse seguindo os traballos de Longstaff–Schwartz [38] e Glasserman [28], obténdose ecuacións de valoración das opcións americanas con risco de contrapartida. Esta metodoloxía considera diferentes instantes de tempo de exercicio; como ocorre coas opcións de tipo Bermuda, unha maior consideración de instantes de exercicio permitirá obter unha mellor valoración. Os resultados obtidos mediante esta técnica comparáronse cos calculados mediante a resolución dos modelos baseados en EDPs, observándose que os primeiros valores están incluídos nos intervalos de confianza obtidos mediante técnicas de tipo Monte Carlo. Con todo, obsérvase cómo o tempo computacional empregado para a resolución das EDPs é menor que o tempo necesario para a valoración de opcións mediante técnicas de Monte Carlo.

Finalmente, estudáronse diferentes casos de opcións europeas e americanas, que mostran o comportamento esperado tanto do valor da opción como dos axustes.

O esquema seguido no traballo foi o seguente:

- O Capítulo 1 consta dunha introdución para pór en contexto a relevancia de considerar o risco de contrapartida na valoración de opcións europeas. A valoración do derivado faise mediante técnicas de cobertura dinámica e con estratexias de non arbitraxe. Neste capítulo considéranse contratos entre dous contrapartes, as cales poden crebar con intensidades de quebra constantes, polo que se obteñen modelos de EDPs lineais e non lineais dependentes dun único factor estocástico, o activo subxacente. Estúdase a existencia e unicidade de solución dos problemas non lineais, considerando o problema lineal como un caso particular. A análise matemática deste problema faise sobre un dominio non acoutado empregando teoría de operadores sectoriais. Proponse diferentes técnicas numéricas para a resolución dos problemas de EDPs obtidos, o método de características combinado con elementos finitos así como un esquema de punto fixo para os problemas non lineais. Introdúcese tamén unha alternativa na valoración do XVA en termos de esperanza mediante técnicas clásicas de Monte Carlo. O capítulo remata con varios resultados que mostran a relevancia de incorporar o risco de contrapartida na valoración do derivado, comparándose os resultados obtidos mediante a resolución das EDPs cos obtidos mediante técnicas de Monte Carlo.

- O Capítulo 2 comenza cunha breve introdución sobre a valoración de opcións americanas incluíndo risco de contrapartida. Do mesmo xeito que no Capítulo 1, as intensidades de quebra son constantes e ambas as partes que interveñen no contrato poden crebar. Utilizando técnicas de cobertura dinámica e estratexias de non arbitraxe dedúcense problemas de complementariedade lineais ou non lineais, segundo a elección que se faga do valor de mercado no instante de quebra, dependentes dun único factor estocástico. Propoñense diferentes técnicas numéricas para a resolución de problemas con obstáculo. Combinado coas técnicas empregadas para o problema de opcións europeas, se implementa un algoritmo de lagranxiano aumentado para resolver problemas con obstáculo. Ademais proponse a valoración de opcións americanas mediante técnicas de Monte Carlo, estendendo o traballo de Longstaff e Schwartz. Ao final do capítulo preséntanse resultados numéricos que mostran o comportamento da opción americana cando se inclúe risco de contrapartida na valoración.

- No Capítulo 3 preséntase a valoración de opcións europeas seguindo un esquema similar ao do Capítulo 1. A principal novidade respecto deste reside na consideración de intensidades de quebra estocásticas. Ademais, considérase un contrato entre dous partes, onde só o investidor ten posibilidade de crebar. Mediante técnicas de cobertura dinámica obtense un modelo dependente de dous factores estocásticos, o activo subxacente e o spread do investidor. Despois da obtención dos modelos, estúdase a existencia e unicidade de solución do problema non lineal. O carácter sectorial do operador correspondente demóstrase para dominios acoutados. Propoñense técnicas numéricas similares ás introducidas no Capítulo 1 e adaptadas a modelos de varias variables para a resolución do problema bidimensional. Finalmente preséntanse os resultados obtidos con ditas técnicas, onde se observa o comportamento do XVA en función do prezo do activo e da probabilidade de quebra do investidor. O comportamento respecto do prezo do activo subxacente é similar ao obtido para os problemas unidimensionais das opcións europeas.

- O Capítulo 4 presenta un esquema similar ao dos capítulos anteriores. Estúdase a valoración de opcións americanas e, do mesmo xeito que no Capítulo 3, só o investidor pode crebar considerandose a intensidade de quebra estocástica. Facendo uso de técnicas de cobertura dinámica e estratexias de ausencia de arbitraxe obtéñense problemas de complementariedade asociados a ecuacións en derivadas parciais lineais e non lineais dependentes de dúas variables espaciais, o activo subxacente e o spread. Estúdase tamén a existencia e unicidade de solución dos devanditos problemas seguindo teorías para inecuacions variacionais semilineares de tipo parabólico. Para a obtención dunha solución numérica, propoñense métodos numéricos similares aos do Capítulo 2 para a resolución de problemas con obstáculo. Os resultados numéricos presentados amosan a variación no valor do derivado debido á incorporación do risco de contrapartida na valoración do mesmo. Neste capítulo tamén se valorou a opción americana considerando risco de contrapartida mediante as técnicas de Monte Carlo detalladas no Capítulo 2 adaptadas a modelos bidimenionais.

Todos os métodos e algoritmos propostos foron implementados nun código baseado en Matlab. Disponse así dunha ferramenta de gran utilidade para a valoración efectiva de opcións europeas e americanas con risco de contrapartida. Dutra banda, os distintos tests realizados amosan a vantaxe de calcular o valor das opcións e os distintos axustes mediante a resolución de modelos baseados en EDPs, fronte aos métodos de Monte Carlo máis utilizados polas compañías financeiras e bancos.

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