PROBLEMS ON APPROXIMATION BY TRIANGLES

by *Peter Braß* * and *Marek Lassak* **
Institut für Informatik, FU Berlin
Takustraße 9, D-14195 Berlin, Germany
brass@inf.fu-berlin.de, lassak@atr.bydgoszcz.pl

I. Introduction

Approximation of planar convex sets by polygons is an old and well-studied subject, started perhaps by Blaschke [5] in 1917. There have been many results, especially on the asymptotic behaviour of approximation by k-gons (as $k \to \infty$), see e.g. the survey by Gruber [19], and recently also on the algorithmic construction of approximating polygons, see e.g. the survey of Alt and Guibas [1]. But even for the simplest case, approximation by triangles, there are still many open problems, which we attempt to survey in this paper. We will be interested in the approximation quality that can be guaranteed in some setting, and the worst approximable sets. We wish to approximate a convex bounded set with nonempty interior (in the following denoted by C) by either a triangle ▷, or a right-angled triangle ◺, or an isosceles triangle △, with respect to the symmetric-difference metric and some other metrics. Important variations to the approximation situation are inner approximation, outer approximation, (Figure 1) and twosided approximation by homothetic pairs (Section IV).

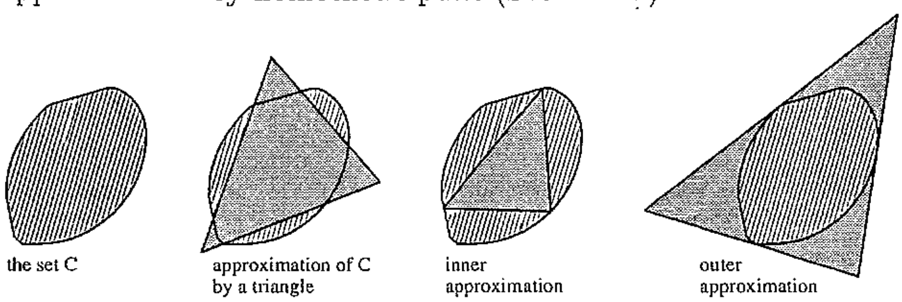

Figure 1

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II. Symmetric-difference Approximation

The symmetric-difference metric is given by

$$d^{\substack{\text{symm}\\ \text{diff}}}(X,Y) := \text{area}(X \setminus Y) + \text{area}(Y \setminus X)\,.$$

For each C we will compare the error of the optimal approximation by a triangle $\inf_{\triangleright} d^{\substack{\text{symm}\\ \text{diff}}}(C,\triangleright)$ to $\text{area}(C)$ in order to avoid this error becoming infinite just by scaling of C, and thus to make the question for the worst-approximable set meaningful.

$d^{\substack{\text{symm}\\ \text{diff}}}$ is the best-studied distance measure in our approximation context, probably because it has applications in packing- and covering-problems, and also because it is in some sense particularly well-behaved, e.g. the sequence of optimal approximation errors for approximation of C by k-gons is for each C a convex sequence (Dowker-type theorems [9,11,12]).

These applications exist mostly for inner and outer approximations, i.e. for inscribed and circumscribed polygons. Thus there has been less work on the most natural question of classical approximation. The key property here is the necessary balance condition for best-approximating polygons: each side of the approximating polygon must have equal lengths inside and outside C. But even the case of the best approximation by a triangle is still open:

Conjecture: For each C there is a triangle $\triangleright$ such that

$$\begin{aligned} d^{\substack{\text{symm}\\ \text{diff}}}(C,\triangleright) &\leq \left(6\arctan\tfrac{\sqrt{3}}{2} - \pi\right)\text{area}(C) \\ &\approx 0.36311\,\text{area}(C)\,. \end{aligned}$$

Ellipses are the only extremal sets.

The corresponding result for inner approximation is already a quite old result:

Theorem: For each C there is a triangle $\triangleright \subset C$ such that

$$d^{\substack{\text{symm}\\ \text{diff}}}(C,\triangleright) \leq \left(1 - \tfrac{3\sqrt{3}}{4\pi}\right)\text{area}(C) \approx 0.586\,\text{area}(C)\,.$$

Ellipses are the only extremal sets.

This was found by Blaschke for triangles and by Sas [30] for k-gons (there is always an inscribed k-gon that covers at least the same portion of the area as a regular k-gon inscribed in a circle). If we prescribe also the direction of a side of the approximating triangle, there is still a $\triangleright$ with $d_{\mathrm{diff}}^{\mathrm{symm}}(C, \triangleright) \leq \frac{5}{8}\,\mathrm{area}(C)$ ([21], extremal case a regular hexagon with the direction parallel to one edge). Another interesting special case is the approximation of k-gons P_k (with small k) by triangles, which was solved for ≤ 6 [13]: the extremal 4- and 5-gons are regular, but the extremal 6-gon is not. This suggests

Problem: What are the smallest numbers c_k such that for each k-gon P_k there is a triangle $\triangleright$ such that $\triangleright \subset P_k$ and

$$d_{\mathrm{diff}}^{\mathrm{symm}}(P_k, \triangleright) \leq c_k\,\mathrm{area}(C)\,,$$

and what are the extremal polygons?

The known values are $c_4 = \frac{1}{2}$, $c_5 = 1 - \frac{1}{\sqrt{5}}$ and $c_6 = \frac{5}{9}$. This problem may be difficult since the similar problems of the minimum diameter of a k-gon with a given area or perimeter are also unsolved for some k (area: [29,16], open for even $k \geq 8$, perimeter: [29] open for $k = 2^l$, frequently rediscovered [32,24,7]) for which they also lead to 'exotic' nonregular k-gons.

In both following conjectures on inner approximation again the circle is believed to be extremal (Figure 2):

Conjecture: For each C there is a right-angled triangle $\lrtriangle \subset C$ with

$$d_{\mathrm{diff}}^{\mathrm{symm}}(C, \lrtriangle) \leq \left(1 - \tfrac{1}{\pi}\right)\mathrm{area}(C)\,.$$

Conjecture: For each C there is an isosceles triangle $\triangle \subset C$ with

$$d_{\mathrm{diff}}^{\mathrm{symm}}(C, \triangle) \leq \left(1 - \tfrac{3\sqrt{3}}{4\pi}\right)\mathrm{area}(C)\,.$$

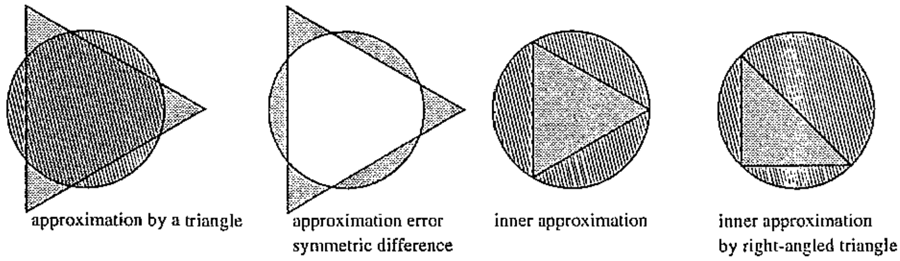


Figure 2

For outer approximation the optimal bound is again an old result [10]:

Theorem: For each C there is a triangle $\triangleright$ such that $C \subset \triangleright$ and

$$d_{\text{diff}}^{\text{symm}}(C, \triangleright) \leq 2\,\text{area}(C)\,.$$

Here parallelograms are extremal (Figure 3), and the same bound holds even if we prescribe the direction of a side of the approximating triangle [10], but here the corresponding result for outer approximation by k-gons is not known.

Problem: What is the smallest number $c_{\lrtriangle}$ such that for each C there is a right-angled triangle $\lrtriangle$ with $C \subset \lrtriangle$ and $d_{\text{diff}}^{\text{symm}}(C, \lrtriangle) \leq c_{\lrtriangle}\,\text{area}(C)$?

Problem: What is the smallest number $c_{\triangle}$ such that for each C there is an isosceles triangle $\triangle$ with $C \subset \triangle$ and $d_{\text{diff}}^{\text{symm}}(C, \triangle) \leq c_{\triangle}\,\text{area}(C)$?

For both constants there is the same upper bound $c_{\lrtriangle}, c_{\triangle} \leq \frac{49}{16} \approx 3.06$ [27], but there is no nontrivial lower bound or conjecture on the extremal sets C.

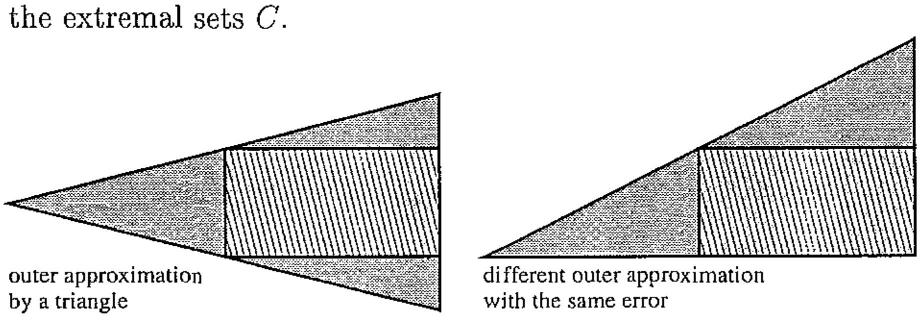


Figure 3

III. Hausdorff Approximation

The Hausdorff metric is defined by

$$\overset{\text{Haus-}}{d^{\text{dorff}}}(X,Y) := \max\Big(\sup_{x\in X}\inf_{y\in Y} d(x,y), \sup_{y\in Y}\inf_{x\in X} d(x,y)\Big).$$

In algorithmic applications it is more important than $\overset{\text{symm}}{d^{\text{diff}}}$, since it is defined for a much larger class of sets (for all compact subsets of a metric space, so e.g. also for discrete point sets, for which $\overset{\text{symm}}{d^{\text{diff}}}$ does not make sense), and most real-world shapes that need to be compared and approximated are not convex discs. But error bounds for $\overset{\text{Haus-}}{d^{\text{dorff}}}$-approximation by k-gons are much harder, and many nice properties of $\overset{\text{symm}}{d^{\text{diff}}}$ do not hold for $\overset{\text{Haus-}}{d^{\text{dorff}}}$ (e.g. there are no Dowker-type theorems).

Again there is a necessary balance condition for the optimal Hausdorff approximation by k-gons: each vertex is at the same distance ε (approximation error) outside C, and each side 'goes ε-deep into C', i.e. if B_ε is the disk of radius ε then each vertex lies on the boundary of the Minkowski sum $C + B_\varepsilon$, and for each edge there is an extreme point (vertex) of C that has its nearest point of the approximating polygon on that edge, in distance ε.

Conjecture: For each C there is a triangle $\triangleright$ such that

$$\overset{\text{Haus-}}{d^{\text{dorff}}}(C, \triangleright) \leq \tfrac{1}{16}\,\mathrm{peri}(C).$$

This is a special case of a conjecture of Popov [28] who conjectured that the worst approximable set for Hausdorff approximation by k-gons is a regular k+1-gon. Here it is even nontrivial to find out the correct multiplicative constant. This was done by Georgiev [15], who proved that among all k+1-gons the worst approximable is the regular k+1-gon. Thus the general conjecture is that there is a k-gon P_k with $\overset{\text{Haus-}}{d^{\text{dorff}}}(P_k, C) \leq \frac{1}{4(k+1)} \tan\frac{\pi}{k+1}\,\mathrm{peri}(C)$. Popov obtained an upper bound of $\frac{1}{18}\sqrt{3}$ for triangles, and $\frac{\sin(\pi/k)}{2k(1+\cos(\pi/k))}$ for k-gons.

Conjecture: For each C there is a triangle $\triangleright \subset C$ with

$$d^{\text{Haus-}}_{\text{dorff}}(C, \triangleright) \leq \alpha\, \mathrm{peri}(C) \approx 0.0968\, \mathrm{peri}(C)\,,$$

where α is a solution of $\frac{4\alpha}{1-4\alpha} = \sin\arctan\frac{1}{2-8\alpha}$, or equivalently $2^{10}\alpha^4 - 2^9\alpha^3 + 2^6\alpha^2 + 2^3\alpha = 1$.

Here we have the same situation: Popov conjectured the extremality of the regular $k{+}1$-gon for approximation by k-gons and proved an upper bound of $\frac{1}{12}\sqrt{3}$ for triangles, $\frac{\sin(\pi/k)}{2k}$ for k-gons [28]; and Ivanov [22] showed that among the $k{+}1$-gons the regular $k{+}1$-gon is indeed worst approximable.

Conjecture: For each C there is a triangle $\triangleright$ such that $C \subset \triangleright$ and

$$d^{\text{Haus-}}_{\text{dorff}}(C, \triangleright) \leq \tfrac{1}{8}\sqrt{2}\, \mathrm{peri}(C) \approx 0.1764\, \mathrm{peri}(C)\,.$$

Here Popov [28] obtained the upper bound $\frac{\sqrt{3}}{6}$ (for approximation by k-gons $\frac{\tan(\pi/k)}{2k}$) and conjectured again that the regular $k{+}1$-gon is worst approximable by k-gons.

A similar question was also treated in the L_1-metric instead of the euclidean, where the square indeed turns out to be extremal for L_1 Hausdorff approximation by triangles [23].

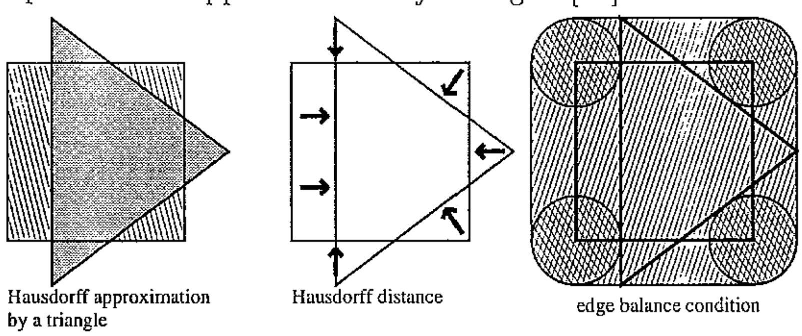


Figure 4

A further variant that is interesting for Hausdorff approximation is approximation by subpolygons, i.e. approximation of polygons by polygons that use only a subset of the vertices. This is a reasonable restriction for algorithmic construction of approximating polygons, since the approximation quality is only slightly worse than inner approximation, and one has to work only on the finite index set instead of finding coordinates of the vertices of the approximating polygon. For symmetric-difference approximation the best inner approximation is always a subpolygon approximation, but this is not true for Hausdorff approximation.

Conjecture: If $\mathcal{P}$ is a class of polygons that is closed under taking of subpolygons, and if there is a polygon that is worst Hausdorff approximable by sub-k-gons in $\mathcal{P}$, then this polygon is a $k{+}1$-gon

This is similar to Popov's conjectures, but does not even restrict to the class of polygons of a given perimeter, but should hold for any class of admissible polygons. The first author has a proof that the worst approximable polygon is always an l-gon with $l \equiv 1 \bmod k$.

IV. Banach-Mazur Related Approximation

The Banach-Mazur distance is a natural similarity measure for two normed spaces of the same dimension. It can be defined by

$$\delta^{\text{Banach}}_{\text{Mazur}}(U, V) := \inf\{\|\phi\|\,\|\phi^{-1}\| \mid \phi\colon U \to V \text{ is a linear bijection}\}.$$

This similarity measure $\delta^{\text{Banach}}_{\text{Mazur}}(U, V)$ is not a metric, but its logarithm $d^{\text{Banach}}_{\text{Mazur}}(U, V) := \ln(\delta^{\text{Banach}}_{\text{Mazur}}(U, V))$ is, and has been much studied in the context of geometric functional analysis. $\delta^{\text{Banach}}_{\text{Mazur}}$ can also be seen as a distance measure of the unit balls of the normed spaces, i.e. of affine equivalence classes of centrally symmetric convex bodies: an alternative definition is

$$\delta^{\text{Banach}}_{\text{Mazur}}(K_1, K_2) := \inf\left\{ \lambda \;\middle|\; \begin{array}{c}\text{there is an affine map } a \text{ and a homo-}\\ \text{thety } h_\lambda \text{ of ratio } \lambda > 0 \text{ such that}\\ a(K_1) \subset K_2 \subset h_\lambda(a(K_1))\end{array} \right\}.$$

In functional analysis this restriction to affine equivalence classes is natural, since they are the isometry classes of the normed spaces; in a geometric context, however, this seems less useful. Here the following related definitions are more natural

$$\delta^{\text{BM-hom}}(K_1, K_2) := \inf\left\{ \frac{\lambda_2}{\lambda_1} \;\middle|\; \begin{array}{c}\text{there are homotheties } h_{\lambda_1} \text{ and } h_{\lambda_2}\\ \text{of ratios } \lambda_1, \lambda_2 > 0 \text{ such that}\\ h_{\lambda_1}(K_1) \subset K_2 \subset h_{\lambda_2}(K_1)\end{array} \right\},$$

$$\delta^{\text{BM-hom}}_{\text{concentric}}(K_1, K_2) := \inf \left\{ \frac{\lambda_2}{\lambda_1} \;\middle|\; \begin{array}{l} \text{there are homotheties } h_{\lambda_1} \text{ and } h_{\lambda_2} \\ \text{of ratios } \lambda_1, \lambda_2 > 0 \text{ such that} \\ \quad h_{\lambda_1}(K_1) \subset K_2 \subset h_{\lambda_2}(K_1) \\ \text{and the centroids of } h_{\lambda_1}(K_1),\ K_2 \\ \text{and } h_{\lambda_2}(K_1) \text{ coincide.} \end{array} \right\} .$$

Both definitions generate distance measures on the homothety equivalence classes of convex sets, which are not metrics, but again their logarithms $\ln \delta^{\text{BM-hom}}$ and $\ln \delta^{\text{BM-hom}}_{\text{concentric}}$ are metrics. Since they are defined on homothety equivalence classes, it is not reasonable to ask for inner and outer approximation like in the previous metrics, and we do not need another size measure (like area(C) or peri(C)) for comparison.

Conjecture: For each C there is a triangle $\triangleright$ such that

$$\delta^{\text{BM-hom}}(C, \triangleright) \leq 1 + \tfrac{1}{2}\sqrt{5} \approx 2.118 \,.$$

Regular pentagons are the only extremal sets.

This conjecture was independently proposed in [13], [2], [26], where upper bounds of 2.25, 2.34 and 2.5, respectively, were shown. In all these proofs the inner triangle $h_{\lambda_1}(\triangleright)$ of this inner and outer approximating homothetic pair is chosen as a maximum area inscribed triangle, but in the conjectured extremal case (regular pentagon) the inner triangle is not a maximum-area triangle.

For centrally symmetric sets C the corresponding problem is long solved: there is always a $\triangleright$ such that $\delta^{\text{BM-hom}}(C, \triangleright) \leq 2$, with parallelograms as extremal sets [20].

Conjecture: For each C there is a triangle $\triangleright$ such that

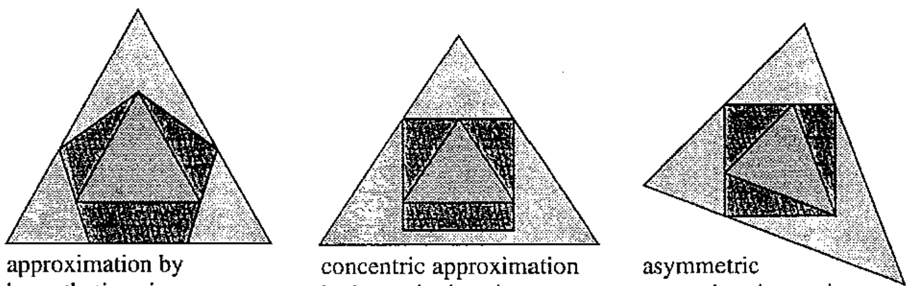
$$\delta^{\text{BM-hom}}_{\text{concentric}}(C, \triangleright) \leq \tfrac{5}{2} \,.$$

Parallelograms are the only extremal sets.

This conjecture was given by Grünbaum [20]. It has been proved for centrally symmetric sets [26] but is still open in the general case.

Another variant is to require only the centroids of the outer and inner homothetic copy to coincide [26]. This does not define a metric like $\delta^{\text{BM-hom}}$ and $\delta^{\overset{\text{BM-hom}}{\text{concentric}}}$, since it is not symmetrical in the two bodies.

Conjecture: For each C there are homothetic triangles $\triangleright_1, \triangleright_2$ such that $\triangleright_1 \subset C \subset \triangleright_2$, the centroids of $\triangleright_1$ and $\triangleright_2$ coincide, and their homothetic ratio is at most $1 + \frac{3}{5}\sqrt{5} \approx 2.341$. Parallelograms are the only extremal sets.



approximation by homothetic pairs — concentric approximation by homothetic pairs — asymmetric approximation variant

Figure 5

Conjecture: For each C there is a right-angled triangle $\lrtriangle$ such that

$$\delta^{\text{BM-hom}}(C, \lrtriangle) \leq 1 + \sqrt{2}\,.$$

Regular $8k$-gons and discs are the only extremal sets.

Problem: What is the smallest number $c_\triangle$ such that for each C there is an isosceles triangle $\triangle$ with $\delta^{\text{BM-hom}}(C, \triangle) \leq c_\triangle$?

For both constants there is the same upper bound $\frac{7}{2}$ [27], but no good conjecture is known for the extremal sets in the case of isosceles triangles. A regular pentagon gives a lower bound $c_\triangle > 1 + \frac{1}{2}\sqrt{5}$, but a slight deformation gives a pentagon which shows that this bound is not extremal (the regular pentagon is stretched along a symmetry axis, the vertex on that axis is moved inward).

V. Non-metric Approximation Problems

There are also a number of ways we can ask for a triangle $\triangleright$ approximating the shape of a set C without involving a metric. Whenever we have a functional f measuring 'size' of convex sets, we can ask for the $\triangleright$ with $\triangleright \subset C$ and $f(\triangleright)$ maximal, and for the $\triangleright$ with $\triangleright \supset C$ and $f(\triangleright)$ minimal. For $f(\cdot) = \text{area}(\cdot)$ we found the optimal answers already in the section on the symmetric-difference metric, but there are many other interesting choices, e.g. perimeter, diameter, width, inradius, circumradius. Some of these are of course trivial, other have already appeared in literature. E.g. it is trivial that each C contains a $\triangleright$ with $\text{diam}(\triangleright) = \text{diam}(C)$, but it is nontrivial that for each C there is a $\triangleright$ such that $C \subset \triangleright$ and $\text{diam}(\triangleright) \leq \sqrt{3}\,\text{diam}(C)$, which is a special case of [14]. For the radius (circumradius) again the answers are simple: each C contains a $\triangleright$ with $\text{radius}(\triangleright) = \text{radius}(C)$, and each C is contained in a $\triangleright$ with $\text{radius}(\triangleright) \leq 2\,\text{radius}(C)$. For the perimeter Eggleston [10] proved that for each C there is a $\triangleright$ with $C \subset \triangleright$ and $\text{peri}(\triangleright) \leq \frac{3\sqrt{3}}{\pi}\,\text{peri}(C)$, with the disc extremal, but the reverse problem is still open:

Problem: What is the largest number c such that for each C there is a triangle $\triangleright \subset C$ with $\text{peri}(\triangleright) \geq c\,\text{peri}(C)$?

For the width it is simple to see that for each C and each $\varepsilon > 0$ there is a $\triangleright$ with $C \subset \triangleright$ and $\text{width}(\triangleright) \leq (1+\varepsilon)\,\text{width}(C)$. The other direction is an open problem:

Conjecture: For each C there is a triangle $\triangleright \subset C$ with

$$\begin{aligned} \text{width}(\triangleright) &\geq \tfrac{2}{1+\frac{1}{\sqrt{3}}\tan\frac{2\pi}{5}}\,\text{width}(C) \\ &\approx 0.720\,\text{width}(C). \end{aligned}$$

Regular pentagons are the only extremal sets.

A lower bound of 0.583 was obtained in [17]. The extremal inscribed triangle here has some vertices on the sides of the pentagon, so it is also an interesting variant to ask for a subpolygon approximation. This conjecture occured in [31]

Conjecture: For each set X there is a subset $Y \subset X$ of at most three points such that

$$\begin{aligned}\text{width}(\text{conv}(Y)) &\geq \tfrac{\sqrt{5}-1}{2}\,\text{width}(\text{conv}(X))\\ &\approx 0.618\,\text{width}(\text{conv}(X)).\end{aligned}$$

The vertices of a regular pentagon are an extremal set.

Here a simple lower bound of $\frac{1}{2}$ is known [31]. For centrally symmetric sets $\frac{\sqrt{5}-1}{2}$ can probably be replaced by $\frac{1}{\sqrt{2}}$.

Another result of this type ist the 'quantitative Steinitz theorem' [3,6]: If C contains a disk of radius r around point x, then there is a sub-k-gon $P_k \subset C$ that contains a disk around x of radius $\frac{\cos\frac{2}{k+1}\pi}{\cos\frac{1}{k+1}\pi}r$, and for subpolygon approximation this is optimal. Without that assumption, certainly a better ratio will be possible. Also this suggests

Problem: What is the largest number c such that for each C there is a triangle $\triangleright \subset C$ with inradius$(\triangleright) \geq c$ inradius(C) ?

A further measure of size for inner approximation based on the aspect ratio was proposed by S. Fekete. If $C_2 \subset C_1$, and there is a point $x \in C_1 \setminus C_2$ from which we see C_2 spanning only a small angle, then C_2 is a bad approximation for C_1. So we define $\mu^{\substack{\text{aspect}\\ \text{ratio}}}(C_1, C_2) := \inf_{x \in C_1 \setminus C_2} \sup_{y_1, y_2 \in C_2} |\sphericalangle y_1 x y_2|$, and ask for a k-gon $P_k \subset C$ that maximizes $\mu^{\substack{\text{aspect}\\ \text{ratio}}}(C, P_k)$. It is important to insist in this definition on $x \notin C_2$, for otherwise the approximation quality is determined by the smallest interior angle of C_1. With this definition, the best-approximating polygons are subpolygons, and the worst aspect ratio is given by an edge $\overline{y_1 y_2}$ of the subpolygon and a vertex x of the polygon cut of by that edge. It is easy to determine here the best approximation by triangles: any C contains a triangle $\triangleright \subset C$ such that $\mu^{\substack{\text{aspect}\\ \text{ratio}}}(C, \triangleright) \geq \frac{\pi}{2}$, and the square does not admit any better approximation. For this one just takes the smallest circle containing C and selects three of the touching points that contain the circle center in their convex hull. The best approximation by k-gons for $k \geq 4$ is still open, although there is a simple lower bound of $\frac{k-2}{k}\pi$:

Conjecture: For each C there is a k-gon $P_k \subset C$ with $\mu^{\text{aspect ratio}}(C, P_k) \geq \frac{k-1}{k+1}\pi$.

A final group of problems in this class can be formulated with the 'relative length'. Given a convex body C, the C-distance of two points a, b is the ratio of their euclidean distance to half the length of the longest chord in C parallel to ab. This is a generalization of the norm distance with a given unit ball to C that are not necessarily centrally symmetric.

Conjecture: For each C there is a triangle $\triangleright \subset C$ such that each side has C-length at least $\frac{1}{2}(\sqrt{5}+1)$ and the regular pentagon is an extremal set.

This was proposed in [25], and in [4] a lower bound of $\frac{4}{3}$ is given. For centrally symmetric C the bounds are better

Conjecture: For each centrally symmetric C there is a triangle $\triangleright \subset C$ such that each side has C-length at least $1 + \frac{1}{\sqrt{2}}$, and the regular octagon is an extremal set.

This was proposed in [8], and in [4] a lower bound of 1.546 is given. The opposite direction is

Conjecture: For each C there is a triangle $\triangleright$ such that $C \subset \triangleright$ and each side of $\triangleright$ has C-length at most 4, and parallelograms are the only extremal sets.

Here is also a nonapproximability conjecture:

Conjecture: A centrally symmetric C can never be approximated by a triangle $\triangleright$ with $C \subset \triangleright$ and each side of $\triangleright$ having a C-length less than 3,

Here exactly 3 is possible for affine regular hexagons.

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[T] Report Requests should be directed to Helmut Alt, Freie Universität Berlin, Fachbereich Mathematik und Informatik, Institut für Informatik, Takustr. 9, D-14195 Berlin, alt@inf.fu-berlin.de.

[A] Report Requests should be directed to Klaus-Peter Löhr, Freie Universität Berlin, Fachbereich Mathematik und Informatik, Institut für Informatik, Takustr. 9, D-14195 Berlin, lohr@inf.fu-berlin.de.