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Introduction

Overview

This dissertation treats questions about the definition of "simplices" inside Riemannian manifolds, the comparison between those simplices and Euclidean ones, as well as Galerkin methods for variational problems on manifolds.

During the last three years, the "Riemannian centre of mass" technique described by Karcher (1977) has been successfully employed to define the notion of a simplex in a Riemannian manifold M of non-constant curvature by Rustamov (2010), Sander (2012) and others. This approach constructs, for given vertices $p_i \in M$, a uniquely defined "barycentric map" $x : \Delta \to M$ from the standard simplex Δ into the manifold, and calls $x(\Delta)$ the "Karcher simplex" with vertices p_i.

However, the question whether x is bijective and hence actually induces barycentric *coordinates* on $x(\Delta)$ remained open for most cases. We show that under shape regularity conditions similar to the Euclidean setting, the distortion induced by x is of the same order as for normal coordinates: dx is almost an isometry (of course, this can only work if Δ is endowed with an appropriately-chosen Euclidean metric), and ∇dx almost vanishes. The estimate on dx could have already been deduced from the work of Jost and Karcher (1982), but it is the combination with the ∇dx estimate which paves the ground for applications of Galerkin finite element techniques.

For example, the construction can be employed to triangulate M and solve problems like the Poisson problem or the Hodge decomposition on the piecewise flat simplicial manifold instead of M. This leads to analogues of the classical estimates by Dziuk (1988) and subsequent authors in the field of surface PDE's (we only mention Hildebrandt *et al.* 2006 and Holst and Stern 2012 at this point), but as no embedding is needed in our approach, the range of the surface finite element method is extended to abstract Riemannian manifolds without modification of the computational scheme. Second, one can approximate submanifolds S inside spaces other than $\mathbb{R}^m$ (for example, minimal submanifolds in hyperbolic space), for which the classical "normal height map" or "orthogonal projection" construction from the above-mentioned literature directly carries over, and the error term generated by the curvature of M is dominated by the well-known error from the principal curvatures of S.

Apart from classical conforming Galerkin methods, there are other discretisation ideas, e. g. the "discrete exterior calculus" (DEC, see Hirani 2003) in which variational problems such as the Poisson problem or the Hodge decomposition can be solved without any reference to some smooth problem. Convergence proofs are less developed in this area, mainly because albeit there are interpolation operators from discrete k-forms to $\mathsf{L}^2\Omega^k$, these interpolations do not commute with the (differing) notions of exterior derivative on both sides. We re-interpret DEC as non-conforming Galerkin schemes.

Results

Let (M,g) be a smooth compact Riemannian manifold. Concerning the simplex definition and parametrisation problem, we obtained the following (for German readers, we also refer to the official abstract on page 113):

⟨**1**⟩ For given points $p_0, \ldots, p_n \in M$ inside a common convex ball, we consider the "barycentric mapping" $x : \Delta \to M$ from the standard simplex into M defined by the Riemannian centre of mass technique. Its image $s := x(\Delta)$ is called the (possibly degenerate) n-dimensional "Karcher simplex" with vertices p_i. If Δ is equipped with a flat metric g^e defined by edge lengths $\mathbf{d}(p_i, p_j) \leq h$, where $\mathbf{d}$ is the geodesic distance in (M,g), and if $\mathrm{vol}(\Delta, g^e) \geq \alpha h^n$ for some $\alpha > 0$ independent of h ("shape regularity"), we give a estimate for the difference $g^e - x^*g$ between the flat and the pulled-back metric of order h^2, as well as a first-order estimate for the difference $\nabla^{g^e} - \nabla^{x^*g}$ between the Euclidean and the pulled-back connection (6.17, 6.23).

⟨**2**⟩ We give estimates for the interpolation of functions $s \to \mathbb{R}$ and $s \to N$, where N is a second Riemannian manifold (7.4, 7.15).

⟨**3**⟩ Starting from the already existing theory of Voronoi tesselations in Riemannian manifolds by Leibon and Letscher (2000) and Boissonnat *et al.* (2011), we define the Karcher–Delaunay triangulation for a given dense and "generic" vertex set (8.8).

⟨**4**⟩ Concerning the Poisson problem on the space of weakly differentiable real-valued functions $\mathsf{H}^1(M, \mathbb{R})$, weakly differentiable real-valued differential forms $\mathsf{H}^1\Omega^k(M)$, and weakly differentiable mappings into a second manifold $\mathsf{H}^1(M,N)$, we prove error estimates for their respective Galerkin approximations (10.13, 10.17, 13.14). The same method gives estimates for the Hodge decomposition in $\mathsf{H}^1\Omega^k(M)$ if appropriate trial spaces as in Arnold *et al.* (2006) are chosen (10.15).

⟨**5**⟩ We give proximity and metric comparison estimates for the "normal height map" or "orthogonal projection map" between a smooth submanifold and its Karcher-simplicial approximation, which is the classical tool for finite element analysis on surfaces in $\mathbb{R}^3$, but this time for submanifolds inside another curved manifold (11.3, 11.18).

⟨**6**⟩ We show that the differential of a Karcher simplex' area functional with respect to variations of its vertices is well-approximated by the area differential of the flat simplex (Δ, g^e) with g^e as above (12.12).

Concerning the convergence analysis of discrete exterior calculus schemes for a simplicial complex:

⟨**7**⟩ We define a (piecewise constant) interpolation $i_k : C^k \to \mathsf{P}^{-1}\Omega^k$ from discrete differential forms to a subspace of $\mathsf{L}^2\Omega^k$, which turns the discrete exterior derivative into a "differential" $\underline{d} : \mathsf{P}^{-1}\Omega^k \to \mathsf{P}^{-1}\Omega^{k+1}$ with Stokes' and Green's formula for simplicial domains. This reduces convergence issues for DEC from simplicial (co-)chains to approximation estimates between the non-conforming trial space $(\mathsf{P}^{-1}\Omega^k, \underline{d})$ and $(\mathsf{H}^1\Omega^k, d)$. We estimate the approximation quality of P^{-1} forms in $\mathsf{H}^1\Omega^k$ (9.19, 9.20) and compare the solutions of variational problems in $\mathsf{P}^{-1}\Omega^k$ and $\mathsf{H}^1\Omega^k$ (10.26–28).

Structure and Method

All the thesis is divided into three parts, one of which introduces notation, the main constructions another, its applications to standard problems in numerical analysis of geometric problems and surfaces PDE's (changing the usual setting from embedded surfaces to abstract (sub-)manifolds) the third. Having in mind that "the introduction of numbers as coordinates [...] is an act of violence" (WEYL 1949, p. 90), we try to stay inside the absolute Riemannian calculus as far as possible. Our main tool are Jacobi fields, which naturally occur when taking derivatives of the exponential map and its inverse. Whereas the standard situation for estimates on a Jacobi field $J(t)$ are given values $J(0)$ and $\dot{J}(0)$, see e. g. JOST (2011, chap. 5), we will deal with Jacobi fields with prescribed start and end value, which is convered by (fairly rough, but satisfying) growth estimates 6.6 and 12.4.

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We know that this thesis, as a time-constraint human work, will be full of smaller or larger mistakes and shortcomings. We strongly hope that none of them will destroy main arguments. For all others, we keep in mind the words of an academic teacher, concerning exercise sheets: "Do not refuse an exercise in which you have found a mistake, but try to find the most interesting and correct exercise in a small neighbourhood of the original one."

Symbol List

We listed here those symbols which occur in several sections without being introduced every time. Symbols with bracketed explanation are also used with a different meaning, which then will be defined in the section. Where it is useful, we added a reference to the definition.

M	manifold, Mg is shortcut for (M, g)
m	dimension of M
g	Riemannian metric on M
P	parallel transport (beside in section 3)
R	curvature tensor of M (1.3)
Γ	Christoffel symbols (1.2), Christoffel operator (1.14)
$\mathbf{d}$	geodesic distance function in M
X_p	gradient of $\frac{1}{2}\,\mathbf{d}(p, \cdot\,)$ (1.22)
x	barycentric mapping (5.4)
inj, cvr	injectivity and convexity radius (5.2)
C_0, C_1	global bound for $\lVert R\rVert$ and $\lVert\nabla R\rVert$ resp.
h	mesh size
ϑ	fullness parameter (3.3)
$C_{0,1}$	$:= C_0 + hC_1$
$C'_{0,1}$	$:= C_{0,1}\vartheta^{-2}$
$\mathfrak{K}$	simplicial complex (4.1)
n	dimension of $\mathfrak{K}$
$\mathfrak{e}, \mathfrak{f}, \mathfrak{s}, \mathfrak{t}$	elements, facets, simplices
r	(realisation operator for simplicial complexes, 4.2)
$\lesssim$	$\leq$ up to a constant that only depends on n
Δ	standard simplex, Laplace–Beltrami operator
e_i	Euclidean basis vector
1_n	$= (1, \dots, 1) \in \mathbb{R}^n$
$\mathbb{1}$	unit matrix
$\mathbb{B}_r(U)$	set of points with distance $< r$ from U
d	differential, exterior derivative
δ	(exterior coderivative, Kronecker symbol)
∂	partial / coordinate derivative, boundary of sets
∇	covariant derivative
D	covariant derivative along curves (except section 3)
L	(weak Laplacian, 2.7), curve length functional
$\lvert\cdot\rvert_{\ell^2}$	canonical Euclidean norm of $\mathbb{R}^n$
$\lvert\cdot\rvert$	pointwise norm on bundles induced by g, volume of sets
$\lVert\cdot\rVert$	pointwise operator norm (1.1)
$\lvert\cdot\rvert$	integrated (or supremum) pointwise g-norm (2.3)
$\lVert\cdot\rVert$	integrated (or supremum) pointwise operator norm
$\vert\!\vert\!\vert\cdot\vert\!\vert\!\vert$	operator norm in function spaces (10.3)

C^k	k-times continuously differentiable functions
L^r	functions whose r'th power is Lebesgue-integrable
$\mathsf{W}^{k,r}$	functions that have k covariant differentials in L^r (2.3)
H^k	$:= \mathsf{W}^{k,2}$ (except section 13)
H^k_0 etc.	functions in H^k etc. with vanishing trace on the boundary
$\mathsf{H}^{1,0}, \mathsf{H}^{0,1}$	forms α with weak $d\alpha$ or $\delta\alpha$ of class L^2 resp.
$\mathsf{H}^{1,1}$	forms α with weak $d\alpha$ and $\delta\alpha$ of class L^2
H^{1+1}	forms with weak $d\alpha$ and $\delta\alpha$ of class $\mathsf{H}^{1,1}$
P	polynomial forms (9.6), functions (10.3), vector fields (12.7)
$\mathfrak{X}$	vector fields of class C^∞
Ω^k	differential k-forms of class C^∞
Ω^k_t, Ω^k_n	diff. forms with vanishing tangential/normal trace on the boundary
$\mathsf{L}^2\mathfrak{X}$ etc.	vector fields of class L^2 etc.
S	submanifold
$TM\|_S$	vector bundle over S with fibres T_pM
$TS^\perp$	normal bundle of S in M
ν	normal on S in M
π	projection
$\boldsymbol{n}$	projection onto normal part
$\boldsymbol{t}$	projection onto tangential part
Φ	normal height map $p \mapsto \exp_p Z$ for normal vector field Z
Φ_t	geodesic homotopy $p \mapsto \exp_p tZ$

Contents

A. Preliminaries

Let us briefly recapitulate basic notions and concepts of the concerned mathematical fields: Riemannian manifolds and variational problems on these, simplex geometry and simplicial complexes. For the quick reader with experience in numerics on surfaces, a short look on the simplex metric in barycentric coordinates (3.11) and our definition of simplicial complexes (4.2) might be of interest.

1. Riemannian Geometry

For this section, we will keep close to the notations of Jost (2011) and Lee (1997).—Let (M, g) or Mg for short be an m-dimensional Riemannian manifold. We write $\langle X, Y\rangle$ or $g\langle X, Y\rangle$ instead of $g(X, Y)$ for $X, Y \in T_pM$, mainly to prevent the use of too many round brackets. Whereas **charts** map open sets in M into $\mathbb{R}^m$, we will mostly use **coordinates** (U, x), i. e. maps x from open sets $U \subset \mathbb{R}^m$ into M that are locally homeomorphisms.

Throughout this thesis, we apply Einstein convention for computations in local coordinates or any other upper-lower index pair. Only when it explicity helps to clarify our statements, we note the evaluation of a vector field X or the metric g at a specific point $p \in M$ as $X|_p$ or $g|_p$ respectively.

Tangent Bundle and Norms. Coordinates (U, x) around p give rise to a basis $\frac{\partial}{\partial x_i}$ **1.1**
or shortly ∂_i of T_pM, and a dual basis dx^i on T_p^*M. The tangent-cotangent isomorphism is denoted by $\flat$ and its inverse by $\sharp$. The natural extension of g to T^*M has coefficients g^{ij} with $g^{ij}g_{jk} = \delta^i_k$ (Kronecker symbol). On higher tensor bundles, g also naturally induces scalar products by $g\langle v \otimes \bar{v}, w \otimes \bar{w}\rangle := g\langle v, w\rangle g\langle \bar{v}, \bar{w}\rangle$ and similar for covector and mixed tensors. The space of smooth vector fields is denotes as $\mathfrak{X}$, the spaces of smooth alternating k-forms as Ω^k. With $\cdot$, we denote the Euclidean scalar product in $\mathbb{R}^n$.

We will denote the norm on all these bundles simply by $|\cdot|$ or $|\cdot|_g$, because we do not see ambiguity here. However, it differs from the operator norm of a tensor denoted as $\|\cdot\|$. Both are equivalent, $\|\cdot\|_g \leq |\cdot|_g \leq c\|\cdot\|_g$ with a constant c that only depends on the dimension m and the rank of the tensor (Golub and van Loan 1983, eqn. 2.2-9). In particular, operator and induced norm agree on 1-forms.

Curvature

In local coordinates (U, x), the metric g is a smooth field of positive definite $m \times$ **1.2**
m-matrices over U. A **connection** ∇ on Mg is given in local coordinates by some

Christoffel symbols $\Gamma^k_{ij} = \Gamma^k_{ji}$ via

$$\nabla_{\partial_i}\partial_j = \Gamma^k_{ij}\partial_k, \qquad \nabla_X Y = (X^i\partial_i Y^k + X^i Y^j \Gamma^k_{ij})\partial_k \tag{1.2a}$$

for vector fields X, Y around p with coordinates $X = X^i\partial_i$ and $Y = Y^i\partial_i$ respectively. It naturally induces a connection on higher tensor bundles, e. g. on the bundle of linear maps $A : T_pM \to T_pM$, by $(\nabla_V A)(W) = \nabla_V(AW) - A(\nabla_V W)$. There is a unique connection that is symmetric and compatible with g, the **Levi–Cività connection** of Mg, whose Christoffel symbols can be computed by

$$\Gamma^k_{ij} = g^{k\ell}(\partial_j g_{i\ell} + \partial_i g_{j\ell} - \partial_\ell g_{ij}). \tag{1.2b}$$

1.3 The **Riemann curvature tensor** R of Mg is defined by

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z. \tag{1.3a}$$

In local coordinates, it has coefficients

$$R^\ell_{ijk} = \partial_i\Gamma^\ell_{jk} - \partial_j\Gamma^\ell_{ik} + \Gamma^n_{jk}\Gamma^\ell_{ni} - \Gamma^n_{ik}\Gamma^\ell_{nj}, \qquad R(\partial_i,\partial_j)\partial_k = R^\ell_{ijk}\partial_\ell \tag{1.3b}$$

and obeys the following (anti-)symmetries:

$$\begin{aligned} \langle R(X,Y)Z,W\rangle &= -\langle R(Y,X)Z,W\rangle, \\ \langle R(X,Y)Z,W\rangle &= -\langle R(X,Y)W,Z\rangle, \\ \langle R(X,Y)Z,W\rangle &= \langle R(Z,W)X,Y\rangle. \end{aligned} \tag{1.3c}$$

Let us agree that ∇ and D bind weaker than linear operators, so D_tAW as above always means $D_t(AW)$, not $(D_tA)W = \dot{A}W$.

1.4 Along smooth curves $c : [a;b] \to M, t \mapsto c(t)$, any connection uniquely induces a **covariant differentiation** D_t along c by

$$\dot{V}(t) := D_tV(t) = (\dot{V}^k + \dot{c}^i V^j \Gamma^k_{ij})\partial_k. \tag{1.4a}$$

A **geodesic** is a curve with vanishing covariant derivative, i. e. $D_t\dot{c} = 0$ or, slightly inprecise, $\nabla_{\dot{c}}\dot{c} = 0$. In coordinates,

$$c^k_{,tt} = -\dot{c}^i\dot{c}^j\Gamma^k_{ij} \tag{1.4b}$$

(note that we use the symbol $\ddot{c}$ only for the covariant derivative of $\dot{c}$, and we denote the coordinate derivative by a comma-separated subscript). If the parametrisation does not matter, we denote a curve c with endpoints $p, q \in M$ as $c : p \rightsquigarrow q$. The geodesic distance $\mathbf{d}(p,q)$ is the length of the shortest geodesic $p \rightsquigarrow q$. A Riemannian manifold is complete if any two points can be joined by a geodesic. For some neighbourhood B of p, we say that B is **convex** if each two points $q, r \in B$ have a unique shortest geodesic $q \rightsquigarrow r$ in M which lies in B (Karcher 1968).

Along a geodesic $c : [a;b] \to M$, there is a **parallel translation** $P^{t,s} : T_{c(s)}M \to T_{c(t)}M$ for every $s, t \in [a;b]$, defined by $P^{t,s}V = W(t)$ for the vector field W along γ

with $W(s) = V$ and $\dot{W} = 0$. Parallel translation is an isometry, as $\frac{\mathrm{d}}{\mathrm{d}t}|W|^2 = \langle \dot{W}, W\rangle = 0$. The derivative of P with respect to a variation of c is computed in 7.8.

As geodesics are unique inside a convex ball B, we will also write $P^{q,p}$ for $q, p \in B$. The unintuitive order of the evaluation points is inspired by the fact that some vector in T_pM enters on the right, and a vector in T_qM comes out on the left.—We remark that in general $P^{r,q}P^{q,p} \neq P^{r,p}$, but instead $P^{p,r}P^{r,q}P^{q,p}$ is the **holonomy** of the loop $p \rightsquigarrow q \rightsquigarrow r \rightsquigarrow p$.

Assumption. Throughout the whole thesis, we will assume that Mg is a compact smooth m-dimensional manifold (without boundary, if not specified) with curvature bounds $\|R\| \leq C_0$ and $\|\nabla R\| \leq C_1$ everywhere. To keep definitions together, we give a "forward declaration": When a radius (or a mesh size) r and a fullness parameter ϑ are defined, we will also use $C_0' := C_0\vartheta^{-2}$ and $C_{0,1} := C_0 + rC_1$, analogously $C_{0,1}'$ (C_1' will not be used). **1.5**

Remark. Up to a factor of $\frac{4}{3}$, the bound $\|R\| \leq C_0$ is the same as requiring that the sectional curvature is bounded, because if all sectional curvatures are bounded by $\pm K$, then $\|R\| \leq \frac{4}{3}K$ (Buser and Karcher 1981, 6.1.1), which is the usual assumption in the works of Karcher, Jost *et al.* Of course, on the other hand $K \leq C_0$.

Second Derivatives

Let $N\gamma_{\alpha\beta}$ and Mg_{ij} be two smooth Riemannian manifolds with coordinates u^α and v^i respectively and $f : N \to M$ be a smooth mapping. Its first derivative is, at each $p \in N$, a linear map $d_pf : T_pN \to T_{f(p)}M$. Of course, the Levi Civita connections of M and N induce a unique way to define the Hessian ∇df. For this purpose, df has to be considered as a section in $E := T^*N \otimes f^*TM$, a bundle over N with fibres $E_p = T_p^*N \times T_{f(p)}M$. We want to give a coordinate expression for this.

Definition. Let M and N be two Riemannian manifolds, $f : N \to M$ smooth. The **Hessian** of f is $\nabla^E df$, a section of $T^*N \otimes T^*N \otimes f^*TM$.

Fact. The connection on the cotangent bundle T^*N is defined by **1.6**

$$d\big(\omega(X)\big) = \omega(\nabla^{TM}X) + (\nabla^{T^*M}\omega)(X) \qquad \text{for } \omega \in \Omega^1(N),\ X \in \mathfrak{X}(N),$$

cf. Jost (2011, eqn. 4.1.20). This gives

$$\begin{aligned} 0 = d\big(du^\alpha(\partial_\beta)\big)(\partial_\gamma) &= du^\alpha(\nabla_{\partial_\gamma}\partial_\beta) + (\nabla_{\partial_\gamma}du^\alpha)(\partial_\beta) \\ &= du^\alpha(\Gamma^\delta_{\beta\gamma}\partial_\delta) + (\nabla_{\partial_\gamma}du^\alpha)(\partial_\beta), \end{aligned}$$

and with $du^\alpha(\partial_\delta) = 1$ if $\alpha = \delta$ and 0 else, this gives that $\nabla_{\partial_\gamma}du^\alpha$ maps a vector ∂_β to $-\Gamma^\alpha_{\beta\gamma}$, so

$$\nabla_{\partial_\gamma}du^\alpha = -\Gamma^\alpha_{\beta\gamma}du^\beta. \tag{1.6a}$$

Vector fields V on M pull back to vector fields f^*V by $(f^*V)|_p = V|_{f(p)}$. The connection ∇^{TM} then induces a connection on f^*TM by

$$\nabla^{f^*TM}_X f^*V = f^*\nabla^{TM}_{df\,X}V. \tag{1.6b}$$

Let us abbreviate $\partial_\alpha := \frac{\partial}{\partial u^\alpha}$ as before, and additionally $\partial_i := f^* \frac{\partial}{\partial v^i}$, and $f^i_{,\alpha} := \frac{\partial f^i}{\partial u^\alpha}$. For example, the usual coordinate representation of df is $df(\partial_\alpha) = f^i_{,\alpha} \frac{\partial}{\partial v^i}$. Therefore,

$$\nabla^{f^*TM}_{\partial_\alpha} \partial_j = f^*(\nabla^{TM}_{f^i_{,\alpha} \frac{\partial}{\partial v^i}} \partial_j) = f^*(f^i_{,\alpha} \Gamma^k_{ij} \tfrac{\partial}{\partial v^k}) = f^i_{,\alpha} \Gamma^k_{ij} \partial_k. \tag{1.6c}$$

The connections on T^*N and f^*TM induce a connection on the product bundle, cf. Jost (2011, eqn. 4.1.23):

$$\nabla^E(\omega \otimes V) = (\nabla^{T^*N}\omega) \otimes V + \omega \otimes (\nabla^{f^*TM} V) \qquad \text{for } \omega \in \Omega^1(N),\ V \in f^*\mathfrak{X}(M). \tag{1.6d}$$

1.7 **Lemma.** *Let $f : N \to M$ be a smooth mapping between Riemannian manifolds, and let $V, W \in T_pN$. Then consider a variation of curves $\gamma(s,t)$ in N with $\partial_t\gamma = W$, $\partial_s\gamma = V$ and $D_s\partial_t\gamma = 0$ (everything is evaluated at $s = t = 0$). Let $c := f \circ \gamma$ be the corresponding variation of curves in M. Then $\partial_t c = dfV$, $\partial_s c = dfW$ and $(\nabla^E df)(V,W) = D_s\partial_t c$. If $dfV \neq 0$ this is*

$$(\nabla^E df)(V, W) = \nabla^{TM}_{dfW} dfV,$$

where V and W are extended such that $\nabla_W V = 0$.

Proof. Inserting $df = f^i_{,\alpha} du^\alpha \otimes \partial_i$ in 1.6d, we have

$$\nabla^E_{\partial_\beta} df = \nabla^{T^*N}_{\partial_\beta}(f^i_{,\alpha} du^\alpha) \otimes \partial_i + f^i_{,\alpha} du^\alpha \otimes \nabla^{f^*TM}_{\partial_\beta} \partial_i.$$

By 1.6a,

$$\nabla^{T^*N}_{\partial_\beta} f^i_{,\alpha} du^\alpha = f^i_{,\alpha\beta} du^\alpha - f^i_{,\alpha} \Gamma^\alpha_{\beta\gamma} du^\gamma$$

and together with 1.6c, this gives

$$\nabla^E_{\partial_\beta} df = (f^i_{,\alpha\beta} du^\alpha - f^i_{,\alpha}\Gamma^\alpha_{\beta\gamma} du^\gamma) \otimes \partial_i + f^i_{,\alpha} du^\alpha \otimes f^j_{,\beta} \Gamma^k_{ij} \partial_k$$

(cf. Jost 2011, eqn. 8.1.19). We conclude that ∇df, taken as bilinear map $T_pN \times T_pN \to T_{f(p)}M$, acts on vectors ∂_β and ∂_δ as

$$\begin{aligned}
\nabla df(\partial_\beta, \partial_\delta) &= [f^i_{,\alpha\beta} du^\alpha(\partial_\delta) - f^i_{,\alpha}\Gamma^\alpha_{\beta\gamma} du^\gamma(\partial_\delta)]\partial_i + f^i_{,\alpha} du^\alpha(\partial_\delta) f^j_{,\beta} \Gamma^k_{ij} \partial_k \\
&= (f^i_{,\delta\beta} - f^i_{,\alpha}\Gamma^\alpha_{\beta\delta})\partial_i + f^i_{,\delta} f^j_{,\beta} \Gamma^k_{ij} \partial_k \\
&= (f^i_{,\delta\beta} - f^i_{,\alpha}\Gamma^\alpha_{\beta\delta} + f^j_{,\delta} f^k_{,\beta} \Gamma^i_{jk})\partial_i.
\end{aligned}$$

This is, as it should be, symmetric in β and δ by the symmetry of $f^i_{,\beta\delta}$ and the Christoffel symbols.

On the other hand, let us compute $D_s\partial_t c$. The derivatives of γ are given by $\partial_t\gamma = \gamma^\alpha_{,t}\partial_\alpha$ and $\partial_s\gamma = \gamma^\beta_{,s}\partial_\beta$. By the chain rule, $\partial_t c = c^i_{,t}\partial_i = \gamma^\alpha_{,t} f^i_\alpha \partial_i$ and $\partial_s c = \gamma^\beta_{,s} f^j_\beta \partial_j$. By 1.4a,

$$D_s\partial_t c = \big((\gamma^\alpha_{,t} f^i_{,\alpha})_{,s} + \gamma^\alpha_{,t} f^j_{,\alpha} \gamma^\beta_{,s} f^k_{,\beta} \Gamma^i_{jk}\big)\partial_i.$$

Now $(\gamma^\alpha_{,t} f^k_{,\alpha})_{,s} = \gamma^\alpha_{,ts} f^k_{,\alpha} + \gamma^\alpha_{,t} f^k_{,\alpha\beta} \gamma^\beta_{,s}$ again by the chain rule. As we have assumed $D_s \partial_t \gamma = 0$, we get $\gamma^\alpha_{,ts} = -\gamma^\beta_{,t} \gamma^\delta_{,s} \Gamma^\alpha_{\beta\delta}$ for every α, so

$$\begin{aligned} D_s \partial_t c = & \quad (-f^i_{,\alpha} \gamma^\beta_{,t} \gamma^\delta_{,s} \Gamma^\alpha_{\beta\delta} + f^i_{,\alpha\beta} \gamma^\alpha_{,t} \gamma^\beta_{,s} + \gamma^\alpha_{,t} f^j_{,\alpha} \gamma^\beta_{,s} f^k_{,\beta} \Gamma^i_{jk}) \partial_i, \\ &= V^\delta W^\beta (-f^i_{,\alpha} \Gamma^\alpha_{\beta\delta} + f^i_{,\delta\beta} + f^j_{,\delta} f^k_{,\beta} \Gamma^i_{jk}) \partial_i, \end{aligned}$$

q. e. d.

Corollary (Jost 2011, *eqns. 4.3.48, 4.3.50*). *If $M = \mathbb{R}$, then the Hessian of a function $f : N \to \mathbb{R}$, applied twice to the tangent of a geodesic γ, is the second derivative of $f \circ \gamma$, and it holds* **1.8**

$$\nabla df(V, W) = \langle \nabla_V \operatorname{grad} f, W \rangle = \langle \nabla_W \operatorname{grad} f, V \rangle = V(Wf) - df(\nabla_V W). \tag{1.8a}$$

Scalings

In most situations, we will try to prove **scale-aware** estimates, i. e. estimates for coordinate expressions or absolute terms where both sides of the inequality scale similar when the coordinates or the diameter of the manifold is scaled (if both sides of the inequality even remain unchanged under rescaling, we call the estimate **scale-invariant**). Therefore, we will need to know the scaling behaviour of vectors and tensors.

Coordinate change, fixed absolute manifold. First, consider the case where the abstract (absolute) geometry of Mg is fixed and only coordinates are changed. A useful application is when coordinates (U, x) are given and the eigenvalues of the matrix g^u_{ij} lie between $\vartheta^2 \mu^2$ and μ^2, but one would like to have eigenvalues in the order of 1 (i. e. between ϑ^2 and 1). This is achieved by coordinates **1.9**

$$y^i = \mu x^i, \qquad \tfrac{\partial}{\partial y^\alpha} = \frac{1}{\mu} \tfrac{\partial}{\partial x^i}.$$

Components of vectors always scale like the coordinates: If $W = w^{i,x} \frac{\partial}{\partial x^i} = w^{i,y} \frac{\partial}{\partial y^i}$, then $w^{i,y} = \mu w^{i,x}$. This scaling indeed fulfills our requirements:

$$g^y_{ij} = g\langle \tfrac{\partial}{\partial y^i}, \tfrac{\partial}{\partial y^j} \rangle = \frac{1}{\mu^2} g\langle \tfrac{\partial}{\partial x^i}, \tfrac{\partial}{\partial x^j} \rangle = \frac{1}{\mu^2} g^x_{ij}$$

The inverse matrix obviously scales with $(g^{ij})^y = \mu^2 (g^{ij})^x$.—The Christoffel symbols and the components of the curvature tensor scale with

$$(\Gamma^k_{ij})^y = \frac{1}{\mu} (\Gamma^k_{ij})^x, \qquad (R^\ell_{ijk})^y = \frac{1}{\mu^2} (R^\ell_{ijk})^x.$$

1.10 **Fixed coordinates, manifold scaling.** Consider a new Riemannian manifold $M\bar{g}$ with $\bar{g} = \mu^2 g$. Then $\operatorname{diam}(M\bar{g}) = \mu \operatorname{diam}(Mg)$ and $\bar{\mathbf{d}}(p,q) = \mu\,\mathbf{d}(p,q)$. The norm of a tensor that is covariant of rank k and contravariant of rank ℓ scales with $\mu^{\ell-k}$. For example, a vector W, a linear form ω and the curvature tensor R scale with

$$|W|_{\bar{g}} = \mu|W|_g, \qquad |\omega|_{\bar{g}} = \frac{1}{\mu}|\omega|_g, \qquad \|R\|_{\bar{g}} = \frac{1}{\mu^2}\|R\|_g.$$

If coordinates (U,x) remain the same, then $\bar{g}_{ij} = \mu^2 g_{ij}$ and $\bar{g}^{ij} = \frac{1}{\mu^2}g^{ij}$, and the Christoffel symbols and tensor components remain fixed:

$$\bar{\Gamma}^k_{ij} = \Gamma^k_{ij}, \qquad \bar{R}^\ell_{ijk} = R^\ell_{ijk}. \tag{1.10a}$$

Now suppose two manifolds Mg and $N\gamma$ with a mapping $f : N \to M$. Consider a scaling μ for M and ν for N. As df can be regarded as a linear form on TN, resulting in a vector in TM, it is natural that the norm of df and ∇df scale as

$$\|df\|_{\bar{\gamma},\bar{g}} = \frac{\mu}{\nu}\|df\|_{\gamma,g}, \qquad \|\nabla df\|_{\bar{\gamma},\bar{g}} = \frac{\mu}{\nu^2}\|\nabla df\|_{\gamma,g}. \tag{1.10b}$$

The scaling behaviour of $\|R\|$ is the reason why we never suppress curvature bounds as “hidden constants”. In fact, most of our results could be simply worked out in balls of radius 1, and their scaling behaviour could be recovered from the curvature bounds and the scaling behaviour of left- and right-hand side operator norms.

1.11 **Coordinate change with manifold scaling.** It might also be useful to use coordinates for $(M, \mu^2 g)$ where the components g_{ij} remain unchanged, for example because they had previously been normalised to have eigenvalues in the order of 1. If a chart (U,x) is known, such coordinates are given by $y^i = \mu x^i$, because vector components also scale as $w^{\alpha,v} = \mu w^{\alpha,u}$ and then

$$|W|^2_{\bar{g}} = w^{\alpha,v}w^{\beta,v}g_{\alpha\beta} = \mu^2 w^{\alpha,u}w^{\beta,u}g_{\alpha\beta} = \mu^2|W|^2_g,$$

as it should. The Christoffel symbols and curvature tensor components scale as

$$(\bar{\Gamma}^\gamma_{\alpha\beta})^v = \frac{1}{\mu}(\Gamma^\gamma_{\alpha\beta})^u, \qquad (\bar{R}^\delta_{\alpha\beta\gamma})^v = \frac{1}{\mu^2}(R^\delta_{\alpha\beta\gamma})^u.$$

If two manifold $N\gamma_{\alpha\beta}$ and Mg_{ij} are scaled with factors μ and ν in this way, resulting in coordinate expressions $v^\alpha = \nu u^\alpha$ for N and $y^i = \mu x^i$ for M, then the coordinate form of f, which was a mapping $U_u \to U_x$, becomes a mapping $\bar{f} : \nu U_u \to \mu U_x, v \mapsto \mu f(v/\nu)$, so by chain rule

$$\bar{f}^i_{,\alpha} = \frac{\mu}{\nu}f^i_{,\alpha}, \qquad \bar{f}^i_{,\alpha\beta} = \frac{\mu}{\nu^2}f^i_{,\alpha\beta}$$

for the components in

$$df = f^i_{,\alpha}du^\alpha \otimes \tfrac{\partial}{\partial x^i}, \qquad d\bar{f} = \bar{f}^i_{,\alpha}dv^\alpha \otimes \tfrac{\partial}{\partial y^i}$$

and $\nabla df = (f^i_{,\delta\beta} - f^i_{,\alpha}\Gamma^\alpha_{\beta\delta} + f^j_{,\delta}f^k_{,\beta}\Gamma^i_{jk})du^\beta \otimes du^\delta \otimes \frac{\partial}{\partial x^i}$ as in the proof of 1.7.

The Exponential Map and Special Coordinates

On a point $p \in M$ (interior, if M has boundary), there is, at least for some small intervall $[-\varepsilon; \varepsilon]$, a unique geodesic for each initial velocity $X \in T_pM$. As the geodesic equation 1.4b is homogenous and the unit ball in T_pM is compact, this is equivalent to the fact that for some small ball B_ε around $0 \in T_pM$, the geodesic c^X with initial velocity $X \in B_\varepsilon$ exists on $[-1; 1]$. As the unit sphere in T_pM is compact, there is some ε that works for any direction X. The **exponential map** is defined to map $B_\varepsilon \to M$, $\exp_p(X) := c^X(1)$. From this mapping, normal and Fermi coordinates can be constructed. The former construction can be found in every Riemannian geometry textbook (e. g. Lee 2003, p. 78).

Normal coordinates around $p \in M$ are coordinates (U, x) with $x(0) = p$ in which straight lines $t \mapsto tv$ are geodesics (arclength-parametrised for $|v|_{\ell^2} = 1$), which implies $g_{ij}(0) = \delta_{ij}$, $\partial_k g_{ij}(0) = 0$ and $\Gamma_{ij}^k(0) = 0$ for all i, j, k. **1.12**

Lemma. *Any orthonormal basis E_i of T_pM induces normal coordinates (B_ε, x) around p via $x : (u^1, \ldots, u^m) \mapsto \exp_p(u^i E_i)$, where ε must be so small that geodesics through p are unique.*

Proof. By homogenity of the geodesic equation 1.4b, the geodesic starting with initial velocity $\dot c(p) = V$ with $V = E_i v^i$ has coordinates

$$c^k(t) = tv^k, \quad \text{so} \quad \dot c^k(t) = v^k \quad \text{and} \quad c^k_{,tt}(t) = 0$$

for all t in the definition interval of c. At the same time, $c^k_{,tt} = -\dot c^i \dot c^j \Gamma^k_{ij}$. As both equalities must hold for *every* $V \in T_pM$, this already implies $\Gamma^k_{ij} = 0$. The correspondence between Γ^k_{ij} and $\partial_k g_{ij}$ is linear and of full rank, so the latter have to vanish, too, *q. e. d.*

Corollary. *$d\exp_p = \mathrm{id}$ at $0 \in T_pM$, that means $d_0(\exp_p)V = V$.* **1.13**

Proof. Consider a geodesic c starting from p with velocity V. As the differential operator $d(\exp_p)$ applied to V can be computed as tangent of this integral curve, $d(\exp_p)V = \dot c(0) = V$, *q. e. d.*

Observation. Then the metric g_{ij} in the parameter domain is **1.14**

$$g_{ij}|_u = \langle dx\, e_i, dx\, e_i\rangle = \langle d_U(\exp_p)E_i, d_U(\exp_p)E_j\rangle, \tag{1.14a}$$

where $U = u^i E_i$. Likewise, the **Christoffel operator** $\Gamma : (v, w) \mapsto \Gamma^k_{ij} v^i w^j \partial_k$ (which is bilinear, but does not behave tensorial under coordinate changes) is computable as pull-back of the connection to the parameter domain: The coordinate expression $\nabla_v w = \nabla^{\mathrm{eucl}}_v w + \Gamma(v, w)$ given in 1.2a can be understood as pull-back $\nabla^{x^* g}$ of the connection onto $\mathbb{R}^m$ (not to be confused with the connection $x^*\nabla^g$ on x^*TM from 1.6b), and such a pull-back is defined by $dx(\nabla^{x^* g}_v w) = \nabla_{dx\, v} dx\, w$. The right-hand side was identified to be $\nabla dx(v, w)$ in 1.7, and so we have

$$d_U(\exp_p)(\Gamma(e_i, e_j)) = \nabla d_U(\exp_p)(E_i, E_j). \tag{1.14b}$$

1.15 **Jacobi Fields.** Let $c(s,t)$ be a smooth variation of geodesics $t \mapsto c(s,t)$. Denote $T := \partial_t c = \dot{c}$ and $J := \partial_s c$. As T and J are coordinate vector fields, $[J,T] = 0$, so $\nabla_J T = \nabla_T J$ or, in other words, $D_s \partial_t c = D_t \partial_s c$. Differentiating the geodesic equation $D_t \dot{c} = \nabla_T T = 0$ gives, by 1.3a,

$$0 = \nabla_J \nabla_T T = \nabla_T \nabla_J T + R(J,T)T = \nabla_T \nabla_T J + R(J,T)T,$$

which is the defining equation for **Jacobi fields**:

$$\ddot{J} = R(T,J)T \tag{1.15a}$$

Conversely, every vector field J along c fulfilling 1.15a gives rise to a variation of geodesics by

$$c(s,t) := \exp_{\exp sJ(0)} t(P\dot{c}(0) + sP\dot{J}(0)), \tag{1.15b}$$

where P is the parallel transport from $c(0)$ to $\exp sJ(0)$ (Jost 2011, thm. 5.2.1).

1.16 **Proposition** (cf. Karcher 1989, eqn. 1.2.5). *Let $c : I \to M$ be a smooth curve and Z be a vector field along c. Then the map $\varphi_t : s \mapsto \exp_{c(s)} tZ(s)$ has derivative $\dot{\varphi}_t(s) = J(t)$ for a Jacobi field J with initial values $J(0) = \dot{c}(s)$, $\dot{J}(0) = \dot{Z}(s)$. In particular, $d_V(\exp_p)W$ is the value $J(1)$ of a Jacobi field along $t \mapsto \exp_p tV$ with initial values $J(0) = 0$ and $\dot{J}(0) = W$.*

Proof. $c(s,t) := \varphi_t(s)$ is a variation of geodesics $t \mapsto c(s,t)$ for every fixed s, so $\partial_s c = \dot{\varphi}_t$ is a Jacobi field, and the values for $t = 0$ are $J(0) = \partial_s c(s,0) = \dot{c}(s)$, and $\dot{J}(0) = D_t \partial_s c(s,0) = D_s \partial_t c(s,0) = D_s Z(s)$ again by 1.13, *q. e. d.*

1.17 **Fermi Coordinates.** Let $c :]a;b[\to M$ be an arclength-parametrised geodesic. Then Fermi or **geodesic normal coordinates** along c are an open neighbourhood U of $0 \in \mathbb{R}^{n-1}$ and coordinates $x :]a;b[\times U \to M$, in which $x(t,0) = c(t)$ and straight lines $s \mapsto c(t) + sv$ with first component $v^0 = 0$ are geodesics (arclength-parametrised for $|v|_{\ell^2} = 1$) perpendicular to c. This implies

$$g_{ij}(t,0) = \delta_{ij}, \quad \Gamma^k_{ij}(t,0) = 0 \qquad \text{for all } t \in]a;b[. \tag{1.17a}$$

If c is not a geodesic, then one can still find coordinates with $g_{ij}(t,0) = \delta_{ij}$, but the Christoffel symbols cannot be controlled. In classical surface geometry, those are called "parallel coordinates" along c (we will not use them).

1.18 **Lemma.** *Let $c :]a;b[\to M$ be a geodesic in M. Any orthonormal basis $E_2, \ldots, E_m$ of $\dot{c}(0)^\perp$ induces Fermi coordinates along c by $x : (t, u^2, \ldots, u^m) \mapsto \exp_{c(t)} (u^i P^{t,0} E_i)$.*

Proof. x is injective because the orthogonal projection onto c is well-defined in a small tube around c, and if a point $q \in M$ projects to $c(t)$, then the connecting geodesic $c(t) \rightsquigarrow q$ determines the components $u^2, \ldots, u^m$ by use of normal coordinates $\dot{c}(t)^\perp \to M$.

By definition of normal coordinates, the claim $g_{ij} = \delta_{ij}$ and $\Gamma^k_{ij} = 0$ along $(t, 0, \ldots, 0)$ is clear for $i,j,k \geq 2$. Because c is arclength-parametrised, $g_{11} = 1$, the orthogonality of $\dot{c}$ and E_i at every $c(t)$ gives $g_{1i} = 0$ for all i. Because $P^{t,0}$ is parallel, $\nabla_{\partial_1} \partial_i = \nabla_{\dot{c}} E_i = 0$ proves the vanishing of the remaining Christoffel symbols, *q. e. d.*

Corollary. *If P is the parallel transport along a geodesic in Mg running through $p \in M$, then for any vector $V \in T_pM$ and a vector field W around p, we have $P\nabla_V W = \nabla_{PV} PW$, and for a vector field V along a geodesic $t \mapsto c(t)$, the* ***fundamental theorem of calculus*** *holds:* 1.19

$$V(t) = P^{t,0}V(0) + \int_0^t P^{t,r}\dot{V}(r)\,\mathrm{d}r. \tag{1.19a}$$

The Distance and the Squared Distance Function

The following properties already occur in Karcher (1989) and Jost and Karcher (1982), but sometimes only hidden inside their proofs. For the same calculations in coordinates, see Ambrosio and Mantegazza (1998).

The geodesic distance $\mathbf{d}(\,\cdot\,,p)$ is a smooth convex function in some small neighbourhood B of p, excluded in p itself. It therefore has a gradient Y_p, and its length is the Lipschitz constant of $\mathbf{d}(\,\cdot\,,p)$, namely 1 everywhere. Additionally, 1.20

$$0 = V\langle Y_p, Y_p\rangle = 2\langle \nabla_V Y_p, Y_p\rangle = 2\langle \nabla_{Y_p} Y_p, V\rangle \qquad \text{for all } V \in T_qM, q \in B,$$

by symmetry 1.8a of the Hessian $\nabla d\,\mathbf{d}$, so Y_p is autoparallel everywhere. The integral curves of Y_p are hence geodesics emanating from p with $d\,\mathbf{d}(\dot\gamma) = 1$, so $\mathbf{d}(\gamma(t), p) = t$ for each such curve. On the other hand, $\mathbf{d}(\,\cdot\,,p)$ is constant on the distance spheres of p, so Y_p is perpendicular to them (**Gauss Lemma**). In normal coordinates $(u^1,\dots,u^m)$ around p, we have $\mathbf{d}(\,\cdot\,,p) = |u|_{\ell^2}$ and hence

$$\mathbf{d}(\,\cdot\,,p)Y_p = u^i\partial_i.$$

Observation. Base and evaluation point can be reversed, and the vector field only changes sign: $Y_p|_q = -P^{q,p}Y_q|_p$, because both are velocities of the arclength-parametrised geodesic $p \rightsquigarrow q$ or $q \rightsquigarrow p$ respectively. 1.21

Lemma. *In a small neighbourhood of p,* 1.22

$$X_p := \operatorname{grad} \tfrac{1}{2}\,\mathbf{d}^2(p,\,\cdot\,) = \mathbf{d}(p,\,\cdot\,)Y_p$$

is an everywhere smooth vector field, its integral lines are (quadratically parametrised) geodesics emanating from p, and $\exp_q(-X_p|_q) = p$, equivalently

$$-X_q|_p = P^{p,q}X_p|_q = (\exp_p)^{-1}q$$

for all q in a convex neighbourhood of p. Loosely speaking, one also writes this as $PX_p = \exp^{-1} p$.

Proof. Let c be the arclength-parametrised geodesic with $c(0) = p$ and $c(\tau) = q$. By definition of exp, we have $\exp_p \dot{c}(0) = q$, as well as $\dot{c}(t) = P^{t,0}\dot{c}(0)$ and $\dot{c}(t) = Y_p|_{c(t)}$ for all t by the Gauss lemma. The switch of base and evaluation point is justified by 1.21, *q. e. d.*

1.23 **Lemma.** *For $V \in T_qM$, where q is in a convex neighbourhood of p, let J be the Jacobi field along $p \rightsquigarrow q$ with $J(0) = 0$ and $J(\tau) = V$. Then*

$$\nabla_V X_p = \tau \dot{J}(\tau), \qquad \nabla^2_{V,V} X_p = \tau D_s \dot{J}(\tau).$$

In particular, if V is parallel to X_p, then $\nabla_V X_p = V$ and $\nabla^2_{V,V} X_p = 0$.

Proof. Let $s \mapsto \delta(s)$ be a geodesic with $\delta(0) = q$ and $\dot{\delta}(0) = V$. Define a variation of geodesics by

$$c(s,t) := \exp_p\big(t(\exp_p)^{-1}\delta(s)\big).$$

Then $\partial_t c$ is an autoparallel vector field and $J := \partial_s c$ a Jacobi field along $t \mapsto c(s,t)$ for every s with boundary values $J(s,0) = 0$ and $J(s,1) = \dot{\delta}(s)$. The t-derivative is

$$\partial_t c(s,t) = P^{t,0}(\exp_p)^{-1}\delta(s) = P^{t,1} X_p|_{\delta(s)}$$

and hence $\dot{J}(t) = D_t \partial_s c(0,t) = D_s \partial_t c(0,t) = D_s X_p|_{c(0,t)} = \nabla_{J(t)} X_p$. Differentiating this once more gives the claim for the second derivative. If V is parallel to X_p, then use $\nabla_Y Y = 0$, *q. e. d.*

Remark. ⟨**a**⟩ Variations of X_p with respect to the base point p will be considered in 12.3.

⟨**b**⟩ Analogously to $(\exp_p)^{-1} = PX_p$, the derivatives of X_p and $\exp_p$ correspond: $\nabla_V X_p$ is the derivative of some Jacobi field with prescribed start and end value, whereas $d_V(\exp_p)W = J(1)$ for a Jacobi field with $J(0) = 0$ and $\dot{J}(0) = W$.

⟨**c**⟩ Although Y_p is not differentiable at p, we have $\nabla X_p = \mathrm{id}$ at p, similar to $d_0 \exp_p = \mathrm{id}$.

⟨**d**⟩ In the notation of Grohs *et al.* (2013), our vector field X_p and its derivative are $X_p|_a = \log(a,p)$ and $\nabla X_p|_a = \nabla_2 \log(a,p)$.

Submanifolds

1.24 **Extrinsic Curvature.** For a smooth k-dimensional submanifold $S \subset M$, we treat T_pS as a linear subspace of T_pM, denote the orthogonal projection $T_pM \to T_pS$ as $\mathfrak{t}$ and the projection onto the normal space $T_pS^\perp$ as $\mathfrak{n}$. The bundle over S with fibres T_pM is denoted as $TM|_S = TS \oplus TS^\perp$ (meaning a fibre-wise sum of vector spaces). The **Weingarten map** or **shape operator** with respect to a normal field ν is $W_\nu := \nabla\nu$, that means $U \mapsto \nabla_U \nu$. The **second fundamental form** with respect to ν is

$$\mathrm{II}_\nu(U,V) := -\langle W_\nu U, V\rangle = \langle \nabla_U V, \nu\rangle \tag{1.24a}$$

because $\langle \nu, V\rangle = 0$ and hence $U\langle \nu, V\rangle = 0$. In particular, $\mathrm{II}_\nu(U,V)$ is in fact tensorial in ν, U and V. Sometimes $\mathrm{II}(U,V) := \mathfrak{n}\nabla_U V$ is also called the second fundamental form in the literature, although it is a bilinear map, not a form. If the orthonormal parallel normal fields $\nu_{k+1}, \ldots, \nu_m$ locally span $TS^\perp$, it holds $\mathrm{II}(U,V) = \nu_i \, \mathrm{II}_{\nu_i}(U,V)$. The covariant derivative induced by $g|_S$ is $\nabla^S = \mathfrak{t}\nabla$, hence $\mathrm{II} = \nabla - \nabla^S$.

Generalised Fermi or **Graph Coordinates.** The "tubular neighbourhood theorem" states that a small neighbourhood $\mathbb{B}_\varepsilon(S)$ of S is diffeomorphic to $S \times B_\varepsilon$ with an open ε-ball $B_\varepsilon \subset \mathbb{R}^{m-k}$ around 0 (BREDON 1993, thm. II.11.4). By explicitely constructing this diffeomorphism, upper bounds on ε can be derived: For $t \in [0;1]$, let **1.25**

$$\Phi_t : \quad TS^\perp \to M, \quad (p, Z) \mapsto \exp_p tZ. \tag{1.25a}$$

If p moves with velocity $\dot{p}$, we know by 1.16 that $d\Phi_t(\dot{p}) = J(t)$ for a Jacobi field with $J(0) = \dot{p}$, $\dot{J}(0) = \nabla_{\dot{p}} Z$.

The case where $Z = \nu$ is parallel along p is particularly interesting. Then Φ_t parametrises the level sets of the distance function from S; we have $\dot{J}(0) = W_\nu \dot{p}$, and the parallel transport of ν along t is normal to the image of Φ_t, so the whole curve fulfills $\dot{J} = W_\nu J$. Therefore, the pull-back metric $\Phi_t^* g(\dot{p}, \dot{p})$ changes with respect to t as $\frac{\mathrm{d}}{\mathrm{d}t} \Phi_t^* g\langle \dot{p}, \dot{p} \rangle = 2g\langle J, \dot{J} \rangle = 2g\langle J, W_\nu J \rangle$, see KARCHER (1989, eqn. 1.2.7). Hence the maximal eigenvalue of W_ν over $\nu \in \mathbb{S}^{m-k} \subset TS^\perp$ and over t bounds ε. We will pursue this more explicitely in 11.8. By $\ddot{J} = \frac{\mathrm{d}}{\mathrm{d}t}(WJ) = \dot{W}J + W\dot{J} = \dot{W}J + W^2 J$, one then obtains a Riccati-type equation for the Weingarten map (KARCHER 1989, eqn. 1.3.1)

$$\dot{W} = R_\nu - W^2 \qquad \text{for} \quad R_\nu = R(\nu, \cdot\,)\nu. \tag{1.25b}$$

Generally, a tangent vector $U \in T_{(p,Z)} TS^\perp$ is induced by a curve $s \mapsto \exp_{p(s)} tZ(s)$, where $\dot{p}$ is tangential to S and $\dot{Z} = t\dot{Z} + n\dot{Z}$. The above-mentioned Jacobi field J can be split into two Jacobi fields $J_p(s) + J_\nu(t)$ with initial values

$$\begin{aligned} J_p(0) &= \dot{p} & J_\nu(0) &= 0 \\ \dot{J}_p(0) &= t\dot{Z} & \dot{J}_\nu(0) &= n\dot{Z}. \end{aligned} \tag{1.25c}$$

The part $t\dot{Z}$ is in fact $t\nabla_{\dot{p}} Z$ (if we assume Z to be extended parallel along t), so it is uniquely determined by $\dot{p}$, and thus U has the representation $(\dot{p}, n\dot{Z})$ in the chart Φ_t. Let ψ be the orthogonal projection $\mathbb{B}_\varepsilon(S) \to S$. As Jacobi fields with orthogonal initial values and velocities stay orthogonal, we have an orthogonal splitting $V = V_p + V_\nu$ for $V \in T_p M$, $p \in \mathbb{B}_\varepsilon(S)$, with $V_p = J_p(1)$, $V_\nu = J_\nu(1)$. This gives a simple representation of $d\psi$, namely $d\psi(V_\nu) = 0$ and $d\psi(V_p) = \dot{p}$. The geometric interpretation of the splitting is

$$V_p = P^{p,\psi(p)} t P^{\psi(p),p} V, \qquad V_\nu = P^{p,\psi(p)} n P^{\psi(p),p} V, \tag{1.25d}$$

that means V_p and V_ν are the orthogonal projections onto PTS and $PTS^\perp$ respectively. This is proven by $\frac{\mathrm{d}^2}{\mathrm{d}t^2}\langle J_p, Z\rangle = 0$ (if Z is extended parallel along t) and the initial conditions $\langle J_p(0), Z\rangle = 0$ and $\langle \dot{J}_p(0), Z\rangle = 0$.

2. Functional Analysis and Exterior Calculus

We will quickly review the Dirichlet problem and the Hodge decomposition in this section. All proofs are reformulations from SCHWARZ (1995), but we tried to take special care that not the vector bundle structure of Ω^k, but only its functional analytical nature has been use (the only exception is 2.16).

Notation. ⟨**a**⟩ We have defined $\mathfrak{X}$ and Ω^k as the spaces of smooth vector fields and k-forms on M. The pointwise scalar product g on all tensor products of TM and T^*M naturally induces an L^2 product on them:

$$\langle\!\langle v, w\rangle\!\rangle := \int_M g\langle v, w\rangle$$

The completion $\mathfrak{X}$ with respect to the L^2 norm will be called $\mathsf{L}^2\mathfrak{X}$, analogously $\mathsf{L}^2\Omega^k$ for the differential forms. The notation $\langle\!\langle\,\cdot\,,\,\cdot\,\rangle\!\rangle$ will only be used for the L^2 scalar product, so all indices like $\langle\!\langle\,\cdot\,,\,\cdot\,\rangle\!\rangle_{\mathsf{L}^2}$, $\langle\!\langle\,\cdot\,,\,\cdot\,\rangle\!\rangle_{\mathsf{L}^2(Mg)}$ and $\langle\!\langle\,\cdot\,,\,\cdot\,\rangle\!\rangle_{\mathsf{L}^2\mathfrak{X}}$ or $\langle\!\langle\,\cdot\,,\,\cdot\,\rangle\!\rangle_{\mathsf{L}^2\Omega^k}$ are only added for ease of reading.

⟨**b**⟩ Let M have a boundary ∂M. The projections $\boldsymbol{t}$ and $\boldsymbol{n}$ from $TM|_{\partial M}$ onto $T\partial M$ and $T\partial M^\perp$ pull back k-forms as $\boldsymbol{t}^*v(V_1, \dots, V_k) = v(\boldsymbol{t}V_1, \dots, \boldsymbol{t}V_k)$ and similarly $\boldsymbol{n}^*\omega$. The spaces of k-forms with vanishing tangential part on ∂M are called Ω^k_t.

Together with the usual **exterior derivative** d, the Ω^k form the smooth **de Rham cochain complex**

$$\Omega^0 \to \dots \to \Omega^n \to 0$$

The **exterior coderivative** δ is, for forms with appropriate boundary conditions, adjoint to d with respect to the L^2 scalar product:

$$\langle\!\langle v, dw\rangle\!\rangle_{\mathsf{L}^2\Omega^{k+1}} = \langle\!\langle \delta v, w\rangle\!\rangle_{\mathsf{L}^2\Omega^k} \qquad \begin{array}{l}\text{for all } v \in \Omega^{k+1}, w \in \Omega^k_t \\ \text{and all } v \in \Omega^{k+1}_n, w \in \Omega^k\end{array} \tag{2.1}$$

The image and the kernel of d in Ω^k are called the spaces of **boundaries** and **cycles**, $\mathfrak{B}^k := \operatorname{im} d|_{\Omega^{k-1}}$ and $\mathfrak{C}^k := \ker d|_{\Omega^k}$. The space of **harmonic** forms is $\mathfrak{H}^k := \mathfrak{C}^k \cap d(\Omega^{k-1}_t)^\perp$. For δ, we have $\mathfrak{B}^*_k$ and $\mathfrak{C}^*_k$ defined analogously, so $\mathfrak{H}^k = \mathfrak{C}^k \cap \mathfrak{C}^*_k$ by 2.1. Denote

$$\mathrm{Lap}(v, w) := \langle\!\langle dv, dw\rangle\!\rangle + \langle\!\langle \delta v, \delta w\rangle\!\rangle, \qquad \mathrm{Dir}(v) := \mathrm{Lap}(v, v). \tag{2.2}$$

2.3 **Definition.** Define the following six norms on each Ω^k:

$$\begin{aligned}
|v|^2_{\mathsf{H}^{1,0}} &:= |v|^2_{\mathsf{L}^2} + |dv|^2_{\mathsf{L}^2}\\
|v|^2_{\mathsf{H}^{0,1}} &:= |v|^2_{\mathsf{L}^2} + |\delta v|^2_{\mathsf{L}^2}\\
|v|^2_{\mathsf{H}^{1,1}} &:= |v|^2_{\mathsf{L}^2} + |dv|^2_{\mathsf{L}^2} + |\delta v|^2_{\mathsf{L}^2} = |v|^2_{\mathsf{L}^2} + \mathrm{Dir}(v)\\
|v|^2_{\mathsf{H}^{1+1}} &:= |v|^2_{\mathsf{H}^{1,1}} + |d\delta v|^2_{\mathsf{L}^2} + |\delta dv|_{\mathsf{L}^2}\\
|v|^2_{\mathsf{H}^1} &:= |v|^2_{\mathsf{L}^2} + |\nabla v|^2_{\mathsf{L}^2}\\
|v|^2_{\mathsf{H}^2} &:= |v|^2_{\mathsf{H}^1} + |\nabla^2 v|^2_{\mathsf{L}^2}
\end{aligned}$$

Let $\mathsf{H}^{1,0}\Omega^k$ etc. be the completion of Ω^k with respect to these norms. The L^r, $\mathsf{W}^{1,r}$ and $\mathsf{W}^{2,r}$ norms are the usual modification of the L^2, H^1 and H^2 norms for exponents $r \neq 2$.

2.4 **Observation.** ⟨**a**⟩ $(\mathsf{H}^{1,0}\Omega, d)$ is a cochain and $(\mathsf{H}^{0,1}\Omega, \delta)$ is a chain Hilbert complex, that means that d or δ are bounded linear operators with $d^2 = 0$ or $\delta^2 = 0$ respectively (to be notationally precise, a Hilbert complex requires d or δ only to be closed operators).

⟨b⟩ The H^1 norm dominates the $\mathsf{H}^{1,0}$ and the $\mathsf{H}^{0,1}$ norm; the H^2 norm dominates all the other norms. For functions, $\mathsf{H}^{0,1} = \mathsf{L}^2$ and $\mathsf{H}^{1,0} = \mathsf{H}^{1,1} = \mathsf{H}^1$.

Remark. SCHWARZ (1995, sec. 1.3) uses a different definition of H^2 which depends on local choices of orthonormal bases. He then uses the H^1 and H^2 norms to control the exterior (co-)derivatives. We consider the use of $\mathsf{H}^{1,0}$ and similar norms a sharper tool for this, as only the actually needed derivatives have to exist. JOST (2011, eqn. 3.4.4) writes H^2 for what we call H^{1+1}.

Fact (BRÜNING and LESCH 1992, corr. 2.6)**.** The spaces $\mathsf{H}^{1,1}\mathfrak{B}^k$ are closed in $\mathsf{H}^{1,1}\Omega^k$ if and only if $\mathsf{H}^{1,1}\mathfrak{B}^*_k$ are closed in $\mathsf{H}^{1,1}\Omega^k$. If this is the case, and if $\mathfrak{H}^k$ is finite-dimensional, then $(\mathsf{H}^{1,1}\Omega^k, d)$ is called a **Fredholm complex.** **2.5**

LAPLACE OPERATOR AND DIRICHLET PROBLEM

Observation. Directly from Green's formula 2.1, one gets for $v \in \mathsf{H}^{1+1}$, $w \in \mathsf{H}^{1,1}$ **2.6**

$$\operatorname{Lap}(v, w) = \langle (d\delta + \delta d)v, w \rangle$$

in either of these four cases:

$$\begin{aligned} &t^*w = 0,\ n^*w = 0 && t^*w = 0,\ t^*\delta v = 0 \\ &n^*w = 0,\ n^*dv = 0 && n^*dv = 0,\ t^*\delta u = 0. \end{aligned}$$

Definition. The **strong Laplacian** is $\Delta := d\delta + \delta d : \mathsf{H}^{1+1}\Omega^k \to \mathsf{L}^2\Omega$. The **weak Laplacian** is $L : \mathsf{H}^{1,1}\Omega^k \to (\mathsf{H}^{1,1}\Omega^k)^*, v \mapsto \operatorname{Lap}(\,\cdot\,, v)$. The **strong Dirichlet problem** is to find $u \in \mathsf{H}^{1+1}\Omega^k$ with **2.7**

$$\Delta u = f, \qquad t^*u = 0,\ t^*\delta u = 0. \tag{2.7a}$$

The **weak Dirichlet problem** is to find $u \in \mathsf{H}^{1,1}\Omega^k_t$ with $Lu = f$ in $(\mathsf{H}^{1,1}\Omega^k_t)^*$, that means

$$\langle du, dv \rangle + \langle \delta u, \delta v \rangle = \langle f, v \rangle \qquad \text{for all } v \in \mathsf{H}^{1,1}\Omega^k_t. \tag{2.7b}$$

Such a u is called a **Dirichlet potential** for f.

Fact (**Dirichlet principle**)**.** A form $u \in \mathsf{H}^{1,1}\Omega^k_t$ is a solution of the weak Dirichlet problem if and only if it minimises $\operatorname{Dir}(v) - \langle f, v \rangle$ over all $v \in \mathsf{H}^{1,1}\Omega^k_t$.

Remark. For sections of smooth vector bundles over M, the trace of the second covariant derivative gives a "metric" Laplace operator $\operatorname{tr} \nabla^2$, connected to our Laplacian or "Laplace–Beltrami" operator by the Weizenböck formula (JOST 2011, thm 4.3.3.), which we do not use, and only mention to avoid confusion. They agree if and only if Mg is flat.

Proposition. *If $u \in \mathsf{H}^{1,1}\Omega^k_t$ is a solution of the weak Dirichlet problem and is in addition contained in $\mathsf{H}^{1+1}\Omega^k$, then it solves the strong Dirichlet problem.* **2.8**

Proof. Suppose $\mathrm{Lap}(u,v) = \langle f, v\rangle$ for all $v \in \mathsf{H}^{1,1}\Omega^k_t$. Then a fortiori this holds for $v \in \mathsf{H}^{1,1}\Omega^k_{t,n}$ and so, by 2.6, $\langle \Delta u, v\rangle = \langle f, v\rangle$ for all $v \in \mathsf{H}^{1,1}\Omega^k_{t,n}$, which shows $\Delta u = f$ by the fundamental lemma. As the vanishing boundary values for v were only needed in the use of Green's formula, not in the L^2 testing, we can infer $\langle \Delta u, v\rangle = \langle f, v\rangle$ for all $v \in \mathsf{H}^{1,1}\Omega^k$ by continuity. But as $\mathsf{H}^{1,1}\Omega^k$ contains functions whose normal trace does not vanish, $\langle \Delta u, v\rangle = \mathrm{Lap}(u,v)$ can only hold if $t\delta u = 0$, *q. e. d.*

Remark. If we had used, for the weak Dirichlet problem, $\mathsf{H}^{1+1}\Omega^k$ for the space of test functions instead of $\mathsf{H}^{1,1}\Omega^k_t$, then the space of Dirichlet potentials for $f = 0$ would agree with $\mathsf{H}^{1,1}\mathfrak{H}^k_t$. But by our definition, this requires the additional assumption $u \in \mathsf{H}^{1+1}\Omega^k$.

2.9 **Observation.** By definition of the spaces involved and Green's formula 2.1, one directly obtains:

$$\begin{aligned} d(\mathsf{H}^{1,0}\Omega^{k-1}_t) &\subset \delta(\mathsf{H}^{1+1}\Omega^{k+1})^\perp & \delta(\mathsf{H}^{0,1}\Omega^{k+1}_n) &\subset d(\mathsf{H}^{1+1}\Omega^{k-1})^\perp \\ \mathsf{H}^{1,0}\mathfrak{C}^k &\supset \delta(\mathsf{H}^{1+1}\Omega^{k+1}_n)^\perp & \mathsf{H}^{0,1}\mathfrak{C}^*_k &\supset d(\mathsf{H}^{1+1}\Omega^{k-1}_t)^\perp \\ \mathsf{H}^{1,1}\mathfrak{H}^k_n &\perp \mathsf{H}^{0,1}\mathfrak{B}^k & \mathsf{H}^{1,1}\mathfrak{H}^k_t &\perp \mathsf{H}^{1,0}\mathfrak{B}^*_k \end{aligned} \tag{2.9a}$$

As $\mathsf{H}^{1,0}\Omega^k$ and $\mathsf{H}^{0,1}\Omega^k$ are completions of spaces with $d^2 = 0$ and $\delta^2 = 0$, this property carries over:

$$d(\mathsf{H}^{1,0}\Omega^k) \subset \mathsf{H}^{1,0}\mathfrak{C}^k, \qquad \delta(\mathsf{H}^{0,1}\Omega^k) \subset \mathsf{H}^{0,1}\mathfrak{C}^*_k. \tag{2.9b}$$

2.10 **Proposition (Poincaré inequality).** *Let $(\mathsf{H}^1\Omega^k, d)$ be a Fredholm complex where the inclusion map $\mathsf{H}^1\Omega^k \to \mathsf{L}^2\Omega^k$ is compact. Then:*

⟨a⟩ Dir *is H^1-coercive on $\mathsf{H}^1(\mathfrak{H}^k)^\perp$, that means there is $C_\boxdot > 0$ with*

$$|v|^2_{\mathsf{H}^1} \le C_\boxdot \,\mathrm{Dir}(v) \qquad \textit{for all } v \in \mathsf{H}^1\Omega^k, v \perp \mathfrak{H}^k.$$

⟨b⟩ *If the trace operator $v \mapsto t^*v$ is a continuous mapping $\mathsf{H}^1\Omega^k(M) \to \mathsf{L}^2\Omega^k(\partial M)$, then* Dir *is H^1-coercive on $(\mathfrak{H}^k_t)^\perp_t$, that means there is $C_\boxdot > 0$ with*

$$|v|^2_{\mathsf{H}^1} \le C_\boxdot \,\mathrm{Dir}(v) \qquad \textit{for all } v \perp \mathfrak{H}^k_t,\ t^*v = 0.$$

⟨c⟩ *If $\mathfrak{A}$ is a closed affine subspace in $\mathsf{H}^1\Omega^k$ that does not include constant forms $\neq 0$, then there is $C_\boxdot > 0$ with*

$$|v|^2_{\mathsf{L}^2} \le C_\boxdot \,|\nabla v|^2_{\mathsf{L}^2} \qquad \textit{for all } v \in \mathfrak{A}.$$

Examples are $\mathfrak{A} = \{u \in \mathsf{H}^1\Omega^k\colon u|_{\partial M} = 0\}$ if M has a boundary, or $\mathfrak{A} = \{u \in \mathsf{H}^1\Omega^k\colon \fint_M u = 0\}$ in cases where the integral $\int_U u$ makes sense. By linear translation, the latter one gives the consequence $|v - \fint_M v|_{\mathsf{L}^2} \le C_\boxdot |\nabla v|_{\mathsf{L}^2}$ for functions $v \in \mathsf{H}^1\Omega^0$.

Proof. It obviously suffices to show the last claim with $|v|^2_{\mathsf{H}^1}$ instead of $|v|^2_{\mathsf{L}^2}$ on the left-hand side. Then the proof always follows the same lines: If the claim is wrong, there has to be a sequence $\langle v_i\rangle \subset \mathsf{H}^1\Omega^k$ with $|v_i|_{\mathsf{H}^1} = 1$, and the right-hand side tends to 0. Because this sequence is bounded in $\mathsf{H}^1\Omega^k$, there has to be a weakly convergent subsequence, which we again denote by $\langle v_i \rangle$. This will suffice to extract a contradiction in all three cases.

ad primum: Because $\mathfrak{H}^k$ is finite-dimensional, it is closed, and so is $(\mathfrak{H}^k)^\perp$, that means $v \perp \mathfrak{H}^k$. At the same time, $\operatorname{Dir}(v) = \lim \operatorname{Dir}(v_i) = 0$, so $v \in \mathsf{H}^{1,1}\mathfrak{H}^k$. Therefore, $v = 0$, but at the same time $|v|_{\mathsf{L}^2} = \lim |v_i|_{\mathsf{L}^2} > 0$ because the imbedding $\mathsf{H}^1\Omega^k \to \mathsf{L}^2\Omega^k$ is compact.

ad sec.: By assumption, $(\mathfrak{H}^k_t)_t$ is closed, so the argument works again, as $t^*v = \lim t^*v_i = 0$.

ad tertium: Here the convergence of the right-hand side means $\nabla v_i \to 0$ strongly in L^2, hence $\nabla v = 0$, so v has to be constant almost everywhere, so $v = 0$ by assumption on $\mathfrak{A}$, *q. e. d.*

Remark. **⟨a⟩** It is common to prove the last part constructively, see 13.7. We are not aware of a constructive proof for the first and second case. **2.11**

⟨b⟩ By a scaling argument, one can see that $C_\boxdot = \tilde{C}_\boxdot \operatorname{diam} M$ with a constant $\tilde{C}_\boxdot$ that does not depend on the size of M.

Proposition. *Situation as in 2.10b. Let $f \in \mathsf{L}^2\Omega^k$ with $f \perp \mathsf{H}^{1,1}\mathfrak{H}^k_t$. Then there is exactly one $u \in (\mathsf{H}^{1,1}\mathfrak{H}^k_t)^\perp_t$ with $\operatorname{Lap}(u, v) = \langle f, v \rangle$ for all $v \in \mathsf{H}^{1,1}\Omega^k_t$.* **2.12**

Proof. By the Lax–Milgram theorem, there is exactly one $u \in (\mathsf{H}^{1,1}\mathfrak{H}^k_t)^\perp_t$ with $\operatorname{Lap}(u, v) = \langle f, v \rangle$ for all $v \in (\mathsf{H}^{1,1}\mathfrak{H}^k_t)^\perp_t$. Now observe that not only $\mathsf{H}^{1,1}\Omega^k_t = \mathsf{H}^{1,1}\mathfrak{H}^k_t \oplus (\mathsf{H}^{1,1}\mathfrak{H}^k_t)^\perp$, but that the second summand must also have zero boundary values, so $\mathsf{H}^{1,1}\Omega^k_t = \mathsf{H}^{1,1}\mathfrak{H}^k_t \oplus (\mathsf{H}^{1,1}\mathfrak{H}^k_t)^\perp_t$. So everything that is missing is to prove this equality also for $v \in \mathsf{H}^{1,1}\mathfrak{H}^k_t$. But that is not difficult: On the one hand, $\operatorname{Lap}(\,\cdot\,, v) = 0$ for such v, on the other $\langle f, v \rangle = 0$ by assumption on f, *q. e. d.*

Remark. **⟨a⟩** For each $u^* \in \mathsf{H}^{1,1}\mathfrak{H}^k_t$, one also has $L(u + u^*) = f$. So the solution is unique up to harmonic components. This non-uniqueness for manifolds of higher genus can indeed be observed in numerics, cf. Arnold *et al.* (2010, section 2.3.3).

⟨b⟩ If f is not orthogonal to $\mathsf{H}^{1,1}\mathfrak{H}^k_t$, then there is an orthogonal projection p of f to this space and a Dirichlet potential u for $f - p$.

Hodge Decompositions

Proposition (**weak Hodge decomposition**, Brüning and Lesch 1992, lemma 2.1)**.** *There is an orthogonal decomposition* **2.13**

$$\mathsf{H}^{1,0}\Omega^k = d(\mathsf{H}^{1,0}\Omega^{k-1}_t) \oplus \mathsf{H}^{1,0}(\mathfrak{C}^k)^\perp \oplus \mathsf{H}^{1,0}\mathfrak{H}^k,$$

where $\mathsf{H}^{1,0}\mathfrak{H}^k := \mathsf{H}^{1,0}\mathfrak{C}^k \cap d(\mathsf{H}^{1,0}\Omega^{k-1}_t)^\perp$.

Proof. By definition of $\mathsf{H}^{1,0}\mathfrak{H}^k$, this sum exhausts $\mathsf{H}^{1,0}\Omega^k$, and the last summand is orthogonal to the two other ones. It only remains to show the orthogonality of $d(\mathsf{H}^{1,0}\Omega^{k-1}_t)$ and $\mathsf{H}^{1,0}(\mathfrak{C}^k)^\perp$. So let $u = da$ with $a \in \mathsf{H}^{1,0}\Omega^{k-1}_t$. Then by 2.9b $u \in \mathsf{H}^{1,0}\mathfrak{C}^k$, hence it is perpendicular to each element in $\mathsf{H}^{1,0}(\mathfrak{C}^k)^\perp$, *q. e. d.*

2.14 **Proposition** (**strong Hodge decomposition**, Brüning and Lesch 1992, cor. 2.5; Schwarz 1995, thm. 2.4.2). *For a Fredholm complex, there is an orthogonal decomposition*

$$\mathsf{H}^{1,1}\Omega^k = d(\mathsf{H}^{1+1}\Omega_t^{k-1}) \oplus \delta(\mathsf{H}^{1+1}\Omega_n^{k+1}) \oplus \mathsf{H}^{1,1}\mathfrak{H}^k.$$

In other words: each $u \in \mathsf{H}^{1,1}\Omega^k$ can be decomposed as $u = da + \delta b + c$ with $\mathfrak{t}a = 0$, $\mathfrak{n}b = 0$, $dc = 0$, and $\delta c = 0$. The parts a and b can be computed as minimisers of $F[u](a) = \langle da, da\rangle - 2\langle da, u\rangle$ over $a \in \mathsf{H}^{1+1}(\mathfrak{C}^{k-1})_t^\perp$ and $G[u](b) = \langle \delta b, \delta b\rangle - 2\langle \delta b, u\rangle$ over $b \in \mathsf{H}^{1+1}(\mathfrak{C}^*_{k+1})_n^\perp$ respectively.

Proof. By the Fredholm property, $\mathsf{H}^{1,1}\mathfrak{H}^k$ is closed and hence convex. By the projection theorem (cf. e. g. Alt 2006, thm. 2.2), $\mathsf{H}^{1,1}\Omega^k$ has an orthogonal decomposition $\mathsf{H}^{1,1}\Omega^k = \mathsf{H}^{1,1}\mathfrak{H}^k \oplus (\mathsf{H}^{1,1}\mathfrak{H}^k)^\perp$. So everything we have to show is

$$\begin{aligned} &\mathsf{H}^{1,1}(\mathfrak{H}^k)^\perp = d(\mathsf{H}^{1+1}\Omega_t^{k-1}) \oplus \delta(\mathsf{H}^{1+1}\Omega_n^{k+1}) \\ \Leftrightarrow \quad &\mathsf{H}^{1,1}\mathfrak{H}^k = \big(d(\mathsf{H}^{1+1}\Omega_t^{k-1}) \oplus \delta(\mathsf{H}^{1+1}\Omega_n^{k+1})\big)^\perp. \end{aligned}$$

The inclusion $\subset$ is clear by the last part of 2.9a. For the other direction consider $u \in \mathsf{H}^{1,1}\Omega^k$ such that

$$\begin{aligned} \langle u, dv\rangle &= 0 &&\text{for all } v \in \mathsf{H}^{1,0}\Omega_t^{k-1}, \\ \langle u, \delta w\rangle &= 0 &&\text{for all } w \in \mathsf{H}^{0,1}\Omega_n^{k+1}. \end{aligned}$$

Because of Green's formula, this means

$$\begin{aligned} \langle \delta u, v\rangle &= 0 &&\text{for all } v \in \mathsf{H}^{1,0}\Omega_t^{k-1}, \\ \langle du, w\rangle &= 0 &&\text{for all } w \in \mathsf{H}^{0,1}\Omega_n^{k+1}. \end{aligned}$$

These v and w suffice to test for $du = 0$ and $\delta u = 0$ in the interior of M, so $u \in \mathsf{H}^{1,1}\mathfrak{H}^k$.

For the variational property, observe that the orthogonal projection da of $u \in \mathsf{H}^{1,1}\Omega^k$ onto $d(\mathsf{H}^{1+1}\Omega_t^{k-1})$ fulfills $\langle u - da, dv\rangle = 0$ for every $v \in \mathsf{H}^{1+1}\Omega_t^{k-1}$, which is exactly the optimality condition for F. The minimiser is unique up to elements of $\mathfrak{C}^k$, for which reason we only seek a in $(\mathfrak{C}^k)^\perp$. The analogous argument applies for G, *q. e. d.*

2.15 **Remark.** By assumption, the ranges of d and δ are closed, so a continuity argument also shows that $\mathsf{L}^2[d(\mathsf{H}^{1,0}\Omega_t^k \oplus \delta(\mathsf{H}^{0,1}\Omega_n^k)]^\perp$ is the L^2 completion $\bar{\mathfrak{H}}^k$ of $\mathfrak{H}^k$, which gives the L^2 Hodge decomposition

$$\mathsf{L}^2\Omega^k = d(\mathsf{H}^{1,0}\Omega_t^{k-1}) \oplus \delta(\mathsf{H}^{0,1}\Omega_n^{k+1}) \oplus \bar{\mathfrak{H}}^k.$$

2.16 **Proposition** (**Hodge–Friedrichs decomposition**, Schwarz 1995, thm 2.4.8). *For a Fredholm complex that is H^{1+1}-regular, that means the Dirichlet potential of an L^2 right-hand side is in $\mathsf{H}^{1+1}\Omega^k$, there is an orthogonal decomposition*

$$\begin{aligned} \mathsf{H}^{1,1}\mathfrak{H}^k &= \mathsf{H}^{1,1}\mathfrak{H}_t^k \oplus \mathsf{H}^{1,1}\mathfrak{H}^k \cap \mathsf{H}^{1,0}\mathfrak{B}_k^*, \\ &= \mathsf{H}^{1,1}\mathfrak{H}_n^k \oplus \mathsf{H}^{1,1}\mathfrak{H}^k \cap \mathsf{H}^{0,1}\mathfrak{B}^k. \end{aligned}$$

Together with 2.14, this gives the decomposition

$$\begin{aligned}\mathsf{H}^{1,1}\Omega^k &= d(\mathsf{H}^{1+1}\Omega_t^{k-1}) \oplus \delta(\mathsf{H}^{1+1}\Omega_n^{k+1}) \oplus \mathsf{H}^{1,1}\mathfrak{H}_t^k \oplus \mathsf{H}^{1,1}\mathfrak{H}^k \cap \mathsf{H}^{1,0}\mathfrak{B}_k^* \\ &= d(\mathsf{H}^{1+1}\Omega_t^{k-1}) \oplus \delta(\mathsf{H}^{1+1}\Omega_n^{k+1}) \oplus \mathsf{H}^{1,1}\mathfrak{H}_n^k \oplus \mathsf{H}^{1,1}\mathfrak{H}^k \cap \mathsf{H}^{0,1}\mathfrak{B}^k.\end{aligned}$$

Proof. We only prove the first decomposition, the second one is literally the same. By the last statement of 2.9a, $\mathsf{H}^{1,1}\mathfrak{H}_t^k \perp \mathsf{H}^{1,0}\mathfrak{B}_k^*$. So if the decomposition exists, it will be orthogonal. We are done if we can show that every $u \in \mathsf{H}^{1,1}\mathfrak{H}^k$ with $u \perp \mathsf{H}^{1,1}\mathfrak{H}_t^k$ is a coboundary, that means there is some $v \in \mathsf{H}^{1+1}\Omega^{k-1}$ with $u = \delta v$.

Therefore, suppose $u \in \mathsf{H}^{1,1}\mathfrak{H}^k$ and $u \perp \mathsf{H}^{1,1}\mathfrak{H}_t^k$. Then by 2.12 there is a Dirichlet potential $w \in \mathsf{H}^{1,1}\Omega_t^k$ for u. Let $v := dw$.

(I.) It holds $u - \delta v \perp \mathsf{H}^{1,1}\mathfrak{H}_t^k$ due to the assumption on u and $\delta v \in \mathsf{H}^{1,0}\mathfrak{B}_k^*$.

(II.) As $\delta u = 0$ and $\delta^2 v = 0$, we have $\delta(u - \delta v) = 0$. Because the Dirichlet problem is H^{1+1}-regular, $u = (d\delta + \delta d)w$ and hence $d(u - \delta v) = d(d\delta w + \delta dw - \delta dw) = d(d\delta w) = 0$. This shows that $u - \delta v \in \mathsf{H}^{1,1}\mathfrak{H}^k$. And because $u - \delta v = d\delta w$ is a boundary, it has vanishing tangent component: Take any $(k-1)$-dimensional domain $U \subset \partial M$. There is a domain $U' \subset M$ with $U = U' \cap \partial M$, and $\int_U t^*(u - \delta v) = \int_{U'} d(u - \delta v) = 0$. For this reason, $u - \delta v \in \mathsf{H}^{1,1}\mathfrak{H}_t^k$.

Now $u - \delta v$ is at the same time in some space and its orthogonal complement, which can only hold if $u - \delta v = 0$, *q. e. d.*

Mixed Form of the Dirichlet Problem in Ω^k

Observation. The Dirichlet problem also has a **mixed form** without coderivatives (Arnold *et al.* 2006, sec. 7.1): If one introduces the auxiliary variable σ, which replaces δu in a weak sense, i.e. which fulfills $\langle \sigma, \tau \rangle = \langle d\tau, u \rangle$ for all $\tau \in \mathsf{H}^{1,0}\Omega_t^{k-1}$, then $Lu = f - p$ can be written as **2.17**

$$\begin{aligned}\langle d\sigma, v\rangle + \langle du, dv\rangle &= \langle f - p, v\rangle && \text{for all } v \in \mathsf{H}^{1,0}\Omega_t^k, \\ \langle \sigma, \tau\rangle - \langle u, d\tau\rangle &= 0 && \text{for all } \tau \in \mathsf{H}^{1,0}\Omega_t^{k-1}, \\ \langle u, q\rangle &= 0 && \text{for all } q \in \mathsf{H}^{1,1}\mathfrak{H}_t^k.\end{aligned}$$

Let $\mathfrak{S} := \mathsf{H}^{1,0}\Omega_t^{k-1} \times \mathsf{H}^{1,0}\Omega_t^k \times \mathsf{H}^{1,1}\mathfrak{H}_t^k$. By computing the Euler–Lagrange equation, one sees that $(\sigma, u, p) \in \mathfrak{S}$ solves the equations above if and only if it is a minimiser of

$$I(\sigma, u, p) := \tfrac{1}{2}\langle \sigma, \sigma\rangle - \langle d\sigma, u\rangle - \tfrac{1}{2}\langle du, du\rangle + \langle f - p, u\rangle.$$

In such a critical point, every $(\sigma, 0, 0) \in \mathfrak{S}$ is a descent direction, and every $(0, v, q) \in \mathfrak{S}$ is an ascent direction, so I has a saddle point at (σ, u, p).

Proposition (Arnold *et al.* 2006, thm. 7.2)**.** *In a Fredholm complex, the weak Dirichlet problem in mixed formulation is well-posed, that means: There is a solution $(\sigma, u, p) \in \mathfrak{S}$ for every $f \in \mathsf{L}^2\Omega^k$, and there is a constant c only depending on M such that $|\sigma, u, p|_{\mathfrak{S}} \leq c|f|_{\mathsf{L}^2}$.* **2.18**

Proof. For $s = (\sigma, u, p)$ and $t = (\tau, v, q)$, let

$$b(\sigma, u, p; \tau, v, q) := \langle \sigma, \tau \rangle - \langle u, d\tau \rangle + \langle d\sigma, v \rangle + \langle du, dv \rangle + \langle p, v \rangle - \langle u, q \rangle, \qquad (2.18a)$$

let $B : \mathfrak{S} \to \mathfrak{S}^*$ be the linear operator with $b(s,t) = \langle Bs, t \rangle_{\mathfrak{S}^*, \mathfrak{S}}$ and $F : t \mapsto \langle f, v \rangle_{\mathsf{L}^2}$. We have to show that there is a solution to the operator equation $B(s) = F$ in $\mathfrak{S}^*$.

There is an inf-sup condition for b: According to 2.13, decompose $u = da + x + y$ with $a \in \mathsf{H}^{1,0}\Omega_t^k \cap \mathfrak{C}^{k-1}$ and $x \perp \mathfrak{C}^k$. Then by 2.10b, we have $|a|_{\mathsf{H}^{1,0}} \le C_\boxdot |a|_{\mathsf{L}^2} \le C_\boxdot |u|_{\mathsf{L}^2}$ and by 2.10a, $|x|_{\mathsf{H}^{1,0}} \le C_\boxdot |dx|_{\mathsf{L}^2} = C_\boxdot |du|_{\mathsf{L}^2}$. With $\tau = \sigma - C_\boxdot^{-2} a$, $v = u + d\sigma + p$ and $q = p - y$, one obtains (all norms are L^2 norms here) $b(\sigma, u, p; \tau, v, q) = |\sigma|^2 - C_\boxdot^{-2}\langle \sigma, a \rangle + C_\boxdot^{-2}|da|^2 + |d\sigma|^2 + |du|^2 + |p|^2 + |y|^2$, which is (for $C_\boxdot \ge 1$) greater than $C_\boxdot^{-2}(|\sigma|^2_{\mathsf{H}^{1,0}} + |u|^2_{\mathsf{H}^{1,0}} + |p|^2_{\mathsf{L}^2})$. As $|\tau, v, q|_\mathfrak{S} \le \alpha |\sigma, u, p|_\mathfrak{S}$ for some $\alpha \in \mathbb{R}$, we have shown that there is a constant γ with

$$\sup_{t \in \mathfrak{S}} \frac{b(s,t)}{|t|_\mathfrak{S}} \ge \gamma |s|_\mathfrak{S} \qquad \text{for all } s \in \mathfrak{S}. \qquad (2.18b)$$

Once this inf-sup-condition is established, the way is well-known (Babuška and Aziz 1972, thm. 5.2.1): Consider the linear operator $B : \mathfrak{S} \to \mathfrak{S}^*$ belonging to b. Because of $|Bs|_{\mathfrak{S}^*} \ge \gamma |s|_\mathfrak{S}$, it must be injective. And as we have that for any $t \in \mathfrak{S}$ there is some $s \in \mathfrak{S}$ with $b(s,t) \neq 0$, the image of B must be the whole space $\mathfrak{S}^*$ by the closed range theorem. Using the inf-sup condition once again, we get that $|\alpha|_{\mathfrak{S}^*} \ge \gamma |B^{-1}\alpha|_\mathfrak{S}$, which shows that B^{-1} is continuous, *q. e. d.*

2.19 **Remark.** The only thing that *cannot* be inferred from our high-level point of view is that $(\mathsf{H}^{1,0}\Omega, d)$ indeed forms a Fredholm complex, and that the imbedding $\mathsf{H}^1\Omega^k \to \mathsf{L}^2\Omega^k$ is compact. The finite-dimensionality of $\mathfrak{H}^k$ and hence the Fredholm property is proven by the inequality $|v|_{\mathsf{H}^1} \le C(\mathrm{Dir}(v) + |v|_{\mathsf{L}^2})$ of Gaffney (1951), see e. g. Schwarz (1995, cor. 2.1.6). The compact embedding is Rellich's inequality (see e. g. Alt 2006, A 6.1), which carries over from the Euclidean case without modification. Both need the vector bundle structure of $(\mathsf{H}^{1,0}\Omega, d)$, but they are obviously also true if $(\mathsf{H}^{1,0}\Omega, d)$ is a complex of finite-dimensional vector spaces. Therefore all proofs above literally carry over to the "discrete exterior calculus" from section 9. A very preliminary version of this attempt of formulation exterior calculus without recurrence to the vector-bundle structure has been given in von Deylen (2012).

2.20 **Definition.** When we speak of **variational problems** in the forthcoming sections, we will always refer to the Hodge decomposition, the Dirichlet problem and other strongly elliptic problems.

3. Geometry of a Single Simplex

As typical domain for the parametrisation of simplices, the numerical community mostly uses the n-dimensional **unit simplex**

$$D := \mathrm{conv}(0, e_1, \dots, e_n) = \{p \in \mathbb{R}^n_{\ge 0} : p \cdot 1_n \le 1\},$$

where 1_n is the vector in $\mathbb{R}^n$ with all entries 1, and e_i is the i'th Euclidean unit vector. We will investigate parametrisation of simplices over D for given edge lengths and see how the Riemannian metric over D changes when those edge lengths are slightly distorted. In contrast, geometers tend to employ the **standard simplex**

$$\Delta := \operatorname{conv}(e_0, \dots, e_n) = \{\lambda \in \mathbb{R}^{n+1}_{\geq 0} \colon \lambda \cdot 1_{n+1} = 1\}$$

for the same purpose (here and in the following, we will use the enumeration $e_0, \dots, e_n$ for the canonical Euclidean basis of $\mathbb{R}^{n+1}$). Although the parametrisation over Δ is not a parametrisation in the strict sense, as not some $\mathbb{R}^n$ itself is used, but some linear subspace of it, we will see that there will be no problems with this additional direction.

The Unit Simplex

Metric on TD. Consider points $p_0, \dots, p_n \in \mathbb{R}^n$ that are supposed to be vertices of a simplex s. As we are only interested in its isometry-invariant properties, we can assume that p_0 is the origin of $\mathbb{R}^n$. Then the matrix $P := [p_1|\cdots|p_n]$ represents a linear map $D \to s$. The first fundamental form has entries **3.1**

$$C_{ij} := (P^t P)_{ij} = p_i \cdot p_j, \qquad i, j = 1, \dots, n,$$

and the volume of s is computable as $\operatorname{vol} s = \frac{1}{n!}(\det C)^{1/2}$. If $\ell_{ij} = |p_i - p_j|$ are the edge lengths of s, we have by the cosine law

$$C_{ij} = \tfrac{1}{2}(\ell_{0i}^2 + \ell_{0j}^2 - \ell_{ij}^2). \tag{3.1a}$$

If only a system of prescribed "edge lengths" $\bar{\ell}_{ij}$ is given, then there is a simplex with such edge lengths if and only if the matrix with entries $\frac{1}{2}(\bar{\ell}_{0i}^2 + \bar{\ell}_{0j}^2 - \bar{\ell}_{ij}^2)$ is positive definite.

Metric on T^*D. Let D_i, $i = 1, \dots, n$, be the **facet** (subsimplex of codimension 1) of D opposite to the vertex e_i. The vector e_i is normal to D_i, and as normal directions transform with P^{-t}, the vectors $v^i := P^{-t} e_i$, $i = 1, \dots, n$, are normal to the facets s_i of s. In other words, P^{-1} has the normals v^i as rows. At the same time, these v^i are the gradients of the barycentric coordinate functions λ^i, defined by the representation $p = \lambda^i p_i$ for any point in s. The length of these gradients decreases as the simplex' height h_i above p_i decreases, more precisely **3.2**

$$v^i = \operatorname{grad} \lambda^i \perp s_i, \qquad |v^i| = \tfrac{1}{h_i}, \tag{3.2a}$$

this in particular implies $|s_i|/|v^i| = \frac{1}{n} \operatorname{vol} s$ for each i. This formula does not only hold for $i = 1, \dots, n$, but also the appropriately-scaled normal v^0 opposite to the origin p_0 is the gradient of λ^0. Define $V := [v^1|\cdots|v^n]$. As the barycentric coefficients sum up to one, $v^0 = -V 1_n$. By definition of V, we have $V^t P = \mathbb{1}$, which means that the v^i and p_j form a "biorthogonal system" (Fiedler 2011, thm. 1.1.2). The matrix

$$\tilde{Q}^{ij} := (V^t V)^{ij} = v^i \cdot v^j, \qquad i, j = 1, \dots, n,$$

is inverse to C_{ij} and hence represents the scalar product of the cotangent space. Clearly $V^t = P\tilde{Q}$, that means $v^k = p_j\langle v^j, v^k\rangle$, so $v^0 = -V1_n = p_j\langle v^j, v^0\rangle$. As p_j also equals the edge vector e_{j0} (due to $p_0 = 0$), this shows the translation-independent form

$$v^i = e_{kj}\, v^k \cdot v^j \qquad \text{for all } i = 0, \ldots, n \tag{3.2b}$$

(where $i = j$ might be included in the summation or not, which does not matter due to $e_{ii} = 0$). In the following, we will only consider the $(n+1) \times (n+1)$ matrix Q, which extends $\tilde{Q}$ by a 0'th row and column:

$$Q^{ij} := v^i \cdot v^j, \qquad i, j = 0, \ldots, n.$$

It is made to have vanishing row-sum and column-sum, i. e. $Q1_{n+1} = 0$ and $1^t_{n+1}Q = 0$. In the special case $n = 2$, we know from PINKALL and POLTHIER (1993, PINKALL nowadays dates the formula back to DUFFIN 1959 or even MACNEAL 1949, but as the formula itself is easy to discover by classical trigonometry, we value the application to computational mathematics higher than the first occurence of two opposite angles' cotangents)

$$|ijk|\, v^j \cdot v^i = \frac{|{*ij}|}{|ij|} = \cot \alpha^k_{ij}, \tag{3.2c}$$

(as is frequently used in discrete exterior calculus, see HIRANI 2003), where $*ij$ is the straight line from ijk's circumcentre to the midpoint of ij, and α^k_{ij} is the angle in ijk opposite to vertex k.

3·3 Definition. We say that s is (ϑ, h)**-small** if all edge lengths ℓ_{ij} are smaller than h and s has volume greater than $\vartheta h^n \sigma_n$, where $\sigma_n := \sqrt{n+1}/(2^{n/2} n!)$ is the volume of the regular n-simplex with unit edge length, i. e. the scaled standard simplex $\frac{1}{\sqrt{2}}\Delta$. In terms of the first fundamental form, $\det C \geq (\vartheta h^n n!\, \sigma_n)^2$.

3·4 Remark. **⟨a⟩** The standard simplex has maximal volume among all n-simplices with the same edge length bounds, so $\vartheta \leq 1$.

⟨b⟩ The parameter ϑ is $1/\sigma_n$ times the **fullness** $\Theta(s)$ in WHITNEY (1957, sec. IV.14). The fullness parameter ϑ from VON DEYLEN *et al.* (2014) is $n!\, \sigma_n \vartheta$ in the notation employed here. It would be equivalent to require a lower bound on the angles between subsimplices.

⟨c⟩ Weaker requirements on the simplex quality that still ensure well-posedness of the interpolation problem, like the famous maximum angle condition of BABUŠKA and AZIZ (1976), are circumstantially treated by SHEWCHUK (2002).

3·5 Lemma. *Let $\underline{\alpha}_n := n!\, \sigma_n n^{1-n}$ for all $n \in \mathbb{N}$. Then the eigenvalues λ_i of C fulfill*

$$\vartheta h\, \underline{\alpha}_n \;\leq\; \sqrt{\lambda_i} \;\leq\; hn.$$

Proof. We have to estimate $\|P\|$ and $\|P^{-1}\|$ from 3.1.—Recall that the n-dimensional unit simplex has interior and boundary measure

$$\mathrm{vol}_n(D) = \frac{1}{n!}, \qquad \mathrm{vol}_{n-1}(\partial D) = \frac{n}{(n-1)!} + \frac{\sqrt{n}}{(n-1)!}$$

(the latter one because the $(n-1)$-dimensional standard simplex $\operatorname{conv}(e_1, \dots, e_n)$ has volume $\frac{\sqrt{n}}{(n-1)!}$). For any n-simplex s, the radius r of the insphere is connected to volume and surface via $\operatorname{vol}_n(s) = \frac{r}{n} \operatorname{vol}_{n-1}(\partial s)$. This can be easily seen by considering the simplices $s_i^* := \operatorname{conv}(s_i, c)$, where s_i is a facet of s and c is the circumcentre of the insphere. These s_i^* all have volume $\operatorname{vol}_n(s_i^*) = \frac{r}{n} \operatorname{vol}_{n-1}(s_i)$, and $\operatorname{vol}_n(s) = \operatorname{vol}_n(s_1^*) + \cdots + \operatorname{vol}_n(s_{n+1}^*)$. Now, solving $\operatorname{vol}_n(D) = \frac{r}{n} \operatorname{vol}_{n-1}(\partial D)$ for r gives

$$r = \frac{1}{n + \sqrt{n}} \geq \frac{1}{2n}.$$

This means that any vector $v \in TD$ with length $\frac{1}{n} \leq 2r$ can be represented as $p - q$ with points $p, q \in \Delta$. Its image in s is $Pp - Pq$, which must be shorter than the diameter of s. So $\|P\| \leq nh$. On the other hand,

$$\lambda_{\min}(nh)^{2n-2} \geq \lambda_{\min}\lambda_{\max}^{n-1} \geq \det C > (\vartheta h^n n!\, \sigma_n)^2$$

q. e. d.

Corollary. *For the norm $|w|_g^2 := w^i w^j C_{ij}$ on $T\Delta$ holds $\vartheta h\, \underline{\alpha}_n |w|_{\ell^2} \leq |w|_g \leq hn|w|_{\ell^2}$. In particular, all edges are longer than $\vartheta h\, \underline{\alpha}_n$. The columns v^i of V form a g-orthonormal basis, and $|v^i|_{\ell^2} \leq (\vartheta h \underline{\alpha}_n)^{-1}$ for all i.* **3.6**

Proof. We have $|w|_g = |C^{1/2} w|_{\ell^2}$, and the extremal eigenvalues of $C^{1/2}$ are $\lambda_{\min}^{1/2}$ and $\lambda_{\max}^{1/2}$, which shows the first claim. The second is clear from $V^t P = \mathbb{1}$ and the fact that C^{-1} has eigenvalues between $(hn)^{-2}$ and $(\vartheta h \underline{\alpha}_n)^{-2}$, *q. e. d.*

Lemma. *Assume two symmetric matrices $C, \bar{C} \in \mathbb{R}^{n \times n}$ with C being positive definite and $|(C - \bar{C})v \cdot v| \leq \varepsilon Cv \cdot v$. Then also $|(C - \bar{C})v \cdot w| \leq \varepsilon |Cv \cdot v|^{1/2} |Cw \cdot w|^{1/2}$.* **3.7**

Proof. The claim is independent of scaling v and w, so let $Cv \cdot v = Cw \cdot w = 1$. We will first show the claim for C-orthogonal vectors and then for linear combinations. So assume $Cv \cdot w = 0$ for the moment. Then $C(v + w) \cdot (v + w) = C(v - w) \cdot (v - w) = 2$, and the parallelogram identity (polarisation formula)

$$4A(v, w) = A(v + w, v + w) - A(v - w, v - w)$$

for any symmetric 2-tensor A gives $4|(C - \bar{C})v \cdot w| \leq \varepsilon C(v + w) \cdot (v + w) + \varepsilon C(v - w) \cdot (v - w) = 4\varepsilon = 4\varepsilon |Cv \cdot v|^{1/2} |Cw \cdot w|^{1/2}$. Now for a linear combination, we obtain $|(C - \bar{C})(v + w) \cdot v|^2 \leq |(C - \bar{C})v \cdot v|^2 + |(C - \bar{C})w \cdot v|^2 \leq 2\varepsilon = \varepsilon |Cv \cdot v|\, |C(v + w) \cdot (v + w)|$, *q. e. d.*

Remark. The polarisation argument is also feasible for higher-order symmetric tensors, as e. g. **3.8**

$$6C(v, v, w) = C(v + w, v + w, v + w) - C(v - w, v - w, v - w) - 2C(w, w, w),$$

which (together with the usual paralellogram identity) shows that all evaluations of a symmetric 3-tensor can be reduced to linear combinations of equal argument evaluations.

3.9 **Lemma.** *Let $C, \bar{C} \in \mathbb{R}^{n\times n}$ be symmetric matrices, where all eigenvalues of C are larger than $\lambda_{\min} > 0$ (in particular, C is positive definite), and $|C_{ij} - \bar{C}_{ij}| \le \varepsilon\lambda_{\min}/n$. Then $|(C-\bar{C})v\cdot w| \le \varepsilon|Cv\cdot v|^{1/2}\,|Cw\cdot w|^{1/2}$.*

Proof. Due to 3.7, the case $v = w$ is sufficient. Then

$$|(C-\bar{C})v\cdot v| = \Big|\sum_{i,j}(C_{ij}-\bar{C}_{ij})v_iv_j\Big| \le \frac{\varepsilon\lambda_{\min}}{n}\sum_{i,j}|v_i||v_j| = \frac{\varepsilon\lambda_{\min}}{n}\Big(\sum_i |v_i|\Big)^2,$$

and the square is, by Jensen's inequality, smaller than $n\sum|v_i|^2 = n|v|^2$. As C is positive definite, $Cv\cdot v \ge \lambda_{\min}|v|^2$, *q. e. d.*

3.10 **Remark.** Compare the classical eigenvalue distortion theorem of Bauer and Fike (1960) in the formulation of Ipsen (1998, theorem 2.1): If C can be orthogonally diagonalised, λ is an eigenvalue of C, then there is an eigenvalue $\bar{\lambda}$ of $\bar{C}$ with $\frac{|\lambda-\bar{\lambda}|}{|\lambda|} \le \|C^{-1}(\bar{C}-C)\|$.

The Standard Simplex

Things are less obvious in barycentric coordinates, for which reason we refer to Fiedler (2011, chapters 1 and 2) for proofs of the following statements.

3.11 **Metric on $T\Delta$.** We drop our assumption $p_0 = 0$ and let $p_0, \dots, p_n$ be arbitrary points in $\mathbb{R}^m$. Their convex hull s has a parametrisation over the standard simplex Δ, represented by the matrix $P_+ = [p_0|\cdots|p_n]$. The Riemannian metric on $T\Delta = \{v \in \mathbb{R}^{n+1} : v\cdot 1_{n+1} = 0\}$ is given by

$$E_{ij} := -\tfrac{1}{2}\left|p_i - p_j\right|^2, \qquad i,j = 0,\dots,n. \tag{3.11a}$$

By 3.1a, we know $C_{ij} = E_{ij} - E_{0i} - E_{0j}$ for $i,j = 1,\dots,n$. The volume of s can be computed as

$$\operatorname{vol} s = \tfrac{2}{n!}(-\det M_+)^{1/2}, \quad \text{where} \quad M_+ = \begin{pmatrix} 0 & -\frac{1}{2}1_{n+1}^t \\ -\frac{1}{2}1_{n+1} & E \end{pmatrix} \in \mathbb{R}^{(n+2)\times(n+2)} \tag{3.11b}$$

is $-\frac{1}{2}$ times the usual **Cayley–Menger** or **extended Menger matrix**. The **volume element** hence is $G := 2(-\det M_+)^{1/2}$.

3.12 **Metric on $T^*\Delta$.** Assume $\det M_+ \neq 0$. The cotangent space on Δ consists of all linear combinations $v_i d\lambda^i$ with $v\cdot 1_{n+1} = 0$. As Q^{ij}, $i,j = 0,\dots,n$, already contains the correct scalar products between *all* possible linear combinations of the $d\lambda^i$, it is in particular the correct representation for the scalar product on $T^*\Delta$. The ambiguity in the choice of g_{ij} and g^{ij} is especially visible in the fact that normally both are inverse matrices, whereas for the choices made here, we only have

$$M_+^{-1} = \begin{pmatrix} 4r^2 & -2q^t \\ -2q & Q \end{pmatrix}, \tag{3.12a}$$

where r is the circumradius and q are the barycentric coordinates of the circumcentre of s.

Remark. FIEDLER (2011) uses the symbols M and M_0 for the objects $-2E$ and $-2M_+$.

Definition. From now on, we will only consider barycentric coordinates on the standard simplex and will always use **3.13**

$$g_{ij} = E_{ij}, \qquad g^{ij} = Q^{ij}, \qquad i,j = 0,\dots,n$$

and say that g is a **(ϑ,h)-small metric** if s is (ϑ,h)-small. The latter one is of course only possible if s is non-degenerate, in particular $n \le m$. Note that for any $c \in \mathbb{R}$, the matrices $g_{ij} + c$ and $g^{jk} + c$ induce the same scalar product on $T\Delta$ and $T^*\Delta$ as g_{ij} and g^{jk} respectively, so we can assume that both are positive definite on $\mathbb{R}^{n+1}$ with the same eigenvector bounds as in 3.5.

Proposition. *Although g_{ij} and g^{jk} are not inverse matrices, the tangent-cotangent isomorphism is given by the usual identities:* **3.14**

$$(v^i\partial_i)^\flat = g_{ij}v^i d\lambda^j, \qquad (\alpha_j d\lambda^j)^\sharp = g^{ij}\alpha_i\partial_j.$$

Proof. By 3.11b and 3.12a, $Q^{ij}E_{jk} = \delta^i_k + q^i$ for all $i,k = 0,\dots,n$. In other words,

$$QE = \mathbb{1} + [q|\cdots|q].$$

But if $\alpha \cdot 1_{n+1} = 0$, then $[q|\cdots|q]\alpha = 0$, irrespective of the entries of q. So on $T\Delta$ (and equally on $T^*\Delta$), both matrices are indeed inverse to each other, *q. e. d.*

Lemma. *Assume real numbers $x, y > 0$ with $|x-y| \le \varepsilon x$ for some $\varepsilon < 1$. Then* **3.15**

$$|x^p - y^p| \le c_p \varepsilon x^p, \qquad \textit{where} \quad c_p = \begin{cases} p\,(1+\varepsilon)^{p-1} & \textit{for } p \ge 1, \\ p\,(1-\varepsilon)^{p-1} & \textit{for } p \in [0;1]. \end{cases}$$

In particular,

$$|\sqrt{x} - \sqrt{y}| \le 2\varepsilon\sqrt{x} \quad \textit{and} \quad |x^2 - y^2| \le 3\varepsilon x^2 \qquad \textit{for } \varepsilon < \tfrac{1}{2}.$$

Proof. Let $f : x \mapsto x^p$. Then $|f(x) - f(y)| \le \max_{\xi\in[x;y]} f'(\xi)|x-y|$. So we only have to find the maximum of f'. Suppose $p < 1$. Here f' is monotonously decreasing. If $x \le y$, then $\max f' = f'(x) = px^{p-1}$. If $y \le x$, then $\max f' = f'(y) = py^{p-1} \le p(1-\varepsilon)^{p-1}x^{p-1}$, and this case dominates the first one.—The argument for $p > 1$ is litterally the same, but with inversed rôles of x and y, *q. e. d.*

Proposition. *Let ℓ_{ij} be the edge lengths of a simplex, defining a (ϑ,h)-small metric g on Δ, and let $\bar\ell_{ij}$ be a second system of desired edge lengths with $|\ell_{ij} - \bar\ell_{ij}| \le \frac{2}{3}\varepsilon n^{-1}\underline{\alpha}_n^2\vartheta^2\ell_{ij}$ where $\varepsilon \le \frac{1}{2}$. Then there is a simplex $\bar s \subset \mathbb{R}^n$ with edge lengths $\bar\ell_{ij}$, and its Riemannian metric $\bar g$ over Δ fulfills $|(g-\bar g)\langle v,w\rangle| \le \varepsilon|v|_g\,|w|_g$.* **3.16**

Proof. For the existence claim, it suffices to show that $\bar g_{ij} = -\frac{1}{2}\bar\ell_{ij}^2$ is positive definite on $T\Delta$. By 3.15 and the assumption, we have $|\ell_{ij}^2 - \bar\ell_{ij}^2| \le 2\varepsilon n^{-1}\underline{\alpha}_n\vartheta^2\ell_{ij}^2$, hence $|E_{ij} - \bar E_{ij}| \le \varepsilon n^{-1}\underline{\alpha}_n^2\vartheta^2h^2$. Now apply 3.9 to get $|(g-\bar g)\langle v,w\rangle| \le \varepsilon|v|_g|w|_g$. In particular, $\bar g$ is positive definite for $\varepsilon < 1$, *q. e. d.*

3.17 **Remark.** Although the proof of 3.9 seems to be fairly rough, the fullness parameter ϑ is essential to 3.16 and cannot be weakened to a bound e. g. on the minimal edge length (which would be equivalent to a minimal angle bound, which is a sharp tool for some interpolation estimates, see Babuška and Aziz 1972). In fact, edge lengths 2, $1+\varepsilon$ and $1+\varepsilon$ of a thin triangle may only be relatively distorted by a factor $\delta \in \left]\frac{1}{1+\varepsilon}; 1+\varepsilon\right[$ to guarantee the existence of a corresponding simplex.

3.18 **Corollary.** *Situation as in 3.16. Then g and $\bar{g}$ are equivalent norms, $(1-\varepsilon)\bar{g}\langle v,v\rangle \leq g\langle v,v\rangle \leq (1+\varepsilon)\bar{g}\langle v,v\rangle$.*

3.19 **Proposition.** *Situation from 3.16. Then the estimate carries over to all higher tensor spaces $T^r_s\Delta$ with constants depending on r and s.*

Proof. Let us first consider the extension of g to bivectors, that means elements of $T^0_2\Delta$. Without regarding the tensor-product structure, we could just say this it is a scalar product with components $g_{ij}g_{k\ell}$, where (i,k) and (j,ℓ) are the indices of the first and second argument respectively. And these components are almost equal to those of $\bar{g}$ (we abbreviate $|g_{ij}-\bar{g}_{ij}| \leq \delta g_{ij}$):

$$\begin{aligned}|g_{ij}g_{k\ell}-\bar{g}_{ij}\bar{g}_{k\ell}| &\leq |g_{ij}g_{k\ell}-\bar{g}_{ij}g_{k\ell}| + |g_{ij}\bar{g}_{k\ell}-\bar{g}_{ij}\bar{g}_{k\ell}| \\ &\leq \delta|g_{ij}g_{k\ell}| + \delta|g_{ij}\bar{g}_{k\ell}| \\ &\leq \delta|g_{ij}g_{k\ell}| + \delta(1+\delta)|g_{ij}g_{k\ell}| \leq 3\delta|g_{ij}g_{k\ell}|.\end{aligned}$$

For the $n^2 \times n^2$ matrices $g_{ij}g_{k\ell}$ and $\bar{g}_{ij}\bar{g}_{k\ell}$, apply 3.9 again, *q. e. d.*

3.20 **Corollary.** *Situation as in 3.16. The volume elements of g and $\bar{g}$ fulfill $|G-\bar{G}| \leq c\,\varepsilon G$ with a constant c depending only on n.*

Proof. The usual proof would be to use $G = 2(-\det M_+)^{1/2}$, together with the first order approximation of the determinant: $\det(F+tA) = \big(1 + t\operatorname{tr} F^{-1}A + O(t^2)\big)\det F$ (a classical reference is Bellmann 1960, pp. 96sqq., although the nicest proof that we know is in Eschenburg and Jost 2007, lemma 8.2.1). However, we want to control more than the first order. So observe that if ∂_i is a coordinate-induced basis of the tangent space, then $G = |\partial_1 \wedge \cdots \wedge \partial_n|_g$, and $\bar{g}-g$ on the space of n-forms is controlled by the previous proposition, *q. e. d.*

3.21 **Proposition.** *Let $g, \bar{g}$ be two metrics with $|(g-\bar{g})\langle v,w\rangle| \leq \varepsilon|v|_g|w|_g$, where $\varepsilon < \frac{1}{2}$. Then the metric g^{ij} on the cotangent space fulfills $|g\langle\alpha,\beta\rangle - \bar{g}\langle\alpha,\beta\rangle| \leq 2\varepsilon\,|\alpha|_g\,|\beta|_g$ for $\alpha,\beta \in T^*\Delta$.*

Proof. Again, we do not use the conventional approach to bound $\bar{M}_+^{-1}$ by the differential of the matrix inversion $\frac{\mathrm{d}}{\mathrm{d}F}F^{-1}(A) = -F^{-1}AF^{-1}$ (Deuflhard and Hohmann, 2003, lemma 2.8). Instead, the definition of $|\cdot|_g$ on $T^*\Delta$ as operator norms will help: For every $v \in T\Delta$, we have $|\alpha|^2_{\bar{g}} \geq (\frac{\alpha(v)}{|v|_{\bar{g}}})^2 \geq \frac{1}{1+\varepsilon}(\frac{\alpha(v)}{|v|_g})^2$, in particular for the v realising $|\alpha|_g$. On the other hand, if w realises $|\alpha|_{\bar{g}}$, then $|\alpha|^2_{\bar{g}} = (\frac{\alpha(w)}{|w|_{\bar{g}}})^2 \leq \frac{1}{1-\varepsilon}(\frac{\alpha(w)}{|w|_g})^2$. So

$$\frac{1}{1+\varepsilon}\,|\alpha|^2_g \quad\leq\quad |\alpha|^2_{\bar{g}} \quad\leq\quad \frac{1}{1-\varepsilon}\,|\alpha|^2_g.$$

Now as $\varepsilon < \frac{1}{2}$ by assumption, we have $\frac{1}{1-\varepsilon} \leq 1+2\varepsilon$ and $\frac{1}{1+\varepsilon} \geq 1-\varepsilon > 1-2\varepsilon$, therefore $\big||\alpha|_g^2 - |\alpha|_{\tilde{g}}^2\big| \leq 2\varepsilon|\alpha|_g^2$, which suffices due to 3.7, *q. e. d.*

4. Simplicial Complexes and Discrete Riemannian Metrics

In computational geometry, it is common to describe simplicial complexes as the union of simplices in Euclidean spaces with appropriate conditions on their intersections. We consider these intersection conditions as tedious and use the more abstract definition via barycentric coordinates, as is usually done in topology. We follow the lines of Munkres (1984), but we repeat the definitions in order to directly deal with abstract simplicial complexes as (almost everywhere smooth) Riemannian manifolds.

Non-Oriented Complexes

Definition. An n-dimensional **combinatorial simplex** (n-simplex) is a set of $n+1$ elements, its ℓ-dimensional **subsimplices** are subsets of cardinality $\ell+1$. An n-dimensional **combinatorial simplicial complex** is a collection $\mathfrak{K} = (\mathfrak{K}^0, \ldots, \mathfrak{K}^n)$, where each $\mathfrak{K}^\ell$ is a collection of ℓ-dimensional simplices such that if $\mathfrak{t}$ is a k-dimensional subsimplex of $\mathfrak{s} \in \mathfrak{K}^\ell$, then $\mathfrak{t} \in \mathfrak{K}^k$. The complex is called **regular** if each simplex is contained in an n-simplex and each $(n-1)$-simplex is the subsimplex of at most two n-simplices. When we speak of simplicial complexes, we always mean regular ones. **4.1**

An $(n-1)$-simplex $\mathfrak{f}$ is called a **boundary simplex** if there is only one $\mathfrak{e} \in \mathfrak{K}^n$ with $\mathfrak{f} \subset \mathfrak{e}$. The $(n-1)$-dimensional complex formed out of these boundary simplices and their subsimplices is called the **boundary complex** $\partial\mathfrak{K}$ of $\mathfrak{K}$.

Notation. We use special notations for the most interesting dimensions (here k is any dimension between 0 and 1, kept fixed inside an argumentation):

vertices	p_i or $i \in \mathfrak{K}^0$	$\mathfrak{t} \in \mathfrak{K}^{k-1}$	facets	$\mathfrak{f} \in \mathfrak{K}^{n-1}$
edges	$ij \in \mathfrak{K}^1$	$\mathfrak{s} \in \mathfrak{K}^k$	elements	$\mathfrak{e} \in \mathfrak{K}^n$

Sometimes we will also use the convention $\mathfrak{t} \in \mathfrak{K}^k$ and $\mathfrak{s} \in \mathfrak{K}^{k+1}$. In every case $\mathfrak{t}$ will be one dimension smaller than $\mathfrak{s}$.

Definition (tom Dieck (2000), p. 63)**.** Let $\mathfrak{s} := \{p_0, \ldots, p_k\}$ be a combinatorial k-simplex. For a function $\lambda : \mathfrak{s} \to \mathbb{R}$, abbreviate $\lambda(p_i)$ as λ^i. The **geometric realisation** of $\mathfrak{s}$ is $r\mathfrak{s} := \{\lambda : \mathfrak{s} \to [0;1] \colon \lambda \cdot 1_{n+1} = 1\}$. For a complex $\mathfrak{K} = (\mathfrak{K}^0, \ldots, \mathfrak{K}^n)$, the realisation is defined as $r\mathfrak{K} := \bigcup_{\mathfrak{e} \in \mathfrak{K}^n} r\mathfrak{e}$. **4.2**

Remark. $\langle$**a**$\rangle$ This definition is equivalent to, but much more elegant than the usual way of "annotating" the vertices of the euclidean standard simplex Δ with the elements of $\mathfrak{K}^0$, considering the disjoint union of $|\mathfrak{K}^n|$ many such annotated simplices and glueing them whenever two sides have equal annotations.

$\langle$**b**$\rangle$ By setting $\lambda^i = 0$ for all unused vertices $p_i \in \mathfrak{K}^0$, the elements of $r\mathfrak{K}$ can naturally be considered as functions $\lambda : \mathfrak{K}^0 \to [0;1]$ with $\operatorname{supp} \lambda = \mathfrak{s}$ for some $\mathfrak{s} \in \mathfrak{K}^k$.

⟨**c**⟩ We say that some property is fulfilled **piecewise** on $r\mathfrak{K}$ if it is fulfilled on each $r\mathfrak{e}$, $\mathfrak{e} \in \mathfrak{K}^n$.

4·3 **Proposition.** *Let $\mathfrak{K}$ be an n-dimensional simplicial complex (regular, as always). Then $r\mathfrak{K}$ is an n-dimensional manifold, which is smooth everywhere except at $(n-2)$-simplices.*

Proof. For each $\mathfrak{f} \in \mathfrak{K}^{n-1}$, belonging to $\mathfrak{e}, \mathfrak{e}' \in \mathfrak{K}^n$, define a chart $x_\mathfrak{f} : r\mathfrak{e} \cup r\mathfrak{e}' \to \mathbb{R}^n$ in the following way: Without loss of generality, assume $\mathfrak{e} = \{p_0, \dots, p_n\}$ and $\mathfrak{e}' = \{p_1, \dots, p_{n+1}\}$. Let $e_1, \dots, e_n$ be the usual euclidean basis vectors in $\mathbb{R}^n$, let e_0 be the origin and $e_{n+1} := \frac{2}{n}(e_1 + \cdots + e_n)$. Then the convex hulls $D := \operatorname{conv}(e_0, \dots, e_n)$ and $D' := \operatorname{conv}(e_1, \dots, e_{n+1})$ are isometric (up to a change of orientation). Now define

$$x_\mathfrak{f}(\lambda) := \lambda^i e_i = \begin{cases} \lambda^0 e_0 + \cdots + \lambda^n e_n & \text{on } r\mathfrak{e}, \\ \lambda^1 e_1 + \cdots + \lambda^{n+1} e_{n+1} & \text{on } r\mathfrak{e}'. \end{cases} \tag{4·3a}$$

This $x_\mathfrak{f}$ is a bijection $r\mathfrak{e} \to D$ and $r\mathfrak{e}' \to D'$. For any other chart $x_{\mathfrak{f}'}$ that also covers $r\mathfrak{e}$, the chart transition is an affine map that maps E to either $\mathfrak{e}$ or $\mathfrak{e}'$, hence $r\mathfrak{K}$ is smooth in the interior of each n-simplex.

Around an $(n-2)$-simplex, we do not give a chart, but we just remark that a topological manifold is sufficiently defined by a finite cover of closed chart domains. Open charts are only needed for the definition of smooth functions.

Note that another choice than $e_0, \dots, e_{n+1}$ would have led to the same differentiable structure on $r\mathfrak{K}$ (as long as the convex hulls are full-dimensional simplices in $\mathbb{R}^n$), *q. e. d.*

4·4 **Observation.** ⟨**a**⟩ Consider $\mathfrak{s} \in \mathfrak{K}^k$. By 4·2, $r\mathfrak{s}$ is a full-dimensional subset of the k-dimensional affine space $\{\lambda : \mathfrak{s} \to \mathbb{R} \colon \lambda \cdot 1_{n+1} = 1\}$, so its tangent space is $T_\lambda r\mathfrak{s} = \{v : \mathfrak{s} \to \mathbb{R} \colon v \cdot 1_{n+1} = 0\}$ at every internal $\lambda \in r\mathfrak{s}$. As $r\mathfrak{s}$ is an affine space, we deliberately drop the foot point λ in most cases, just as we do with $T\Delta$.

⟨**b**⟩ The obvious linear isomorphism $\Delta \to r\mathfrak{s}$ is $e_i \mapsto rp_i$. This means that $\lambda \in \Delta$ and $v \in T\Delta$ are mapped to $\lambda^i rp_i$ and $v^i rp_i$.

⟨**c**⟩ The realisation of the boundary complex is the boundary of the realisation: $\partial r\mathfrak{K} = r\partial\mathfrak{K}$. In particular, $r\mathfrak{K}$ is a manifold without boundary iff each $(n-1)$-simplex in $\mathfrak{K}$ belongs to two n-simplices.

4·5 **Definition** (Wardetzky 2006 or, similar but shorter, Hildebrandt *et al.* 2006)**.** Define a **differentiable structure** on $r\mathfrak{K}$ by the requirement that some function is smooth (or of class $\mathsf{C}^{k,\alpha}$) if it has this smoothness property piecewise and is continuous up to the boundary. Consequently, define H^k as the completion of C^k with respect to the H^k norm.

4·6 **Definition** (Bobenko *et al.* 2010)**.** Let $\mathfrak{K}$ be a simplicial complex. A function $\ell : \mathfrak{K}^1 \to \mathbb{R}_{\geq 0}$ with the property that C_{ij} from 3·1a is positive semidefinite for each $\mathfrak{e} \in \mathfrak{K}^n$ is called a **discrete Riemannian metric**. In particular, ℓ fulfills the triangle inequality on each $\mathfrak{t} \in \mathfrak{K}^2$.

On each $T_\lambda r\mathfrak{s}$, $\mathfrak{s} \in \mathfrak{K}^k$, the discrete Riemannian metric ℓ induces a usual Riemannian metric $g_\ell\langle v, w\rangle = v^i w^j g_{ij}$ by $g_{ij} := -\frac{1}{2}\ell^2_{ji}$, cf. 3.11a. As this metric does not change with λ, $r\mathfrak{s}$ will be flat. When we deal with a piecewise flat metric, we always assume that it is defined via a discrete Riemannian metric.

Observation. $\langle$**a**$\rangle$ Let $\mathfrak{t}$ be a facet of $\mathfrak{s}$. The restriction of ℓ to edges in $\mathfrak{t}$ is a discrete Riemannian metric for itself, and its induced Riemannian metric $g_{\ell,\mathfrak{t}}$ on $r\mathfrak{t}$ is the restriction of g_ℓ. So the glueing of two supersimplices $\mathfrak{s}, \mathfrak{s}'$ of $\mathfrak{t}$ along $\mathfrak{t}$ is done isometrically with respect to g_ℓ. **4.7**

$\langle$**b**$\rangle$ Consequently, every set $U \subset r\mathfrak{K}$ that does not contain any $(n-2)$-simplex is flat. In fact, also $(n-2)$-simplices might be included if they have some flat neighbourhood, which is equivalent to requiring that their internal dihedral angles as defined by Cheeger *et al.* (1984, p. 412) sum up to 1. In this sense, curvature of piecewise flat spaces is concentrated in the $(n-2)$-simplices.

$\langle$**c**$\rangle$ If p_i are points in Euclidean space with $\ell_{ij} = |p_i - p_j|_{\ell^2}$, then g_ℓ coincides with the pull-back metric g_s of $s := \operatorname{conv}(p_0, \dots, p_k)$ to the standard simplex Δ. Hence, $x^\Delta : (r\mathfrak{s}, g_\ell) \to (\Delta, g_s)$, $\lambda \mapsto \lambda^i e_i$ and $x^s : (r\mathfrak{s}, g_\ell) \to (s, \ell^2)$, $\lambda \mapsto \lambda^i p_i$ are both isometries.

$\langle$**d**$\rangle$ In the construction 4.3a, one may use points q_i with distances $|q_i - q_j|_{\ell^2} = \ell_{ij}$ instead of the points e_i (of course, $|q_0 - q_{n+1}|_{\ell^2}$ does not undergo any restriction). Up to Euclidean isometries, these q_i are unique. This defines an atlas $\{x_\mathfrak{f} \colon \mathfrak{f} \in \mathfrak{K}^{n-1}\}$ of isometries.

$\langle$**e**$\rangle$ Consider a triangle $\{p_i, p_j, p_k\} \in \mathfrak{K}^2$, shortly written as ijk. By the usual trigonometric formulas incorporating only edge lengths, one can define angles α^k_{ij} opposite to the edge ij and area $|ijk|$ on the basis of ℓ only, without using g_ℓ. The metric $\mathbf{d}_\ell$ on $r\mathfrak{K}$ obtained by the requirement

$$|jk\lambda| = \lambda^i |ijk|, \qquad |ki\lambda| = \lambda^j |ijk|, \qquad |ij\lambda| = \lambda^k |ijk|.$$

is the same as the metric induced by g_ℓ. The generalisation of this approach to higher dimensions is of course feasible and natural, but notationally tedious.

Proposition. *Let $r\mathfrak{K}$ be a realised simplicial complex with a piecewise flat metric g. Consider two adjacent elements $\mathfrak{e} = \{p_0, \dots, p_n\}$ and $\mathfrak{e}' = \{p_1, \dots, p_{n+1}\} \in \mathfrak{K}^n$ with common subsimplex $\mathfrak{f}$. Then for any λ in the interior of $r\mathfrak{f}$, the differential of the transition map $Tr\mathfrak{e} \to Tr\mathfrak{e}'$ has dual* **4.8**

$$\begin{aligned} (d\tau_{\mathfrak{e}',\mathfrak{e}})^\flat : T^*r\mathfrak{e} \to T^*r\mathfrak{e}', \quad d\lambda^i &\mapsto d\lambda^i \qquad \text{for } i = 1, \dots, n, \\ d\lambda^0 &\mapsto -\tfrac{|d\lambda^0|}{|d\lambda^{n+1}|} d\lambda^{n+1}. \end{aligned} \tag{4.8a}$$

Proof. It is clear that the common differentials $d\lambda^1, \dots, d\lambda^n$ remain unchanged. Under an isometric embedding as in 4.7d, $(d\lambda^0)^\sharp$ and $(d\lambda^{n+1})^\sharp$ are normal to the common facet (cf. 3.2), pointing in opposite directions, which gives $\frac{1}{|d\lambda^0|} d\lambda^0 = -\frac{1}{|d\lambda^{n+1}|} d\lambda^{n+1}$, *q. e. d.*

Remark. $\langle$**a**$\rangle$ We have chosen to give $(d\tau_{\mathfrak{s}',\mathfrak{s}})^\flat$ and not $d\tau_{\mathfrak{s},\mathfrak{s}'}$ just to obtain a nicer formula. One could as well say

$$\operatorname{grad}\lambda^0 \mapsto \tfrac{|d\lambda^0|}{|d\lambda^{n+1}|}\operatorname{grad}\lambda^{n+1}.$$

$\langle$**b**$\rangle$ Formally, $\tau_{\mathfrak{c}',\mathfrak{c}}$ is only defined on $r\mathfrak{c} \cup r\mathfrak{c}'$, where it is the identity. But the charts $x^\Delta_\mathfrak{c}$ and $x^\Delta_{\mathfrak{c}'}$ from 4.7c can be extended to some neighbourhood of the standard simplex, as $(r\mathfrak{c}, g_\ell)$ and $(r\mathfrak{c}', g_\ell)$ are glued isometrically.

Oriented Complexes

4.9 **Definition** („Nanu, Sie kennen $\mathfrak{K}_{\mathrm{or}}$ noch nicht?"). Let V be a set. Define an equivalence relation $\sim$ on the set V^{n+1} of $(n+1)$-tuples over V by $a \sim b$ iff there is a permutation with positive sign that maps a into b. Let $[a_0, \dots, a_n]$ be the equivalence class of $a \in V^n$. The quotient of V^{n+1} under $\sim$ is called the set of **oriented** k**-simplices** with vertices in V and is denoted by $[V^n]$.

If $[b] \in [V^k]$ is an oriented simplex, its **facets** are the oriented $(k-1)$-simplices obtained by dropping one of its vertices: $[b_0, \dots, \widehat{b_i}, \dots, b_k] < [b_0, \dots, b_k]$. The **subsimplices** of $[b]$ are obtained by dropping one or more vertices. If dimensions do not matter, we also abbreviate $[a] < \cdots < [b]$ as $[a] < [b]$ if $[a]$ is a subsimplex of $[b]$.

An oriented **simplicial complex** $\mathfrak{K}_{\mathrm{or}}$ of dimension n with vertex set V is a collection $\mathfrak{K}^0_{\mathrm{or}}, \dots, \mathfrak{K}^n_{\mathrm{or}}$, where $\mathfrak{K}^k_{\mathrm{or}} \subset [V^k]$, such that $[a] < [b]$ for some $[b] \in \mathfrak{K}^k_{\mathrm{or}}$ implies $[a] \in \mathfrak{K}^{k-1}_{\mathrm{or}}$. The complex is **regular** if no vertex occurs twice in any simplex, each simplex is contained in an n-dimensional simplex, each $(n-1)$-simplex is the boundary of exactly one n-simplices, and each two n-simplices in $\mathfrak{K}^n_{\mathrm{or}}$ have different vertex sets.

If $\mathfrak{K}_{\mathrm{or}}$ is a regular orientable simplicial complex, we denote the corresponding complex made out of non-oriented simplices by $\mathfrak{K}$. The **realisation** of an oriented complex $\mathfrak{K}_{\mathrm{or}}$ is defined as $r\mathfrak{K}_{\mathrm{or}} := r\mathfrak{K}$.

4.10 **Remark.** $\langle$**a**$\rangle$ There are exactly two distinct oriented simplices with the same set of vertices $a_0, \dots, a_n$, which we write $[a_0, \dots, a_n]$ and $[a_0, \dots, a_n]^-$. As non-oriented simplices were defined as sets, each non-oriented simplex corresponds to two oriented simplices. So the last condition on a regular complex says that not $[a]$ and $[a]^- \in [V^n]$ can belong to an n-dimensional complex at the same time.

$\langle$**b**$\rangle$ The vertices of a non-degenerate euclidean simplex $s = \operatorname{conv}(p_0, \dots, p_n) \subset \mathbb{R}^m$ can be ordered such that $P = [p_1 - p_0| \cdots |p_n - p_0]$ as in 3.1 has positive determinant. This is what we call the **canonical** orientation of $\{p_0, \dots, p_n\}$. (On the other hand, if $p_0, \dots, p_n$ are not taken out of some oriented space, there is no canonical choice.)

4.11 **Proposition.** *$r\mathfrak{K}_{\mathrm{or}}$ is an orientable piecewise smooth manifold for any regular oriented simplicial complex $\mathfrak{K}_{\mathrm{or}}$.*

Proof. We will show that if we use only those charts from the proof of 4.3 that respect the orientation of n-simplices, we obtain an oriented atlas of $r\mathfrak{K}_{\mathrm{or}}$:

Suppose there are two simplices $\mathfrak{s}, \mathfrak{s}' \in \mathfrak{K}^n_{\mathrm{or}}$ that share n vertices, say $p_1, \dots, p_n$. Then because $\mathfrak{t} := [p_1, \dots, p_n]$ can only be contained in one of them, we can assume that

$$\mathfrak{s} = [p_0, p_1, \dots, p_n], \qquad \mathfrak{s}' = [p_{n+1}, p_1, \dots, p_n]^-$$

for two vertices $p_0, p_{n+1} \in \mathfrak{K}^0_{\mathrm{or}}$. Now let $x_\mathfrak{t}$ be the chart as in 4.3a. Obviously, $[e_0, \dots, e_n]$ and $[e_{n+1}, e_1, \dots, e_n]^-$ are both canonically oriented. As there was no choice in this construction, every other chart that covers $\mathfrak{s}$ must also map $r\mathfrak{s}$ to a euclidean simplex with this orientation, therefore every transition map is orientation-preserving, *q. e. d.*

Barycentric Subdivision

Definition. Let $\mathfrak{K}$ be a simplicial complex, regular as usual, and $\mathfrak{K}^* := \mathfrak{K}^1 \cup \dots \cup \mathfrak{K}^n$ be the set of all its simplices. An (ascending) k-**flag** in $\mathfrak{K}$ is a set $\mathfrak{a} := \{\mathfrak{a}_0, \dots, \mathfrak{a}_k\} \subset (\mathfrak{K}^*)^{k+1}$ such that, if its elements are ordered by magnitude, $\mathfrak{a}_i \subset \mathfrak{a}_{i+1}$. In other words, a k-flag is a sequence of $k+1$ nested simplices. If $\mathfrak{a}_i \in \mathfrak{K}^{n_i}$, we also write $\mathfrak{a} = (\langle n_0 \rangle, \dots, \langle n_k \rangle)$, meaning that $\langle j \rangle$ is a “generic j-simplex”. **4.12**

Remark. **⟨a⟩** The notation $\langle i \rangle$ is uncommon, but not more ambigous than other authors' notations such as σ^i. Our notation is made to save double subscripts. **4.13**

⟨b⟩ Of course, flags are simplices, only in some special complex. But having a different name will (hopefully) prevent confusion. The term “flag” is borrowed from algebra, where it signifies sequences of nested linear spaces, whereas set theory mostly speaks of “ascending chains” for nested sets. But the term “chain” already has a canonical meaning in simplicial homology theory, and in section 9 we need to use both at a time.

⟨c⟩ All elements of a k-flag lie in a common n-simplex. An n-flag contains exactly one k-simplex for each k.

Example. Suppose $\mathfrak{K}$ consists of one triangle ijk, its edges and its vertices. Then the 0-flags are the elements of $\mathfrak{K}^*$ (to be totally precise, the 0-flags are singletons containing elements of $\mathfrak{K}^*$). The 1-flags are of the form $(\langle 0 \rangle, \langle 1 \rangle)$, that means combinations of a vertex and an edge containing it, or of the form $(\langle 0 \rangle, \langle 2 \rangle)$, i. e. a vertex and the triangle, or $(\langle 1 \rangle, \langle 2 \rangle)$, an edge and the triangle: **4.14**

$$\{i, ij\}, \{i, ik\}, \{i, ijk\}, \{ij, ijk\}, \{ik, ijk\} \quad \text{and similar for the vertices } j \text{ and } k.$$

The 1-flags $(\langle 0 \rangle, \langle 1 \rangle)$ are interpreted as straight line segments from the point $r\langle 0 \rangle$ to a point $\lambda_{\langle 1 \rangle}$ somewhere on the edge $\langle 1 \rangle$, and the flags $(\langle 1 \rangle, \langle 2 \rangle)$ connect the points $\lambda_{\langle 1 \rangle}$ to the “barycentre” $\lambda_{\langle 2 \rangle}$. The 2-flags consist of a vertex, an edge containing this vertex, and the triangle, they are all of the form $(\langle 0 \rangle, \langle 1 \rangle, \langle 2 \rangle)$:

$$\{i, ij, ijk\}, \{i, jk, ijk\} \quad \text{and similar for other vertices.}$$

Definition. The **(barycentric) subdivision** $\operatorname{sd}\mathfrak{K}$ of the complex $\mathfrak{K}$ is a complex of the same dimension whose k-simplices are the k-flags in $\mathfrak{K}$. **4.15**

Suppose there is some $\lambda_\mathfrak{s} \in r\mathfrak{s}$ given for each $\mathfrak{s} \in \mathfrak{K}^*$. Because of 4.13c, the mapping

$$r(\operatorname{sd}\mathfrak{K})^0 \to r\mathfrak{K}, \quad r\{\mathfrak{s}\} \mapsto \lambda_\mathfrak{s}$$

can be uniquely extended to a continuous, piecewise affine mapping $i : r(\operatorname{sd}\mathfrak{K}) \to r\mathfrak{K}$, mapping the realisation of a flag $r(\mathfrak{a}_0, \dots, \mathfrak{a}_k)$ to the convex hull of $\lambda_{\mathfrak{a}_0}, \dots, \lambda_{\mathfrak{a}_l}$. If ℓ is a discrete Riemannian metric on $\mathfrak{K}$, then $r(\operatorname{sd}\mathfrak{K})$ can be endowed with the induced metric $\ell_{\{\mathfrak{s}\},\{\mathfrak{s}'\}} = |\lambda_{\mathfrak{s}} - \lambda_{\mathfrak{s}'}|_{g_\ell}$, and i becomes an isometry. Let $r' := i \circ r$ be the "realisation of $\operatorname{sd}\mathfrak{K}$ in $r\mathfrak{K}$".

If $\mathfrak{K}_{\mathrm{or}}$ is an oriented complex, one can obviously define an oriented subdivision by considering the n-flags as tuples instead of sets and using the orientation induced by r'.

4.16 **Observation.** There are several obvious conclusions from the fact that $r'\{\mathfrak{a}_0, \dots, \mathfrak{a}_k\} = \operatorname{conv}(\lambda_{\mathfrak{a}_0}, \dots, \lambda_{\mathfrak{a}_k})$. Most prominently, one can decompose the realisation of a k-simplex $r\mathfrak{t}$ into the realisations of k-flags ending at $\mathfrak{t}$. The boundary $\partial(r\mathfrak{s})$ of a realised $(k+1)$-simplex $\mathfrak{s}$ is covered by (the realisation of) k-flags ending at facets of $\mathfrak{s}$:

$$r\mathfrak{t} = \bigcup_{\langle k\rangle=\mathfrak{t}} r'(\langle 0\rangle, \dots, \langle k\rangle), \qquad \partial(r\mathfrak{s}) = \bigcup_{\langle k\rangle\subset\mathfrak{s}} r'(\langle 0\rangle, \dots, \langle k\rangle).$$

Definition. For $\mathfrak{s} \in \mathfrak{K}^k$, aggregate the n-flags containing $\mathfrak{s}$ in the **neighbourhood** $U(\mathfrak{s})$ of $\mathfrak{s}$ and the $(n-k)$-flags starting with $\mathfrak{s}$ in the **dual cell** $*\mathfrak{s}$:

$$U(\mathfrak{s}) := \bigcup_{\langle k\rangle=\mathfrak{s}} r'(\langle 0\rangle, \dots, \langle n\rangle), \qquad r(*\mathfrak{s}) := \bigcup_{\langle k\rangle=\mathfrak{s}} r'(\langle k\rangle, \dots, \langle n\rangle). \tag{4.16a}$$

4.17 **Observation.** **⟨a⟩** The flags occuring in $U(\mathfrak{s})$ must obviously be different from the flags occuring in $U(\mathfrak{s}')$ for $\mathfrak{s} \neq \mathfrak{s}'$, so these neighbourhoods form a covering of $r\mathfrak{K}$ with disjoint interior for each k.

⟨b⟩ The set of all n-flags "running through $\mathfrak{s}$" can be represented as a product of two flag sets: k-flags ending at $\mathfrak{s}$, whose union is $r\mathfrak{s}$, and the $(n-k)$-flags beginning at $\mathfrak{s}$, whose union is $r(*\mathfrak{s})$. For us, the latter is just a way to write this union, we will not define the combinatorial dual of $\mathfrak{K}$. The interested reader is referred to MUNKRES (1984, § 64).

⟨c⟩ The boundary of a neighbourhood consists of those flags where $\langle k\rangle = \mathfrak{t}$ is left out:

$$\partial U(\mathfrak{s}) = \bigcup_{\langle k\rangle=\mathfrak{s}} r'(\langle 0\rangle, \dots, \widehat{\langle k\rangle}, \dots, \langle n\rangle).$$

This can be seen as follows: The boundary of any n-flag $\mathfrak{a}$ consists of $(n-1)$-flags $\mathfrak{a}'$ where any one of the elements in $\mathfrak{a}$ is left out. The boundary of the union $U(\mathfrak{s})$ now consists of those facets $r'\mathfrak{a}'$ where some element is left out *and* there is no other n-flag belonging to $U(\mathfrak{s})$ on the other side of $r'\mathfrak{a}'$. This second condition is satisfied only if $\mathfrak{s}$ is left out, because if $\langle i\rangle \neq \mathfrak{s}$ is left out, there is another flag $(\langle 0\rangle', \dots, \langle n\rangle')$ running through $\mathfrak{s}$ with $\langle i\rangle' \neq \langle i\rangle$.

4.18 **Lemma.** *Let $r\mathfrak{K}$ be a realised simplicial complex with piecewise flat metric, and let $\lambda_{\mathfrak{s}}$ be the circumcentre of $r\mathfrak{s}$ for each $\mathfrak{s} \in \mathfrak{K}^*$. Then for each n-flag $(\langle 0\rangle, \dots, \langle n\rangle)$, the vectors $v_{\langle i\rangle,\langle i+1\rangle} := \lambda_{\langle i+1\rangle} - \lambda_{\langle i\rangle}$ are perpendicular to $r\langle i\rangle$ and thus pairwise orthogonal.*

Proof. Consider the two-dimensional case: If λ_{ijk} is the circumcentre of $r(ijk)$, then $|v_{i,ijk}| = |v_{j,ijk}|$. The "circumcentre" of the edge ij is $\lambda_{ij} = \frac{1}{2}(ri + rj)$. So we have two equilateral triangles $(\lambda_i, \lambda_{ij}, \lambda_{ijk})$ and $(\lambda_j, \lambda_{ij}, \lambda_{ijk})$, which must hence have the same angle $\pi/2$ at λ_{ij}. The same argument applies in higher dimensions: If $\lambda_\mathfrak{t}$ is the barycentre of $r\mathfrak{t}$, then all $v_{i,\mathfrak{t}}$ have the same length. If $\mathfrak{t}$ is a facet of $\mathfrak{s}$, then all triangles $(i, \lambda_\mathfrak{t}, \lambda_\mathfrak{s})$ are equilateral and hence have the same angle at $\lambda_\mathfrak{t}$. This can only be (because the vectors $v_{i,\mathfrak{t}}$ span the supporting plane of $r\mathfrak{t}$) if $v_{\mathfrak{t},\mathfrak{s}}$ is perpendicular to the supporting plane of $\mathfrak{t}$, *q. e. d.*

Corollary. *If the complex is* ***well-centred****, i. e. if the circumcentre $\lambda_\mathfrak{s}$ always lies inside $r\mathfrak{s}$, then the volume of a k-flag $\mathfrak{a} \in (\mathrm{sd}\,\mathfrak{K})^k$ can be computed as* **4.19**

$$|r'\mathfrak{a}| = \frac{1}{k!}|v_{\mathfrak{a}_0,\mathfrak{a}_1}| \cdots |v_{\mathfrak{a}_{k-1},\mathfrak{a}_k}| \quad = \quad \frac{1}{k!}|v_{\mathfrak{a}_0,\mathfrak{a}_1} \wedge \cdots \wedge v_{\mathfrak{a}_{k-1},\mathfrak{a}_k}|.$$

Together with 4.17b, we get for an n-dimensional complex

$$|\mathfrak{s}|\,|{*\mathfrak{s}}| = \binom{n}{k} |U(\mathfrak{s})| \qquad \text{for } \mathfrak{t} \in \mathfrak{K}^k,$$

where we have written $|\mathfrak{s}|$ instead of $|r\mathfrak{s}|$ for short, as we will always do in the following (no ambiguity will occur, as the magnitude $k+1$ of $\mathfrak{s}$ is always indicated by saying $\mathfrak{s} \in \mathfrak{K}^k$).

Remark. This last volume equation is, to the best of our knowledge, not yet used in discrete exterior calculus, but frequently in Regge calculus, see e. g. Miller *et al.* (2013), and its use for discrete calculus seems to date back to Miller (1997).

B. Main Constructions

5. The Karcher Simplex: Definition

Notation. Let $\mathbf{d}$ be the geodesic distance on M. For points $p_0, \ldots, p_n \in M$ and the n-dimensional standard simplex Δ, consider the function

$$E : M \times \Delta \to \mathbb{R}, \quad (a, \lambda) \mapsto \lambda^0\, \mathbf{d}^2(a, p_0) + \cdots + \lambda^n\, \mathbf{d}^2(a, p_n).$$

Convexity. Let $\operatorname{cvr} M$ be the largest radius such that all geodesic balls $\mathbb{B}_{\operatorname{cvr} M}(p)$, $p \in M$, are convex in the sense of 1.4. It can be estimated by **5·1**

$$\operatorname{cvr} M \le \tfrac{1}{2} \min\left\{\tfrac{\pi}{\sqrt{C_0}}, \operatorname{inj} M\right\},$$

where $\operatorname{inj} M$ is the injectivity radius of the manifold (Cheeger and Ebin 1975, thm. 5.14), and $\mathbf{d}(p, \cdot\,)$ is convex in $\mathbb{B}_{\operatorname{cvr} M}(p)$. Consequently, for a smaller ball $B := \mathbb{B}_{\frac{1}{2} \operatorname{cvr} M}(p)$, all functions $\mathbf{d}(q, \cdot\,)$, $q \in M$, are convex in B.

Remark. One knows that $\operatorname{inj} M \ge \min\{$minimal distance between conjugate points$\}, \{\tfrac{1}{2}$min. length of a closed geodesic$\}$, and because conjugate points need to have distance greater than $\pi/\sqrt{C_0}$ by Rauch's comparison theorem (which is true also if the sectional curvature is somewhere negative, so the restriction of nowhere positive sectional curvature in Cheeger and Ebin 1975, corr. 5.7, is only historically determined and factually unneccessary),

$$\operatorname{cvr} M \le \tfrac{1}{2}\Big\{\tfrac{\pi}{\sqrt{C_0}},\ \tfrac{1}{2}\text{min. length of a closed geodesic}\Big\}.$$

The probably best known estimate for the latter term in arbitrary dimension is

$$L(\gamma) \ge 2\pi \frac{\operatorname{vol} M}{\operatorname{vol} \mathbb{S}^m} \Big(\frac{\sqrt{C_0}}{\sinh(\sqrt{C_0}\operatorname{diam} M)}\Big)^{m-1}$$

for a closed geodesic γ (Heintze and Karcher 1978), where $\mathbb{S}^m$ is the m-dimensional unit sphere in $\mathbb{R}^{m+1}$. For even dimensions, Klingenberg (1959) showed $L(\gamma) \ge 2\pi\sqrt{C_0}$ for orientable and $L(\gamma) \ge \pi\sqrt{C_0}$ for non-orientable M.

Observation. Local minimisers of $E(\,\cdot\,, \lambda)$ for fixed λ are zeroes of the section $F : M \times \Delta \to TM$, **5·2**

$$F(a, \lambda) := \lambda^i X_i|_a \qquad \text{with } X_i = \tfrac{1}{2} \operatorname{grad} \mathbf{d}^2(\,\cdot\,, p_i) \text{ from 1.22.}$$

If the points p_i lie in a common ball $B := \mathbb{B}_{\frac{1}{2} \operatorname{cvr} M}(p)$ for some $p \in M$, then $E(\,\cdot\,, \lambda)$ is convex, and hence there is a unique minimiser in B. But this can be sharpened:

Proposition. *Let $p_0, \ldots, p_n$ be contained in a ball $B = \mathbb{B}_r(q) \subset M$ with $r \le \frac{1}{2} \operatorname{cvr} M$. Then for each λ in the standard simplex, $E(\,\cdot\,, \lambda)$ has a global minimiser in B.* **5·3**

Proof. Let x be the minimiser of $E(\,\cdot\,,\lambda)$ over $a \in B$. The key observation is that this function cannot have more than one minimiser in $\mathbb{B}_{2r}(M)$, as KENDALL (1990, thm. 7.3) has shown. His approach uses convex functions on M; but AFSARI (2011) has given a direct proof: $F(\,\cdot\,,\lambda)$ points inwards on the boundary $\partial\mathbb{B}_{2r}(p)$, and the Hessian of E is positive definite at all critical points. So by the Poincaré–Hopf index theorem, F needs to have exactly one zero.

Outside $\mathbb{B}_{2r}(p)$, there cannot be any minimisers: If $a \in M \setminus \mathbb{B}_{2r}(q)$, then $\mathbf{d}(a,p_i) > r \geq \mathbf{d}(x,p_i)$ and hence $E(a,\lambda) > E(x,\lambda)$, *q. e. d.*

Assumption. From now on, we only consider $p_0,\dots,p_n$ that lie in a common ball B of radius smaller than $\frac{1}{2}\operatorname{cvr} M$, in particular $C_0 r^2 \leq \frac{\pi^2}{4}$.

5·4 **Definition.** For a given $\lambda \in \Delta$, let $x(\lambda)$ be the minimizer of $E(\,\cdot\,,\lambda)$ in B. We call this map $x : \Delta \to M$ the **barycentric mapping** with respect to vertices p_i, and its image $s := x(\Delta)$ the corresponding **Karcher simplex**.

5·5 **Remark.** **⟨a⟩** In case M is the Euclidean space, x is just the canonical parametrisation $\lambda \mapsto \lambda^i p_i$, because $\mathbf{d}^2(p,a) = |a-p|^2_{\ell^2}$ gives $X_i|_a = a - p_i$.

⟨b⟩ For $\lambda^i = 0$, the value $x(\lambda)$ is independent of p_i. So the subsimplices of the standard simplex are mapped to “Karcher subsimplices” which only depend on the vertices of the subsimplex.

⟨c⟩ If e_i is the i-th Euclidean basis vector of $\mathbb{R}^{n+1}$, then $x(te_j + (1-t)e_i) = \gamma(t)$, where γ is the unique shortest geodesic with $\gamma(0) = p_i$ and $\gamma(1) = p_j$.

⟨d⟩ Because x is continuous, the Karcher subsimplices form the boundary of a Karcher simplex: $\partial(x(\Delta)) = x(\partial\Delta)$.

⟨e⟩ Concerning the definition of x, we will not make use of the fact that all λ^i are positive beside in 6.16. It was only needed to have an easy access to the well-definedness of the minimiser. SANDER (2013) showed that negative weights also lead to a well-defined minimum if the p_i are contained in a ball whose radius is bounded by a constant depending on $\operatorname{inj} M$, the curvature of M and $\max|\lambda^i - \lambda^j|$.

5·6 **Proposition.** *If all p_i lie in a totally geodesic submanifold S, then $x(\Delta) \subset S$. Therefore the usual notion of simplices in spaces of constant curvature as convex hull of the vertices (cf. e. g. THURSTON 1997, ex. 3.3.6) is recovered.*

Proof. Let $k_i := \frac{1}{2}\mathbf{d}^2(\,\cdot\,,p_i)$. If B is convex, then so is $B \cap S$. So $E(\lambda,\,\cdot\,)$ is convex on $B \cap S$, and hence there is a unique minimizer a of $E(\lambda,\,\cdot\,)|_S$, so there are coefficients λ^i with $\lambda^i \operatorname{grad}(k_i|_S) = 0$ at a. As S is totally geodesic in M, it holds $\operatorname{grad}(k_i|_S) = (\operatorname{grad} k_i)|_S = X_i|_S$. Hence $\lambda^i X_i = 0$ at the point $a \in S$, *q. e. d.*

5·7 **Proposition.** *Define bundle maps σ, A_v (for $v \in \mathbb{R}^{n+1}$) by*

$$\begin{aligned} \sigma|_\lambda :\quad & T_\lambda\Delta \to T_{x(\lambda)}M, & v &\mapsto -\, v^i X_i|_{x(\lambda)}, \\ A_v|_\lambda :\quad & T_{x(\lambda)}M \to T_{x(\lambda)}M, & V &\mapsto v^i \nabla_V X_i|_{x(\lambda)}. \end{aligned}$$

If x is smooth at $\lambda \in \Delta$, then its first and second derivative there fulfill

$$A_\lambda dx\, v - \sigma(v) = 0, \tag{5.7a}$$

$$A_\lambda \nabla dx(v,w) + A_w dx\, v + A_v dx\, w + \lambda^i \nabla^2_{dx\, w, dx\, v} X_i = 0. \tag{5.7b}$$

Proof. Similar to the proof of the implicit function theorem: The derivatives of F in a direction $(V, v) \in T_{x(\lambda)}M \times T_\lambda \Delta$ are

$$\nabla_{(V,v)} F = \lambda^i \nabla_V X_i + v^i X_i = A_\lambda V - \sigma_\lambda(v).$$

Now consider a curve $\gamma : t \mapsto \lambda + tv$ with derivative $\dot\gamma(t) = v$. We have $F(x(\gamma(t)), \gamma(t)) = 0$, and differentiating this gives, like in the usual proof of the implicit function theorem, $0 = D_t F = \nabla_{(dx\, v, v)} F$. This shows the first claim.

The second claim is a totally analogous computation, but involves second covariant derivatives $\nabla^2_{V,W} := \nabla_V \nabla_W - \nabla_{\nabla_V W}$. We differentiate F once more and obtain

$$\begin{aligned}\nabla_{(W,w)} \nabla_{(V,v)} F &= w^i \nabla_V X_i + v^i \nabla_W X_i + \lambda^i \nabla_W \nabla_V X_i \\ &= w^i \nabla_V X_i + v^i \nabla_W X_i + \lambda^i \nabla^2_{W,V} X_i + \lambda^i \nabla_{\nabla_W V} X_i\end{aligned}$$

Again, F neither changes in direction $(dx\, v, v)$ nor in direction $(dx\, w, w)$, so we get

$$\begin{aligned}0 &= w^i \nabla_{dx\, v} X_i + v^i \nabla_{dx\, w} X_i + \lambda^i \nabla^2_{dx\, w, dx\, v} X_i + \lambda^i \nabla_{\nabla_{dx\, w} dx\, v} X_i \\ &= A_w dx\, v + A_v dx\, w + \lambda^i \nabla^2_{dx\, w, dx\, v} X_i + A_\lambda \nabla_{dx\, w} dx\, v.\end{aligned}$$

And because Δ is flat, we have $\nabla_{dx\, w} dx\, v = \nabla dx(v, w)$ by 1.7, *q. e. d.*

Remark. One can consider λ as a point measure on M that assigns the mass λ^i to the point p_i. For a general probability measure μ on M, Karcher (1977) speaks of the minimiser of **5.8**

$$E_\mu(a) := \int_M \mathbf{d}^2(a, p)\, \mathrm{d}\mu(p)$$

as **Riemannian centre of mass**, but the subsequent literature has mostly called it the **Karcher mean** with respect to the measure μ (cf. e.g. Jermyn 2005, probably initiated by Kendall 1990). The concept seems to go back to Cartan (see the historic overview in Afsari 2011), but has not been used by others until the work of Grove and Karcher (1973).

Karcher himself used the centre of mass to retrace the standard mollification procedure of Gauss kernel convolution in the case of functions that map into a manifold. Considering the centre of mass as a function from an interesing finite-dimensional space of measures into M, as we use it, has been done by Rustamov (2010), and the barycentric coordinates we deal with have been used by Sander (2012, 2013) and Grohs *et al.* (2013). All these emphasise the possibility to glue the Karcher simplices along corresponding facets, but do not investigate the distortion properties of this mapping. Recently, we were informed that Dyer and Wintraecken (Rijksuniversiteit Groningen) have also proven a result similar to 6.17 by Topogonov's angle comparison theorems. However, this approach seems not to deliver an analogue of 6.22.

In contrast, there is a large literature for "barycentric coordinates" on general convex polygons in the plane, cf. Warren *et al.* (2007); Meyer *et al.* (2002) and references therein.

6. Approximation of the Geometry

Estimates for Jacobi Fields

It is well-known that locally, Jacobi fields grow approximately linearly:

$$|J(t) - P^{t,0}(J(0) + t\dot{J}(0))| \leq C_0 t^2 |\dot{\gamma}|^2 \big(|J(0)| + \tfrac{1}{4} t |\dot{J}(0)|\big) \qquad \text{for } C_0 t^2 \leq \tfrac{\pi^2}{4}. \tag{6.1}$$

In fact, Jost (2011, thm. 5.5.3) proves that the left-hand side is smaller than $|J(0)|$ $(\cosh ct - 1) + \frac{\mathrm{d}}{\mathrm{d}t}|J|(0)(\frac{1}{c}\sinh ct - t)$ for $c = \sqrt{C_0}$. By Taylor expansion and $\frac{\mathrm{d}}{\mathrm{d}t}|J| \leq |\dot{J}|$, this estimate is weakened to our form. Clearly, if the values $J(0)$ and $J(\tau)$ are given, one can expect $\dot{J}$ to behave like $\frac{1}{\tau}(J(\tau) - J(0))$, but as Richard Dedekind (1893, p. 11) said, "nothing that is provable ought to be believed without proof in science."

6.2 **Situation.** Suppose $\gamma : [0;\tau] \to M$ is an arclength-parametrised geodesic with $\gamma(0) = p$ and $\gamma(\tau) = q$, and $V \in T_q M$. Let $s \mapsto \delta(s)$ be a geodesic with $\delta(0) = \gamma(\tau)$ and $\dot{\delta}(0) = V$. Define a variation of geodesics by

$$c(s,t) := \exp_p\big(\tfrac{t}{\tau}(\exp_p)^{-1}\delta(s)\big).$$

Then $T := \partial_t c$ is an autoparallel vector field and $J := \partial_s c$ a Jacobi field along $t \mapsto c(s,t)$ for every s with boundary values $J(s,0) = 0$ and $J(s,\tau) = \dot{\delta}(s)$.

6.3 **Proposition.** *Situation as in 6.2. Define $V(s,t) := P^{t,\tau}\dot{\delta}(s)$ and $\ell(s) := \tau|T|(s)$, the distance from p to $\delta(s)$. (By construction, $|V(s,t)|$ is constant in s and t, and $|T(s,t)|$ is constant in t, so we drop the unneeded arguments.) If $C_0\ell^2(s) < \frac{\pi^2}{4}$ for all s, then*

$$\begin{aligned} \big|J(s,t) - \tfrac{t}{\tau}V(s,t)\big| &\leq 2C_0\ell^2(s)|V|, \\ |\dot{J}(s,t) - \tfrac{1}{\tau}V(s,t)| &\leq \tfrac{3}{2}C_0\ell(s)\,|T|(s)|V|, \\ |\ddot{J}(s,t)| &\leq C_0|T|^2(s)|V|. \end{aligned}$$

If the derivatives of R up to order k are bounded by constants, then so are the t-derivatives of J up to order $k+2$.

Proof. From the usual Jacobi field estimates, e. g. Jost (2011, thm. 5.5.1), we get that $|J|$ is increasing for all $t < \tau$ in case $C_0\ell^2 < \frac{\pi^2}{4}$. By the Jacobi equation 1.15a, this already shows the last claim. Now observe $J(s,0) - 0\dot{J}(s,0) = 0$ and

$$\Big|D_t\left(J(s,t) - t\dot{J}(s,t)\right)\Big| = t\,|\ddot{J}(s,t)| \leq C_0 t\,|T|^2(s)|V|.$$

So the vector field $U : t \mapsto J(s,t) - t\dot{J}(s,t)$ vanishes at $t = 0$, and we have bounded its derivative. The fundamental theorem of calculus 1.19a gives

$$|\dot{J}(s,t) - tJ(t,s)| \leq \tfrac{1}{2}C_0 t^2 |T|^2(s)|V|. \tag{6.3a}$$

By $J(s,\tau) = V(s,\tau)$, we have

$$|V(s,\tau) - \tau\dot{J}(s,\tau)| \leq \tfrac{1}{2}C_0\,\ell^2(s)\,|V|.$$

Now $|P^{t,\tau}\dot J(s,\tau) - \dot J(s,t)| \le (\tau - t)\max|\ddot J|$ by the mean value theorem, and thus

$$\begin{aligned} |V(s,t) - \tau\dot J(s,t)| &\le |P^{t,\tau}V(s,\tau) - \tau P^{t,\tau}\dot J(s,\tau)| + \tau|P^{t,\tau}\dot J(s,\tau) - \dot J(s,t)| \\ &\le \tfrac12 C_0|V|\,\ell^2(s) \qquad\qquad + C_0(\tau-t)\tau|V|\,|T|^2(s) \\ &\le \tfrac32 C_0\ell^2(s)|V|. \end{aligned}$$

This proves the comparison between $\dot J$ and $\frac{1}{\tau}V$. For the comparison to J, consider

$$\begin{aligned} \big|J(s,t) - \tfrac{t}{\tau}V(s,t)\big| &\le |J(s,t) - t\dot J(s,t)| + t|\dot J(s,t) - \tfrac{1}{\tau}V| \\ &\le \tfrac12 C_0 t^2|V|\,|T|^2(s) \;+ \tfrac32 C_0 t\tau|V|\,|T|^2(s) \\ &\le 2C_0\ell^2(s)|V|. \end{aligned}$$

The statement about higher derivatives of J is justified by the fact that one can easily give linear ODE's for them by differentiating the Jacobi equation, e. g. $\dddot J + R(\dot J, T)T + \dot R(J,T)T = 0$ as $\dot T = 0$, *q. e. d.*

Remark. These estimates are scale-aware with respect to reparametrisations of γ: If t is replaced by λt, then also τ becomes $\lambda\tau$, whereas $|T|$ becomes $\frac{1}{\lambda}|T|$. So $\frac{t}{\tau}$ and hence the whole first inequality in 6.3 is scale-independend. As $\dot J = \nabla_T J$ (loosely speaking), the second inequality scales with $1/\lambda$ and the third one with $1/\lambda^2$. **6.4**

Lemma. *Consider some C^2 function $U : [0;\tau] \to \mathbb{R}^m$ satisfying the linear second-order differential equation $\ddot U = AU + B$ with smooth time-dependent data $A(t) \in \mathbb{R}^{m\times m}$ and $B(t) \in \mathbb{R}^m$ as well as boundary conditions $U(0) = U(\tau) = 0$. Then, provided that $\|A(t)\|\tau^2 \le 1$ everywhere, it holds* **6.5**

$$|\dot U(t)| \le 3|B|\tau, \qquad |U(t)| \le 6|B|t(\tau - t).$$

Proof (by David Glickenstein). Denote the maxima of $\|A\|$ and $|B|$ over $[0;\tau]$ as a and b respectively. As U is C^2, there is an upper bound K for $|U|$ on $[0;\tau]$, attained at $t = \vartheta$. As this point is critical for $|U|^2$, we have $\langle U, \dot U\rangle = 0$ there. So

$$K^2 + \vartheta^2|\dot U(\vartheta)|^2 = |U - t\dot U|^2(\vartheta) = \Big|\int_0^\vartheta t\ddot U\,\mathrm{d}t\Big|^2 \le \Big|\int t(aK+b)\Big|^2 = \big(\tfrac12\vartheta^2(aK+b)\big)^2.$$

This shows $K \le \frac12\vartheta^2(aK+b)$, so $K \le \tau^2 b$ by assumtion and hence $|\ddot U| \le 2b$. (Note that this argument, which first roughly bounds $|U|$ and then re-inserts this bound into the differential inequality to get a sharper estimate, is the same as in 11.17sq.) Furthermore, the inequality chain also shows $\vartheta|\dot U(\vartheta)| \le b\vartheta^2$, which means $|\dot U(\vartheta)| \le b\tau$. For other values of t, we have $|\dot U(t)| \le b\tau + \int|\ddot U(t)| \le 3b\tau$ and, by integrating once more, $|U(t)| \le 3b\tau t$ as well as $|U(t)| \le 3b\tau(\tau - t)$, whose minimum is dominated by $6bt(\tau - t)$, *q. e. d.*

6.6 Proposition. *Sitation as in 6.2, $C_0\ell^2(s) \le \frac{\pi^2}{4}$. Then*

$$|D_sJ(s,t)| \le 90C_{0,1}(s)\frac{t(\tau-t)}{\tau}\,|V|^2\,|T|(s) \le 90C_{0,1}(s)\,t\,|V|^2\,|T|(s), \qquad |D_s\dot J(s,t)| \le 50C_{0,1}(s)\,|V|^2\,|T|(s).$$

with $C_{0,1}(s) := C_0 + \ell(s)C_1$. If derivatives of R up to order k are bounded by constants $C_1,\dots,C_k$, then $\tau|D^k_{s\dots s}\dot J| \le c(C_0,\dots,C_k)\,|V|^2$. Under reparametrisations of γ as in 6.4, the first estimate remains unchanged, the second one scales with $\frac{1}{\lambda}$.

Proof. Our approach is to derive some differential equation for $D_sJ = \nabla_J J$, which has boundary values $D_sJ(s,0) = 0$ and $D_sJ(s,\tau) = 0$ for all s because $J(s,0) = 0$ is constant in s and $J(s,\tau) = \dot\delta(s)$ is the tangent of a geodesic.

ad primum: Because J and T are coordinate vector fields, 1.3a gives $D_sD_tU = D_tD_sU + R(J,T)U$ for every vector field U, so we have

$$\begin{aligned} D_s\ddot J = D_sD_tD_t\partial_s c &= D_tD_sD_t\partial_s c + R(J,T)\dot J \\ &= D_tD_tD_s\partial_s c + D_tR(J,T)J + R(J,T)\dot J \\ &= D^2_{tt}D_sJ + \dot R(J,T)J + R(\dot J,T)J + 2R(J,T)\dot J \end{aligned}$$

whereas the (negative) left-hand side is, due to the Jacobi equation,

$$-D_s\ddot J = D_s\big(R(J,T)T\big) = (D_sR)(J,T)T + R(D_sJ,T)T + R(J,\dot J)T + R(J,T)\dot J$$

(note $D_sT = D_tJ = \dot J$). From now on, we consider J and $\dot J$ as being part of the given data (which is allowed, as we have already sufficiently described their behaviour in 6.3). So we have a linear second-order ODE for $U := D_sJ$:

$$\ddot U = AU + B,$$

where both sides scale with $1/\lambda^2$ under reparametrisation, and the norm of A is bounded through $\|A\| \le C_0|T|^2(s)$. For ease of notation, we will thus assume that we consider a t-line with $|T|(s) = 1$ and rescale our results afterwards. By assumption on the smallness of τ,

$$\begin{aligned} |B| &\le 2C_1|J|^2 + 5C_0|J|\,|\dot J| \\ &\le 2C_1|V|^2 + 5C_0|V|(\tfrac{1}{\tau} + \tfrac{3}{2}C_0\tau)|V| \\ &\le 15C_{0,1}\,\tfrac{1}{\tau}|V|^2. \end{aligned}$$

Now consider Fermi coordinates along $c(s,\cdot\,)$ as in 1.17 to obtain an ODE in Euclidean space. For any smooth vector field $V = V^i\partial_i$, the covariant derivative in direction $T = \partial_t c$ is just $\nabla_T V = V^i_{,1}\partial_i$. Hence, our ODE has the coordinate expression

$$U^i_{,11}\partial_i = (A^i_jU^j + B^i)\partial_i.$$

As we only need to know the values of U on $x = (t,0,\dots,0)$, this gives a euclidean differential equation for the components U^i of the same form as above. The claim on U is then contained in 6.5.

ad sec.: With $U = D_s J$ as above, we have $D_s \dot{J} = \dot{U} + R(J,T)J$ and thus

$$|D_s \dot{J}| \le |\dot{U}| + C_0 |J|^2 \le 45 C_{0,1} |V|^2 + 4 C_0 |V|^2.$$

ad tertium: For higher s-derivatives, one can proceed by induction: The statement is true for $k = 0, 1$, as we have shown above. By analogous computations, one can control $D^k_{s \dots s} \dot{J}$ by a linear second-order ODE, in which all lower derivatives might enter as "given data". This data is bounded by a constant, and hence the solution will be bounded as well, *q. e. d.*

Estimates for Normal Coordinates

Situation. Fix some $p \in M$ and consider normal coordinates around p as in 1.12: **6.7**

$$x : (u^1, \dots, u^m) \mapsto \exp_p u^i E_i$$

for some orthonormal basis E_i of $T_p M$. Recall from 1.14 that the deviation of metric and connection from its Euclidean counterparts can be described by differential and Hessian of the exponential map. In the following, we let $r := \mathbf{d}(p, \cdot\,)$ be the geodesic distance to p.

Lemma. *Situation as above, $C_0 r^2 \le \frac{\pi^2}{4}$. Then $|g_{ij} - \delta_{ij}| \le C_0 r^2$.* **6.8**

Proof. By 3.7, it suffices to consider $i = j$. So we only have to show $\big| |dx\, e_i|^2 - 1 \big| \le C_0 r^2$. By means of 1.14a, this amounts to control $\big| |d_U(\exp_p) E_i| - |E_i| \big|$. From 1.16, we know that $d_U(\exp_p) E_i$ is the terminal value $J(1)$ of a Jacobi field with $J(0) = 0$ and $\dot{J}(0) = E_i$. Now $\big| |d_U(\exp_p) E_i| - |E_i| \big| \le |d_U(\exp_p) E_i - P E_i| \le \frac{1}{4} C_0 r^2$ by 6.1, and the squared norms thus cannot differ by more than $2(1 + \frac{\pi^2}{16}) \cdot \frac{1}{4} C_0 r^2 \le 0.809 C_0 r^2$ due to 3.15, *q. e. d.*

Lemma. *Situation as above, $C_0 r^2 \le \frac{\pi^2}{4}$. Then $\|\Gamma\| \le 10 C_0 r + 5 C_1 r^2$.* **6.9**

Proof. Again, the case $i = j$ is sufficient. Additionally, we will only prove the claim for $r = 1$. The correct scaling is then automatically enforced by 1.10. So let $T, V \in T_p M$ be unit vectors, assume $C_0 \le \frac{\pi^2}{4}$, and consider a variation of geodesics $c(s,t) = \exp_p t(T + sV)$. As the exponential mapping has no radial distortion, we may assume $V \perp T$ wthout loss of generality. This delivers us a Jacobi field $J(s, \cdot\,) = \partial_s c(s, \cdot\,)$ for each s, and 1.14b tells us that $d_T(\exp_p)(\Gamma(v,v)) = \nabla d_T(\exp_p)(V,V) = \nabla_J J(0,1) = D_s \partial_s c(0,1)$ for $V = v^i E_i$. As observed in 6.6, the vector field $U := D_s \partial_s(0, \cdot\,)$ along $c(0, \cdot\,)$ obeys the linear second-order ODE

$$\ddot{U} = R(T,U)T + R(\dot{J}, J)T + 3R(T,J)\dot{J} + R(T,\dot{J})J + \dot{R}(T,J)J + (D_s R)(T,J)T$$

where the obvious notation T for $\partial_t c$ has been used. So again we have $\ddot{U} = AU + B$ with $\|A\| \le C_0$ and $|B| \le 2C_1 |J|^2 + 5 C_0 |J| |\dot{J}|$, but this time as an initial-value problem with $U(0) = \dot{U}(0) = 0$. Denoting the supremum over $|B|$ by b again, the norm $|U|$ will be dominated by the solution of $\ddot{u} = c^2 u + b$, $c = \sqrt{C_0}$, which is $\frac{b}{2c^2} \mathrm{e}^{-ct} (\mathrm{e}^{ct} - 1)^2$, which itself is smaller than $\frac{5}{8}b$ for $ct \le \frac{\pi}{2}$. This means $|\nabla d_T(\exp_p)(V,V)| \le \frac{5}{8}b$, and our task is to estimate B against $V = \dot J(0)$.

From 6.1, we get $|J(t)| \le (1+\frac{1}{4}C_0t^2)t|\dot J(0)| \le (1+\frac{\pi^2}{16})t|\dot J(0)|$ for all $t \le 1$. On the other hand, 6.3a gives $t|\dot J(t)| \le (1+\frac{1}{2}C_0t^2)|J(t)|$, and combining both leads us to the rought, but sufficient estimate $|\dot J(t)| \le (1+\frac{\pi^2}{8})(1+\frac{\pi^2}{16})|\dot J(0)|$. So we have, as $V = \dot J(0)$ is of unit length,

$$|B| \le (1+\tfrac{\pi^2}{16})^2 C_1 + (1+\tfrac{\pi^2}{8})(1+\tfrac{\pi^2}{16})^2 C_0. \tag{6.9a}$$

So far, we have only estimated the norm of $\nabla d_T(\exp_p)(V,V) = d_T(\exp_p)(\Gamma(v,v))$ by $\frac{5}{8}b$, and this needs to be compared to $\Gamma(v,v)$. By 1.16, the former is the value $Z(1)$ of a Jacobi field Z along $c(0,\cdot)$, and the latter is $\dot Z(0)$. Using 6.1 for Z, we obtain $|\dot Z(1)| \le (1-\frac{\pi^2}{16})^{-1}|Z(1)|$, and by inserting this into 6.9a, we finally get

$$|\Gamma(e_i,e_i)| \le \frac{\frac{5}{8}|B|}{1-\frac{\pi^2}{16}} \le \frac{\frac{5}{8}(1+\frac{\pi^2}{16})^2}{1-\frac{\pi^2}{16}}C_1 + \frac{\frac{5}{8}(1+\frac{\pi^2}{8})(1+\frac{\pi^2}{16})^2}{1-\frac{\pi^2}{16}}C_0 \le 5C_1 + 10C_0,$$

q. e. d.

6.10 Remark. ⟨**a**⟩ As one can easily see in the proof, our numerical constants are by no means optimal. A sharper result, but with much more technical effort, has been given by Kaul (1976). This author also deals with the case that the sectional curvature might be asymmetrically bounded between c_0 and C_0, whereas we are only interested in the simpler case $c_0 = -C_0$.

⟨**b**⟩ Considering the Christoffel symbols as objects that store "derivative information" for the metric, the classical procedure of numerical analysis would have been to first estimate the Christoffel symbols and then integrate this to obtain a bound for the metric tensor. It is a specific property of the g_{ij} that they can be bounded by a right-hand side which includes fewer terms than the bound for their derivatives.

⟨**c**⟩ Under scaling of Mg as in 1.11, the estimate 6.9 scales like $\frac{1}{\mu}$, and 6.8 is scale-independent. The assumptions in both propositions are scale-independent.

⟨**d**⟩ Regarding 1.3b rises the question if derivatives of R are actually needed to bound $\|\Gamma\|$. In fact they *are* needed in normal coordinates (de Turck and Kazdan 1981, ex. 2.3), but not in harmonic coordinates, which would lead to estimates that only depend on C_0r^2 (*loc. cit.*, thm. 2.1). As Bemelmans *et al.* (1984) remarked, the metric g can be infinitesimally abridged by a short time of Ricci flow, and the new metric $\bar g$ has $\|\nabla^i \bar R\| \le \bar C_i(C_0)$ for all i. Furthermore, a bound on ∇R will be needed in 6.6 anyway, so we decided to take normal coordinates, which make it easier to give explicit numerical constants in the estimates.

6.11 Conclusion. *In a normal coordinate ball (B,u) of radius r with $C_0r^2 < 1$ and $2r < \operatorname{inj} M$, g and the Euclidean standard metric are equivalent, and*

$$\big||V|_{g(u)} - |V|_{\ell^2}\big| \le C_0|u|^2|V|_{\ell^2}, \qquad \|\Gamma(u)\| \le 10C_0|u| + 5C_1|u|^2. \tag{6.11a}$$

Corollary. *In a Fermi coordinate tube of radius r with $C_0 r^2 < 1$ and $2r < \operatorname{inj} M$, g and the Euclidean standard metric are equivalent, and* **6.12**

$$\big||V|_{g(t,u)} - |V|_{\ell^2}\big| \le C_0|u|^2|V|_{\ell^2}, \qquad \|\Gamma(t,u)\| \le 10C_0|u| + 5C_1|u|^2. \tag{6.12a}$$

Lemma. *Let g and g^e be two Riemannian metrics with $\big||v|_g - |v|_{g^e}\big| \le \varepsilon|v|_{g^e}$, $\varepsilon < 1$. Then the curve lengths and geodesic distances with respect to g and g^e fulfill* **6.13**

$$|L_g(c) - L_{g^e}(c)| \le \varepsilon L_{g^e}(c), \qquad |\mathbf{d}_g(p,q) - \mathbf{d}_{g^e}(p,q)| \le \varepsilon\, \mathbf{d}_{g^e}(p,q).$$

Proof. The first claim is proven in the obvious way by integrating $\big||\dot c|_g - |\dot c|_{g^e}\big| \le \varepsilon|\dot c|_{g^e}$ along c. The second claim is a combination with $L_g(c) \le L_g(c^e)$ and $L_{g^e}(c^e) \le L_{g^e}(c)$ if c and c^e are the distance-realising geodesics for g and g^e respectively, *q. e. d.*

Approximation of the Metric

Corollary. *Let q be in a convex neighbourhood of p, $\ell := \mathbf{d}(p,q)$ with $C_0\ell^2 \le \frac{\pi^2}{4}$, and let $U \in T_qM$ be an arbitrary direction. Then* **6.14**

$$|\nabla_V X_p - V| \le \tfrac{3}{2}C_0\ell^2|\pi_Y^\perp V| \le \tfrac{3}{2}C_0\ell^2|V|,$$
$$|\nabla^2_{V,V}X_p| \le 50(C_0 + \ell C_1)\ell\,|\pi_Y^\perp V|^2.$$

Here $\pi_Y^\perp$ is the orthogonal projection onto the orthogonal complement of $Y_p|_q$ in T_qM.

Proof. Direct consequence of 6.3 and 6.6 together with 1.23, *q. e. d.*

Remark. With $|U|$ instead of $|\pi_Y^\perp U|$, but with a smaller constant, the first claim is directly proven in Jost and Karcher (1982, also cf. Karcher 1977, A.5.4). For the improvement, see Kaul (1976). An exact computation of $\nabla d\exp$ for symmetric spaces is given by Fletcher (2013).

Lemma. *Let $A : V \to V$ be an endomorphism of a normed vector space V with $\|\operatorname{id} - A\| \le \varepsilon < 1$. Then $\|\operatorname{id} - A^{-1}\| \le \varepsilon/(1-\varepsilon)$.* **6.15**

Proof. By the Neumann series (Alt 2006, ex. 3.7):

$$A^{-1} = \sum_{i=0}^{\infty}(\operatorname{id} - A)^i, \quad \text{so} \quad \|\operatorname{id} - A^{-1}\| \le \sum_{i=1}^{\infty}\varepsilon^i = \frac{\varepsilon}{1-\varepsilon},$$

q. e. d.

Lemma. *Let $p_0, \dots, p_n$ be distinct points inside a convex ball of radius h and x be their barycentric mapping. If $6C_0h^2 \le 1$, then for a tangent vector $v \in T_\lambda\Delta$ at any $\lambda \in \Delta$ and σ as in Proposition 5.7,* **6.16**

$$\big|dx\, v - \sigma(v)\big| \le 2C_0h^2\,|\sigma(v)|.$$

Proof. By 5.7, $d_\lambda x\, v = (A_\lambda)^{-1}\sigma|_\lambda(v)$ and hence $\bigl|dx\, v - \sigma(v)\bigr| \le \|A_\lambda^{-1} - \mathrm{id}\,\|\,|\sigma(v)|$. By 6.14, one has $|\nabla_V X_i - V| \le \frac{3}{2}C_0\, \mathbf{d}^2(\,\cdot\,, p_i)|V|$ for all tangent vectors V, or, in terms of operator norms, $\|\nabla X_i - \mathrm{id}\,\| \le \frac{3}{2}C_0\,\mathbf{d}^2(\,\cdot\,,p_i) \le \frac{3}{2}C_0 h^2$. Thus, as $\lambda \cdot 1_{n+1} = 1$ and $\lambda^i \ge 0$,

$$\|A_\lambda - \mathrm{id}\,\| = \|\lambda^i(\nabla X_i - \mathrm{id})\| \le |\lambda^i|\,\|\nabla X_i - \mathrm{id}\,\| \le \tfrac{3}{2}C_0 h^2.$$

Now if $6C_0h^2 \le 1$, then $1 - \frac{3}{2}C_0h^2 \ge \frac{3}{4}$, and the claim follows from 6.15, *q. e. d.*

Notation. We write $a \lesssim b$ if there is some constant c which only depends on n such that $a \le c\,b$ (saying "$a \le b$ up to a constant."). Equivalently, we will also write $a = O(b)$. We in particular remark that our suppressed constants do not depend on the geometry parameters.

6.17 Theorem. *Let $p_0, \dots, p_n$ be distinct points inside a convex ball and x be their barycentric mapping. Let g^e be the flat Riemannian metric on Δ induced by geodesic distances $\mathbf{d}(p_i, p_j)$. Suppose g^e is (ϑ, h)-small, $3nC_0'h^2 < \underline{\alpha}_n^2$ with $\underline{\alpha}_n$ from 3.5. Then it holds for tangent vectors $v, w \in T_\lambda\Delta$*

$$|(x^*g - g^e)\langle v, w\rangle| \lesssim C_0' h^2 |v|\,|w|. \tag{6.17a}$$

The norms on the right-hand side can be either x^*g or g^e norms, as both are equivalent.

Proof. Note that the assumption on h includes the requirements of 6.16 and 6.13. Due to 3.7, it suffices to show the claim for $v = w$. Consider a point $\lambda \in \Delta$ with image $a = x(\lambda)$. We first compare x^*g to the Euclidean metric of the simplex $\bar{s}_a = \mathrm{conv}(X_i|_a) \subset T_aM$, and compare this metric to g^e afterwards.

Parametrise $\bar{s}_a$ in the canonical way over the unit simplex via $\bar{x} : \lambda^i e_i \mapsto \lambda^i X_i|_a$. Now clearly $d\bar{x} = \sigma$ from 5.7. The metric of $\bar{s}_a$ is the induced metric of the surronding vector space, namely $g|_a$. Now use 6.16 to get

$$\begin{aligned}|(x^*g|_a)\langle v,v\rangle^{1/2} - (\bar{x}^*g|_a)\langle v,v\rangle^{1/2}| &= \bigl||dx\,v|_{g|_a} - |d\bar{x}\,v|_{g|_a}\bigr| \\ &\le |dx\,v - d\bar{x}\,v|_{g|_a} \le 2C_0h^2|d\bar{x}\,v|_{g|_a} = 2C_0h^2|v|_{\bar{x}^*g}.\end{aligned}$$

And of course, the same is true for the squared norms by 3.15: $|(x^*g - \bar{x}^*g)\langle v, v\rangle| \le 6C_0h^2|v|^2_{\bar{g}^e}$. Hence we have successfully compared x^*g to the euclidean metric of $\bar{s}_a$. If we can show that s and $\bar{s}_a$ have almost equal metrics, we are done with 3.16.

The edge lengths of $\bar{s}_a$ are $|X_i - X_j|_{g|_a}$, and the edge lengths of s are the geodesic distance between $p_i = \exp_a(X_i)$ and $p_j = \exp_a(X_j)$. By 6.13, we have for their edge lengths ℓ_{ij} and $\bar{\ell}_{ij}$

$$|\ell_{ij} - \bar{\ell}_{ij}| = \bigl|\mathbf{d}(p_i,p_j) - |X_i - X_j|\bigr| \le C_0h^2\,\mathbf{d}(p_i,p_j) = C_0h^2\ell_{ij},$$

so g^e and $\bar{g}^e$ match 3.16 with $\frac{2}{3}\varepsilon n^{-1}\underline{\alpha}_n^2\vartheta^2 = C_0h^2$, *q. e. d.*

6.18 Corollary. $h\vartheta|\cdot|_{\ell^2} \lesssim |\cdot|_g \lesssim h|\cdot|_{\ell^2}$.

Definition. We say that points $p_1, \dots, p_n \in M$ lie in (ϑ, h)-**close position**, if there is some $p \in x(\Delta)$ such that $\bar{g}^e_{ij} = -\frac{1}{2}|X_i - X_j|^2_{g|_p}$ defines a (ϑ, h)-small metric in the notation of 3.3. (Note that this can only be if $n \leq m$.)

Corollary. *Each collection of points $p_0, \dots, p_n$ in (ϑ, h)-close position that fulfill $3nC_0'h^2 \leq \underline{\alpha}_n$ defines an injective barycentric map.* **6.19**

Remark. As x^*g and g are equivalent metrics, there is a self-adjoint automorphism J of $T_\lambda\Delta$ such that $x^*g\langle v, w\rangle = g^e\langle Jv, w\rangle$, as has been empoyed by Holst and Stern (2012, thm. 3.8). For a comparison to the metric distortion tensor A of Wardetzky (2006) and the A_h of Dziuk (1988) et al., see 11.15. **6.20**

Approximation of Covariant Derivatives

Remark. The second-order approximation qualities of a parametrisation would usually be measured by bounds on the Christoffel symbols. However, our definition of g^{ij} is not exactly the inverse matrix of g_{ij}, so the usual definition 1.2b would not work. **6.21**

Instead, we employ the idea from Kaul (1976) to consider the operator $\Gamma = \nabla^{x^*g} - \nabla^{g^e}$, which would have the coordinate expression $\Gamma(V, W) = \Gamma^k_{ij} V^i W^j \partial_k$ in a usual n-dimensional chart. Recall from 6.7 that ∇^{x^*g} is defined by $dx \nabla^{x^*g}_v w = \nabla_{dx\,v} dx\, w$ We suppress the g subscript for norms.

Theorem. *Situation as in 6.17. Then $\|\nabla dx\|_{\Delta g^e, Mg} \lesssim C'_{0,1}h$.* **6.22**

Proof. Due to 3.7, it suffices to show the theorem for $v = w$. Similar to $\|A_\lambda^{-1} - \mathrm{id}\,\| \lesssim C_0h^2$ we have, as the v^i sum up to zero,

$$|A_v(V)| = |v^i \nabla_V X_i - \textstyle\sum v^i V| \leq |v^i||\nabla_V X_i - V| \leq \frac{3}{2}|v|_{\ell^1}\, C_0 h^2 |V|_g.$$

Now we use 5.7b and again that $\|A_\lambda^{-1}\| \lesssim 1 + C_0h^2$:

$$\begin{aligned} \frac{1}{1 + C_0h^2}|\nabla dx(v, v)| &\leq 2|A_v(dx\,v)| + |\lambda^i \nabla^2_{dx\,v, dx\,v} X_i| \\ &\lesssim C_0h^2 |v|_{\ell_1}\, |dx\,v| + C_{0,1}h|dx\,v|^2 \lesssim C_{0,1}h^2|v|_{\ell^2}\, |v|, \end{aligned}$$

where $\nabla^2 X_i$ has been estimated by 6.14, *q. e. d.*

Corollary. $|(\nabla^{x^*g} - \nabla^{g^e})_v w| \lesssim C'_{0,1}h|v|\,|w|$. **6.23**

Proof. It suffices to consider v and w with constant coefficients, so $\nabla^{g^e}_v w = 0$. By definition of ∇^{x^*g} and 1.7, $|\nabla^{x^*g}_v w|_{x^*g} = |\nabla^g_{dx\,v} dx\,w|_g = |\nabla dx(v, w)|$, and so the preceding theorem applies, *q. e. d.*

Corollary. *For $\lambda, \mu \in \Delta$, it holds $\left|d_\lambda x\, v - P d_\mu x\, v\right| \lesssim C'_{0,1}h\,|\lambda - \mu|\,|v|$.* **6.24**

Proof. By the fundamental theorem 1.19a,

$$d_\lambda x\, v = P d_\mu x\, v + \int_\gamma P \nabla_{dx\,\dot\gamma} dx\, v = P d_\mu x\, v + \int P \nabla dx(\dot\gamma, v)$$

for a curve $\gamma : \lambda \rightsquigarrow \mu$, *q. e. d.*

6.25 **Remark.** ⟨**a**⟩ The piecewise flat metric $g^e_{ij} = -\frac{1}{2}\ell^2_{ij}$ is nothing more than the "most natural candidate" for a constant Riemannian metric. Any other g^e that is a second-order approximation of $g|_a$ for some $a \in s$ would give the same result. The particulary interesting observation is that g can be approximated up to second order by a piecewise *constant* metric, whereas an arbitrary function would require a piecewise linear function for a similar approximation order.

⟨**b**⟩ The convergence result for the connection does *not* mean that if Mg is triangulated over a sequence of finer and finer simplicial complices $r\mathfrak{K}_h$, the connections of $x^*_h g$ and g^e_h would converge. In fact, g^e_h would be always piecewise flat, so the connection would vanish and hence can never approximate the connection of a curved Mg. This global impossibility is consistent with our simplex-wise convergence result because the connection for g^e on two adjacent simplices cannot be compared to each other, as the metric is not continuous across the simplex boundary. The connection ∇^{g^e} can henceforth not be connected to the derivative of g^e globally, but only in those matters which make sense in this situation, e. g. higher derivatives of real-valued functions as in 7.2b.

⟨**c**⟩ The convergence of Lipschitz–Killing curvatures from Cheeger *et al.* (1984) applies for our situation, although their triangulation is defined in a slighly different way, see 8.9b. It is a convergence in measure of rate $h^{1/2}$. For submanifolds of Euclidean space, Cohen-Steiner and Morvan (2006) give a convergence order h in measure, but we did not check if their arguments can be carried over to our setting.

⟨**d**⟩ The metric approximation result is similar to, and of the same order as the usual one using orthogonal projection of a triangular surface onto some nearby smooth surface (cf. Dziuk 1988). We will reproduce this conventional approach for the approximation of submanifolds in section 11.

7. Approximation of Functions

Goal. In this section, we want to apply the Karcher mean construction and the results from the previous section to functions between manifolds: First, we consider the case of functions $(\Delta, x^*g) \to \mathbb{R}$, where the preimage is to be approximated by (Δ, g^e). After that, we will consider the case of approximation in the image, which means we will interpolate some function $y : \Delta \to M$ by the Karcher simplex parametrisation x with prescribed values $x(e_i) = y(e_i)$.

Approximation in the Preimage

7.1 **Situation.** Suppose x^*g and g^e are two Riemannian metrics on Δ, that g^e is flat and that 6.17 for the metric as well as 6.23 for the Christoffel operator holds. Vector and operator norms, if not explicitely qualified, are taken with respect to one of these equivalent norms.

7.2 **Proposition.** *Situation as in 7.1. For given smooth $u : \Delta \to \mathbb{R}$,*

$$|\mathrm{grad}^{x^*g}\, u - \mathrm{grad}^{g^e}\, u| \lesssim C'_0 h^2 |\mathrm{grad}^{x^*g}\, u|, \tag{7.2a}$$

$$\|\nabla^{x^*g} du - \nabla^{g^e} du\| \lesssim C'_{0,1} h |du| \tag{7.2b}$$

Remark. It is easier to estimate the *operator* norm $\|\nabla^{x^*g}du - \nabla^{g^e}du\|$, although we will actually need the induced norm $|\nabla^{x^*g}du - \nabla^{g^e}du|$ for bilinear forms or bi-covectors. Recall that the equivalence constant for these two norms only depends on the dimension, which will be neglected as usual, so $\|\cdot\|_g \le |\cdot|_g \lesssim \|\cdot\|_g$ on any tensor bundle over TM.

Proof. ad primum: Represent $du = u_i d\lambda^i$. In the notation of 3.13, we have $\mathrm{grad}^{x^*g}\, u = Q^{ij}u_i\partial_j$ and $\mathrm{grad}^{g^e}\, u = (Q^e)^{ij}u_i\partial_j$. So with $\bar u = (u_1, \dots, u_n)$,

$$\begin{aligned}|\mathrm{grad}^{x^*g}\, u - \mathrm{grad}^{g^e}\, u|_g^2 &= E(Q-Q^e)\bar u \cdot (Q-Q^e)\bar u \\ &\lesssim (C_0'h^2)^2 EQ\bar u \cdot Q\bar u \\ &= (C_0'h^2)^2 Q\bar u \cdot \bar u = (C_0'h^2\,|du|)^2.\end{aligned}$$

ad sec.: By 1.2a, the difference between two connections only depends on their Christoffel symbols. Extend the vectors $v, w \in T_\lambda\Delta$ to vector fields with constant coefficients. As g^e is flat, this gives $\nabla^{g^e}v = 0$ and $\nabla^{g^e}w = 0$. Now by 1.8a,

$$(\nabla^{g^e}du - \nabla^{x^*g}du)(v,w) = du(\nabla_v^{g^e}w) - du(\nabla_v^{x^*g}w) = du((\nabla^{g^e} - \nabla^{x^*g})_v w)$$

and together with 6.23,

$$\begin{aligned}|(\nabla^{x^*g}du - \nabla^{g^e}du)(v,w)| &= |du(\nabla_v^{x^*g}w - \nabla_v^{g^e}w)| \le |du|\,|\Gamma(v,w)| \\ &\le |du| C_{0,1}'h|v|\,|w|,\end{aligned}$$

q. e. d.

Proposition. *Situation as in 7.1. The $\mathsf{W}^{k,p}$-norms, $k = 0,1,2$, with respect to x^*g and g^e are equivalent for every $p \in [1;\infty[$:* **7.3**

$$|u|^p_{\mathsf{L}^p(\Delta x^*g)} = |u|^p_{\mathsf{L}^p(\Delta g^e)}(1 + O(C_0'h^2)), \tag{7.3a}$$

$$|du|^p_{\mathsf{L}^p(\Delta x^*g)} = |du|^p_{\mathsf{L}^p(\Delta g^e)}(1 + O(C_0'c_ph^2)), \tag{7.3b}$$

$$|du|^p_{\mathsf{W}^{1,p}(\Delta x^*g)} = |du|^p_{\mathsf{W}^{1,p}(\Delta g^e)}(1 + O(C_{0,1}'c_ph)), \tag{7.3c}$$

with c_p from 3.15. The same holds, without power p and factor c_p, for the $\mathsf{W}^{k,\infty}$ norms.

Remark. Note that the estimates speak about $|\cdot|^p$ instead of $|\cdot|$. This means that the estimates become worse for $p \to \infty$. Therefore, an additional argument for the case $p = \infty$ is needed.

Proof. Case $k = 0$: The Lebesgue norms on Δx^*g and Δg^e only differ by their volume elements G and G^e, which fulfills the claimed equivalences thanks to 3.20. So

$$\Big|\int_\Delta |u|^pG - \int_\Delta |u|^pG^e\Big| \lesssim C_0'h^2 \int_\Delta |u|^pG.$$

In the L^∞ norm, there is nothing to show, as both norms agree.

Case $k=1$: Here an approximation of the volume element and the gradient norm enter:

$$\begin{aligned}\Big|\int g\langle du,du\rangle^{p/2}G-\int g^e\langle du,du\rangle^{p/2}G^e\Big|&\\ \le\Big|\int g\langle du,du\rangle^{p/2}(G-G^e)\Big|+c_p\Big|\int(g-g^e)\langle du,du\rangle^{p/2}G^e\Big|&\\ \lesssim C_0'c_ph^2\int g\langle du,du\rangle^{p/2}G,&\end{aligned}$$

because $c_p\ge 1$. For the L^∞ norm of du, it suffices to observe that if $|d_{\lambda^*}u|_{g^e}$ is maximal among all $\lambda\in\Delta$, then $|d_{\lambda^*}u|_{g^e}\lesssim(1+O(C_0'h^2))|d_{\lambda^*}u|_g\le(1+O(C_0'h^2))\max_\lambda|d_\lambda u|_g$.

Case $k=2$: We do not have an estimate of our usual form $|x-y|\le\varepsilon|x|$ for the Hessian, but the proof of 3.15 also admits this situation:

$$\begin{aligned}\big||\nabla^g du|_g^p-|\nabla^{g^e}du|_g^p\big|&\le c_p|\nabla^g du|_g^{p-1}\big||\nabla^g du|_g-|\nabla^{g^e}du|_g\big|\\&\le c_p|\nabla^g du|_g^{p-1}|du|_g\|\Gamma\|\\&\le c_p\big(\tfrac{p-1}{p}|\nabla^g du|_g^p+\tfrac{1}{p}|du|_g^p\big)\|\Gamma\|\\&\le c_p(|\nabla^g du|_g^p+|du|_g^p)\|\Gamma\|,\end{aligned}$$

thanks to Young's inequality (Alt 2006, eqn. 1–11). Now one needs approximations of the volume form, the norm on covectors and bi-covectors from 3.19, as well as of the Hessian:

$$\begin{aligned}&\Big|\int|\nabla^g du|_g^pG-\int|\nabla^{g^e}du|_{g^e}^pG^e\Big|\\&\le\int\big||\nabla^g du|_g^p-|\nabla^{g^e}du|_g^p\big|G+\int\big||\nabla^{g^e}du|_g^p-|\nabla^{g^e}du|_{g^e}^p\big|G+\int|\nabla^{g^e}du|_{g^e}^p(G-G^e)\\&\lesssim\int\big||\nabla^g du|_g^p-|\nabla^{g^e}du|_g^p\big|G\qquad+\qquad C_0'c_ph^2\int|\nabla^{g^e}du|_g^pG\\&\lesssim(C_0'c_ph^2+C_{0,1}'c_ph)\int|\nabla^g du|^pG+C_{0,1}'c_ph\int|du|_g^pG,\end{aligned}$$

q. e. d.

7.4 **Theorem.** *Situation as in 7.1. For a C^2 function $u:\Delta\to\mathbb{R}$, let $u_h:\Delta\to\mathbb{R}$ be its Lagrange interpolation, that means u_h is linear and $u_h(e_i)=u(e_i)$. Then*

$$|u-u_h|_{\mathsf{L}^\infty(\Delta)}+h|d(u-u_h)|_{\mathsf{L}^\infty(\Delta)}\lesssim h^2\vartheta^{-1}\|\nabla^{g^e}du\|_{\mathsf{L}^\infty(\Delta g^e)}.$$

The right-hand side can be replaced by $h^2\vartheta^{-1}(1+C_{0,1}'h)\|\nabla^{x^*g}du\|_{\mathsf{L}^\infty(\Delta x^*g)}$.

Proof. If we were only interested in this interpolation of real-valued functions, the easiest method of proof would be to use the interpolation estimates in Euclidean space. But when we come to mappings into a second manifold in 7.9, these methods would not be applicable without further work. Therefore we decided to use a more "geometric" approach.

ad primum: Let $\mu \in \Delta$ be an arbitrary point, consider the tangent $e_{ij} = e_j - e_i$ to the geodesic $\gamma_{ij} : e_j \rightsquigarrow e_i$ and

$$r_1 : \lambda \mapsto (d_\lambda u - d_\lambda u_h)e_{ij}.$$

This scalar-valued function has a zero along the geodesic γ_{ij}, because $r_1 \circ \gamma_{ij}$ is the map $t \mapsto (du - du_h)(\dot\gamma_{ij}) = \frac{\mathrm{d}}{\mathrm{d}t}|u - u_h|(\gamma(t))$, and $|u - u_h|$ is zero at both endpoints of γ_{ij}. Let $\nu \in \Delta$ be the position of this extremum.

Now let γ be the geodesic $\nu \rightsquigarrow \mu$ and $\psi(t) := r_1(\gamma(t)) = (d_{\gamma(t)}u - d_{\gamma(t)}u_h)e_{ij}$. Then

$$\dot\psi(t) = g^e\langle \operatorname{grad}^{g^e}(u - u_h), \nabla^{g^e}_{\dot\gamma} e_{ij}\rangle + g^e\langle \nabla^{g^e}_{\dot\gamma} \operatorname{grad}^{g^e}(u - u_h), e_{ij}\rangle.$$

The first summand vanishes because e_{ij} is parallel with respect to g^e, and the second one is $\nabla^{g^e} d(u - u_h)(e_{ij}, \dot\gamma)$ due to 1.7. So

$$|\dot\psi(t)| \le h \|\nabla^{g^e} d(u - u_h)\|_{g^e} \, |\dot\gamma|_{g^e} \qquad \text{for all } t, \tag{7.4a}$$

and because u_h is linear, $\nabla^{g^e} du_h = 0$. Hence $|\psi(t)| \le \int |\dot\psi(s)| \, \mathrm{d}s \le h^2 \, \|\nabla^{g^e} du\|_{\mathsf{L}^\infty(\gamma_{ij} g^e)}$.

If E_k form an orthonormal basis, then $|du|^2 = \sum du(E_k)^2$ Because of 3.6, the E_k have an expression in the e_{ij} with coefficients smaller than $1/\vartheta h$, which gives

$$|du - du_h|_{g^e|_\mu} \lesssim h\vartheta^{-1} \, \|\nabla du\|_{\mathsf{L}^\infty(\Delta, g^e)}.$$

As μ was chosen arbitrarily, this holds for every point in Δ.

ad sec.: Now consider a new arbitrary point $\mu \in \Delta$, the function

$$r_0 : \lambda \mapsto |u(\lambda) - u_h(\lambda)|^2$$

and a geodesic $\gamma : e_i \rightsquigarrow \lambda$ for some vertex e_i of Δ. Then let $\varphi(t) := r_0(\gamma(t))$. As r_0 vanishes at the interpolation points, we have $\varphi(0) = 0$, and everywhere $|\dot\varphi(t)| = |d(u - u_h)\dot\gamma| \le |d(u - u_h)|_{g^e} \, |\dot\gamma|_{g^e} \lesssim h\vartheta^{-1}|\dot\gamma|_{g^e} \, \|\nabla^{g^e} du\|_{\mathsf{L}^\infty(\Delta, g^e)}$ and thus $|\varphi(t)| \le \int |\dot\varphi(s)| \, \mathrm{d}s \lesssim h^2\vartheta^{-1} \, \|\nabla^{g^e} du\|_{\mathsf{L}^\infty(\Delta, g^e)}$.

ad tertium: The last statement is a direct application of 7.3, *q. e. d.*

Corollary. *The same result also applies for the* L^p *norms:* **7.5**

$$|u - u_h|_{\mathsf{L}^p(\Delta)} + h \, |d(u - u_h)|_{\mathsf{L}^p(\Delta)} \lesssim h^2\vartheta^{-1} \, \|\nabla^{g^e} du\|_{\mathsf{L}^p(\Delta g^e)}.$$

The right-hand side can be replaced by $h^2\vartheta^{-1}(1 + C'_{0,1}h) \, \|\nabla^{g^e} du\|_{\mathsf{L}^p(\Delta x^* g)}$.

Proof. Only the estimate 7.4a has to be refined by the "Hölder 1-trick", a common application of Hölder's inequality (Alt 2006, lemma 1.10): Suppose some function $a \in \mathsf{L}^\infty(\Delta)$ is estimated pointwise by $|a(\lambda)| \le \int_{\gamma[\lambda]} b$, where the integration path

$\gamma[\lambda] : e_0 \rightsquigarrow \lambda$ is of size h. Then as in the most "basic" proof (there are others, cf. 2.10b) of the Poincaré inequality (ADAMS 1975, sec. 6.26),

$$
\begin{aligned}
|a|^p_{\mathsf{L}^p(\Delta)} &= \int_\Delta |a|^p \le \int_\Delta \Big(\int_{\gamma[\lambda]} b\,1 \Big)^p \\
&\le \int_\Delta \Big(\int_{\gamma[\lambda]} b^p \Big) \Big(\int 1 \Big)^{p/q} \le \int_\Delta \Big(\int_{\gamma[\lambda]} b^p \Big) h^{p/q}.
\end{aligned}
\tag{7.5a}
$$

Then compute the Δ integral by first integrating over the subsimplex Δ_0 opposite to the vertex e_0 and then over the ray $e_0 \rightsquigarrow \mu \in \Delta_0$. Then $\lambda = t\mu + (1-t)e_0$ for some t between 0 and 1, and each function $c \in \mathsf{L}^\infty(\mathbb{R})$ with $c \ge 0$ fulfills $\int_0^r \int_0^t c(s)\,\mathrm{d}s\,\mathrm{d}t \le r \int_0^r c(s)\,\mathrm{d}s$, we have

$$\int_\Delta \int_{\gamma[\lambda]} b^p \le h \int_\Delta b^p. \tag{7.5b}$$

Then observe $\frac{p}{q} = p-1$, so we have $|a|^p_{\mathsf{L}^p(\Delta)} \le h^p |b|^p_{\mathsf{L}^p(\Delta)}$ for such a function a. As there does not occur any L^∞ term in the final estimate, it remains valid for $a, b \in \mathsf{L}^p(\Delta)$, *q. e. d.*

APPROXIMATION IN THE IMAGE

Remark. For curves in M, there are already interpolation estimates for high-order (quasi-) interpolation methods by WALLNER and DYN (2005) and GROHS (2013).

During the finishing of this thesis, GROHS *et al.* (2013) have given a very elaborate estimate for higher-order "polynomial" interpolation using the Karcher mean construction. We decided to nevertheless publish our proof here, as we hope that our approach gives more geometric intuition, involves simpler constants, and is used in sections 11–13.

7.6 Situation. In the following, we assume that Δ carries a (ϑ, h)-small Euclidean metric g^e (which is not assumed to come from geodesic distances in M). We consider a smooth function $y : \Delta g^e \to Mg$ (and assume that $y(\Delta)$ lies in a convex ball of radius r as in 6.11 with $C_0 r^2 < 1$) and define x to be the barycentric mapping with respect to the vertices $y(e_i)$. We will usually write x and y instead of $x(\lambda)$ and $y(\lambda)$.

7.7 Lemma. *Situation as in 7.6. Let P be the parallel transport $T_yM \to T_xM$. Consider $\mathbf{d}(x,y)$ and $\mathbf{d}^2(x,y)$ as functions $\Delta \to \mathbb{R}$. Then*

$$
\begin{aligned}
d(\mathbf{d}^2(x,y))v &= 2g|_x\langle X_y, (dx - Pdy)v\rangle, \\
d(\mathbf{d}(x,y))v &= g|_x\langle Y_y, (dx - Pdy)v\rangle,
\end{aligned}
$$

with X_p, Y_p as in 1.22.

Proof. From 1.22, we know the gradients of $\mathbf{d}$ and $\mathbf{d}^2$ if only one of the two arguments is varying. Then for $\varphi : M \times M \to \mathbb{R}$, $(p,q) \mapsto \mathbf{d}^2(p,q)$, we have for tangent vectors $V \in T_pM$ and $W \in T_qM$ that

$$d\varphi(V,W) = g|_p\langle V, X_q\rangle + g|_q\langle W, X_p\rangle.$$

Now $\mathbf{d}^2(x, y)$ is the concatenation of the map $\lambda \mapsto (x, y)$, which has derivative $v \mapsto (dx\, v, dy\, v)$ with φ, so

$$d(\mathbf{d}^2(x, y))v = g|_x \langle X_y, dx\, v\rangle + g|_y \langle X_x, dy\, v\rangle$$

As X_y is the starting tangent of the geodesic $x \rightsquigarrow y$ parametrised over $[0; 1]$, we have $PX_x = -X_y$, and P is an isometry, so $g|_y \langle X_x, dy\, v\rangle = -g|_x \langle X_y, P dy\, v\rangle$, *q. e. d.*

Lemma. *Let $c(s, t)$ be a smooth variation of curves $c(s, \cdot\,)$, and let $P_s^{b,a} : T_{c(s,a)}M \to T_{c(s,b)}M$ be the parallel transport along these curves. Then* **7.8**

$$D_s P_s^{b,a} = \int_a^b P_s^{b,t} R(\partial_t, \partial_s) P_s^{t,a}\, \mathrm{d}t$$

(note that the integrand is always a linear map $T_{c(s,a)}M \to T_{c(s,b)}M$, the integration is therefore defined without problems) and hence $\|D_s P_s^{b,a}\| \le C_0 \int_{c(s,\,\cdot\,)} |\partial_s|$.

Proof. Consider a vector field $V(s) \in T_{c(s,a)}M$, and let $V(s, t) := P_s^{t,a} V(s)$. As first step, observe that $D_s(P_s^{b,a} V(s)) = P_s^{b,a}(D_s V(s)) + (D_s P_s^{b,a})V(s)$. This formula seems obvious, but actually requires a little argumentation: It symbolically resembles $\nabla(AV) = (\nabla A)V + A(\nabla V)$ for linear bundle maps A from 1.2, but as P mediates between different tangent spaces for preimage and image, the ∇ operator is not the same on both sides. Instead, consider the function $f : c(s, a) \mapsto c(s, b)$ between the a and the b isoline A and B respectively. The parallel transport from a to b is a mapping $T_x M \to T_{f(x)}M$ and hence an element of $TM|_A \otimes f^*(TM|_B)$. As in 1.6b, the induced connection on this bundle is given by

$$\nabla_{\partial_s}(\omega \otimes f^* V) = (\nabla_{\partial_s}\omega) \otimes f^* V + \omega \otimes f^* \nabla_{df(\partial_s)} V,$$

and indeed $df(\partial_s c(s, a)) = \partial_s c(s, b)$, giving the Leibniz rule for PV. On the other hand, the fundamental theorem of calculus gives

$$D_s(V(s, b)) = P_s^{b,a}(D_s V(s, a)) + \int_a^b P_s^{b,t}(D_t D_s V(s, t))\, \mathrm{d}t.$$

Because $V(s, \cdot\,)$ is parallel, the vector field in the integrand is

$$D_t D_s V(s, t) = D_s D_t V(s, t) + R(\partial_t, \partial_s)V(s, t) = 0 + R(\partial_t, \partial_s) P_s^{t,a} V(s).$$

Because $V(s)$ is independent of t, it can be pulled out of the integral.—The second claims results from

$$\|D_s P_s^{b,a}\| \le \int_a^b \|R\|\, |\partial_t|\, |\partial_s|\, \|P_s^{b,t}\|\, \|P_s^{t,a}\|\, \mathrm{d}t$$

and $\|P_s^{t,t'}\| = 1$ everywhere because parallel transport is isometric, *q. e. d.*

Remark. Generally, it is well-known that the curvature tensor can be characterised as infinitesimal version of holonomy, i. e. the parallel transport along a closed curve (see e. g. PETERSEN 2006, sec 8.6). We found this specific version in RANI (2009, lemma 3.2.2). The estimate can obviously be sharpened by replacing $|\partial_t||\partial_s|$ by $|\partial_t \wedge \partial_s|$, see BUSER and KARCHER (1981, 6.2.1).

7.9 **Lemma.** *Situation as in 7.6. Then if* $\mathbf{d}(x,y) \le \rho$ *everywhere in* Δ, *we have at every vertex* e_i

$$\|d_{e_i}x - d_{e_i}y\|_{\Delta g^e, Mg} \lesssim h\vartheta^{-1}\left(\|\nabla dy\|_{\mathsf{L}^\infty(\Delta g^e, Mg)} + C_{0,1}\rho\|dy\|^2_{\mathsf{L}^\infty(\Delta g^e, Mg)}\right).$$

Proof. First, consider v to be an edge vector $e_j - e_i$, so $c : t \mapsto e_i + tv$ parametrises the ij edge over $[0;1]$. Then choose Fermi coordinates $(t, u^2, \dots, u^m)$ along an arclength-parametrised version of $\gamma := x \circ c$. As γ itself is not parametrised by arclength, it has coordinates $\gamma(t) = (\frac{t}{\alpha}, 0, \dots, 0)$ with $\alpha = \mathbf{d}(p_i, p_j)$. The image of c under y is another curve δ which intersects γ at p_i and p_j, so

$$\delta(0) = \gamma(0), \qquad \delta(1) = \gamma(1).$$

By the intermediate value theorem, each component $\dot\gamma^i - \dot\delta^i$ must have a zero at some $\tau^i \in [0;1]$. As $D_t\dot\gamma = 0$ and $\Gamma^k_{ij} = 0$ along γ, the second derivatives $\gamma^i_{,tt}$ of the components vanish, too. So

$$|(\dot\gamma^i - \dot\delta^i)(0)| \le \int_0^{\tau^i} |\delta^i_{,tt}|\,\mathrm{d}t \le \tau^i\,|\delta^i_{,tt}|_{\mathsf{L}^\infty([0;1])}.$$

By 1.7, $D_t\delta = \nabla dy(v,v)$, and together with $D_t\delta = (\delta^i_{,tt} + \delta^j_{,t}\delta^k_{,t}\Gamma^i_{jk})\partial_i$ from 1.4a, we have

$$|\delta_{,tt}|_g \le |\nabla dy(v,v)|_g + |dy\,v|^2_g \max\|\Gamma\|$$

By 6.12a, we have $|dy\,v| = |\dot\delta|_g \lesssim (1 + C_{0,1}\rho^2)|\dot\delta|_{\ell^2}$, which means that both norms are equivalent for small ρ. Similarly, $\max\|\Gamma\| \lesssim C_{0,1}\rho$. Together with $|v|_{g^e} = \alpha \le h$, we have

$$|(dx - dy)v|_{g|_{p_i}} \lesssim h\left(\|\nabla dy\|_{\mathsf{L}^\infty([0;a]g^e, Mg)} + C_{0,1}\rho\|dy\|^2_{\mathsf{L}^\infty(cg^e, Mg)}\right)|v|_{g^e}.$$

This shows the claimed estimate for edge vectors. And some general v that is not tangent to an edge can be represented as linear combination of edge tangents e_i, and all coefficients v^i are estimated from above by $|v|_g/\vartheta$ up to a constant, *q. e. d.*

7.10 **Remark.** ⟨**a**⟩ For triangles, the fullness parameter ϑ controls the minimum angle at each vertex. This is exactly the parameter that enters in the last argument, so there is a direct geometry meaning of the factor ϑ^{-1}.

⟨**b**⟩ There are also coordinate-free methods to prove 7.9, but we did not find any method that is "so intrinsic" that no curvature term like $C_{0,1}h\rho\|dy\|$ comes in. For example, one could transport $\dot\delta$ and $\dot\gamma$ both to the vertex $p = y(e_i)$ and do all comparisons there. Then the estimate $\delta_{,tt} - \ddot\delta$ is not needed anymore, but some ∇P and the holonomy $P^{\gamma(t),\delta(t)} - P^{\gamma(t),p}P^{p,\delta(t)}$ have to be estimated by 7.8 and 13.4.

⟨**c**⟩ The term in parentheses on the right-hand side of 7.9 is what GROHS *et al.* (2013) estimate by their "smoothness descriptor". Our computation shows that the nonlinear lower-order term $|dy|^2$ only enters with an additional distance factor ρ.

Proposition. *Situation as in 7.6. Then if* $\mathbf{d}(x, y) \leq \rho$ *everywhere in* Δ, **7.11**

$$\|dx - Pdy\|_{\mathsf{L}^\infty(\Delta g^e, Mg)} \lesssim h\vartheta^{-1} \left(\|\nabla dy - P\nabla dx\|_{\mathsf{L}^\infty(cg^e, Mg)} + C_{0,1}\rho \|dy\|^2_{\mathsf{L}^\infty(cg^e, Mg)} \right).$$

Proof. Let us prove the claim at some $p = x(\mu)$. Consider some vector $v \in T\Delta$, and let $V := (dx - Pdy)v$. Along a geodesic $\gamma = x \circ c : p_i \rightsquigarrow p$, which comes from a curve $c : e_i \rightsquigarrow \mu$ in Δ, we have by the fundamental theorem 1.19a

$$V|_p = \tilde{P}^{1,0} V|_{p_i} + \int_0^1 \tilde{P}^{1,t} D_t V|_{\gamma(t)} \, \mathrm{d}t,$$

where $\tilde{P}$ is the parallel transport along γ (not to be confused with the parallel transport P along geodesics $y \rightsquigarrow x$). Inside the integral, we have $D_t V = \nabla_{dx\,\dot{c}} V = \nabla_{dx\,\dot{c}} dx\, v - \nabla_{dx\,\dot{c}}(Pdy)v$. As in the proof of 7.8, define a mapping $f : x(\lambda) \mapsto y(\lambda)$. Then $df(dx\, w) = dy\, w$ and hence $\nabla_{dx\,\dot{c}}(Pdy)v = P\nabla_{dy\,\dot{c}} dy\, v + (\nabla_{dx\,\dot{c}} P)(dy\, v)$. Together, this gives

$$\begin{aligned} D_t V &= \nabla_{dx\,\dot{c}} dx\, v - P\nabla_{dy\,\dot{c}} dy\, v - (\nabla_{dx\,\dot{c}} P) dy\, v \\ &= \nabla dx(\dot{c}, v) - P\nabla dy(\dot{c}, v) - (\nabla_{dx\,\dot{c}} P) dy\, v. \end{aligned}$$

By 7.8, we have $|\nabla_{dx\,\dot{c}} P| \leq C_0 \rho \max|\partial_s|$, where ρ is again the maximum distance between x and y, and ∂_s is the vector field defined in the proof above and has values $dx\,\dot{c}$ and $dy\,\dot{c}$ at the endpoints x and y. Thus

$$|D_t V|_g \leq \|\nabla dy - P\nabla dx\| |\dot{c}|_{g^e} |v|_{g^e} + C_0 \rho \max|\partial_s| \, \|dy\| \, |v|_{g^e}$$

Now observe $|\partial_s| \lesssim |dy\,\dot{c}|$, which gives

$$|D_t V|_g \lesssim h\big(\|\nabla dy - P\nabla dx\| + C_0 \rho \|dy\|^2\big) |v|_{g^e} |\dot{c}|_{g^e}.$$

By $|V|_{g|_p} \leq |V|_{g|_{p_i}} + \max|D_t V|$, the claim is proven with help of 7.9 for the initial value at p_i, *q. e. d.*

Proposition. *Situation as in 7.6. If* $C_{0,1}\vartheta^{-1} h \|dy\|^2$ *is small, then* **7.12**

$$|\mathbf{d}(x, y)|_{\mathsf{L}^\infty(\Delta)} \lesssim h^2 \vartheta^{-1} \|\nabla dy - P\nabla dx\|_{\mathsf{L}^\infty(cg^e, Mg)}.$$

Proof. Consider any point $\lambda \in \Delta$ and a geodesic $c : e_i \rightsquigarrow \lambda$. Then, with 7.7,

$$\mathbf{d}(x(\lambda), y(\lambda)) = \int d(\mathbf{d}(x, y))\dot{c} \leq \int |(dx - Pdy)\dot{c}|_g \leq h \|dx - Pdy\|,$$

everywhere, and this norm is estimated by 7.11: There is a constant α such that

$$\frac{1}{1 - \alpha C_{0,1} h \vartheta^{-1} \|dy\|^2} \, \mathbf{d}(x(\lambda), y(\lambda)) \leq h\vartheta^{-1} \|\nabla dy - P\nabla dx\|,$$

and the assumption means that the fraction is greater than, say, $\frac{1}{2}$, *q. e. d.*

7.13 **Remark.** ⟨**a**⟩ The smallness assumption on $C_{0,1}\vartheta^{-1}h\|dy\|^2$ is reasonable because the interesting situation is when the domain is decomposed into finer and finer simplicial complexes. In this case $h \to 0$, whereas (given that the subdivision is performed intelligently) ϑ can be bounded from below independent of h, and $C_{0,1}$ as well as $\|dy\|$ are independent of this refinement (here it is important that $\|dy\|$ is taken with respect to g^e on Δ, not ℓ^2).

⟨**b**⟩ The estimates are scale-aware: When Δ is scaled by $\bar{g}^e = \nu^2 g^e$ and M is scaled by $\bar{g} = \mu^2 g$ like in 1.10b, then both sides of the estimates 7.11 and 7.12 scale similarly, namely like $\frac{\mu}{\nu}$ and like μ respectively: In fact, $\bar{r} = \mu r$, $\bar{h} = \nu h$, $\|R\|_{\bar{g}} = \frac{1}{\mu^2}\|R\|_g$, and derivatives of x and y scale like in 1.10b: $\|dx\|_{\bar{g}^e,\bar{g}} = \frac{\mu}{\nu}\|dx\|_{g^e,g}$, $\|\nabla dx\|_{\bar{g}^e,\bar{g}} = \frac{\mu}{\nu^2}\|\nabla dx\|_{g^e,g}$ and similar for y.

7.14 **Conclusion.** *Taking 7.12 and 7.11 together, we get*

$$|\mathbf{d}(x,y)|_{\mathsf{L}^\infty(\Delta)} + h\,\|dx - Pdy\|_{\mathsf{L}^\infty(\Delta g^e, Mg)} \lesssim h^2\vartheta^{-1}\,\|\nabla dy - P\nabla dx\|_{\mathsf{L}^\infty(\Delta g^e, Mg)}.$$

7.15 **Theorem.** *Let N and M be Riemannian manifolds with curvature bounds C_0 and C_1 as usual, and $y : N \to M$ be a given smooth function. Suppose $p_0, \dots, p_n \in N$ are given points in (ϑ, h)-close position with h so small that their barycentric mapping $\Delta \to s$ is injective, where $s \subset N$ is the Karcher simplex with respect to vertices p_i, and furthermore suppose that the barycentric mapping $\Delta \to M$ with respect to vertices $y(p_i)$ is well-defined. Then if $C'_{0,1}h\|dy\|^2_{\mathsf{L}^\infty}$ is small in comparison to the dimensions, there is a function $y_h : s \to M$ interpolating y at the p_i, with*

$$|\mathbf{d}(y_h,y)|_{\mathsf{L}^\infty(s)} + h\,\|dy_h - Pdy\|_{\mathsf{L}^\infty(s,M)} \lesssim h^2\vartheta^{-1}\,\|\nabla dy_h - P\nabla dy\|_{\mathsf{L}^\infty(s,M)}.$$

Proof. By 6.19, there is a bijective barycentric mapping $x_N : \Delta \to s$ with $e_i \mapsto p_i$. If $x_M : \Delta \to M$ is the barycentric mapping with respect to vertices $y(p_i)$, which have distance less than $h\|dy\|$, set $y_h := x_M \circ x_N^{-1}$. The the estimate is a combination of 7.14 and 7.3, *q. e. d.*

7.16 **Remark.** ⟨**a**⟩ One could have proven the intermediate estimates 7.9 and 7.11 for scaled versions of Δ and M, for example with ℓ^2 instead of g^e or $\operatorname{diam} M \leq 1$. But we did not consider the situation above complicated enough to justify a separate scaling argument. But if one likes, the argument obviously could have been executed for Δ and M having both unit size. Then 7.13b is the equivalent of the usual "transformation from the reference element".

⟨**b**⟩ The step from the L^∞ estimate 7.14 to an L^p estimate works exactly as in 7.5, so we save paper by not repeating all the integrals.

⟨**c**⟩ The estimate could be considered as "incomplete work", as the right-hand side still contains a ∇dx term. We decided not to estimate it by 6.22 to make clear that the right-hand side tends to zero if y is an "almost barycentric" map.

⟨**d**⟩ For a "higher-order" interpolation, Grohs *et al.* (2013) use basis functions $\varphi^i : \Delta \to M$ of higher order, not just $\varphi^i(\lambda) = \lambda^i$ as we did, that fulfill $\varphi^1 + \dots + \varphi^k = 1$ and

$\varphi^i(\mu_j) = \delta^i_j$ for control points $\mu_j \in \Delta$. Then the interpolation with respect to points $p_i = y(\mu_i)$ is the minimiser of $\varphi^i\,\mathbf{d}^2(p_i,\,\cdot\,)$.

If one chooses the φ^i such that there are $k+1$ control points on each edge, as is usually done (e. g. for the quadratic basis functions $\lambda^i(2\lambda^i - 1)$ and $4\lambda^i\lambda^j$, $i \neq j$), then 7.9 and 7.11 can obviously be interated to give

$$\|\nabla^\ell dx - P\nabla^\ell dy\| \lesssim h^{k-\ell}\vartheta^{\ell-k}\|\nabla^k dx - \nabla^k P dy\| + \text{ curvature terms.}$$

The only point of difficulty is to show that $\|\nabla^k dx\|$ is actually bounded by $\|dy\|$ and the geometry, which needs quite some computation, as it involves many derivative norms $\|\nabla^j X_i\|$, but is provable along the same lines.

8. The Karcher–Delaunay Triangulation

Notation. The term "triangulation" is used in various senses in (discrete) geometry, topology, and computational mathematics. We will use it only in the topological meaning as a map $r\mathfrak{K} \to M$ for some simplicial complex $\mathfrak{K}$. (To obtain the correct homology, one usually requires this map to be bijective. We will construct this map and give conditions for its injectivity, but we will also call it a triangulation without these conditions.) The partition of a space M into topological disks will in contrast be referred to as a "tesselation".

Goal. In this section, we want to explore how a simplicial structure can be imposed on a "δ-dense" point set in a manifold. Conversely, if the simplicial structure and the vertex set are given, the resulting triangulation weill be considered in section 11.

Situation. In the following, the usual assumption from 1.5 that M is closed (i. e. complete and without boundary) is essential. Let $V \subset M$ be a set of finitely many, but at least m points in M and $\delta > 0$ be such that each δ-ball in M contains at least one point from V. We say that V is **δ-dense** in M. **8.1**

Definition (Leibon and Letscher 2000)**.** Situation as in 8.1. Let $p \in V$. The **Voronoi cell** of p is the set of points in M that are nearer to p than to any other point in V: **8.2**

$$V_p := \{a \in M\colon\ \mathbf{d}(a,p) \le \mathbf{d}(a,q) \text{ for all } q \in V\}$$

These sets cover M, overlapping only on their boundaries. The cover $\{V_p\colon p \in V\}$ is called the **Voronoi tesselation** of M with vertex set V.

The **bisector** B_{pq} of p and $q \in V$ is the set of points which have equal distance to p and q, but larger distance to all other points in V:

$$B_{pq} := \{a \in M\colon\ \mathbf{d}(a,p) = \mathbf{d}(a,q) \le \mathbf{d}(a,r) \text{ for all } r \in V\}$$

Obviously, $B_{pq} = V_p \cap V_q$. Similarly, the bisector $B_{\mathfrak{s}}$ of a set $\mathfrak{s} \subset V$ is defined as $\bigcap_{p\in\mathfrak{s}} V_p$. The set V is said to be **generic** if each non-empty bisector $B_{\mathfrak{s}}$ is a disk-type submanifold (with boundary) of codimension $|\mathfrak{s}| - 1$ and $B_{\mathfrak{s}}$ is empty for $|\mathfrak{s}| > m+1$.

8.3 **Proposition.** *Situation as in 8.1, $p \in V$. Then V_p has diameter less than 2δ. If $2\delta \leq \operatorname{cvr} M$, then V_p is a topological ball.*

Proof. ad primum: No point in V_p has distance greater than δ from p, so the claim is simply the triangle inequality.

ad sec.: If geodesics starting from p are unique, then V_p is star-shaped, cf. 5.1, *q. e. d.*

8.4 **Remark.** **⟨a⟩** Let $\mathfrak{s} \subset V$ with nonempty $B_\mathfrak{s}$. Locally, $B_\mathfrak{s}$ is a smooth submanifold, and its tangent space in a non-boundary point $a \in B_\mathfrak{s}$ is

$$\{W \in T_a M\colon g\langle W, X_p - X_q\rangle = 0 \text{ for all } p, q \in \mathfrak{s}\},$$

where X_p and X_q are the gradients of squared distances as in 5.2. In fact, a curve γ with image in $B_\mathfrak{s}$ fulfills $\mathbf{d}^2(\gamma(t), p) - \mathbf{d}^2(\gamma(t), q) = 0$ everywhere, which has derivative $g\langle \dot\gamma, X_p - X_q\rangle$.

⟨b⟩ Note that V_p will in general not be convex because V_p and V_q could only be both convex if B_{pq} were totally geodesic. Beem (1975) showed that all bisectors are totally geodesic if and only if M has constant sectional curvature.

⟨c⟩ Generally, the properties of topological spheres in Riemannian manifolds are treated by Karcher (1968): A topological sphere that does not meet its cut locus cuts M in two open sets, some "interior" ball and some "outside". A set $B \subset M$ is convex if and only if each $p \in \partial B$ has a "geodesic support plane", that is a subspace $H_p \subset T_p M$ of codimension 1, such that all starting directions $(\exp_p)^{-1} a$ for points $a \in B$ lie on the same side of H_p.

⟨d⟩ Boissonnat *et al.* (2011) remark that genericity of a point set, which can be achieved in Euclidean space by an arbitrarily small perturbation of a degenerate point set, is not always removable by infinitesimally small changes of V. For this reason, we assume a generic V and disregard the question of sharp conditions that ensure this. The problem is currently treated in detail by Dyer and Wintraecken (Rijksuniversiteit Groningen).

8.5 **Proposition.** *Situation as in 8.1 with generic V and $2\delta < \operatorname{cvr} M$. Then $\mathfrak{K}^\ell := \{\mathfrak{s} \subset V\colon B_\mathfrak{s}$ is non-empty, $\mathfrak{s} \in \mathfrak{K}^{\ell-1}\}$, $\ell = 0, \dots, m$, define a regular simplicial complex without boundary, called the* ***Delaunay complex*** *for M, with vertex set $\mathfrak{K}^0 = V$.*

Proof. The only property for a simplicial complex, that some $\mathfrak{t} \subset \mathfrak{s}$ with cardinality k is contained in the set of k-simplices $\mathfrak{K}^k$, is clear, because $B_\mathfrak{t} \subset B_\mathfrak{s}$ for $\mathfrak{t} \subset \mathfrak{s}$.

It remains to show that $\mathfrak{K}$ is regular and has no boundary. An $(m-1)$-simplex $\mathfrak{t}$ cannot be part of more than two m-simplices, because a non-constraint bisector $\{a \in M\colon\ \mathbf{d}(a,p) = \mathbf{d}(a,q)\}$ divides M into two distinct sets. On the other hand, there cannot be only one m-simplex containing $\mathfrak{t}$, because boundaries of the Voronoi cells can only occur where two cells meet if M has no boundary for itself. And each ℓ-simplex belongs to an $(\ell+1)$-simplex: Let $\mathfrak{t} \in \mathfrak{K}^\ell$. Because $B_\mathfrak{t}$ is a topological disk of dimension $n - \ell$, it must have a boundary, which in turn can only consist of bisectors $B_\mathfrak{s}$ with $\mathfrak{t} \subset \mathfrak{s}$, *q. e. d.*

Remark. Note that the following situations are ruled out by our assumptions: **8.6**

⟨**a**⟩ an m-dimensional sphere with $V = \{p_0, \dots, p_m\}$, because B_V would consist of two points of equidistance, which is not a 0-ball, but a 0-disk. However, the definition of Voronoi regions would be feasible, but its dual would consist of two m-simplices with the same vertices, and our notation does not allow to distinguish between them.

⟨**b**⟩ $m+2$ equidistant points $V = \{p_0, \dots, p_{m+1}\}$ in an m-dimensional manifold, because this V is not generic: In fact, B_V would be the point of equidistance, but this set should be empty, as $|V| > m+1$.

⟨**c**⟩ The counterexample of BOISSONNAT *et al.* (2011, pp. 38sqq.), because the bisector $B_{\{p,u,v,w\}}$ is not empty.

Definition. Situation as in 8.1 with generic V and $2\delta < \operatorname{cvr} M$. Let $\mathfrak{K}$ be the complex from 8.5. For $\mathfrak{c} \in \mathfrak{K}^n$, let $x_\mathfrak{c}$ be the mapping from 5.4. As $x_\mathfrak{c}|_{r\mathfrak{f}}$ only depends on $\mathfrak{f}$ for $\mathfrak{f} \subset \mathfrak{c}$, this piecewise definition gives a well-defined mapping $x : r\mathfrak{K} \to M$, called the **Karcher–Delaunay triangulation** of M with vertex set V. **8.7**

Proposition. *The Karcher–Delaunay triangulation is indeed a triangulation in the usual sense: Situation as in 8.1 with generic V and $2\delta < \operatorname{cvr} M$. If δ is so small that the requirements of 6.19 are met on each Karcher simplex, then x is bijective.* **8.8**

Proof. The map x is surjective because its image is non-empty and has no boundary in M. By 6.19 each $x_\mathfrak{c}$ is injective, and as the Karcher simplices do not overlap except on their boundaries, so is x, *q. e. d.*

Remark. ⟨**a**⟩ If M is not closed, but compact and with boundary, the construction is of course feasible, but will only be bijective if there are also points on the boundary and the boundary is aligned with their Karcher–Delaunay triangulation. **8.9**

⟨**b**⟩ The construction of CHEEGER *et al.* (1984) seems similar, but (of course) does not use our barycentric mapping. It starts with a triangulation $x : r\mathfrak{K} \to M$, considers finer and finer subdivisions $s : r\mathfrak{K}' \to r\mathfrak{K}$ of the complex, and then compares the metric $(s \circ x)^* g$ on $r\mathfrak{K}'$ to the piecewise flat metric induced by edge lengths $\ell_{ij} = \mathbf{d}(x \circ s(ri), x \circ s(rj))$, for edges $ij \in (\mathfrak{K}')^1$ in the subdivided complex.

⟨**c**⟩ We know, however, that BURAGO *et al.* (2013) state that "it is now clear that in dimensions beyond three polyhedral structures are too rigid to serve as discrete models of Riemannian spaces with curvature bounds", but nevertheless there will certainly be rigidity results for spaces of piecewise constant curvature without counterparts in the smooth category, we are not convinced that the references they give support this statement in its full generality.

9. A PIECEWISE CONSTANT INTERPOLATION OF DEC

Goal. As second main construction of this thesis, we will now give an interpretation of the discrete exterior calculus as piecewise constant differential forms, which turns variational problems in the simplicial cohomology (C^k, ∂^*) into problems in a complex

$(\mathsf{P}^{-1}\Omega^k, \underline{d})$. The main question will be the connection between $\underline{d}$ and the usual exterior derivative d on $\mathsf{H}^{1,0}\Omega^k$. The introductional definitions are the basics of simplicial homology as they can be found in any topology textbook, e. g. MUNKRES (1984) or HATCHER (2001).

DISCRETE EXTERIOR CALCULUS (DEC)

9.1 **Definition.** Let R be a ring with neutrals 0 and 1, and let $\mathfrak{K}_{\mathrm{or}}$ be a regular n-dimensional oriented simplicial complex. For any simplex $\mathfrak{s} \in \mathfrak{K}^k_{\mathrm{or}}$, let $\chi_\mathfrak{s} : \mathfrak{K}^k_{\mathrm{or}} \to R$ be defined by $\chi_\mathfrak{s}(\mathfrak{s}) = 1$ and $\chi_\mathfrak{s}(\mathfrak{s}') = 0$ for any $\mathfrak{s}' \neq \mathfrak{s}$.

Consider the R-module $\tilde{C}_k(\mathfrak{K})$ that is spanned by all $\chi_\mathfrak{s}$, $\mathfrak{s} \in \mathfrak{K}^k_{\mathrm{or}}$. Let $C_k(\mathfrak{K})$, the space of **k-chains over $\mathfrak{K}$** (with coefficients in R), be the quotient of $\tilde{C}_k(\mathfrak{K})$ under the identification of $\chi_{\mathfrak{s}^-}$ and $-\chi_\mathfrak{s}$. Its dual space $C^k(\mathfrak{K})$, the R-module of all homomorphisms $C_k(\mathfrak{K}) \to R$, is called the space of **k-cochains over $\mathfrak{K}$** (with coefficients in R). Let $f^\mathfrak{s}$ be the generators of $C^k(\mathfrak{K})$ dual to $\delta_\mathfrak{s}$, that means $f^\mathfrak{s}(\chi_\mathfrak{s}) = 1$ and $f^\mathfrak{s}(\chi_{\mathfrak{s}'}) = 0$ for $\mathfrak{s} \neq \mathfrak{s}'$. In the following, we will only use $R = \mathbb{R}$.

The **boundary operator** is the linear map $C_k(\mathfrak{K}) \to C_{k-1}(\mathfrak{K})$, defined on the generators by

$$\partial\chi_{[p_0,\dots,p_k]} := (-1)^i\chi_{[p_0,\dots,\hat{p}_i,\dots,p_k]}$$

(as usual, summation over i is intended), where $\hat{p}_i$ means that this vertex is omitted. With respect to the basis $\chi_\mathfrak{s}$, we write ∂ in coefficients:

$$\partial\chi_\mathfrak{s} = \partial_\mathfrak{s}^\mathfrak{t}\chi_\mathfrak{t} \qquad \text{for } \mathfrak{s} \in \mathfrak{K}^k,$$

where summation over $\mathfrak{t} \in \mathfrak{K}^{k-1}$ is intended. For the whole section, we will sum about indices occuring twice in a product, irrespective if they are superscripts oder subscripts. Volume terms like $|\mathfrak{s}|$ or $|U(\mathfrak{s})|$ do not count for this, as $\mathfrak{s}$ is no sub- or superscript in them. The **co-boundary operator** ∂^* is the dual of ∂, i. e. a map $C^{k-1}(\mathfrak{K}) \to C^k(\mathfrak{K})$ uniquely characterised by $\partial^*\alpha(c) = \alpha(\partial c)$ for all $\alpha \in C^{k-1}(\mathfrak{K})$ and all $c \in C_k(\mathfrak{K})$.

9.2 **Remark.** **⟨a⟩** By a direct computation, or by common linear algebra knowledge, one obtains that the matrix representation of ∂^* is the transposed of the matrix representation of ∂. In other words, $(\partial^*)^\mathfrak{t}_\mathfrak{s} = \partial^\mathfrak{t}_\mathfrak{s}$ for $\partial^* f^\mathfrak{t} = (\partial^*)^\mathfrak{t}_\mathfrak{s} f^\mathfrak{s}$.

⟨b⟩ It would be very natural to write $\delta_\mathfrak{s}$ instead of $\chi_\mathfrak{s}$, because $\chi_\mathfrak{s}$ actually *is* the Kronecker delta on $\mathfrak{K}^k$. But we will already have some operator δ acting on differential forms, we will define some δ for cochains and some $\underline{\delta}$ on piecewise constant forms. In the whole following section, we will not use the Kronecker symbol.

⟨c⟩ The use of functions $\chi_\mathfrak{s}$ as generators of $C_k(\mathfrak{K})$ is only one possible definition. The other frequently encountered possibility is to speak of "formal linear combinations" of the $\mathfrak{s}$ themselves (e. g. HATCHER 2001 and HIRANI 2003 use this definition). Logically, there is no difference between both definitions, as the only strict way to define "formal linear combinations" is to use the characteristic functions $\chi_\mathfrak{s}$. However, the existence of both approaches introduces an unpleasant notational ambiguity that may disturb a quick reader: Linear maps from simplices to R are chains in our notation, whereas they represent *co*chains in the other. Our notation has the advantage to employ $\mathfrak{s}$ only as sub- and superscript, but not as term, which allows for usual summation convention.

Lemma. *Let $\mathfrak{K}$ consist of one single n-simplex. Then the boundary map $\partial_k : C_k \to C_{k-1}$, which can be written as $\binom{n+1}{k+1} \times \binom{n+1}{k}$-matrix, has rank $\binom{n}{k}$.* **9·3**

Proof. The matrix size just stems from counting the elements in $\mathfrak{K}^k$, which arise from choosing $k+1$ vertices out of $n+1$.

The rank of ∂_k is proven by induction over k, starting with $k = n$. Here the statement is that $\partial_n : C_n \to C_{n-1}$ has rank one, which is true because $\partial_n \neq 0$. For any $k < n$, the rank-nullity theorem gives that the rank of ∂_k can be computed as dimension $\binom{n+1}{k+1}$ of its image space minus the dimension of its kernel, which is the rank of ∂_{k+1} because the k'th homology group of the simplex vanishes. And by assumption, ∂_{k+1} has rank $\binom{n}{k+1}$, which gives rank $\partial_k = \binom{n+1}{k+1} - \binom{n}{k+1} = \binom{n}{k}$, *q. e. d.*

Short introduction to discrete exterior calculus (DEC). The discrete exterior calculus (DESBRUN *et al.* 2005, HIRANI 2003) attempts to build a simple and useable finite-dimensional version of the de Rham cohomology based on an intelligent interpretation of simplical cohomology. It calls ∂^* the **discrete exterior derivative** d, which gives that $d : C^k(\mathfrak{K}) \to C^{k+1}(\mathfrak{K})$ acts as **9·4**

$$df^{\mathfrak{t}} = d^{\mathfrak{t}}_{\mathfrak{s}} f^{\mathfrak{s}} \qquad \text{with } d^{\mathfrak{t}}_{\mathfrak{s}} = \partial^{\mathfrak{t}}_{\mathfrak{s}}. \tag{9.4a}$$

If points $\lambda_{\mathfrak{s}} \in r\mathfrak{s}$ for all simplices $\mathfrak{s}$ and numbers $a_k \in \mathbb{R}$ are given, leading to dual cells $r(*\mathfrak{s})$ as in 4.16a, it defines the scalar product of two discrete k-forms as

$$\langle \alpha_{\mathfrak{s}} f^{\mathfrak{s}}, \beta_{\mathfrak{s}'} f^{\mathfrak{s}'} \rangle_{C^k} := a_k \alpha_{\mathfrak{s}} \beta_{\mathfrak{s}} \tfrac{|*\mathfrak{s}|}{|\mathfrak{s}|}. \tag{9.4b}$$

Remark. The numbers a_k do usually not occur in the definition of the scalar product, but we will see that they must be chosen as $a_k = \binom{n}{k}$ to obtain a correspondence to piecewise constant forms. The points $\lambda_{\mathfrak{s}}$ are classically chosen to be the circumcentres of the $r\mathfrak{s}$.

The coderivative δ is supposed to be dual to d with respect to this scalar product, that means $\langle \alpha, d\beta \rangle_{C^k} = \langle \delta\alpha, \beta \rangle_{C^{k-1}}$ for all $\alpha \in C^k(\mathfrak{K})$ and all $\beta \in C^{k-1}(\mathfrak{K})$. Spelling out both sides for $\alpha = f^{\mathfrak{s}}$ and $\beta = f^{\mathfrak{t}}$ gives

$$d^{\mathfrak{t}}_{\mathfrak{s}} \tfrac{|*\mathfrak{s}|}{|\mathfrak{s}|} a_k = \delta^{\mathfrak{s}}_{\mathfrak{t}} \tfrac{|*\mathfrak{t}|}{|\mathfrak{t}|} a_{k-1}, \qquad \Leftrightarrow \qquad \delta^{\mathfrak{s}}_{\mathfrak{t}} = \frac{a_k}{a_{k-1}} \frac{|*\mathfrak{s}|\,|\mathfrak{t}|}{|*\mathfrak{t}|\,|\mathfrak{s}|} \partial^{\mathfrak{t}}_{\mathfrak{s}}. \tag{9.4c}$$

Other definitions are obvious: A form α is called harmonic if $(\delta d + d\delta)\alpha = 0$ etc.

Piecewise Constant Differential Forms

Situation. Let $\mathfrak{K}_{\mathrm{or}}$ be an oriented regular n-dimensional simplicial complex with a discrete Riemannian metric g, let $\mathfrak{K}$ be the corresponding non-oriented complex, and let $\lambda_{\mathfrak{s}}$ for any simplex $\mathfrak{s}$ define subdivision neighbourhoods $U(\mathfrak{s})$. **9·5**

Definition. Situation as in 9.5. Let $\mathsf{P}^0\Omega^k$ be the space of $\mathsf{L}^\infty\Omega^k$ forms that are constant in $U(\mathfrak{s})$ for each $\mathfrak{s} \in \mathfrak{K}^k$. **9·6**

Any simplex $\mathfrak{s} \in \mathfrak{K}^k$ has a volume k-form $\mathrm{dvol}_{r\mathfrak{s}}$ which can be extended to a constant k-form in whole $U(\mathfrak{s})$. Denoting the extension also as $\mathrm{dvol}_{r\mathfrak{s}}$, let

$$\omega^{\mathfrak{s}} := \begin{cases} \mathrm{dvol}_{r\mathfrak{s}} & \text{in } U(\mathfrak{s}) \\ 0 & \text{elsewhere,} \end{cases} \qquad \text{and} \quad \mathsf{P}^{-1}\Omega^k := \operatorname{span}\{\omega^{\mathfrak{s}} : \mathfrak{s} \in \mathfrak{K}^k\} \subset \mathsf{P}^0\Omega^k.$$

9·7 **Example.** Situation as in 9.5, dimension $n = 2$. Consider two triangles $rijk$ and $rjil$, which together contain the subdivision neighbourhood $U(ij)$. Then ω^{ij} is the flattened unit vector vector field in direction $ri - rj$ in $U(ij)$ and zero elsewhere, ω^i is the characteristic function of $U(i)$, and similarly ω^{ijk} is the volume form of $r\mathfrak{K}$ in $rijk$ and the zero 2-form elsewhere.

9·8 **Observation.** All basis elements have pointwise unit length with respect to the metric induced on the tensor bundles by g, and have distinct support up to null sets, so the L^2 scalar product has diagonal form in the basis $\omega^{\mathfrak{s}}$:

$$\langle \alpha_{\mathfrak{s}}\omega^{\mathfrak{s}}, \beta_{\mathfrak{s}'}\omega^{\mathfrak{s}'} \rangle_{\mathsf{L}^2\Omega^k} = |U(\mathfrak{s})|\, \alpha_{\mathfrak{s}}\beta_{\mathfrak{s}} \tag{9.8a}$$

Definition. Situation as in 9.5. Let $\underline{d} : \mathsf{P}^{-1}\Omega^{k-1} \to \mathsf{P}^{-1}\Omega^k$ be defined by

$$\underline{d}\omega^{\mathfrak{t}} = \underline{d}^{\mathfrak{t}}_{\mathfrak{s}}\omega^{\mathfrak{s}}, \qquad \underline{d}^{\mathfrak{t}}_{\mathfrak{s}} := \frac{|\mathfrak{t}|}{|\mathfrak{s}|}\partial^{\mathfrak{t}}_{\mathfrak{s}}. \tag{9.8b}$$

Let $\underline{\delta} : \mathsf{P}^{-1}\Omega^k \to \mathsf{P}^{-1}\Omega^{k-1}$ be defined by

$$\underline{\delta}\omega^{\mathfrak{s}} = \underline{\delta}^{\mathfrak{s}}_{\mathfrak{t}}\omega^{\mathfrak{t}}, \qquad \underline{\delta}^{\mathfrak{s}}_{\mathfrak{t}} = \frac{|U(\mathfrak{s})|}{|U(\mathfrak{t})|}\underline{d}^{\mathfrak{t}}_{\mathfrak{s}}. \tag{9.8c}$$

9·9 **Proposition.** *Situation as in 9.5. The maps $\underline{d}$ and $\underline{\delta}$ fulfill the*

$$\text{“discrete Stokes' formula”} \qquad \int_{r\mathfrak{s}} \underline{d}\alpha = \int_{\partial r\mathfrak{s}} \alpha, \tag{9.9a}$$

$$\text{“discrete Green's formula”} \quad \langle \underline{d}\alpha, \beta \rangle_{\mathsf{L}^2\Omega^k} = \langle \alpha, \underline{\delta}\beta \rangle_{\mathsf{L}^2\Omega^{k-1}} \tag{9.9b}$$

for all $\mathfrak{s} \in \mathfrak{K}^k_{\mathrm{or}}$, $\alpha \in \mathsf{P}^{-1}\Omega^{k-1}$, and $\beta \in \mathsf{P}^{-1}\Omega^k$. In particular, $\underline{d}^2 = 0$.

Proof. ad primum: For any $\alpha \in \Omega^{k-1}$, we have $\int_{\partial r\mathfrak{s}} \alpha = \partial^{\mathfrak{t}'}_{\mathfrak{s}} \int_{r\mathfrak{t}'} \alpha$. Now let $\alpha = \omega^{\mathfrak{t}}$ for some $\mathfrak{t} \in \mathfrak{K}^{k-1}$. Then we have

$$\int_{\partial r\mathfrak{s}} \omega^{\mathfrak{t}} = \partial^{\mathfrak{t}'}_{\mathfrak{s}} \int_{r\mathfrak{t}'} \omega^{\mathfrak{t}} = \partial^{\mathfrak{t}}_{\mathfrak{s}}\, |\mathfrak{t}|$$

(without summation over $\mathfrak{t}$). On the other hand,

$$\int_{r\mathfrak{s}} \underline{d}\omega^{\mathfrak{t}} = \int_{r\mathfrak{s}} \underline{d}^{\mathfrak{t}}_{\mathfrak{s}'}\omega^{\mathfrak{s}'} = \underline{d}^{\mathfrak{t}}_{\mathfrak{s}}\, |\mathfrak{s}|\,.$$

ad sec.: If one spells out both scalar products with help of 9.8a for $\alpha = \omega^{\mathfrak{t}} \in \mathsf{P}^{-1}\Omega^{k-1}$ and $\beta = \omega^{\mathfrak{s}} \in \mathsf{P}^{-1}\Omega^k$, one gets $\underline{\delta}^{\mathfrak{s}}_{\mathfrak{t}}\, |U(\mathfrak{t})| \overset{!}{=} \underline{d}^{\mathfrak{t}}_{\mathfrak{s}}\, |U(\mathfrak{s})|$ for 9.9b to hold, *q. e. d.*

Remark. The discrete Green's formula 9.9b holds without assumption on the bound- **9.10**
ary values because the weights $|U(\mathfrak{s})|$ and $|U(\mathfrak{t})|$ already incorporate the smaller extent of $\underline{\delta}$. For a correct treatment of boundary conditions in variational problems, one would have to modify 9.8c. We decided to investigate the original DEC setup here.

Proposition. *Situation as in 9.5. The map $i_k : C^k \to \mathsf{P}^{-1}\Omega^k$, $f^{\mathfrak{s}} \mapsto \frac{1}{|\mathfrak{s}|}\omega^{\mathfrak{s}}$ is a cochain* **9.11**
map, i. e. each square in the following diagram commutes:

$$\begin{tikzcd}
C^0 \arrow[r, "d"'] \arrow[d, "i_0"'] & C^1 \arrow[r, "d"'] \arrow[d, "i_1"'] & \cdots \arrow[r, "d"'] & C^n \arrow[d, "i_n"'] \\
\mathsf{P}\Omega^0 \arrow[r, "\underline{d}"] & \mathsf{P}\Omega^1 \arrow[r, "\underline{d}"] & \cdots \arrow[r, "\underline{d}"] & \mathsf{P}\Omega^n
\end{tikzcd}$$

If $a_k = \binom{n}{k}$, it is an isometry for each k and a chain map, i. e. each square in the following diagram commutes:

$$\begin{tikzcd}
C^0 \arrow[d, "i_0"'] & C^1 \arrow[l, "\delta"] \arrow[d, "i_1"'] & \cdots \arrow[l, "\delta"] & C^n \arrow[l, "\delta"] \arrow[d, "i_n"'] \\
\mathsf{P}\Omega^0 & \mathsf{P}\Omega^1 \arrow[l, "\underline{\delta}"'] & \cdots \arrow[l, "\underline{\delta}"'] & \mathsf{P}\Omega^n \arrow[l, "\underline{\delta}"']
\end{tikzcd}$$

Proof. The isometry property is clear by the expressions 9.4b and 9.8a for the scalar product of C^k and $\mathsf{P}^{-1}\Omega^k$ respectively. The properties $\underline{d}i_{k-1} = i_k d$ and $\underline{\delta}i_k = i_{k-1}\delta$ only need to be checked for basis elements, so it suffices to show

$$\underline{d}^{\mathfrak{t}}_{\mathfrak{s}}\tfrac{1}{|\mathfrak{t}|} = d^{\mathfrak{t}}_{\mathfrak{s}}\tfrac{1}{|\mathfrak{s}|}, \qquad \underline{\delta}^{\mathfrak{s}}_{\mathfrak{t}}\tfrac{1}{|\mathfrak{s}|} = \delta^{\mathfrak{s}}_{\mathfrak{t}}\tfrac{1}{|\mathfrak{t}|} \qquad \text{for all } \mathfrak{s} \in \mathfrak{K}^k, \mathfrak{t} \in \mathfrak{K}^{k-1}.$$

The first one is obvious from definitions 9.4a and 9.8b. The second one comes from 9.4b, as

$$\underline{\delta}^{\mathfrak{s}}_{\mathfrak{t}} = \frac{|\mathfrak{t}|\,|U(\mathfrak{s})|}{|\mathfrak{s}|\,|U(\mathfrak{t})|}\partial^{\mathfrak{s}}_{\mathfrak{t}} = \frac{\binom{n}{k}\ \ |{*\mathfrak{s}}|}{\binom{n}{k-1}\,|{*\mathfrak{t}}|}\partial^{\mathfrak{s}}_{\mathfrak{t}} = \frac{a_{k-1}\ \binom{n}{k}\ \ |\mathfrak{s}|}{a_k\ \binom{n}{k-1}\,|\mathfrak{t}|}\delta^{\mathfrak{s}}_{\mathfrak{s}},$$

q. e. d.

Remark. It might seem a little bit queer to use piecewise constant forms for this construction and not the elementary forms introduced by Whitney (1957, sec. IV.27)

$$\tilde{\omega}^{[p_0 \cdots p_k]} = k!\,\lambda^i d\lambda_0 \wedge \cdots \widehat{d\lambda^i} \cdots \wedge d\lambda^k,$$

which would also make i a cochain map. The reason is that we did not succeed to find any relation between the L^2 scalar product of Whitney's elementary forms and the DEC scalar product 9.4b. This means that although there is a worked-out interpolation estimate for the space spanned by $\tilde{\omega}^{\mathfrak{s}}$, $\mathfrak{s} \in \mathfrak{K}^k$, by Dodziuk (1976), it gives no possibility to compare solutions of variational problems that were computed using the DEC scalar product.

Proposition. *Suppose that $r\mathfrak{K}g$ is a piecewise flat, (ϑ, h)-small and absolutely well-* **9.12**
centred realised simplicial complex, that means all circumcentres $\lambda_{\mathfrak{s}}$ have barycentric

coordinates $\lambda^i_{\mathfrak{s}} > \alpha$*, and that the circumradii are bounded by* βh*. Then if* $\bar{g}$ *is a second piecewise flat metric with* $|(g-\bar{g})\langle v,w\rangle| \le ch^2|v||w|$*, it holds for* $c' := \frac{c\beta}{\alpha\vartheta}$*:*

$$|\omega^{\mathfrak{s}}_g - \omega^{\mathfrak{s}}_{\bar{g}}|_{\mathsf{L}^2} \lesssim c'h^2\,|\omega^{\mathfrak{s}}_g|_{\mathsf{L}^2}$$
$$|\underline{d}_g\omega^{\mathfrak{s}}_g - \underline{d}_{\bar{g}}\omega^{\mathfrak{s}}_{\bar{g}}|_{\mathsf{L}^2} \lesssim c'h^2\,|\underline{d}_g\omega^{\mathfrak{s}}_g|_{\mathsf{L}^2}$$
$$|\langle\alpha_{\mathfrak{s}}\omega^{\mathfrak{s}}_g, \beta_{\mathfrak{s}'}\omega^{\mathfrak{s}'}_g\rangle_g - \langle\alpha_{\mathfrak{s}}\omega^{\mathfrak{s}}_{\bar{g}}, \beta_{\mathfrak{s}'}\omega^{\mathfrak{s}'}_{\bar{g}}\rangle_{\bar{g}}| \lesssim c'h^2\langle\alpha_{\mathfrak{s}}\omega^{\mathfrak{s}}_g, \beta_{\mathfrak{s}'}\omega^{\mathfrak{s}'}_g\rangle_g$$

Proof. The difference between $\underline{d}_g$ and $\underline{d}_{\bar{g}}$ is easiest, because it only involves simplex volumes like $|\mathfrak{s}|_g$ and $|\mathfrak{s}|_{\bar{g}}$. These are close to each other by 3.20. The approximation of the scalar product involves comparison between the neighbourhood volumes $|U_g(\mathfrak{s})|_g$ and $|U_{\bar{g}}(\mathfrak{s})|_{\bar{g}}$. These can be estimated if we know how the circumcentres are distorted. By 3.12a, these are controlled by the distortion of the Cayley–Menger matrix inverse M_+^{-1}, and inverses of symmetric matrices are treated by 3.21 (which we apply to M_+ instead of g):

$$|q^i - \bar{q}^i| \lesssim ch^2 r|\mathrm{grad}_g\,\lambda^i|$$

(where r is the circumradius with respect to g) because $4r^2$ and $|v^i|^2 = |\mathrm{grad}_g\,\lambda^i|^2$ are the corresponding diagonal entries of M_+^{-1}. By assumption, this is smaller than $c'h^2|q^i|$, *q. e. d.*

Connection to the BV Derivative

Goal. Recall that piecewise constant functions possess distributional derivatives, which are $(n-1)$-dimensional measures concentrated on the jump sets. Their analogue for differential forms are the currents from geometric measure theory. (In order to avoid "currential derivative" or similar terms, we will speak of BV derivatives.) If our definition of discrete exterior derivatives is meaningful, it should be connected to this sort of derivative. In fact, the BV derivative of a piecewise constant k-form α also fulfills Stokes' theorem if the jump set is transversal to the integration domain. But as their support is $(n-1)$-dimensional, we will see that its scaling behaviour does match the one of full-dimensional $(k+1)$-forms such as $\underline{d}\alpha$.

9.13 **Definition.** The **comass** of a k-covector α is the absolute value of its largest component, equivalently: the maximum over all applications of α to simple unit k-vectors:

$$\|\alpha\|_* = \max \alpha(e_{i_0} \wedge \cdots \wedge e_{i_k}).$$

For completeness, we also define that the **mass** of a k-vector is the norm dual to the comass: $\|v\|_* = \max_{\|\alpha\|_*=1} \alpha(v)$. A differential form $\alpha \in \mathsf{L}^1_{\mathrm{loc}}\Omega^k(M)$ has locally **bounded variation** (is locally of BV) if

$$\sup_{\substack{\beta\in\mathsf{C}^1_0\Omega^{k+1}(U)\\ \|\beta\|_*\le 1}} \langle\alpha,\delta\beta\rangle \quad \text{is finite} \qquad \text{for all } U \subset\subset M, \tag{9.13a}$$

where of course $\mathsf{C}^1_0\Omega^k(U)$ denotes the space of continuously differentiable k-forms on M with compact support inside U. The space of k-forms with locally bounded variation is called $\mathsf{BV}_{\mathrm{loc}}\Omega^k$. The globalisation to the space $\mathsf{BV}\Omega^k$ is as usual.

Fact (cf. EVANS and GARIEPY 1992, thm. 5.1). For each $\alpha \in \mathsf{BV}_{\mathrm{loc}}\Omega^k$, there is a Borel-regular measure μ on M and a μ-integrable $(k+1)$-form $d^{\mathsf{BV}}\alpha$ such that **9.14**

$$\langle\!\langle \alpha, \delta\beta \rangle\!\rangle = \int_M \langle d^{\mathsf{BV}}\alpha, \beta\rangle \, \mathrm{d}\mu \qquad \text{for all } \beta \in \mathsf{C}^1_0\Omega^k. \tag{9.14a}$$

We will mostly write $\langle\!\langle d^{\mathsf{BV}}\alpha, \beta\rangle\!\rangle$ as abbreviation of the right-hand side.

Remark. This formulation of the **BV structure theorem** is the one normally used for functions of bounded variation. For differential forms, one calls the supremum in 9.13a the **mass** of the current (linear form on Ω^k) $\beta \mapsto \langle\!\langle \alpha, \delta\beta\rangle\!\rangle$, and then observes that every current of finite mass is "representable by integration" in the meaning of the theorem (FEDERER 1969, sec. 4.1.7, or MORGAN 2000, sec. 4.3B). To obtain uniqueness of $d^{\mathsf{BV}}\alpha$, one usually requires it to have unit-mass everywhere, and the pointwise scaling then comes from μ. As we are only interested in $d^{\mathsf{BV}}\alpha$ for $\alpha \in \mathsf{P}^{-1}\Omega^k$, it will be more adequate to use a non-unit-length $(k+1)$-form and the volume form of $\partial U(\mathfrak{s})$, $\mathfrak{s} \in \mathfrak{K}^k$, for μ.

For the proof of 9.14, we refer to EVANS and GARIEPY (*loc. cit.*), because it only consists of the observation that $\beta \mapsto \langle\!\langle \alpha, \delta\beta\rangle\!\rangle$ has a norm-preserving continuation to $\mathsf{C}^0_0\Omega^k$, and the application of Riesz' representation theorem.

Proposition. *For the basis elements $\omega^{\mathfrak{t}}$ of $\mathsf{P}^{-1}\Omega^k$, the* BV *derivative is given by $\mu = \mathrm{dvol}_{\partial U(\mathfrak{t})}$ and $d^{\mathsf{BV}}\omega^{\mathfrak{t}} = \nu \wedge \omega^{\mathfrak{t}}$, where ν is the outer normal on $U(\mathfrak{t})$.* **9.15**

Proof. If $\beta \in \mathsf{C}^1_0\Omega^{k+1}$, the product $\langle \omega^{\mathfrak{t}}, \delta\beta\rangle$ is supported only in $U(\mathfrak{t})$, where we can apply the classical Green's formula because the integrand is smooth. So

$$\langle\!\langle d^{\mathsf{BV}}\omega^{\mathfrak{t}}, \beta\rangle\!\rangle \overset{(9.14a)}{=} \langle\!\langle \omega^{\mathfrak{t}}, \delta\beta\rangle\!\rangle = \int_{\partial U(\mathfrak{t})} \omega^{\mathfrak{t}} \wedge *\beta + \langle\!\langle d\omega^{\mathfrak{t}}, \beta\rangle\!\rangle = \int_{\partial U(\mathfrak{t})} \langle \nu \wedge \omega^{\mathfrak{t}}, \beta\rangle \, \mathrm{dvol}_{\partial U(\mathfrak{t})}, \tag{9.15a}$$

the last equality by usual multilinear algebra and $d\omega^{\mathfrak{t}} = 0$ almost everywhere, *q. e. d.*

Proposition. *There is a variant of Stokes' theorem for the* BV *derivative of* P^{-1} *forms: If we define* **9.16**

$$\int_{r\mathfrak{s}} d^{\mathsf{BV}}\omega^{\mathfrak{t}} := \int_{r\mathfrak{s}\cap\partial U(\mathfrak{t})} \omega^{\mathfrak{t}}, \tag{9.16a}$$

then

$$\int_{r\mathfrak{s}} d^{\mathsf{BV}}\alpha = \int_{\partial r\mathfrak{s}} \alpha \qquad \textit{for all } \alpha \in \mathsf{P}^{-1}\Omega^k,\ \mathfrak{s} \in \mathfrak{K}^k. \tag{9.16b}$$

Proof. The homotopy formula is easy for constant forms: If $d\alpha = 0$, then $0 = \int_U d\alpha = \int_{\partial U} \alpha$, hence $\int_A \alpha = \pm \int_B \alpha$ if the integration domains A and B bound a common $(k+1)$-dimensional domain U, the sign depending on the orientation of B. This is the case for

$$\partial(r\mathfrak{s} \cap U(\mathfrak{t})) \quad = \quad \partial r\mathfrak{s} \cap U(\mathfrak{t}) \quad \cup \quad r\mathfrak{s} \cap \partial U(\mathfrak{t}).$$

So the formula is clear for $\alpha = \omega^{\mathfrak{t}}$ by definition of the "domain integral" over $r\mathfrak{s}$, and by linearity, it hence holds for all $\alpha \in \mathsf{P}^{-1}\Omega^k$, *q. e. d.*

Remark. ⟨**a**⟩ The notation 9.16a might seem to obscure the actual integration process over a subdomain, but we would like to emphasise the analogue to $\underline{d}\alpha$, which fulfills the same Stokes formula.

⟨**b**⟩ For a smoothly bounded $(k+1)$-dimensional integration domain U and differential forms $\alpha \in \Omega^k$, $\beta \in \Omega^{k-1}$, it is always true that

$$\int_U \alpha = \int_U \langle \alpha, \mathrm{dvol}_U \rangle \mathrm{dvol}_U, \qquad \int_{\partial U} \beta = \pm \int_{\partial U} \langle \nu \wedge \beta, \mathrm{dvol}_U \rangle \mathrm{dvol}_{\partial U},$$

where the sign is the same as in $\mathrm{dvol}_U = \pm \nu \wedge \mathrm{dvol}_{\partial U}$. Therefore, the notation 9.16a can also be interpreted as

$$\int_{r\mathfrak{s}} d^{\mathsf{BV}} \alpha = \int_{r\mathfrak{s} \cap \partial U(\mathfrak{t})} \langle d^{\mathsf{BV}} \alpha, \mathrm{dvol}_{r\mathfrak{s}} \rangle \mathrm{dvol}_{r\mathfrak{s} \cap \partial U(\mathfrak{t})}$$

with $d^{\mathsf{BV}} \omega^{\mathfrak{t}} = \nu \wedge \omega^{\mathfrak{t}}$ as in 9.15. The notational problem is mainly that the BV derivative is supported on a codimension-1-set, which makes the integral in Green's formula $(n-1)$-dimensional instead of n-dimensional, and the left-hand side integral in Stokes' formula k-dimensional instead of $(k+1)$-dimensional. Unfortunately, we do not know a common notation covering both.

⟨**c**⟩ The formula stays correct (with an appropriate notational adaption) for SBV forms (introduced by DE GIORGI and AMBROSIO 1988, as overview we refer to AMBROSIO *et al.* 2000) which are BV forms whose derivative measure μ consists of parts μ_{ac} and μ_s which are absolutely continuous with respect to the n-dimensional and to the $(n-1)$-dimensional Hausdorff measure in $r\mathfrak{K}$, if μ_s is supported on a set that is transversal to the integration domain U. For the proof, one can use the approximation of $\alpha \in \mathsf{SBV}\Omega^k$ by convolution with smooth Gaussian kernels. If the jump set of α, i. e. the support of μ_s, is transveral to U, then the convergence is uniform almost everywhere on ∂U, and so the integrals $\int_{\partial U} \alpha_i$ of the mollified forms α_i tend to $\int_{\partial U} \alpha$ and give a well-defined interpretation of $\int_U d^{\mathsf{BV}} \alpha$.

⟨**d**⟩ This means that for $\beta \in \mathsf{L}^\infty \Omega^{k+1}$ which is smooth inside each $U(\mathfrak{s})$, $\mathfrak{s} \in \mathfrak{K}^{k+1}$, we have

$$\langle\!\langle \alpha, \delta\beta \rangle\!\rangle = \sum_{\mathfrak{s}} \int_{\partial U(\mathfrak{s})} \alpha \wedge *\beta + \langle\!\langle d^{\mathsf{BV}} \alpha, \beta \rangle\!\rangle \qquad \text{for all } \alpha \in \mathsf{P}^{-1}\Omega^k.$$

9.17 **Proposition.** $\langle\!\langle d^{\mathsf{BV}} \alpha, \beta \rangle\!\rangle = \frac{n}{k+1} \langle\!\langle \underline{d}\alpha, \beta \rangle\!\rangle$ *for all* $\alpha \in \mathsf{P}^{-1}\Omega^k$ *and all* $\beta \in \mathsf{P}^0\Omega^{k+1}$.

Proof. Due to 9.15a, it suffices to consider, for each $\mathfrak{s} \in \mathfrak{K}^{k+1}$ and each $\mathfrak{s} \in \mathfrak{K}^k$,

$$\langle\!\langle d^{\mathsf{BV}} \omega^{\mathfrak{t}}, \beta \rangle\!\rangle_{U(\mathfrak{s})} = \int_{U(\mathfrak{s}) \cap \partial U(\mathfrak{t})} \omega^{\mathfrak{t}} \wedge *\beta$$

which can be spelled out by using the $(n-1)$-flags $\mathfrak{a}$ in $U(\mathfrak{s}) \cap \partial U(\mathfrak{t})$:

$$\int_{U(\mathfrak{s}) \cap \partial U(\mathfrak{t})} \omega^{\mathfrak{t}} \wedge *\beta = \sum_{\mathfrak{a}} \int_{\Delta^{n-1}} \omega^{\mathfrak{t}} \wedge *\beta = \frac{1}{(n-1)!} \sum_{\mathfrak{a}} (\omega^{\mathfrak{t}} \wedge *\beta)(b_{\mathfrak{a}}),$$

where $b_{\mathfrak{a}}$ is the pull-back of an orthonormal basis of $r'\mathfrak{a}$. The flags occuring in this sum are of the form $(\langle 0\rangle, \dots, \hat{\mathfrak{t}}, \mathfrak{s}, \dots, \langle n\rangle)$, cf. 4.17c. Using the vectors $v_{\langle i\rangle,\langle i+1\rangle}$ from 4.18, we have inside each $r'\mathfrak{a}$

$$\begin{aligned} b_{\mathfrak{a}} &= v_{\langle 0\rangle,\langle 1\rangle} \wedge \dots \wedge \qquad v_{\langle k-1\rangle,\mathfrak{s}} \qquad \wedge v_{\mathfrak{s},\langle k+2\rangle} \wedge \dots \wedge v_{\langle n-1\rangle,\langle n\rangle} \\ &= v_{\langle 0\rangle,\langle 1\rangle} \wedge \dots \wedge (v_{\langle k-1\rangle,\mathfrak{t}} + v_{\mathfrak{t},\mathfrak{s}}) \wedge v_{\mathfrak{s},\langle k+2\rangle} \wedge \dots \wedge v_{\langle n-1\rangle,\langle n\rangle}, \end{aligned}$$

the factors in the latter product are all mutually perpendicular. Now observe

$$(\omega^{\mathfrak{t}})^\sharp = \frac{v_{\langle 0\rangle,\langle 1\rangle} \wedge \dots \wedge v_{\langle k-1\rangle,\mathfrak{t}}}{|v_{\langle 0\rangle,\langle 1\rangle} \wedge \dots \wedge v_{\langle k-1\rangle,\mathfrak{t}}|}, \qquad (*\omega^{\mathfrak{s}})^\sharp = \frac{v_{\mathfrak{s},\langle k+2\rangle} \wedge \dots \wedge v_{\langle n-1\rangle,\langle n\rangle}}{|v_{\mathfrak{s},\langle k+2\rangle} \wedge \dots \wedge v_{\langle n-1\rangle,\langle n\rangle}|}.$$

By orthogonality of all vectors in $b_{\mathfrak{a}}$, the application $(\omega^{\mathfrak{t}} \wedge *\beta)(b_{\mathfrak{a}})$, usually comprising all permutations of the factors, splits as

$$\begin{aligned} (\omega^{\mathfrak{t}} \wedge *\beta)(b_{\mathfrak{a}}) &= \omega^{\mathfrak{t}}(v_{\langle 0\rangle,\langle 1\rangle} \wedge \dots \wedge v_{\langle k-1\rangle,\mathfrak{t}})\,(*\beta)(v_{\mathfrak{s},\langle k+2\rangle} \wedge \dots \wedge v_{\langle n-1\rangle,\langle n\rangle}) \\ &= |v_{\langle 0\rangle,\langle 1\rangle} \wedge \dots \wedge v_{\langle k-1\rangle,\mathfrak{t}}| \quad \langle \beta, \omega^{\mathfrak{s}}\rangle |v_{\mathfrak{s},\langle k+2\rangle} \wedge \dots \wedge v_{\langle n-1\rangle,\langle n\rangle}|. \end{aligned}$$

Summation over all flags $(\langle 0\rangle, \dots, \hat{\mathfrak{t}}, \mathfrak{s}, \dots, \langle n\rangle)$ then gives, by 4.19,

$$\int\limits_{U(\mathfrak{s})\cap\partial U(\mathfrak{t})} \omega^{\mathfrak{t}} \wedge *\beta = \frac{k!(n-k-1)!}{(n-1)!}\,|\mathfrak{t}|\,|*\mathfrak{s}|\langle \omega^{\mathfrak{s}}, \beta\rangle = \frac{n}{k+1}\,\frac{|\mathfrak{t}|}{|\mathfrak{s}|}|U(\mathfrak{s})|\langle \omega^{\mathfrak{s}}, \beta\rangle,$$

q. e. d.

Approximation Estimate for $\mathsf{P}^{-1}\Omega^k$

Lemma. *Let us denote the set of multiindices $I = (i_1, \dots, i_k)$ with $i_j < i_{j+1}$ and $1 \le i_j \le n$ for all j, by $\binom{n}{k}$ (which in fact is its cardinality). Suppose $\mathfrak{K}$ is a simplicial complex with only one n-simplex $\mathfrak{e}$ with a non-degenerate flat metric. Then the $\binom{n}{k} \times \binom{n+1}{k+1}$ matrix* **9.18**

$$M_{(k)}(\mathfrak{e}) := \Big(\int\limits_{r\mathfrak{e}} \omega^{\mathfrak{t}}(v_I)\Big)_{I\in\binom{n}{k},\mathfrak{t}\in\mathfrak{K}^k}$$

has full rank $\binom{n}{k}$, where $v_{\{i_1,\dots,i_k\}} := \bigwedge v_{i_j}$ for an arbitrary basis v_j of $Tr\mathfrak{e}$.

Proof. The choice of v_j does not matter, because a change of this basis only results in a multiplication with a non-singular $\binom{n}{k} \times \binom{n}{k}$-matrix from the left. Furthermore, it suffices to show that the matrix $\tilde{M}_{(k)} := (\int \mathrm{dvol}_{r\mathfrak{t}}(v_I))_{I,\mathfrak{t}}$ has full rank, because it only differs from $M_{(k)}$ by factors depending on the volumes $|U(\mathfrak{t})|$, which must be non-vanishing for at least $\binom{n}{k}$ of the $\mathfrak{t}$ (which happens if the circumcentre lies on a facet of $\mathfrak{t}$). Now if we choose $v_i = ri - r0$, we can transform the situation onto the unit simplex D with vertices $0, e_1, \dots, e_n$, where then $v_i = e_i$, and the volume forms of simplices $\mathfrak{t}$ containing the vertex 0 have a particularly easy expression:

$$\mathrm{dvol}_{r\mathfrak{t}} = dx^{i_1} \wedge \dots \wedge dx^{i_k} \qquad \text{for } \mathfrak{t} = \{0, i_1, \dots, i_k\}.$$

So $\mathrm{dvol}_{r(\{0\}\cup I)}(v_{I'}) = 1$ if $I = I'$ and 0 else, hence these rows of $\tilde{M}_{(k)}$ are already linearly independent, *q. e. d.*

9.19 **Proposition.** *Suppose $r\mathfrak{K}$ is a simplicial complex with a piecewise flat and (ϑ, h)-small metric, $\alpha \in \mathsf{H}^{1,1}\Omega^k$. Then there are $\alpha^0, \alpha^1 \in \mathsf{P}^{-1}\Omega^k$ and, if $\lambda^i_{\mathfrak{s}} > 0$ for all components of the $\lambda_{\mathfrak{s}}$ defining subdivision neighbourhoods $U(\mathfrak{s})$, there is $\alpha^2 \in \mathsf{P}^{-1}\Omega^k$, such that the L^2 norms of α^0, $\underline{d}\alpha^1$ and $\underline{\delta}\alpha^2$ are estimated by the corresponding norms of α up to a constant, and (with the Poincaré constant $\tilde{C}_{\boxdot}$ from 2.11b)*

$$\begin{aligned} \langle\!\langle \alpha - \alpha^0, \beta \rangle\!\rangle &\lesssim \tilde{C}_{\boxdot} h |\alpha| |\nabla\beta| && \text{for all } \beta \in \mathsf{H}^1\Omega^k, \\ \langle\!\langle d\alpha - \underline{d}\alpha^1, \beta \rangle\!\rangle &\lesssim \tilde{C}_{\boxdot} h |d\alpha| |\nabla\beta| && \text{for all } \beta \in \mathsf{H}^1\Omega^{k+1}, \\ \langle\!\langle \delta\alpha - \underline{\delta}\alpha^2, \beta \rangle\!\rangle &\lesssim \tilde{C}_{\boxdot} h |\delta\alpha| |\nabla\beta| && \text{for all } \beta \in \mathsf{H}^1\Omega^{k-1}. \end{aligned}$$

Proof. Let us introduce a space $\mathsf{P}^{-1}_{elw}\Omega^k$ of "elementwise" P^{-1} forms, spanned by

$$\omega^{\mathfrak{s},\mathfrak{e}} := \begin{cases} \mathrm{dvol}_{r\mathfrak{s}} & \text{in } U(\mathfrak{s}) \cap r\mathfrak{e} \\ 0 & \text{elsewhere} \end{cases} \qquad \text{for } \mathfrak{s} \subset \mathfrak{e}, \mathfrak{e} \in \mathfrak{K}^k.$$

Then we can find $\tilde{\alpha}^0$, $\tilde{\alpha}^1$ and $\tilde{\alpha}^2$ such that, for each $\mathfrak{e} \in \mathfrak{K}^n$,

$$\begin{aligned} \int_{r\mathfrak{e}} \langle \tilde{\alpha}^0, \bar{\beta} \rangle &= \int_{r\mathfrak{e}} \langle \alpha, \bar{\beta} \rangle && \text{for all constant } \bar{\beta} \in \Omega^k, \\ \int_{r\mathfrak{e}} \langle \underline{d}\tilde{\alpha}^1, \bar{\beta} \rangle &= \int_{r\mathfrak{e}} \langle d\alpha, \bar{\beta} \rangle && \text{for all constant } \bar{\beta} \in \Omega^{k+1} \end{aligned} \tag{9.19a}$$

and similarly for $\underline{\delta}\alpha^2$. In fact, writing $\tilde{\alpha}^0 = \tilde{\alpha}^0_{\mathfrak{t},\mathfrak{e}}\omega^{\mathfrak{t},\mathfrak{e}}$ and inserting $\beta = v^\flat_I$ from above with $I \in \binom{n}{k}$, the equations for determining coefficients $\tilde{\alpha}^0_{\mathfrak{t},\mathfrak{e}}$ have $M_{(k)}$ as system matrix, which has full rank by 9.18. The integral $\int \langle \underline{d}\tilde{\alpha}^1, \bar{\beta} \rangle$ reduces to a boundary integral by 9.17 which does not include $\underline{d}$ anymore, this boundary integral is invariant under affine transformations, and the problem is solvable on the unit simplex D. If all $\lambda^i_{\mathfrak{s}}$ are positive, the $\underline{\delta}^{\mathfrak{t}}_{\mathfrak{s}}$ are just a row- and column-rescaling of $\underline{d}^{\mathfrak{s}}_{\mathfrak{t}}$ by non-zero factors.

As a second step, let $\alpha^0 = \alpha^0_{\mathfrak{t}}\omega^{\mathfrak{t}} \in \mathsf{P}^{-1}\Omega^k$ be defined by the condition that, for each $\mathfrak{t} \in \mathfrak{K}^k$ and all $\mathfrak{s} \in \mathfrak{K}^{k+1}$,

$$\begin{aligned} \int_{U(\mathfrak{t})} \langle \alpha^0, \bar{\beta} \rangle &= \int_{U(\mathfrak{t})} \langle \tilde{\alpha}^0, \bar{\beta} \rangle && \text{for all constant } \bar{\beta}' \in \Omega^k, \\ \int_{U(\mathfrak{s})} \langle \underline{d}\alpha^1, \bar{\beta} \rangle &= \int_{U(\mathfrak{s})} \langle \underline{d}\tilde{\alpha}^1, \bar{\beta} \rangle && \text{for all constant } \bar{\beta}' \in \Omega^{k+1} \end{aligned}$$

(and similarly for $\underline{\delta}\alpha^2$), which means averaging $\tilde{\alpha}^0$ over all parts $U(\mathfrak{t}) \cap r\mathfrak{e}$ with $\mathfrak{t} \subset \mathfrak{e}$. These forms have the desired properties: Norm-preservation is clear by construction. For the approximation, observe that $\beta \in \Omega^k$ can be replaced by some $\bar{\beta}$ that is constant in each element $r\mathfrak{e}$, and the error is estimated by the Poincaré inequality 2.10c: $|\beta - \bar{\beta}|_{\mathsf{L}^2} \leq \tilde{C}_{\boxdot} h |\nabla\beta|_{\mathsf{L}^2}$. And similarly it can be replaced by some $\bar{\beta}'$ that is constant in each neighbourhood $U(\mathfrak{t})$. This gives

$$\begin{aligned} \langle\!\langle \alpha - \alpha^0, \beta \rangle\!\rangle &= \langle\!\langle \alpha - \tilde{\alpha}^0, \beta \rangle\!\rangle + \langle\!\langle \tilde{\alpha}^0 - \alpha^0, \beta \rangle\!\rangle \\ &\lesssim \langle\!\langle \alpha - \tilde{\alpha}^0, \bar{\beta} \rangle\!\rangle + \langle\!\langle \tilde{\alpha}^0 - \alpha^0, \bar{\beta}' \rangle\!\rangle + \tilde{C}_{\boxdot} h |\alpha| |\nabla\beta|, \end{aligned}$$

and the scalar products vanish by choice of $\tilde{\alpha}^0$ and α^0. Exactly the same computation is feasible for $\langle d\alpha - \underline{d}\alpha^1, \beta\rangle$ and $\langle \delta\alpha - \underline{\delta}\alpha^2, \beta\rangle$, *q. e. d.*

Proposition. *For a complex consisting of only one n-simplex $\mathfrak{e}$, let $\underline{d}_{(k)}(\mathfrak{e})$ be the $\binom{n+1}{k+2} \times \binom{n+1}{k+1}$ matrix representation $(\underline{d}^{\mathfrak{t}}_{\mathfrak{s}})_{\mathfrak{s}\in\mathfrak{K}^{k+1},\mathfrak{t}\in\mathfrak{K}^k}$ of $\underline{d} : \mathsf{P}^{-1}\Omega^k \to \mathsf{P}^{-1}\Omega^{k+1}$.* **9.20**

Suppose $\mathfrak{K}$ is a simplicial complex with a piecewise flat and (ϑ, h)-small metric. Assume that the $\binom{n+1}{k+1} \times \binom{n+1}{k+1}$-matrix

$$\begin{pmatrix} M_{(k)}(\mathfrak{e}) \\ M_{(k+1)}(\mathfrak{e})\underline{d}_{(k)}(\mathfrak{e}) \end{pmatrix} \quad \text{has full rank for each } \mathfrak{e} \in \mathfrak{K}^n. \tag{9.20a}$$

Let $\tilde{C}_{\boxdot}$ be the Poincaré constant from 2.11b. Then for each $\alpha \in \Omega^k$, there is $\bar{\alpha} \in \mathsf{P}^{-1}\Omega^k$ with $|\bar{\alpha}|_{\mathsf{L}^p} \lesssim |\alpha|_{\mathsf{L}^p}$, $|\underline{d}\bar{\alpha}|_{\mathsf{L}^p} \lesssim |d\alpha|_{\mathsf{L}^p}$ and

$$\begin{aligned} \langle \alpha - \bar{\alpha}, \beta\rangle &\lesssim \tilde{C}_{\boxdot} h |\alpha||\nabla\beta| \qquad \text{for all } \beta \in \mathsf{H}^1\Omega^k \\ \langle d\alpha - \underline{d}\bar{\alpha}, \beta\rangle &\lesssim \tilde{C}_{\boxdot} h |d\alpha||\nabla\beta| \qquad \text{for all } \beta \in \mathsf{H}^1\Omega^{k+1}. \end{aligned} \tag{9.20b}$$

Proof. The assumption 9.20a guarantees that we can find one single $\tilde{\alpha} \in \mathsf{P}^{-1}_{elw}\Omega^k$ fulfilling both equations in 9.19a at the same time, *q. e. d.*

Remark. ⟨**a**⟩ We did not succeed to verify 9.20a in the general case, but there are at least no structural obstructions for it to hold: $M_{(k)}$ has full rank $\binom{n}{k}$, and $\underline{d}_{(k)}$, which is the transposed of ∂_{k+1} from 9.3 with rows scaled by $|\mathfrak{t}|$ and columns scaled by $|\mathfrak{s}|$, has rank $\binom{n}{k+1}$, which add up to $\binom{n+1}{k+1}$. **9.21**

⟨**b**⟩ Using an L^p Poincaré inequality (Evans and Gariepy 1992, thm. 4.5.2) instead of 2.10c leads to $\langle\alpha - \alpha^0, \beta\rangle \le \tilde{C}_{\boxdot,p} h |\alpha|_{\mathsf{L}^p} |\nabla\beta|_{\mathsf{L}^p}$ and similar for $d\alpha - \underline{d}\alpha^1$ and $\delta\alpha - \underline{\delta}\alpha^2$.

⟨**c**⟩ The estimates 9.20b are formulated as H^{-1} norm estimates. By inserting mollified characteristic forms or forms with small support and $|\int\beta| = 1$ ("Dirac forms"), one can localise the convergence.

⟨**d**⟩ We do not call 9.19 and 9.20 "interpolation estimates", as α^0, α^1, α^2 or $\bar{\alpha}$ may have nothing to do with α pointwise, but only in integral mean. In contrast to interpolation, the integral of α and α^0 over smaller spaces like boundaries of the $U(\mathfrak{t})$ will in general not converge, as 9.17 shows. For an example, see also 10.30. The section title "interpolation of DEC" does not refer to interpolation of smooth functions, but to the process of extending the simplicial definitions in 9.4 to L^∞ forms in 9.11.

C. Applications

10. Real-Valued Variational Problems

Situation. Using the results of the preceding sections, we do not speak of a manifold and its triangulation, but directly suppose that $M = r\mathfrak{K}$ is a realised n-dimensional regular simplicial complex (compact, as usual), endowed with a piecewise flat and (ϑ, h)-small Riemannian metric g^e, as well as with a smooth metric g fulfilling 6.17 and 6.23 with $C'_{0,1}h < 1$. Except for 10.18sqq., we assume that if M has a boundary, it follows the boundary of the Karcher simplices. Therefore, the homeomorphism property of x remains unchanged. 10.1

Remark. ⟨**a**⟩ Obviously, the spaces $\mathsf{C}^{k,\alpha}$ of strongly differentiable and Lipschitz functions for g and g^e (defined in the classical meaning for g and by 4.5 for g^e) are different, but as the Sobolev norms for differentiation orders $k = 0, 1, 2$ are equivalent, the spaces $\mathsf{W}^{k,r}(Mg)$ and $\mathsf{W}^{k,r}(Mg^e)$ coincide. 10.2

⟨**b**⟩ The convergence of curve length and geodesic distance, treated in Hildebrandt *et al.* (2006, sec. 4.1) for the case of embedded surfaces, is already covered by 6.13 in our setting.

The Dirichlet Problem for Functions

Goal. In this section, we will deal with approximations of the Dirichlet problem, that is solving a weak version of $\Delta u = f$, where Δ is the Laplace–Beltrami operator of M. The Laplacian of k-differential forms will be dealt in the subsequent section.

As first step, we will give a short review of the usual proof for convergence of Galerkin approximations to the Dirichet problem as can be found for instance in Braess (2007). In the second step, we will add the usual error terms resulting from the "variational crime" to use g^e instead of g. This is standard in the FE theory for geometric PDE's initiated by Dziuk (1988), but often not separated from the error of the first step.

Definition. Situation as in 10.1. Denote by H^1_0 the space of weakly differentiable functions with vanishing trace on ∂M and by P^1 the space of globally continuous, piecewise linear functions (here, "linear" of course means usual linearity in the parameter domain $r\mathfrak{c}$, $\mathfrak{c} \in \mathfrak{K}^n$), by P^1_0 the same but with vanishing boundary values. The norm of an operator on function spaces will be denoted by $\|\cdot\|$. 10.3

Definition. For $v, w \in \mathsf{P}^1$, recall the definition $\mathrm{Lap}(u, v) := \langle du, dv\rangle_{\mathsf{L}^2(Mg)}$ from 2.2 and that the (homogeneous) weak g-Dirichlet problem is the task to find $u \in \mathsf{H}^1_0(M)$ such that $\mathrm{Lap}(u, v) = \langle f, v\rangle_{Mg}$ for all $v \in \mathsf{H}^1_0$, we shortly write $Lu = f$ with an operator $L : \mathsf{H}^1_0 \to (\mathsf{H}^1_0)^*$. The g-**Galerkin solution** to the Dirichlet problem with 10.4

respect to the trial space P^1_0 is the solution u_h to $\mathrm{Lap}(u_h, v) = \langle f, v\rangle_{Mg}$ for all $v \in \mathsf{P}^1_0$. Naturally, there is also the notion of a g^e-Galerkin solution.

10.5 **Remark.** By 2.12, we know that the Dirichlet problem has no solution for general $f \in \mathsf{L}^2$, but only for $f \perp \mathfrak{H}$, and the solution is unique up to harmonic components, in other words: there is a unique solution in $\mathfrak{H}^\perp$. But the space of harmonic functions is one-dimensional, consisting only of the constant functions—and these are ruled out by the boundary value requirements.

10.6 **Fact** (SCHWARZ 1995, also cf. 2.19). The de Rham complex $(\mathsf{H}^{1,0}\Omega, d)$ of a smooth compact Riemannian manifold is a Fredholm complex, so the Dirichlet problem is uniquely solvable, and $|du|_{\mathsf{L}^2} \le C_\boxdot |f|_{\mathsf{L}^2}$ with the Poincaré constant $C_\boxdot$ from 2.10b. This means that L^{-1} is a bounded linear operator.

If ∂M is piecewise smooth or convex (that means, convex where it is not smooth), then M is **H^2-regular**, i.e. there is a constant $C_\boxdot$ depending on M, but not on f, with $|u|_{\mathsf{H}^2} \le C_\boxdot |f|_{\mathsf{L}^2}$, that means that $\|L^{-1}\|_{\mathsf{L}^2,\mathsf{H}^2} \le C_\boxdot$ in this case.

10.7 **Lemma** (**Céa**). *Situation as in 10.1. Let u be the Dirichlet potential and u_h be the g-Galerkin solution to $f \in \mathsf{L}^2$. Then u_h is the orthogonal projection of u onto P^1_0 with respect to* $\mathrm{Lap}(\,\cdot\,,\,\cdot\,)$.

Proof. As $\mathsf{P}^1_0 \subset \mathsf{H}^1_0$, also u fulfills $\mathrm{Lap}(u, v) = \langle f, v\rangle_{\mathsf{L}^2}$ for all $v \in \mathsf{P}^1_0$ by which u_h was defined. So we have the so-called "Galerkin orthogonality" $\mathrm{Lap}(u - u_h, v) = 0$ for all such $v \in \mathsf{P}^1_0$, which is the characterising property of the projection error, *q. e. d.*

10.8 **Corollary.** *Situation as in 10.1. Let Π be the orthogonal projection $\mathsf{H}^1_0 \to \mathsf{P}^1_0$ with respect to* $\mathrm{Lap}(\,\cdot\,,\,\cdot\,)$ *and $\Pi^\perp := \mathrm{id} - \Pi$ be the projection error. Then for any k for which both sides are defined,*

$$|u - u_h|_{\mathsf{H}^k} \le |f|_{\mathsf{L}^2}\, \|\Pi^\perp L^{-1}\|_{\mathsf{L}^2,\mathsf{H}^k}.$$

10.9 **Proposition.** *Situation as in 10.1 with dimension $n \le 3$. Then $\|\Pi^\perp\|_{\mathsf{H}^2,\mathsf{H}^1} \lesssim \vartheta^{-1}h$, and if additionally M is H^2-regular, then*

$$|u - u_h|_{\mathsf{H}^1} \lesssim C_\boxdot h\vartheta^{-1} |f|_{\mathsf{L}^2}.$$

Proof. It suffices to show that there is *one* $u_h \in \mathsf{P}^1$ with $|u - u_h|_{\mathsf{H}^1} \lesssim \vartheta^{-1}h|u|_{\mathsf{H}^2}$, then the projection of u will produce a smaller error than this u_h. As usual, we take u_h to be the Lagrange interpolation of u (which is well-defined, as $\mathsf{H}^2 \subset \mathsf{C}^0$ in dimension ≤ 3, cf. ADAMS 1975, Theorem 5.4.C). And this interpolation estimate is exactly 7.5, *q. e. d.*

10.10 **Proposition** (**Aubin–Nitsche**). *Situation as in 10.1. Then $\|\Pi^\perp L^{-1}\|_{\mathsf{L}^2,\mathsf{L}^2} \le \|\Pi^\perp L^{-1}\|^2_{\mathsf{L}^2,\mathsf{H}^1}$. Under the same conditions as in 10.9,*

$$|u - u_h|_{\mathsf{L}^2} \lesssim C_\boxdot^2 h^2 \vartheta^{-2} |f|_{\mathsf{L}^2}.$$

Proof. First, note that for a right-hand side g, the solution $L^{-1}g$ is characterised by $\langle g, v\rangle_{Mg} = \mathrm{Lap}(L^{-1}g, v)$ for all $v \in \mathsf{H}^1_0$. Now for a right-hand side $f \in \mathsf{L}^2$, consider

$$\begin{aligned}
|\Pi^\perp L^{-1} f|_{\mathsf{L}^2(Mg)} &= \sup_{g \in \mathsf{L}^2} \frac{\langle \Pi^\perp L^{-1} f, g\rangle_{\mathsf{L}^2}}{|g|_{\mathsf{L}^2}} \\
&= \sup \frac{\mathrm{Lap}(\Pi^\perp L^{-1} f, L^{-1} g)}{|g|_{\mathsf{L}^2}} \overset{(*)}{=} \sup \frac{\mathrm{Lap}(\Pi^\perp L^{-1} f, \Pi^\perp L^{-1} g)}{|g|_{\mathsf{L}^2}} \\
&\leq |\Pi^\perp L^{-1} f|_{\mathsf{H}^1} \,\|\Pi^\perp L^{-1}\|_{\mathsf{L}^2,\mathsf{H}^1},
\end{aligned}$$

where we have used in $(*)$ that Π and hence $\Pi^\perp$ is a Lap-orthogonal projection, *q. e. d.*

Remark. It would of course be possible to consider other interpolation procedures than just nodal Lagrange interpolation, for example averaged Taylor polynomials as in Brenner and Scott (2002, section 4.1), which would circumvent the dimension restrictions. However, the emphasis of this thesis lies more on the different possible applications of the Karcher simplex construction than on optimal results for the Dirichlet problem. **10.11**

Lemma. *Situation as in 10.1. Let* $F(v) := \langle v, f\rangle_{M,g}$*, and let* Lap^e *and* F^e *be defined similar to* Lap *and* F*, but with* g^e *instead of* g *everywhere. Then* $|(\mathrm{Lap} - \mathrm{Lap}^e)\,(v,w)| \lesssim C_0' h^2 |dv|_{\mathsf{L}^2} |dw|_{\mathsf{L}^2}$ *and* $|(F - F^e)v| \lesssim C_0' h^2 |v|_{\mathsf{L}^2}$. **10.12**

Proof. Exactly as in 7.3, *q. e. d.*

Remark. In the understanding of Hildebrandt *et al.* (2006), the "weak Laplacian" L_g is a mapping $\mathsf{H}^1 \to (\mathsf{H}^1)^*$, $L_g u : v \mapsto \mathrm{Lap}(u, v)$. In this setting, 10.12 can be seen as a convergence result for the weak Laplacians: $\|L_g - L_{g^e}\|_{\mathsf{H}^1,(\mathsf{H}^1)^*} \lesssim C_0' h^2$.

Proposition. *Situation as in 10.1 with* H^2*-regular* M*. Let* $u_h, u_h^e \in \mathsf{P}^1_0$ *be the Galerkin solutions to* $L_g u = F$ *and* $L_{g^e} u^e = F^e$*. Then* **10.13**

$$|u_h - u_h^e|_{\mathsf{L}^2} + C_{\boxdot} |du_h - du_h^e|_{\mathsf{L}^2} \lesssim C_0' C_{\boxdot}^2 h^2 |f|_{\mathsf{L}^2}.$$

Proof. During this proof, $|\cdot|$ always means $|\cdot|_{\mathsf{L}^2(Mg)}$. Let us first consider the derivative term on the left-hand side: For some v with $|v| = 1$, we have

$$\begin{aligned}
|du_h - du_h^e| &= \mathrm{Lap}(u_h - u_h^e, v) \\
&\leq |\mathrm{Lap}(u_h, v) - \mathrm{Lap}^e(u_h^e, v)| + |\mathrm{Lap}^e(u_h^e, v) - \mathrm{Lap}(u_h^e, v)| \\
&\leq |(F - F^e)v| + |(\mathrm{Lap}^e - \mathrm{Lap})(u_h^e, v)| \\
&\lesssim C_0' h^2 |f|\,|v| + C_0' h^2 |du_h^e|\,|dv|
\end{aligned}$$

Then use $|du_h^e| \leq C_{\boxdot} |f|$ from 10.6. For the estimate of $|u_h - u_h^e|$, use the Poincaré inequality again, *q. e. d.*

Remark. As in the euclidean setting, the proofs carry over to an arbitrary continuous, strongly H^1_0-elliptic bilinear form on H^1 instead of Lap.

Variational Problems in Ω^k

10.14 **Assumption.** Situation as in 10.1. For $k = 0, \dots, n$, let there be finite-dimensional subspaces $\mathsf{P}\Omega^k$ of $\mathsf{H}^{1,0}\Omega^k$ (or $\mathsf{H}^{0,1}\Omega^k$, if needed) with L^2 and $\mathsf{H}^{1,1}$ approximation order h analogous to 7.4:

$$\min_{v_h \in \mathsf{P}\Omega^k} |v - v_h|_{\mathsf{L}^2} + |dv - dv_h|_{\mathsf{L}^2} + \underbrace{|\delta v - \delta v_h|_{\mathsf{L}^2}}_{\text{only for 10.15a}} \leq \alpha h |v|_{\mathsf{H}^2}$$

and similar for $t^*v = t^*v_h = 0$ or $nv = nv_h = 0$. Furthermore, assume that the Dirichlet problem is H^2-regular and the Hodge decomposition $u = da + \delta b + c$ is H^1-regular, which means $|da|_{\mathsf{H}^1} \lesssim |u|_{\mathsf{H}^1}$ etc. We abbreviate $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathsf{L}^2(Mg^e)}$ as $\langle\,\cdot\,,\,\cdot\,\rangle_e$.

10.15 **Proposition.** *Assume 10.14. Let $u = da + \delta b + c$ be the Hodge decomposition of $u \in \mathsf{H}^{1,1}\Omega^k$, which can be computed as $a = \operatorname{argmin} F[u]$ over $a \in \mathsf{H}^{1,1}\Omega_t^{k-1}$ and $b = \operatorname{argmin} G[u]$ over $b \in \mathsf{H}^{1,1}\Omega_n^{k+1}$ as in 2.14. If $a_h = \operatorname{argmin} F[u]$ over $a_h \in \mathsf{P}\Omega_t^{k-1}$ and $b_h = \operatorname{argmin} G[u]$ over $b_h \in \mathsf{P}\Omega_n^{k+1}$, then*

$$|da - da_h|_{\mathsf{L}^2} + |\delta b - \delta b_h|_{\mathsf{L}^2} \leq \alpha h |u|_{\mathsf{H}^1}. \tag{10.15a}$$

If $u = da_e + \delta b_e + c_e$ is the Hodge decomposition with respect to g^e, and if $a_{h,e}$ and $b_{h,e}$ are defined similiarly, then

$$|da - da_e|_{\mathsf{L}^2} + |\delta b - \delta b_e|_{\mathsf{L}^2} + |c - c_e|_{\mathsf{L}^2} \lesssim C_0' h^2 |u|_{\mathsf{L}^2}, \tag{10.15b}$$

$$|da_h - da_{h,e}|_{\mathsf{L}^2} + |\delta b_h - \delta b_{h,e}|_{\mathsf{L}^2} \lesssim C_0' h^2 |u|_{\mathsf{L}^2}. \tag{10.15c}$$

Proof. ad primum: By the Euler–Lagrange equation $\langle da, dv\rangle = \langle u, dv\rangle$ for all $v \in \mathsf{H}^{1,1}\Omega_t^{k+1}$ and $\langle da_h, dv\rangle = \langle u, dv\rangle$ for all $v \in \mathsf{P}\Omega_t^{k+1}$, we know that da_h is the L^2-best approximation of da in $d(\mathsf{P}\Omega_t^k)$, which is smaller than $\alpha h|\nabla da|$ by assumption.

ad sec.: If $\langle da_{h,e}, dv\rangle_e - \langle u, dv\rangle_e = 0$, then $\langle da_{h,e}, dv\rangle - \langle u, dv\rangle \lesssim C_0' h^2(|da_{h,e}| + |u|)|v|$ and hence $\langle da_h - da_{h,e}, dv\rangle \lesssim C_0' h^2(|da_{h,e}| + |u|)|v|$ for all $v \in \mathsf{P}\Omega_t^k$. The same calculation is valid for $da_h - da_{h,e}$ instead of $da - da_e$. The $c - c_e$ estimate comes out as the remainder, *q. e. d.*

10.16 **Remark.** **⟨a⟩** 10.15b is our analogue of thm. 3.4.6 in Wardetzky (2006).

⟨b⟩ In general, there will be no exact finite-dimensional Hodge decomposition in $\mathsf{P}\Omega^k$, as we have not required any connection between $d(\mathsf{P}\Omega^k)$ and $\mathsf{P}\Omega^{k+1}$. There is a Hodge decomposition in the space of Whitney forms with convergence proven by Dodziuk (1976, thm. 4.9). Variational problems in a specific space $\mathsf{P}^{-1}\Omega^k$ of piecewise constant forms will be treated in 10.24sqq.

⟨c⟩ The FEEC setting of Arnold *et al.* only has a weak Hodge decomposition $u = da_h + \tilde{b}_h + c_h$ of $u \in \mathsf{P}\Omega^k$ as in 2.13, but as its parts are also orthogonal projections, there is an estimate

$$|da_h - da_{h,e}|_{\mathsf{L}^2} + |\tilde{b}_h - \tilde{b}_{h,e}|_{\mathsf{L}^2} + |c_h - c_{h,e}|_{\mathsf{L}^2} \lesssim C_0' h^2 |u|_{\mathsf{L}^2}$$

corresponding to 10.15b.

Mixed form of Dirichlet problem. ARNOLD *et al.* (2006, 2010) have shown how to construct finite-dimensional subcomplexes $(\mathsf{P}\Omega, d)$ of $(\mathsf{H}^{1,0}\Omega, d)$ and solve the mixed Dirichlet problem therein. HOLST and STERN (2012) have extended this to the situation where the domain of the Sobolev space and the finite-dimensional approximation are endowed with different, but close Riemannian metrics g and g^e, which leads to the situation that the inclusion map $\mathsf{P}\Omega^k(Mg^e) \to \mathsf{H}^{1,0}\Omega^k(Mg)$ is not norm-preserving anymore, but only an almost-isometric map. Their setting directly applies to Finite Element computations on the Karcher–Delaunay triangulation: **10.17**

Proposition (HOLST and STERN 2012, thm. 3.10). *Assume 10.14, and use the notation from 2.17sq. For $f \in \mathsf{L}^2\Omega^k$, let $(\sigma, u, p) \in \mathsf{P}\mathfrak{S}$ and $(\sigma_e, u_e, p_e) \in \mathsf{P}\mathfrak{S}_e$ be the solution of the mixed formulation 2.18 of the Dirichlet problem in Mg and Mg^e respectively, where $\mathsf{P}\mathfrak{S} = \mathsf{P}\Omega^k_t \times \mathsf{P}\Omega^{k+1}_t \times \mathsf{P}\mathfrak{H}^k_t$ is a stable choice of trial spaces from* ARNOLD et al. *(2006, eqn 7.14), and $\mathsf{P}\mathfrak{S}_e$ differs from $\mathsf{P}\mathfrak{S}$ only by the last factor $(\mathsf{P}\mathfrak{H}^k_e)_t$, the harmonic trial functions with respect to g^e. Then*

$$|\sigma - \sigma_e|_{\mathsf{H}^1} + |u - u_e|_{\mathsf{H}^1} + |p - p_e|_{\mathsf{L}^2} \lesssim \frac{C_0'}{\gamma} h^2 |f|_{\mathsf{L}^2},$$

where γ is the inf-sup constant as in 2.18 (but over $\mathsf{P}\Omega^k$).

Proof. The solution $s = (\sigma, u, p)$ with respect to the "correct" scalar product g fulfills $b(s,t) = F(t)$ for every test triple $t = (\tau, v, q) \in \mathsf{P}\mathfrak{S}$. On the other hand, the distorted solution s_e fulfills $b_e(s_e, t_e) = F_e(t_e)$ for all $t_e \in \mathsf{P}\mathfrak{S}_e$ with the obvious definition of b_e and F_e. As the trial spaces only differ in the last term q, we have

$$b_e(s_e, t_e) = b_e(s_e, t) + \langle u_e, q_e - q\rangle_e = b_e(s_e, t) + \langle u_e, q\rangle_e$$

(because $u_e \perp \mathsf{P}\mathfrak{H}^k_e$). Now observe $\langle u_e, q\rangle = \langle \pi u_e, q\rangle$, where π is the orthogonal projection onto $\mathsf{P}\mathfrak{H}^k$, and by 10.16c the projection of a $\mathsf{P}\mathfrak{H}^k_e$ element onto $\mathsf{P}\mathfrak{H}^k$ is small. Hence

$$\langle u_e, q\rangle_e = \langle u_e, q\rangle + (\langle u_e, q\rangle_e - \langle u_e, q\rangle) \lesssim C_{0,1}' h^2 |u|\,|q|.$$

Weakening the right-hand side, we obtain $|b_e(s_e, t) - F_e(t)| \lesssim C_0' h^2 |s|\,|t|$. By the scalar product comparison 10.12, also $|b(s_e, t) - F(t)| \lesssim C_0' h^2 |s|\,|t|$, and taking this together with $b(s,t) = F(t)$, we have

$$b(s - s_e, t) \lesssim C_0' h^2 |s|\,|t|.$$

Now, by the inf-sup-condition 2.18b, $\gamma |s - s_e| \leq \sup_t b(s - s_e, t)/|t|$, *q. e. d.*

Dirichlet Problems with Curved Boundary

The case that the analytical and the computational domain actually coincide is not the only interesting problem. When for example a Dirichlet problem on the unit disk in hyperbolic space is considered, a Karcher triangulation with respect to the whole hyperbolic space will not exactly cover the unit disk. But the treatment of such a boundary approximation is standard in Finite Element theory, and the main task is to carefully inspect which arguments have to be modified because they rely on the Euclidean structure of the domain. We give a presentation according to Dörfler and Rumpf (1998) and do not treat the difference between g and g^e, as this comparison can be done separately by using 10.13 after 10.21.

The usual setup for boundary approximation is that a domain Ω is replaced by a simplicial domain Ω_h whose boundary vertices lie on $\partial\Omega$. By $(n-1)$-dimensional interpolation estimates, one then gets that $\partial\Omega$ and $\partial\Omega_h$ are only $\lesssim h^2\kappa$ far apart, where κ bounds the curvature of $\partial\Omega$ and h the mesh size of Ω_h. We translate this, for $\Omega \subset M$, into the following

10.18 **Situation.** Let $M = r\mathfrak{K}$ be a piecewise flat and (ϑ, h)-small realised simplicial complex. Let $\Omega \subset M$ be a full-dimensional domain and $\Omega_h = r\bar{\mathfrak{K}}$ a realised full-dimensional subcomplex, connected by a "normal graph map" $\Phi : \partial\Omega \to \partial\Omega_h$, $p \mapsto \exp_p \mathbf{d}\,\nu$, where $\mathbf{d} : \partial\Omega \to \mathbb{R}$ is Lipschitz-continuous and ν is the outer normal on $\partial\Omega$, with the following properties: First, the retraction inverse $(p,t) \mapsto \exp_t t\,\mathbf{d}\,\nu$ is injective (to ensure that no topology change may happen). Seond, it is "short" in the send that $|\mathbf{d}| \le \alpha h^2$, Lip $\mathrm{d} \le \alpha h \le 1$, and $|\nabla d\,\mathbf{d}| \le \alpha$ (where $\mathbf{d}$ is smooth) for some $\alpha \in \mathbb{R}$. Let all principal curvatures of $\partial\Omega$ in M be bounded by κ. This implies that for small h the norms of $d\Phi$ and $\nabla d\Phi$ are bounded, see section 11.

10.19 **Lemma.** *Situation as in 10.18. If $v \in \mathsf{H}^1(\Omega_h)$, then $|v|_{\mathsf{L}^2(\Omega_h\setminus\Omega)} \lesssim \alpha h^2 |dv|_{\mathsf{L}^2(\Omega_h\setminus\Omega)}$ and $|v|_{\mathsf{L}^2(\partial\Omega\cap\Omega_h)} \lesssim \sqrt{\alpha}h|dv|_{\mathsf{L}^2(\Omega_h\setminus\Omega)}$ for small h.*

Proof. ad primum: It suffices to show the claim for smooth v. Consider $\lambda \in \Omega_h \setminus \Omega$. As $\mathbf{d}(\lambda, \partial\Omega_h) \lesssim \alpha h^2$, there is a curve $\gamma[\lambda] : \mu \rightsquigarrow \lambda$ for some $\mu \in \partial\Omega_h$ with length $\lesssim \alpha h^2$. If h is small, this curve can be supposed to be a straight line lying entirely in one simplex of $\bar{\mathfrak{K}}$. As $v(\mu) = 0$,

$$v(\lambda) = \int\limits_{\gamma[\lambda]} dv\,\dot\gamma. \tag{10.19a}$$

Suppose γ is arclength-parametrised. Now we can again apply the arguments from the proof of 7.5 (keeping in mind that $\gamma[\lambda]$ has length h there, but αh^2 here):

$$\int\limits_{\Omega_h\cap\Omega} |v|^2 \overset{(10.19a)}{\le} \int\limits_{\Omega_h\cap\Omega} \Big(\int\limits_{\gamma[\lambda]} |dv|\Big)^2 \overset{(7.5a)}{\lesssim} \alpha h^2 \int\limits_{\Omega_h\cap\Omega}\int\limits_{\gamma[\lambda]} |dv|^2 \overset{(7.5b)}{\lesssim} \alpha^2 h^4 |dv|^2_{\mathsf{L}^2(\Omega_h\setminus\Omega)}$$

ad sec.: Because $\partial\Omega_h$ is a graph over $\partial\Omega$, the inverse is also true: $\partial\Omega$ is a graph (usually not normal) over $\partial\Omega_h$, so we can introduce coordinates in which a simplex $\mathfrak{f}$ of $\partial\Omega_h$ lies in the x_m-plane and $\partial\Omega$ is parametrised by $(x_1, \ldots, x_{m-1}) \mapsto (x_1, \ldots, x_{m+1}, \rho)$. Then

$$\begin{aligned}|v|^2_{\mathsf{L}^2(\partial\Omega\cap r\mathfrak{f})} &= \int\limits_{\mathfrak{t}} |v|^2\sqrt{1+|d\rho|^2} \overset{(11.10b)}{\lesssim} \int\limits_{r\mathfrak{f}} |v|^2 \\ &\overset{(10.19a)}{\lesssim} \int\limits_{r\mathfrak{f}} \Big(\int\limits_{\gamma[\lambda]} |dv|\Big)^2 \overset{(7.5a)}{\lesssim} \alpha h^2 |dv|^2_{\mathsf{L}^2(\Omega_h\setminus\Omega)},\end{aligned}$$

q. e. d.

Lemma. *Situation as in 10.18. For $v : M \to \mathbb{R}$, which is H^2 continuous in Ω and $M \setminus \Omega$, let $[v]$ be the jump of v across $\partial\Omega$. If h is small, there is a continuous extension $\bar{u}$ of $u \in \mathsf{H}^2(\Omega)$ onto $\Omega \cup \Omega_h$ such that $\bar{u}|_\Omega = u$, $|\bar{u}|_{\mathsf{H}^2(\Omega_h \setminus \Omega)} \lesssim |u|_{\mathsf{H}^2(\Omega)}$ and $|[d\bar{u}\,\nu]|_{\mathsf{L}^2(\partial\Omega \cap \Omega_h)} \lesssim |u|_{\mathsf{H}^1(\Omega)}$.* **10.20**

Proof. By assumption, all points in $\Omega_h \setminus \Omega$ are covered by the homotopy

$$\Phi_t : \quad p \mapsto \exp_p t\nu,$$

where at each $p \in \partial\Omega \cap \Omega_h$, the parameter t is chosen within $]0; \mathbf{d}(p)]$ (in particular, points with negative $\mathbf{d}(p)$ are excluded, as they would parametrise $\Omega \setminus \Omega_h$ instead of $\Omega_h \setminus \Omega$). For an image point of Φ_t, set $\bar{u}(\exp_p t\nu) := u(\exp_p -t\nu)$, the reflection along $\partial\Omega$. This $\bar{u}$ is continuous, and $[d\bar{u}\,\nu] = \pm 2du\,\nu$. The H^2 norm-preservation follows from the assumptions on Φ (but note that $\bar{u}$ is not H^2 in $\Omega_h \cup \Omega$ due to the jump on $\partial\Omega$, even though Φ_t is smooth), *q. e. d.*

Proposition. *Situation as in 10.18. Let $u \in \mathsf{H}^2_0(\Omega)$ be the solution of $Lu = f$ with respect to Ω, and let $u_h \in \mathsf{P}^1_0(\Omega_h)$ be the Galerkin solution over Ω_h for an extension of the right-hand side f by zero onto $\Omega_h \setminus \Omega$. Then $|du - du_h|_{\mathsf{L}^2(\Omega)} \lesssim \sqrt{\alpha}h|u|_{\mathsf{H}^2(\Omega)}$ for small h, where u_h has been extended by zero in $\Omega \setminus \Omega_h$.* **10.21**

Proof. Let $\bar{u}$ be the extension of u from 10.20. Assume we can show

$$|\bar{u} - u|_{\mathsf{H}^1(\Omega_h)} \lesssim |\bar{u} - v|_{\mathsf{H}^1(\Omega_h)} + \alpha h^2 |\bar{u}|_{\mathsf{H}^2(\Omega_h \setminus \Omega)} + \sqrt{\alpha}h\,|[d\bar{u}(\nu)]|_{\mathsf{L}^2(\partial\Omega \cap \Omega_h)} \quad (10.21a)$$

for every $v \in \mathsf{P}^1(\Omega_h)$. Then the claim is proven by 7.5 and 10.20. Supposed $v \in \mathsf{P}^1$, observe that in $|dv - du_h| = \sup \langle dv - du_h, dw\rangle / |dw|$, it suffices to take $w \in \mathsf{P}^1$. So we have

$$\begin{aligned} |d\bar{u} - du_h|_{\mathsf{L}^2(\Omega_h)} &\le |d\bar{u} - dv| + |dv - du_h| = |d\bar{u} - dv| + \sup_{w \in \mathsf{P}^1} \frac{\langle dv - du_h, dw\rangle}{|dw|} \\ &= |d\bar{u} - dv| + \sup_{w \in \mathsf{P}^1} \frac{\langle dv - d\bar{u}, dw\rangle + \langle d\bar{u} - du_h, dw\rangle}{|dw|} \\ &\le 2|d\bar{u} - dv| + \sup_{w \in \mathsf{P}^1} \frac{\langle d\bar{u} - du_h, dw\rangle}{|dw|}. \end{aligned}$$

And now, if $\bar{f}$ is the extension of f by $\bar{f} = 0$ in $\Omega_h \setminus \Omega$,

$$\begin{aligned} \langle d\bar{u} - du_h, dw\rangle_{\mathsf{L}^2(\Omega_h)} &= \int_{\Omega_h \cap \Omega} \langle d\bar{u}, dw\rangle - fw + \int_{\Omega_h \setminus \Omega} \langle d\bar{u}, dw\rangle - \bar{f}w \\ &= \int_{\Omega_h \cup \Omega} \underbrace{(-\Delta u - f)}_{=0} w + \int_{\partial\Omega_h \cap \Omega} w\,du\,\nu + \int_{\Omega_h \setminus \Omega} -w\Delta u + \int_{\partial(\Omega_h \setminus \Omega)} w\,d\bar{u}(-\nu), \end{aligned}$$

as $-\nu$ is the outer normal of $\Omega_h \setminus \Omega$. So this gives

$$\langle d\bar{u} - du_h, dw\rangle_{\mathsf{L}^2(\Omega_h)} \le |\Delta\bar{u}|_{\mathsf{L}^2(\Omega_h \setminus \Omega)} |w|_{\mathsf{L}^2(\Omega_h \setminus \Omega)} + |[du(\nu)]|_{\mathsf{L}^2(\partial\Omega \cap \Omega_h)} |w|_{\mathsf{L}^2(\partial\Omega \cap \Omega_h)},$$

which shows, together with 10.19 for the w norms, the claimed estimate 10.21a, *q. e. d.*

Heat Flow

Goal. As a short outlook on Galerkin methods for parabolic problems, we consider the approximation of heat flow under perturbations of metric. We decided to exclude the general convergence theory (see e. g. THOMÉE 2006, chap. 1) and concentrate on the difference between Galerkin approximations with respect to g and g^e.

10.22 **Proposition.** *Situation as in 10.1. For a time interval* $[0;a]$*, let* $u_h, u_{h,e}$ *be the time-continuous Galerkin approximation to the heat flow with initial value* $u_0 \in \mathsf{P}^1_0$ *and right-hand side* $f \in \mathsf{L}^\infty([0;a], \mathsf{L}^2(Mg))$ *for metrics* g *and* g^e *respectively, that means*

$$\langle \dot u_h, v\rangle + \langle du_h, dv\rangle = \langle f, v\rangle \qquad \text{for all } v \in \mathsf{P}^1, \qquad u_h|_{t=0} = u_0,$$
$$\langle \dot u_{h,e}, v\rangle_e + \langle du_{h,e}, dv\rangle_e = \langle f, v\rangle_e \qquad \text{for all } v \in \mathsf{P}^1, \qquad u_{h,e}|_{t=0} = u_0,$$

where $\langle\,\cdot\,,\,\cdot\,\rangle_e$ *is the abbreviation for* $\langle\,\cdot\,,\,\cdot\,\rangle_{Mg^e}$*. Then their difference can be estimated by*

$$|u_h - u_{h,e}|_{\mathsf{L}^\infty(\mathsf{L}^2)} \lesssim C_0' C_\boxdot h^2 |u_0|_{\mathsf{H}^1} + C_0' C_\boxdot h^2 \Big(\int |f(t)|^2_{\mathsf{L}^2} \Big)^{1/2}.$$

Proof. The proof follows the line of the usual convergence proof for parabolic problems as in THOMÉE (2006, thm. 1.2): Consider $\varepsilon := u_h - u_{h,e}$. By the defining equations for u_h and $u_{h,e}$, we have

$$\langle \dot\varepsilon, v\rangle + \langle d\varepsilon, dv\rangle = \langle f, v\rangle - \langle \dot u_{h,e}, v\rangle - \langle du_{h,e}, dv\rangle.$$

By 7.3 and 10.12, we have

$$|\langle \dot u_{h,e}, v\rangle - \langle du_{h,e}, dv\rangle - \langle f, v\rangle| \lesssim C_{0,1}' h^2 \big(|\dot u_{h,e}|\,|v| + |du_{h,e}|\,|dv| + |f|\,|v|\big),$$

where all norms are L^2 norms. So we have for $v = \varepsilon$, together with the Poincaré constant $C_\boxdot$ from 2.10c,

$$\tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}|\varepsilon|^2 + |d\varepsilon|^2 \lesssim C_0' C_\boxdot h^2 \big(|\dot u_{h,e}| + |du_{h,e}| + |f|\big)|d\varepsilon|.$$

Then Young's inequality gives $2cab \le c^2a^2 + b^2$, hence we obtain a separated summand $|d\varepsilon|^2$ on the right-hand side, which can be cancelled (the suppressed constant belongs to c):

$$\tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}|\varepsilon|^2 \lesssim (C_0' C_\boxdot h^2)^2 \big(|\dot u_{h,e}|^2 + |du_{h,e}|^2 + |f|^2\big)$$

Integration over $[0;a]$ gives, as $\varepsilon|_{t=0} = 0$,

$$|\varepsilon|^2 \le (C_0' C_\boxdot h^2)^2 \int |\dot u_{h,e}|^2 + |du_{h,e}|^2 + |f|^2.$$

From the usual regularity theory for parabolic problems (THOMÉE 2006, eqn. 1.20, case $m = 0$), we know that $\int(|\dot u|^2 + |u|^2_{\mathsf{H}^1}) \lesssim |u_0|_{\mathsf{H}^1} + \int |f|^2$, which shows the desired estimate for ε, *q. e. d.*

Proposition. *Situation as above. Let $u_h^n, u_{h,e}^n$ be the Galerkin approximation to the heat flow with implicit Euler time discretisation with respect to g and g^e respectively, that means* 10.23

$$\langle \bar\partial u_h^n, v\rangle + \langle du_h^n, dv\rangle = \langle f, v\rangle \quad \text{for all } v \in \mathsf{P}_0^1, \qquad u_h^0 = u_0,$$
$$\langle \bar\partial u_{h,e}^n, v\rangle_e + \langle du_{h,e}^n, dv\rangle_e = \langle f, v\rangle_e \quad \text{for all } v \in \mathsf{P}_0^1, \qquad u_{h,e}^0 = u_0$$

for the backward difference quotient $\bar\partial v^n := \frac{1}{\tau}(v^n - v^{n-1})$. Then their difference at time $t = n\tau$ can be estimated by $|du_h - du_{h,e}^n|_{\mathsf{L}^2} \lesssim Kh^2 t$, where K depends on the geometry, $|f|_{\mathsf{L}^\infty(\mathsf{L}^2)}$ and $|u_0|_{\mathsf{H}^1}$.

Proof. As before, let $\varepsilon^n := u_h^n - u_{h,e}^n$. Then

$$\langle \bar\partial \varepsilon^n, v\rangle + \langle d\varepsilon, dv\rangle = \langle f, v\rangle - \langle \bar\partial u_{h,e}^n, v\rangle - \langle du_{h,e}, dv\rangle,$$

and the right-hand side is bounded by

$$\begin{aligned}|\langle \bar\partial u_{h,e}^n, v\rangle - \langle \bar\partial u_{h,e}^n, v\rangle_e| + |\langle du_{h,e}, dv\rangle - \langle du_{h,e}, dv\rangle_e| + |\langle f, v\rangle - \langle f, v\rangle_e|&\\ \lesssim C_0' h^2 \big(|\bar\partial u_{h,e}^n|\,|v| + |du_{h,e}|\,|dv| + |f|\,|v|\big)&\\ \lesssim C_0' C_{\boxdot} h^2 \big(|\bar\partial u_{h,e}^n| + |du_{h,e}| + |f|\big)\,|dv|.&\end{aligned}$$

Denote the whole term in parentheses as Λ. As before, it is bounded in terms of the given data. Then again the choice $v = \varepsilon^n$ gives

$$|\varepsilon^n|^2 - \langle \varepsilon^{n-1}, \varepsilon^n\rangle + |d\varepsilon^n|^2 \lesssim C_0' C_{\boxdot} h^2 \tau \Lambda |d\varepsilon^n|^2$$

and so

$$|\varepsilon^n|^2 + |d\varepsilon^n|^2 \lesssim C_0' C_{\boxdot} h^2 \tau \Lambda |d\varepsilon^n|^2 + C_{\boxdot}^2 |d\varepsilon^{n-1}|\,|d\varepsilon^n|.$$

And of course $|d\varepsilon^n|^2$ is smaller than the last left-hand side, which gives $|d\varepsilon^n| \lesssim C_{0,1}' C_{\boxdot} h^2 \tau \Lambda + C_{\boxdot}^2 |d\varepsilon^{n-1}|$. Then the claim follows by induction over n, *q. e. d.*

Discrete Exterior Calculus

Observation. As we have noticed in 2.19, all variational problems from section 2 are uniquely solvable in $(\mathsf{P}^{-1}\Omega^k, \underline{d})$ like in (Ω^k, d) by the construction of $\mathsf{P}^{-1}\Omega^k$ as a (co-)chain complex. As $(\mathsf{P}^{-1}\Omega^k, \underline{d})$ just a gentle way of writing the simplicial cochain complex (C^k, ∂^*), its (co-)homology is isomorphic to the de Rham complex' one (a short direct proof, called "the theorem of de Rham", is given in Whitney 1957, sec. iv.29, although de Rham 1931 proved isomorphy to singular, not simplicial cohomology). Therefore, we can hope for approximating smooth solutions of variational problems by ones in $\mathsf{P}^{-1}\Omega^k$. 10.24

Situation. Let $r\mathfrak{K}$ be a realised oriented regular n-dimensional simplicial complex without boundary with a piecewise flat, (ϑ, h)-small metric g. Let $\lambda_{\mathfrak{s}}$, $\mathfrak{s} \in \mathfrak{K}^*$, be the simplices' circumcentres, and suppose $\lambda_{\mathfrak{s}}^i > 0$ for all their components (i. e. $r\mathfrak{K}g$ is well-centred). Assume 9.20a and that the Hodge decomposition $u = da + \delta b + c$ is H^1-regular, meaning $|da|_{\mathsf{H}^1} \lesssim |u|_{\mathsf{H}^1}$ etc. We use the Poincaré inequality in the form 2.11b. 10.25

10.26 **Proposition.** *Situation as in 10.25. For a function $f \in \mathsf{H}^1$, let $u \in \mathsf{H}^2$ be the solution of the Poisson problem $\langle du, dv \rangle = \langle f, v \rangle$ for all $v \in \mathsf{H}^1$, and let $u_h \in \mathsf{P}^{-1}\Omega^0$ be the solution of $\langle du_h, dv_h \rangle = \langle f, v_h \rangle$ for all $v_h \in \mathsf{P}^{-1}\Omega^0$. Then*

$$\langle du - \underline{d}u_h, \underline{d}v_h \rangle \lesssim \tilde{C}_{\boxdot} h(|\nabla f||v_h| + |\nabla du||\underline{d}v_h|) \qquad \text{for all } v_h \in \mathsf{P}^{-1}\Omega^k.$$

Proof. Let v and v_h be connected by 9.20b. Then

$$\langle du - \underline{d}u_h, \underline{d}v_h \rangle = \langle du, dv \rangle - \langle \underline{d}u_h, \underline{d}v_h \rangle + \langle du, \underline{d}v_h - dv \rangle = \langle f, v - v_h \rangle + \langle du, \underline{d}v_h - dv \rangle,$$

and both terms can be estimated as claimed, *q. e. d.*

10.27 **Proposition.** *Situation as in 10.25. Let $u = da + \delta b + c$ be the Hodge decomposition of $u \in \mathsf{H}^{1,1}\Omega^k$, and let $\bar{u} = \underline{d}a_h + \underline{\delta}b_h + c_h$ be the Hodge decomposition of its L^2-orthogonal projection onto $\mathsf{P}^{-1}\Omega^k$. Then*

$$\begin{aligned} \langle da - \underline{d}a_h, \underline{d}v_h \rangle &\lesssim \tilde{C}_{\boxdot} h |u|_{\mathsf{H}^1} |\underline{d}v_h|_{\mathsf{L}^2} \\ \langle \delta b - \underline{\delta}b_h, \underline{\delta}v_h \rangle &\lesssim \tilde{C}_{\boxdot} h |u|_{\mathsf{H}^1} |\underline{\delta}v_h|_{\mathsf{L}^2} \end{aligned} \qquad \text{for all } v_h \in \mathsf{P}^{-1}\Omega^k.$$

Proof. We know that da is characterised by $\langle da, dv \rangle = \langle u, dv \rangle$ for all $v \in \mathsf{H}^{1,0}\Omega^k$. Naturally, a_h is characterised by $\langle \underline{d}a_h, \underline{d}v_h \rangle = \langle \bar{u}, \underline{d}v_h \rangle$ for all $v_h \in \mathsf{P}^{-1}\Omega^k$, but the right-hand side is $\langle u, \underline{d}v_h \rangle$ if $\underline{d}u$ is the orthogonal projection onto P^{-1}. So we can proceed exactly like before, but using 9.19 to connect only dv and $\underline{d}v_h$ instead of v and v_h:

$$\langle da - \underline{d}a_h, \underline{d}v_h \rangle = \langle da, dv \rangle - \langle \underline{d}a_h, \underline{d}v_h \rangle + \langle da, \underline{d}v_h - dv \rangle = \langle u, dv - \underline{d}v_h \rangle + \langle da, \underline{d}v_h - dv \rangle,$$

the $|\nabla da|$ produced by the latter term can be estimated by $|u|_{\mathsf{H}^1}$ by assumption. The same procedure is feasible for δb and $\underline{\delta}b_h$ (where another test form v can be employed such that δv is close to $\underline{\delta}v_h$), *q. e. d.*

10.28 **Proposition.** *Define $\mathfrak{S}^1 := \mathsf{H}^1\Omega^{k-1} \times \mathsf{H}^1\Omega^k \times \mathsf{H}^1\mathfrak{H}^k$ and $\mathsf{P}^{-1}\mathfrak{S} := \mathsf{P}^{-1}\Omega^{k-1} \times \mathsf{P}^{-1}\Omega^k \times \mathsf{P}^{-1}\mathfrak{H}^k$. Suppose $s = (\sigma, u, p) \in \mathfrak{S}^1$ is a solution of the Poisson problem in mixed form as in 2.17, and $s_h = (\sigma_h, u_h, p_h) \in \mathsf{P}^{-1}\mathfrak{S}$ is the solution of the corresponding finite-dimensional problem. Then for all $t_h = (\tau_h, v_h, q_h) \in \mathsf{P}^{-1}\mathfrak{S}$,*

$$b(s - s_h, t_h) \lesssim \tilde{C}_{\boxdot} h(|\nabla f|_{\mathsf{L}^2} + |\nabla s|_{\mathsf{L}^2})|t_h|_{\mathsf{H}^{1,0}},$$

where the left-hand side is of course not to be taken literally as in 2.18a, but with P^{-1} exterior derivatives for s_h and t_h, i. e. consisting of terms like $\langle du - \underline{d}u_h, \underline{d}v_h \rangle$ etc.

Proof. In the spirit of 10.26, we start with

$$b(s - s_h, t_h) = b(s, t) - b(s_h, t_h) + b(s, t - t_h).$$

As before, the first two terms are $\langle f, v - v_h \rangle$, which is well-controlled by the right-hand side of the claim. In $b(s, t - t_h)$, there are many easy terms, which we do not explicitely discuss once more. Only $\langle u, q - q_h \rangle = \langle u, q_h \rangle$ is iteresting. Estimating it actually means bounding the difference between $\mathsf{H}^1\mathfrak{H}^k$ and $\mathsf{P}^{-1}\mathfrak{H}^k$. Let $u = da + \delta b$ be the Hodge decomposition of u (which does not contain a harmonic term), and choose $\underline{d}a_h$, $\underline{\delta}b_h$ close to them in the sense of 9.19. Then, as $q_h \in \mathsf{P}^{-1}\mathfrak{H}^k$,

$$\langle u, q_h\rangle = \langle da + \delta b, q_h\rangle = \langle da - \underline{d}a_h, q_h\rangle + \langle \delta b - \underline{\delta}b_h, q_h\rangle \lesssim \tilde{C}_{\boxdot} h(|\nabla da| + |\nabla db|)|q_h|,$$

q. e. d.

Remark. ⟨**a**⟩ In 10.27, we cannot say anything about $c - c_h$ (yet), as we would need to control the terms in $\langle c - c_h, v_h\rangle = \langle \underline{d}a_h - da, v_h\rangle + \langle \underline{\delta}b_h - \delta b, v_h\rangle$, but we only have control over the scalar product with $\underline{d}v_h$ or $\underline{\delta}v_h$ respectively. **10.29**

⟨**b**⟩ As remarked in 9.10, the correct treatment of variational boundary value problems in $\mathsf{P}^{-1}\Omega^k$ would require a modification of $\underline{\delta}$ at boundary simplices.

⟨**c**⟩ Employing P^{-1} forms also as test functions is unsatisfactory, as they are no classical objects to test with, so the results are not easily comparable to usual estimates. However, we are not sure which forms would be the right ones to test with: Perhaps forms that are "almost constant" in some small, but non-shrinking region would be good to obtain an average value for such a region. $\mathsf{H}^{1,1}$ forms are not the right candidates: If for example in the Poisson problem $\langle du - \underline{d}u_h, dv\rangle$ converged for all "well-behaving" $v \in \mathsf{H}^1$, then also $|du|^2 - |\underline{d}u_h|^2$ would converge, which is not the case:

Example. Consider an equilateral triangle mesh in the xy plane with unit edge length, rotated such that one of the edges is parallel to the x-axis. Now consider the constant vector field $v = (1,0)$, and let us compute its $\mathsf{P}^{-1}\Omega^1$ approximation according to 9.20: In a triangle ijk, where ij points in x direction, we have $\int_{ijk}\langle v_h, (1,0)\rangle = \frac{1}{3}\,|ijk|\,(\alpha_{ij} + \frac{1}{2}\alpha_{ik} + \frac{1}{2}\alpha_{kj})$. So $\alpha_{ij} = 2$ and $\alpha_{ik} = \alpha_{kj} = 1$ gives the correct integral mean (the scalar product with $(0,1)$ is obvious). There are other combinations matching $⨍_{ijk} v$ (for example the "obvious" choice $\alpha_{ij} = 3$ and $\alpha_{ik} = \alpha_{kj} = 0$), but this one also captures its exterior derivative: As v is constant, $dv = 0$, and $\underline{d}v_h = 0$ for $v_h = 2\omega^{ij} + \omega^{ik} + \omega^{kj}$. **10.30**

This vector field v is also the gradient of the function $f : (x,y) \mapsto x$. If $f_h \in \mathsf{P}^{-1}\Omega^0$ has the same values as f on all vertices, then $\underline{d}f_h = v_h$. The homogeneous Dirichlet problem in $\mathsf{P}^{-1}\Omega^0$ is equivalent to the Dirichlet problem in P^1 by 3.2c, for which reason f_h will be the P^{-1} harmonic function with prescribed boundary values $f_h|_{\partial\mathfrak{K}}$, no matter where the boundary is drawn. Thus, f and f_h with derivatives v and v_h are the solutions compared in 10.26. As $|v| = 1$ everywhere, we have $|df|^2_{\mathsf{L}^2(ijk)} = |ijk|$ in each triangle, but $|\underline{d}f_h|^2_{\mathsf{L}^2(ijk)} = \frac{1}{3}\,|ijk|\,(4+1+1) = 2\,|ijk|$. The discrepancy $|\underline{d}f_h|^2_{\mathsf{L}^2(ijk)} = 2\,|df|^2_{\mathsf{L}^2(ijk)}$ does not shrink with smaller and smaller edge lengths, so the Dirichlet energy of the P^{-1} approximations will always be twice as large as of the analytical solution.

11. Approximation of Submanifolds

Extrinsic and Intrinsic Karcher Triangulation

Definition. Let the piecewise smooth n-dimensional submanifold $S \subset M$ be given by a bijective triangulation $y : r\mathfrak{K} \to S$ with vertices $p_i = y(ri)$. If $y|_{r\mathfrak{c}}$ for each $\mathfrak{c} \in \mathfrak{K}^n$ **11.1**

is a barycentric mapping with respect to the induced metric $g|_S$, then y is called an **intrinsic** Karcher triangulation. If moreover each $y|_{r\mathfrak{c}}$ is also a barycentric mapping with respect to the metric of M, then y is called an **extrinsic Karcher triangulation**.

Goal. The possibility for the existence of an intrinsic Karcher triangulation has been dealt with in section 8. The question of this section will now be how well an extrinsic Karcher triangulation, induced by the same complex $\mathfrak{K}$ and the same vertex set $\{p_i\}$, approximates S. Note that such an extrinsic Karcher triangulation is always an interpolation of the given triangulation of S in the sense of 7.14.

11.2 **Proposition.** *Let y be a piecewise (ϑ, h)-small intrinsic Karcher triangulation y of S with $\|W_\nu\| \leq \kappa$ for all Weingarten maps W_ν. Suppose that all vertices $p_i = y(ri)$, $i \in \mathfrak{c}$, lie in a common convex ball with respect to g for each $\mathfrak{c} \in \mathfrak{K}^n$. Then for small edge lengths $\ell_{ij} := \mathbf{d}_S(p_i, p_j)$ and $\bar{\ell}_{ij} := \mathbf{d}(p_i, p_j)$ with respect to $g|_S$ and g, it holds $|\ell_{ij} - \bar{\ell}_{ij}| \lesssim \kappa h \vartheta^{-1} \bar{\ell}_{ij}$.*

Proof. There exists an extrinsic Karcher triangulation x of some set $S' \subset M$ with the same combinatorics and vertices as y (that means: interpolating y) because the vertices of each simplex are contained in a convex ball. We do not know if S' is a manifold, because the fullness of the extrinsic simplices is not clear *a priori*, but will be a result of the length estimate.

Let us show the claim for the edge $\gamma : e_i \rightsquigarrow e_j$ in $r\mathfrak{c}$ with tangent $v := \dot{\gamma}$. The estimate 7.9 can be extended to the whole edge:

$$|(dx - Pdy)v| \lesssim h\vartheta^{-1} \frac{|(\nabla dx - P\nabla dy)(v,v)|}{|dy\, v|}$$

As edges are mapped to geodesics, $\nabla dx(v, v) = 0$ and $t\nabla dy(v, v) = 0$. And as $\langle \nabla_{dy\,v} dy\, v, \nu\rangle = -\langle \nabla_{dy\,v}\nu, dy\, v\rangle$ for any normal ν to S, we have $|\nabla dy(v, v)| \leq \kappa |dy\, v|^2$. So

$$|\ell_{ij} - \bar{\ell}_{ij}| \leq \int \big||dx\, v| - |dy\, v|\big| \leq \int |(dx - Pdy)v| \leq \kappa h \vartheta^{-1} \int |dy\, v|,$$

q. e. d.

11.3 **Corollary.** *Situation as before, additionally $\|W_\nu\| + h\|\nabla W_\nu\| \leq \kappa$ for all Weingarten maps. Let $\bar{\ell}_{ij}$ also induce a (ϑ, h)-full metric g^e on $r\mathfrak{K}$. Then for small h,*

$$|(y^*g - g^e)\langle v, w\rangle| \lesssim (C_0 h^2 + \kappa h \vartheta^{-1})|v|\,|w|, \tag{11.3a}$$

$$|\nabla_v^{y^*g} w - \nabla_v^{g^e} w| \lesssim \tilde{C}'_{0,1} h |v|\,|w|, \tag{11.3b}$$

where $\tilde{C}_{0,1} = (C_{0,1} + \kappa^2)\vartheta^{-1}$. The second estimate also holds for any other piecewise flat metric g^e on $r\mathfrak{K}$.

Proof. The metric estimate comes from the edge length comparison above, and the connection estimate from 6.23 does not depend on the chosen metric, as long as it is flat. Due to the Gauß equation (e. g. Jost 2011, thm. 4.7.2), the intrinsic curvature tensor of S is bounded by $C_0 + \|W_\nu\|^2$ and its derivative by $C_1 + \|W_\nu\|\|\nabla W_\nu\|$, *q. e. d.*

Remark. The observation that 11.3b also holds for any other piecewise flat metric on $\mathfrak{K}$ means that if g is approximated up to second order by a better-suited approximation of edge lengths ℓ_{ij} than just $\bar{\ell}_{ij}$, then the approximation of the connection remains unchanged. **11.4**

Nevertheless, taken as it is, 11.3 says that a simple interpolation of a given triangulation, just as in Euclidean space, is not the best candidate for geometry approximation. Henceforth, the rest of this section is devoted to the normal graph mapping, which reveals better approximation properties.

General Properties of Normal Graphs

Definition. Let $S \subset M$ be an n-dimensional compact boundaryless smooth submanifold. A second submanifold $S' \subset M$ is said to be a **normal graph** over S if there is a normal vector field Z on S such that **11.5**

$$\Phi : a \mapsto \exp_a Z|_a \tag{11.5a}$$

is a bijective mapping $S \to S'$. Where we need it, we will also consider the "smooth transition" $S \rightsquigarrow S'$ via the homotopy

$$\Phi_t : a \mapsto \exp_a tZ|_a. \tag{11.5b}$$

Parallel transport along $t \mapsto \Phi_t(p)$ from $\Phi_a(p)$ to $\Phi_b(p)$ will be denoted by $P^{b,a}$.

Remark. **⟨a⟩** The term "normal graph" or "normal height map" is mostly used in the context of triangular approximation of surfaces in $\mathbb{R}^3$, e.g. in Hildebrandt *et al.* (2006). In the context of manifold-valued pde's, it is more common to consider the **geodesic homotopy** Φ_t, see 13.6d, which also Grohs *et al.* (2013) use. In particular, their control of distortion *along* Φ_t is equivalent to our control of the distortion *by* Φ. **11.6**

⟨b⟩ Here, as usual, we do not want to treat global properties of M, so we always tacitly assume $|Z| < \operatorname{inj} M$.

⟨c⟩ Any other n-dimensional submanifold $S' \subset M$ that is near enough to have a bijective orthogonal projection $S' \to S$ can be represented as a normal graph over S. Here "normal projection" means mapping some $p \in S'$ onto the point $q \in S$ minimising $\mathbf{d}(p, q)$. The largest ε such that the orthogonal projection $\mathbb{B}_\varepsilon(S) \to S$ is well-defined is called the **reach** of S (introduced by Federer 1959, def. 4.1, for a recent overview see Thäle 2008). Another formulation for the same thing is that

$$\tilde{\Phi} : TS^\perp \to M, (p, Z) \to \exp_p Z$$

is a diffeomorphism from $O_\varepsilon := \{\nu \in TS^\perp : |\nu| < \varepsilon\}$ onto its image.

⟨d⟩ It is well-known (see e.g. Hildebrandt 2012, eqn 1.11) that for $M = \mathbb{R}^m$, the map Φ is locally a diffeomorphism if $|Z| \|W_\nu\| < 1$ for all Weingarten maps W_ν. (Note that this bound can only capture the local geometry of S but cannot see if some part of S that is intrinsically far from a point $p \in S$ comes close to p in the surrounding space M.) Different to the usual argument involing the curvature radius of S and osculating spheres, one can use Jacobi fields as in 1.25 (we use the notation from there) to see this:

We already know $d\Phi_t\,\dot p = J(t)$ for a Jacobi field with $J(0) = \dot p$ and $\dot J(0) = \nabla_{\dot p}\nu$, so $\frac{\mathrm{d}}{\mathrm{d}t}\Phi_t^* g\langle \dot p, \dot p\rangle = g\langle J, \dot J\rangle$. Now let ν be a unit normal field. Then $\dot J = W_\nu J$. If $\dot p$ is the eigenvector in direction of the largest eigenvalue κ, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_t^* g\langle \dot p, \dot p\rangle = 2\kappa(t)|J(t)|,$$

where $\kappa(t)$ is the eigenvalue of the Weingarten map W_ν in direction $\dot p$ at $\Phi_t(p)$. In our case $R = 0$, the Riccati equation 1.25b gives $\dot W_\nu = -W_\nu^2$, so the eigenvalues κ_i also evolve by $\dot\kappa_i = -\kappa_i^2$. This differential equation has solution $\kappa_i(t) = (t - \frac{1}{\kappa_i(0)})^{-1} = \frac{\kappa_i(0)}{\kappa_i(0)t-1}$, hence

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_t^* g\langle \dot p, \dot p\rangle = \frac{2\kappa}{\kappa t - 1}\Phi_t^* g\langle \dot p, \dot p\rangle.$$

This is solved by $\Phi_t^* g\langle \dot p, \dot p\rangle = (1-\kappa t)^2 g\langle \dot p, \dot p\rangle$, so g is positive definite for $\kappa t < 1$.

⟨**e**⟩ If we represent $Z = f^i\nu_i$ with parallel unit normal vector fields ν_i and scalar functions f^i, then $\nabla Z = df^i \otimes \nu_i + f^i\nabla\nu_i$, which splits into tangential and normal parts

$$\mathfrak{t}\nabla Z = f^i\nabla\nu_i, \qquad \mathfrak{n}\nabla Z = df^i \otimes \nu_i.$$

Hence, although $\|\nabla Z\|$ usually shrinks slower than $|Z|$ for $|f| \to 0$, the tangential part is $\|\mathfrak{t}\nabla Z\| \lesssim \kappa|f|$, where κ is an upper bound for the Weingarten maps W_ν of S. Similary, $\|\mathfrak{t}\nabla^2 Z\| \lesssim \kappa|df| + \kappa'|f|$ if κ' bounds all $\|\nabla W_\nu\|$.

11.7 **Situation.** Let S' be given as a normal graph over S by a vector field Z with $\mathbf{d} := |Z| \le \varepsilon^2$ and $\|d\,\mathbf{d}\,\| \le \varepsilon$ everywhere, and let the Weingarten maps of S be bounded by $\|W_\nu\| + \varepsilon\|\nabla W_\nu\| \le \kappa$. This means $\|\nabla Z\| \lesssim \kappa\varepsilon$. For simplicity, assume $\kappa \le \kappa^2$.

11.8 **Proposition.** *Situation as in 11.7. The map $\Phi : a \mapsto \exp_a Z$ is locally a diffeomorphism if $|\mathbf{d}|(\kappa + \sqrt{C_0}) < 1$ everywhere.*

Proof. Let us suppose Z has unit length at some point and see for which t the map Φ_t is locally a diffeomorphism. Note that 1.25b gives $|\dot\kappa_i| \le C_0 + |\kappa_i^2|$ for any eigenvalue of a Weingarten map. The equation $\dot u = -C_0 - u^2$ leads to a subsolution

$$u(t) = \sqrt{C_0}\tan\left(\sqrt{C_0}t - \arctan\tfrac{\kappa_i(0)}{\sqrt{C_0}}\right).$$

Regarding $\frac{\mathrm{d}}{\mathrm{d}t}\Phi_t^* g\langle \dot p, \dot p\rangle \le \kappa(t)\Phi_t^* g\langle \dot p, \dot p\rangle$, which has positive solutions (for positive initial data) as long as κ is bounded, it suffices to know where the first pole of $\kappa_i(t)$ can occur. The first pole of u (which must also bound the position of the first pole of κ_i) is where $\sqrt{C_0}t - \arctan\frac{\kappa_i(0)}{\sqrt{C_0}} = \pm\frac{\pi}{2}$. Now a simple function inspection shows

$$\frac{1}{1+s} < \tfrac{\pi}{2} + \arctan s, \qquad -\frac{1}{1+s} > -\tfrac{\pi}{2} + \arctan s,$$

so there will be no pole as long as $|\sqrt{C_0}t| < (1 + \frac{\kappa}{\sqrt{C_0}})^{-1}$, *q. e. d.*

Observation. By 1.16, the differential of Φ_t is $d\Phi_t V = J(t)$ for the Jacobi field with $J(0) = V$, $\dot J(0) = \nabla_V Z$. By 6.1, this means for $Q_t : V \mapsto V + t\nabla_V Z$ **11.9**

$$|d\Phi_t V - P^{t,0} Q_t V| \lesssim C_0\, \mathbf{d}^2\, t^2 (1 + \kappa\varepsilon t)|V| \tag{11.9a}$$

and

$$|d\Phi_s V - P^{t,0} V| \lesssim \kappa\varepsilon |V| + C_0\, \mathbf{d}^2\, t^2 (1 + \kappa\varepsilon t)|V|. \tag{11.9b}$$

This estimate is scale-invariant with respect to scaling of Z: If $Z' = \alpha Z$, then $\Phi'_{t/\alpha} = \Phi_t$. So $tZ = t'Z'$ is scale-invariant, and the estimate only contains tZ, never Z alone.

But due to 11.6e, this distortion happens mostly in normal direction, the tangential change is of higher order:

$$\|\mathit{t}(d\Phi - P)\| \lesssim \kappa\, \mathbf{d} + C_0\, \mathbf{d}^2 \tag{11.9c}$$

Proposition. *Situation as in 11.7. Consider some point $p \in \operatorname{reach} S$ with projection $\psi(p)$ onto S. If $p = \exp_{\psi(p)} \mathbf{d}\,\nu$ for some unit normal vector ν with $\|W_\nu\| \le \kappa$, and if $\kappa\varepsilon < \frac{1}{2}$, the orthogonal projection ψ satisfies* **11.10**

$$\|d\psi - Q_\nu^{-1}\, \mathit{t} P^{p,\psi(p)}\| \lesssim C_0\, \mathbf{d}^2, \qquad \|d\psi - \mathit{t} P^{p,\psi(p)}\| \lesssim \kappa\, \mathbf{d} + C_0\, \mathbf{d}^2, \tag{11.10a}$$

where Q_ν is the linear map $T_pS \to T_pM$, $V \mapsto V + \mathbf{d}\, \nabla_V \nu$. If $\mathbf{d}\,\nu$ is replaced by some other normal vector field Z with $\exp_{\psi(p)} Z = p$, and Q is replaced by $V \mapsto V + \nabla_V Z$, it holds

$$\|d\psi - (\mathit{t}Q)^{-1} \mathit{t} P^{p,\psi(p)}\| \lesssim C_0\, \mathbf{d}^2, \qquad \|d\psi - \mathit{t} P^{p,\psi(p)}\| \lesssim \kappa\, \mathbf{d} + C_0\, \mathbf{d}^2\,. \tag{11.10b}$$

Proof. Let us first show that $\mathit{t}Q$ does not depend on how Z is chosen at points $\neq \psi(p)$. If $Z = \mathbf{d}\,\nu$ in a neighbourhood of $\psi(p)$, where ν is a parallel unit normal field and $\mathbf{d}$ is constant, then Z is parallel, and so $\nabla Z = \mathit{t}\nabla Z = \mathbf{d}\, W_\nu$. For any other Z, $\mathit{t}\nabla Z$ stays the same, and only some part $\mathit{n}\nabla Z \neq 0$ is added. That means $\mathit{t}Q_\nu = Q_\nu$ on $T_{\psi(p)}S$. So we will only prove 11.10b.

ad primum: Observe that the operator $\mathit{t}Q : T_pS \to T_pS$ fulfills $\|\mathit{t}Q - \mathrm{id}\,\| \lesssim \kappa\varepsilon < \frac{1}{2}$ by assumption, hence is invertible with $\|(\mathit{t}Q)^{-1} - \mathrm{id}\,\| \lesssim \frac{\kappa\varepsilon}{1-\kappa\varepsilon}$ by 6.15, which gives $\|(\mathit{t}Q)^{-1}\| \lesssim 1 + \frac{\kappa\varepsilon}{1-\kappa\varepsilon} = \frac{1}{1-\kappa\varepsilon} < 2$. Hence the claim is proven if we can show $\|\mathit{t}(Qd\psi - P)\| \lesssim C_0\, \mathbf{d}^2$.

Consider some vector $V \in T_pM$ and split $V = V_p + V_\nu$ as in 1.25c. Then $\mathit{t}Qd\psi(V) = \dot p + \mathit{t}\nabla_{\dot p} Z = J_p(0) + \dot J_p(0)$ on the one hand, and $\mathit{t}PV = PV_p = PJ_p(1)$ by 1.25d on the other. So 6.1 gives

$$|J_p(1) - P(J_p(0) + \dot J_p(0))| \lesssim C_0\, \mathbf{d}^2\, |\dot p| \lesssim C_0\, \mathbf{d}^2\, |V|.$$

ad sec.: We have $\|d\psi - \mathit{t}P\| \le \|d\psi - (\mathit{t}Q)^{-1}\mathit{t}P\| + \|(\mathit{t}Q)^{-1}\mathit{t} - \mathit{t}\|$. The first norm has been estimated above, and the second is $\lesssim \kappa\, \mathbf{d}$ because $\|\mathit{t}Q\mathit{t} - \mathit{t}\| \lesssim \kappa\, \mathbf{d}$ due to 11.6e and the boundedness of $(\mathit{t}Q)^{-1}$, *q. e. d.*

Remark. For $C_0 = 0$, this (exact) representation of the projection differential is the one in Wardetzky (2006, thm. 3.2.1) and Morvan and Thibert (2004, lemma 4).

11.11 **Proposition.** *Situation as in 11.7. Let* $\mathfrak{t}'$ *be the orthogonal projection* $TM|_{S'} \to TS'$. *Then for small* ε, *we have* $\|P\mathfrak{t}' - \mathfrak{t}P\| \lesssim \kappa\varepsilon + C_0\,\mathbf{d}^2$. *This means that the angles* $\measuredangle(T_{\Phi(p)}S', PT_pS)$ *and* $\measuredangle((T_{\Phi(p)}S')^{\perp}, PT_pS^{\perp})$ *between the corresponding tangent and normal spaces must be bounded by this factor, too. Therefore, normals* ν_i *to* S *can be extended to normal fields* $\nu_{i,t}$ *along* Φ_t *with* $|\nu_{i,t} - P^{t,0}\nu_i| \lesssim \kappa\varepsilon + C_0\,\mathbf{d}^2$.

Proof. For the time of this proof, let us write the terminal value $J(1)$ of a Jacobi field along $t \mapsto \Phi_t(p)$ with initial values $J(0) = \dot{q}$ and $\dot{J}(0) = \dot{\nu}$ as $T(\dot{q}, \dot{\nu})$. Linearity of the Jacobi equation translates into linearity of T. In this notation, the splitting from 1.25c says that a vector $V \in TM|_{S'}$ can be represented as $V = T(\dot{p}, \mathfrak{t}\dot{\nu}) + T(0, \mathfrak{n}\dot{\nu})$. We argue that its projection $\mathfrak{t}'V$ onto TS' is almost $T(\dot{p}, \nabla_{\dot{p}}Z)$.

In fact, all tangent vectors on S' have the form $T(\dot{q}, \nabla_{\dot{q}}Z)$ for some $\dot{q} \in TS$. Now consider

$$|V - T(\dot{q}, \nabla_{\dot{q}}Z)|^2 = |T(\dot{p} - \dot{q}, \dot{\nu} - \nabla_{\dot{q}}Z)|^2.$$

This is minimal among all $\dot{r}$ if $\mathfrak{t}'V = T(\dot{q}, \nabla_{\dot{q}}Z)$, this means its norm has vanishing derivative in direction $(\dot{r}, \nabla_{\dot{r}}Z)$. Because T is linear, this gives

$$0 = \langle T(\dot{p} - \dot{q}, \dot{\nu} - \nabla_{\dot{q}}Z), T(\dot{r}, \nabla_{\dot{r}}Z)\rangle \qquad \text{for all } \dot{r} \in T_pS.$$

Now recall that $T(U, W) = P(U + \mathbf{d}\,W) + O(C_0\,\mathbf{d}^2)$, hence this is

$$= \langle \dot{p} - \dot{q}, \dot{r} + \mathbf{d}\,\nabla_{\dot{r}}Z\rangle + \mathbf{d}\langle \dot{\nu} - \nabla_{\dot{q}}Z, \dot{r}\rangle + \mathbf{d}^2\langle \dot{\nu} - \nabla_{\dot{q}}Z, \nabla_{\dot{r}}Z\rangle + O(C_0\,\mathbf{d}^2).$$

If $\dot{p} = \dot{q}$, the first term vanishes, and (using that $\kappa\,\mathbf{d}$ is small) the remaining ones are estimated from above by $\kappa\,\mathbf{d}\,|V|\,|\dot{r}|$. Because the minimisation is well-conditioned at this position, the optimal $\dot{q}$ is $\dot{p} + O((\kappa\,\mathbf{d} + C_0\,\mathbf{d}^2)|V|)$.

Now recall from 1.25d that $P\mathfrak{t}PV = T(\dot{p}, \mathfrak{t}\nabla_{\dot{p}}Z)$, which gives that the claim $|P\mathfrak{t}'V - \mathfrak{t}PV| = |(\mathfrak{t}' - P\mathfrak{t}P)V| = |T(0, \mathfrak{n}\nabla_{\dot{p}}Z)| + O((\kappa\,\mathbf{d} + C_0\,\mathbf{d}^2)|V|) \lesssim (\kappa\varepsilon + C_0\,\mathbf{d}^2)|V|$ is just the usual Jacobi field estimate 6.1, *q. e. d.*

11.12 **Corollary.** *Omitting the last paragraph of the proof, one gets* $\|\mathfrak{t}P\mathfrak{t}' - \mathfrak{t}P\| \lesssim \kappa\,\mathbf{d} + C_0\,\mathbf{d}^2$.

Remark. This is analogous to the classical statement $\|P(P_h - \mathbb{1})P\| \lesssim \mathbf{d}$ up to constants depending on the geometry from Dziuk *et al.*, where P is the projection onto TS and P_h the projection onto TS'.

Geometric Distortion by the Graph Mapping

11.13 **Lemma.** *Situation as in 11.7. Then for* $Q : U \mapsto U + \nabla_U Z$,

$$|\langle QU, QV\rangle - \langle U, V\rangle| \lesssim (\kappa^2\,\mathbf{d} + C_0\,\mathbf{d}^2)\,|U|\,|V| \qquad \textit{for all } U, V \in T_pS.$$

Proof. Just because $\langle U + \nabla_U Z, V + \nabla_V Z\rangle - \langle U, V\rangle = \langle \nabla_U Z, V\rangle + \langle \nabla_V Z, U\rangle + \langle \nabla_U Z, \nabla_V Z\rangle$ and 11.9c, *q. e. d.*

11.14 **Conclusion.** *Situation as in 11.7. Pulled back to* S, *the* S' *metric* $\Phi^*g|_p\langle U, V\rangle = g|_{\Phi(p)}\langle d\Phi\,U, d\Phi\,V\rangle$ *fulfills*

$$\big|(\Phi^*g - g)\langle U, V\rangle\big| \lesssim (\kappa^2\,\mathbf{d} + C_0\,\mathbf{d}^2)\,|U|\,|V|\,.$$

Proof. Is a direct application of 11.9c and 11.13. We especially remark that the difference between $g|_p$ and $g|_{\Phi(p)}$ does not need to be handled explicitely, as $P^{p,\Phi(p)}$ is an isometry with respect to these two metrics, *q. e. d.*

Remark. As Φ^*g and g are equivalent metrics, $A := d\Phi^t d\Phi$ (where $d\Phi^t$ denotes the g-adjoint of $d\Phi$) is a self-adjoint automorphism of T_pS such that $\Phi^*g\langle U,V\rangle = g\langle AU,V\rangle$, called the **metric distortion tensor** by Wardetzky (2006, p. 53). In the numerical literature, it is common not to compare the Riemannian metrics, but to estimate directly $\|\frac{G^e}{G}A - \mathrm{id}\|$, which already includes the volume element change (cf. the proof of 7.3), see Dziuk (1988); Demlow (2009); Heine (2005). **11.15**

For a comparison with the tensor J from 6.20, consider $\Psi := \Phi^{-1}$. If M is the Euclidean space $\mathbb{R}^n$ and S' is a piecewise flat submanifold, its metric $g^e := g|_{S'}$ is piecewise flat. The metric $g|_S$ pulls back to a metric Ψ^*g on S', and there is J such that $\Psi^*g\langle U,V\rangle = g^e\langle JU,V\rangle$. So the transformations A and J perform inverse tasks.

Proposition. *Situation as in 11.7. For a given vector U and a vector field V on S, define the "connection distortion" $W := \nabla_{d\Phi_t U} d\Phi_t V - d\Phi_t(t\nabla_U V)$. This vector field obeys the differential equation* **11.16**

$$\ddot{W} = R(Z,W)Z + \dot{F} \qquad \text{for } F := R(Z, d\Phi_t U)d\Phi_t V + \nabla^2_{d\Phi_t U, d\Phi_t V} Z$$

with initial values $W(0) = 0$ and $\dot{W}(0) = F(0)$.

Proof. Let us abbreviate $U^t := d\Phi_t U$, $V^t := d\Phi_t V$, and denote the parallel translation of Z along $t \mapsto \exp_p tZ$ also as Z. Let $K := \nabla_{U^t} V^t$ and $J := d\Phi_t(t\nabla_U V)$. Then we want to determine $W = K - J$.

By 1.25, J is a Jacobi field, i. e. $\ddot{J} = R(Z,J)Z$. An inhomogeneous Jacobi equation describes K:

$$\ddot{K} = R(Z,K)Z + D_t R(Z,U^t)V^t + D_t \nabla^2_{U^t,V^t} Z.$$

In fact, consider a variation $\gamma(r,s)$ of geodesics (in M), i. e. we assume that $s \mapsto \gamma(r,s)$ is a geodesic for each fixed s, with $\partial_s\gamma(0,0) = V$ and $\partial_r\gamma(0,0) = U$. Transport this along t as $c(r,s,t) := \exp_{\gamma(r,s)} tZ|_{\gamma(r,s)}$. Then we want to determine $K = D_r\partial_s$, so we consider

$$\begin{aligned}\ddot{K} = D_t D_t D_r \partial_s &= D_t D_r D_t \partial_s + D_t R(\partial_t, \partial_r)\partial_s \\ &= D_t D_r D_s \partial_t + D_t R(\partial_t, \partial_r)\partial_s,\end{aligned}$$

the first term of which is

$$\begin{aligned}\nabla_Z \nabla_{U^t} \nabla_{V^t} Z &= \nabla_Z \nabla^2_{U^t,V^t} Z + \nabla_Z \nabla_{\nabla_{U^t} V^t} Z \\ &= \nabla_Z \nabla^2_{U^t,V^t} Z + \nabla_{\nabla_{U^t} V^t} \nabla_Z Z + R(Z, \nabla_{U^t} V^t), Z \\ &= D_t \nabla^2_{U^t,V^t} Z \;+\; \qquad 0 \qquad + R(Z,K)Z.\end{aligned}$$

The initial value is computed in exactly the same way, *q. e. d.*

11.17 **Proposition.** *Situation as before. If $C_0|Z|^2 + \|\nabla Z\| \leq \frac{1}{2}$, then*

$$|\nabla_{d\Phi\, U} d\Phi\, V - d\Phi\; t\nabla_U V| \lesssim |U||V|(\|\nabla^2 Z\| + C_0|Z|) + C_0|\nabla_U V||Z|^2.$$

If we only consider the tangential part tW, then even

$$|t\nabla_{d\Phi\, U} d\Phi\, V - d\Phi\; t\nabla_U V| \lesssim |U||V|\big(\|t\nabla^2 Z\| + \|\nabla Z\| + C_0|Z|^2\big) + C_0|\nabla_U V||Z|^2. \tag{11.17a}$$

Proof. Preparatory step one: Let us first establish the boundedness of $W = K - J$ and show $|W| \lesssim |\nabla_U V| + at$, where $a := |U^t||V^t|(\|\nabla^2 Z\| + C_0|Z|)$: The t-derivative of K is, as $\partial_t = Z$,

$$\begin{aligned}\nabla_Z \nabla_{U^t} V^t &= \nabla_{U^t} \nabla_Z V^t + R(Z, U^t)V^t \\ &= \nabla_{U^t} \nabla_{V^t} Z + R(Z, U^t)V^t \quad = \quad \nabla^2_{U^t, V^t} Z + \nabla_{\nabla_{U^t} V^t} Z + R(Z, U^t)V^t,\end{aligned}$$

so $\frac{\mathrm{d}}{\mathrm{d}t}|K| \leq |\dot K| \leq a + \|\nabla Z\|\,|K|$. As $|Z|$ is short by assumption, we have $|U^t| \lesssim |U|$ and $|V^t| \lesssim |V|$. The differential inequality of the form $\dot u \leq a + bu$ gives $u \leq (u_0 + \frac{a}{b})\mathrm{e}^{bt} - \frac{a}{b}$. For $bt \leq \frac{1}{2}$, this function is dominated by $u_0 + 2bt(u_0 + \frac{a}{b}) \leq 2(u_0 + at)$.

For the bound on J, we have $|J| \leq |J(0)| + t|\dot J(0)|$ as usual, and $J(0) = \nabla_U V$, $\dot J(0) = \nabla_Z J(0)$ shows that these terms are already contained in the K estimate.

Preparatory step two: Now let us show

$$\Big|W(t) - \int_0^t P^{t,\tau} F(\tau)\,\mathrm{d}\tau\Big| \lesssim C_0 t^2 |Z|^2 (|\nabla_U V| + at).$$

The proof idea is from Jost (2011, thm 5.5.2). Let $A := \int_0^t PF$. This is a vector field fulfilling $\ddot A = \dot F$ with the same initial values as W, namely $A(0) = 0$ and $\dot A(0) = F(0)$. Furthermore, let $w : [0; t] \to \mathbb{R}$ be the solution of $\ddot w = C_0|Z|^2|W|$ with initial values $w(0) = \dot w(0) = 0$. Then, for some parallel vector field E along t, define $v := (\langle W - A, E\rangle - w)/t$ and obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}(\dot v t^2) = \frac{\mathrm{d}}{\mathrm{d}t}\big((\langle \dot W - \dot A, E\rangle - \dot w)t - \langle W - A, E\rangle + w\big) = (\langle \ddot W - \ddot A, E\rangle - \ddot w)t \leq 0.$$

This means that $\dot v t^2 \leq 0$, hence $\dot v \leq 0$. Now $\langle W - A, E\rangle - v$ has a double root at $t = 0$, so $v(0) = 0$ and thus $v \leq 0$ everywhere. And because E was arbitrary, this already means

$$|W - A| \leq w.$$

Therefore we are done if we can bound u by the right-hand side of the proposition. But as we know that $|W| \lesssim |\nabla_U V| + at$, we can simply integrate $\ddot w = C_0|Z|^2|W|$ twice and obtain the desired estimate (the argument is the same as in the proof of 6.5).

ad primum: The final estimate for $|W|$ comes from $|\int PF| \leq \int |F|$, together with $|F(t)| \lesssim |F(0)| \leq a$ for $t \leq 1$, because the same holds for U^t and V^t, and the norm of $\nabla^2 Z|_{\Phi_t(p)}$ is the same as the norm of $\nabla^2 Z|_p$ because Z is parallel along $t \mapsto \Phi_t(p)$.

ad sec.: For the estimate of $|tW|$ let t^t be the orthogonal projection onto the tangent space of $\Phi_t(S)$ and consider

$$t^t W(t) - \int_0^t t^t P^{t,\tau} F(\tau)\,\mathrm{d}\tau = t^t W(t) - \int_0^t P^{t,0} t^t P^{0,\tau} F(\tau) + (t^t P^{t,\tau} - P^{t,0} t P^{0,\tau}) F(\tau)\,\mathrm{d}\tau.$$

Then $|tPF| \lesssim |U||V|\|t\nabla^2 Z\|$, and the projection difference is estimated by $t\|\nabla Z\| + C_0 t^2 |Z|^2$ in 11.11, *q. e. d.*

Theorem. *Let $y : r\mathfrak{K} \to S$ be the triangulation of a smooth submanifold $S \subset M$ with Weingarten maps W_ν bounded by $\|W_\nu\| + h\|\nabla W_\nu\| \leq \kappa$ and $x : r\mathfrak{K} \to S'$ an extrinsic Karcher triangulation with the same vertices p_i and $y = \psi(x)$. Suppose g^e is a (ϑ, h)-small piecewise flat metric on $r\mathfrak{K}$ induced by edge lengths $\mathbf{d}_{S'}(p_i, p_j) = \mathbf{d}(p_i, p_j)$. Then for small h, it holds $\mathbf{d}(x, y) \lesssim h^2 \vartheta^{-1} \|\nabla dx - P\nabla dy\|_{\mathsf{L}^\infty}$ and* **11.18**

$$|(y^*g - g^e)\langle v, w\rangle| \lesssim (\kappa^2\,\mathbf{d} + C_0\,\mathbf{d}^2)|v||w|,$$
$$|\nabla_v^{y^*g} w - \nabla_v^{g^e} w| \lesssim \kappa h \vartheta^{-2} \|\nabla dx - P\nabla dy\|_{\mathsf{L}^\infty} |v||w| + ho.,$$

where "ho." stands for higher-order terms whose coefficients depend on C_0, κ, ϑ, $|v|$, $|w|$ and $|\nabla_v^{g^e} w|$. The norm on the left-hand side may be induced by either x^*g, y^*g, or g^e, because all three are equivalent.

Proof. By the estimate 7.12 for $\mathbf{d}(x, y)$, S' is a normal graph over S for small h with $|Z| = \mathbf{d}(x, y) \lesssim h^2 \vartheta^{-1} \|\nabla dx - P\nabla dy\|$. Morally, it is clear that $\|\nabla Z\|$ must be controlled by $\|\nabla dx - P\nabla dy\|$, too. In fact, we can precisely compute this for $V = dy\, v$:

$$\nabla_V Z = \nabla_{dy\,v}(-X_x|_y) = \dot{J}(1)$$

for a Jacobi field along $x \rightsquigarrow y$ with $J(0) = dx\, v$ and $J(1) = dy\, v$ by combining 1.23 and 12.3 (we postpone the computation to the next section because it is more relevant there), so by 6.1 $|\nabla_V Z - (dx - Pdy)v| \lesssim C_0\, \mathbf{d}^2(x, y)$, hence 7.14 gives

$$\|\nabla Z\| \lesssim h\vartheta^{-1} \|\nabla dx - P\nabla dy\| + C_0\, \mathbf{d}^2.$$

Now because $x = \Phi \circ y$ and hence $dx = d\Phi dy$, 11.14 gives

$$|(x^*g - y^*g)\langle v, w\rangle| \lesssim (\kappa^2\,\mathbf{d} + C_0\,\mathbf{d}^2)|v|_{y^*g}|w|_{y^*g}.$$

The comparison of x^*g and g^e is done in 6.17.—Analogously, we only compare ∇^{y^*g} to ∇^{x^*g} and refer to 6.23 for the rest: For a vector v and a vector field w, 11.17 gives (together with 11.6e)

$$|\nabla^{S'}_{dx\,v} dx\, w - d\Phi \nabla^S_{dy\,v} dy\, w| \lesssim \kappa h^2 \vartheta^{-1} \|\nabla dx - P\nabla dy\| + ho.$$

By definition of the pull-back connection, $\nabla^{S'}_{dx\,v} dx\, w = dx \nabla_v^{x^*g} w$ and thus

$$|d\Phi dy(\nabla_v^{x^*g} w - \nabla_v^{y^*g} w)| \lesssim \kappa h^2 \vartheta^{-1} \|\nabla dx - P\nabla dy\| + ho.$$

Together with $\|d\Phi^{-1}\| \lesssim 1$ and $\|dy^{-1}\| \lesssim 1/h\vartheta$, this shows the claim, *q. e. d.*

The Weak Shape Operator

11.19 **Lemma.** *Let $S \subset M$ be a smooth submanifold with boundary ∂S in M. Let U be a smooth vector field on S, not neccessarily tangential to S. Then U may be extended to a vector field on M in such a way that $\operatorname{div}^M tU = \operatorname{div}^S tU$ and $\operatorname{div}^M nU = -\langle U, H\rangle$, where $H = \nabla^M_{e_i} e_i$ for any orthonormal basis e_i of T_pS is the* ***mean curvature vector*** *of S.*

Proof. It suffices to find a local extension of U to some small neighbourhood of S. Let $e_1, \dots, e_n, \nu_{n+1}, \dots, \nu_m$ be an orthonormal basis of T_pM. Then $\operatorname{div}^M U = \langle \nabla^M_{e_i} U, e_i\rangle + \langle \nabla^M_{\nu_j} U, \nu_j\rangle$. If U is extended parallel in normal direction, the latter term vanishes.

Regarding the tangential part, observe $\langle \nabla^M_{e_i} tU, e_i\rangle = \langle t\nabla^M_{e_i} tU, e_i\rangle$, and because $t\nabla^M = \nabla^S$, this is $\operatorname{div}^S tU$.

Now consider $nU = \alpha^j \nu_j$. Again, if U is constant in normal direction, $\operatorname{div}^M nU = \langle (\partial_i \alpha^j)\nu_j, e_i\rangle + \langle \alpha^j \nabla^M_{e_i} \nu_j, e_i\rangle$, the former term vanishes, the latter one is $\alpha^j \langle \nabla^M_{e_i} \nu_j, e_i\rangle = -\alpha^j \langle \nabla^M_{e_i} e_i, \nu_j\rangle$, *q. e. d.*

11.20 **Lemma.** *Let $S \subset M$ be a smooth submanifold with boundary ∂S in M. Then for smooth vector fields V and W on M,*

$$\int_S \langle W, \nu\rangle \operatorname{div}^M V + \langle W, \nu\rangle\langle V, H\rangle + \langle \nabla^M_V \nu, W\rangle + \langle \nu, \nabla^M_V W\rangle = \int_{\partial S} \langle W, \nu\rangle\langle V, \tau\rangle,$$

where τ is the outer normal of ∂S in S.

Proof. Let $f := \langle W, \nu\rangle$. By the divergence theorem (Lee 2003, thm. 14.23), we have

$$\int_S \operatorname{div}^S(fV) = \int_{\partial S} \langle fV, \tau\rangle.$$

Now by product rule and 11.19, $\operatorname{div}^S(fV) = f \operatorname{div}^S V + Vf = f \operatorname{div}^M V - f\langle V, H\rangle + Vf$, *q. e. d.*

11.21 **Corollary.** *Let $S \subset M$ be a smooth submanifold without boundary. The operators $s_\nu, \sigma_\nu : \mathfrak{X}(TM|_S) \times \mathfrak{X}(TM|_S) \to \mathbb{R}$, defined by*

$$s_\nu(V, W) := \int_S \langle \nabla^M_V \nu, W\rangle + \langle W, \nu\rangle\langle V, H\rangle, \quad \sigma_\nu(V, W) := -\int_S \langle W, \nu\rangle \operatorname{div}^M V + \langle \nu, \nabla^M_V W\rangle$$

coincide for smooth S and each normal field ν. On tangential vector fields, $s_\nu(V, W) = \sigma_\nu(V, W) = -\int \mathrm{II}_\nu(V, W)$. If S were only piecewise smooth, σ_ν would still be well-defined. It is called the ***weak shape operator*** *of S.*

11.22 **Proposition.** *Situation as in 11.7. Then there are normal fields ν for S and ν' for S' such that σ_ν approximates the weak shape operator $\sigma'_{\nu'}$ of S', which is*

$$\sigma'_{\nu'}(V, W) = -\int_{S'} \langle W, \nu'\rangle \operatorname{div}^M V + \langle \nu', \nabla^M_V W\rangle,$$

up to first order: $|(\sigma_\nu(V, W) - \sigma'_{\nu'}(PV, PW)| \lesssim \varepsilon |V|_{\mathsf{H}^1(TM|_S)} |W|_{\mathsf{H}^1(TM|_S)}$.

Proof. The derivatives of V and W are not distorted at all, $P\nabla_V W = \nabla_{PV} PW$, see 1.17, hence $\operatorname{div}^M PV = \operatorname{div}^M V$. According to 11.11, the normals ν and ν' can be chosen such that they do not differ by more than ε, *q. e. d.*

Remark. This is our analogue of the weak shape operator convergence result from Hildebrandt and Polthier (2011, thm. 8) (similar, but more extensive, Hildebrandt 2012, thm. 2.4). However, Hildebrandt defines $\sigma'_{\nu'}$ with $\operatorname{div}^{S'}$ instead of div^M, which did not meet our own calculations.

12. Variation of Karcher Simplex Volume

Notation. Different from other sections, we will mostly write $\operatorname{vol} S$ instead of $|S|$ for the volume of some set $S \subset M$, but where we find it meaningful, we will mix both notations.

Goal. Consider a full-dimensional Karcher simplex $x(r\mathfrak{c})$ with vertices $p_0, \dots, p_m$. If p_i is moved with velocity $\dot{p}_i = dx\, u_i$, leading to a family x_t of barycentric mappings, we would like to compute the derivative $\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0} \operatorname{vol}_g(x_t(r\mathfrak{s}))$ of a subsimplex volume. In a second step, we want to show that this derivative is close to $\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0} \operatorname{vol}_{g^e}(r\mathfrak{s}_t)$, where the vertices e_i of $r\mathfrak{s}$ are moved with velocity u_i.

Fact (see e. g. Jost 2011, eqn. 4.8.3). Let $\Phi_t : S \to S^t$ be a normal variation of the smooth submanifold $S \subset M$ with a velocity $Z = \partial_t \Phi_t$ that has compact support. If H is the mean curvature vector of S, then **12.1**

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \operatorname{vol} S^t = -\int_S \operatorname{div}^S Z \overset{(11.20)}{=} \int_S \langle Z, H\rangle,$$

where the last equality only holds if Z vanishes at the boundary of S, whereas the divergence formula is also correct for velocities with support up to the boundary.

Situation. Let $\mathfrak{K}$ be a regular m-dimension simplicial complex, $x : r\mathfrak{K} \to M$ be a Karcher triangulation with respect to vertices $p_i \in M$ such that $d_{ri} x_{\mathfrak{c}}$ is bijective for every vertex of every element and x induces metrics x^*g and g^e as usual, where g^e is (ϑ, h)-small, fulfilling 6.17 and 6.23. Let $\bar{\mathfrak{K}}$ be an n-dimensional subcomplex of $\mathfrak{K}$ such that $S := x(r\bar{\mathfrak{K}})$ is an n-dimensional submanifold of M. **12.2**

Variation of Karcher Triangulations

Lemma. *Let $p \in M$ be some point and X be its half squared distance gradient from 5.2. Recall from 1.23 that if the evaluation point a is moved on a curve $a(r)$, then $\nabla_{\dot{a}(r)} X = \sigma \dot{J}_r(\sigma)$, where J_r is a Jacobi field along the geodesic $p \rightsquigarrow a(r)$ with $J_r(0) = 0$ and $J_r(\sigma) = \dot{a}(r)$.* **12.3**

Now if p moves on a curve $p(t)$, this induces a new variation vector field $\dot{X}_{p(t)} := D_t X_{p(t)}$. It satisfies $\dot{X}_{p(t)}|_a = \sigma \dot{J}_t(\sigma)$, where J_t is a Jacobi field along $p(t) \rightsquigarrow a$ with boundary values $J_t(0) = \dot{p}(t)$ and $J_t(\sigma) = 0$. Combining both variations, we obtain $\nabla_{\dot{a}(r)} \dot{X}_{p(t)} = \sigma D_r \dot{J}_r(\sigma) = \sigma D_t \dot{J}_r(\sigma)$

Remark. ⟨**a**⟩ Note that $\dot J$ and $\dot X$ denote derivatives with respect to different directions. In this section, the parameter r takes the rôle of t in section 6.

⟨**b**⟩ In the notation of Grohs *et al.* (2013), $\dot X = \nabla_1 \log(a,p)$.

Proof. ad primum: Without restriction, we can assume that $p(t)$ and $a(r)$ describe geodesics, as no second derivatives of them enter. Analogous to 1.23, consider a variation of geodesics

$$c(r,s,t) := \exp_{p(t)}\left(\tfrac{s}{\sigma}(\exp_{p(t)})^{-1}a(r)\right).$$

As in 1.23, one can see that

$$X_{p(t)}|_{c(r,s,t)} = \tfrac{s}{\sigma}\partial_s c(r,s,t), \qquad J_r = \partial_r, \qquad J_t = \partial_t$$

because ∂_r and ∂_t are Jacobi fields with the desired values at $s=0$ and $s=\sigma$.

ad sec.: We have $\frac{1}{\sigma}\nabla_{\dot a(r)}\dot X_{p(t)} = D_r D_t \partial_s c(r,s,\sigma) = D_t D_r \partial_s c(r,s,\sigma) + R(\partial_r,\partial_t)\partial_s$, and because $c(r,s,\sigma) = a(r)$ is independent of t, we have $\partial_t = 0$ there. And the former term is $D_t D_s \partial_r c(r,s,\sigma) = D_t \dot J_r(\sigma)$, *q. e. d.*

12.4 **Lemma.** *Situation as before. Abbreviate* $\ell := \mathbf{d}(a(r),p(t))$, $V := \dot a(r)$, $W := \dot p(t)$ *and* $T := \partial_s$. *Then if* $C_0\ell^2 < \frac{\pi^2}{4}$,

$$\begin{aligned}|D_t J_r| &\le 90 C_{0,1}(r)\tfrac{s(\sigma-s)}{\sigma}\,|V|\,|W|\,|T|(r)\\ &\le 90 C_{0,1}(r)\,s\,|V|\,|W|\,|T|(r),\end{aligned} \qquad |D_t \dot J_r| \le 50 C_{0,1}(r)\,|V|\,|W|\,|T|(r).$$

Proof. The claim is similar to 6.6, and so is the proof: We again have to find a differential equation for $U := D_t J_r$ to apply 6.5. By the usual laws of covariant differentiation and $\partial_r = J_r$, $\partial_t = J_t$, we have

$$\begin{aligned}D_t \ddot J_r = D_t D_s D_s \partial_r &= D_s D_t D_s \partial_r + R(J_t,T)\dot J_r\\ &= D_s D_s D_t \partial_r + D_s R(J_t,T)J_r + R(J_t,T)\dot J_r\\ &= D^2_{ss}U + (D_sR)(J_t,T)J_r + R(\dot J_t,T)J_r + 2R(J_t,T)\dot J_r.\end{aligned}$$

On the other hand, using the Jacobi equation,

$$-D_t\ddot J_r = D_t R(J_r,T)T = (D_tR)(J_r,T)T + R(U,T)T + R(J_r,\dot J_t)T + R(J_r,T)\dot J_t.$$

So with $A := -R(\,\cdot\,,T)T$, we have $\ddot U = AU + B$, where $\|A\| \le C_0|T|^2$. Let us assume that we consider some geodesic with $|T| = 1$ (as usual, the correct power of $|T|$ follows from a scaling argument). Then, because 6.3 holds for J_r as well as for J_t,

$$\begin{aligned}|B| &= |(D_sR)(J_t,T)J_r + (D_tR)(J_r,T)T\\ &\qquad + R(\dot J_t,T)J_r + 2R(J_t,T)\dot J_r + R(J_r,\dot J_t)T + R(J_r,T)\dot J_t|\\ &\le 2C_1|V|\,|W| + 15C_0\tfrac{1}{\sigma}|V|\,|W|.\end{aligned}$$

The claim on $D_t\dot J_r$ follows from $D_t\dot J_r = D_tD_s\partial_r = D_sD_t\partial_r + R(\partial_t,\partial_r)\partial_s$, *q. e. d.*

Lemma. *Notation as before, and A_λ as in 5.7. For varying vertices $p_i(t) \in M$, let x_t be the corresponding Karcher triangulations, and $V = dx\, v$. Then* **12.5**

$$A_\lambda \dot{x}_t|_\lambda = -\lambda^i \dot{X}_i|_{x_t(\lambda)}, \qquad A_\lambda \nabla_V \dot{x} = -v^i \dot{X}_i - \lambda^i \nabla^2_{V,\dot{x}} X_i - \lambda^i \nabla_V \dot{X}_i - A_v \dot{x}.$$

Proof. ad primum: Along $t \mapsto x_t(\lambda)$, consider the vector field $U(t) := \lambda^i X_i(t)|_{x_t(\lambda)}$. As has been stated in 5.2, this vector field vanishes for all r, so $\dot{U} = 0$. For those who believe, the shorthand proof is

$$\begin{aligned} 0 = D_t|_{t=0}(\lambda^i X_i(t)|_{x_t}) &= \lambda^i D_t|_{t=0}(X_i(t)|_{x_0}) + \lambda^i D_t|_{t=0}(X_i(0)|_{x_t}) \\ &= \lambda^i (\dot{X}_i + \nabla_{\dot{x}_0} X_i)|_{x_0}, \end{aligned} \tag{12.5a}$$

a rather uncommon use of the chain rule. For all others, this argument is justified by a calculation in coordinates: Let $\lambda^i X_i = U^j \partial_j$ and $\dot{x}(t) = \dot{x}^j(t)\partial_j$ be the needed coordinate representation. As $U^k(t) = 0$ for all k and all t, we have $\dot{U}^k(t) = 0$, and this is by usual Euclidean chain rule $\partial_t U^k(t, x^1(t), \dots, x^n(t)) + \partial_\ell U^k(t, x^1(t), \dots, x^n(t))\dot{x}^\ell(t)$. And if all U^k vanish, we can add $\Gamma^k_{\ell j} U^j \dot{x}^\ell$ without harm, which gives

$$0 = (\partial_t U^k + \partial_\ell U^k \dot{x}^\ell + \Gamma^k_{\ell j} U^j \dot{x}^\ell)\partial_k = D_t U + \nabla_{\dot{x}} U.$$

ad sec.: Differentiating 12.5a once more leads to

$$0 = \lambda^i \nabla_V \nabla_{\dot{x}} X_i + \nabla_V(\lambda^i \dot{X}_i) = \lambda^i \nabla^2_{V,\dot{x}} X_i + \lambda^i \nabla_{\nabla_V \dot{x}} X_i + v^i \dot{X}_i + \lambda^i \nabla_V \dot{X}_i + v^i \nabla_{\dot{x}} X_i,$$

q. e. d.

Proposition. *Situation as in 12.2, and let the variation of p_i be given by $\dot{p}_i(t) = dx\, w_i$ for some vector $w_i \in T\mathbb{R}^n$. Then for $u := \lambda^i w_i$, we have* **12.6**

$$|\dot{x} - dx\, u| \lesssim C'_{0,1} h |\dot{x}|, \qquad |\nabla_{dx\, v}\dot{x} - dx \nabla^{g^e}_v u| \lesssim C'_{0,1} h |u|\, |\dot{x}|.$$

Proof. ad primum: At the vertex e_i of the standard simplex, $d_{e_i} x\, w$ and the variation $\dot{p}_i$ agree. At another point $\lambda \in \Delta$,

$$d_\lambda x\, w_i = P d_{e_i} x\, w_i + O(C'_{0,1} h^2 |w_i|) = P\dot{p}_i + O(C'_{0,1} h^2 |w_i|)$$

by 6.24, which means $dx\, u = \lambda^i P \dot{p}_i + O(C'_{0,1} h |u|)$ by definition of u, and

$$\begin{aligned} \dot{x} &= -\lambda^i \dot{X}_i + O(C'_0 h^2 |\dot{x}|) && \text{by 12.5} \\ &= \lambda^i P \dot{p}_i + O(C'_0 h^2 |\dot{x}|) && \text{by 12.3 and 6.3.} \end{aligned}$$

ad sec.: The derivative of u is, by Euclidean calculus, just $\nabla^{g^e}_v u = v^i w_i$, so the latter term is $dx \nabla^{g^e}_v u = v^i dx\, w_i$. The covariant derivative $\nabla^2 X_i$ in 12.5 is estimated by 6.14, and the $\nabla_V \dot{X}$ term by 12.4, so we have

$$\begin{aligned} |A_\lambda \nabla_{dx\, v}\dot{x} - v^i dx\, w_i| &\le v^i |\dot{X}_i + dx\, w_i| + \lambda^i |\nabla^2_{V,\dot{x}} X_i| + \lambda^i |\nabla_V \dot{X}_i| + |A_v \dot{x}| \\ &\lesssim C_0 h^2 v^i |\dot{p}_i| + C'_{0,1} h |v|\, |\dot{x}|, \end{aligned}$$

q. e. d.

Discrete Vector Fields

12.7 **Definition.** For a piecewise barycentric mapping $x : r\mathfrak{K} \to M$, let

$$\bar{\mathsf{P}}\mathfrak{X}_x := TM|_{x(\mathfrak{K}^0)} = \bigsqcup_{p_i, i \in \mathfrak{K}^0} T_{p_i}M$$

be the disjoint union of all vertex tangent spaces. For $\bar{U} = (U_i) \in \bar{\mathsf{P}}\mathfrak{X}_x$, define a piecewise interpolation: It induces a variation of x by defining $x_t[\bar{U}]$ to be the piecewise barycentric mapping with respect to vertices $\exp_{p_i} tU_i$ (where we keep t so small that $x(\lambda)$ and $x_t(\lambda)$ stay in a convex neighbourhood of each other). We call $\bar{U} \mapsto U := \dot{x}_t[\bar{U}]|_{t=0}$ the **P^1-interpolation** of $\bar{U}$ and

$$\mathsf{P}^1\mathfrak{X}_x := \{U : \bar{U} \in \bar{\mathsf{P}}\mathfrak{X}_x\}$$

the set of piecewise smooth, globally continuous test vector fields.

12.8 **Observation.** As a finite sum of vector spaces with scalar products g and g^e, $\bar{\mathsf{P}}\mathfrak{X}_x$ carries the natural inner products

$$\ell^2 g\langle \bar{V}, \bar{W} \rangle = \sum_i g\langle V_i, W_i \rangle, \qquad \ell^2 g^e \langle \bar{V}, \bar{W} \rangle = \sum_i g^e \langle v_{\mathfrak{c},i}, w_{\mathfrak{c},i} \rangle,$$

whereas $\mathsf{P}^1\mathfrak{X}_x$ has the scalar products $\mathsf{L}^2 g$ and $\mathsf{L}^2 g^e$ that are induced from $\mathsf{L}^2\mathfrak{X}(TM)$:

$$\mathsf{L}^2 g\langle V, W \rangle = \int_{x(r\mathfrak{K})} g\langle V, W \rangle, \qquad \mathsf{L}^2 g^e \langle V, W \rangle = \sum_{\mathfrak{c} \in \mathfrak{K}^n} \int_{r\mathfrak{c}} g^e \langle \bar{v}, \bar{w} \rangle,$$

where $V = dx\,\bar{v}$ and $W = dx\,\bar{w}$. As both are isomorphic finite-dimensional vector spaces, all these norms are equivalent. The equivalence constants between $\ell^2 g$ and $\ell^2 g^e$ and between $\mathsf{L}^2 g$ and $\mathsf{L}^2 g^e$ are the ones from 6.17a and 7.3a, whereas the equivalence constant between $\mathsf{L}^2 g^e$ and $\ell^2 g^e$ depends on the maximal and minimal simplex volume.

12.9 **Definition.** Situation as in 12.2, and $\bar{U} \in \bar{\mathsf{P}}\mathfrak{X}_x$. Inside every simplex $\mathfrak{c} \in \mathfrak{K}^m$, the vector U_i, $i \in \mathfrak{c}$, can be represented as $U_i = d_{ri}x_{\mathfrak{c}}\, w_i^{\mathfrak{c}}$ (without any summation). Define a piecewise linear, globally discontinuous vector field $u|_{r\mathfrak{c}} := \lambda^i w_i^{\mathfrak{c}}$ and a piecewise smooth vector field $\bar{u}$ by requiring $dx\,\bar{u} = U$ everywhere.

12.10 **Conclusion.** *By definition, $dx\nabla_v^{x^*g}\bar{u} = \nabla_{dx\,v}\dot{x}_t[\bar{U}]$. From 12.6, we hence know that*

$$|u - \bar{u}| \lesssim C'_{0,1}h|u|, \qquad |\nabla_v^{x^*g}\bar{u} - \nabla_v^{g^e}u| \lesssim C'_{0,1}h|v||u|,$$

where all norms are $|\cdot|_{x^*g}$ norms. The same estimates hold for the jump $[dx\,u]_{\mathfrak{f}} = dx_{\mathfrak{c}}\lambda^i w_i^{\mathfrak{c}} - dx_{\mathfrak{c}'}\lambda^i w_i^{\mathfrak{c}'}$ of u across a facet $\mathfrak{f} = \mathfrak{c} \cap \mathfrak{c}'$.

Area Differentials

Observation. Situation as in 12.2. If the vertices p_i of a Karcher triangulation vary smoothly with velocity $(U_i) \in \bar{\mathsf{P}}\mathfrak{X}_x$, the area change of $S = x(r\bar{\mathfrak{K}})$ is also smooth, hence has a differential **12.11**

$$d\,\mathrm{vol}_{\bar{\mathfrak{K}}} : \bar{\mathsf{P}}\mathfrak{X}_x \to \mathbb{R}. \tag{12.11a}$$

The volume is additive in $\mathfrak{s} \in \bar{\mathfrak{K}}^n$, and the variations of different vertices are linearly independent, so it suffices to compute the differential $d\,\mathrm{vol}^i_{\mathfrak{s},g} : T_{ri}r\mathfrak{K} \to \mathbb{R}$ of $|x(r\mathfrak{s})|_g$ with respect to the variation of $x(ri)$, $i \in \mathfrak{s}$. Correspondingly, let $d\,\mathrm{vol}^i_{\mathfrak{s},g^e}$ be the analogous differential of $|\mathfrak{s}|_{g^e}$.

Remark. We do not think that a notational distinction between this area differential and the volume form from 9.6 is neccessary. For readers who disagree, we remark that in 12.11a, the d denotes a differential and is hence written in italics, whereas it is upright as part of the volume form dvol.

Proposition. *Situation as in 12.2. Then* $|d\,\mathrm{vol}^i_{\mathfrak{s},g} - d\,\mathrm{vol}^i_{\mathfrak{s},g^e}| \lesssim C'_{0,1}h\,|\mathfrak{s}|_{g^e}$. **12.12**

Proof. By 12.1, $d\,\mathrm{vol}^i_{\mathfrak{s},g}(w) = -\int \mathrm{div}^S Z$, where $Z = \dot{x}$ is induced by the vertex variation $\dot{p}_i = dx\,w$. If $\tilde{v}_j$ form a g-orthogonal basis of $Tr\mathfrak{s}$, this is $\int\langle\nabla_{dx\,\tilde{v}_j}\dot{x}, dx\,\tilde{v}_j\rangle$. By the comparison of volume elements for g and g^e in 3.20,

$$d\,\mathrm{vol}^j_{\mathfrak{s},g}(w) = -\int\limits_{x(r\mathfrak{s}),g} g\langle\nabla_{dx\,\tilde{v}_j}\dot{x}, dx\,\tilde{v}_j\rangle = -(1 + O(C'_0h^2))\int\limits_{x(r\mathfrak{s}),g^e} g\langle\nabla_{dx\,\tilde{v}_j}\dot{x}, dx\,\tilde{v}_j\rangle,$$

and, noting that there is a g^e-orthonormal basis v_j of $Tr\mathfrak{s}$ with $|v_j - \tilde{v}_j| \lesssim C'_0h^2$ by 3.6, the integrand is

$$\begin{aligned} g\langle\nabla^g_{dx\,\tilde{v}_j}\dot{x}, dx\,\tilde{v}_j\rangle &= x^*g\langle\nabla^{g^e}_{\tilde{v}_j}u, \tilde{v}_j\rangle + O(C'_{0,1}h|u|) && \text{by 12.6} \\ &= x^*g\langle\nabla^{g^e}_{v_j}u, v_j\rangle + O(C'_{0,1}h|u|) + O(C'_0h^2\|\nabla u\|) \\ &= g^e\langle\nabla^{g^e}_{v_j}u, v_j\rangle + O(C'_{0,1}h|u|) + O(C'_0h^2\|\nabla u\|), \end{aligned}$$

and the last right-hand-side term is $\mathrm{div}^{(r\mathfrak{s},g^e)}\,u$, *q. e. d.*

Remark. **⟨a⟩** It is common knowledge that $d\,\mathrm{vol}^i_\Delta = d\lambda^i|\Delta|$, proven by inserting $\mathrm{div}\,u = d\lambda^i(w)$ for $u = \lambda^i w$ into 12.1. Classically, one says for triangles (Polthier 2002, eqn. 4.3) that the gradient of the area functional with respect to vertex variations is the $\frac{\pi}{2}$ rotation of the opposite edge vector, which is exactly $(d\lambda^i)^\sharp$. **12.13**

One has to take care to transfer this to the subsimplex situation, because $\mathrm{div}^{(r\mathfrak{s},g^e)}\,u = 0$ if w is perpendicular to $r\mathfrak{s}$, so one needs a form $d\lambda^i_\mathfrak{s}$ that acts like $d\lambda^i$ on $Tr\mathfrak{s}$ and vanishes on $Tr\mathfrak{s}^\perp$. For example, if $r\mathfrak{s} = \mathrm{conv}(e_0, \ldots, e_n)$ and $v_0 = \mathrm{grad}\,\lambda^0 \in T\Delta$,

$$d\lambda^i_\mathfrak{s}(w) = d\lambda^i\Big(w - \frac{\langle w_0, w\rangle}{|v_0|^2}v_0\Big) = w^i - \tfrac{w^0}{|d\lambda^0|^2}\langle d\lambda^i, d\lambda^0\rangle.$$

It is easier to transfer the gradient $v_{\mathfrak{s}}^i$ of λ^i in $r\mathfrak{s}$ to $Tr\mathfrak{c}$, as its vector components stay the same: $\operatorname{div}^{(\mathfrak{s},g^e)} u = g^e\langle v_{\mathfrak{s}}^i, w\rangle$. By 3.2a, this $v_{\mathfrak{s}}^i$ is characterised as the vector in $Tr\mathfrak{s}$ that is perpendicular to the facet $\mathfrak{s} \setminus \{i\}$ opposite i with lengths h_i^{-1}, the reciprocal of $r\mathfrak{s}$'s height above $\mathfrak{s} \setminus \{i\}$ from 3.2a.

⟨b⟩ For computational purposes, 12.12 is insatisfactory, as only $d\operatorname{vol}_{\mathfrak{s},g^e}(u)$ is numerically accessible, not $d\operatorname{vol}_{\mathfrak{s},g^e}(\bar{u})$. But this is only an easy combination with 12.10, which will be spelled out for the area gradients in the following paragraphs.

Area Gradients

12.14 Observation. The area differentials $d\operatorname{vol}_{\mathfrak{s},g}$ and $d\operatorname{vol}_{\mathfrak{s},g^e}$ can be expressed as gradient with respect to different norms on $\bar{\mathsf{P}}\mathfrak{X}_x$. The gradients with respect to $\ell^2 g$ and $\ell^2 g^e$ correspond to the "mean curvature vector" of Polthier (2002), whereas the gradients with respect to or $\mathsf{L}^2 g$ and $\mathsf{L}^2 g^e$ give the construction from Dziuk (1991).

12.15 Corollary. *If $H_{\ell^2 g} \in T_{p_i}M$ is the gradient of $|x(r\mathfrak{s})|_g$ with respect to a variation of p_i, and $H_{\ell^2 g^e} = |\mathfrak{s}|_{g^e} \operatorname{grad}\lambda^i$ is the corresponding gradient of $|\mathfrak{s}|_{g^e}$, then $|dx\, H_{\ell^2 g^e} - H_{\ell^2 g}| \lesssim C'_{0,1} h |H_{\ell^2 g^e}|$.*

12.16 Definition. The **discrete mean curvature vector** $H_{\mathsf{L}^2 g} \in \mathsf{P}^1\mathfrak{X}_x$ is the solution of $\mathsf{L}^2 g\langle H_{\mathsf{L}^2 g}, V\rangle = \operatorname{dvol}_{\bar{\mathfrak{K}},g}(V)$ for all $V \in \mathsf{P}^1\mathfrak{X}$. The approximate mean curvature vector $H_{\mathsf{L}^2 g^e} \in \mathsf{P}^1\mathfrak{X}_x$ is the solution of $\mathsf{L}^2 g^e\langle H_{\mathsf{L}^2 g^e}, V\rangle = d\operatorname{vol}_{\bar{\mathfrak{K}},g^e}(V)$ for all $V \in \mathsf{P}^1\mathfrak{X}$.

12.17 Observation. The differentials of the right-hand sides can be represented as L^2 products of linear maps: If vectors $Z_i = dx\, w_i \in T_{p_i}M$ induce a variation $\dot{x} = dx\,\bar{z}$ of x, and if we define $z = \lambda^i w_i$, as well as $d\lambda : e_i \mapsto \operatorname{grad}\lambda^i$, then

$$\mathsf{L}^2 g\langle H_{\mathsf{L}^2 g}, Z\rangle = \langle dx, \nabla^{x^* g}\bar{z}\rangle_{\mathsf{L}^2 g(Tr\mathfrak{K}\otimes x^*TM)}, \tag{12.17a}$$

$$\mathsf{L}^2 g^e\langle H_{\mathsf{L}^2 g^e}, Z\rangle = \langle d\lambda, \nabla^{g^e} z\rangle_{\mathsf{L}^2 g^e(Tr\mathfrak{K}\otimes x^*TM)}. \tag{12.17b}$$

To see 12.17a, we take an orthonormal basis $E_i = dx\, y_i$ in $\operatorname{div}^S Z = \langle \nabla_{E_i} Z, E_i\rangle$. Then we have an integrand of the form $\langle \alpha(y_i), \beta(y_i)\rangle$ for two linear maps α, β. A computation in coordinates easily shows that this is $\langle \alpha, \beta\rangle$.

For 12.17b, the computation is even simpler: $z = \lambda^i v_i$ (without summation) for the gradient v_i of λ^i has derivative $\nabla z = v_i \otimes d\lambda^i$, which maps e_k to v_k, so $\operatorname{div} z = v_i^i = \langle \nabla_{e_i} z, \operatorname{grad}\lambda^i\rangle$.

12.18 Proposition. *Situation as before. Then $|H_{\mathsf{L}^2 g} - H_{\mathsf{L}^2 g^e}|_{\mathsf{L}^2} \lesssim C'_{0,1} h(1 + h|H_{\mathsf{L}^2 g^e}|_{\mathsf{L}^2})$.*

Proof. Similar to 10.13: The functionals on the right-hand side of 12.17b only differ by a factor of $1 + O(C'_{0,1}h|S|)$, and the bilinear forms on the left fulfill $|\mathsf{L}^2 g\langle U,V\rangle - \mathsf{L}^2 g^e\langle U,V\rangle| \lesssim C'_0 h^2 |U|\,|V|$, so

$$\begin{aligned}
|H_{\mathsf{L}^2 g} - H_{\mathsf{L}^2 g^e}|^2_{\mathsf{L}^2 g} &= \mathsf{L}^2 g\langle H_{\mathsf{L}^2 g} - H_{\mathsf{L}^2 g^e}, H_{\mathsf{L}^2 g} - H_{\mathsf{L}^2 g^e}\rangle \\
&\le |\mathsf{L}^2 g\langle H_{\mathsf{L}^2 g}, H_{\mathsf{L}^2 g} - H_{\mathsf{L}^2 g^e}\rangle - \mathsf{L}^2 g^e\langle H_{\mathsf{L}^2 g^e}, H_{\mathsf{L}^2 g} - H_{\mathsf{L}^2 g^e}\rangle| \\
&\qquad + |(\mathsf{L}^2 g - \mathsf{L}^2 g^e)\langle H_{\mathsf{L}^2 g^e}, H_{\mathsf{L}^2 g} - H_{\mathsf{L}^2 g^e}\rangle| \\
&\lesssim |(d\operatorname{vol}_{\bar{\mathfrak{K}},g} - d\operatorname{vol}_{\bar{\mathfrak{K}},g^e})(H_{\mathsf{L}^2 g} - H_{\mathsf{L}^2 g^e})| \\
&\qquad + C'_0 h^2 |H_{\mathsf{L}^2 g^e}|\,|H_{\mathsf{L}^2 g} - H_{\mathsf{L}^2 g^e}| \\
&\lesssim C'_{0,1} h |H_{\mathsf{L}^2 g} - H_{\mathsf{L}^2 g^e}|(1 + h|H_{\mathsf{L}^2 g^e}|),
\end{aligned}$$

q. e. d.

13. The Manifold-Valued Dirichlet Problem

Goal. Let $N\gamma$ and Mg be two smooth compact Riemannian manifolds of dimension n and m. We have seen in 8.8 that a sufficiently dense generic point set in N gives us an almost-isometry $N \to r\mathfrak{K}$, and in 7.14 that smooth functions $r\mathfrak{K} \to M$ can be interpolated by piecewise barycentric mappings $r\mathfrak{s} \to M$, $\mathfrak{s} \in \mathfrak{K}^n$, that are globally continuous. Now it is a natural attempt to consider Galerkin approximations e. g. to the Dirichlet problem in $\mathsf{H}^1(N, M)$: Suppose N has a Lipschitz boundary that can be resolved by the triangulation. Given a smooth harmonic function $y : N \to M$, let $x : r\mathfrak{K} \to M$ be the function that minimises the Dirichlet energy among all functions that are piecewise barycentric mappings and agree with y at the boundary vertices. How well is y then approximated by x?

A popular example for such functions y is the minimal submanifold problem: If $S \subset M$ is a minimal submanifold, then the identity mapping $S \to M$ is harmonic (Jost 2011, eqn. 4.8.12). One already sees that to keep the notation consistent among the chapters, we denote smooth harmonic functions etc. by y and the interpolation in the sense of 7.6 by x.

Remark. The results presented in this section are generally the same as in Grohs *et al.* (2013), but although their interpolation procedure is the same as ours (but including higher-order interpolation, see 7.16d), their functional analytic approach is slightly different: They do not use the distance measure ρ_1 and its corresponding Poincaré inequality, but consider functionals that are (in a weak sense) convex along geodesic homotopies.

The General Galerkin Approach

Definition. The **Dirichlet energy** of $y \in \mathsf{C}^1(N, M)$ is **13.1**

$$\operatorname{Dir}(y) := \frac{1}{2} \int_N |dy|^2.$$

Recall that the norm on $TN \otimes y^*TM$ induced by γ and g has a representation in local coordinates u^α for N and v^i for M as $|dy|^2 = \gamma^{\alpha\beta} g_{ij} y^i_{,\alpha} y^j_{,\beta}$.

Proposition (**Dirichlet principle**, Jost 2011, eqn. 8.1.13). *$y \in \mathsf{C}^1(N, M)$ is a critical point for* Dir *with given boundary values iff $\langle dy, \nabla V\rangle = 0$ for all compactly supported vector fields $V \in \mathfrak{X}(y^*TM)$.* **13.2**

Proof. Every compactly supported vector field V along y induces a variation $y_t := \exp_y(tV)$ of y that does not change the boundary values. By the usual calculus of variations, $\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0} \operatorname{Dir}(y_t) = \langle dy, D_t dy\rangle$. So we only have to compute the t-derivative of $dy_t = y^i_{,\alpha}(t)\partial_i \otimes du^\alpha$ with the notation from 1.6. As $D_t = \nabla_V$ in the usual sloppy notation and $D_t(du^\alpha) = 0$, we have at the origin of normal coordinates in N

$$\begin{aligned} D_t dy_t = D_t(y^i_{,\alpha}\partial_i) \otimes du^\alpha &= V^i_{,\alpha}\partial_i \otimes du^\alpha + y^i_{,\alpha}\nabla_V \partial_i \otimes du^\alpha \\ &= V^i_{,\alpha}\partial_i \otimes du^\alpha + y^i_{,\alpha} V^j \Gamma^k_{ij}\partial_k \otimes du^\alpha \end{aligned}$$

On the other hand, regarding 1.6c,

$$\nabla V = \nabla_{\partial_\alpha}(V^i\partial_i) \otimes du^\alpha = V^i_{,\alpha}\partial_i \otimes du^\alpha + V^i y^j_{,\alpha}\Gamma^k_{ij}\partial_k \otimes du^\alpha,$$

q. e. d.

13·3 **Definition.** ⟨**a**⟩ For $N = r\mathfrak{K}$, let $\mathsf{P}^1(N,M)$ be the space of piecewise barycentric, globally continuous mappings (which obviously depends on the simplicial structure and not only on its manifold structure, but we do not explicitely denote this). Obviously the domain of Dir can be extended to include also piecewise smooth mappings, so every $v \in \mathsf{P}^1(N,M)$ has finite Dirichlet energy.

⟨**b**⟩ For $a, b \in M$, denote $a \sim b$ if there is a unique shortest geodesic $a \rightsquigarrow b$ in M. Say that $x, y : N \to M$ are **close** of $x(p) \sim y(p)$ for almost $p \in N$.

⟨**c**⟩ On $\mathsf{C}^1(N,M)$, define the L^r metric $\rho_{0,r}$ and the $\mathsf{H}^{1,r}$ "distance measure" $\rho_{1,r}$ by

$$\rho_{0,r}(x,y) := \Big(\int_N \mathbf{d}^r(x(p), y(p))\,\mathrm{d}p\Big)^{1/r},$$

$$\rho_{1,r}(x,y) := \Big(\int_N \begin{Bmatrix} \|d_p x - P d_p y\|^r & \text{if } x(p) \sim y(p) \\ \infty & \text{else} \end{Bmatrix} \mathrm{d}p\Big)^{1/r},$$

with the usual modification for $r = \infty$. We abbreviate $\rho_0 := \rho_{0,2}$ and $\rho_1 := \rho_{1,2}$. Let $\mathsf{H}^1(N,M)$ be the completion of $\mathsf{C}^1(N,M)$ with respect to $\rho_0 + \rho_1$.

13·4 **Lemma.** *Let γ be a closed curve in a convex region of M, and let P be the parallel transport along γ. If $C_0 L^2(\gamma) < \pi^2$, then $\|P_\gamma - \mathrm{id}\,\| \le \frac{1}{2} C_0 L^2(\gamma)$.*

Proof. The parallel transport is continuous with respect to L^∞ convergence in the space of loops $[0;1] \to M$, so it suffices to show the claim for smooth γ. To fix notation, let us say $\gamma : [0;1] \to M$, $\gamma(0) = \gamma(1) = p$. As this curve lies entirely in a convex region, it can be represented as $\gamma(t) = \exp_p V(t)$ with a vector field $V : [0;1] \to T_p M$. Define a homotopy $c(s,t) := \exp_p sV(t)$ between γ and the lazy loop. Denote the s-parameter lines by c_t and the t-parameter lines by γ_s. By 7.8, we have

$$P_\gamma - \mathrm{id} = \int_0^1 \int_0^1 P_s^{1,t} R(\dot c_t, \dot\gamma_s) P_s^{t,0}\,\mathrm{d}t\,\mathrm{d}s$$

The coordinate vectors $\dot c_t = \partial_s c$ and $\dot\gamma_s = \partial_t c$ can be explicitely computed: $\partial_s c(s,t) = P_t^{s,0} V(t)$ because c_t is a geodesic with initial velocity $V(t)$, and $\partial_t c(s,t) = J_t(s)$ for a Jacobi field J_t along c_t with values $J_t(0) = 0$, $J_t(1) = \dot\gamma(t)$ and $\dot J_t(0) = \dot V(t)$ by 1.16 (any two of these conditions determine J_t uniquely). So we have $|\dot c_t| = |V|$ and $|\dot\gamma_s| \le |\dot\gamma|$ because the Jacobi field grows monotonously in s, see the proof of 6.3. Because the parallel transports along γ_s are isometries, we obtain

$$\|P_\gamma - \mathrm{id}\,\| \le \int\!\!\int C_0 |\dot c_t|\,|\dot\gamma_s| \le C_0 L(\gamma) \max|V|.$$

And $|V(t)|$ is the distance from p to $\gamma(t)$, which cannot be larger than $\frac{1}{2}L(\gamma)$, *q. e. d.*

13·5 **Proposition ("triangle inequality").** *If $x, y, z \in \mathsf{C}^1(N,M)$ with $\rho_{0,\infty}(x,y) + \rho_{0,\infty}(y,z) \le \ell$, then*

$$\rho_1(x,z) \le \rho_1(x,y) + \rho_1(y,z) + \tfrac{1}{2} C_0 \ell^2 \,\mathrm{Dir}^{1/2}(z)$$

Proof. Pointwise, we have

$$\|dx - P^{x,z}dz\| \leq \|dx - P^{x,y}dy\| + \|dy - P^{y,z}dz\| + \|P^{x,z} - P^{x,y}P^{y,z}\|\,\|dz\|.$$

The difference between the parallel transports is the holonomy of the loop $x \rightsquigarrow y \rightsquigarrow z \rightsquigarrow x$, which is smaller than $\frac{1}{2}C_0\ell^2$ by 13.4. Now the claim is a simple application of Minkowski's inequality in $\mathsf{L}^2(M, \mathbb{R})$, *q. e. d.*

Remark. ⟨**a**⟩ That $\rho_{0,r}$ is indeed a metric is proven with the same argument as the usual Minkowsky inequality, see e. g. ALT (2006, lemma 1.18). In contrast, 13.5 gives only a distorted triangle inequality for ρ_1. Nevertheless, $\mathsf{H}^1(N, M)$ can be defined as the completion of $\mathsf{C}^1(N, M)$ with respect to $\rho_0 + \rho_1$, because this term does not disturb the usual completion construction for metric spaces, see e. g. ALT (2006, no. 0.20), and we never need to use the triangle inequality explicitely. **13.6**

⟨**b**⟩ Because of

$$|\mathrm{Dir}(x) - \mathrm{Dir}(y)| \lesssim \rho_1^2(x, y)$$

(the hidden constant comes from the comparison of $|dx - Pdy|$ with $\|dx - Pdy\|$), the definition above ensures that every $u \in \mathsf{H}^1(N, M)$ indeed has finite Dirichlet energy. Nevertheless, not all functions with finite Dirichlet energy are contained in our definition of $\mathsf{H}^1(N, M)$, but only those that are limits of smooth function sequences. The usual counterexample is the function $u \mapsto \frac{u}{|u|}$ from the unit ball to the unit sphere minimises Dir in dimension $m \geq 3$ and larger (HILDEBRANDT *et al.* 1977, sec. 6; general regularity theory is given in SCHOEN and UHLENBECK 1982). So the usual definition of $\mathsf{W}^{1,2}(N, M)$ as

$$\{y \in \mathsf{W}^{1,2}(N, \mathbb{R}^k) \colon y(p) \in M \text{ a. e.}\},$$

where M is embedded in $\mathbb{R}^k$, neccessarily has the drawback that $\mathsf{C}^1(N, M)$ is not dense in $\mathsf{W}^{1,2}(N, M)$ in dimension 3 and larger (SCHOEN and UHLENBECK 1983, sec. 4). In allusion to MEYERS and SERRIN (1964), JOST (1988, p. 266) states this as $\mathsf{H}^1(N, M) \neq \mathsf{W}^{1,2}(N, M)$. By our use of $\mathsf{H}^1(N, M)$, we restrict ourselves to functions that *can* be smoothly approximated. This space is well-suited for approximation questions, but the wrong one to show existence of solutions. For an overview over difficulties and pitfalls of the harmonic mapping problem, we refer to the survey of JOST (1988).

⟨**c**⟩ Consequently, two functions $x, y \in \mathsf{C}^1(N, M)$ are close iff $\rho_1(x, y)$ is finite.

⟨**d**⟩ If $x, y \in \mathsf{C}^1(N, M)$ are close, the geodesics $x(p) \rightsquigarrow y(p)$ give rise to a **geodesic homotopy** $h : x \rightsquigarrow y$, i. e. a smooth mapping $N \times [0; 1] \to M$, $(p, s) \mapsto h_s(p)$, such that $h_0 = x$ and $h_1 = y$ and $s \mapsto h_s(p)$ is a geodesic for any p. It minimises the energy

$$E(h) := \int_N \int_0^1 |\tfrac{\mathrm{d}}{\mathrm{d}s} h_s(p)|^2 \,\mathrm{d}s\,\mathrm{d}p$$

over all homotopies in the same class (JOST 2011, lemma 8.5.1). In fact, $E(h) = \rho_0^2(x, y)$ if h is the geodesic homotopy $x \rightsquigarrow y$ and x, y are close, because $|\frac{\mathrm{d}}{\mathrm{d}s} h_s(p)|$ is independent of s in this case. This is also called the L^2-**width** of the geodesic homotopy (KOKAREV 2013, the older literature mostly uses the L^∞-width $\rho_{0,\infty}$ from SIEGEL and WILLIAMS 1984).

13.7 **Proposition (Poincaré inequality).** *Suppose ∂N is smooth, all Weingarten maps of ∂N with respect to N are bounded by $\|W_\nu\| \leq \kappa$ everywhere, and no point in N has distance larger than r to ∂N. Then*

$$|f|^2_{\mathsf{L}^2(N)} \leq 2rC_N\,|f|^2_{\mathsf{L}^2(\partial N)} + 4r^2\,|df|^2_{\mathsf{L}^2(N)} \qquad \textit{with } C_N := \mathrm{e}^{r\max(\kappa,\sqrt{C_0})}.$$

Proof. Without regarding the constants, it would be very easy to reduce this case to the Poincaré inequality for vanishing boundary values 2.10c. As a very personal attitude, we would like to circumvent the contradiction argument there. Let us first consider a positive C^1 function $g : N \to \mathbb{R}$.

As Mantegazza and Mennucci (2003, prop. 3.5) have shown, the distance field $\mathbf{d} := \mathbf{d}(\,\cdot\,, \partial N)$ is C^1 except on an $(n-1)$-dimensional set S (in fact, they deal with the distance field of an arbitrary submanifold K for boundaryless N, but the case of $K = \partial N$ is also possible). By the coarea formula (Evans and Gariepy 1992, thm. 3.4.2), $\int g$ can be computed by integration over the t-level sets $N^t := \{p \in N \setminus S\colon \mathbf{d} = t\}$, where points in S can be omitted because it is a null set:

$$\int\limits_N g = \int\limits_0^r \Big(\int\limits_{N^t} g\Big)\,\mathrm{d}t$$

(note that $|\operatorname{grad}\mathbf{d}| = 1$, so there is no additional weighting factor). There are smooth homotopies h^t retracting each level set N^t to the boundary, defined on a subset N^t_0 of ∂N, with $h^t_0 = \mathrm{id}$ and $h^t_t(N^t_0) = N^t$, following the gradient field of $\mathbf{d}$. The intermediate mappings h^t_s cover sets $N^t_s \subset N^s$, and the N^t integral can be computed by the fundamental theorem of calculus for $a(s) = \int_{N^t_s} g$ as $a(t) = a(0) + \int_0^s \dot a(s)\,\mathrm{d}s$. The derivative of the integrals is composed of the integrand's change along s-lines and the changing of the volume element:

$$\frac{\mathrm{d}}{\mathrm{d}s}\int\limits_{N^t_s} g = \int\limits_{N^t_s} dg(\dot h^t_s) + \int\limits_{N^t_s} g\operatorname{tr} W_s,$$

where $\dot h^t_s$ denotes the s-derivative of the homotopy and W_s is the Weingarten operator of the distance set N^s from 1.25b. Here we have used that $\tau(s) := \operatorname{tr} W_s$ is the derivative of the volume element (Karcher 1989, eqn. 1.5.4 or, in a more general setup, Delfour and Zolesio 2011, eqn. 9.4.17). Now by 1.25b, the function $\tau(s)$ obeys $\dot\tau \leq C_0 - \tau^2$ with initial value $\tau(0) \leq \kappa$ by assumption. This differential inequality delivers us $\tau \leq K := \max(\kappa, \sqrt{C_0})$. (Note that not the absolute value of τ can be bounded, only τ itself—in fact $\tau \to -\infty$ where dh^t_s becomes singular.) So we have

$$\frac{\mathrm{d}}{\mathrm{d}s}\int\limits_{N^t_s} g \leq \int\limits_{N^t_s} |dg| + K\int\limits_{N^t_s} g$$

or $\dot a \leq b + Ka$ with b being the integral over $|dg|$. This differential inequality has the supersolution $a(0)\mathrm{e}^{Kt} + \int_0^t b(s)\,\mathrm{d}s$, which hence is a bound for $a(t)$. That means

$$\int_N g \le \int_0^r a(t)\,\mathrm{d}t \le r\mathrm{e}^{Kr} \int_{\partial N} g + r \int_N |dg|.$$

Now for $f \in \mathsf{H}^1$, let $g = f^2$. The latter term becomes $|d(f^2)| = 2f|df|$, and its integral is estimated by $2|f||df|$ by Hölder. Then apply Young's inequality $uv \le \delta u^2 + \frac{1}{4\delta}v^2$ with $\delta = \frac{1}{4r}$ to obtain

$$r|d(f^2)|_{\mathsf{L}^2} \le \tfrac{1}{2}|f|^2_{\mathsf{L}^2} + 2r^2|df|^2_{\mathsf{L}^2},$$

q. e. d.

Corollary. *Situation as before. Suppose $x, y \in \mathsf{H}^1(N, M)$ are close maps with $\mathbf{d}(x, y)(p) \le \varepsilon$ for all boundary points $p \in \partial N$. Defining $C'_N := C_N\sqrt{r}$, it holds $\rho_0(x, y) \lesssim C'_N\varepsilon + r\rho_1(x, y)$.* **13.8**

Proof. Consider the function $f := \mathbf{d}(x, y) : N \to \mathbb{R}$. It has differential $df(V) = g\langle Y_y, (dx - Pdy)V\rangle$ by 7.7, and hence $|df| \le \|dx - Pdy\|$, *q. e. d.*

Remark. **⟨a⟩** The Poincaré inequality in the form above also holds for differential forms, with the covariant derivative on the right-hand side. In fact, consider $u \in \mathsf{H}^1\Omega^k$ and $f := |u|$. Then, because ∇ is metric, $|df| = \langle\nabla u, u\rangle/|u| \le |\nabla u|$.

⟨b⟩ By the same method of proof, the Poincaré inequality of Kappeler *et al.* (2003, thm. 0.4) can be significantly shortened. They prove that if N, M are closed and compact and M has negative sectional curvature, then any two homotopic mappings $x, y \in \mathsf{C}^1(N, M)$ satisfy $\rho_0(x, y) \lesssim 1 + \mathrm{Dir}(x)^{1/2} + \mathrm{Dir}(y)^{1/2}$.

Situation. For simplicity, we assume $N = r\mathfrak{K}$ (otherwise, concatenate the results below with 10.12). Suppose the metric γ of N is piecewise $(\frac{1}{2}, h)$-small, so that we can omit the fullness parameter. If $y : N \to M$ is a smooth function, we assume that its piecewise barycentric interpolation x is close to y, which is the case for small enough C_0h^2. **13.9**

Proposition (Galerkin orthogonality). *Situation as in 13.9. Let $y \in \mathsf{H}^1(N, M)$ be a critical point of* Dir *with respect to compactly supported variations, and let $x \in \mathsf{P}^1(N, M)$ be a critical point of* Dir *with respect to variations $W \in \mathsf{P}^1\mathfrak{X}_x$ as in 12.7 that vanish at boundary vertices, such that $x(p_i) = y(p_i)$ on all boundary vertices. Then if x and y are close,* **13.10**

$$\langle\!\langle dx - Pdy, \nabla W\rangle\!\rangle = 0 \qquad \textit{for all } W \in \mathsf{P}^1\mathfrak{X}_x, W|_{\partial N} = 0.$$

Proof. Because x and y are close, the parallel transport induces a bundle isomorphism $x^*TM \to y^*TM$. Because piecewise smooth vector fields are in H^1 and the variation on the whole boundary vanishes if it vanishes on the vertices (recall that the barycentric mapping on a subsimplex only depends on the vertices of this subsimplex), we obtain that PW is an admissible variation field along y for all $W \in \mathsf{P}^1\mathfrak{X}_x$. Therefore $\langle\!\langle Pdy, \nabla W\rangle\!\rangle = 0$ by 13.2, and similarly $\langle\!\langle dx, \nabla W\rangle\!\rangle = 0$, *q. e. d.*

13.11 **Corollary.** *Situation as before. Then* $|dx - Pdy|_{\mathsf{L}^2} \leq \inf_{W \in \mathsf{P}^1\mathfrak{X}_x} |dx - Pdy - \nabla W|_{\mathsf{L}^2}$, *because* $|dx - Pdy|^2 = \langle dx - Pdy - \nabla W, dx - Pdy\rangle \leq |dx - Pdy - \nabla W|\,|dx - Pdy|$.

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13.12 **Lemma.** *Situation as in 13.9. Let* V *be an* H^1 *vector field along* x. *Then for any* $i \in \mathfrak{K}^0$, *there is a variation* $p_i(t)$ *of* $x(ri)$ *such that the vector field* $\dot{X}_i$ *from 12.5 satisfies* $|V - \dot{X}_i|_{\mathsf{L}^2(s_i)} \lesssim h(1 + C_{0,1}h)|\nabla V|_{\mathsf{L}^2(s_i)} + C_{0,1}h^2|V|_{\mathsf{L}^2(s_i)}$, *where* s_i *denotes the star of* ri, *i. e. the union of all simplices* $r\mathfrak{s}$ *with* $i \in \mathfrak{s}$.

Proof. Abbreviate $p := x(ri)$ and write X instead of X_i for the time of the proof. In the first step, let us consider a smooth vector fields V. Choose the variation of $x(ri)$ such that $\dot{p}(0) = V|_p$. Then by 12.5, $\dot{X} = V$ at p and hence

$$(V - \dot{X})|_q = \int\limits_\gamma P\nabla_{\dot\gamma}(V - \dot{X}) = \int\limits_\gamma P\nabla_{\dot\gamma} V - \int\limits_\gamma P\nabla_{\dot\gamma}\dot{X},$$

where $\gamma : p \rightsquigarrow q$. The second integral should disappear in the result. By 12.4, we have $|\nabla_{\dot\gamma}\dot{X}| \lesssim C_{0,1}h|\dot\gamma|\,|\dot{p}|$. So we end up with $|\dot{p}|$, which is a point evaluation of V and hence undesired. Express $\dot{p} = V|_p = PV|_{\gamma(t)} - \int P\nabla_{\dot\gamma}V$, where the integral only runs from 0 to t. Then

$$|V - \dot{X}|_{g|_q} \leq \int \|\nabla V\| + C_{0,1}h\int\Big(|V| + \int\|\nabla V\|\Big) \lesssim (1 + C_{0,1}h)\int\limits_\gamma \|\nabla V\| + C_{0,1}h\int\limits_\gamma |V|,$$

Squaring both sides and applying Hölder's inequality as in 7.5b gives

$$\int\limits_{s_i} |V - \dot{X}|^2 \lesssim h(1 + C_{0,1}h)\int\limits_{s_i}\|\nabla V\|^2 + C_{0,1}h^2\int\limits_{s_i}|V|^2.$$

So for a smooth vector field V, we have constructed an interpolation. The best approximation in L^2 must of course also fulfill this inequality. And by continuity of the L^2-orthogonal projection, this holds for every vector field of class H^1, *q. e. d.*

13.13 **Proposition.** *Situation as in 13.9. Let* Q *be an* H^1 *section of* $T^*N \otimes x^*TM$. *Then there is some* $W \in \mathsf{P}^1\mathfrak{X}_x$ *with* $|Q - \nabla W|_{\mathsf{L}^2} \lesssim h|\nabla Q|_{\mathsf{L}^2} + C_{0,1}h^2|Q|$. *The hidden constant depends on* n, m *and* $|N|$.

Proof. It suffices to show the claim for the $\mathsf{L}^2(N, M)$ operator norm in the left-hand side instead of L^2 norm:

$$\|Q - \nabla W\|_{\mathsf{L}^2} \overset{!}{\lesssim} h|\nabla Q|_{\mathsf{L}^2} + C_{0,1}h^2|Q|. \qquad (13.13a)$$

In fact, let v be the unit vector field on N realising $\|Q\|$ everywhere. Then $v \in \mathsf{L}^2(x^*TM)$ and hence $|Q|^2 = \int |Q|^2 \lesssim \int \|Q\|^2 = \int |Qv|^2 \lesssim \|Q\|^2|v|^2 \lesssim \|Q\|^2|N|^2$. So let us prove 13.13a.

In any simplex, Q can be applied to vectors $ri - rj$ and their linear combinations. Choose a norm-preserving H^1 extension of Q such that it can also be applied to vectors ri. Then define, on each star s_i, a vector field $V_i := Qri$. Then $Qv = v^i V_i$ for any $v \in TN|_{s_i}$. Now let $\dot{X}_i$ be the L^2 best approximation to $-V_i$ on s_i. Then

$$\begin{aligned}\Big(\int |Qv + v^i \dot{X}_i|^2\Big)^{1/2} &= \Big(\int (v^i)^2 |V_i + \dot{X}_i|^2\Big)^{1/2}\\ &\le |v^i|\,|V_i + \dot{X}_i|\\ &\lesssim |v^i|\big(h(1+C_{0,1}h)\,|\nabla V_i| + C_{0,1}h^2\,|V_i|\big)\\ &\lesssim |v|\big(h(1+C_{0,1}h)\,\|\nabla Q\| + C_{0,1}h^2\,\|Q\|\big).\end{aligned}$$

Now recall that $|A_\lambda \nabla_{dx\,v}\dot{x} + v^i \dot{X}_i| \lesssim C_{0,1}h|dx\,v|\,|\dot{x}|$ from 12.5 in combination with 6.14 and 12.4, and $\|A_\lambda - \mathrm{id}\,\| \lesssim C_0 h^2$ from 6.16. This gives $|\nabla_{dx\,v}\dot{x} + v^i \dot{X}_i| \lesssim C_0 h^2\,|\nabla_{dx\,v}\dot{x}| + C_0 h^2\,|\dot{X}_i|\,|v^i|$. So we have for $W := \dot{x} \in \mathsf{P}^1\mathfrak{X}_x$

$$\begin{aligned}|\nabla_{dx\,v}W - Qv| &\le |\nabla_{dx\,v}W + v^i \dot{X}_i| + |v^i \dot{X}_i + Qv|\\ &\lesssim C_{0,1}h^2\,|v|\,\|\nabla W\| + h(1+C_{0,1}h)\,\|\nabla Q\| + C_{0,1}h^2\,\|Q\|.\end{aligned}$$

Because ∇W is almost an L^2 best approximation of Q, we have $\|\nabla W\| \lesssim \|Q\|$, which completes the proof, *q. e. d.*

Theorem. *Situation as in 13.10. Then* **13.14**

$$\rho_0(x,y) + \rho_1(x,y) \lesssim \rho_{0,\partial N}(x,y) + h\,|\nabla dy| + C'_{0,1}h,$$

where the hidden constant depends on m and the geometry of N.

Proof. Applying 13.13 to $Q = dx - Pdy$, there is $W \in \mathsf{P}^1\mathfrak{X}_x$ with

$$|dx - Pdy - \nabla W| \lesssim h\,|\nabla dx - \nabla Pdy| + C_{0,1}h^2\,|dx - Pdy|.$$

Because $C_{0,1}h^2$ is assumed to be small, say $\le \frac{1}{2}$, we can neglect the latter term. Due to 6.22, $|dx| \lesssim C'_{0,1}h|N|$, and $\nabla Pdy - P\nabla dy$ can be shown to be bounded with an argument like in 7.12 (spelled out in detail, this amounts to a rought L^∞ estimate for $\mathbf{d}(x,y)$, which is provable by a suitable modification of the standard first-order L^∞ estimate as in Braess 2007, p. 89). Then the claim is proven by 13.11 and 13.8, *q. e. d.*

D. Outlook

There are several research directions that would naturally continue the course of this dissertation, but which could not be further investigated due to time constraints:

⟨1⟩ Whereas the weak formulation and approximation of extrinsic curvature is obviously bound to the embedding of a submanifold, the weak form of Ricci curvature as in Fritz (2013) could be formulated intrinsically.

⟨2⟩ The measure-valued approximation of Lipschitz–Killing curvatures of submanifolds in Euclidean space from Cohen-Steiner and Morvan (2006) could possibly be carried over to situations where the surrounding space itself has curvature.

⟨3⟩ The level set approach (Osher and Sethian 1988, Osher and Fedkiw 2003) that was used to approximate PDE's on surfaces (Dziuk and Elliott 2008), surface flows (Deckelnick and Dziuk 2001) or, as combination of both, PDE's on evolving surfaces (Dziuk and Elliott 2013, sec. 8) can directly be carried over to submanifolds.

⟨4⟩ Assumption 9.20a has to be verified, perhaps under additional conditions. The testing with P^{-1} forms in 10.26–28 should be sharpened or at least re-interpreted with the use of more classical test functions.

⟨5⟩ Our definition of the barycentric mapping x is implicit and needs to know gradients of the squared distance function $\mathbf{d}^2$. The exact computation of geodesic distances is very expensive, and the task to find fast and accurate approximations is a current research problem, cf. Crane *et al.* (2013), Campen *et al.* (2013) and references therein. The use of any of these $\mathbf{d}^2$ approximations to compute the barycentric mapping would lead to a computationally feasible approximation of x.

⟨6⟩ After this, or restricted to 3-manifolds where geodesic distances can be exactly computed (or sufficiently well approximated), the minimal surface algorithms from Brakke (1992), Pinkall and Polthier (1993), Renka and Neuberger (1995), and Dziuk and Hutchinson (1999) can be applied, for example in hyperbolic three-space $\mathbb{H}^3$, the product $\mathbb{H}^2 \times \mathbb{R}$ of hyperbolic 2-space and the real line, or products with twisted metrics.

⟨7⟩ Variational methods in shape space, as have been dealt by Rumpf and Wirth (2011), can be extended e. g. to the computation of minimal submanifolds (whose dimension can be freely chosen) or multi-dimensional regression.

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Zusammenfassung

⟨1⟩ Sei (M, g) eine unberandete, kompakte Riemannsche Mannigfaltigkeit und Δ das n-dimensionale Standardsimplex. Für $n+1$ gegebene Punkte $p_i \in M$ betrachten wir mit Karcher (1977) die Funktion

$$E : M \times \Delta \to \mathbb{R}, \quad (a, \lambda) \mapsto \lambda^0 \, \mathbf{d}^2(a, p_0) + \cdots + \lambda^n \, \mathbf{d}^2(a, p_n),$$

worin $\mathbf{d}$ der geodätische Abstand in M sei. Liegen alle p_i in einem hinreichend kleinen geodätischen Ball, so ist $x : \lambda \mapsto \operatorname{argmin}_a E(a, \lambda)$ eine wohldefinierte Funktion $\Delta \to M$ (5.3). Wir nennen $s := x(\Delta)$ das Karcher-Simplex mit Ecken p_i. Auf Δ sei eine flache Riemannsche Metrik g^e durch Vorgabe von Seitenlängen $\mathbf{d}(p_i, p_j)$ definiert. Wenn alle Seitenlängen kleiner als h sind und $\operatorname{vol}(\Delta, g^e) \geq \alpha h^n$ für ein $\alpha > 0$ ist, so zeigen wir in 6.17 und 6.23

$$|(x^* g - g^e)\langle v, w\rangle| \leq ch^2 |v||w|, \qquad |(\nabla^{x^* g} - \nabla^{g^e})_v w| \leq ch|v||w| \tag{A.1a}$$

mit einer nur vom Krümmungstensor R von (M, g) und ϑ abhängigen Konstanten c. Daraus folgen mit wenig Aufwand Interpolationsabschätzungen für Funktionen $u : s \to \mathbb{R}$ (7.4) und $y : s \to N$ für eine zweite Riemannsche Mannigfaltigkeit N (7.15). Auch erlaubt diese Simplexdefinition, auf Grundlage der Voronoi-Zerlegung von Leibon und Letscher (2000) eine Karcher-Delaunay-Triangulierung zu definieren (8.7).

Daher können wir im folgenden ganz (M, g) als trianguliert annehmen. Auf jedem Simplex ist g durch eine Metrik g^e mit A.1a approximiert, und schwach differenzierbare Funktion $u \in \mathsf{H}^1(M, g)$ lassen sich durch stückweise polynomielle $u_h \in \mathsf{P}^1(M)$ approximieren. In der üblichen Weise (Dziuk 1988, Holst und Stern 2012) lassen sich daher Variationsprobleme wie das Poissonproblem (10.13, 10.17, 13.14) oder die Hodge-Zerlegung (10.15) in $\mathsf{H}^1(M, g)$ mit denjenigen in $\mathsf{H}^1(M, g^e)$ und ihren Galerkin-Approximationen in $\mathsf{P}^1(M)$ vergleichen.

Anknüpfend an die gängige Finite-Elemente-Theorie für Probleme auf Untermannigfaltigkeiten des $\mathbb{R}^m$ lassen sich auch Untermannigfaltigkeiten $S \subset M$ durch Karcher-Simplexe approximieren. Der dabei auftretende Geometriefehler ist gleich dem für Untermannigfaltigkeiten des $\mathbb{R}^m$ zuzüglich eines Terms ch^2 (11.18).

⟨2⟩ Sei M die geometrische Realisierung eines simplizialen Komplexes $\mathfrak{K}$. Die simpliziale Kohomologie $(C^k(\mathfrak{K}), \partial^*)$ ist von Desbrun und Hirani (2003, 2005) als diskretes äußeres Kalkül (DEC) interpretiert worden. Wir definieren Räume $\mathsf{P}^{-1}\Omega^k \subset \mathsf{L}^\infty\Omega^k$ und äußere Differentiale und geben eine isometrische Kokettenabbildung $C^k \to \mathsf{P}^{-1}\Omega^k$ an (9.11). Damit ist die Berechnung von Variationsproblemen im diskreten äußeren Kalkül auf Variationsprobleme in einem Raum von nicht-konformen Ansatz-Differentialformen zurückgeführt. Wir untersuchen die Approximationseigenschaften von $\mathsf{P}^{-1}\Omega^k$ in $\mathsf{H}^1\Omega^k$ (9.19, 9.20) und vergleichen die Lösungen von Variationsproblemen in ihnen (10.26–28).