SERIE B — INFORMATIK

Probability that n random points are in convex position◇

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Abstract

We show that n random points chosen independently and uniformly from a parallelogram are in convex position with probability

$$\left(\frac{\binom{2n-2}{n-1}}{n!}\right)^2.$$

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1 The result

A finite set of points in the plane is called *convex* if its points are vertices of a convex polygon. In this paper we show the following result.

Theorem 1 *The set A of n random points chosen independently and uniformly from a parallelogram S is convex with probability*

$$\left(\frac{\binom{2n-2}{n-1}}{n!} \right)^2 .$$

A large part of studies in stochastic geometry deals with the convex hull C of a set of n points placed independently and uniformly in a fixed convex body K in $\mathbb{R}^d$. Typical questions are: How many vertices does C have? What is the volume of C? What is the surface area of C? See [WW] for a survey. In this paper we settle one very special case – the probability that C has n vertices in the case K is a parallelogram. It is interesting that our approach is purely combinatorial, with no use of integration. We think that our method based on an approximation of the uniform distribution in a square by a large grid might have other applications. However, it is already not clear how to apply our method for K a triangle or in three dimensions.

In this section we prove Theorem 1, and in the next section we mention some applications of Theorem 1.

Proof of Theorem 1. Let $n > 2$ be a fixed integer. Since a proper affine transformation transfers the uniform distribution on S onto the uniform distribution on a square, we may and shall assume that S is a square. We shall approximate the square S by a grid whose size tends to infinity.

Let m be a positive integer (denoting the size of the grid). Partition the (axis–parallel) square S by $m-1$ horizontal and by $m-1$ vertical lines into m^2 squares $S_1, \ldots, S_{m^2}$ of equal size. The centers of the squares $S_1, \ldots, S_{m^2}$ form a square grid $m \times m$. Every point of A lies in each of the squares $S_1, \ldots, S_{m^2}$ with the same probability $1/m^2$. Move every point of A to the center of the square S_i in which it lies, and denote the obtained multiset by $A(m)$. It is not difficult to see that

$$\text{Prob}(A \text{ is convex}) = \lim_{m \to \infty} \text{Prob}(A(m) \text{ is convex}).$$

Thus,

$$\text{Prob}(A \text{ is convex}) = \lim_{m \to \infty} \text{Prob}(R_m \text{ is convex}),$$

where, for every $m \geq 1$, R_m is a multiset of n points chosen randomly and independently from the square grid $G_m = \{(i,j) : i,j = 1,2,\ldots,m\}$ (each point of G_m is always taken with the same probability $1/m^2$).

Let $\mathcal{M}(G_m)$ be the set of all multisets of size n with elements from G_m, and let $\mathcal{C}(G_m)$ be the set of all convex n-element subsets of G_m. It is easy to see that

$$\mathrm{Prob}(A \text{ is convex}) = \lim_{m\to\infty} \mathrm{Prob}(R_m \text{ is convex}) = \lim_{m\to\infty} \frac{|\mathcal{C}(G_m)|}{|\mathcal{M}(G_m)|} = \lim_{m\to\infty} \frac{|\mathcal{C}(G_m)|}{\binom{m^2}{n}}.$$

In the sequel we shall estimate the size of $\mathcal{C}(G_m)$.

Every convex set $R \in \mathcal{C}(G_m)$ is uniquely defined by the smallest axis–parallel rectangle $Q(R)$ containing R and by the set $V(R)$ of the n integer vectors forming the boundary of the convex hull of R oriented in counterclockwise order.

Let $X(R)$ and $Y(R)$ be the multisets of the first and of the second coordinates of vectors in $V(R)$, respectively. Formally,

$$X(R) = \bigcup_{(x,y)\in V(R)} \{x\}, \qquad Y(R) = \bigcup_{(x,y)\in V(R)} \{y\}.$$

Let $\mathcal{C}'(G_m)$ be the set of all convex sets $R \in \mathcal{C}(G_m)$ such that $0 \notin X(R) \cup Y(R)$ and that the directions of the n^2 vectors (x, y) formed by all the n^2 pairs $x \in X(R), y \in Y(R)$ are distinct. Thus, in particular, the multisets $X(R)$ and $Y(R)$ are sets for any $R \in \mathcal{C}'(G_m)$. It is not difficult to see that

$$\lim_{m\to\infty} \frac{|\mathcal{C}'(G_m)|}{|\mathcal{C}(G_m)|} = 1.$$

Therefore,

$$\mathrm{Prob}(A \text{ is convex}) = \lim_{m\to\infty} \frac{|\mathcal{C}(G_m)|}{\binom{m^2}{n}} = \lim_{m\to\infty} \frac{|\mathcal{C}'(G_m)|}{\binom{m^2}{n}}.$$

In the estimation of the size of $\mathcal{C}'(G_m)$ we use an auxiliary set $\mathcal{S}$ defined by

$$\mathcal{S} = \{(X(R), Y(R), Q(R)) : R \in \mathcal{C}'(G_m)\}.$$

The following construction shows that, for every $(X, Y, Q) \in \mathcal{S}$, there are exactly $n!$ sets $R \in \mathcal{C}'(G_m)$ with $(X(R), Y(R), Q(R)) = (X, Y, Q)$:

Take any of the $n!$ one-to-one correspondences $f : X \to Y$ between X and Y, and define a set V of n vectors by $V = \{(x, f(x)) : x \in X\}$. Due to the definitions of $\mathcal{C}'(G_m)$ and $\mathcal{S}$, vectors in V have distinct directions and, consequently, form the (counterclockwise oriented) boundary of the convex hull of a unique set $R \in \mathcal{C}'(G_m)$ fitting into the rectangle Q.

Thus,

$$|\mathcal{C}'(G_m)| = n! \cdot |\mathcal{S}|$$

and

$$\mathrm{Prob}(A \text{ is convex}) = \lim_{m\to\infty} \frac{|\mathcal{C}'(G_m)|}{\binom{m^2}{n}} = \lim_{m\to\infty} \frac{n! \cdot |\mathcal{S}|}{\binom{m^2}{n}}.$$

It remains to estimate the size of the set $\mathcal{S}$ which is done in the sequel technical part of the proof.

For $(X, Y, Q) \in \mathcal{S}$, partition each of the two sets X and Y into two subsets containing elements with the same sign:

$$X^+ = \{x \in X : x > 0\}, \qquad X^- = \{x \in X : x < 0\},$$

$$Y^+ = \{y \in Y : y > 0\}, \qquad Y^- = \{y \in Y : y < 0\}.$$

Suppose that each of the sets X^+, X^-, Y^+, Y^- is ordered in an arbitrary way. Denote $s = |X^+|$ and $t = |Y^+|$. Thus,

$$X^+ = \{x_1, \ldots, x_s\}, \qquad X^- = \{x_{s+1}, \ldots, x_n\},$$

$$Y^+ = \{y_1, \ldots, y_t\}, \qquad Y^- = \{y_{t+1}, \ldots, y_n\}.$$

For every $(X, Y, Q) \in \mathcal{S}$, where $Q = \{(x, y) : a_1 \le x \le a_2,\ b_1 \le y \le b_2\}$, the orders on the sets X^+, X^-, Y^+, Y^- uniquely determine four sets D^-, E^-, D^+, E^+ of integers from the set $\{1, 2, \ldots, m\}$ in the following way:

$$D^+ = \{a_1 + \sum_{i=1}^{k} x_i : k = 0, 1, \ldots, s\}, \quad D^- = \{a_2 + \sum_{i=s+1}^{k} x_i : k = s, s+1, \ldots, n\},$$

$$E^+ = \{b_1 + \sum_{i=1}^{k} y_i : k = 0, 1, \ldots, t\}, \quad E^- = \{b_2 + \sum_{i=t+1}^{k} x_i : k = t, t+1, \ldots, n\}.$$

Note that the sets D^-, E^-, D^+, E^+ satisfy the following conditions:

$$|D^+| + |D^-| = n + 2,\ a_1 = \min D^+ = \min D^-,\ a_2 = \max D^+ = \max D^-, \quad (1)$$
$$|E^+| + |E^-| = n + 2,\ b_1 = \min E^+ = \min E^-,\ b_2 = \max E^+ = \max E^-. \quad (2)$$

For any $(X, Y, Q) \in \mathcal{S}$, we obtain $|X^+|!|X^-|!|Y^+|!|Y^-|!$ different 4-tuples of sets D^-, E^-, D^+, E^+ corresponding to different orders on the sets X^+, X^-, Y^+, Y^-. Denote the set of all these 4-tuples (D^-, E^-, D^+, E^+) by $\mathcal{F}(X, Y, Q)$. Thus,

$$\begin{aligned} |\mathcal{F}(X, Y, Q)| &= |X^+|!|X^-|!|Y^+|!|Y^-|! \\ &= (|D^+| - 1)!(|D^-| - 1)!(|E^+| - 1)!(|E^-| - 1)!, \end{aligned}$$

where (D^-, E^-, D^+, E^+) is an arbitrary 4-tuple in $\mathcal{F}(X, Y, Q)$. For $0 \le i \le n - 2$ and $0 \le j \le n - 2$, we say that a 4-tuple (D^-, E^-, D^+, E^+) of sets of integers has property $\mathcal{P}_{i,j}$ if

$\mathcal{P}_{i,j}$: $|D^+| = i + 2$, $|E^+| = j + 2$, and the sets D^-, E^-, D^+, E^+ satisfy (1) and (2) for some $1 \le a_1 < a_2 \le m$ and $1 \le b_1 < b_2 \le m$.

There are $\binom{n-2}{i}\binom{m}{n} \cdot \binom{n-2}{j}\binom{m}{n}$ 4-tuples (D^-, E^-, D^+, E^+) with $\mathcal{P}_{i,j}$ and $|D^+ \cap D^-| = |E^+ \cap E^-| = 2$. It follows that there are $(1 + o(1)) \cdot \binom{n-2}{i}\binom{m}{n} \cdot \binom{n-2}{j}\binom{m}{n}$ 4-tuples (D^-, E^-, D^+, E^+) with $\mathcal{P}_{i,j}$. (Throughout the proof, $o(1)$ denotes functions of m

which tend to 0 as m tends to infinity.) Most of them (i.e., a $(1-o(1))$-fraction of them) lie in the disjoint union

$$\bigcup_{(X,Y,Q)\in\mathcal{S}} \mathcal{F}(X,Y,Q).$$

Thus,

$$|\mathcal{S}| = \sum_{(X,Y,Q)\in\mathcal{S}} 1 = \sum_{(X,Y,Q)\in\mathcal{S}} \frac{|\mathcal{F}(X,Y,Q)|}{|X^+|!|X^-|!|Y^+|!|Y^-|!} =$$

$$= \sum_{i=0}^{n-2}\sum_{j=0}^{n-2} \frac{(1-o(1))\cdot(1+o(1))\binom{n-2}{i}\binom{m}{n}\binom{n-2}{j}\binom{m}{n}}{(i+1)!(n-i-1)!(j+1)!(n-j-1)!} =$$

$$= (1+o(1))\binom{m}{n}^2 \cdot \sum_{i=0}^{n-2}\sum_{j=0}^{n-2} \frac{\binom{n}{n-i-1}\cdot\binom{n}{n-j-1}}{(n!)^2}\binom{n-2}{i}\binom{n-2}{j} =$$

$$= (1+o(1))\binom{m}{n}^2 \frac{1}{(n!)^2}\left(\sum_{i=0}^{n-2}\binom{n}{n-i-1}\binom{n-2}{i}\right)\left(\sum_{j=0}^{n-2}\binom{n}{n-j-1}\binom{n-2}{j}\right) =$$

$$= (1+o(1))\binom{m}{n}^2 \frac{1}{(n!)^2}\binom{2n-2}{n-1}^2.$$

Hence,

$$\mathrm{Prob}(A \text{ is convex}) = \lim_{m\to\infty} \frac{n!\cdot|\mathcal{S}|}{\binom{m^2}{n}} = \lim_{m\to\infty} \frac{(1+o(1))\binom{m}{n}^2\frac{1}{n!}\binom{2n-2}{n-1}^2}{\binom{m^2}{n}} =$$

$$= \left(\frac{\binom{2n-2}{n-1}}{n!}\right)^2.$$

□

2 Applications and related results

In this section we sketch some applications of our result.

1. Replacing the parallelogram by a convex body. It is known that, for every bounded convex body K in the plane, there are two parallelograms, one containing K and one contained in K, whose areas differ from the area of K at most by a constant factor (e.g., see [Ba] for analogous results). Using this result and Theorem 1, it is not difficult to show that there are two positive constants c_1 and c_2 such that the set of n points chosen independently and uniformly from an arbitrary convex body is convex with probability at least $\left(\frac{c_1}{n}\right)^n$ and at most $\left(\frac{c_2}{n}\right)^n$.

2. The expected area of a random triangle. It is not difficult to show that

$$\mathrm{Prob}(A \text{ is convex}) + 4 \cdot E[\text{Area of T}] = 1,$$

where A is a set of four random points selected independently and uniformly from a convex body S of area 1, and T is a triangle with random vertices selected also independently and uniformly from S. If S is a parallelogram, Theorem 1 yields that the expected area of T is

$$\frac{1-(5/6)^2}{4} = \frac{11}{144},$$

which was also shown in [He] by a different method.

3. Convex subsets of a random set. The author originally considered Theorem 1 in connection with the following result.

Theorem 2 *Let A be a set of n random points chosen independently and uniformly from a parallelogram. Let $c(A)$ be the largest convex subset of A. Set $h = 2^{4/3}e \approx 6.85$. Then $c(A) \geq \lambda n^{1/3}$ with probability smaller than $\left(\frac{h}{\lambda}\right)^{3\lambda n^{1/3}}$, for any $\lambda \geq h$.*

Proof. Let $\lambda \geq h$. For simplicity, assume that $\lambda n^{1/3}$ is an integer. The set A contains $\binom{n}{\lambda n^{1/3}}$ subsets of size $\lambda n^{1/3}$. According to Theorem 1, each of them is convex with probability

$$\left(\frac{\binom{2\lambda n^{1/3}-2}{\lambda n^{1/3}-1}}{(\lambda n^{1/3})!}\right)^2.$$

It follows that the expected number of convex independent subsets of A of size $\lambda n^{1/3}$ is at most

$$\binom{n}{\lambda n^{1/3}} \cdot \left(\frac{\binom{2\lambda n^{1/3}-2}{\lambda n^{1/3}-1}}{(\lambda n^{1/3})!}\right)^2 < \frac{n^{\lambda n^{1/3}}}{\left(\frac{\lambda n^{1/3}}{e}\right)^{\lambda n^{1/3}}} \cdot \left(\frac{4^{\lambda n^{1/3}}}{\left(\frac{\lambda n^{1/3}}{e}\right)^{\lambda n^{1/3}}}\right)^2 =$$

$$= \left(\frac{4^2 e^3}{\lambda^3}\right)^{\lambda n^{1/3}} = \left(\frac{h}{\lambda}\right)^{3\lambda n^{1/3}}.$$

Consequently, A contains a convex independent subset of size $\geq \lambda n^{1/3}$ with probability smaller than $\left(\frac{h}{\lambda}\right)^{3\lambda n^{1/3}}$. $\square$

One application of Theorem 2 on so–called dense sets may be found in the author's PhD. thesis [Va].

By a more careful handling with the result of Theorem 1, one can prove that, for any $\varepsilon > 0$ and any sufficiently large $n \geq n(\varepsilon)$,

$$(h/2-\varepsilon)n^{1/3} \leq c(A) \leq hn^{1/3}$$

holds with a high probability.

4. Construction of random convex sets. Emo Welzl pointed out that the above proof of Theorem 1 yields a fast way how to construct a random convex set of size n in a square. Let M_n be the set of all n-element subsets of a square S, and let μ be the probabilistic measure on M_n corresponding to a choice of n points selected independently and uniformly from the square S. Let C_n be the set of all convex n-element subsets of S. Theorem 1 gives $\mu(C_n) = \left(\binom{2n-2}{n-1}/n!\right)^2$. The measure $\mu' = \mu/(\mu(C_n))|_{C_n}$ is a probabilistic measure on C_n. With respect to μ', a random convex set $A \in C_n$ can be constructed in a straightforward way by repeated choosings of an n-point random subset A of S with respect to μ, until the set A is convex. However, this procedure has the expected running time at least $\Omega(R \cdot \left(n!/\binom{2n-2}{n-1}\right)^2) = \Omega(R \cdot (n/4e)^{2n+1})$, where R is the time required for finding a random real number uniformly distributed in the interval $[0,1]$. The above proof of Theorem 1 yields a procedure which constructs a random convex set with respect to μ' essentially faster, in time $\mathcal{O}(n \log n + n \cdot R + P(n))$, where R is as above and $P(n)$ is the time required for constructing a random permutation of the set $\{1, 2, \ldots, n\}$. Of course, the argument also applies for any parallelogram.

5. The limit shape of a random convex set. Scale and shift the square grid $n \times n$ so that it fits into the square $S = \{(x,y) : -1 \leq x, y \leq 1\}$, and consider the set $K(n)$ of all its convex subsets. Bárány [Bá] proved that for every $\varepsilon > 0$ there exists n_0 such that for every $n \geq n_0$ the following holds: if we randomly choose an element A of $K(n)$, each with the same probability, then the Hausdorff distance between the boundary of the convex hull of A and the curve $\{(x,y) : \sqrt{1-|x|} + \sqrt{1-|y|} = 1\}$ is smaller than ε with a high probability. (Hausdorff distance between two sets is the maximum distance of a point in any of the two sets to the other set.)

It is interesting that random convex sets have the same limit shape. Consider the square $S = \{(x,y) : -1 \leq x, y \leq 1\}$ again, and define C_n and μ' as in the above paragraph *"Construction of random convex sets"*. With a help of the above proof of Theorem 1, it can be shown that for every $\varepsilon > 0$ there exists n_0 such that for every $n \geq n_0$ the following holds: if we randomly choose an element A of C_n with respect to the measure μ', then the Hausdorff distance between the boundary of the convex hull of A and the curve $\{(x,y) : \sqrt{1-|x|} + \sqrt{1-|y|} = 1\}$ is smaller than ε with a high probability.

Let us note that the limit shape curve of the boundary of the convex hull of n random points chosen independently and uniformly inside any planar convex body K is (obviously) the perimeter of K.

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