

Gaussification and Entanglement Distillation of Continuous-Variable Systems: A Unifying Picture

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Distillation of entanglement using only Gaussian operations is an important primitive in quantum communication, quantum repeater architectures, and distributed quantum computing. Existing distillation protocols for continuous degrees of freedom are only known to converge to a Gaussian state when measurements yield precisely the vacuum outcome. In sharp contrast, non-Gaussian states can be deterministically converted into Gaussian states while preserving their second moments, albeit by usually reducing their degree of entanglement. In this work—based on a novel instance of a noncommutative central limit theorem—we introduce a picture general enough to encompass the known protocols leading to Gaussian states, and new classes of protocols including multipartite distillation. This gives the experimental option of balancing the merits of success probability against entanglement produced.

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Entangled quantum states are the fundamental resources that enable quantum key distribution, quantum communication, and instances of distributed quantum computing. Real physical systems are affected by decoherence and nonideal apparatus that degrades the quality of experimentally preparable quantum states. However, entanglement distillation protocols provide a means of converting many copies of partially entangled states into a smaller number of more entangled states [1]. When entanglement is required over very long distances, distillation can be implemented at regular intervals called repeater nodes [2–4]. Photonic systems that carry entanglement in continuous degrees of freedom are difficult to manipulate arbitrarily. However, so-called *Gaussian* operations are more easily implemented by a combination of beam splitters, phase shifters, and squeezers. Furthermore, preparation of Gaussian states is routine in many laboratories, and such states are especially useful for numerous quantum information tasks.

Unfortunately, Gaussian operations are quite limited in their capacity to distill entanglement. In particular, a series of no-go theorems have shown that with only Gaussian resources, it is impossible to increase entanglement [5]. These results can be circumvented when non-Gaussian resources are available. Given an appropriate resource, Gaussian operations can simultaneously increase the entanglement and make the state more Gaussian, a process we refer to as the “*Gaussification protocol*” (GP) [6] (for steps towards experimental realization of this and related protocols, see Refs. [7]). Alternatively, the theorem can also be circumvented by using a non-Gaussian operation implemented by photon detectors. Entanglement distillation can be achieved by a combination of photon subtraction, a de-Gaussifying operation, and, subsequently, Gaussification [6,8].

While these techniques give hope for simple realizations of quantum information protocols, they do not exploit the richness of Gaussian operations. Specifically, GP utilizes only projections onto the vacuum. These projections are feasible if reliable detectors are available that distinguish zero from one or more photons. However, strictly speaking these detectors do not fall within the realm of Gaussian devices. Performing eight-port homodyne detection [9] and postselecting on the vacuum outcome achieves the same projection, but this will have zero success probability when postselecting *exactly* on the vacuum measurement outcomes. In this work, we prove Gaussification for a wide class of truly Gaussian protocols with nonzero, and tunable, success probabilities for multimode states. All known

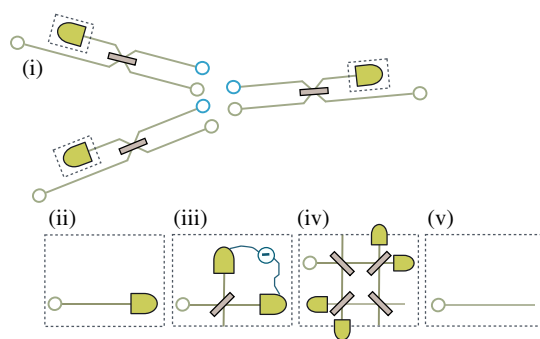


FIG. 1 (color online). (i) A single step of the general class of protocols (GG) considered, illustrated for three parties. This embodies the known Gaussifier (GP) entanglement distillation schemes based on projections onto the vacuum, or the extremality protocol (EP) mapping unknown states onto Gaussian ones with the same second moments. The Gaussian projection can be arbitrary [including (ii) vacuum projection, (iii) homodyning, (iv) eight-port homodyning, (v) tracing out], and schemes with finite widths of acceptance.

feasible distillation protocols, including our protocols, are iterative and consume a number of copies exponential in the iterations required. As such, our scheme's capability to increase success probability significantly improves the prospects of distillation and repeater implementations with only modest resources. Our techniques also open up the perspective of directly distilling into multipartite Gaussian states.

In addition to the practical applications of our results, our analysis provides a more intuitive explanation of the phenomena of Gaussification. In the original distillation protocols [6], the process of Gaussification is quite mysterious, but it is very apparent in the protocol of Ref. [10] which provides an alternative method that uses no measurements, referred to as the “*extremality protocol*” (EP), as it is used to show the extremality of Gaussian states with respect to several properties. Yet EP can convert many non-Gaussian states into a more Gaussian state while conserving the expectation value of observables quadratic in position and momentum. Although the EP does Gaussify, its capacity for increasing entanglement is restrained by its deterministic nature. The proof of EP elegantly employs the central limit theorem [11] that explains the ubiquity of Gaussian distributions in classical statistics and nature itself. Our approach unifies GP and EP within a comprehensive theory of “*general Gaussification*” (GG) founded on a noncommutative central limit theorem and so provides an intuitive mechanism for Gaussification. In addition to bipartite entanglement distillation, our approach reveals whole new classes of protocols illustrated in Fig. 1 and discussed throughout.

We consider GG protocols that can be implemented iteratively, with the $(n+1)$ th iteration as follows: (i) Take two copies of an m -mode state ρ_n shared between m parties; (ii) Each party applies a 50:50 beam splitter transformation between their pair of modes; (iii) Every party makes a Gaussian measurement on the output of one beam-splitter port; (iv) The parties compare measurement results and postselect such that the operation implemented is Gaussian; (v) The output state ρ_{n+1} is used for the next iteration.

Formal description of GG protocols.—There are $2m$ modes involved in the protocol, and we label annihilation operators, $\hat{a}_{j,k}$, with two indices; the index $j = 1, 2, \dots, m$ labels the respective party and $k = 1, 2$ the copy at a particular node. The beam splitters, in the Heisenberg picture, perform

$$U\hat{a}_{j,k}U^\dagger = (\hat{a}_{j,1} + (-1)^k\hat{a}_{j,2})/\sqrt{2}. \quad (1)$$

The measurements at step (iii) can be homodyne, eight-port homodyne, or any other Gaussian measurements projecting onto a state $\Pi_{\mathbf{m}}$ for measurement outcome \mathbf{m} . We are interested in Gaussian protocols that postselect on a set of measurement outcomes, and mix over all accepting outcomes,

$$\rho_{n+1} \propto \int d\mathbf{m} P(\mathbf{m}) \text{tr}^2[U(\rho_n \otimes \rho_n)U^\dagger(\mathbb{1} \otimes \Pi_{\mathbf{m}})], \quad (2)$$

where tr^2 denotes a partial trace over the second copy, with $k = 2$. The integral over measurement outcomes is weighted by $P(\mathbf{m})$. The weights $P(\mathbf{m}) = 0$ and $P(\mathbf{m}) = 1$ correspond to a rejection and an acceptance, respectively, but we also allow for probabilistic strategies where $P(\mathbf{m})$ gives the probability of acceptance. The protocol is described by a single operator we call the *filter*

$$\Pi = \int d\mathbf{m} P(\mathbf{m}) \Pi_{\mathbf{m}}. \quad (3)$$

We allow for arbitrary Gaussian filters, Π , that are invertible and proportional to a fully separable Gaussian state with vanishing first moments and finite energy. Certain interesting cases, such as GP, EP, and protocols using precise homodyne detection are included as limits within this family of filters, so keeping full generality. With this notation

$$\rho_{n+1} \propto \text{tr}^2[U(\rho_n \otimes \rho_n)U^\dagger(\mathbb{1} \otimes \Pi)]. \quad (4)$$

Phase space.—Before we give our results, we review phase space representations for the position and momentum observables of an m -mode system, labeled as

$$\hat{\mathbf{R}} = (\hat{R}_1, \hat{R}_2, \dots, \hat{R}_{2m-1}, \hat{R}_{2m}) = (\hat{X}_1, \hat{P}_1, \dots, \hat{X}_m, \hat{P}_m),$$

where $\hat{X}_j = (\hat{a}_j^\dagger + \hat{a}_j)/\sqrt{2}$ and $\hat{P}_j = i(\hat{a}_j^\dagger - \hat{a}_j)/\sqrt{2}$ for $j = 1, \dots, m$. The canonical commutation relations between the coordinates are embodied in the symplectic matrix Σ . The *covariance matrix* of an operator A records the second moments of these observables.

$$(\Gamma_A)_{j,k} = \text{tr}(\{\hat{R}_j - \langle \mathbf{d}_A \rangle_j, \hat{R}_k - \langle \mathbf{d}_A \rangle_k\}_+ A), \quad (5)$$

where $\{\cdot, \cdot\}_+$ denotes the anticommutator and first moments are $\langle \mathbf{d}_A \rangle_j = \text{tr}(\hat{R}_j A)$. Furthermore, we make use of characteristic functions, $\chi_A: \mathbb{R}^{2m} \rightarrow \mathbb{C}$, that encode all the information of A as $\chi_A(\mathbf{r}) = \text{tr}(D(\mathbf{r})A)$, where $D(\mathbf{r})$ is the displacement operator, $D(\mathbf{r}) = \exp(i\mathbf{r} \cdot \hat{\mathbf{R}})$. Such a function is said to be Gaussian when

$$\chi_A(\mathbf{r}) = \exp(i\mathbf{r} \cdot \mathbf{d}_A - \mathbf{r}^T \Gamma_A \mathbf{r}/4). \quad (6)$$

If the operator A is a physical state, its covariance matrix will be real. However, an instrumental tool in our analysis is that we work with $A = \sigma_n = \rho_n \Pi / \text{tr}(\rho_n \Pi)$. Indeed, we will not employ characteristic functions of states satisfying the conditions of Bochner's theorem [11,12], but of more general objects, hence leading to more general complex-valued functions. Since Π is invertible, σ_n uniquely defines a quantum state ρ_n .

A new noncommutative quantum central limit theorem.—With these definitions at hand we can state our first result, with a stronger form of convergence demonstrated later.

Theorem 1 (Convergence of general Gaussifier protocols).—Consider an initial state $\rho = \rho_0$, with associated operator $\sigma = \rho\Pi/\text{tr}(\rho\Pi)$ such that the following conditions are satisfied: (i) $\mathbf{d}_\sigma = 0$; (ii) $|\chi_\sigma(\mathbf{r})| \leq 1$ for all \mathbf{r} ; (iii) the covariance matrix

$$\Gamma_{\rho_\infty} = (\Gamma_\Pi - i\Sigma)(\Gamma_\Pi - \Gamma_\sigma)^{-1}(\Gamma_\Pi + i\Sigma) - \Gamma_\Pi, \quad (7)$$

exists and is positive definite. Let ρ_∞ denote the Gaussian state with covariance matrix Γ_{ρ_∞} . GG with filter Π causes ρ_n to weakly converge to ρ_∞ in the following sense: If $|x\rangle$ and $|y\rangle$ are eigenvectors of Π , then for all $\epsilon > 0$ there exists a n_ϵ such that for all $n > n_\epsilon$

$$|\langle x|\rho_n|y\rangle/\text{tr}(\rho_n\Pi) - \langle x|\rho_\infty|y\rangle/\text{tr}(\rho_\infty\Pi)| < \epsilon. \quad (8)$$

Proof of the statement.—Much of the basic structure of the proof follows an argument of a noncommutative quantum central limit theorem for quantum states [10–13]. The problem once one allows for Gaussian measurements is that the characteristic function of the output is unwieldy indeed. We circumvent this problem by a bold step: We put in an additional filter Π at the output “by hand”, in order to exploit symmetry, at the expense of having to consider χ_{σ_n} of different objects, σ_n . This will then lead to the desired result. From Eq. (4) we have,

$$\chi_{\sigma_{n+1}}(\mathbf{r}) \propto \text{tr}[(D(\mathbf{r}) \otimes \mathbb{1})U(\rho_n \otimes \rho_n)U^\dagger(\Pi \otimes \Pi)]. \quad (9)$$

Using the cyclicity of the trace and Eq. (1)

$$\chi_{\sigma_{n+1}}(\mathbf{r}) \propto \text{tr}[D(\mathbf{r}/\sqrt{2})^{\otimes 2}\rho_n^{\otimes 2}U^\dagger\Pi^{\otimes 2}U]. \quad (10)$$

Next, we recall that Gaussian states with zero first moments commute with beam splitters, $U\Pi^{\otimes 2} = \Pi^{\otimes 2}U$, such that

$$\begin{aligned} \chi_{\sigma_{n+1}}(\mathbf{r}) &\propto \text{tr}\{[D(\mathbf{r}/\sqrt{2})\rho_n\Pi]^{\otimes 2}\} \propto \text{tr}[D(\mathbf{r}/\sqrt{2})\sigma_n]^2, \\ &= \chi_{\sigma_n}(\mathbf{r}/\sqrt{2})^2 = \chi_\sigma(\mathbf{r}/\sqrt{N})^N, \end{aligned} \quad (11)$$

where in the last equality $N = 2^{n+1}$, iterating the formula. The key to the simplicity of this formula is to consider convergence of σ_n rather than directly ρ_n . By introducing an additional projector within the trace, symmetry allows us to commute through the beam-splitter unitaries, which is the essential simplifying step. To find the limiting characteristic function, we consider a given phase space point \mathbf{r}_0 and the function $f_{\sigma_{n+1},\mathbf{r}_0}:\mathbb{R} \rightarrow \mathbb{C}$ defined as $f_{\sigma_{n+1},\mathbf{r}_0}(t) = \chi_{\sigma_{n+1}}(t\mathbf{r}_0)$. In the spirit of a *classical central limit theorem* [10,11,13] but for non-Hermitian operators we write

$$f_{\sigma,\mathbf{r}_0}\left(\frac{t}{\sqrt{N}}\right)^N \propto \left(1 - t^2 \frac{\mathbf{r}_0^T \Gamma_\sigma \mathbf{r}_0}{4N} + o\left(\frac{t^2}{N}\right)\right)^N, \quad (12)$$

which converges assuming that second moments are finite, $f(0) = 1$ and $|f(t)| \leq 1$ for all t . This last condition, which is always satisfied for classical characteristic functions, may be violated for non-Hermitian σ_n . However, provided $|\chi_\sigma(\mathbf{r})| \leq 1$ for all \mathbf{r} , we find [14] in the limit of large n that

$$\lim_{n \rightarrow \infty} f_{\sigma_{n+1}}(t) \propto \exp(-t^2 \mathbf{r}_0^T \Gamma_\sigma \mathbf{r}_0 / 4), \quad (13)$$

pointwise in t . Setting $t = 1$ shows pointwise convergence of $\chi_{\sigma_{n+1}}$ for each phase space point \mathbf{r}_0 . Furthermore, following the reasoning of Refs. [12,13], this entails that any trace class operator B that is absolutely integrable such that $\int |\chi_B(\mathbf{r})| d\mathbf{r} < \infty$, has a convergent expectation value $\text{tr}(B\sigma_n) \rightarrow \text{tr}(B\sigma_\infty)$ (see Ref. [14]). Setting $B = \lambda_y^{-1}|y\rangle\langle x|$ where $\Pi|y\rangle = \lambda_y|y\rangle$, then for large n

$$\frac{\langle x|\rho_n|y\rangle}{\text{tr}(\rho_n\Pi)} = \frac{\langle x|\sigma_n|y\rangle}{\lambda_y} \rightarrow \frac{\langle x|\sigma_\infty|y\rangle}{\lambda_y} = \frac{\langle x|\rho_\infty|y\rangle}{\text{tr}(\rho_\infty\Pi)}. \quad (14)$$

All that remains is to show that Gaussian σ_∞ entails a Gaussian ρ_∞ . First we observe (see Ref. [14]) that the product of two operators has a characteristic equation

$$\chi_{\sigma_\infty}(\mathbf{q}) \propto \int \chi_{\rho_\infty}(\mathbf{r})\chi_\Pi(\mathbf{q} - \mathbf{r}) \exp(-i\mathbf{r}^T \sigma \mathbf{q} / 2) d\mathbf{r}. \quad (15)$$

For Gaussian χ_Π and χ_{ρ_∞} , the integral is a multivariate Gaussian integral that evaluates (see Ref. [14]) to another Gaussian with the covariance matrix taking the form of a Schur complement [15],

$$\Gamma_\sigma = \Gamma_\Pi - (\Gamma_\Pi + i\Sigma)(\Gamma_{\rho_\infty} + \Gamma_\Pi)^{-1}(\Gamma_\Pi - i\Sigma). \quad (16)$$

Rearranging this formula for Γ_{ρ_∞} gives us the covariance matrix for the convergent state ρ_∞ in terms of Γ_σ and Γ_Π as in the theorem.

Examples.—We have introduced a broad class of protocols for which our theorem indicates Gaussification. However, for concreteness it is helpful to keep in mind a simple class of protocols. Consider when each party performs eight-port homodyne measurements that project onto a coherent state. When the k th party obtains outcome α_k projecting onto coherent state vector $|\alpha_k\rangle$, we declare the iteration a success with probability $P = \exp(-|\alpha|^2/2c^2)$. The degree of postselection is quantified by a real variance c . It follows that the filter is indeed proportional to a Gaussian state with covariance matrix $\Gamma_\Pi = \Delta^{-1}\mathbb{1}$, with $\Delta = (1 + c^2)^{3/2}$. This class of protocols is important as it contains GP and EP as limits $\Delta \rightarrow 1$ and $\Delta \rightarrow 0$, respectively.

We now consider the degree of entanglement that is achieved by applying our protocol with eight-port homodyne measurements. First we consider the well-studied bipartite state vector $|\Psi_\lambda\rangle \propto |0,0\rangle + \lambda|1,1\rangle$, and present the log-negativity [16] of the convergent Gaussian in Fig. 2. Varying the parameter Δ interpolates between the entanglement achieved by GP and EP, with increased yield compensating for reductions in entanglement. We also analyzed a tripartite entangled state vector $|\Phi_\mu\rangle \propto |0,0,0\rangle + \mu(|1,1,0\rangle + |1,0,1\rangle + |0,1,1\rangle)$, having up to two photons in three modes, and for the log-negativity summed over all 3 bipartitions. In fact, in this way, one

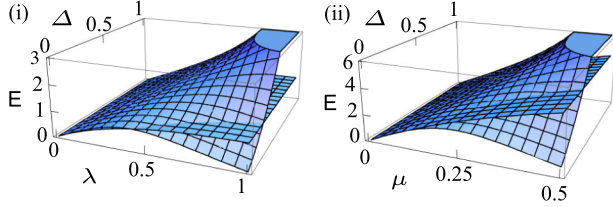


FIG. 2 (color online). The degree of entanglement of the initial state ρ in terms of the log-negativity versus the target Gaussian ρ_∞ for (i) bipartite state vectors $|\Psi_\lambda\rangle$ and (ii) tripartite state vectors $|\Phi_\mu\rangle$. In both figures the variable $0 < \Delta < 1$ controls the degree of postselection. The curve varying with Δ is the entanglement of ρ_∞ , and the curve constant in Δ is the initial entanglement of ρ .

can straightforwardly engineer multipartite hybrid distillation protocols for quantum networks, giving rise to primitives in repeater architectures where entanglement is shared across many repeater nodes. These would overcome the known limitations providing a road block against entirely Gaussian continuous-variable repeater networks [17].

Strong convergence.—Our previous theorem proves a convergence result identical to that of Ref. [6] and GP. However, it would often be preferable to have convergence of $\langle x|\rho_n|y\rangle$ to $\langle x|\rho_\infty|y\rangle$ in a stronger sense (without the addition factors of $\text{tr}(\rho_n\Pi)^{-1}$ and $\text{tr}(\rho_\infty\Pi)^{-1}$ and as a convergence in trace-norm). For most physically relevant (see Ref. [14]) instances of initial quantum states, we now show that this is indeed the case. We will make use of expectation values of normally order operators,

$$\alpha_n^{\mathbf{x},\mathbf{y}} = \text{tr}[V(\otimes_{k=1}^m \hat{a}_{j,k}^{x_j})^\dagger (\otimes_{k=1}^m \hat{a}_{j,k}^{y_j}) V^\dagger \sigma_n], \quad (17)$$

where $\mathbf{x}, \mathbf{y} \in \mathbb{N}^m$ and V is the Gaussian unitary such that $V^\dagger \Pi V$ is a thermal state.

Theorem 2 (Strong convergence).—In addition to Theorem 1, if for all $\mathbf{x}, \mathbf{y} \in \mathbb{N}^m$ the expectations values of normally ordered operators satisfy, $|\alpha_0^{\mathbf{x},\mathbf{y}}| \leq \alpha_\infty^{\mathbf{x},\mathbf{y}}$ for $\alpha_\infty^{\mathbf{x},\mathbf{y}} = |\alpha_\infty^{\mathbf{x},\mathbf{y}}|$. It follows that for all $\epsilon > 0$ there exists an n_ϵ such that for all $n > n_\epsilon$, we have $\|\rho_n - \rho_\infty\| < \epsilon$, where $\|\cdot\|$ is the trace norm.

The conditions of the theorem are stated technically, but physically prevent overpopulation of higher Fock numbers. E.g., it is easy to check these conditions are met in all the low photon examples analyzed in Fig. 2. We begin by first showing $\text{tr}(\Pi\rho_n)$ converges to $\text{tr}(\Pi\rho_\infty)$, which in terms of σ is

$$\text{tr}(\Pi^{-1}\sigma) = \text{tr}(\Pi^{-1}\rho\Pi)/\text{tr}(\rho\Pi) = \text{tr}(\rho\Pi)^{-1}. \quad (18)$$

The inverse filter, Π^{-1} , has an exponential form that can be Taylor expanded and, using the bosonic commutation relations, normally ordered such that $\text{tr}(\Pi^{-1}\sigma_n) = \sum_{\mathbf{x},\mathbf{y}} q_{\mathbf{x},\mathbf{y}} \alpha_n^{\mathbf{x},\mathbf{y}}$, where $q_{\mathbf{x},\mathbf{y}} \geq 0$. Hence, when the conditions

of our theorem hold, we conclude $\text{tr}(\Pi^{-1}\sigma_0) \leq \text{tr}(\Pi^{-1}\sigma_\infty)$, and we proceed by showing this holds for all σ_n . Iteratively we have

$$\alpha_{n+1}^{\mathbf{x},\mathbf{y}} = \sum_{\mathbf{u} \leq \mathbf{x}, \mathbf{v} \leq \mathbf{y}} C_{\mathbf{u},\mathbf{v}}^{\mathbf{x},\mathbf{y}} \alpha_n^{\mathbf{u},\mathbf{v}} \alpha_n^{\mathbf{x}-\mathbf{u}, \mathbf{y}-\mathbf{v}}, \quad (19)$$

where $C_{\mathbf{u},\mathbf{v}}^{\mathbf{x},\mathbf{y}}$ is a combinatorial quantity that is non-negative and real. Under our assumptions the absolute values obey

$$|\alpha_{n+1}^{\mathbf{x},\mathbf{y}}| \leq \sum_{\mathbf{u} \leq \mathbf{x}, \mathbf{v} \leq \mathbf{y}} C_{\mathbf{u},\mathbf{v}}^{\mathbf{x},\mathbf{y}} \alpha_{\infty}^{\mathbf{u},\mathbf{v}} \alpha_{\infty}^{\mathbf{x}-\mathbf{u}, \mathbf{y}-\mathbf{v}} = \alpha_{\infty}^{\mathbf{x},\mathbf{y}}, \quad (20)$$

so initially satisfying our conditions entails satisfaction for all n . Hence, for all n we deduce $\text{tr}(\Pi^{-1}\sigma_n) \leq \text{tr}(\Pi^{-1}\sigma_\infty)$ and equivalently $\text{tr}(\rho_n\Pi) \geq \text{tr}(\rho_\infty\Pi)$. Next we bound $\text{tr}(\rho_n\Pi)$ from above. Consider a finite rank projector P that commutes with Π , then $B = \Pi^{-1}P$ is trace class and has an absolutely integrable characteristic function. Hence, for arbitrarily small $\delta > 0$ there exists an n_δ such that for $n > n_\delta$

$$\text{tr}(B\sigma_n) - \text{tr}(B\sigma_\infty) = \frac{\text{tr}(P\rho_n)}{\text{tr}(\Pi\rho_n)} - \frac{\text{tr}(P\rho_\infty)}{\text{tr}(\Pi\rho_\infty)} \geq -\delta. \quad (21)$$

Since $\text{tr}(P\rho_n) \leq 1$, and P can be chosen so $\text{tr}(P\rho_\infty) = 1 - \delta'$ is arbitrarily close to unity, we conclude $\text{tr}(\Pi\rho_n)^{-1} - \text{tr}(\Pi\rho_\infty)^{-1} \geq -\epsilon$, where $\epsilon = \delta + \delta'/\text{tr}(\rho_\infty\Pi)^{-1}$ is again small. This inequality rearranges to $\text{tr}(\Pi\rho_n) \leq (\text{tr}(\Pi\rho_\infty)^{-1} - \epsilon)^{-1}$, giving an arbitrarily tight bound from above. Combined with our lower bound we conclude that $\text{tr}(\Pi\rho_n)$, under the stated assumptions, converges to $\text{tr}(\Pi\rho_\infty)$. As such $\langle x|\rho_n|y\rangle$ converges to $\langle x|\rho_\infty|y\rangle$, and as is well known this entails trace-norm convergence [12].

Summary.—We have introduced a framework for constructing a range of new protocols for entanglement distillation and manipulation. At the same time, this work provides a unified framework for existing protocols leading to Gaussian states: Notably, the mysterious emergence of Gaussian states in distillation schemes is once again related to an instance of a quantum central limit theorem, albeit for a much broader class of protocols than previously considered. This framework also allows us to look at Gaussification in experimentally realistic acceptance windows, and to trade-off different figures of merit against each other. Such trade-off control is essential as previous proposals are so heavily postselective that over several iterations the success probability would reduce dramatically, whereas our protocols offer an arbitrarily good chance success. Potential for future research is broad as a unique protocol is defined by every separable Gaussian state. For example, our techniques can be applied to homodyne detection protocols for either multimode entanglement distillation, or single mode squeezing enhancement.

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