Boundaries of bounded unstable manifolds in gradient Sturm systems are Schoenflies spheres

– in memoriam Klaus Kirchgässner –

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Abstract

Let v be a hyperbolic equilibrium of a smooth finite-dimensional gradient or gradient-like dynamical system. Assume that the unstable manifold W of v is bounded, with topological boundary $\Sigma = \partial W := (clos W)\setminus W$. Then Σ need not be homeomorphic to a sphere, or to any compact manifold. However, consider the PDE

$$u_t = u_{xx} + f(x, u, u_x)$$

of gradient Sturm type, i.e. scalar reaction-advection-diffusion equations with separated boundary conditions on a bounded interval. Then the eigenprojection $P\Sigma$ of Σ onto the unstable eigenspace of v is homeomorphic to a sphere. In particular this excludes complications like lens spaces and Reidemeister torsion. Excluding Schoenflies complications like Alexander horned spheres, we also show that both the interior domain PW of $P\Sigma$ and the one-point compactified exterior domain in the tangential eigenspace are homeomorphic to open balls. Our results are based on Sturm nodal properties.

1 Introduction

As a specific Sturm system we consider the scalar partial differential equation (PDE)

$$u_t = u_{xx} + f(x, u, u_x) \tag{1.1}$$

on the unit interval $0 < x < 1$. Subscripts t, x indicate partial derivatives of solutions $u = u(t, x)$. PDEs of the form (1.1) model the standard Brownian heat equation $u_t = u_{xx}$ with nonlinear, x-dependent effects f of reaction and advection type. Numerous examples from population biology, spatially heterogeneous chemical reactions, stochastic processes of interacting particles, viscous hyperbolic balance laws, and many other applied areas, appear as such models in their most simplistic PDE form. See for example the survey [FiSche03] and the references there.
To be specific, we impose Neumann boundary conditions

$$u_x = 0 \tag{1.2}$$

at $x = 0, 1$. Any other linear separated boundary conditions, i.e. of Dirichlet or Robin type, would work equally well with only minor modifications. We explicitly exclude periodic boundary conditions $x \in S^1 = \mathbb{R}/\mathbb{Z}$, for reasons detailed below. Throughout we assume $f \in C^2$ is twice continuously differentiable. Standard analytic semigroup theory provides a strongly continuous, compact semiflow $u(t) := u(t, \cdot) = T(t)u_0 \in X^\alpha$ for Cauchy initial data $u_0 \in X^\alpha$. Here $X = L^2$, and X^α, $0 \leq \alpha < 1$, are the domains of the fractional powers $(-\partial_x^2)^\alpha$ with the graph norm. We choose $3/4 < \alpha < 1$ to ensure the embedding $X^\alpha \hookrightarrow C^1$ is continuous. See for example [He81, Pa83, SeYo02] for details.
We will occasionally consider solutions $u(t) = T(t)u_0$ of (1.1), (1.2) in backwards time $t < 0$. This will abbreviate the property $T(-t)u(t) = u_0$ of $u(t)$, where $-t > 0$. Although several such $u(t)$ might exist, in general, backwards uniqueness holds true in our specific parabolic setting; see for example [Fr64]. Backwards existence fails miserably, of course, for general $u_0 \in X^\alpha$, except in very particular circumstances, often of finite dimension. Prominent interesting examples where even global backwards existence does hold are unstable manifolds and global attractors, as outlined below. Equilibria, i.e. time-independent solutions $u(t) \equiv v$, are trivial examples.
Let $v = v(x)$ denote an equilibrium of the PDE (1.1), (1.2), i.e. a solution of the ODE boundary value problem

$$0 = v_{xx} + f(x, v, v_x), \tag{1.3}$$

for $0 < x < 1$, again with Neumann boundary

$$v_x = 0 \tag{1.4}$$

at $x = 0, 1$. We call v *hyperbolic* if $\mu = 0$ is not an eigenvalue of the linearization

$$\mu\ \varphi = \varphi_{xx} + b\ \varphi_x + a\ \varphi \tag{1.5}$$

on $0 < x < 1$, of course under Neumann boundary $\varphi_x = 0$, for $x = 0, 1$. Here $b = b(x) := f_p(x, v(x), v_x(x))$ and $a = a(x) := f_u(x, v(x), v_x(x))$ depend on x, in general, via the partial derivatives f_u, f_p of $f = f(x, u, p)$. Clearly (1.5) is a Sturm-Liouville eigenvalue problem, and all eigenvalues μ are real. In particular $\mu = 0$ is the only candidate for a purely "imaginary" eigenvalue and our definition of hyperbolicity of the equilibrium coincides with the usual definition for dynamical systems. For later use we observe that all eigenvalues $\mu_0 > \mu_1 > \ldots \to -\infty$ of (1.5) are algebraically simple, with associated eigenfunctions $\varphi_0, \varphi_1, \ldots$, normalized in X^α, and with the sign convention

$$\varphi_k(0) > 0. \tag{1.6}$$

The present paper aims at a precise geometric description of the finite-dimensional unstable manifold W associated to the hyperbolic equilibrium v – up to, and including, its topological boundary $\partial W :=\ \operatorname{clos}(W)\backslash W$. Quite differently from the vast geometric possibilities in general nonlinear dynamics, we will prove that $\Sigma = \partial W$ is a *Schoenflies sphere*. More precisely Σ is not only homeomorphic to a sphere, but its interior W is a ball, and so is its exterior after one-point compactification – both of course in a suitable global finite-dimensional coordinatization. See theorem 1.1 below.
Two structural ingredients are essential to our result: the variational structure of (1.1), (1.2), and its nonlinear Sturm property. We explain these ingredients next.
The Sturm aspect which appears in linear guise in (1.5) becomes crucial as a nodal property or *zero number* z, in the nonlinear version. For any continuous $\varphi : [0, 1] \to \mathbb{R}$ let $0 \le z(\varphi) \le \infty$ denote the number of strict sign changes of φ; only for φ identically zero let $z(\varphi) := -1$. More precisely $z(\varphi)$ is the supremum of all k such that we can find $x_0 < x_1 < \ldots < x_k$ with strict sign alternation $\varphi(x_j)\varphi(x_{j+1}) < 0$, for all $j = 0, \ldots, k-1$. Standard Sturm theory, for example, asserts $z(\varphi_k) = k$ for all eigenfunctions. We will also use the notation

$$z(\varphi) = k_\pm \tag{1.7}$$

to indicate that $z(\varphi) = k$ and $\pm\varphi(0) > 0$. In particular $z(\varphi_k) = k_+$, by our sign convention. A parabolic PDE version of (1.5), already studied by Sturm [St36] for $a = a(x), b = b(x)$, is

$$\varphi_t = \varphi_{xx} + b\ \varphi_x + a\ \varphi; \tag{1.8}$$

see also [Po33]. For time dependent coefficients $a = a(t, x),\ b = b(t, x)$, assumed continuous here for simplicity, the most comprehensive generalization to date is due to [An88] as follows.

Let $\varphi(t) := \varphi(t,\cdot) \in X^\alpha, t \geq 0$, abbreviate the solution of (1.8) with initial condition $\varphi(0,\cdot) = \varphi_0 \in X^\alpha \backslash \{0\}$. Then

$$
\begin{array}{lll}
 & \text{(i)} & z(\varphi(t)) < \infty \text{ for any } t > 0; \\
(1.9) & \text{(ii)} & t \mapsto z(\varphi(t)) \text{ is nonincreasing with } t; \\
 & \text{(iii)} & t \mapsto z(\varphi(t)) \text{ drops strictly at } t = t_0 > 0, \text{if and only if} \\
 & & \varphi(t_0) \text{ possesses a multiple zero } x_0 \in [0,1], \text{ i.e. } \varphi(t_0,x_0) = \varphi_x(t_0,x_0) = 0.
\end{array}
$$

This implies, for example, that $t \mapsto z(\varphi(t))$ can drop only finitely often, after it has become finite. For all other times t, the spatial profile $x \mapsto \varphi(t,x)$ possesses only simple zeros.

With $\varphi := u_x$ this idea had been brought to bear in a nonlinear context of $f = f(u)$ by Matano [Mat82] under the name of *lap-number*. Many papers have followed his footsteps. In our setting it is most suitable to consider differences

$$\varphi := u^2 - u^1 \tag{1.10}$$

of any two distinct solutions u^1, u^2 of the PDE (1.1), (1.2). Indeed such φ satisfy equations of the form (1.8) with suitable time-dependent coefficients a, b which depend on the choice of u^1, u^2. In this setting we call (1.9)(i)-(iii) the *Sturm property* of the PDE (1.1), (1.2). Sturm himself used the Sturm property of (1.8) to show $k \leq z(\varphi) \leq \ell$ for any nontrivial linear combination φ of the eigenfunctions $\varphi_k, \varphi_{k+1}, \ldots, \varphi_\ell$; see [St36].
Delayed monotone feedback equations and Jacobi systems are tow other types of systems which are known to possess the Sturm property. *Jacobi systems*, as introduced and studied by [FuOl88] take the ODE form

$$\dot{u}_j = f_j(u_{j-1}, u_j, u_{j+1}) \tag{1.11}$$

for $j = 1, \ldots, N$. Here all f_j are assumed to possess strictly positive partial derivatives with respect to their off-diagonal entries $u_{j\pm1}$. "Neumann" boundary conditions are

$$u_0 := u_1, \qquad u_{N+1} := u_N. \tag{1.12}$$

Other separated linear boundary conditions like "Dirichlet" $u_0 = u_{N+1} = 0$ or mixed Robin type are possible. Periodic boundary conditions $j = 1$ $(mod\ N)$ with cyclic $u_{N+1} := u_1,\ u_0 := u_N$ also possess the Sturm property. For the cyclic variant of *cyclic monotone feedback systems* see for example [MPSm90, Sm95] and the many references there.
Delayed monotone feedback equations

$$\dot{u}(t) = f(u(t), u(t-1)) \tag{1.13}$$

have been studied extensively by [MP88, FiMP89a, MPSe96a, MPSe96b] and others. For constant sign of the partial derivatives of f with respect to the delayed feedback entry $u(t-1)$, the Sturm property prevails.

Under separated linear boundary conditions, like the Neumann case (1.2), the parabolic PDE (1.1) also possesses a *variational structure*, in addition to the Sturm property (1.9), (1.10). More precisely, there exists a strict Lyapunov function $V = V(u)$ of the form

$$V(u) := \int_0^1 g(x, u, u_x) dx \tag{1.14}$$

with suitable g. In fact, the time derivative takes the form

$$\frac{d}{dt} V(u(t)) = -\int_0^1 h(x, u, u_x) u_t^2 \, dx \tag{1.15}$$

with uniformly positive h. In particular V decreases strictly monotonically, except at equilibria. See [Ze68] and also the elegant approach of [Mat88] for details. The special case $f(x, u, p) = f(x, u)$ of advection-free nonlinearities f is well-known: $g(x, u, p) := \frac{1}{2}p^2 - F(x, u)$, where $F_u = f$ defines a primitive function F of f, and $h \equiv 1$. By LaSalle's invariance principle and compactness of the semiflow $T(t)$, for $t > 0$, the Lyapunov function V ensures that ω-limit sets consist entirely of equilibria. An analogous property holds for Jacobi systems (1.11) under separated linear boundary conditions.

We caution our reader that periodic boundary conditions for the PDE (1.1) or Jacobi systems (1.11) preclude the existence of a strict Lyapunov function V in general. Indeed rotating wave periodic solutions may arise for suitable $f := f(u, p)$ in the PDE (1.1), and in the cyclic Jacobi system (1.11) for suitable $f_i = f(u_{i-1}, u_i, u_{i+1})$. Delayed monotone feedback equations (1.13) also support time periodic solutions. Therefore we will exclude these non-variational cases, here and below, and focus again on the variational PDE (1.1), (1.2) with Sturm property (1.9) and Lyapunov function (1.14) in the present paper.

To formulate our main result, theorem 1.1 below, we consider an unstable hyperbolic equilibrium solution $u(t, x) = v(x)$ of the parabolic PDE (1.1), (1.2). Let $i(v) \geq 1$ denote the unstable dimension of v. In other words, the eigenvalues $\mu_k,\ k \geq 0$, of the linearization (1.5) at v satisfy

$$\mu_0 > \cdots > \mu_{i(v)-1} > 0 > \mu_{i(v)} > \cdots \to -\infty. \tag{1.16}$$

¿From the variational view point of the Lyapunov function V, the equilibrium v is a critical point of V with Morse index $i(v) \geq 1$.

Following [He81], the hyperbolic equilibrium v comes equipped with a differentiable unstable manifold W of dimension $i(v)$, much like in finite-dimensional dynamics. The manifold W consists of all $u_0 \in X^\alpha$ which possess a global past history $u(t,\cdot) \in X^\alpha$, for all $-\infty < t \leq 0$, such that $u(0) = u_0$ and

$$\lim_{t\to-\infty} u(t) = v \tag{1.17}$$

holds in X^α. In fact backwards convergence of $u(t) \in W\backslash\{v\}$ to v is exponential and asymptotically tangent to some eigenfunction φ_k, for some $k = 0, \ldots, i(v)-1$, i.e.

$$\lim_{t\to-\infty} \frac{u(t)-v}{|u(t)-v|_\alpha} = \pm\varphi_k \tag{1.18}$$

in X^α.
More detailed, [BrFi86] have investigated the properties of differentiable *fast unstable submanifolds* W^i of W with dimension $1 \leq i \leq i(v)$. See also lemma 2.1 below. Here W^i is characterized by backwards asymptotic tangency (1.18), for some $k = 0, \ldots, i-1$. Of course $W = W^{i(v)}$ in this notation. Moreover the tangent space to W^i at v is given by

$$E^i := \mathrm{span}\{\varphi_0, \ldots, \varphi_{i-1}\} \tag{1.19}$$

for any $i = 1, \ldots, i(v)$. Let P^i denote the eigenprojection onto E^i,

$$\ker\ P^i = \mathrm{span}\ \{\varphi_i, \varphi_{i+1}, \ldots\}. \tag{1.20}$$

By general invariant manifold theory, the projection

$$P^i : W^i \to E^i \tag{1.21}$$

is a local diffeomorphism in a neighborhood of the hyperbolic equilibrium $v \in W^i$. Here we have translated v to $v \equiv 0$, without loss of generality, by rewriting (1.1) as a PDE for $u - v$ of the same general form, but with $f(x, 0, 0) \equiv 0$.
Due to the variational structure and the Sturm property of our specific Sturm system (1.1), (1.2) we will establish the following result of Schoenflies type on the global topology of the fast unstable manifolds W^i, and in particular on the unstable manifold $W = W^{i(v)}$ itself.

Theorem 1.1 *Let $v \equiv 0$ be an unstable hyperbolic equilibrium of the Sturm system* (1.1), (1.2)*, with Morse index alias unstable dimension $i(v) \geq 1$. Let $i \in \{1, \ldots, i(v)\}$ be arbitrary, and assume the fast unstable manifold W^i of dimension i to be bounded in X^α. Let $\overline{W}^i :=$ closW^i denote the closure of W^i in X^α with topological boundary $\Sigma^{i-1} = \partial W := \overline{W}^i\backslash W^i$.*

Let P^i denote the eigenprojection onto the tangent space E^i of W^i at $v \equiv 0$, given by the span of the eigenfunctions $\varphi_0, \dots, \varphi_{i-1}$ of the linearization (1.5) *of* (1.1), (1.2) *at $v \equiv 0$. Then the eigenprojection*

$$P^i : \quad \overline{W}^i \to \bar{B}^i := P^i \overline{W}^i \subseteq E^i \cong \mathbb{R}^i \tag{1.22}$$

is a homeomorphism onto its image.
Moreover there exists a homeomorphism

$$h : \quad \mathbb{R}^i_\infty \to \mathbb{R}^i_\infty \tag{1.23}$$

of the one-point compactification $\mathbb{R}^i_\infty = \mathbb{R}^i \cup \{\infty\}$ of $\mathbb{R}^i$ such that the homeomorphic images

$$\begin{array}{rcl} h\ P^i W^i &=& B \\ h\ P^i \overline{W}^i &=& \bar{B} \\ h\ P^i \Sigma^{i-1} &=& S \end{array} \tag{1.24}$$

are the open/closed unit balls $B/\bar{B}$ and the unit sphere S in $\mathbb{R}^i \subset \mathbb{R}^i_\infty$, respectively. Similarly, the exteriors correspond to their unit exterior counterpart balls:

$$\begin{array}{rcl} h(\mathbb{R}^i_\infty \backslash P^i W^i) &=& \mathbb{R}^i_\infty \backslash B \\ \\ h(\mathbb{R}^i_\infty \backslash P^i \overline{W}^i) &=& \mathbb{R}^i_\infty \backslash \bar{B}. \end{array} \tag{1.25}$$

We compare our result with the case of standard gradient systems

$$\dot{u} = -V_u(u) \tag{1.26}$$

of scalar smooth or analytic Morse functions $V : \mathbb{R}^N \to \mathbb{R}$ in finite dimension N. We will only address the generic case of only hyperbolic equilibria, alias critical points of V, with transverse intersections of all stable and unstable manifolds, to meticulously avoid any nongeneric pathologies. In view of the results by Palis and Smale [Pa69, PaSm70, PdM82], C^0 structural stability ensues, under compactness assumptions. The resulting differences to Sturm systems like the PDE (1.1), (1.2) can then be ascribed to the sole influence of the Sturm properties (1.9), (1.10). We consider the planar case $N = 2$ first; see figure 1.1. On the left we indicate the flow in the closure $\overline{W}$ of the two-dimensional unstable manifold W of the trivial equilibrium $v := C \equiv 0$ for the *Chafee-Infante problem* (1.1), (1.2), where the nonlinearity

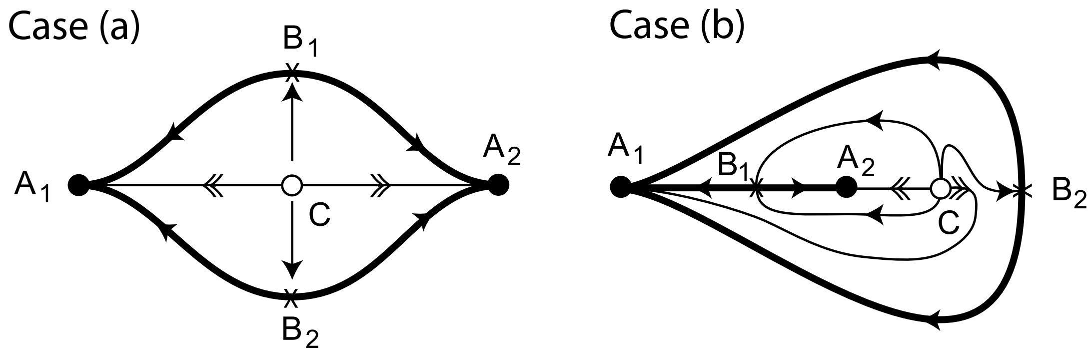

Figure 1.1: *Two planar unstable manifolds W of an equilibrium C with Morse index 2; stable equilibria A_1, A_2 with Morse index 0 and saddles B_1, B_2 of Morse index 1 on the boundary $\partial W := \overline{W} \backslash W$ (fat). The one-dimensional fast unstable manifold W^1 of C is indicated by double arrows. Case (a) left: the Chafee-Infante problem (1.1), (1.2) with $f = \lambda u(1-u^2)$, $\pi^2 < \lambda < 4\pi^2$, $C = 0$, of Sturm type. Case (b) right: a planar gradient flow example with suitable real analytic $V(u)$. Note how ∂W is homeomorphic to a circle in the Chafee-Infante Sturm case (a), but not in case (b).*

$$f = \lambda u(1 - u^2) \tag{1.27}$$

is assumed to be a symmetric cubic. The parameter λ is chosen between π^2 and $4\pi^2$ to ensure the Morse index $i(C)$ equals 2. Note how the fast unstable manifold W^1 of C is an interval with zero-sphere boundary $\Sigma^0 = \partial W^1 = \{A_1, A_2\}$ given by the spatially homogeneous sinks $A_1 \equiv -1, A_2 \equiv +1$. The boundary $\Sigma^1 := \partial W$ of the full unstable manifold $W = W^2$ of C is homeomorphic to the unit circle/1-sphere S^1. This picture already appears in [He81]. See also [ChIn74] and the work by Conley and Smoller in [Sm83] for pioneering further details on this problem.

Next consider a gradient flow (1.26) of type (b), as on the right of figure 1.1. Evidently the generating Lyapunov function V can be chosen real analytic, and the example is structurally stable due to [Pa69]. The fast unstable manifold W^1 of $v := C$ has zero-sphere boundary $\partial W^1 = \{A_1, A_2\}$ again. The boundary $\Sigma^1 := \partial W$ of the full unstable manifold $W = W^2$ of C, however, is not a circle. Rather ∂W is homeomorphic to a circle with an interior spike. The circle is the one-dimensional unstable manifold of the saddle B_2, with boundary given by the single sink $\{A_1\}$. The additional spike is the one-dimensional unstable manifold of the saddle B_1, with zero-sphere boundary $\{A_1, A_2\}$. The interior spike is attached to the circle at A_1. Admittedly ∂W is homotopy equivalent to the circle S^1, but not homeomorphic.

Clearly example (b) does not satisfy the conclusions of theorem 1.1 and therefore fails to appear in the class of Sturm systems.
In [BrFi89] uniqueness of heteroclinic orbits between equlibria of adjacent Morse indices was observed. Here a global solution $u(t,x), t \in \mathbb{R}$, of (1.1), (1.2) is called *heteroclinic* from equilibrium v_- to v_+, in symbols $u : v_- \rightsquigarrow v_+$, if

$$\lim_{t\to\pm\infty} u(t) = v_\pm \tag{1.28}$$

in X^α. By inspection of figure 1.1 we find the numbers of heteroclinic orbits between equilibria of adjacent Morse indices indicated in Table 1.1. While their uniqueness is confirmed in the Chafee-Infante case (a) of Sturm type, in the upper right triangle of Table 1.1, such uniqueness fails in the lower left triangle of the non-Sturm case (b). Indeed there are two heteroclinic orbits from the planar source C to the saddle B_1 and two heteroclinics from the saddle B_2 to sink A_1, in case (b).

(b)\ (a)	C	B_1	B_2	A_1	A_2
C	-	1	1	-	-
B_1	2	-	-	1	1
B_2	1	-	-	1	1
A_1	-	1	2	-	-
A_2	-	1	-	-	-

Table 1.1: Numbers of heteroclinic orbits between equilibria of adjacent Morse indices in figure 1.1. Upper right : Sturm case (a). Lower left: non-Sturm case (b)

In fact the one-dimensional case $i = 1$ of theorem 1.1 is equivalent to the elementary uniqueness of monotone heteroclinic orbits between adjacent equilibria. Indeed the fastest unstable manifold W^1 of any hyperbolic equilibrium v is one-dimensional and tangent to the positive first eigenfunction φ_0 at v. Moreover comparison principles show that (1.1), (1.2) is a *strongly monotone dynamical system* in the sense of [Hi88, Mat86b, Mat87]. Of course this kind of monotonicity corresponds to the fact that

$$z(u^2(t) - u^1(t)) = 0 \tag{1.29}$$

is preserved for any $t \geq t_0$, once it holds at $t = t_0$; see the monotonicity property (1.9)(ii) of the zero number z. In particular $\partial W^1 = \{v^\pm\}$ must consist of exactly two equilibria

$$v^+(x) > v(x) > v^-(x), \tag{1.30}$$

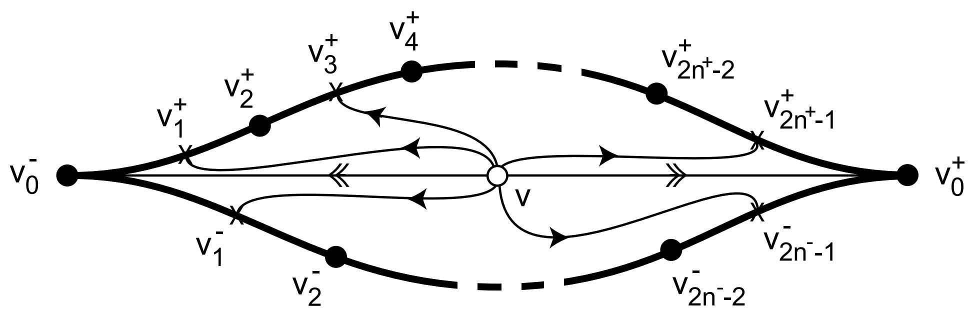

Figure 1.2: *Two dimensional closures $\overline{W}$ of unstable manifolds have circle boundaries. Equilibria $v_k^\pm$ with even indices k are sinks $i(v_k^\pm) = 0$, odd k are saddles $i(v_k^\pm) = 1$, and v is a source, $i(v) = 2$.*

for all $0 \leq x \leq 1$. The heteroclinic orbits $u^\pm(t, x)$ from v to $v^\pm$ satisfy

$$\lim_{t \to -\infty} \frac{u^\pm(t) - v}{|u^\pm(t) - v|_\alpha} = \pm\varphi_0 \tag{1.31}$$

in X^α; see (1.18). In particular strong monotonicity implies strict monotonicity of $t \mapsto \pm u^\pm(t, x)$, for any fixed $0 \leq x \leq 1$. This proves theorem 1.1, for $i = 1$.

Since the particular case $i = 1$ uses strong monotonicity, only, but not the full subtle force of the Sturm properties (1.9), (1.10), it extends to scalar reaction-diffusion-advection equations in more than one space dimensions. Flow embedding theorems in the spirit of [Po95, PrRy98a, PrRy98b, DaPo99], on the other hand, are able to realize flows like the one in figure 1.1(b), in the closure of fast unstable manifolds W^2, and thus provide counterexamples to theorem 1.1 in space dimensions at least two, of x.

For Sturm systems, the planar case $i(v) = \dim W = 2$ has already been addressed in [FiRo08]; see sections 3, 4 there. Even though the results there were only derived for planar global Sturm attractors, as defined below, they carry over to the present case. See figure 1.2. The fast unstable manifold W^1, where $z(u - v) = 0_\pm$, is drawn horizontally, with limiting equilibria $v_0^\pm$. The remaining parts of the boundary $\partial W \cong S^1$ are two graphs over W^1 with alternating saddles and sinks $v_k^\pm$, $k = 1, \ldots, 2n^\pm - 1$ on each graph. The equilibria there satisfy $z(v_k^\pm - v) = 1_\pm$. Moreover $z(v_k^\pm - v_j^\pm) = 0$ for $v_j^\pm \neq v_k^\pm$ of the same $\pm$ superscript. See [FiRo08] for further details. The Chafee-Infante case of figure 1.1(a) corresponds to $n^\pm = 1$, where the two boundary graphs over W^1 are the unstable manifolds of just a single saddle $v_1^\pm$, each.

After this planar Sturm excursion let us resume our discussion of the general variational Morse case. Suppose the boundary ∂W of the unstable manifold W of a hyperbolic equilibrium v is in fact a compact manifold M. Obviously $M = \partial W$ need not be of dimension $i(v) - 1 = \dim W - 1$. Already the *height flow* on the unit sphere $\overline{W} = S^{i(v)} \subset \mathbb{R}^{i(v)+1}$, with $V(u) := u_{i(v)+1}$ and the gradient V_u taken inside $S^{i(v)}$ provides such an example in $N = i(v) + 1$ dimensions. Indeed v is the positive unit vector $e_{i(v)+1}$, and $\partial W = \{-e_{i(v)+1}\}$ is not a sphere of dimension $i(v) - 1$.

Lens spaces $L = L(p; q_1, \dots, q_n), n \geq 2$, are quotients of the unit sphere S^{2n-1} in $\mathbb{C}^n$ under free linear actions of cyclic groups $\mathbb{Z}_p$, for any $p \geq 2$. They provide examples of more intriguing boundaries $\partial W = L$. The free $\mathbb{Z}_p$ action for $q_k \in \{1, \dots, p-1\}$, each co-prime to p, is given by the diagonal action of the $\mathbb{Z}_p$ generator

$$z_k \mapsto \exp(2\pi i q_k / p) z_k, \tag{1.32}$$

for $k = 1, \dots, n$. Then the lens space L is defined as the orbit space

$$L(p; q_1, \dots, q_n) := S^{2n-1}/\mathbb{Z}_p \tag{1.33}$$

of the free $\mathbb{Z}_p$ action. Non-homeomorphic lens spaces for different choices of $q_1, \dots, q_n$ may be homotopy equivalent, thus with identical homology, but can be distinguished by Reidemeister torsion, [Hat02].

As a nontrivial model for the closure $\overline{W}$ of the unstable manifold W of the trivial equilibrium $v = 0$ of a gradient flow (1.26) we let $\overline{W}$ be the closed unit ball in $\mathbb{R}^{2n} = \mathbb{C}^n$, but with points on the boundary S^{2n-1} identified to the lens space $L = L(p; q_1, \dots, q_n)$ via the free $\mathbb{Z}_p$-action. In polar coordinates $u = r\vartheta, \vartheta \in S^{2n-1}, V = V(r, \vartheta)$ on $\overline{W}$ the gradient flow (1.26) becomes

$$\begin{aligned} \dot{r} &= -V_r \\ \dot{\vartheta} &= -\frac{1}{r}\, V_\vartheta \end{aligned} \tag{1.34}$$

For simplicity, we consider V of the particular form

$$V = \exp(R(r) + \Phi(\vartheta)). \tag{1.35}$$

To allow for the boundary identification with the lens space L, we first choose $\Phi(\vartheta)$ invariant under the $\mathbb{Z}_p$-action on $\vartheta \in S^{2n-1}$. With

$$\exp R(r) := 1 - \frac{1}{2} r^2 (1 - \frac{1}{2} r^2), \tag{1.36}$$

the gradient flow (1.35) becomes

$$
\begin{aligned}
\dot{r} &= V \cdot r(1-r^2); \\
\dot{\vartheta} &= -\frac{1}{r} V \cdot \Phi_\vartheta .
\end{aligned}
\tag{1.37}
$$

The positive Euler multiplier $V > 0$ does not affect time orbits. Equivariance of (1.26), (1.35) under the $\mathbb{Z}_p$-action and time invariance of the unit sphere $r = 1$ provides the unstable manifold W of the trivial, totally unstable equilibrium $v \equiv 0$ to possess lens space boundary, as claimed. By the strong Whitney embedding theorem the above construction can be embedded into a total dimension $N \geq 4n$, [Hi76], keeping the Morse index $i(v \equiv 0) = \dim W = 2n$ unchanged.

If desired, this potential V can also be adapted such that all critical points are hyperbolic with transverse intersections of stable and unstable manifolds. If all eigenvalues μ_k of the linearization are required to de distinct, this will of course require a slight breaking of the $\mathbb{Z}_p$-equivariance of V. By structural stability this will not affect the topology of the unstable manifold W with lens space boundary $\partial W = L(p; q_1, \ldots, q_n)$.

Other examples with manifold boundary ∂W arise from free linear actions of noncyclic finite groups T on S^{2n-1}. See [Wo11] for a classification, and [Hat02] for references and a summary with particular attention to the case of S^3.

Suppose that the boundary ∂W of the unstable manifold W of an unstable hyperbolic equilibrium v is already known to be homeomorphic to the unit sphere S of dimension $i(v) - 1$. For $i(v) \geq 3$ this does not imply, directly, that W is homeomorphic to the open unit ball B of dimension $i(v)$. This difficulty is called the *Schoenflies problem*. The Alexander horned 3-sphere [Al24] is a striking illustration of a fundamental obstacle, in the C^0 category. For smooth embeddings of spheres S, and in the piecewise linear case, such counterexamples do not arise except possibly in dimension $i(v) = 4$. Under an extension assumption on the homeomorphic embedding of S, called flat embedding, the Schoenflies problem does not arise either; see [Bro60, Maz59, Mo60]. Conceivably, hyperbolicity and transversality on ∂W may suffice to resolve the Schoenflies problem for general gradient flows (1.26). In the general case of theorem (1.1), however, which does not rely on such additional generic assumptions we prefer to pursue a direct proof which emphasizes the pervasive geometric power of the Sturm properties (1.9), (1.10),

Many previous results on parabolic PDEs and on delay equations have focussed on descriptions of the *global attractor* $\mathcal{A}$. See for example [BaVi92, ChVi02, Ed&al94, Ha88, Ha&al02, La91, Ra02, Te88], for general background information on global attractors. Basically a dissipativeness assumption is imposed on f such that solutions $u(t, \cdot) = T(t)u_0$ enter and stay inside a large ball $|u|_\alpha \leq C$ after some time $t \geq t_0 = t_0(u_0)$ which may depend on u_0. For compact C^1 semigroups, finite box-counting dimension of $\mathcal{A}$ ensues. Moreover $\mathcal{A}$ is nonempty, maximal compact and invariant, attracts all bounded sets, and consists of all global solutions $u(t,\cdot), t \in \mathbb{R}$, which are uniformly bounded by C.

For dissipative Sturm systems we call the global atractor $\mathcal{A}$ a *Sturm attractor*. Due to the variational structure (1.14), (1.15), the Sturm attractor $\mathcal{A} = \mathcal{A}_f$ of (1.1), (1.2) consists of equilibria and their heteroclinic orbits, only. This is immmediate, because the Lyapunov function V also ensures α-limit sets of bounded global orbits $u(t,\cdot), t \in \mathbb{R}$, to consist entirely of equilibria. In particular the closure $\overline{W}$ of the unstable manifold W of any hyperbolic equilibrium v is contained in the Sturm attractor,

$$\overline{W} \subseteq \mathcal{A}_f. \tag{1.38}$$

Let $\mathcal{E} \subseteq \mathcal{A}_f$ denote the set of all equilibria. If all equilibria are hyperbolic, then $\mathcal{E}_f$ is finite by compactness of $\mathcal{A}_f$, and

$$\mathcal{A}_f = \bigcup_{v \in \mathcal{E}} W(v) \tag{1.39}$$

is the union of the unstable manifolds $W(v)$ of the equilibria $v \in \mathcal{E}$.

Let $N := \max\{i(v);\ v \in \mathcal{E}\}$ denote the maximal Morse index in $\mathcal{E}$. For $f = f(u)$ of class C^2 and independent of x and u_x it was shown in [Jo89, Bru90] that the L^2-orthogonal eigenprojection

$$P^N := \mathcal{A}_f \to E^N = \text{span}\ \{\varphi_0, \ldots, \varphi_{N-1}\} \tag{1.40}$$

is injective. Moreover the global attractor is a C^1 graph over the compact range of P^N. Here $\varphi_0, \ldots, \varphi_{N-1}$ can in fact be chosen to be eigenfunctions of *any* Sturm-Liouville problem (1.5), not necessarily coming from a linearization. The Fourier choice $\varphi_k(x) := \cos(k\pi x)$ is perfectly acceptable. These results are based on the Sturm nodal property (1.9), (1.10). A generalization to $f = f(x, u, u_x)$ of class C^2 was established in [Ro91], for separated boundary conditions. Under periodic boundary conditions $x \in \mathbb{R}/\mathbb{Z}$ and for Lipschitz f [MaNa97] have shown that the global attractor is still a C^0 graph. They have also obtained Lipschitz graphs, albeit under a somewhat peculiar Lipschitz constraint like $|f_p| \leq 2\pi$ on the circle $x \in \mathbb{R}/\mathbb{Z}$.

These results are stronger than the mere homeomorphisms of theorem 1.1 in the regularity aspect. Moreover they address the whole Sturm attractor $\mathcal{A}_f$ rather than just a single unstable manifold W or its closure $\overline{W}$. On the other hand, no geometric information on $\mathcal{A}_f$ or its P_N projection is obtained.

Already the planar Chafee-Infante example of figure 1.1(a) shows that the boundary circle $\partial W(C) \cong S^1$ can be Hölder continuous, at best, but not Lipschitz. Indeed Hölder cusps arise at A_1, A_2, of Hölder exponent $\mu_0/\mu_1 > 0$, where $\mu_k > 0$ denote the eigenvalues at A_1, A_2. We do not pursue this regularity question here.

In a continuing investigation of Sturm attractors, the present authors have focussed on a much more restricted geometric aspect. We only addressed the rather delicate global question of which equilibria, exactly, possess heteroclinic connections to which other equilibria. This question dates back to previous work by Conley and Smoller, [CoSm83, Sm83], and has been pursued extensively by [He85, BrFi88, BrFi89, FuRo91, FiRo96, FiRo99, FiRo00, FiRo09a, FiRo08, FiRo09b, Fi&al04].

In that line of research, theorem 1.1 is a first attempt at lifting these essentially combinatorial results to a geometric level.

As a cautionary remark we add that a geometric description of the closure of the *heteroclinic set* between two hyperbolic equilibria v_1 and v_2, i.e. of the transverse intersection between their unstable and stable manifolds, has remained elusive. Only the uniqueness result on heteroclinic orbits between equilibria of adjacent Morse index can be seen as a proof that their heteroclinic set is an interval with S^0 boundary $\{v_1, v_2\}$. In the planar case $i(v_1) = i(v_2) + 2$ we believe the heteroclinic set to be a disc with S^1-boundary. The boundary should consist of the unique heteroclinics between v_1, v_2, on the one hand, and two further equilibria $v_\pm$ of intermediate Morse index $i(v_\pm) = i(v_2) + 1$, on the other hand. See figure 1.2 for the typical case. This is also indicated by the quadrangulation procedure in planar connection graphs; see [FiRo08]. The snow-ball principle of [BrFi89] is based on the same idea. Higher dimensional cases $i(v_1) \geq i(v_2) + 3$ have not been published, to our knowledge, in spite of substantial encouragement from our advisors.

An outline of the proof of theorem 1.1, and of the paper, is as follows. We proceed by induction on the dimension $1 \leq i \leq i(v)$ of the fast unstable manifold W^i tangent to the eigenspace $E^i = \mathrm{span}\{\varphi_0, \ldots, \varphi_{i-1}\}$ of eigenvalues $\mu_0 > \cdots > \mu_{i-1} > 0$ at the hyperbolic equilibrium $v \equiv 0$; see (1.16)–(1.21). We have already dealt with the strictly monotone case $i = 1$; see (1.29)–(1.31). We therefore consider the theorem proved for $W^0, \ldots, W^{i-1}$ and complete the induction step for W^i. As was explained repeatedly we may assume $v \equiv 0$ without loss of generality.

In section 2 we prepare the induction step for the boundary $\Sigma^{i-1} := \partial W^i$, which proceeds in Mayer-Vietoris style. Lemma 2.1 collects some known results on the fast unstable manifolds W^i. We then decompose the sphere candidate Σ^{i-1} into two closed hemisphere candidate caps $\Sigma^{i-1}_\pm$,

$$\Sigma^{i-1} = \Sigma^{i-1}_+ \cup \Sigma^{i-1}_-, \qquad \Sigma^{i-1}_\pm := \omega(\Sigma^{i-1}_{\delta\pm}); \tag{1.41}$$

where $\Sigma_\pm^{i-1}$ are obtained as ω-limit sets of two carefully flattened small upper and lower hemispherical closed protocaps $\Sigma_{\delta\pm}^{i-1}$ around the equilibrium $v = 0$. The notion of ω-limit set is recalled in (2.17). The protocaps $\Sigma_{\delta\pm}^{i-1}$ are constructed over the faster unstable submanifold W^{i-1} in W^i which divides their union equatorially; see (2.18) – (2.20). In lemma 2.2 we establish that

$$z(u^2 - u^1) < i - 1 \tag{1.42}$$

for distinct u^κ in the same flattened protocap $\Sigma_{\delta\pm}^{i-1}$. It is a rather delicate matter, achieved only in lemma 3.6, to extend this property to the ω-limiting hemisphere candidates $\Sigma_\pm^{i-1}$. Of course the Sturm property (1.9) is crucial here.

More broadly, section 3 is devoted to injectivity results for projections P^j on $\overline{W}^j$ and, most importantly, for P^{i-1} on $\Sigma_\pm^{i-1}$. See lemmas 3.2 and 3.6. Together with the surjectivity results of lemma 4.1, which only use Brouwer degree, this will prove that the homeomorphic ranges

$$P^{i-1}\Sigma_+^{i-1} = P^{i-1}\Sigma_-^{i-1} = P^{i-1}\overline{W}^{i-1} =: \tilde{B}^{i-1} \tag{1.43}$$

all coincide. This allows us to write the homeomorphic images $P^i\Sigma_\pm^{i-1}$ and $P^i\overline{W}^{i-1}$ in the eigenspace E^i as graphs of scalar functions $\eta_\pm$ and η_e, respectively, over the same domain $\tilde{B}^{i-1} \subseteq E^{i-1}$. The functions $\eta_\pm, \eta_e$ simply denote the remaining φ_{i-1}-component of the respective projections P^i to E^i, for any given projection P^{i-1} to E^{i-1}.

In lemma 4.2 we then establish that

$$\eta_- < \eta_e < \eta_+ \tag{1.44}$$

in $\tilde{B}^{i-1}$, except at the boundary $\partial\tilde{B}^{i-1} = P^{i-1}\Sigma^{i-2}$ where all three functions coincide. By the induction hypothesis, however, the homeomorphic image $\tilde{B}^{i-1} = P^{i-1}\overline{W}^{i-1}$ is already a ball with Schoenflies boundary sphere $\partial\tilde{B}^{i-1} = P^{i-1}\Sigma^{i-2}$. This proves that the hemisphere candidates $\Sigma_\pm^{i-1}$ are indeed otherwise disjoint hemispheres over their joint equator

$$\begin{aligned} \Sigma_+^{i-1} \ \cup \ \Sigma_-^{i-1} &= \Sigma^{i-1}, \\ \Sigma_+^{i-1} \ \cap \ \Sigma_-^{i-1} &= \Sigma^{i-2}, \end{aligned} \tag{1.45}$$

of Mayer-Vietoris style.

To prove the ball part, in section 5, we first show that $P^i\overline{W}$, in (1.43), is contained in the closed ball $\bar{B}^i$ which consists of the boundary $(i-1)$-sphere $S^{i-1} := P^i\Sigma^{i-1} = P^i\partial W$ and its interior open ball $B^i = \text{ int } \mathrm{S}^{i-1} \subseteq \mathbb{R}^i$. We then proceed indirectly to show surjectivity

$$P^i : \quad \overline{W} \to \bar{B}^i. \tag{1.46}$$

In fact we construct an impossible retraction $\rho : \bar{B} \to S^{i-1}$ if surjectivity fails, and thus reach a contradiction. This completes the proof of theorem 1.1

Acknowledgment: We are indebted to the late Floris Takens for cautioning us against the Reidemeister intricacy in the study of unstable manifolds of gradient systems. We are also grateful to Björn Sandstede and Matthias Wolfrum for sustained two-fold advice and encouragement: insisting that the problem was quite easy, but not spoiling us by too many hints to take the excitement out of it. Finally, we gratefully acknowledge mutual hospitality during extensive productive visits.
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2 Zero numbers on invariant manifolds

We recall and adapt some results from the literature on invariant manifolds in Sturm systems. In lemma 2.1 we recall the analysis of zero numbers on fast stable and unstable manifolds from [BrFi86]. We then introduce the flattened small upper and lower hemispherical protocaps $\Sigma^{i-1}_{\delta\pm}$ around the equilibrium $v \equiv 0$, which generate the hemisphere candidates $\Sigma^{i-1}_{\pm}$ as their ω-limit sets. In lemma 2.2 we show $z(u^2 - u^1) < i - 1$ on the protocaps $\Sigma^{i-1}_{\delta\pm}$. We also show $\Sigma^{i-1}_{\pm} \cap \overline{W}^{i-1} = \Sigma^{i-2}$ to prepare for the final Mayer-Vietoris hemisphere decomposition

$$\begin{aligned} \Sigma^{i-1} &= \Sigma^{i-1}_{+} \cup \Sigma^{i-1}_{-}, \\ \Sigma^{i-2} &= \Sigma^{i-1}_{+} \cap \Sigma^{i-1}_{-} \end{aligned} \tag{2.1}$$

which will only be achieved in lemma 4.2.

Lemma 2.1 [BrFi86] *Let $v \equiv 0$ be a hyperbolic equilibrium of the Sturm PDE* (1.1), (1.2), *with unstable dimension $i(v) \geq 1$, eigenvalues $\mu_0 > \ldots > \mu_{i(v)-1} > 0 > \mu_{i(v)} > \ldots$, eigenfunctions $\varphi_0, \ldots, \varphi_{i(v)-1}, \varphi_{i(v)}, \ldots$, complementary eigenspaces $E^i = \operatorname{span}\{\varphi_0, \ldots, \varphi_{i-1}\}$, $E_i = \operatorname{span}\{\varphi_i, \varphi_{i+1} \ldots\}$,*

$$X^\alpha = E^i \oplus E_i \tag{2.2}$$

and associated complementary eigenprojections $P^i, P_i := \mathrm{id} - P^i$.
Then for every integer $1 \leq i \leq i(v)$, for every $\varepsilon_0 > 0$ small enough, and $0 < \varepsilon < \varepsilon_0$, there exists $\delta > 0$ such that the following holds on open balls $B_\delta, B_\varepsilon \subset X^\alpha$ and $B^j_\delta \subseteq E^j, B_{j,\varepsilon} \subseteq E_j$ of radii δ, ε in the X^α-norm $|\cdot|_\alpha$.

Let $u(t) \in B_{\varepsilon_0}$ *be a solution of* $(1.1), (1.2)$ *for all* $t \leq 0$. *Then* $u(t)$ *converges to* v *and is asymptotically tangent to some* φ_k, *i.e.*

$$\lim_{t \to -\infty} u(t)/|u(t)|_\alpha = \pm\varphi_k \tag{2.3}$$

for some $0 \leq k < i(v)$. *Similarly*

$$\lim_{t \to +\infty} u(t)/|u(t)|_\alpha = \pm\varphi_k \tag{2.4}$$

holds for global forward solutions $u(t) \in B_{\varepsilon_0}, t \geq 0$, *and some* $k \geq i(v)$. *The same two statements* $(2.3), (2.4)$ *hold, more generally, if we replace* $u(t)$ *by the difference* $u^2(t) - u^1(t)$ *of any two non-identical solutions* $u^1(t), u^2(t) \in B_{\varepsilon_0}$, *for* $t \to \mp\infty$ *respectively.*
In the unstable case $0 \leq i < i(v)$ *there exists a* C^1*-map*

$$\sigma^i : \quad E^i \supset B^i_\delta \to B_{i,\varepsilon} \subset E_i, \tag{2.5}$$

such that

$$(\sigma^i)'(0) = 0 \tag{2.6}$$

and the local fast unstable manifold $W^i_{\text{loc}} :=$ graph $\sigma^i \subset B_\varepsilon \subset X^\alpha$ *consists of* $v \equiv 0$ *and all* $v \neq u_0 \in B_\varepsilon$ *with* $P^i_{u_0} \in B^i_\delta$ *and a global past history* $u(t) \in B_\varepsilon, t \leq 0$, *tangent to* E^i, *i.e.*

$$\lim_{t \to -\infty} u(t)/|u(t)|_\alpha = \pm\varphi_k \tag{2.7}$$

for some $0 \leq k < i$. *Moreover*

$$z(u^2 - u^1) < i \tag{2.8}$$

for any two distinct $u^1, u^2 \in W^i_{\text{loc}}$.
Similarly, in the stable case $i \geq i(v)$ *there exist* ε_0, δ *and a* C^1*-map*

$$\eta_i : E_i \supset B_{i,\delta} \to B^i_\varepsilon \subset E^i \tag{2.9}$$

such that the local fast stable manifold $W_{i,\text{loc}} =$ graph $\sigma_i \subset B_\varepsilon \subset X^\alpha$ *consists of* $v \equiv 0$ *and all* $v \neq u_0 \in B_\varepsilon$ *with* $P_i u_0 \in B_{i,\delta}$ *and a global forward solution* $u(t) \in B_\varepsilon,\ t \geq 0$, *tangent to* E_i, *i.e.*

$$\lim_{t \to +\infty} u(t)/|u(t)|_\alpha = \pm\varphi_k \tag{2.10}$$

for some $k \geq i$. *Moreover*

$$z(u^2 - u^1) \geq i \tag{2.11}$$

for any two non-identical $u^1, u^2 \in W_{i,\mathrm{loc}}$.

Proof:

The proof of statements (2.5) – (2.11) of lemma 2.1 proceeds via semigroup theory along the lines of [BrFi86], where the case $f = f(u)$ was presented in detail, in a slightly different formulation. Aysmptotic tangencies (2.7), (2.10) had been replaced by exponential decay estimates at rates $\mu_{i-1} - \tilde{\varepsilon}$ and $-(\mu_i + \tilde{\varepsilon})$, respectively. Therefore the tangencies (2.7) hold in $W^{k+1}_{\mathrm{loc}} \backslash W^k_{\mathrm{loc}}$ with $W^0_{\mathrm{loc}} := \{v\}$ and $W^{i(v)}_{\mathrm{loc}} = W_{\mathrm{loc}}$ the usual local unstable manifold of v. Similarly the tangencies (2.10) hold in $W_{k,\mathrm{loc}} \backslash W_{k+1,\mathrm{loc}}$ with $W_{i(v),\mathrm{loc}}$ the usual local stable manifold $W^s_{\mathrm{loc}}(v)$. The asymptotic tangency statement (2.3) for single trajectories $u(t)$ is then straightforward in the finite-dimensional unstable manifolds, writing

$$W^{i(v)} \backslash \{v\} = \bigcup_{k=0}^{i(v)-1} (W^{k+1}_{\mathrm{loc}} \backslash W^k_{\mathrm{loc}}). \tag{2.12}$$

Statement (2.10), in the stable case, needs a separate argument to exclude super-exponential decay and to ensure

$$\bigcap_{k \geq i(v)} W_{k,\mathrm{loc}} = \{v\}. \tag{2.13}$$

This fact has been established by [An88]; see also the precursors in [He85, An86]. The arguments in [An88], which reduce the convergence question to a linear problem without advection term, apply equally well to cover the generalization to differences $u(t) := u^2(t) - u^1(t)$ which are not identically zero. In particular the asymptotics (2.7), (2.10) for differences $u = u^2 - u^1$ imply constraints (2.8), (2.11) for the zero numbers.

$\bowtie$

Recall from (1.8) that the linear case $f(x, u, p) = b(x)p + c(x)u$ was already considered by Sturm as early as 1836; see [St36]. The fast unstable manifolds $W^i = E^i$ and stable manifolds $W_i = E_i$ then coincide with their tangent eigenspaces: $\eta^i = 0,\ \sigma_i = 0$. Via the explicit solution 82

$$u(t) = \sum_{i \geq 0} u_{0,i} \varphi_i e^{\mu_i t} \tag{2.14}$$

with constant real coefficients $u_{0,i}$ the asymptotic statements of lemma 2.1 follow; for example the asymptotic shape φ_k for $t \to -\infty$ in (2.7) is given by the maximal index $k < i(v)$ for which $u_{0,k} \neq 0$. Analogously (2.10) holds for the minimal such $k \geq i(v)$. Similarly

$$j \leq z(u_0) \leq k \tag{2.15}$$

for all nontrivial linear combinations $u_0 \in \text{span}\{\varphi_j, \ldots, \varphi_k\}$: following Sturm's original argument we just invoke the Sturm property (1.9), (1.10) for $\varphi = u(t)$ and compare the dominant eigenfunctions in (2.14) for $t \to \pm\infty$.

After these preliminaries we introduce the main players in our proof of theorem 1.1. As in section 1 we consider the fast unstable manifolds $W^i, 1 \leq i \leq i(v)$, of the hyperbolic trivial equilibrium $v \equiv 0$ with Morse index $i(v)$. The hemisphere candidate caps $\Sigma_\pm^{i-1}$ for the Mayer-Vietoris type decomposition

$$\begin{array}{rcccl} \partial W^i &=& \Sigma^{i-1} &=& \Sigma_+^{i-1} \cup \Sigma_-^{i-1} \\ \partial W^{i-1} &=& \Sigma^{i-2} &=& \Sigma_+^{i-1} \cap \Sigma_-^{i-1} \end{array} \tag{2.16}$$

introduced in (1.41) are defined as ω-limit sets $\omega(\Sigma_{\delta+}^{i-1})$ of carefully constructed small closed upper and lower flattened protocaps $\Sigma_{\delta\pm}^{i-1}$ of radius $\delta > 0$ around $v = 0$ in W^i; see also figure 2.1.

For the convenience of our reader, we recall the notion of the *ω-limit set* for sets $M \subset X^\alpha$ of initial conditions. The set

$$\begin{array}{rl} \omega(M) := & \bigcap\limits_{t_0 \geq 0} \text{clos}\,(\bigcup\limits_{t \geq t_0} T(t)M) = \\ = & \{u \mid u = \lim\limits_{n\to\infty} u^n(t_n) \;\text{ for suitable }\; u_0^n = u^n(0) \in M,\; t_n \to \infty\} \end{array} \tag{2.17}$$

denotes the ω-limit set of M, provided that $T(t)u_0$ is defined for all $u_0 \in M, t \geq 0$.

To construct the flattened hemispherical protocaps $\Sigma_{\delta\pm}^{i-1}$ in W_{loc}^i we work in the coordinates

$$u = \eta_0\varphi_0 + \ldots + \eta_{i-1}\varphi_{i-1} + o(|\eta|^2), \tag{2.18}$$

of the tangent space $E^i = \text{ span }\{\varphi_0, \ldots, \varphi_{i-1}\}$ to W_{loc}^i at $v = 0$. The faster unstable manifold W_{loc}^{i-1} can be written as the graph of a function

$$\eta_{i-1} = \eta_e(\hat\eta) = \eta_e(\eta_0, \ldots, \eta_{i-2}) \tag{2.19}$$

where $\hat\eta = (\eta_0, \ldots, \eta_{i-2})$ and $\eta_e:\; E^{i-1} \cong \mathbb{R}^{i-1} \supseteq \{|\hat\eta| \leq \delta\} \to \mathbb{R}$, in these coordinates. The index e foreshadows the eventual equatorial role of W^{i-1} in the eventual Schoenflies sphere Σ^{i-1}. We define the upper protocap $\Sigma_{\delta+}^{i-1}$ as the graph of the C^1-function

$$\eta_{i-1} = \eta_e(\hat\eta) + (\delta^2 - |\hat\eta|^2)^2 \tag{2.20}$$

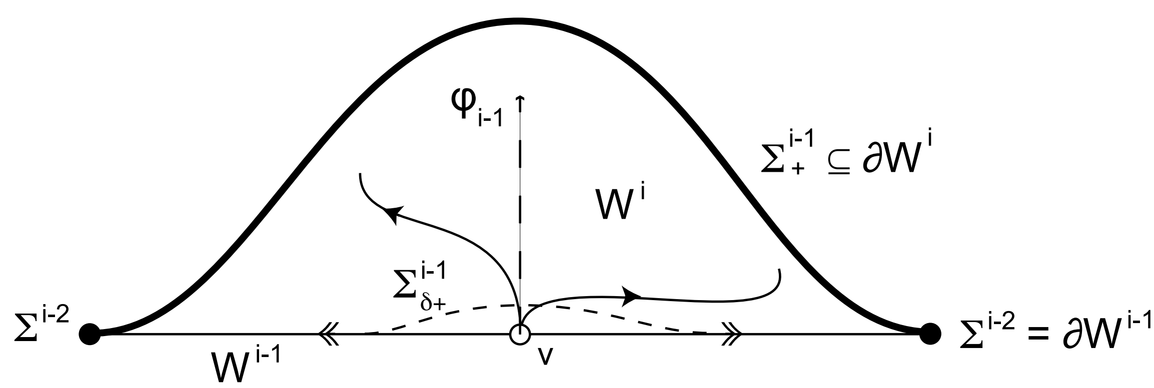

Figure 2.1: *Upper half of fast unstable manifold W^i of $v = 0$: horizontal faster unstable part W^{i-1} (double arrows), upper protocap hemisphere $\Sigma^{i-1}_{\delta+}$ (dashed), upper cap boundary $\Sigma^{i-1}_{+} \subseteq \Sigma^{i-1} = \partial W^i$ (fat), and boundary Schoenflies (sphere $\Sigma^{i-2} = \partial W^{i-1}$ (dots). The slowest unstable tangent vector φ_{i-1} at v is vertical (dashed).*

with Euclidean norm $|\hat\eta|$. The domain is the open ball $|\hat\eta| \leq \delta$. The flattened lower hemispherical protocap $\Sigma^{i-1}_{\delta-}$ is constructed analogously with $-(\delta^2 - |\hat\eta|^2)^2$ in (2.20), instead. Henceforth we choose $\delta > 0$ in our construction so small that the assertion (2.21) lemma 2.2 below will hold.
By tangency of W^j to the eigenspace E^j at $v = 0$, the protocaps $\Sigma^{i-1}_{\delta\pm}$ are indeed homeomorphic to true hemispheres. Likewise Σ^{i-2} is homeomorphic to a Schoenflies sphere in E^{i-1}, by the induction hypothesis of our proof of theorem 1.1. For the objects $\Sigma^{i-1}, \Sigma^{i-1}_{\pm}$, however, the sphere jargon is only a suggestive terminology which still remains to be justified with mathematical rigor.

Lemma 2.2 *With the above notation and under the assumptions of theorem* 1.1 *the following holds true for all* $1 \leq i \leq i(v)$ *on the fast unstable manifolds* W^i *in the unstable manifold* $W = W^{i(v)} = W^u(v)$ *of* $v \equiv 0$*. Let* $\delta > 0$ *be chosen small enough.*
Then, for any two (distinct) elements u^1, u^2 *of the same protocap* $\Sigma^{i-1}_{\delta+}$ *or* $\Sigma^{i-1}_{\delta-}$*, we have*

$$z(u^2 - u^1) < i - 1, \tag{2.21}$$

provided that $\delta > 0$ *is chosen small enough.*

Proof:
We proceed indirectly and assume sequences of distinct $u_0^{1,n}, u_0^{2,n}$ to exist in protocaps $\Sigma^{i-1}_{\delta_n+} \subseteq W^i_{\mathrm{loc}}$, for $\delta_n \to 0$, such that $z(u_0^{2,n} - u_0^{1,n}) \geq i - 1$, and hence

$$z(u_0^{2,n} - u_0^{1,n}) = i - 1. \tag{2.22}$$

We normalize the differences and, because W^i is finite-dimensional and tangent to E^i, pass to a convergent subsequence

$$\tilde{u}_0^n := (u_0^{2,n} - u_0^{1,n})/|u_0^{2,n} - u_0^{1,n}|_\alpha \to \tilde{u}_0 \neq 0. \tag{2.23}$$

By construction (2.18)–(2.20) of the flattened protocaps $\Sigma_{\delta+}^{i-1}$ we have $\tilde{u}_0 \in E^{i-1}$.
A slight difficulty arises because $\tilde{u}_0(x)$ might possess degenerate zeros which perturb to any number of strict sign changes for $u_0^{2,n}(x) - u_0^{1,n}(x)$. This difficulty can be overcome via backwards time extensions

$$u^{1,n}(t,x), u^{2,n}(t,x), \quad \text{and} \quad \tilde{u}^n(t,x), \tag{2.24}$$

under the linearized PDE in E^i and the original semilinear PDE in W^i, respectively, as follows. Fix some time $t = -\tau < 0$ such that all zeros of $x \mapsto \tilde{u}(-\tau, x)$ are simple. This is possible because the zero number can drop only finitely often; see Sturm properties (1.9)(i)–(iii). Backwards solutions of the ODE on $W^i \subseteq X^\alpha$ depend continuously on initial conditions, and $\delta \to 0$ makes that nonlinear ODE converge to its linearization. Therefore the same simple zeros of $\tilde{u}(-\tau, \cdot)$, and only those, are inherited by the normalized quotients $\tilde{u}^n(-\tau, \cdot)$ and hence by the differences $u^{2,n}(-\tau, \cdot) - u^{1,n}(-\tau, \cdot)$. Suppressing superscripts n, the Sturm property (1.9)(ii) implies

$$i - 1 = z(u_0^2 - u_0^1) \leq z(u^2(-\tau, \cdot) - u^1(-\tau, \cdot)) = z(\tilde{u}(-\tau, \cdot)) < i - 1. \tag{2.25}$$

In the last inequality we have used that $\tilde{u}_0 \in E^{i-1}$ implies $\tilde{u}(-\tau, \cdot) \in E^{i-1}$, and $z < i - 1$ in $E^{i-1} = \operatorname{span}\{\varphi_0, \ldots, \varphi_{i-2}\}$. The contradiction (2.25) completes the indirect proof of the lemma.

$\bowtie$

3 Injectivity

Let the assumptions of theorem 1.1 remain in effect throughout the following sections.
In this section we collect injectivity properties of eigenprojections P^j onto eigenspaces E^j, when restricted to the closure $\overline{W}^i = W^i \cup \Sigma^{i-1}$ of the fast unstable manifold and to the hemisphere candidate caps $\Sigma_\pm^{i-1} = \omega(\Sigma_{\delta\pm}^{i-1})$ of the trivial hyperbolic equilibrium $v \equiv 0$. Here $j = i$ and $j = i-1$ for $\overline{W}^i$ and $\Sigma_\pm^{i-1}$, respectively. See lemmas 3.2 and 3.6. The string of lemmas (3.5)–(3.7) prepares the main injectivity result of lemma 3.6 on the hemisphere candidate caps $\Sigma_\pm^{i-1}$ as follows. In lemma 3.3 we study how all orbits in $W^i\backslash\{v\}$ traverse the protocap barrier

$$\Sigma_\delta^{i-1} := \Sigma_{\delta+}^{i-1} \cup \Sigma_{\delta-}^{i-1}. \tag{3.1}$$

Lemma 3.4 then establishes the equatorial role of the faster unstable manifold W^{i-1} with Schoenflies boundary $\Sigma^{i-2} = \partial\overline{W}^{i-1}$ for the hemisphere cap candidates $\Sigma_\pm^{i-1}$, as

$$\Sigma^{i-2} = \Sigma_\pm^{i-1} \cap \overline{W}^{i-1}. \tag{3.2}$$

Lemma 3.5 establishes how distinct limiting elements $u^\kappa = \lim_{n\to\infty} u^{\kappa,n}(t_n^\kappa) \in \Sigma_\pm^{i-1} = \omega(\Sigma_{\delta\pm}^{i-1})$, $\kappa = 1,2$, of the same protocap $\Sigma_{\delta\pm}^{i-1}$ can actually be obtained by one and the same choice $t_n^\kappa = t_n$ of stepping times, all starting at $u^{\kappa,n}(0) \in \Sigma_{\delta\pm}^{i-1}$. This is the most important technical step in the proof of the final injectivity lemma 3.6 on the hemisphere candidate caps $\Sigma_\pm^{i-1}$. Surjectivity properties, the other main ingredient to our proof of theorem 1.1, will be collected in section 4.
We begin with an elementary observation on general invariant sets $M \subseteq X^\alpha$ which relates injectivity of P^j to the zero number $z(u^2 - u^1)$ of the difference of points $u^\kappa \in M,\ \kappa = 1,2$.

Lemma 3.1 *Let $M \subseteq X^\alpha$ and assume*

$$z(u^2 - u^1) < j \tag{3.3}$$

for any two distinct $u^\kappa \in M$. Then the restricted eigenprojection

$$P^j : M \to E^j = \text{span}\,\{\varphi_0, \ldots, \varphi_{j-1}\} \tag{3.4}$$

is injective on M.
Similarly, we obtain injectivity of the complementary eigenprojection

$$P_j : M \to E_j = \text{span}\,\{\varphi_j, \varphi_{j+1}, \ldots\} \tag{3.5}$$

under the complementary assumption on distinct $u^\kappa \in M$,

$$z(u^2 - u^1) \geq j. \tag{3.6}$$

If M is also compact, then the injective projections P^j are homeomorphisms onto the image P^jM.

Proof:
The proof of claim (3.4) is by contraposition: suppose

$$P^ju^2 = P^ju^1 \tag{3.7}$$

for two distinct $u^\kappa \in M$. In other words, $0 \neq \varphi := u^2 - u^1 \in E_j = \text{ span } \{\varphi_j, \ldots\}$ is in the Sturm-Liouville eigenspace $E_j = \ker P^j$ complementary to E^j. Essentially by Sturm's original argument [St36] this implies

$$z(\varphi) \geq j; \tag{3.8}$$

see (2.14), (2.15). This negation of assumption (3.3) proves injectivity of (3.4). The proof of (3.6) is analogous.
The homeomorphism claim, i.e. continuity of the inverse $(P^j)^{-1} : \ P^jM \to M$, is also immediate: by compactness of M the continuous projection $P^j = ((P^j)^{-1})^{-1}$ maps closed, viz compact, sets to compact sets – which are closed. This proves the lemma.

$\bowtie$

This lemma has been crucial, already, in the results of [Bru90, Jo89, Ro91, MaNa97] on global attractors, and in the Poincaré-Bendixson results of [FiMP89b, Na90]. It is worth mentioning that the assumption (3.3) on M does not depend on the particular choice of a Sturm-Liouville problem for the eigenprojection (3.4). For example we might choose L^2-orthogonal Fourier decomposition, with $\varphi_\kappa(x) = \cos(\kappa\pi x)$.
We explicitly caution our alert reader that the "intermediate assumption"

$$j \leq z(u^2 - u^1) < k \tag{3.9}$$

on M *does not imply injectivity* of the intermediate eigenprojection

$$P_j^k : M \to E_j^k := \text{ span } \{\varphi_j, \ldots, \varphi_{k-1}\}. \tag{3.10}$$

Counterexamples with $j = 2$, $k = 4$, $\varphi_j := \cos(j\pi x)$, and $M = \text{ span } \{\varphi\}$ for suitable $\varphi \in \text{ span } \{\varphi_1, \varphi_4\}$ with $z(\varphi) = 3$ are easily constructed. Indeed $P_j^kM = \{0\}$. This phenomenon may present a serious obstacle, in fact, to a simple geometric description of the set of heteroclinic orbits $u \colon v_- \rightsquigarrow v_+$ with $k = i(v_-)$ and $j = i(v_+)$.

We can now collect injectivity results for our proof of theorem 1.1 under the assumptions there. We begin with the eigenprojection P^i on the closure $\overline{W}^i$ of the fast unstable manifold W^i of the unstable equilibrium and with P^{i-1} on the protocaps $\Sigma^{i-1}_{\delta\pm}$. We recall (2.18)–(2.20) and figure 2.1 for definitions of the flattened protocaps $\Sigma^{i-1}_{\delta\pm} \subseteq W^i$.

Lemma 3.2 *The eigenprojections*

$$P^i : \quad \overline{W}^i \quad \to E^i \tag{3.11}$$

$$P^{i-1} : \quad \overline{W}^{i-1} \quad \to E^{i-1} \tag{3.12}$$

$$P^{i-1} : \quad T(t)\Sigma^{i-1}_{\delta\pm} \quad \to E^{i-1} \tag{3.13}$$

are all injective, separately but not jointly in the $\pm$ case, for $\delta > 0$ fixed small enough, and for all $t \geq 0$.

Proof:
By lemma 3.1, injectivity claim (3.11) for P^i on $M := \overline{W}^i$ holds true, provided we show

$$z(u^2 - u^1) < i \tag{3.14}$$

for distinct $u^\kappa \in \overline{W}^i$. The latter fact was established on W^i_{loc} in lemma 2.1; see (2.8). It extends to the fast unstable manifold W^i by the Sturm property of the forward semiflow $T(t), t \geq 0$. It extends to the forward and backward invariant closure $\overline{W}^i$, indirectly. Suppose $z(u^2 - u^1) \geq i$ for two distinct $u^\kappa \in \Sigma^{i-1} = \partial W^i = \overline{W}^i \backslash W^i$. By the Sturm property (1.9)(ii) the same holds true at any negative time $t = -\tau < 0$. Starting there, instead, we may assume all zeros of $x \mapsto u^2(x) - u^1(x)$ to be simple. This extends to approximating elements $W^i \ni u^{\kappa,n} \to u^\kappa,\ \kappa = 1, 2$, and contradicts (3.14) in W^i. Therefore (3.14) and injectivity (3.11) extend to the closure $\overline{W}^i$. Obviously this also proves (3.12).
To prove injectivity (3.13) on the hemisphere cap $T(t)\Sigma^{i-1}_{\delta+}$ we recall injectivity of P^{i-1} on $\overline{W}^{i-1}$, by (3.12) or the induction hypothesis. Consider $t = 0$ first. The flattened closed hemisphere $\Sigma^{i-1}_{\delta+}$ intersects $W^{i-1}_{\mathrm{loc}} \subset \overline{W}^{i-1}$ in the slightly deformed standard sphere $\bar{S}^{i-1}_{\delta+} \cap \overline{W}^{i-1} = \{|\hat{\eta}| = \delta,\ \eta_{i-1} = \eta_e(\hat{\eta})\}$, in the local coordinates $\hat{\eta} = (\eta_0, \ldots, \eta_{i-2}),\ \eta_{i-1}$ of (2.18)–(2.20). Moreover

$$P^{i-1} : \quad \Sigma^{i-1}_{\delta+} \to E^{i-1} \tag{3.15}$$

is injective, by lemma 3.1 and zero number property (2.21) of lemma 2.2 on $\Sigma^{i-1}_{\delta+}$. For any fixed $t > 0$, injectivity (3.15) of P^{i-1} persists on $T(t)\Sigma^{i-1}_{\delta+}$ by dropping of the zero number.

This proves (3.13) and the lemma. $\bowtie$

To prepare for injectivity of P^{i-1} on the hemisphere cap candidates $\Sigma_{\pm}^{i-1} = \omega(\Sigma_{\delta\pm}^{i-1})$, which we only achieve in lemma 3.6 below, we now study the behavior of the protosphere

$$\Sigma_{\delta}^{i-1} := \Sigma_{\delta+}^{i-1} \cup \Sigma_{\delta-}^{i-1} \quad \subseteq W^{i} \tag{3.16}$$

under the (semi-)flow $T(t)$ on W^i. Geometrically Σ_{δ}^{i-1} is a sphere only in the equatorial faster unstable manifold W^{i-1}, flattened in the remaining φ_{i-1}-direction of W^i_{loc}. Since $P^i : \overline{W}^i \to E^i$ is injective, by lemma 3.2, we can use coordinates $\eta_0\varphi_0 + \ldots + \eta_{i-1}\varphi_{i-1} \in E^i$ on $\overline{W}^i$ globally, from now on. Therefore we study the eigenprojection $P^iT(t)\Sigma_{\delta}^{i-1} \subseteq E^i$ of the flowing protosphere.

Lemma 3.3 *Under the assumptions of theorem 1.1, which remain in effect throughout, the protosphere Σ_{δ}^{i-1} around $v = 0$ in W^i has the following properties.*

(i) *$P^iT(t)\Sigma_{\delta}^{i-1} \subset E^i$ is a homeomorhic Schoenflies sphere embedding, for any real time t;*

(ii) *there exists $t_0 > 0$ such that the snapshots $P^iT(t)\Sigma_{\delta}^{i-1}$ remain interior to the Schoenflies snapshot*

$$\tilde{\Sigma}_{m}^{i-1} := P^iT(mt_0)\Sigma_{\delta}^{i-1} \tag{3.17}$$

for $t \leq (m-1)t_0$, and exterior for $t \geq (m+1)t_0$.

(iii) *In particular the snapshots are pairwise disjoint and form an expanding system of nested embedded Schoenflies spheres, for $m \in \mathbb{Z}$.*

Proof:
Claim (i) is immediate because the protosphere $\Sigma_{\delta}^{i-1} \subset W^{i-1}$ itself provides a homeomorphic Schoenflies sphere embedding $P^i\Sigma_{\delta}^{i-1} \subseteq E^i$ by explicit construction; see (2.17)–(2.18). Indeed $T(t)$ is a homeomorphism and P^i projects all of the compact closure $\overline{W}^i$ homeomorphically onto its image, by injectivity lemma 3.2 and by lemma 3.1. Via the open neighborhood $P^iW^i \subset E^i$ of the sphere embeddings $P^iT(t)\Sigma_{\delta}^{i-1}$ we therefore obtain embeddings as Schoenflies spheres.
Obviously claim (iii) is a special case of claim (ii). To prove claim (ii) we first note that any of the above homeomorphic Schoenflies sphere embedding in E^i decomposes the complement into two open connected components by Brouwer degree: the interior, which contains $v = 0$, and the exterior. By homotopy to $t = 0$, the flow homeomorphisms $T(t)$ on W^i preserve this decomposition, for any real time t.

Now choose $t_0 > 0$ large enough such that $P^i T(t)\Sigma_\delta^{i-1}$ is in the interior of the Schoenflies sphere Σ_δ^{i-1}, for all $t \leq -t_0$. Such a t_0 exists because the unstable hyperbolic equilibrium $v = 0$ is asymptotically stable within the unstable manifold $W^u \supseteq W^i$, in backwards time direction, and because the finite-dimensional protosphere Σ_δ^{i-1} is compact.
By this construction of t_0, Schoenflies spheres $P^i T(t)\Sigma_\delta^{i-1}$ remain interior to $\tilde{\Sigma}_0^{i-1}$ for $t \leq -t_0$. This proves the interior part of claim (ii), for $m = 0$.

To prove the exterior part, for $m = 0$, we only note that any Schoenflies sphere $\tilde{\Sigma}$ is in the exterior of another Schoenflies sphere $\hat{\Sigma}$ if, and only if, $\hat{\Sigma}$ is in the interior of $\tilde{\Sigma}$. In symbols we denote this relation as $\hat{\Sigma} \lhd \tilde{\Sigma}$. We have already proved

$$P^i T(-t)\Sigma_\delta^{i-1} \lhd \tilde{\Sigma}_0^{i-1} = P^i T(0)\Sigma_\delta^{i-1}, \tag{3.18}$$

for all $t \geq t_0$. Since the flow homeomorphism $T(t)$ on W^i preserves the interior/exterior relation $\lhd$ of Schoenflies spheres, (3.18) implies

$$\tilde{\Sigma}_0^{i-1} = P^i T(0)\Sigma_\delta^{i-1} \lhd P^i T(t)\Sigma_\delta^{i-1}, \tag{3.19}$$

for all $t \geq t_0$. This proves the exterior part of claim (ii), for $m = 0$.
The general case of claim (ii) is now obvious, if we apply the flow homeomorphism $T(mt_0)$, $m \in \mathbb{Z}$, to both sides of (3.18) and (3.19). This proves the lemma.

$\bowtie$

The above lemma formalizes the gate keeper property of the protosphere Σ_δ^{i-1} in W^i: any trajectory $u(t) \in W^i \backslash \{v\}$ has to traverse the gate Σ_δ^{i-1} after lingering and loitering there for less than a time $2t_0$:

$$W^i \backslash \{v\} = \bigcup_{t\in\mathbb{R}} T(t)\Sigma_\delta^{i-1}. \tag{3.20}$$

Constructing the flattened protosphere gate Σ_δ^{i-1} more carefully, transverse to the flow in Σ_δ^{i-1}, we could have reduced loitering time to zero. But why take the trouble and suppress loitering entirely?
As a first application of the gatekeeper protosphere Σ_δ^{i-1} and its protocap constituents $\Sigma_{\delta+}^{i-1} \cup \Sigma_{\delta-}^{i-1} = \Sigma_\delta^{i-1}$, we study the intersection of the hemisphere candidate caps $\Sigma_\pm^{i-1}$ with the closure $\overline{W}^{i-1}$ of the faster unstable manifold W^{i-1} with boundary sphere Σ^{i-2}.

Lemma 3.4

$$\omega(\Sigma_\delta^{i-1}) = \Sigma_+^{i-1} \cup \Sigma_-^{i-1} = \Sigma^{i-1}; \tag{3.21}$$

$$\Sigma_\pm^{i-1} \cap \overline{W}^{i-1} = \Sigma^{i-2}. \tag{3.22}$$

Proof:
We recall the definitions of the boundaries $\Sigma^{j-1} = \overline{W}^j \backslash W^j$ and the hemisphere candidate caps $\Sigma_\pm^{i-1} = \omega(\Sigma_{\delta\pm}^{i-1})$.
To prove claim (3.21) we observe that $\Sigma^{i-1} = \omega(\Sigma_\delta^{i-1})$ by gatekeeper lemma 3.3. Indeed $\Sigma^{i-1} := \overline{W}^i \backslash W^i$ because all of $P^i W^i$ is swept out by the snapshots $\tilde{\Sigma}_{m,\delta}^{i-1}$, $m \to \infty$. This implies (3.21) because

$$\begin{aligned} \Sigma^{i-1} = \omega(\Sigma_\delta^{i-1}) &= \omega(\Sigma_{\delta+}^{i-1} \cup \Sigma_{\delta-}^{i-1}) = \\ &= \omega(\Sigma_{\delta+}^{i-1}) \cup \omega(\Sigma_{\delta-}^{i-1}) = \Sigma_+^{i-1} \cup \Sigma_-^{i-1}, \end{aligned} \tag{3.23}$$

by definition (3.16) of the protosphere Σ_δ^{i-1} and definition (2.17) of ω-limit sets.
To prove claim (3.22) next, we first show $\Sigma^{i-2} \subseteq \Sigma_\pm^{i-1} \cap \overline{W}^{i-1}$. Obviously $\Sigma^{i-2} := \overline{W}^{i-1} \backslash W^{i-1} \subseteq \overline{W}^{i-1}$. Applying gatekeeper lemma 3.3 to $\Sigma^{i-2} := \overline{W}^{i-1} \backslash W^{i-1}$, this time, we also obtain

$$\Sigma^{i-2} = \omega(\Sigma_{\delta\pm}^{i-1} \cap W^{i-1}) \subseteq \omega(\Sigma_{\delta\pm}^{i-1}) =: \Sigma_\pm^{i-1}. \tag{3.24}$$

In the first equality of (3.24) we have used that the δ-sphere $\Sigma_{\delta+}^{i-1} \cap W^{i-1} = \{|\hat{\eta}| = \delta\}$ in $W^{i-1} = \{\eta_{i-1} = \eta_e(\hat{\eta})\}$ sweeps out all of W^{i-1} under $T(t)$, just like the flattened protosphere Σ_δ^{i-1} does in W^i.
It remains to show the converse claim $\Sigma_\pm^{i-1} \cap \overline{W}^{i-1} \subseteq \Sigma^{i-2} := \overline{W}^{i-1} \backslash W^{i-1}$. We proceed indirectly and assume the existence of $u_0 \in \Sigma_\pm^{i-1} \cap W^{i-1} = \omega(\Sigma_{\delta\pm}^{i-1}) \cap W^{i-1}$. By the gatekeeper property (ii) of lemma 3.3 there exists $m \in \mathbb{Z}$ such that $P^i u_0$ is in the interior of some Schoenflies snapshot $\tilde{\Sigma}_m^{i-1}$, whereas $P^i \Sigma^{i-1} = P^i \omega(\Sigma_\delta^{i-1})$ is in the exterior. This contradicts $u_0 \in \Sigma_\pm^{i-1}$, proves claim (3.22), and completes the lemma. $\bowtie$
We now establish how two (distinct) elements

$$u^\kappa = \lim_{n\to\infty} \tilde{u}(t_n^\kappa) \in \Sigma_\pm^{i-1} = \omega(\Sigma_{\delta\pm}^{i-1}),\ \kappa = 1, 2, \tag{3.25}$$

of the boundary hemisphere candidate caps $\Sigma_\pm^{i-1}$ are actually isochronous limits $t_n^1 = t_n^2$, of suitably chosen trajectories $u^{\kappa,n}(t)$ with $u_0^{\kappa,n} := u^{\kappa,n}(0) \in \Sigma_{\delta\pm}^{i-1}$.

Lemma 3.5 *Let* $u^1, u^2 \in \Sigma_+^{i-1} = \omega(\Sigma_{\delta+}^{i-1})$.

Then there exist two sequences $u_0^{\kappa,n} \in \Sigma_{\delta+}^{i-1}$ *and a* single sequence $t_n \to +\infty$ *such that*

$$u^\kappa = \lim_{n\to\infty} u^{\kappa,n}(t_n) \tag{3.26}$$

holds for the trajectories $u^{\kappa,n}(t) := T(t)u_0^{\kappa,n}$*, simultaneously for* $\kappa = 1$ *and* $\kappa = 2$.
The same statement holds true for $u^\kappa \in \Sigma_-^{i-1} = \omega(\Sigma_{\delta-}^{i-1})$.

Proof:
We consider $u^\kappa \in \Sigma_+^{i-1} = \omega(\Sigma_{\delta+}^{i-1})$, only, the case of Σ_-^{i-1} being identical. By definition (2.17) of ω-limit sets we have non-isochronous trajectories $\tilde{u}^{\kappa,n}(t_n^\kappa) \in T(t_n^\kappa)\Sigma_{\delta+}^{i-1} \subseteq W^i$ which approximate $u^\kappa \in \Sigma_+^{i-1} \subseteq \overline{W}^i$ as in (3.25), for $\kappa = 1, 2$. To construct the isochronous trajectories $u^{\kappa,n}(t_n)$ of (3.26) consider any small $\varepsilon > 0$. It is then sufficient to determine $m \in \mathbb{Z}$ such that the same protosphere snapshot

$$\tilde{\Sigma}_m^{i-1} := \; P^i T(mt_0)\Sigma_\delta^{i-1} = P^i T(mt_0)(\Sigma_{\delta+}^{i-1} \cup \Sigma_{\delta-}^{i-1}) \tag{3.27}$$

contains elements, both, in the ε-neighborhood of $P^i u^1$ and $P^i u^2$. Here we use the projection $P^i : \overline{W}^i \to E^i$, which is homeomorphic onto its image by lemma 3.2, to determine convergence for $\varepsilon \to 0$. We also use the separation

$$T(t)\Sigma_{\delta+}^{i-1} \cap T(t)\Sigma_{\delta-}^{i-1} = (T(t)\Sigma_{\delta\pm}^{i-1}) \cap W^{i-1} \tag{3.28}$$

which is inherited from $t = 0$ and allows us to attribute the approximations $u^{\kappa,n} \in T(mt_0)\Sigma_\delta^{i-1}$ of $u^\kappa \in \Sigma_+^{i-1}$ to $T(mt_0)\Sigma_{\delta+}^{i-1}$.
Given $\varepsilon > 0$ we first choose $n_0 \in \mathbb{N}$ large enough such that

$$\text{dist }(P^i \tilde{u}^{\kappa,n}(t_n^\kappa), P^i u^\kappa) < \varepsilon, \tag{3.29}$$

for all $n \geq n_0$ and, always, for $\kappa = 1, 2$ alike. We then choose $n_\kappa > n_0$ such that

$$t_{n_\kappa}^\kappa \geq 3t_0 + \max(t_{n_0}^1, t_{n_0}^2). \tag{3.30}$$

With such a gap of length at least $3t_0$ we can now find $m \in \mathbb{Z}$ such that

$$t_{n_0}^\kappa \leq (m-1)t_0 < (m+1)t_0 \leq t_{n_\kappa}^\kappa. \tag{3.31}$$

By our gatekeeper lemma 3.3(ii) this implies that $P^i\tilde{u}^{\kappa,n}(t)$ are inside the Schoenflies sphere $\tilde{\Sigma}_m^{i-1}$ for $t = t_{n_0}^\kappa$, but outside for $t = t_{n_\kappa}^\kappa$. The two straight lines from $P^i\tilde{u}^{\kappa,n}(t_{n_0}^\kappa)$ to $P^i\tilde{u}^{\kappa,n}(t_{n_\kappa}^\kappa)$ are entirely in the ε-neighborhoods of $P^i u^\kappa$, respectively, for $\kappa = 1, 2$. Each line must contain an element $P^i u^{\kappa,\varepsilon}(mt_0)$ of the same Schoenflies sphere $\tilde{\Sigma}_m^{i-1}$. For $\varepsilon_n \to 0$

the times $t_n := m(\varepsilon_n)$ to provide the isochronous approximants $u^{\kappa,n}(t_n) := u^{\kappa,\varepsilon_n}(m(\varepsilon_n)t_0) \in T(m(\varepsilon_n)t_0)\Sigma^{i-1}_{\delta+}$, as claimed in (3.26). This proves the lemma.

$\bowtie$

We are now ready to prove the culminating injectivity lemma of this section, which asserts injectivity of not just P^i but P^{i-1} on the hemisphere candidate caps $\Sigma^{i-1}_\pm$. Together with surjectivity onto the interior of the equator $P^{i-1}\Sigma^{i-2}$ this will show that $\Sigma^{i-1}_\pm$ are indeed hemisphere caps, in section 4.

Lemma 3.6

(i) Let $u^\kappa \in \Sigma^{i-1}_+$ for $\kappa = 1, 2$. Then

$$z(u^2 - u^1) < i - 1. \tag{3.32}$$

The same statement holds true for $u^\kappa \in \Sigma^{i-1}_-$.

(ii) Each of the two continuous projections

$$P^{i-1}: \quad \Sigma^{i-1}_\pm \to E^{i-1} \tag{3.33}$$

is injective.

Proof: By lemma 3.1, claim (i) implies claim (ii). It only remains to prove (3.32) of claim (i) for any two distinct elements u^1, u^2 of Σ^{i-1}_+, the proof for Σ^{i-1}_- being identical.
As in the proof of lemma 2.2 we proceed indirectly and suppose

$$z(u^2 - u^1) \geq i - 1. \tag{3.34}$$

Possibly after a negative time step $t = -\tau < 0$ in the forward and backward invariant ω-limit set $\Sigma^{i-1}_+ = \omega(\Sigma^{i-1}_{\delta+})$, as in (2.24), we may assume all zeros of $u^2 - u^1$ to be simple. By the Sturm property (1.9)(ii), assumption (3.34) is indeed preserved under such a backward time step.
By Lemma 3.5 there exist isochronously approximating trajectories $u^{\kappa,n}(t_n) \to u^\kappa$ which start at $u^{\kappa,n}_0 \in \Sigma^{i-1}_{\delta+}$ in the upper protocap, for both $\kappa = 1, 2$. Because all zeros of $u^2 - u^1$ are now simple, we conclude

$$\begin{aligned} i - 1 \;\; &\leq \;\; z(u^2 - u^1) = z(u^{2,n}(t_n) - u^{1,n}(t_n)) \leq \\ &\leq \;\; z(u^{2,n}_0 - u^{1,n}_0) < i - 1 \end{aligned} \tag{3.35}$$

for large enough n. Here we have used Sturm property (1.9)(ii) for the second inequality, and lemma 2.2, (2.21) for the protocap elements $u^{\kappa,n}_0 \in \Sigma^{i-1}_{\delta+}$ in the last inequality.
The contradiction (3.35) to (3.34) proves the lemma.

$\bowtie$

4 Surjectivity

In this section we prove the sphere part of theorem 1.1: the projection

$$P^i\Sigma^{i-1} \subseteq E^i = \text{ span } \{\varphi_0, \ldots, \varphi_{i-1}\} \tag{4.1}$$

is homeomorphic to a Schoenflies sphere. By induction hypothesis this is already proved for $P^{i-1}\Sigma^{i-2} \subseteq E^{i-1}$. In lemma 3.4 we have written

$$\Sigma^{i-1} = \Sigma_+^{i-1} \cup \Sigma_-^{i-1} \tag{4.2}$$

as a union of the hemisphere cap candidates $\Sigma_\pm^{i-1}$ which intersect with the faster unstable manifold closure $\overline{W}^{i-1}$ only at its boundary $\Sigma^{i-2} = \partial W^{i-1} = \overline{W}^{i-1} \backslash W^{i-1}$. See also definition (1.41) of $\Sigma_\pm^{i-1}$ as ω-limits of the protocaps $\Sigma_{\delta\pm}^{i-1}$.
In lemma 4.1 below we show that each restricted eigenprojection $P_\pm^{i-1}$ of P^{i-1},

$$P_\pm^{i-1}: \qquad E^i \supseteq P^i\Sigma_\pm^{i-1} \to \bar{B}^{i-1} \subseteq E^{i-1}, \tag{4.3}$$

is a homeomorphism onto the Schoenflies ball $\bar{B}^{i-1}$ with boundary sphere $S^{i-2} := P^{i-1}\Sigma^{i-2}$. We will then denote the inverses of $P_\pm^{i-1}$ by

$$\begin{array}{rrcl} \eta^\pm: & \bar{B}^{i-1} & \to & P^i\Sigma_\pm^{i-1}; \\ & (\eta_0, \ldots, \eta_{i-2}) & \mapsto & (\eta_0, \ldots, \eta_{i-2}, \eta_{i-1}^\pm). \end{array} \tag{4.4}$$

Obviously $\eta_{i-1}^\pm$ are determined by continuous scalar functions $\eta_\pm := \eta_{i-1}^\pm$,

$$\eta_\pm: \quad \bar{B}^{i-1} \to \mathbb{R}. \tag{4.5}$$

The equatorial ball section $\bar{B}^{i-1} := P^i\overline{W}^{i-1}$ of $\bar{B}^i := P^i\overline{W}^i$ is given by the graph of yet another continuous scalar function

$$\eta_e: \quad \bar{B}^{i-1} \to \mathbb{R}. \tag{4.6}$$

See (2.19) for the local version of η_e, which extends globally to $\bar{B}^{i-1} = P^{i-1}\overline{W}^{i-1}$ by injectivity lemma 3.2, (3.12).
In lemma 4.2 we show that

$$\eta_- \leq \eta_e \leq \eta_+ \tag{4.7}$$

in $\bar{B}^{i-1}$. The inequalities turn out to be strict, except at the boundary $S^{i-2} = P^{i-1}\Sigma^{i-2} = \partial\bar{B}^{i-1}$, where all three functions coincide. In particular this finally establishes a hemisphere decomposition

$$\begin{array}{rcl} \Sigma^{i-1} & = & \Sigma_+^{i-1} \cup \Sigma_-^{i-1} \\ \Sigma^{i-2} & = & \Sigma_+^{i-1} \cap \Sigma_-^{i-1} \end{array} \tag{4.8}$$

of Mayer-Vietoris style, with equator Σ^{i-2}, as announced in (1.45).
Therefore the graphs of $\eta_\pm$ define a homeomorphism h of $P^i\Sigma^{i-1}$ to the standard $(i-1)$-sphere $S \subseteq E^i$. Indeed h has been already constructed, for the equatorial ball of S in E^{i-1}, by the induction hypothesis on the Schoenflies sphere Σ^{i-2}, and only has to be lifted to the graph of η_e, via η_e. The intervals $\eta_- \leq \eta_{i-1} \leq \eta_+$ are mapped to the corresponding fibers over the equatorial ball of S, extending from the lower to the upper standard hemisphere. This proves the sphere part of theorem 1.1, up to lemmas 4.1 and 4.2 below.

Lemma 4.1 *The two eigenprojections P^{i-1}, restricted as*

$$P_\pm^{i-1}: \qquad P^i\Sigma_\pm^{i-1} \to E^{i-1}, \tag{4.9}$$

are homeomorphisms onto their shared range $\bar{B}^{i-1} := P^i\overline{W}^{i-1}$, as claimed in (4.3)*.*

Proof:
Injectivity of $P_\pm^{i-1}$ in (4.9) follows from injectivity of

$$P^{i-1}: \qquad \Sigma_\pm^{i-1} \to E^{i-1} \tag{4.10}$$

as proved in lemma 3.6; see (3.33). Indeed $P_\pm^{i-1}P^i\Sigma_\pm^{i-1} = P^{i-1}\Sigma_\pm^{i-1}$ inherits injectivity.
To prove surjectivity, i.e. $\bar{B}_\pm^{i-1} = P^{i-1}\Sigma_\pm^{i-1}$ we first show

$$\bar{B}^{i-1} \subseteq P^{i-1}\Sigma_\pm^{i-1}. \tag{4.11}$$

We proceed with the + case, without loss.
We apply Brouwer degree in the ball $\bar{B}^{i-1} = P^{i-1}\overline{W}^{i-1}$ with Schoenflies sphere boundary S^{i-2}. Fix any $\hat{\eta}_* = (\eta_0, \ldots, \eta_{i-2})$ such that, by slight abuse of notation,

$$\hat{\eta} := \eta_0\varphi_0 + \ldots + \eta_{i-2}\varphi_{i-2} \in B^{i-1} := \bar{B}^{i-1}\backslash S^{i-2}. \tag{4.12}$$

Since $\Sigma_+^{i-1} = \omega(\Sigma_{\delta+}^{i-1})$, by (2.17), it is sufficient to show

$$\hat{\eta}_* \in P^{i-1}T(t)\Sigma_{\delta+}^{i-1} \tag{4.13}$$

for all (large enough) $t \geq 0$.
We define a continuous homotopy $\mathbf{H} : [0,\infty) \times \bar{B}^{i-1} \to E^{i-1}$ as follows. For $\hat{\eta} \in \bar{B}^{i-1}$ with $|\hat{\eta}| \leq \delta$ we define

$$\mathbf{H}(t,\hat{\eta}) := P^{i-1}T(t)u_0, \tag{4.14}$$

where $u_0 \in \Sigma_{\delta+}^{i-1}$ is the unique initial value in the upper protocap for which $P^{i-1}u_0 = \hat{\eta}$; see lemma 3.2, (3.13). For $\hat{\eta} \in \bar{B}^{i-1}$ with $|\hat{\eta}| \geq \delta$ we use $u_0 \in \overline{W}^{i-1}$ in the faster unstable manifold instead; see lemma 3.2, (3.12). Note continuity of the homotopy $\mathbf{H}$, because the protocap $\Sigma_{\delta+}^{i-1}$ intersects $\overline{W}^{i-1}$ precisely at $|\hat{\eta}| = \delta$. Obviously $H(0,\cdot) = \mathrm{id}$, on S^{i-2}. For all $t \geq 0$, we have $\mathbf{H}(t, S^{i-2}) = S^{i-2} = P^{i-1}\Sigma^{i-2}$, by flow-invariance of Σ^{i-2}. Therefore $\hat{\eta}_* \notin \mathbf{H}(t, S^{i-2})$, for any $t \geq 0$. We can therefore apply homotopy invariance of the Brouwer degree $\deg(\mathbf{H}, \bar{B}^{i-1}, \hat{\eta}_*)$ and conclude

$$\deg(\mathbf{H}(t,\cdot), \bar{B}^{i-1}, \hat{\eta}_*) = \deg(\mathbf{H}(0,\cdot), \bar{B}^{i-1}, \hat{\eta}_*) = 1. \tag{4.15}$$

In particular $\hat{\eta}_* \in \mathbf{H}(t, \bar{B}^{i-1})$, for any $t \geq 0$.
To complete the proof of claim (4.11) we choose any sequence $t_0 \leq t_n \to \infty$ large enough, so that

$$\hat{\eta}_* \notin P^{i-1}T(t)u_0, \tag{4.16}$$

for any $t \geq t_0$ and any $u_0 \in \overline{W}^{i-1}$ such that $|P^{i-1}u_0| \geq \delta$. This is certainly possible because $\hat{\eta}_* \notin S^{i-1} = P^{i-1}\Sigma^{i-2} = P^{i-1}\omega(\overline{W}^{i} \cap \Sigma_{\delta+}^{i-1})$: the sphere $|\hat{\eta}| = \delta$ in $\overline{W}^{i-1}$ recedes to the boundary Σ^{i-2} of $\overline{W}^{i-1}$. Therefore $H(t_n, \hat{\eta}^n) = \hat{\eta}_*$ implies $P^{i-1}T(t_n)u_0^n = \hat{\eta}_*$ for some $u_0^n \in \Sigma_{\delta+}^{i-1}$ with $P^{i-1}u_0^n = \hat{\eta}^n$. For $n \to \infty$ we obtain a convergent subsequence

$$T(t_n)u_0^n \to u \in \Sigma_{+}^{i-1} = \omega(\Sigma_{\delta+}^{i-1}) \tag{4.17}$$

such that

$$P^{i-1}u = \hat{\eta}_* = P^{i-1}T(t_n)u_0^n. \tag{4.18}$$

This proves claim (4.11).
To prove, conversely, that

$$P^{i-1}\Sigma_{+}^{i-1} \subseteq \bar{B}^{i-1} \tag{4.19}$$

we argue indirectly. Suppose there exists $u \in \Sigma_{+}^{i-1} = \omega(\Sigma_{\delta+}^{i-1})$ such that $P^{i-1}u$ is strictly outside the Schoenflies sphere $S^{i-2} = \partial\bar{B}^{i-1} = \bar{B}^{i-1} \backslash B^{i-1}$. Then the same holds true for some points in the protocap $T(t)\Sigma^{i-1}_{\delta +}$, and certain sufficiently large $t = t_n \to \infty$. We now invoke injectivity of P^{i-1} on $T(t)\Sigma^{i-1}_{\delta +}$ to see that

$$P^{i-1}T(t)\Sigma^{i-1}_{\delta +} \;\subseteq\; P^{i-1}T(t)(\Sigma^{i-1}_{\delta +} \cap \overline{W}^{i-1}) \;\subseteq\; \bar{B}^{i-1}. \tag{4.20}$$

Indeed, the homeomorphisms $P^{i-1}T(t)$ must map the interior of the protocap $\Sigma^{i-1}_{\delta +}$ to the interior. Passing to the ω-limit proves claim (4.19).
Together (4.19) and (4.12) prove surjectivity of P^{i-1}_{+}. The case of P^{i-1}_{-} is analogous, and the lemma is proved.

$\bowtie$

Lemma 4.2 *The functions $\eta_\pm, \eta_e : \bar{B}^{i-1} \to \mathbb{R}$ defined in* (4.3)–(4.6) *satisfy claim* (4.7)*, i.e. $\eta_- \leq \eta_e \leq \eta_+$ on $\bar{B}^{i-1}$, with equality at the boundary $S^{i-2} = \partial \bar{B}^{i-1}$, only. In particular we have a hemisphere decomposition of Mayer-Vietoris type*

$$\Sigma^{i-1} = \Sigma^{i-1}_{+} \cup \; \Sigma^{i-1}_{-}, \tag{4.21}$$

$$\Sigma^{i-2} = \Sigma^{i-1}_{+} \cap \; \Sigma^{i-1}_{-}. \tag{4.22}$$

as announced in (1.45).

Proof:
By construction and lemmas 3.4, 3.6, 4.1, graph $\eta_\pm = P^i \Sigma^{i-1}_\pm$ are the homeomorphic P^i-images of the upper and lower hemisphere candidates $\Sigma^{i-1}_{+} \cup \Sigma^{i-1}_{-} = \Sigma^{i-1}$; see (3.21). This proves (4.21).
By lemma 3.4, (3.22) we also know

$$\Sigma^{i-1}_{\pm} \cap \overline{W}^{i-1} = \partial \overline{W}^{i-1} = \Sigma^{i-2}. \tag{4.23}$$

Since $\overline{W}^{i-1} =$ graph η_e, this implies $\eta_\pm = \eta_e$ on the boundary sphere $S^{i-2} = P^{i-1}\Sigma^{i-2} = \partial \bar{B}^{i-1}$.
Now fix any $\hat{\eta}_*$ in the interior B^{i-1} of the reference ball $\bar{B}^{i-1} = P^{i-1}\overline{W}^{i-1} = P^{i-1}\Sigma^{i-1}_{\pm}$; see lemma 4.1. It only remains to show

$$\eta_+(\hat{\eta}_*) > \eta_e(\hat{\eta}_*); \tag{4.24}$$

the arguments for $\eta_-(\hat{\eta}_*) < \eta_e(\hat{\eta}_*)$ are analogous.
To show (4.24) we first note time invariance of $W^i \backslash W^{i-1}$. Moreover we can choose a trajectory $u(t) \in W^i \backslash W^{i-1}$, with projection

$$P^i u(t) =: (\hat{\eta}(t), \eta_{i-1}(t)), \tag{4.25}$$

such that $u(\tau)$ is close enough to $\tilde{u} \in \Sigma_+^{i-1}$ with $P^i\tilde{u} = (\hat{\eta}_*, \eta_+(\hat{\eta}_*))$ to satisfy

$$\sigma(t) := \text{ sign } (\eta_{i-1}(t) - \eta_e(\hat{\eta}(t))) = \text{ sign } (\eta_+(\hat{\eta}_*) - \eta_e(\hat{\eta}_*)) \neq 0 \tag{4.26}$$

at $t = \tau$. Indeed (4.17), (4.18) in the proof of lemma 4.1 show that this is possible for some $u(0) := u_0^n \in \Sigma_{\delta+}^{i-1}$ and $\tau := t_n$ such that, in addition

$$P^{i-1}u(\tau) \ = \ \hat{\eta}_* \ = \ P_u^{i-1}. \tag{4.27}$$

In this construction we have used $\eta_+(\hat{\eta}_*) \neq \eta_e(\hat{\eta}_*)$ already.
By backward invariance of $W^i \backslash W^{i-1}$ we have $\sigma(t) \neq 0$, for all $t \leq 0$. Hence

$$\sigma(t) \equiv \sigma \tag{4.28}$$

does not depend on t. To prove claim (4.24) we consider the limit of $\sigma(t)$, for $t \to -\infty$. By lemma 2.1 and because $u(0) \in \Sigma^{i-1}\delta+ \backslash W^{i-1}$ we have

$$\lim_{t\to-\infty} \frac{u(t) - v}{|u(t) - v|_\alpha} = +\varphi_{i-1}, \tag{4.29}$$

in fact with $v \equiv 0$. Moreover $\eta_e(\hat{\eta}) = o(|\hat{\eta}|)$ for $\hat{\eta} \to 0$, because W^{i-1} is tangent to the eigenspace E^{i-1} at $v = 0$. Therefore (4.29) and the faster exponential decay of the $\hat{\eta}(t)$ components, for $t \to -\infty$, imply

$$\sigma = \lim_{t\to-\infty} \frac{\eta_{i-1}(t) - \eta_e(\hat{\eta}(t))}{|\eta_{i-1}(t) - \eta_e(\hat{\eta}(t))|} = \lim_{t\to-\infty} \frac{\eta_{i-1}(t)}{|\eta_{i-1}(t)|} = +1, \tag{4.30}$$

by (4.29). This proves claim (4.24), via (4.26), (4.28), and the lemma. $\bowtie$

5 Interior balls

In the previous sections 3 and 4 we have represented

$$P^i\Sigma^{i-1} = P^i\Sigma_+^{i-1} \cup P^i\Sigma_-^{i-1} \tag{5.1}$$

as the union of two graphs, $P^i\Sigma_\pm^{i-1} =$ graph $\eta_\pm$. Here

$$\eta_\pm : \qquad \bar{B}^{i-1} = P^{i-1}\overline{W}^{i-1} \to \mathbb{R} \tag{5.2}$$

indicate the φ_{i-1}-component of points on the projected hemispheres $P^i\Sigma_\pm^{i-1}$, and $\eta^\pm(\hat{\eta}) = (\hat{\eta}, \eta_\pm(\hat{\eta}))$ represent the inverses of the homeomorphic projections

$$P_\pm^{i-1} : \qquad P^i\Sigma^{i-1} \to \bar{B}^{i-1}. \tag{5.3}$$

In the present section we complete the proof of theorem 1.1 by showing surjectivity of the injective projection

$$P^i : \overline{W}^i \to [\eta_-, \eta_+] \subset E^i. \tag{5.4}$$

Here we represent elements of $E^i = \text{ span } \{\varphi_0, \ldots, \varphi_{i-1}\}$ by their φ_j-coordinates $(\hat{\eta}, \eta_{i-1})$ and define the graph segment

$$[\eta_-, \eta_+] := \{(\hat{\eta}, \eta_{i-1});\ \hat{\eta} \in \bar{B}^{i-1}, \eta_-(\hat{\eta}) \leq \eta_{i-1} \leq \eta_+(\hat{\eta})\}. \tag{5.5}$$

By lemma 4.2 and by the induction hypothesis we already know $\eta_- < \eta_+$ except at the Schoenflies sphere boundary $\hat{\eta} \in S^{i-2} = \partial\bar{B}^{i-1} = P^{i-1}\Sigma^{i-2}$, where $\eta_- \equiv \eta_+$. Therefore $P^i\overline{W}^i$ is a Schoenflies ball, and theorem 1.1 is proved by induction on i, once surjectivity of (5.4) is established in lemma 5.1.

Lemma 5.1 *The eigenprojection P^i projects the closure $\overline{W}^i$ of the fast unstable manifold W^i of the hyperbolic equilibrium $v \equiv 0$ onto the graph segment $[\eta_-, \eta_+]$, i.e.*

$$P^i\overline{W}^i = [\eta_-, \eta_+]. \tag{5.6}$$

Proof:
We first recall that $P^i\partial W^i = P^i\Sigma^{i-1} = P^i\Sigma_+^{i-1} \cup P^i\Sigma_-^{i-1}$ is the union of the graphs of $\eta_\pm$. It remains to prove

$$P^iW^i = ((\eta_-, \eta_+)) \tag{5.7}$$

where $((\eta_-, \eta_+))$ denotes the interior of $[\eta_-\eta_+]$, characterized by strict inequalities in (5.5). We first prove

$$P^iW^i \supseteq ((\eta_-, \eta_+)) \tag{5.8}$$

indirectly. Suppose there exists $\eta = (\hat{\eta}, \eta_{i-1}^0) \in ((\eta_-, \eta_+))\backslash P^iW^i$. Then $\eta_-(\hat{\eta}) < \eta_{i-1}^0 < \eta_+(\hat{\eta})$, and likewise $\eta_-(\hat{\eta}) < \eta_e(\hat{\eta}) < \eta_+(\hat{\eta})$ by lemma 4.2. Let $I \subseteq ((\eta_-, \eta_+))$ denote the closed interval of $(\hat{\eta}, \eta_{i-1})$ with end-points $\eta^1 = (\hat{\eta}, \eta_{i-1}^0)$ and $\eta^2 = (\hat{\eta}, \eta_e(\hat{\eta}))$. Then $\eta^1 \notin P^iW^i$ versus $\eta^2 \in P^iW^{i-1}$.
Therefore the closed interval I from η^1 to η^2 contains a boundary point

$$\eta^c \in I \cap P^i(\overline{W}^i \backslash W^i) = I \cap P^i\Sigma^{i-1}. \tag{5.9}$$

But the intersection on the right hand side of (5.9) is empty. Indeed

$$\begin{aligned} I \cap P^i\Sigma^{i-1} &\subseteq I \cap \ (\text{graph } \eta_+ \cup \text{graph } \eta_-) \\ &\subseteq ((\eta_-, \eta_+)) \cap (\text{graph } \eta_+ \cup \text{graph } \eta_-) = \emptyset \end{aligned} \tag{5.10}$$

by construction. This proves part (5.8) of equality (5.7).
To prove, conversely, that

$$P^iW^i \subseteq ((\eta_-, \eta_+)) \tag{5.11}$$

we first observe that P^iW^{i-1}, the graph of η_e on the open domain B^{i-1}, is contained in $((\eta_-, \eta_+))$, again by lemma 4.2. In particular $0 \equiv v \in ((\eta_-, \eta_+))$. It remains to show

$$P^i(W^i \backslash \{v\}) \subseteq ((\eta_-, \eta_+)). \tag{5.12}$$

From gatekeeper lemma 3.3 we recall

$$W^i \backslash \{v\} = \bigcup_{t \in \mathbb{R}} T(t)\Sigma^{i-1}_{\delta +}; \tag{5.13}$$

see (3.20). This implies the remaining claim (5.12), provided we show

$$P^iT(t)\Sigma^{i-1}_{\delta +} \subseteq ((\eta_-, \eta_+)). \tag{5.14}$$

For $t = 0$, i.e. for the flattened protosphere $\Sigma^{i-1}_{\delta +}$ itself, this follows from $v \in ((\eta_-, \eta_+))$ and openness of $((\eta_-, \eta_+))$ provided $\delta > 0$ is chosen small enough. By continuity of $\eta_{i-1}(t)$ along any trajectory $u(t) \in W^i$, and because injectivity lemma 3.2 implies $P^iW^i \cap \{\eta_-, \eta_+\} = P^iW^i \cap P^i\partial W^i = \emptyset$, claim (5.14) extends to all real times t. This proves (5.12), (5.11), (5.7), the lemma, and completes the proof of theorem 1.1. $\bowtie$

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