SERIE B — INFORMATIK

Ramsey-Remainder

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Abstract

We investigate the following Ramsey-type problem. Given a natural number k, determine the smallest integer $rr(k)$ such that, if n is sufficiently large with respect to k, and S is any set of n points in general position in the plane, then all but at most $rr(k)$ points of S can be partitioned into convex sets of sizes $\geq k$. We provide estimates on $rr(k)$ which are best possible if a classic conjecture of Erdős and Szekeres on the Ramsey number for convex sets is valid. We also prove that in many types of combinatorial structures, the corresponding "Ramsey-remainder" $rr(k)$ is equal to the off-diagonal Ramsey number $r(k, k-1)$ minus 1.

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1 Introduction

In a general setting, Ramsey theory states that any sufficiently large combinatorial structure of a suitable type contains a "regular" substructure of a relatively large size. A combinatorial structure is defined by an underlying set (in our case finite) and some structure on this set. By a combinatorial structure we will often mean just its underlying set and the size of a combinatorial structure will mean the number of elements of the underlying set. The structure on the underlying set also determines which substructures (subsets) are regular. Regularity has a different meaning in different types of combinatorial structures.

Let $\mathbf{S}$ be a class of combinatorial structures. A typical (finite) Ramsey-type theorem says that for any positive integer k there is an integer $r(k)$ such that any structure $S \in \mathbf{S}$ with at least $r(k)$ elements contains a regular substructure with k elements. Ramsey theory studies the numbers $r(k)$ – their existence and their values – for different classes of combinatorial structures. A survey on current directions in Ramsey theory is given in a nice recent book [NR 90].

All classes $\mathbf{S}$ of combinatorial structures which we study have two properties:
(i) $\mathbf{S}$ satisfies a Ramsey-type theorem (i. e., the numbers $r(k)$ exist).
(ii) The regularity is hereditary, i. e., if $S \in \mathbf{S}$ is regular then all subsets of S are regular.

The first property is necessary for obtaining nontrivial results in our study. The second property is quite natural and will be used in all considered cases.

Problem 1 *Given a positive integer k, find the minimum number $rr(k)$ such that any sufficiently large combinatorial structure (set) belonging to the class $\mathbf{S}$ can be decomposed into regular sets of size $\geq k$ and a remainder (arbitrary set) of size $\leq rr(k)$.*

Observe that property (ii) implies $rr(k) < r(k)$. It is not clear, however, how close the two functions $rr(k)$ and $r(k)$ are to each other. Our aim is to derive tight estimates on $rr(k)$ for various sorts of combinatorial structures. It turns out in each studied case that $rr(k)$ has an equivalent interpretation in terms of "off-diagonal" Ramsey numbers.

From the techniques needed to determine $rr(k)$, it seems to us that the class of structures deserving most interest from this respect is the one of planar point sets, where "regular substructure" means "vertex set of a convex polygon." The entire Section 3 is devoted to the solution of this problem. We prove that $rr(k)$ is equal to a certain Ramsey-type number (Theorem 18), and give a lower bound (Claim 17) which is best possible if a nearly 35-years-old conjecture of Erdős and Szekeres is valid. (Namely, the conjecture states that every set of $2^{n-2}+1$ points in the plane contains the vertex set of a convex n-gon.)

In Section 2 we discuss the Ramsey-remainder for several other types of combinatorial structures (edge-colored graphs and hypergraphs, sequences of reals, sequences of natural numbers, and partially ordered sets). We show that in each of them the determination of $rr(k)$ is equivalent to finding some specific Ramsey number. Those results are stated in Section 2 and proved in Section 4.

2 Results

Let there be given a class $\mathbf{S}$ of combinatorial structures with its subclass $\mathbf{R}$ of regular structures, and let k be a positive integer. Denote by $\mathbf{R}_k$ the class of regular structures of size $\geq k$. *The Ramsey-remainder* $rr(S,k)$ of a combinatorial structure $S \in \mathbf{S}$ is defined as the minimum number of elements of a subset $S' \subseteq S$ such that the set $S - S'$ can be partitioned into regular sets belonging to $\mathbf{R}_k$. *The Ramsey-remainder* $rr(k)$ of $\mathbf{S}$ is then defined as

$$rr(k) = \lim_{n\to\infty} \max\{rr(S,k) : S \in \mathbf{S}, |S| = n\},$$

where $|S|$ denotes the number of elements in S. In this paper we study the existence and the values of $rr(k)$ for different classes $\mathbf{S}$ of combinatorial structures.

Now we survey those classes of combinatorial structures which we will study. For each class we recall a Ramsey-type theorem with a bibliographic note, and present our theorem giving the value of the Ramsey-remainder. Theorems about the Ramsey-remainders are proven in Section 4.

A. Two-colored complete graphs

Let $\mathbf{S}$ be now the class of all finite 2-colored complete graphs whose edges are colored red and blue. A vertex set $V' \subseteq V$ will be called *red* if it induces a (complete) subgraph of G all of whose edges are red. Analogously we define *blue* vertex sets. If $V' \subseteq V$ is red or blue then we say that it is *monochromatic*. So each one-point subset of V is both blue and red, and each two-point subset of V is monochromatic. A graph $G \in \mathbf{S}$ is called *regular* if all its edges are colored by the same color (i. e., its vertex set is monochromatic).

We say that a set is a k-set if it contains k elements. Here is a classical Ramsey-type theorem.

Theorem 2 *(Ramsey* [R 30]*)* *For any two positive integers k and l there is an integer $n(k,l)$ such that the vertex set of any 2-colored complete graph on $n(k,l)$ vertices contains a red k-set or a blue l-set.*

The smallest number $n(k,l)$ satisfying Theorem 2 will be denoted as usual by $r(k,l)$, and we set $r(k) = r(k,k)$.

Theorem 3 *Let* $\mathbf{S}$ *be the class of all finite 2-colored graphs. Then*

$$rr(k) = r(k, k-1) - 1,$$

for any positive integer k.

Theorem 3 can be restated as follows. Some set of all but at most $r(k,k-1)-1$ vertices of any sufficiently large 2-colored complete graph G can be partitioned into sets of size $\geq k$ each inducing a (complete) subgraph of G all of whose edges are colored by the same color. Moreover, the number $r(k,k-1)-1$ cannot be replaced by a smaller one.

B. Partially ordered sets (posets)

Let **S** be now the class of all finite partially ordered sets (posets). A poset from **S** is called *regular* if it is either a chain or an antichain, i. e., either any two elements are comparable or any two elements are incomparable.

Theorem 4 *(Dilworth* [D 50]*) Let k and l be two positive integers. Any poset with at least $(k-1)(l-1)+1$ elements contains a chain of k elements or an antichain of l elements. This result is best possible. In the Ramsey notation, $r(k,l) = (k-1)(l-1)+1$ and $r(k) = r(k,k) = (k-1)^2+1$.*

Theorem 5 *Let* **S** *be the class of all finite posets. Then*

$$rr(k) = r(k,k-1)-1 = (k-1)(k-2),$$

for any positive integer k.

C. Sub-t-colored complete graphs

Let $K_m = (V,E)$ denote the complete graph on m vertices and let $f : E \to \mathrm{N}$ be an assignment of the edges to positive integers. Then f is called a *sub-t-coloring of* K_m if $|f^{-1}(i)| \leq t$ for all $i \in N$, i. e., each color appears at most t times in K_m. We say that a vertex set $Y \subseteq V$ induces a *rainbow subgraph* of K_m in the coloring f if each edge in the induced subgraph $K_m|_Y = \{e \in E : e \subseteq Y\}$ has a distinct color. Alon, Caro, Rödl, and the second author proved the following unpublished result.

Theorem 6 *Let t be any fixed positive integer. Then for every positive integer k there is an integer $m(k)$ such that, for any $m \geq m(k)$, the graph K_m colored by any sub-t-coloring contains a rainbow subgraph on k vertices. For the minimum value $r(k)$ of $m(k)$ the following bounds hold:*

$$ctk^2/\log k \leq r(k) \leq c'tk^2,$$

for some positive constants c and c'.

Now the rainbow subgraphs are considered to be the *regular* substructures. The following theorem about the Ramsey-remainder has already been proven by Caro and the second author.

Theorem 7 *(Caro, Tuza* [CT 88]*) Let* **S** *be the class of all finite sub-t-colored complete graphs, where $t \geq 1$ is a fixed integer. Then*

$$rr(k) = 0,$$

for any positive integer k.

D. Sequences of real numbers

Let **S** be now the class of all finite sequences of pairwise distinct real numbers.

Theorem 8 *(Erdős, Szekeres* [ES 35]*) Let k and l be two positive integers. Then any sequence of at least $(k-1)(l-1)+1$ pairwise distinct real numbers contains an increasing subsequence of length k or a decreasing subsequence of length l. This result is best possible. In the Ramsey notation, $r(k,l) = (k-1)(l-1)+1$.*

The class of *regular* structures is now the class of monotone sequences from **S**.

Theorem 9 *Let* **S** *be the class of all finite sequences of pairwise distinct real numbers. Then*

$$rr(k) = r(k, k-1) - 1 = (k-1)(k-2),$$

for any positive integer k.

E. Increasing sequences of natural numbers

Let **S** be now the class of all finite increasing sequences of natural numbers.

Theorem 10 *(Erdős* [E 47]*) Let k and l be two positive integers. Any increasing sequence of at least $(k-1)(l-1)+1$ natural numbers contains a subsequence of length k in which any number is divisible by all predecessors or a subsequence of length l in which any number is divisible by none of others. This result is best possible. In the Ramsey notation, $r(k,l) = (k-1)(l-1)+1$.*

The class of *regular* structures is now the class of all sequences belonging to **S** in which either each number is divisible by all predecessors or each number is not divisible by any other one.

Theorem 11 *Let* **S** *be the class of all finite increasing sequences of natural numbers. Then*

$$rr(k) = r(k, k-1) - 1 = (k-1)(k-2),$$

for any positive integer k.

F. $\mathbf{K_t}$−free graphs

Let $t \geq 2$ be an integer. Call a graph K_t*−free* if it does not contain a complete subgraph on t vertices. By *regular* structures (graphs) we will mean graphs with no edges. The Ramsey-type result is now Theorem 2 above. In the following Theorem 12, the number $r(k, t-1)$ is the classical Ramsey number taken from Theorem 2.

Theorem 12 *Let* **S** *be a class of all finite* K_t*-free graphs. Then*

$$rr(k) = r(k, t-1) - 1,$$

for any positive integer k.

G. Hypergraphs

Now we generalize Theorems 2 and 3 for hypergraphs. A *u-uniform hypergraph* $H = (V, E)$ is a pair of a finite set V and an arbitrary set $E \subseteq \binom{V}{u}$ of u-element subsets of V. Elements of V are called *vertices,* elements of E are called *edges.* A u-uniform hypergraph $H = (V, E)$ is called *complete* if the set E contains all u-element subsets of V. A complete u-uniform hypergraph $H = (V, E)$ whose edges are colored by c colors is called *regular (monochromatic)* if all its edges are colored by the same color.

Let c and u be two integers greater than 1. Define a Ramsey-type number $r_{c,u}(k)$ as the minimum number $r \geq u$ satisfying the following condition: For an r-element set V_r, there is no coloring of subsets of V_r of size $\leq u$ by c colors $0, 1, \ldots, c-1$ such that

(i) the empty set has color 0,

(ii) no 1-element subset has color 0,

(iii) for any $j \in \{0, 1, \ldots, u-1\}$ there exists no set $V \subseteq V_r$ of size $k-j$ such that all subsets of V of size $u, u-1, \ldots, u-j$ are colored by the same color.

One can show by a standard technique that the number $r_{c,u}(k)$ is always finite. Here is a Ramsey-type theorem and a theorem about the Ramsey-remainder for hypergraphs.

Theorem 13 *Let* c, u, k *be three integers greater than 1. Then any sufficiently large u-uniform complete hypergraph whose edges are colored by c colors contains a monochromatic induced subhypergraph of size* k.

Theorem 14 *Let* c *and* u *be two integers greater than 1 and let* **S** *be the class of all finite u-uniform complete hypergraphs whose edges are colored by c colors. Then*

$$rr(k) = r_{c,u}(k) - 1,$$

for any positive integer $k \geq u$.

H. Planar point sets

The last investigated class of combinatorial structures is the class of planar point sets, where the regular sets are the convex sets. Since this class has some specific features, we investigate it separately in Section 3.

3 Planar point sets

Let $\mathbf{S}$ be the class of all finite point sets in general position in the plane. (A set of points in the plane is *in general position* if no three points lie on a line.) A set $C \in \mathbf{S}$ is called *convex* if the points of C are vertices of a convex polygon. In this section we study the Ramsey-remainder of $\mathbf{S}$, where the class of regular sets is the class of convex sets. In 1935 the first author and Szekeres proved the Ramsey theorem for $\mathbf{S}$.

Theorem 15 *(Erdős, Szekeres* [ES 35]*) For any positive integer n there is a positive integer $N(n)$ such that any set of at least $N(n)$ points in general position in the plane contains a convex subset of size n.*

Denote by $r(n)$ the smallest integer $N(n)$ satisfying the Erdős-Szekeres theorem. The first author and Szekeres [ES 35], [ES 60] also proved Theorem 16 and stated Conjecture A.

Theorem 16 $\qquad 2^{n-2} + 1 \leq r(n) \leq \binom{2n-4}{n-2} + 1.$

Conjecture A ([ES 60]): $\qquad r(n) = 2^{n-2} + 1.$

Conjecture A is known to be true for $n \leq 5$.

Now we make some preparation before we present Theorem 18 about the value of the Ramsey-remainder for $\mathbf{S}$. Consider the plane with the Cartesian coordinate system. Let $\mathbf{S}^*$ be the subclass of $\mathbf{S}$ containing the sets of $\mathbf{S}$ with no pair of points with the same x-coordinate. We say that points $[x_1, y_1], \ldots, [x_n, y_n], x_1 < x_2 < \ldots < x_n$, form

(i) *a convex sequence* if, for any $1 \leq i < j < k \leq n$, the point $[x_j, y_j]$ lies below the line $[x_i, y_i]\ [x_k, y_k]$,

(ii) *a concave sequence* if, for any $1 \leq i < j < k \leq n$, the point $[x_j, y_j]$ lies above the line $[x_i, y_i]\ [x_k, y_k]$.

A sequence of length n is called a *n-sequence.*

For three positive integers k, l, l', define $m(k, l, l')$ as the maximum size of a set $S \in \mathbf{S}^*$ with no convex k-set, with no convex l-sequence, and with no concave l'-sequence. In the proof of the following claim we use a proof technique from [ES 35].

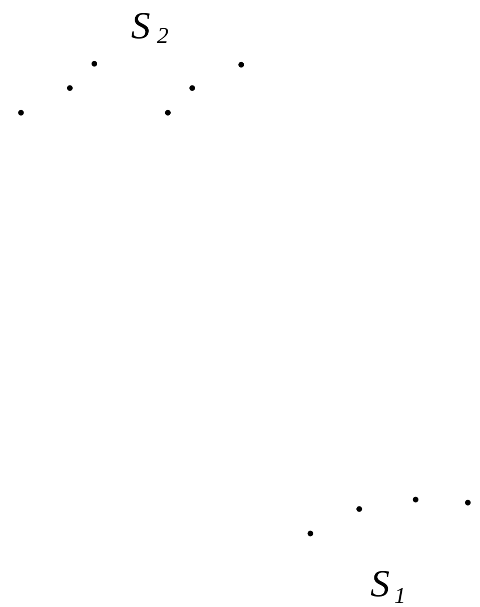

Figure 1: The sets S_1 and S_2 (for $k=6,\ l=4,\ l'=5$)

Claim 17 *If $k \geq l \geq 2$, $k \geq l' \geq 2$, and $k \leq l+l'-2$, then*

$$m(k,l,l') \geq \sum_{i=k-l'}^{l-2} \binom{k-2}{i}.$$

Proof. We proceed by induction on $\Delta = l+l'-k-2$. If $\Delta = 0$ then $m(k,l,l')$ is the maximum size of a set $S \in \mathbf{S}^*$ with no convex l-sequence and with no concave l'-sequence (since a convex k-set for $k=(l+l'-2)$ contains a convex l-sequence or a concave l'-sequence). The first author and Szekeres [ES 35] proved $m(k,l,l') = \binom{l+l'-4}{l-2}$ in this case. Hence, $m(l+l'-2,l,l') = \binom{l+l'-4}{l-2} = \sum_{i=l-2}^{l-2}\binom{k-2}{i} = \sum_{i=k-l'}^{l-2}\binom{k-2}{i}$.

Now let $\Delta = l+l'-k-2 > 0$ and suppose that the claim holds for every smaller value of $l+l'-k-2$. There exist two sets $S_1, S_2 \in \mathbf{S}^*$ of size $m(k,l-1,l')$ and $m(k,l,k-l+2)$ with no convex k-set, with no convex sequence of length $l-1$ and l, and with no concave sequence of length l' and $k-l+2$, respectively.

We translate the sets S_1 and S_2 so that (see Fig. 1)
(i) any point of S_2 has a smaller x-coordinate than any point of S_1,
(ii) S_2 lies entirely above any line containing a pair of points of S_1,
(iii) S_1 lies entirely below any line containing a pair of points of S_2.

Now the set $S = S_1 \cup S_2$ contains no convex k-set, no convex l-sequence, and no concave l'-sequence. The size of S is equal to $m(k,l-1,l') + m(k,l,k-l+2)$ which, according to the induction hypothesis, is at least

$$\sum_{i=k-l'}^{l-3} \binom{k-2}{i} + \sum_{i=l-2}^{l-2} \binom{k-2}{i} = \sum_{i=k-l'}^{l-2} \binom{k-2}{i}.$$

$\square$

In the light of Claim 17 we formulate the following version of Conjecture A.

Conjecture A^***:** The bound in Claim 17 is exact, i.e.,

$$m(k,l,l') = \sum_{i=k-l'}^{l-2} \binom{k-2}{i}.$$

Let us show that Conjectures A and A^* are equivalent. If Conjecture A^* is true then in particular $m(k,k,k) = \sum_{i=0}^{k-2} \binom{k-2}{i} = 2^{k-2}$ which implies Conjecture A.

Suppose now that Conjecture A^* is false. Thus $m(k,l,l') > \sum_{i=k-l'}^{l-2} \binom{k-2}{i}$, for some $k,\ l,\ l',\ k \geq l \geq 2,\ k \geq l' \geq 2,\ k \leq l+l'-2$. Using the technique of the proof of Claim 17, one can derive that

$$m(k,l+1,l') \geq m(k,l,l') + m(k,l+1,k-l+1) > \sum_{i=k-l'}^{l-2} \binom{k-2}{i} + \sum_{i=l-1}^{l-1} \binom{k-2}{i} =$$

$$\sum_{i=k-l'}^{l-1} \binom{k-2}{i},$$

$$m(k,l+2,l') \geq m(k,l+1,l') + m(k,l+2,k-l) > \sum_{i=k-l'}^{l} \binom{k-2}{i},$$

$$\ldots\ldots,$$

$$m(k,k,l') \geq m(k,k-1,l') + m(k,k,2) > \sum_{i=k-l'}^{k-2} \binom{k-2}{i},$$

$$m(k,k,l'+1) \geq m(k,k,l') + m(k,k-l'+1,l'+1) > \sum_{i=k-l'-1}^{k-2} \binom{k-2}{i},$$

$$\ldots\ldots,$$

$$m(k,k,k) \geq m(k,k,k-1) + m(k,2,k) > \sum_{i=0}^{k-2} \binom{k-2}{i} = 2^{k-2}.$$

Then $r(k) = m(k,k,k)+1 > 2^{k-2}+1$ and Conjecture A is false. Thus Conjectures A and A^* are equivalent.

Here is a theorem about the Ramsey-remainder.

Theorem 18 *For any $k \geq 2$, $rr(k) = m(k,k-2,k)$. In other words, for any integer $k \geq 2$ there is an integer n such that any set $S \in \mathbf{S}$ of at least n points can be partitioned into convex sets of size $\geq k$ and a set of size $\leq m(k,k-2,k)$, and this upper bound is best possible.*

If Conjecture A (or Conjecture A^*) is true then $m(k,k-2,k) = \sum_{i=0}^{k-4} \binom{k-2}{i} = 2^{k-2} - k + 1$. It would mean that the difference between the Ramsey number and the Ramsey-remainder $r(k) - rr(k) = k = \Theta(\log r(k))$ is considerably smaller than in all the other cases described in Section 2.

Before the proof of Theorem 18 we prove a lemma. In the sequel, by *an unbounded region* of the plane we mean a region which doesn't lie entirely between any pair of parallel lines.

Lemma 19 *Let S be a finite set of points in general position in the plane. Let $\mathcal{A}$ be the set of the $\binom{|S|}{2}$ lines containing the pairs of points of S. Let T be a finite set of points lying inside a region R of $\mathcal{A}$ and out of the convex hull of S. Then there is a set $S' \cup T'$ combinatorially equivalent to $S \cup T$ (S' is mapped onto S and T' onto T in the equivalence) such that the points of T' lie in an unbounded region of $\mathcal{A}'$, where $\mathcal{A}'$ is the arrangement of the $\binom{|S'|}{2} = \binom{|S|}{2}$ lines containing the pairs of points of S'.*

Proof. Let S and T be two point sets in the plane $P \subset \mathrm{E}^3$ satisfying the assumptions of the lemma. Find three parallel lines p, q, r satisfying the following five conditions (see Fig. 2):

(i) the lines p, q, r contain no point of $S \cup T$,

(ii) r separates S and T,

(iii) all points of T lie between q and r,

(iv) q intersects the interior of the region R,

(v) q lies between p and r.

In the sequel, we transform the plane in the way that is usually referred to as "to send the line q to infinity". Let ϱ be the plane orthogonal to P such that $P \cap \varrho = p$. Let C be a point of $\mathrm{E}^3 - P$ whose orthogonal projection to P lies on q. Define a projection $\pi : H(q,r) \mapsto \rho$, where $H(q,r)$ is the open halfplane of P with boundary line q and containing r, as follows. For a point $A \in H(q,r)$, the point $\pi(A)$ is defined as the intersection of the line AC with the plane ϱ.

We show that π projects the set $S \cup T$ onto a set $\pi(S \cup T) = S' \cup T'$ with the required properties. A point $A \in H(q,r)$ lies in the convex hull of points

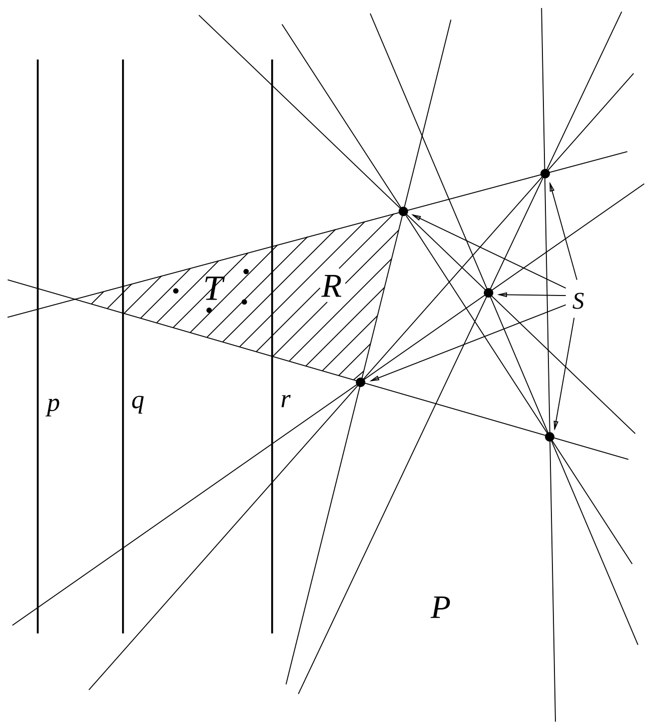

Figure 2: The lines $p,\ q,\ r$ in the plane P

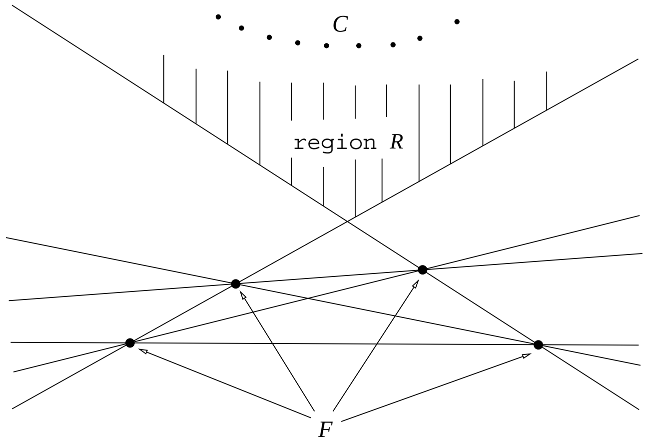

Figure 3: The set $S = F \cup C$

$A_1, A_2, \dots, A_n \in H(q,r)$ if and only if the point $\pi(A) \in \mathbf{S}$ lies in the convex hull of the points $\pi(A_1), \pi(A_2), \dots, \pi(A_n) \in \varrho$. Therefore $S' \cup T'$ is combinatorially equivalent to $S \cup T$. The region $R \cap H(q,r)$ is projected onto a region $R' = \pi(R \cap H(q,r))$ of $\mathcal{A}'$ containing T'. The region R' is unbounded, however, because the placement of images of points close to the line q in the projection π is unbounded. $\square$

Proof of Theorem 18.
Lower bound. Let $r = m(k, k-2, k)$. So there exists a set $F \in \mathbf{S}^*$ of size r that contains no convex k-set and no convex $(k-2)$-sequence. Consider the arrangement $\mathcal{A}$ of the $\binom{r}{2}$ lines connecting the pairs of points of F. Let R be the region of $\mathcal{A}$ containing the "point" $[0, +\infty]$. So R is the unbounded region of the arrangement $\mathcal{A}$ that lies in the direction of the y-axis above the set F. For any $n \geq r$, we consider a set S of n points which is a union of the set F and a set C of $n-r$ points which lie in R and form a convex sequence. Moreover, we assume that the set F lies entirely below every line containing two points of C (see Fig. 3).

Now we show that the Ramsey-remainder of $S = F \cup C$ is at least $|F| = r$. Let K be a convex subset of S such that $K \cap F \neq \emptyset$. If $K \cap C = \emptyset$ then $|K| < k$ since F contains no convex k-set. If $K \cap C \neq \emptyset$ then $K \cap F$ and $K \cap C$ form a convex and a concave sequence, respectively. It gives $|K| = |K \cap F| + |K \cap C| \leq (k-3) + 2 < k$. So no convex subset of S of size $\geq k$ contains any point of F. Therefore $rr(k) \geq rr(S,k) \geq |F| = r$.

Upper bound. Denote $s = r(k)$. Let $S \in \mathbf{S}$ be a set of at least $r(k \cdot (s^4 + s))$ points in the plane. So S contains a convex set C of size $k \cdot (s^4 + s)$. We take an arbitrary maximum partial partition of the set $S - C$ into convex $k-$sets. Let $S' \subseteq S - C$ be the set of the remaining points. S' contains no convex k-set; thus $|S'| \leq s - 1$. Now we take an arbitrary maximum partial partition of the set $S' \cup C$ into convex k-sets containing at least one point of S'. Let $S'' \cup C'$ be the set of the remaining points, where $S'' \subseteq S'$ and $C' \subseteq C$. Observe that $|C'| \geq |C| - (k-1)|S'| \geq$

$k \cdot (s^4 + s) - (k-1)(s-1) > k \cdot s^4$ and note that the set $S - S''$ can be partitioned into convex sets of size $\geq k$. We will show that $|S''| \leq m(k, k-2, k)$ which implies $rr(\mathbf{S}, k) \leq m(k, k-2, k)$.

Consider the arrangement $\mathcal{A}$ of the $\binom{|S''|}{2}$ lines containing the pairs of points of S''. It partitions the plane into $\binom{\binom{|S''|}{2}+1}{2} + 1 < \frac{s^4}{8}$ regions. According to the pigeonhole principle, there is a set C'' of $8k$ points of C' lying in one region R of $\mathcal{A}$. Further we will consider only the set $S'' \cup C''$.

Suppose first that the region R lies inside the convex hull $conv(S'')$. Since R is a region of $\mathcal{A}$, it is contained in some triangle T with vertices in S''. Since $|C''| = 8k > 3(k-3)$, some vertex of T and some $k-1$ (consecutive) points of C'' form a convex k-set, a contradiction with the maximality of the partial partition on $S' \cup C$. Thus the region R lies outside $conv(S'')$, and according to Lemma 19 we can assume that the region R is unbounded.

Rotate the plane so that $S'' \cup C'' \in \mathbf{S}^*$ and that the region R contains the "point" $[0, +\infty]$. After such a rotation the set S'' contains neither a convex k-set nor a convex $(k-2)$-sequence (otherwise there would be a convex subset of $C'' \cup S''$ of size k containing two points of S''). Therefore $|S''| \leq m(k, k-2, k)$. $\square$

4 Proofs of Ramsey-remainder theorems

In this section we prove Theorems 3, 5, 9, 11, 12, and 14. Theorem 7 has already been proven in [CT 88] (Theorem 4.6).

Proof of Theorem 3.

Lower bound. Let $r = r(k, k-1) - 1$. We will show that $rr(k) \geq r$. Since $r < r(k, k-1)$, there exists a two-coloring of the complete graph $K_r = (V_r, E_r)$ on r vertices such that V_r contains no blue k-subset and no red $(k-1)$-subset. We will extend the coloring of the graph K_r into an arbitrary large complete graph so that no monochromatic k-set will contain any vertices of K_r.

For any $n \geq r$ we define a 2-coloring of the complete graph $K_n = (V, E)$ on n vertices as follows. Let $V = V_r \cup V_{n-r}$ be a partition of the vertex set V into two sets of size r and $n - r$, respectively. We let V_r to coincide with the vertex set of the above graph K_r and color the edges of K_n with both vertices in V_r as in K_r. The edges of K_n with both vertices in V_{n-r} will be colored blue, and all edges between V_r and V_{n-r} will be colored red (see Fig. 4).

Now let M be any monochromatic subset of V such that $M \cap V_r \neq \emptyset$. If M is blue then $M \subseteq V_r$; if M is red then $|M \cap V_r| \leq k-2$ and $|M \cap V_{n-r}| \leq 1$. In both cases M contains at most $k-1$ vertices. It follows that no (partial) partition of V into monochromatic sets of size $\geq k$ can use any vertex of V_r. Therefore

$$rr(k) \geq \lim_{n\to\infty} rr(K_n, k) \geq \lim_{n\to\infty} |V_r| = r.$$

Upper bound. Let $K_n = (V, E)$ be a 2-colored complete graph on n vertices, where $n \geq r((k-1)(r(k) + k - 3) + 1)$. We will show that the vertex set V can be

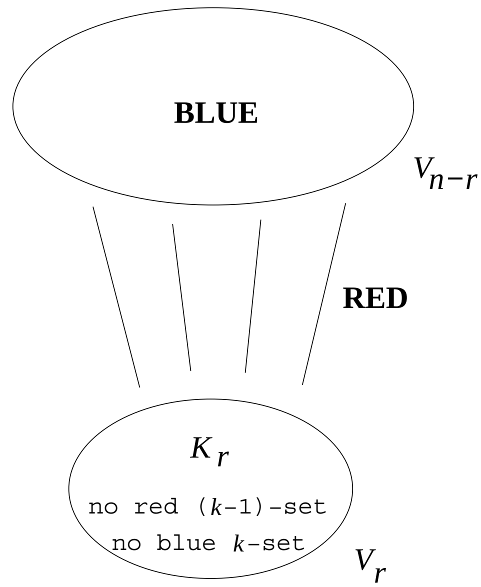

Figure 4: 2-colored complete graph with a large Ramsey-remainder

partitioned into monochromatic subsets of size $\geq k$ and a subset of less than $r(k, k-1)$ vertices. Such a partition immediately gives the upper bound of Theorem 3.

Since $n \geq r((k-1)(r(k)+k-3)+1)$, the set V contains a monochromatic, say blue, subset B of size $(k-1)(r(k)+k-3)+1$. Take an arbitrary maximum partial partition of the set $V-B$ into monochromatic k-subsets. Let $C \subset V-B$ be the set of remaining vertices. So C contains no monochromatic k-subset and $|C| < r(k)$. Then take an arbitrary maximum partial partition of the set $B \cup C$ into monochromatic k-sets, each containing at least one vertex of C. The set of remaining vertices will be some set $B' \cup C'$, where $B' \subseteq B$ and $C' \subseteq C$. Note that the set $V - B' \cup C'$ is already partitioned into monochromatic k-sets. Obviously, $|B'| \geq |B| - (k-1)|C| \geq (k-1)(r(k)+k-3)+1-(k-1)(r(k)-1) = (k-1)(k-2)+1$.

We will prove that C' contains no red $(k-1)$-subset. Suppose on the contrary that K is a red $(k-1)$-set in C'. The maximality of the partial partition on $B \cup C$ implies that each vertex of C' is connected by a blue edge with at most $k-2$ vertices of B'. So there are at most $(k-1)(k-2)$ blue edges between K and B'. The above inequality $|B'| \geq (k-1)(k-2)+1$ implies now that some vertex v of B' is connected with all $k-1$ vertices of K by a red edge. Thus $K \cup \{v\}$ is a red k-set containing some points of C, a contradiction.

So C' contains no red $(k-1)$-subset and no blue k-subset. Thus $|C'| < r(k, k-1)$. Adding the monochromatic (blue) set B' to the above partition of $V - B' \cup C'$ into monochromatic k-sets we obtain a required partition of V with the remaining set C' of size less than $r(k, k-1)$. □

Similarities to the above proof of Theorem 3 allow us to omit some details in the

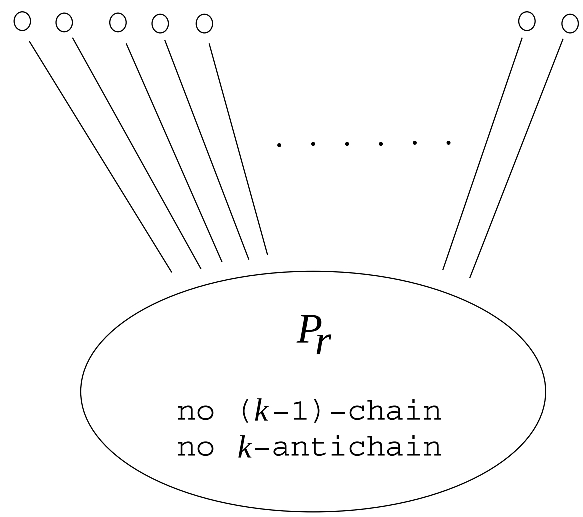

Figure 5: Poset with a large Ramsey-remainder

following proofs.

Proof of Theorem 5.

Lower bound. Let $r = r(k, k-1) - 1 = (k-1)(k-2)$ and let P_r be a poset with r elements containing no chain of $k-1$ elements and no antichain of k elements.

Let $n \geq r$. Add $n - r$ pairwise incomparable elements to the poset P_r, each of them greater than any element of P_r (see Fig 5). In this way we obtain a poset P of size n with Ramsey-remainder $rr(P, k) \geq |P_r| = r$. It implies $rr(k) \geq r$.

Upper bound. Let P_n be a poset of size n, where $n \geq r((k-1)(r(k)+k-3)+1)$. We color a complete graph $G = (V, E)$, where V is the set of elements of P_n, as follows. An edge will be colored red if it connects two elements comparable in P_n. Otherwise an edge is colored blue. Chains now correspond to the red sets, antichains correspond to the blue sets. We can show in the way as in the proof of Theorem 3 that the set V can be partitioned into monochromatic subsets of size $\geq k$ and a subset of fewer than $r(k, k-1)$ vertices. Such a partition immediately gives the upper bound of Theorem 5. $\square$

Proof of Theorem 9.

Lower bound. Let $r = r(k, k-1) - 1 = (k-1)(k-2)$ and let S_r be a sequence of r real numbers containing no increasing subsequence of length $k-1$ and no decreasing subsequence of length k.

For any $n \geq r$, concatenate the sequence S_r with a decreasing sequence of $n-r$ real numbers each of them greater than any element of S_r (see Fig. 6). In such a way we obtain a sequence S of length n with Ramsey-remainder $rr(S, k) \geq |S_r| = r$. It implies $rr(k) \geq r$.

Upper bound. Let S_n be a sequence of n distinct real numbers, where $n \geq$

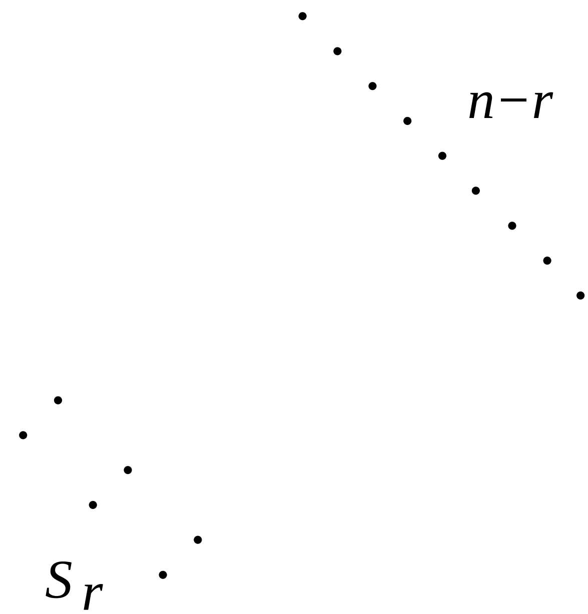

Figure 6: Sequence of reals with a large Ramsey-remainder (for $k = 4$)

$r((k-1)(r(k)+k-3)+1)$. We color a complete graph $G = (V, E)$, where V is the set of elements of S_n, as follows. An edge will be colored red if it connects two elements forming an increasing subsequence of S_n. Otherwise an edge is colored blue. Monotone sequences in S_n now correspond to monochromatic subsets of V. We can show in the way as in the proof of Theorem 3 that the set V can be partitioned into monochromatic subsets of size $\geq k$ and a subset of fewer than $r(k, k-1)$ vertices. Such a partition immediately gives the upper bound of Theorem 9. $\square$

Proof of Theorem 11.

Lower bound. Let $r = r(k, k-1) - 1 = (k-1)(k-2)$ and let S_r be a sequence of r natural numbers containing no subsequence of length $k-1$ in which every number is divisible by all predecessors and no subsequence of length k in which no number is divisible by any other.

For any $n \geq r$, concatenate the sequence S_r with an increasing sequence of $n - r$ natural numbers which are divisible by a common multiple of the numbers of S_r but are not divisible by each other. In such a way we obtain a sequence S of length n with Ramsey-remainder $rr(S, k) \geq |S_r| = r$. It implies $rr(k) \geq r$.

Upper bound. Let S_n be an increasing sequence of n natural numbers, where $n \geq r((k-1)(r(k)+k-3)+1)$. We color the complete graph $G = (V, E)$, where V is the set of elements of S_n, as follows. An edge will be colored red if it connects a number with its multiple. Otherwise an edge is colored blue. Regular sequences in S_n now correspond to monochromatic subsets of V. We can show in the way as in the proof of Theorem 3 that the set V can be partitioned into monochromatic subsets of size $\geq k$ and a subset of fewer than $r(k, k-1)$ vertices. Such a partition immediately gives the upper bound of Theorem 11. $\square$

Proof of Theorem 12.

Lower bound. Let $r = r(k, t-1) - 1$ and let the complete graph K_r be colored by red and blue colors so that there is no red vertex set of size $t-1$ and no blue vertex set of size k. Let G_r be the graph on the same vertex set as K_r containing just those edges which are red in K_r. Then G_r is a K_{t-1}-free graph on r vertices containing no independent set of k vertices.

Let $n \geq r$. Add $n - r$ vertices to the graph G_r and join each of them by an edge just with the r original vertices of G_r. In such a way we obtain a graph G of size n with Ramsey-remainder $rr(G, k) \geq |G_r| = r$. It implies $rr(k) \geq r$.

Upper bound. Let $G_n = (V, E)$ be a K_t-free graph on n vertices, where $n \geq r((k-1)(r(k,t) + t - 2))$. We will show that the vertex set V can be partitioned into independent subsets of size $\geq k$ and a subset of fewer than $r(k, t-1)$ vertices. Such a partition immediately gives the upper bound of Theorem 12.

Let K be a complete graph on the same vertex set V as G_n. An edge of K will be colored red if it is an edge of G_n; otherwise an edge will be colored blue. Sets of vertices of G_n inducing a complete graph now correspond to the red sets, and independent sets correspond to the blue sets. Since $n \geq r((k-1)(r(k,t) + t - 2))$, the set V contains a monochromatic subset B of size $(k-1)(r(k,t)+t-2)$. Assume $k \geq 3$ and $t \geq 3$ (otherwise Theorem 12 is trivially true). Then $|B| \geq t$ and B must be blue. Take an arbitrary maximum partial partition of the set $V - B$ into blue k-sets. Let $C \subset V - B$ be the set of remaining vertices. So C contains no blue k-subset and $|C| < r(k,t)$. Then take an arbitrary maximum partial partition of the set $B \cup C$ into monochromatic k-sets, each containing at least one vertex of C. The set of remaining vertices will be some set $B' \cup C'$, where $B' \subseteq B$ and $C' \subseteq C$. Note that the set $V - (B' \cup C')$ is already partitioned into blue k-sets. Obviously, $|B'| \geq |B| - (k-1)|C| \geq (k-1)(r(k,t)+t-2) - (k-1)(r(k,t)-1) = (k-1)(t-1) \geq (k-2)(t-1) + 1$.

We will prove that C' contains no red $(t-1)$-subset. Suppose on the contrary that K is a red $(t-1)$-set in C'. The maximality of the partial partition on $B \cup C$ implies that each vertex of C' is connected by a blue edge with at most $k-2$ vertices of B'. So there are at most $(t-1)(k-2)$ blue edges between K and B'. The above inequality $|B'| \geq (k-2)(t-1)+1$ implies now that some vertex v of B' is connected with all vertices of K by a red edge. Thus, $K \cup \{v\}$ is a red t-set, a contradiction.

So C' contains no red $(t-1)$-subset and no blue k-subset. Thus, $|C'| < r(k, t-1)$. Adding the blue set B' to the above partition of $V - (B' \cup C')$ we obtain a required partial partition of V with the remaining set C' of size less than $r(k, t-1)$. $\square$

Proof of Theorem 14. (sketch)

Lower bound. Let $r = r_{c,u}(k) - 1$ and let V_r be an r-element set in which the subsets of size $\leq u$ are colored by c colors $0, 1, \ldots, c-1$ so that the properties (i)-(iii) described before Theorem 13 are satisfied.

Let $n \geq r$. Take an n-element set $V = V_r \cup V_{n-r}$, where $|V_{n-r}| = n - r$, and color each u-element subset of V (an edge of the complete u-uniform hypergraph H on the vertex set V) by the color of the set $V \cap V_r$ in the above coloring. It follows from the properties (i)-(iii) that every monochromatic (regular) subset of V of size $\geq k$ is disjoint from V_r. Thus, $rr(H, k) \geq |V_r| = r$ and $rr(k) \geq r$.

Upper bound. Let $H_n = (V, E)$ be a complete u-uniform hypergraph on n vertices, where n is large with respect to k. We will show that the vertex set V can be partitioned into monochromatic subsets of size $\geq k$ and a subset of less than $r_{c,u}(k)$ vertices. Such a partition immediately gives the upper bound of Theorem 14.

Since n is "large", the set V contains a "relatively large" monochromatic subset B. Without loss of generality, let the edges with all vertices in B be colored by color 0. Take an arbitrary maximum partial partition of the set $V - B$ into monochromatic k-subsets. Let $C \subseteq V - B$ be the set of remaining vertices. So C contains no monochromatic k-subset and $|C| < r(k)$, where $r(k)$ is the Ramsey number corresponding to Theorem 13. Then take an arbitrary maximum partial partition of the set $B \cup C$ into monochromatic k-sets, each containing at least one vertex of C. The set of remaining vertices will be some set $B' \cup C'$, where $B' \subseteq B$ and $C' \subseteq C$. Note that the set $V - (B' \cup C')$ is already partitioned into monochromatic k-sets. The partial partition of $B \cup C$ has contained at most $|C| < r(k)$ monochromatic k-sets. Thus B' is still "relatively large" because $|B'| \geq |B| - (k-1)|C| > |B| - k \cdot r(k)$.

Using standard Ramsey-type techniques one can show that the "relatively large" set B' contains a subset B'' of size k whose vertices are undistinguishable in relation to the set C'. It means that the color of an edge $e \in E$ such that $e \subseteq B'' \cup C'$ depends only on the intersection $e \cap C'$ (i.e., if two edges $e, e' \in E$ are subsets of $B'' \cup C'$ such that $e \cap C' = e' \cap C'$ then e and e' have the same color). Now we can color any subset S of C' of size $\leq u$ by the color of the edges $e \in E, e \cap C' = S$. Such a coloring satisfies the above properties (i)-(iii), therefore $|C'| < r_{c,u}(k)$. It gives the required upper bound because the set $V - C' = (V - (B' \cup C')) \cup B'$ can be partitioned into monochromatic sets of size $\geq k$. $\square$

5 Concluding remarks

1. We studied the numbers $rr(k) = \lim_{n\to\infty} g_k(n)$, where $g_k(n) = \max\{rr(S, k) : S \in \mathbf{S}, |S| = n\}$. We didn't investigate from which n the number $g_k(n)$ starts to be identically equal to $rr(k)$. Our proof gives $g_k(n) = rr(k)$ only for enormously large numbers n. Caro and the second author [CT 88] solved this question for the class of sub-t-colored complete graphs (see Theorem 7). They showed in this case $g_k(n) = rr(k) = 0$ for $n \geq n(k)$, where $n(k) = \Theta(k^3)$, which is best possible.

2. We didn't study partitions into substructures of size *exactly* k either. For two positive integers n and k, denote $s(n, k) = n - rr(k) (\text{mod } k) \in \{0, 1, \ldots, k-1\}$. The following two claims follow from the proofs of Theorems 3, 5, 9, 11, 12, 14, and 18.

Claim 20 *Let* $\mathbf{S}$ *be any class described in subsections A.-H. of Section 2. Then, for any sufficiently large* n*, any structure (set)* $S \in \mathbf{S}$ *can be partitioned into regular substructures (subsets) of size* k *and a remaining set of size* $rr(k) + s(n,k)$.

Claim 21 *Let* $\mathbf{S}$ *be any class described in subsections A.-H. of Section 2. Then, for any sufficiently large* n*, any structure (set)* $S \in \mathbf{S}$ *can be partitioned into regular substructures (subsets) of size* k*,* $s(n,k)$ *regular substructures (subsets) of size* $k+1$*, and a remaining set of size* $rr(k)$.

3. The value of the Ramsey-remainder is not known to us for the class $\mathbf{S}^*$ defined in Section 3, where the regular structures are the point sets forming a convex or a concave sequence. We can only show that in this case $rr(k) \geq r(k,k-1) - 1 + r(k-1,k-1) - 1 = \binom{2k-5}{k-3} + \binom{2k-6}{k-3}$. Let us recall from [ES 35] that the corresponding Ramsey numbers are known, namely $r(k,l) = \binom{k+l-4}{k-2} + 1$. In particular, $r(k) = r(k,k) = \binom{2k-4}{k-2} + 1$.

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References

[CT 88] Y. Caro and Zs. Tuza, Decompositions of partially ordered sets into chains and antichains of given size, Order 5 (1988), 245-255.

[D 50] R. P. Dilworth, A decomposition theorem for partially ordered sets, Ann. Math. 51 (1950), 161-166.

[E 47] P. Erdős, Some remarks on the theory of graphs, Bull. Am. Math. Soc. 53 (1947), 292-294.

[ES 35] P. Erdős and G. Szekeres, A combinatorial problem in geometry, Compositio Math. 2 (1935), 463-470.

[ES 60] P. Erdős and G. Szekeres, On some extremum problems in elementary geometry, Ann. Univ. Sci. Budapest 3/4 (1960/61), 53-62.

[NR 90] J. Nešetřil and V. Rödl (eds.), Mathematics of Ramsey theory, Springer-Verlag (1990).

[R 30] F. P. Ramsey, On a problem of formal logic, Proc. Lond. Math. Soc., II. Ser. 30 (1930), 264-286.