Electronic Journal of Differential Equations, Vol. 2018 (2018), No. 101, pp. 1–18.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

EXISTENCE OF SOLUTIONS TO NONHOMOGENEOUS DIRICHLET PROBLEMS WITH DEPENDENCE ON THE GRADIENT

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Communicated by Vicentiu D. Radulescu

Abstract. The goal of this article is to explore the existence of positive solutions for a nonlinear elliptic equation driven by a nonhomogeneous partial differential operator with Dirichlet boundary condition. This equation a convection term and thereaction term is not required to satisfy global growth conditions. Our approach is based on the Leray-Schauder alternative principle, truncation and comparison approaches, and nonlinear regularity theory.

1. Introduction

Given a bounded domain $\Omega \subset \mathbb{R}^N$ with C^2-boundary $\partial\Omega$, $1 < p < +\infty$, a continuous function $a : \mathbb{R}^N \to \mathbb{R}^N$, and a nonlinear function $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$, we consider the following nonlinear nonhomogeneous Dirichlet problem involving a convection term:

$$\begin{gathered} -\operatorname{div} a(Du(z)) = f(z, u(z), Du(z)) \quad \text{in } \Omega, \\ u(z) = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

with $u(z) > 0$ in Ω.

In this article, the function $a : \mathbb{R}^N \to \mathbb{R}^N$ is assumed to be continuous and strictly monotone, also satisfies certain regularity and growth conditions listed in hypotheses (H1) below. It is worth to mention that these conditions are mild and incorporate in our framework many classical operators of interest, for example the p-Laplacian, the (p, q)-Laplacian (that is, the sum of a p-Laplacian and a q-Laplacian with $1 < q < p < \infty$) and the generalized p-mean curvature differential operator. The forcing term depends also on the gradient of the unknown function (convection term). For this reason we are not able to apply variational methods directly on equation (1.1).

For problems with convection terms we mention the following works: Figueiredo-Girardi-Matzeu [8], Girardi-Matzeu [23] (semilinear equations driven by the Dirichlet Laplacian), Faraci-Motreanu-Puglisi [6], Huy-Quan-Khanh [25], Iturriaga-Lorca-Sanchez [26], Ruiz [36] (nonlinear equations driven by the Dirichlet p-Laplacian),

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2010 *Mathematics Subject Classification.* 35J92, 35P30.
Key words and phrases. Nonhomogeneous p-Laplacian operator; nonlinear regularity; Dirichlet boundary condition; convection term; truncation; Leray-Schauder alternative.

Submitted December 8, 2017. Published May 2, 2018.

Averna-Motreanu-Tornatore [2], Faria-Miyagaki-Motreanu [7], Tanaka [37] (equations driven by the Dirichlet (p,q)-Laplacian) and finally Gasiński-Papageorgiou [22] (Neumann problems driven by a differential operator of the form $\operatorname{div}(a(u)Du)$).

Unlike the aforementioned works, in this paper, the convection term f does not have any global growth condition. Instead we suppose that $f(z,\cdot,y)$ admits a positive root (zero) and all the other conditions refer to the behaviour of the function $x \mapsto f(z,x,y)$ near zero locally in $y \in \mathbb{R}^N$. Our approach is based on the Leray-Schauder alternative principle, truncation and comparison techniques, nonlinear regularity theory and it is closely related to the paper Bai-Gasiński-Papageorgiou in [3], where the Robin boundary value problem was considered. Finally for other problems with a general nonhomogeneous operator we refer to Gasiński-Papageorgiou [14, 15, 19, 21], Papageorgiou-Rădulescu [30, 31, 33, 34] and for particular cases of a nonhomogeneous operator we refer to Gasiński-Papageorgiou [13, 16] ($p(z)$-Laplacian) and Gasiński-Papageorgiou [17, 18], and for (p,q)-Laplacian, Papageorgiou-Rădulescu [32].

2. Notation and preliminaries

In the study of problem (1.1), we will use the Sobolev space $W_0^{1,p}(\Omega)$ as well as the ordered Banach space $C_0^1(\overline{\Omega}) = \{u \in C^1(\bar{\Omega}) : u(z) = 0 \text{ on } \partial\Omega\}$ which has positive (order) cone

$$C_0^1(\overline{\Omega})_+ = \{u \in C_0^1(\overline{\Omega}) : u(z) \geq 0 \text{ in } \overline{\Omega}\}.$$

The interior of this cone contains the set

$$D_+ = \{u \in C_0^1(\overline{\Omega}) : u(z) > 0 \text{ in } \Omega\}.$$

Then, we give the following notation, which will be used in the sequel. For $x \in \mathbb{R}$, we denote $x^{\pm} = \max\{\pm x, 0\}$. Likewise, for $u \in W_0^{1,p}(\Omega)$ fixed, we use the notation $u^{\pm}(\cdot) = u(\cdot)^{\pm}$. We have that

$$u^{\pm} \in W_0^{1,p}(\Omega), \quad u = u^+ - u^-, \quad |u| = u^+ + u^-.$$

For $u \in W_0^{1,p}(\Omega)$ such that $u(z) \geq 0$ for a.a. $z \in \Omega$, we define

$$[0,u] = \{h \in W_0^{1,p}(\Omega) : 0 \leqslant h(z) \leqslant u(z) \text{ for a.a. } z \in \Omega\}.$$

Now we present the conditions on the map $a(y)$. Assume that $\zeta \in C^1(0,\infty)$ is such that

$$0 < \widehat{c} \leqslant \frac{\zeta'(t)t}{\zeta(t)} \leqslant c_0 \quad \text{and} \quad c_1 t^{p-1} \leqslant \zeta(t) \leqslant c_2(1 + |t|^{p-1}) \quad \forall t > 0, \tag{2.1}$$

for some $c_1, c_2 > 0$.

The hypotheses on the map $y \mapsto a(y)$ are as follows:

(H1) $a : \mathbb{R}^N \to \mathbb{R}^N$ is such that $a(y) = a_0(|y|)y$ for all $y \in \mathbb{R}^N$ with $a_0(t) > 0$ for all $t > 0$ and

(i) $a_0 \in C^1(0,\infty)$, $t \mapsto a_0(t)t$ is strictly increasing on $(0,\infty)$ and

$$\lim_{t \to 0^+} a_0(t)t = 0 \quad \text{and} \quad \lim_{t \to 0^+} \frac{a_0'(t)t}{a_0(t)} = c > -1;$$

(ii) there exists $c_3 > 0$ such that

$$|\nabla a(y)| \leqslant c_3 \frac{\zeta(|y|)}{|y|} \quad \text{for all } y \in \mathbb{R}^N \setminus \{0\};$$

(iii) for all $y \in \mathbb{R}^N \setminus \{0\}$ and $\xi \in \mathbb{R}^N$,

$$(\nabla a(y)\xi, \xi)_{\mathbb{R}^N} \geqslant \frac{\zeta(|y|)}{|y|}|\xi|^2;$$

(iv) denoting $G_0(t) = \int_0^t a_0(s)s\,ds$, we can find $q \in (1, p)$ satisfying

$$t \mapsto G_0(t^{1/q}) \text{ is convex on } \mathbb{R}_+ = [0, +\infty),$$

$$\lim_{t \to 0^+} \frac{qG_0(t)}{t^q} = c^* > 0,$$

$$0 \leqslant pG_0(t) - a_0(t)t^2 \text{ for all } t > 0.$$

Remark 2.1. Conditions (H1)(i), (ii) and (iii) are required by the nonlinear regularity theory of Lieberman [28] and the nonlinear strong maximum principle of Pucci-Serrin [35].

Example 2.2. The following maps satisfy hypotheses (H1) (see Papageorgiou-Rădulescu [29]).

(a) $a(y) = |y|^{p-2}y$ with $1 < p < \infty$. The operator $\operatorname{div}(a(Du))$ reduces to the p-Laplace differential operator

$$\Delta_p u = \operatorname{div}(|Du|^{p-2}Du) \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

(b) $a(y) = |y|^{p-2}y + |y|^{q-2}y$ with $1 < q < p < \infty$. The map $\operatorname{div}(a(Du))$ corresponds to the (p, q)-Laplace differential operator

$$\Delta_p u + \Delta_q u \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Such operators arise in problems of mathematical physics (see Cherfils-Il'yasov [4]).

(c) $a(y) = (1 + |y|^2)^{\frac{p-2}{2}}y$ with $1 < p < \infty$. The operator $\operatorname{div}(a(Du))$ corresponds to the generalized p-mean curvature differential operator

$$\operatorname{div}((1 + |Du|^2)^{\frac{p-2}{2}}Du) \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

(d)

$$a(y) = \begin{cases} 2|y|^{\gamma-2}y, & \text{if } |y| < 1, \\ |y|^{p-2}y + |y|^{q-2}y & \text{if } 1 < |y|, \end{cases}$$

where $1 < q < p$, $\gamma = \frac{p+q}{2}$.

On the other hand, we use hypotheses (H1) to indicate that G_0 is strictly increasing and strictly convex. Also, we denote

$$G(y) = G_0(|y|) \quad \text{for all } y \in \mathbb{R}^N.$$

We have

$$\nabla G(y) = G_0'(|y|)\frac{y}{|y|} = a_0(|y|)y = a(y) \quad \text{for all } y \in \mathbb{R}^N \setminus \{0\}.$$

So, G is the primitive of a, it is convex with $G(0) = 0$. Hence, one has

$$G(y) = G(y) - G(0) \ \leqslant \ (a(y), y)_{\mathbb{R}^N} \quad \text{for all } y \in \mathbb{R}^N. \tag{2.2}$$

Such hypotheses were also considered in recent the works of Gasiński-O'Regan-Papageorgiou [10], Papageorgiou-Rădulescu [29, 30, 31] and Bai-Gasiński-Papageorgiou [3].

Under hypotheses (H1)(i), (ii) and (iii), we have the following lemma, which summarizes some of important properties for the map $a(\cdot)$.

Lemma 2.3 ([38, Lemma 3])**.** *Assume that the map $a(\cdot)$ satisfies hypotheses* (H1) (i), (ii), (iii)*. Then the following statements hold*

(a) *$y \mapsto a(y)$ is continuous and strictly monotone (hence maximal monotone);*
(b) *$|a(y)| \leqslant c_4(1+|y|^{p-1})$ for all $y \in \mathbb{R}^N$, for some $c_4 > 0$;*
(c) *$(a(y), y)_{\mathbb{R}^N} \geqslant \frac{c_1}{p-1}|y|^p$ for all $y \in \mathbb{R}^N$, where c_1 is given in* (2.1)*.*

We have the following bilateral growth restrictions on the primitive G is established.

Lemma 2.4. *Assume that the map $a(\cdot)$ satisfies hypotheses* (H1) (i), (ii), (iii)*. Then, there exists $c_5 > 0$ such that*

$$\frac{c_1}{p(p-1)}|y|^p \leqslant G(y) \leqslant \ c_5(1+|y|^p) \quad \text{for all } y \in \mathbb{R}^N.$$

Let $W^{-1,p'}(\Omega)$ be the dual space of the Sobolev space $W_0^{1,p}(\Omega)$. We denote the duality brackets between $W^{-1,p'}(\Omega)$ and $W_0^{1,p}(\Omega)$ by $\langle \cdot, \cdot \rangle$. Also, we introduce a nonlinear operator $A : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ corresponding to map $a(\cdot)$ defined by

$$\langle A(u), h\rangle = \int_\Omega (a(Du), Dh)_{\mathbb{R}^N}\, dz \quad \text{for all } u, h \in W_0^{1,p}(\Omega).$$

Next proposition summarizes some properties of the operator A (see Gasiński-Papageorgiou [12] for a more general version).

Proposition 2.5. *Assume that (H1)*(i), (ii) *and* (iii) *are fulfilled. Then, the map $A : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ is continuous, bounded (thus is, maps bounded sets in $W_0^{1,p}(\Omega)$ to bounded sets in $W^{-1,p'}(\Omega)$), monotone (hence maximal monotone too) and of type $(S)_+$, i.e.,*

if $u_n \xrightarrow{w} u$ in $W_0^{1,p}(\Omega)$ and $\limsup_{n\to+\infty}\langle A(u_n), u_n - u\rangle \leqslant 0$, then $u_n \to u$ in $W_0^{1,p}(\Omega)$.

For $1 < q < +\infty$, we consider the nonlinear eigenvalue problem

$$\begin{gathered} -\Delta_q u(z) = \widehat{\lambda}|u(z)|^{q-2}u(z) \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega. \end{gathered}$$

The number $\widehat{\lambda}$ such that the above Dirichlet problem admits a nontrivial solution $\widehat{u}$ is called an eigenvalue of $-\Delta_q$ with Dirichlet boundary condition, also the nontrivial solution $\widehat{u}$ is an eigenfunction corresponding to $\widehat{\lambda}$. From Faraci-Motreanu-Puglisi [6], we can see that there exists a smallest eigenvalue $\widehat{\lambda}_1(q) > 0$ such that

- $\widehat{\lambda}_1(q)$ is positive, isolated and simple (that is, if $\widehat{u}, \widehat{v}$ are eigenfunctions corresponding to $\widehat{\lambda}_1(q)$, then $\widehat{u} = \xi\widehat{v}$ for some $\xi \in \mathbb{R} \setminus \{0\}$).
- the following variational characterization holds

$$\widehat{\lambda}_1(q) = \inf\Big\{ \frac{\int_\Omega |Du|^q\, dx}{\int_\Omega |u|^q\, dx} : u \in W_0^{1,q}(\Omega) \text{ with } u \neq 0 \Big\}.$$

In what follows, we denote by $\widehat{u}_1(q)$ the positive eigenfunction normalized as $\|\widehat{u}_1(q)\|_q^q = \int_\Omega |u|^q\,dx = 1$, which is associated to $\widehat{\lambda}_1(q)$. One has $\widehat{u}_1(q) \in D_+$. Additionally, we know that if u is an eigenfunction corresponding to an eigenvalue $\widehat{\lambda} \neq \widehat{\lambda}_1(q)$, then $u \in C_0^1(\overline{\Omega})$ changes sign (see Lieberman [27, 28]).

Let $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$. The function f is called to be *Carathéodory*, if

- for all $(x, y) \in \mathbb{R} \times \mathbb{R}^N$, $z \mapsto f(z, x, y)$ is measurable on Ω;
- for a.a. $z \in \Omega$, $(x, y) \mapsto f(z, x, y)$ is continuous.

Such a function is automatically jointly measurable (see Hu-Papageorgiou [24, p. 142]).

For the convection term f in problem (1.1), we assume that

(H2) $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function such that $f(z, 0, y) = 0$ for a.a. $z \in \Omega$, all $y \in \mathbb{R}^N$ and
  (i) there exists $\eta > 0$ such that
$$f(z, \eta, y) = 0 \quad \text{for a.a. } z \in \Omega, \text{ all } y \in \mathbb{R}^N,$$
$$f(z, x, y) \geqslant 0 \quad \text{for a.a. } z \in \Omega, \text{ all } 0 \leqslant x \leqslant \eta, \text{ all } y \in \mathbb{R}^N,$$
$$f(z, x, y) \leqslant \widetilde{c}_1 + \widetilde{c}_2 |y|^p \quad \text{for a.a. } z \in \Omega, \text{ all } 0 \leqslant x \leqslant \eta, \text{ all } y \in \mathbb{R}^N,$$
  with $\widetilde{c}_1 > 0$, $\widetilde{c}_2 < \frac{c_1}{p-1}$;
  (ii) for every $M > 0$, there exists $\eta_M \in L^\infty(\Omega)$ such that
$$\eta_M(z) \geqslant c^* \widehat{\lambda}_1(q) \quad \text{for a.a. } z \in \Omega, \ \eta_M \not\equiv c^* \widehat{\lambda}_1(q),$$
$$\liminf_{x \to 0^+} \frac{f(z, x, y)}{x^{q-1}} \geqslant \eta_M(z) \quad \text{uniformly for a.a. } z \in \Omega, \text{ all } |y| \leqslant M$$
  (here $q \in (1, p)$ and c^* are as in hypothesis (H1)(iv));
  (iii) there exists $\xi_\eta > 0$ such that for a.a. $z \in \Omega$, all $y \in \mathbb{R}^N$ the function
$$x \mapsto f(z, x, y) + \xi_\eta x^{p-1}$$
  is nondecreasing on $[0, \eta]$, for a.a. $z \in \Omega$, all $y \in \mathbb{R}^N$ and
$$\lambda^{p-1} f(z, \frac{1}{\lambda} x, y) \leqslant f(z, x, y), \tag{2.3}$$
$$f(z, x, y) \leqslant \lambda^p f(z, x, \frac{1}{\lambda} y)$$
  for a.a. $z \in \Omega$, all $0 \leqslant x \leqslant \eta$, all $y \in \mathbb{R}^N$ and all $\lambda \in (0, 1)$.

Remark 2.6. Because the goal of the present paper is to explore the existence of nonnegative solutions, so for $x \leqslant 0$, without loss of generality, we may assume that
$$f(z, x, y) = 0 \quad \text{for a.a. } z \in \Omega, \text{ all } y \in \mathbb{R}^N.$$
Note that (2.3) is satisfied if for example, for a.a. $z \in \Omega$, all $y \in \mathbb{R}^N$, the function $x \mapsto \frac{f(z,x,y)}{x^{p-1}}$ is nonincreasing on $(0, +\infty)$.

Example 2.7. The following function satisfies hypotheses (H2). For the sake of simplicity we drop the z-dependence:
$$f(x, y, z) = \begin{cases} (x^{r-1} - x^{s-1})|y|^p & \text{if } 0 \leqslant x \leqslant 1, \\ (x^\tau \ln x)|y|^p & \text{if } 1 < x, \end{cases}$$
with $1 < r < s < q < p$ and $\tau > 1$.

Finally we recall the well known Leray-Schauder alternative principle (see e.g., Gasiński-Papageorgiou [11, p. 827]), which will play important role to establish our main results.

Theorem 2.8. *Let X be a Banach space and $C \subseteq X$ be nonempty and convex. If $\vartheta : C \to C$ is a compact map, then exactly one of the following two statements is true:*

(a) *ϑ has a fixed point;*
(b) *the set $S(\vartheta) = \{u \in C : u = \lambda\vartheta(u),\ \lambda \in (0,1)\}$ is unbounded.*

3. Positive solutions

In this section, we explore a positive solution to nonlinear nonhomogeneous Dirichlet problem (1.1). To this end, for $v \in C_0^1(\overline{\Omega})$ fixed, we first consider the following intermediate Dirichlet problem

$$\begin{gathered} -\operatorname{div} a(Du(z)) = f(z,u(z),Dv(z)), \quad \text{in } \Omega, \\ u(z) = 0, \quad \text{on } \partial\Omega. \end{gathered} \tag{3.1}$$

Now, we apply truncation and perturbation approaches to prove that (3.1) has at least one positive solution. So, we turn our attention to consider the following truncation-perturbation Dirichlet problem

$$\begin{gathered} -\operatorname{div} a(Du(z)) + \xi_\eta u(z)^{p-1} = \widehat{f}(z,u(z),Dv(z)), \quad \text{in } \Omega, \\ u(z) = 0, \quad \text{on } \partial\Omega, \end{gathered} \tag{3.2}$$

where $\widehat{f} : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is the corresponding truncation-perturbation of convection term f with respect to the second variable, defined by

$$\widehat{f}(z,x,y) = \begin{cases} f(z,x,y) + \xi_\eta (x^+)^{p-1} & \text{if } x \leqslant \eta, \\ f(z,\eta,y) + \xi_\eta \eta^{p-1} & \text{if } \eta < x. \end{cases} \tag{3.3}$$

Remark 3.1. Recall that f is a Carathéodoty function (see hypotheses (H2)). It is obvious that the truncation-perturbation $\widehat{f}$ is a Carathéodoty function as well.

It is obvious that if a function $u : \Omega \to \mathbb{R}$ with $u = 0$ on $\partial\Omega$ and $0 \le u(z) \le \eta$ for a.a. $z \in \Omega$ is a solution of problem (3.2), then u is also a solution of problem (3.1). Using this fact, we will now prove the existence of a positive solution for problem (3.1).

Proposition 3.2. *Assume that* (H1) *and* (H2) *are satisfied. Then problem* (3.1) *has a positive solution u_v such that*

$$u_v \in [0,\eta] \cap D_+.$$

Proof. To prove the existence of a nontrivial solution, we introduce the C^1-functional $\widehat{\varphi}_v : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\widehat{\varphi}_v(u) = \int_\Omega G(Du)\,dz + \frac{\xi_\eta}{p}\|u\|_p^p - \int_\Omega \widehat{F}_v(z,u)\,dz$$

for all $u \in W_0^{1,p}(\Omega)$, where $\widehat{F}_v$ is given by

$$\widehat{F}_v(z,x) = \int_0^x \widehat{f}(z,s,Dv(z))\,ds.$$

Combining Lemma 2.4 and definition of $\widehat{f}$ (see (3.3)), we conclude that the functional $\widehat{\varphi}_v$ is coercive. On the other hand, the Sobolev embedding theorem and the convexity of G reveal that the functional $\widehat{\varphi}_v$ is sequentially weakly lower semicontinuous. Therefore, it allows us to use the Weierstrass-Tonelli theorem to find $u_v \in W_0^{1,p}(\Omega)$ such that

$$\widehat{\varphi}_v(u_v) = \inf_{u \in W_0^{1,p}(\Omega)} \widehat{\varphi}_v(u). \tag{3.4}$$

We take $M := \sup_{z\in\overline{\Omega}} |Dv(z)|$ and then use hypothesis (H2)(ii) to obtain that for any $\varepsilon > 0$ fixed, there exists $\delta \in (0, \eta]$ satisfying

$$f(z,x,y) \geqslant (\eta_M(z) - \varepsilon)x^{q-1} \quad \text{for a.a. } z \in \Omega, \text{ all } x \in [0,\delta], \text{ all } |y| \leqslant M;$$

this results in

$$\widehat{f}(z,x,Dv(z)) \geqslant (\eta_M(z) - \varepsilon)x^{q-1} + \xi_\eta x^{p-1} \quad \text{for a.a. } z \in \Omega, \text{ all } x \in [0,\delta]$$

(see (3.3)). Also, we can calculate

$$\widehat{F}_v(z,x) \geqslant \frac{1}{q}(\eta_M(z) - \varepsilon)x^q + \frac{\xi_\eta}{p}x^p \quad \text{for a.a. } z \in \Omega, \text{ all } x \in [0,\delta]. \tag{3.5}$$

Note that $G(y) = G_0(|y|)$ for all $y \in \mathbb{R}^N$ and $\lim_{t\to 0^+} \frac{qG_0(t)}{t^q} = c^* > 0$ (see (H1)(iv)), so

$$G(y) \leqslant \frac{c^* + \varepsilon}{q}|y|^q \quad \text{for all } |y| \leqslant \delta. \tag{3.6}$$

As $\widehat{u}_1(q) \in D_+$, we can take $t \in (0,1)$ small enough such that

$$t\widehat{u}_1(q)(z) \in [0,\delta], \quad t|D\widehat{u}_1(q)(z)| \leqslant \delta \quad \text{for all } z \in \overline{\Omega}. \tag{3.7}$$

Obviously, we can obtain

$$\begin{aligned}
\widehat{\varphi}_v(t\widehat{u}_1(q)) &\leqslant \frac{c^* + \varepsilon}{q}t^q\widehat{\lambda}_1(q) - \frac{t^q}{q}\int_\Omega (\eta_M(z) - \varepsilon)\widehat{u}_1(q)^q\,dz \\
&\leqslant \frac{t^q}{q}\Big(\int_\Omega (c^*\widehat{\lambda}_1(q) - \eta_M(z))\widehat{u}_1(q)^q\,dz + \varepsilon(\widehat{\lambda}_1(q) + 1)\Big)
\end{aligned} \tag{3.8}$$

(recall that $\|\widehat{u}_1(q)\|_q = 1$). From $\eta_M(z) \geqslant c^*\widehat{\lambda}_1(q)$ for a.a. $z \in \Omega$, $\eta_M \not\equiv c^*\widehat{\lambda}_1(q)$ (see (H2)(ii)) and $\widehat{u}_1(q) \in D_+$, it yields

$$r_0 = \int_\Omega (\eta_M(z) - c^*\widehat{\lambda}_1(q))\widehat{u}_1(q)^q\,dz > 0.$$

So, (3.8) becomes

$$\widehat{\varphi}_v(t\widehat{u}_1(q)) \leqslant \frac{t^q}{q}(-r_0 + \varepsilon(\widehat{\lambda}_1(q) + 1)).$$

Now, we pick $\varepsilon \in (0, \frac{r_0}{\widehat{\lambda}_1(q)+1})$ to obtain $\widehat{\varphi}_v(t\widehat{u}_1(q)) < 0$. This means that

$$\widehat{\varphi}_v(u_v) < 0 = \widehat{\varphi}_v(0),$$

hence $u_v \neq 0$. Therefore, we have proved the existence of a nontrivial solution to problem (3.1).

Next, we show that u_v is nonnegative. Equality (3.4) indicates $\widehat{\varphi}_v'(u_v) = 0$, hence

$$\begin{aligned}
&\langle A(u_v), h\rangle + \xi_\eta \int_\Omega |u_v|^{p-2}u_v h\,dz \\
&= \int_\Omega \widehat{f}(z,u_v,Dv)h\,dz \quad \text{for all } h \in W_0^{1,p}(\Omega).
\end{aligned} \tag{3.9}$$

Inserting $h=-u_v^- \in W_0^{1,p}(\Omega)$ into (3.9) to obtain

$$-\langle A(u_v), u_v^-\rangle - \xi_\eta \int_\Omega |u_v|^{p-2}u_v u_v^-\,dz = -\int_\Omega \widehat{f}(z,u_v,Dv)u_v^-\,dz,$$

thus (see (3.3) and (H2)),

$$\langle A(u_v), u_v^-\rangle + \xi_\eta ||u_v^-||_p^p \le 0.$$

Combining with Lemma 2.3 and (3.3), we calculate

$$\frac{c_1}{p-1}\|Du_v^-\|^p + \xi_\eta \|u_v^-\|_p^p \leqslant 0,$$

which gives $u_v \geqslant 0$ and $u_v \neq 0$.

Furthermore, we shall illustrate that $u_v \in [0,\eta]$. Putting $h=(u_v-\eta)^+ \in W_0^{1,p}(\Omega)$ into (3.9), we obtain

$$\begin{aligned}&\langle A(u_v), (u_v-\eta)^+\rangle + \xi_\eta \int_\Omega u_v^{p-1}(u_v-\eta)^+\,dz,\\ &= \int_\Omega \big(f(z,\eta,Dv)+\xi_\eta\eta^{p-1}\big)(u_v-\eta)^+\,dz = \int_\Omega \xi_\eta\eta^{p-1}(u_v-\eta)^+\,dz\end{aligned}$$

(see (3.3) and condition (H2)(i)). We use the fact that $A(\eta)=0$, to obtain

$$\langle A(u_v)-A(\eta), (u_v-\eta)^+\rangle + \xi_\eta \int_\Omega (u_v^{p-1}-\eta^{p-1})(u_v-\eta)^+\,dz \leqslant 0.$$

However, the monotonicity of A implies $u_v \leqslant \eta$. Until now, we have verified that

$$u_v \in [0,\eta]\setminus\{0\}. \tag{3.10}$$

Finally, we demonstrate the regularity of u_v, more precisely we will show that $u_v \in D_+$. It follows from (3.3), (3.9) and (3.10) that

$$\langle A(u_v), h\rangle = \int_\Omega f(z,u_v,Dv)h\,dz \quad \text{for all } h \in W_0^{1,p}(\Omega),$$

which gives

$$\begin{gathered}-\operatorname{div} a(Du_v(z)) = f(z,u_v(z),Dv(z)) \quad \text{for a.a. } z\in\Omega,\\ u_v(z)=0 \quad \text{on } \partial\Omega.\end{gathered} \tag{3.11}$$

From (3.11) and Papageorgiou-Rădulescu [30], we have

$$u_v \in L^\infty(\Omega).$$

However, using the regularity results from Lieberman [28] (see also Fukagai-Narukawa [9]), we have

$$u_v \in C_0^1(\overline{\Omega})\setminus\{0\}.$$

To conclude, we have $u_v \in [0,\eta]\cap C_0^1(\overline{\Omega})\setminus\{0\}$. Moreover, we can use the nonlinear maximum principle, see Pucci-Serrin [35]), to conclude directly that $u_v \in D_+$. □

From the proof of Proposition 3.2, we know that problem (3.1) has a solution $u_v \in [0,\eta]\cap D_+$. Next, we will prove that problem (3.1) has a smallest positive solution in the order interval $[0,\eta]$. In what follows, we denote

$$S_v = \{u \in W_0^{1,p}(\Omega) : u\neq 0,\ u\in[0,\eta] \text{ is a solution of } (3.1)\}.$$

Proposition 3.2 implies

$$\emptyset \neq S_v \subseteq [0,\eta]\cap D_+.$$

Let p^* be the critical Sobolev exponent corresponding to p, i.e.,

$$p^* = \begin{cases} \frac{Np}{N-p} & \text{if } p < N, \\ +\infty & \text{if } N \leqslant p. \end{cases}$$

For $\varepsilon > 0$ and $r \in (p, p^*)$ fixed, from hypotheses (H2)(i) and (ii), there exists $c_6 = c_6(\varepsilon, r, M) > 0$ (recall that $M := \sup_{z\in\overline{\Omega}} |Dv(z)|$) such that

$$f(z, x, Dv(z)) \geqslant (\eta_M(z) - \varepsilon)x^{q-1} - c_6 x^{r-1} \tag{3.12}$$

for a.a. $z \in \Omega$, and all $0 \leqslant x \leqslant \eta$. This unilateral growth restriction on $f(z, \cdot, Dv(z))$ drives us to consider another auxiliary Dirichlet problem as follows:

$$\begin{gathered} -\operatorname{div} a(Du(z)) = (\eta_M(z) - \varepsilon)u(z)^{q-1} - c_6 u(z)^{r-1} \quad \text{in } \Omega, \\ u(z) = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{3.13}$$

with $u(z) > 0$ in Ω.

Proposition 3.3. *If hypotheses* (H1) *holds, then for all $\varepsilon > 0$, auxiliary problem* (3.13) *admits a unique positive solution $u^* \in D_+$.*

Proof. First we show the existence of positive solutions for problem (3.13). To do so, consider the C^1-functional $\psi : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\begin{aligned} \psi(u) = &\int_\Omega G(Du)\,dz + \frac{1}{p}\|u^-\|_p^p - \frac{1}{q}\int_\Omega (\eta_M(z) - \varepsilon)(u^+)^q\,dz \\ &+ \frac{c_6}{r}\|u^+\|_r^r \quad \text{for all } u \in W_0^{1,p}(\Omega). \end{aligned}$$

From the facts $G(0) = 0$, $u = u^+ - u^-$ and [11, Proposition 2.4.27], we have

$$\int_\Omega G(Du)\,dz = \int_\Omega G(Du^+)\,dz + \int_\Omega G(-Du^-)\,dz.$$

So, from Lemma 2.4 we have

$$\begin{aligned} \psi(u) \geqslant &\frac{c_1}{p(p-1)}\|Du^+\|_p^p + \frac{c_6}{r}\|u^+\|_r^r + \frac{c_1}{p(p-1)}\|Du^-\|_p^p + \frac{1}{p}\|u^-\|_p^p \\ &- \frac{1}{q}\int_\Omega (\eta_M(z) - \varepsilon)(u^+)^q\,dz, \end{aligned}$$

hence (see, Poincaré inequality, e.g. [11, Theorem 2.5.4, p.216])

$$\psi(u) \geqslant c_7\|u\|^p - c_8(\|u\|^q + 1),$$

for some $c_7, c_8 > 0$. Since $q < p$, it is clear that ψ is coercive. We use the compactness of embedding $W_0^{1,p}(\Omega) \subseteq L^p(\Omega)$ and the convexity of G again, to conclude that ψ is sequentially weakly lower semicontinuous. By the Weierstrass-Tonelli theorem, we get $u^* \in W_0^{1,p}(\Omega)$ such that

$$\psi(u^*) = \inf_{u\in W_0^{1,p}(\Omega)} \psi(u). \tag{3.14}$$

Using the same method as in the proof of Proposition 3.2, we can take $t \in (0, 1)$ and $\varepsilon > 0$ small enough to obtain $\psi(t\widehat{u}_1(q)) < 0$. This implies (see (3.14))

$$\psi(u^*) < 0 = \psi(0),$$

so, $u^* \neq 0$.

The equality (3.14) implies $\psi'(u^*)=0$. For $h\in W_0^{1,p}(\Omega)$, one has

$$\begin{aligned}\langle A(u^*),h\rangle-\int_\Omega((u^*)^-)^{p-1}h\,dz=\int_\Omega(\eta_M(z)-\varepsilon)((u^*)^+)^{q-1}h\,dz\\ -c_6\int_\Omega((u^*)^+)^{r-1}h\,dz.\end{aligned}\tag{3.15}$$

Taking $h=-(u^*)^-\in W_0^{1,p}(\Omega)$ into (3.15), we use Lemma 2.3 again to obtain

$$\frac{c_1}{p-1}\|D(u^*)^-\|_p^p+\|(u^*)^-\|_p^p\leqslant 0.$$

So, we have $u^*\geqslant 0$ and $u^*\neq 0$. Therefore, (3.15) reduces to

$$\langle A(u^*),h\rangle=\int_\Omega(\eta_M(z)-\varepsilon)(u^*)^{q-1}h\,dz-c_6\int_\Omega(u^*)^{r-1}h\,dz$$

for all $h\in W_0^{1,p}(\Omega)$, this means

$$\begin{gathered}-\operatorname{div}a(Du^*(z))=(\eta_M-\varepsilon)(u^*)(z)^{q-1}-c_6(u^*)(z)^{r-1}\quad\text{for a.a. } z\in\Omega,\\ u^*(z)=0\quad\text{on }\partial\Omega.\end{gathered}\tag{3.16}$$

As in the proof of Proposition 3.2, using the nonlinear regularity theory, we have

$$u^*\in C_0^1(\Omega)_+\setminus\{0\}.$$

Next we shall verify that u^* is the unique positive solution to problem (3.13). For this goal, we consider the integral functional $j:L^1(\Omega)\to\overline{\mathbb{R}}=\mathbb{R}\cup\{+\infty\}$ defined by

$$j(u)=\begin{cases}\int_\Omega G(Du^{1/q})\,dz & \text{if } u\geqslant 0,\ u^{1/q}\in W_0^{1,p}(\Omega),\\ +\infty & \text{otherwise,}\end{cases}$$

where the effective domain of the functional j is denoted by

$$\operatorname{dom}j=\{u\in L^1(\Omega):j(u)<+\infty\}.$$

We will show that the integral functional j is convex. Let $u_1,u_2\in\operatorname{dom}j$ and $u=(1-t)u_1+tu_2$ with $t\in[0,1]$. [5, Lemma 1] states that the function $u\to|Du^{1/q}|^q$ is convex, so we have

$$|Du^{1/q}(z)|\leqslant\Big((1-t)|Du_1(z)^{1/q}|^q+t|Du_2(z)^{1/q}|^q\Big)^{1/q}\quad\text{for a.a. } z\in\Omega.$$

The monotonicity of G_0 and the convexity of $t\mapsto G_0(t^{1/q})$ (see hypothesis (H1)(iv)) ensure that

$$\begin{aligned}G_0(|Du^{1/q}(z)|)&\leqslant G_0\big(\big((1-t)|Du_1(z)^{1/q}|^q+t|Du_2(z)^{1/q}|^q\big)^{1/q}\big)\\ &\leqslant(1-t)G_0(|Du_1(z)^{1/q}|)+tG_0(|Du_2(z)^{1/q}|)\end{aligned}$$

for a.a. $z\in\Omega$. Which leads to

$$G(Du^{1/q}(z))\leqslant(1-t)G(Du_1(z)^{1/q})+tG(Du_2(z)^{1/q})\quad\text{for a.a. } z\in\Omega,$$

thus the map j is convex.

Suppose that $\widetilde{u}^*$ is another positive solution of (3.13). As we did for u^*, we can check that $\widetilde{u}^*\in C_0^1(\Omega)_+\setminus\{0\}$. For $h\in C_0^1(\overline{\Omega})$ fixed and $|t|$ small enough, we obtain

$$u^*+th\in\operatorname{dom}j\quad\text{and}\quad\widetilde{u}^*+th\in\operatorname{dom}j.$$

Recalling that j is convex, it is evidently Gâteaux differentiable at u^* and at $\widetilde{u}^*$ in the direction h. Further, we apply the chain rule and the nonlinear Green's identity (see Gasiński-Papageorgiou [11, p. 210]) to obtain

$$\begin{aligned} j'(u^*)(h) &= \frac{1}{q}\int_\Omega \frac{-\operatorname{div} a(Du^*)}{(u^*)^{q-1}} h\,dz \quad \text{for all } h \in C_0^1(\overline{\Omega}), \\ j'(\widetilde{u}^*)(h) &= \frac{1}{q}\int_\Omega \frac{-\operatorname{div} a(D\widetilde{u}^*)}{(\widetilde{u}^*)^{q-1}} h\,dz \quad \text{for all } h \in C_0^1(\overline{\Omega}). \end{aligned}$$

Putting $h = (u^*)^q - (\widetilde{u}^*)^q$ into the above inequalities and then subtracting the resulting equalities, it follows from the monotonicity of j' (since j is convex) that

$$\begin{aligned} 0 &\leqslant \frac{1}{q}\int_\Omega \Big(\frac{-\operatorname{div}(Du^*)}{(u^*)^{q-1}} - \frac{-\operatorname{div} a(D\widetilde{u}^*)}{(\widetilde{u}^*)^{q-1}}\Big)((u^*)^q - (\widetilde{u}^*)^q)\,dz \\ &= \frac{c_6}{q}\int_\Omega \big((\widetilde{u}^*)^{r-q} - (u^*)^{r-q}\big)\big((u^*)^q - (\widetilde{u}^*)^q\big)\,dz \end{aligned}$$

(see (3.13)), so, from $q < p < r$, we conclude that $u^* = \widetilde{u}^*$. This proves that $u^* \in C_0^1(\Omega)_+ \setminus \{0\}$ is the unique positive solution for problem (3.13). We are now to apply the nonlinear maximum principle, see Pucci-Serrin [35]), again to obtain $u^* \in D_+$. □

Proposition 3.4. *If hypotheses* (H1) *and* (H2) *hold, then $u^* \leqslant u$ for all $u \in S_v$.*

Proof. Let $u \in S_v$. We now introduce the following Carathéodory function $e : \Omega \times \mathbb{R} \to \mathbb{R}$

$$e(z,x) = \begin{cases} (\eta_M(z) - \varepsilon)(x^+)^{q-1} - c_6(x^+)^{r-1} + \xi_\eta (x^+)^{p-1} & \text{if } x \leqslant u(z), \\ (\eta_M(z) - \varepsilon)u(z)^{q-1} - c_6 u(z)^{r-1} + \xi_\eta u(z)^{p-1} & \text{if } u(z) < x. \end{cases} \tag{3.17}$$

Also, we denote

$$E(z,x) = \int_0^x e(z,s)\,ds$$

and consider the C^1-functional $\tau : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\tau(u) = \int_\Omega G(Du)\,dz + \frac{\xi_\eta}{p}\|u\|_p^p - \int_\Omega E(z,u)\,dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

By the definition of e (see (3.17)), we see that τ is coercive. Also, it is sequentially weakly lower semicontinuous. Invoking the Weierstrass-Tonelli theorem, we can find $\widetilde{u}^* \in W_0^{1,p}(\Omega)$ such that

$$\tau(\widetilde{u}^*) = \inf_{v \in W_0^{1,p}(\Omega)} \tau(v). \tag{3.18}$$

As before, since $q < p < r$, we have

$$\tau(\widetilde{u}^*) < 0 = \tau(0),$$

which implies $\widetilde{u}^* \neq 0$. From (3.18), we have $\tau'(\widetilde{u}^*) = 0$, which means

$$\langle A(\widetilde{u}^*), h\rangle + \xi_\eta \int_\Omega |\widetilde{u}^*|^{p-2}\widetilde{u}^* h\,dz = \int_\Omega e(z,\widetilde{u}^*) h\,dz \tag{3.19}$$

for all $h \in W_0^{1,p}(\Omega)$. Putting $h = -(\widetilde{u}^*)^- \in W_0^{1,p}(\Omega)$ into the above equality and then using Lemma 2.3, we have

$$\frac{c_1}{p-1}\|D(\widetilde{u}^*)^-\|_p^p + \xi_\eta \|(\widetilde{u}^*)^-\|_p^p \leq 0$$

(see (3.17)), so $\widetilde{u}^* \geqslant 0$ and $\widetilde{u}^* \neq 0$.

On the other hand, inserting $h = (\widetilde{u}^* - u)^+ \in W_0^{1,p}(\Omega)$ into (3.19), we obtain

$$\begin{aligned}
&\langle A(\widetilde{u}^*), (\widetilde{u}^* - u)^+\rangle + \xi_\eta \int_\Omega (\widetilde{u}^*)^{p-1}(\widetilde{u}^* - u)^+ \, dz \\
&= \int_\Omega \big((\eta_M(z) - \varepsilon)u^{q-1} - c_6 u^{r-1} + \xi_\eta u^{p-1}\big)(\widetilde{u}^* - u)^+ \, dz \\
&\leqslant \int_\Omega f(z, u, Dv)(\widetilde{u}^* - u)^+ \, dz + \xi_\eta \int_\Omega u^{p-1}(\widetilde{u}^* - u)^+ \, dz \\
&= \langle A(u), (\widetilde{u}^* - u)^+\rangle + \xi_\eta \int_\Omega u^{p-1}(\widetilde{u}^* - u)^+ \, dz
\end{aligned}$$

(see (3.12), (3.17), and recall that $u \in S_v$). Therefore, we have

$$\langle A(\widetilde{u}^*) - A(u), (\widetilde{u}^* - u)^+\rangle + \xi_\eta \int_\Omega \big((\widetilde{u}^*)^{p-1} - u^{p-1}\big)(\widetilde{u}^* - u)^+ \, dz \leqslant 0.$$

Using the monotonicity of A, we deduce $\widetilde{u}^* \leqslant u$. So, we have verified that

$$\widetilde{u}^* \in [0, u] \setminus \{0\}. \tag{3.20}$$

Taking into account (3.17) and (3.20), we rewrite (3.19) as

$$\langle A(\widetilde{u}^*), h\rangle = \int_\Omega \big((\eta_M(z) - \varepsilon)(\widetilde{u}^*)^{q-1} - c_6(\widetilde{u}^*)^{r-1}\big) h \, dz$$

for all $h \in W_0^{1,p}(\Omega)$. This combined with Proposition 3.3 gives $\widetilde{u}^* = u^*$, so $u^* \leqslant u$, which completes the proof. $\square$

Applying Proposition 3.4, we shall show that problem (3.1) admits a smallest positive solution $\widehat{u}_v \in [0, \eta] \cap D_+$.

Proposition 3.5. *If* (H1) *and* (H2) *are fulfilled, then problem* (3.1) *admits a smallest positive solution* $\widehat{u}_v \in [0, \eta] \cap D_+$.

Proof. Invoking [24, Lemma 3.10 p. 178], we can find a decreasing sequence $\{u_n\}_{n\geqslant 1} \subseteq S_v$ such that

$$\inf S_v = \inf_{n\geqslant 1} u_n. \tag{3.21}$$

For all $n \geqslant 1$, we have

$$\langle A(u_n), h\rangle = \int_\Omega f(z, u_n, Dv) h \, dz \quad \text{for all } h \in W_0^{1,p}(\Omega), \tag{3.22}$$

however, from Proposition 3.4, one has

$$u^* \leqslant u_n \leqslant \eta. \tag{3.23}$$

Then by hypothesis (H2)(i) and Lemma 2.3, we have that the sequence $\{u_n\}_{n\geqslant 1} \subseteq W_0^{1,p}(\Omega)$ is bounded. Passing to a subsequence, we may assume that

$$u_n \xrightarrow{w} \widehat{u}_v \quad \text{in } W_0^{1,p}(\Omega) \quad \text{and} \quad u_n \to \widehat{u}_v \quad \text{in } L^p(\Omega). \tag{3.24}$$

Choosing $h = u_n - \widehat{u}_v \in W_0^{1,p}(\Omega)$ for (3.22), we pass to the limit as $n \to \infty$ and then apply (3.24) to get

$$\lim_{n\to+\infty} \langle A(u_n), u_n - \widehat{u}_v\rangle = 0,$$

but the $(S)_+$-property of A (see Proposition 2.5), results in

$$u_n \to \widehat{u}_v \quad \text{in } W_0^{1,p}(\Omega). \tag{3.25}$$

Passing to the limit as $n \to +\infty$ in (3.22) and using (3.25) to reveal

$$\langle A(\widehat{u}_v), h\rangle = \int_\Omega f(z, \widehat{u}_v, Dv)h\,dz \quad \text{for all } h \in W_0^{1,p}(\Omega).$$

On the other hand, taking the limit as $n \to +\infty$ in (3.23), we conclude that

$$u^* \leqslant \widehat{u}_v \leqslant \eta.$$

From the above inequality, it follows that

$$\widehat{u}_v \in S_v \quad \text{and} \quad \widehat{u}_v = \inf S_v,$$

which completes the proof. □

Now, we consider the set

$$C = \{u \in C_0^1(\overline{\Omega}) : 0 \leqslant u(z) \leqslant \eta \text{ for all } z \in \overline{\Omega}\}$$

and introduce the mapping $\vartheta : C \to C$ given by

$$\vartheta(v) = \widehat{u}_v.$$

It is obvious that a fixed point of map ϑ is also a positive solution to problem (1.1). Therefore, next, we focus our attention to produce a fixed point for ϑ. Here our approach will apply the Leray-Schauder alternative principle (see Theorem 2.8). To do so, we will need the following lemma.

Lemma 3.6. *If* (H1) *and* (H2) *are satisfied, then for any sequence* $\{v_n\}_{n\geqslant 1} \subseteq C$ *with* $v_n \to v$ *in* $C_0^1(\overline{\Omega})$*, and* $u \in S_v$*, there exists a sequence* $\{u_n\} \subseteq C_0^1(\overline{\Omega})$ *with* $u_n \in S_{v_n}$ *for* $n \geqslant 1$*, such that* $u_n \to u$ *in* $C_0^1(\overline{\Omega})$*.*

Proof. Let $\{v_n\}_{n\geqslant 1} \subseteq C$ be such that $v_n \to v$ in $C_0^1(\overline{\Omega})$, and $u \in S_v$. First, we consider the nonlinear Dirichlet problem

$$\begin{gathered} -\operatorname{div} a(Dw(z)) + \xi_\eta |w(z)|^{p-2}w(z) = \widehat{f}(z, u(z), Dv_n(z)) \quad \text{in } \Omega, \\ w(z) = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{3.26}$$

Since $u \in S_v \subseteq [0,\eta] \setminus \{0\}$, from (3.3) and hypothesis (H2)(i), we see that

$$\widehat{f}(\cdot, u(\cdot), Dv_n(\cdot)) \not\equiv 0 \quad \text{for all } n \geqslant 1,$$
$$\widehat{f}(z, u(z), Dv_n(z)) \geqslant 0 \quad \text{for a.a. } z \in \Omega \text{ and all } n \geqslant 1.$$

It is obvious that problem (3.26) has a unique positive solution $u_n^0 \in D_+$. It follows from (3.3), the fact that $u \in S_v \subseteq [0,\eta] \setminus \{0\}$, and hypotheses (H2)(i), (iii) that

$$\begin{aligned} &\langle A(u_n^0), (u_n^0 - \eta)^+\rangle + \xi_\eta \int_\Omega (u_n^0)^{p-1}(u_n^0 - \eta)^+\,dz \\ &= \int_\Omega (f(z, u, Dv_n) + \xi_\eta u^{p-1})(u_n^0 - \eta)^+\,dz \\ &\leqslant \int_\Omega (f(z, \eta, Dv_n) + \xi_\eta \eta^{p-1})(u_n^0 - \eta)^+\,dz \\ &= \int_\Omega \xi_\eta \eta^{p-1}(u_n^0 - \eta)^+\,dz, \end{aligned}$$

hence, from $A(\eta)=0$, we have

$$\langle A(u_n^0)-A(\eta),(u_n^0-\eta)^+\rangle+\xi_\eta\int_\Omega((u_n^0)^{p-1}-\eta^{p-1})(u_n^0-\eta)^+\,dz\leqslant 0.$$

However, the monotonicity of A implies $u_n^0\leqslant\eta$. So, we conclude that

$$u_n^0\in[0,\eta]\setminus\{0\}\quad\forall n\geqslant 1.$$

Moreover the nonlinear regularity theory of Lieberman [28], and the nonlinear maximum principle of Pucci-Serrin [35]) imply that

$$u_n^0\in[0,\eta]\cap D_+\quad\forall n\geqslant 1. \tag{3.27}$$

We have

$$\begin{gathered}-\operatorname{div}a(Du_n^0(z))+\xi_\eta((u_n^0(z))^{p-1}-u(z)^{p-1})=f(z,u(z),Dv_n(z))\quad\text{for a.a. } z\in\Omega,\\ u_n^0(z)=0\quad\text{on }\partial\Omega.\end{gathered} \tag{3.28}$$

From (3.27)–(3.28), Lemma 2.3 and hypothesis (H2)(i), we conclude that the sequence $\{u_n^0\}_{n\geqslant1}$ is bounded in $W_0^{1,p}(\Omega)$. So, on account of the nonlinear regularity theory of Lieberman [28], we can find $\beta\in(0,1)$ and $c_9>0$ such that

$$u_n^0\in C^{1,\beta}(\overline{\Omega})\quad\text{and}\quad\|u_n^0\|_{C^{1,\beta}(\overline{\Omega})}\leqslant c_9\quad\forall n\geqslant 1.$$

The compactness of the embedding $C^{1,\beta}(\overline{\Omega})\subseteq C^1(\overline{\Omega})$ implies that there exists a subsequence $\{u_{n_k}^0\}_{k\geqslant1}$ of the sequence $\{u_n^0\}_{n\geqslant1}$ such that

$$u_{n_k}^0\to\widetilde{u}^0\quad\text{in } C^1(\overline{\Omega})\quad\text{as } k\to+\infty.$$

Using this fact and (3.28), we have

$$\begin{gathered}-\operatorname{div}a(D\widetilde{u}^0(z))+\xi_\eta((\widetilde{u}^0(z))^{p-1}-u(z)^{p-1})=f(z,u(z),Dv(z))\quad\text{for a.a. } z\in\Omega,\\ \widetilde{u}^0(z)=0\quad\text{on }\partial\Omega.\end{gathered} \tag{3.29}$$

Recall that $u\in S_v$, so (3.1) holds. Taking into account (3.1) and (3.29), we have

$$\langle A(\widetilde{u}^0)-A(u),h\rangle+\xi_\eta\int_\Omega(\widetilde{u}^0(z)^{p-1}-u(z)^{p-1})h\,dz=0$$

for all $h\in W_0^{1,p}(\Omega)$. Additionally, we insert $h=(\widetilde{u}^0-u)^+$ and $h=-(u-\widetilde{u}^0)^+$ into the above equality to obtain

$$\widetilde{u}^0=u\in S_v.$$

So, for the original sequence $\{u_n^0\}_{n\geqslant1}$, one has

$$u_n^0\to u\quad\text{in } C_0^1(\overline{\Omega})\quad\text{as } n\to+\infty.$$

Next, we consider the nonlinear Dirichlet problem

$$\begin{gathered}-\operatorname{div}a(Dw(z))+\xi_\eta|w(z)|^{p-2}w(z)=\widehat{f}(z,u_n^0(z),Dv_n(z))\quad\text{in }\Omega,\\ w(z)=0\quad\text{on }\partial\Omega.\end{gathered}$$

As before, we verify that the above problem admits a unique solution such that

$$u_n^1\in[0,\eta]\cap D_+\quad\forall n\geqslant 1.$$

We apply nonlinear regularity theory of Lieberman [28] again to obtain

$$u_n^1\to u\quad\text{in } C_0^1(\overline{\Omega})\quad\text{as } n\to+\infty.$$

Repeating this procedure, we construct a sequence $\{u_n^k\}_{k,n\geqslant 1}$ such that

$$\begin{gathered} -\operatorname{div} a(Du_n^k(z)) + \xi_\eta u_n^k(z)^{p-1} = \widehat{f}(z, u_n^{k-1}(z), Dv_n(z)) \quad \text{in } \Omega, \\ u_n^k(z) = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{3.30}$$

for all $n, k \geqslant 1$ with

$$u_n^k \in [0,\eta] \cap D_+ \quad \forall n, k \geqslant 1, \tag{3.31}$$

$$u_n^k \to u \quad \text{in } C_0^1(\overline{\Omega}) \quad \text{as } n \to +\infty \quad \forall k \geqslant 1. \tag{3.32}$$

For $n \geqslant 1$ fixed, as above, we know that the sequence $\{u_n^k\}_{k\geqslant 1} \subseteq C_0^1(\overline{\Omega})$ is relatively compact. Therefore, there has a subsequence $\{u_n^{k_m}\}_{m\geqslant 1}$ of the sequence $\{u_n^k\}_{k\geqslant 1}$ satisfying

$$u_n^{k_m} \to \widetilde{u}_n \quad \text{in } C_0^1(\overline{\Omega}) \quad \text{as } m \to +\infty.$$

This and (3.32) imply

$$\begin{gathered} -\operatorname{div} a(D\widetilde{u}_n(z)) + \xi_\eta \widetilde{u}_n(z)^{p-1} = \widehat{f}(z, \widetilde{u}_n(z), Dv_n(z)) \quad \text{for a.a. } z \in \Omega, \\ \widetilde{u}_n(z) = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{3.33}$$

The uniqueness of the solution of (3.33) deduces that for the original sequence we have

$$u_n^k \to \widetilde{u}_n \quad \text{in } C_0^1(\overline{\Omega}) \quad \text{as } k \to +\infty.$$

However, from (3.31), we obtain

$$\widetilde{u}_n \in [0,\eta] \cap D_+ \quad \forall n \geqslant 1,$$

but from (3.32) and the double limit lemma (see Aubin-Ekeland [1] or Gasiński-Papageorgiou [20, p. 61]), we have $\widetilde{u}_n \in [0,\eta] \cap D_+ \quad \forall n \geqslant n_0$. Consequently,

$$\widetilde{u}_n \in S_v \quad \forall n \geqslant n_0 \quad \text{and} \quad \widetilde{u}_n \to u \quad \text{in } C_0^1(\overline{\Omega}),$$

which completes the proof of the Lemma. □

Remark 3.7. Actually, if we introduce the set-valued mapping $\mathcal{S} : C^1(\overline{\Omega}) \to 2^{C^1(\overline{\Omega})}$ by

$$\mathcal{S}(v) = S_v,$$

then by the above lemma, we conclude that the mapping $\mathcal{S}$ is lower semicontinuous.

Applying this lemma, we will prove that the map $\vartheta : C \to C$ defined by $\vartheta(v) = \widehat{u}_v$ is compact.

Proposition 3.8. *If hypotheses* (H1) *and* (H2) *are fulfilled, then the map* $\vartheta : C \to C$ *is compact.*

Proof. To end this, we shall show that ϑ is continuous and maps bounded sets in C to relatively compact subsets of C.

First, for the part of continuity of ϑ, let $v \in C$ and $\{v_n\}_{n\geqslant 1} \subseteq C$ be such that $v_n \to v$ in $C_0^1(\overline{\Omega})$, and denote $\widehat{u}_n = \vartheta(v_n)$ for $n \geqslant 1$. So, we get

$$\begin{gathered} -\operatorname{div} a(D\widehat{u}_n(z)) = f(z, \widehat{u}_n(z), Dv_n(z)) \quad \text{for a.a. } z \in \Omega, \\ \widehat{u}_n(z) = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{3.34}$$

with $\widehat{u}_n \in [0,\eta]$ for all $n \geqslant 1$. It is easy to check that $\{\widehat{u}_n\}_{n\geqslant 1} \subseteq W_0^{1,p}(\Omega)$ is bounded. So, it follows from Lieberman [28] that there exist $\beta \in (0,1)$ and $c_{10} > 0$ satisfying

$$\widehat{u}_n \in C^{1,\beta}(\overline{\Omega}) \quad \text{and} \quad \|\widehat{u}_n\|_{C^{1,\beta}(\overline{\Omega})} \leqslant c_{10} \quad \forall n \geqslant 1.$$

Without loss of generality, we may assume that

$$\widehat{u}_n \to \widehat{u} \quad \text{in } C_0^1(\overline{\Omega}) \quad \text{as } n \to +\infty. \tag{3.35}$$

Passing to the limit in (3.34), it yields

$$\begin{gathered} -\operatorname{div} a(D\widehat{u}(z)) = f(z, \widehat{u}(z), Dv(z)) \quad \text{for a.a. } z \in \Omega, \\ \widehat{u}(z) = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{3.36}$$

By taking $M > \sup_{n\geqslant 1} \|v_n\|_{C^1(\overline{\Omega})}$, we apply Proposition 3.4 to obtain $u^* \leqslant \widehat{u}_n \quad \forall n \geqslant 1$, hence, convergence (3.35) implies

$$u^* \leqslant \widehat{u} \in C_0^1(\overline{\Omega})_+. \tag{3.37}$$

We now assert that $\widehat{u} = \vartheta(v)$. Invoking Lemma 3.6, we can take a sequence $\{u_n\} \subseteq C_0^1(\overline{\Omega})$ with $u_n \in S_{v_n}$, $n \geqslant 1$ and

$$u_n \to \vartheta(v) \quad \text{in } C_0^1(\overline{\Omega}) \quad \text{as } n \to +\infty. \tag{3.38}$$

By the definition of ϑ, we have

$$\widehat{u}_n = \vartheta(v_n) \leqslant u_n \quad \forall n \geqslant 1.$$

This combined with (3.35) and (3.38) gives $\widehat{u} \leqslant \vartheta(v)$. Recalling that (3.37), we obtain

$$\widehat{u} = \vartheta(v),$$

therefore, ϑ is continuous.

Next we will verify that ϑ maps bounded sets in C to relatively compact subsets of C. Assume that $B \subseteq C$ is bounded in $C_0^1(\overline{\Omega})$. As before, we know that the set $\vartheta(B) \subseteq W_0^{1,p}(\Omega)$ is bounded. On the other hand, we apply the nonlinear regularity theory of Lieberman [28] and the compactness of the embedding $C_0^{1,s}(\overline{\Omega}) \subseteq C_0^1(\overline{\Omega})$ (with $0 < s < 1$) to reveal that the set $\vartheta(B) \subseteq C_0^1(\overline{\Omega})$ is relatively compact, thus ϑ is compact. □

Now we give the main result of this article.

Theorem 3.9. *If* (H1) *and* (H2) *are satisfied, then problem* (1.1) *admits a positive solution* $\widehat{u}$*, more precisely,*

$$\widehat{u} \in [0, \eta] \cap D_+.$$

Proof. Let $U(\vartheta)$ be the set defined by

$$U(\vartheta) = \{u \in C : u = \lambda\vartheta(u),\ 0 < \lambda < 1\}.$$

For any $u \in U(\vartheta)$, we have $\frac{1}{\lambda}u = \vartheta(u)$, so

$$\langle A(\frac{1}{\lambda}u), h\rangle = \int_\Omega f(z, \frac{u}{\lambda}, Du)h\, dz \quad \text{for all } h \in W_0^{1,p}(\Omega). \tag{3.39}$$

Inserting $h = \frac{u}{\lambda} \in W_0^{1,p}(\Omega)$ into (3.39) and taking into account Lemma 2.3, we calculate

$$\begin{aligned} \frac{c_1}{p-1}\|D(\frac{u}{\lambda})\|_p^p &\leqslant \int_\Omega f(z, \frac{u}{\lambda}, Du)\frac{u}{\lambda}\, dz \leqslant \int_\Omega f(z, u, Du)\frac{u}{\lambda^p}\, dz \\ &\leqslant \int_\Omega f(z, u, D(\frac{u}{\lambda}))u\, dz \leqslant \int_\Omega \big(\widetilde{c}_1 + \widetilde{c}_2|D(\frac{u}{\lambda})|^p\big)\, dz \end{aligned}$$

where the last three inequalities are obtained by using (2.3), (H2)(iii), and (H2)(i), respectively. Considering the inequality $\widetilde{c}_2 < \frac{c_1}{p-1}$ (see hypothesis (H2)(i)), one has

$$\|D(\frac{u}{\lambda})\|_p \leqslant c_{11} \quad \text{for all } \lambda \in (0,1),$$

for some $c_{11} > 0$. Hence, we have

$$\{\frac{u}{\lambda}\}_{u\in U(\vartheta)} \subseteq W_0^{1,p}(\Omega) \text{ is bounded.} \tag{3.40}$$

From (3.39) we have

$$\begin{gathered} -\operatorname{div} a(D(\frac{u}{\lambda})(z)) = f(z, \frac{u}{\lambda}(z), Du(z)) \quad \text{for a.a. } z \in \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{3.41}$$

However, condition (H2)(iii) ensures that

$$f(z, \frac{u}{\lambda}, Du) \leqslant \lambda^p f(z, \frac{u}{\lambda}, D(\frac{u}{\lambda})) \quad \text{for a.a. } z \in \Omega. \tag{3.42}$$

Then from (3.40)–(3.42) and the nonlinear regularity theory of Lieberman [28], we have

$$\|\frac{u}{\lambda}\|_{C_0^1(\overline{\Omega})} \leqslant c_{12} \quad \text{for all } u \in U(\vartheta),$$

for some $c_{12} > 0$, thus $U(\vartheta) \subseteq C_0^1(\overline{\Omega})$ is bounded.

Recall that ϑ is compact, see Proposition 3.8, we are now in a position to apply the Leray-Schauder alternative theorem (see Theorem 2.8), to look for a function $\widehat{u} \in C$ such that

$$\widehat{u} = \vartheta(\widehat{u}).$$

Consequently, we know that $\widehat{u} \in [0,\eta] \cap D_+$ is a positive solution of (1.1). $\square$

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