Interval Reductions and Extensions of Orders: Bijections to Chains in Lattices

Stefan Felsner[1], Jens Gustedt[2] and Michel Morvan[3]

[1] *Freie Universität Berlin, Fachbereich Mathematik und Informatik, Takustr. 9, 14195 Berlin, Germany*
E-mail: felsner@inf.fu-berlin.de

[2] *Technische Universität Berlin, Fachbereich Mathematik Strae des 17. Juni 136, MA 611 10623 Berlin, Germany*
E-mail: gustedt@math.tu-berlin.de

[3] *LIAFA Université Paris 7 Denis Diderot Case 7014 - 2, place Jussieu, 75251 Paris Cedex 05, France*
E-mail: morvan@liafa.jussieu.fr

Abstract. We discuss bijections that relate families of chains in lattices associated to an order P and families of interval orders defined on the ground set of P. Two bijections of this type have been known:

(1) The bijection between maximal chains in the antichain lattice $\mathcal{A}(P)$ and the linear extensions of P.

(2) A bijection between maximal chains in the lattice of maximal antichains $\mathcal{A}_M(P)$ and minimal interval extensions of P.

We discuss two approaches to associate interval orders to chains in $\mathcal{A}(P)$. This leads to new bijections generalizing Bijections 1 and 2. As a consequence we characterize the chains corresponding to weak-order extensions and minimal weak-order extensions of P.

Seeking for a way of representing interval reductions of P by chains we came up with the separation lattice $\mathcal{S}(P)$. Chains in this lattice encode an interesting subclass of interval reductions of P. Let $\mathcal{S}_M(P)$ be the lattice of maximal separations in the separation lattice. Restricted to maximal separations the above bijection specializes to a bijection which nicely complements 1 and 2.

(3) A bijection between maximal chains in the lattice of maximal separations $\mathcal{S}_M(P)$ and minimal interval reductions of P.

Mathematics Subject Classifications (1991). 06A07, 06A06.

Key Words. Chains, lattice of antichains, bijection, linear extension, interval extension, interval reduction, weak order.

20. April 1998

1 Introduction

1.1 Overview

Several lattices can be associated to a partial order P and reflect different aspects of the structure of P. The most prominent examples are

- the Dedekind-MacNeille $\mathcal{L}(P)$ completion of P,
- the lattice of antichains $\mathcal{A}(P)$, and
- the lattice of maximal antichains $\mathcal{A}_M(P)$.

We consider families of chains in such lattices related to P, namely families of chains that bijectively correspond to certain families of orders related to P. Examples for such families of orders are

- interval extensions of P,
- interval reductions of P, and
- interval orders that are essential for P, in particular essential weak-orders. (See Section 1.3 for the definition of "essential").

In the sequel we give several such bijections. We begin with a classical result appearing in early work of Bonnet, Monjardet, Pouzet and Stanley [13]. It connects chains in the lattice of antichains $\mathcal{A}(P)$ of an order P with linear extensions.

Bijection 1 *There is a bijection between the linear extensions of P and maximal chains in $\mathcal{A}(P)$.*

Some years ago Habib et. al. [7] gave a theorem which is similar in spirit. It connects chains in the lattice of maximal antichains $\mathcal{A}_M(P)$ of an order P with minimal interval extensions.

Bijection 2 *There is a bijection between the minimal interval extensions of P and maximal chains in $\mathcal{A}_M(P)$.*

In Section 1.3 we define what it means for an interval representation or interval order to be essential for P or strongly-essential for P. We relate these notions to extensions and reductions of P. In Subsection 1.5 we digress to prove a theorem about interval order extensions: Restrict the extension lattice of an n-element oredered set to those extensions which are interval orders. It is shown, Theorem 1, that the covers of this restriction are covers of the extension lattice, i.e., correspond to the addition of a single comparability.

In Section 2 we characterize classes of interval orders corresponding bijectively to chains in $\mathcal{A}(P)$. First, Bijection 3 relates these chains to essential weak-orders. This extends to a characterization of those chains corresponding to weak-order extensions, Proposition 1, and minimal weak-order extensions, Proposition 2. Bijection 4 is a second bijection involving chains in $\mathcal{A}(P)$. It maps the chains to strongly-essential representations of interval extensions of suborders of P. This extends to a characterization of those chains corresponding to strongly-essential interval extensions of P, Proposition 3.

In Section 3 we show that Bijection 4 specializes to Bijection 2 when restricted to maximal chains in $\mathcal{A}_M(P)$. In particular we obtain a new and more transparent proof for Bijection 2.

If the order P is an interval order Bijection 2 is the well known characterization of interval orders by the sequence of the maximal antichains. Another characterization of interval orders involving a linear order on the predecessor and successor sets of single elements motivates the definition of two new lattices associated to an order P. In Section 4 we define and study the lattice of separations $\mathcal{S}(P)$ and the lattice of maximal separations $\mathcal{S}_M(P)$. Interval orders are characterized by the property that $\mathcal{S}_M(P)$ is a chain. We also characterize the lattice $\mathcal{S}_M(P)$ for N-free orders P.

Bijection 5, in Section 5, relates chains in $\mathcal{S}(P)$ and essential representations of interval reductions of P. In Proposition 4 we characterize a class of chains generating each essential interval reduction exactly once.

In Section 6 we restrict Bijection 5 to maximal chains of maximal separations. Bijection 6 is a bijection between maximal chains in $\mathcal{S}_M(P)$ and maximal interval reductions of P. Note that bijections 2 and 6are both relate maximal chains in some lattice associated to P to extremal interval extensions or reductions. When restricted to interval orders both bijections reduce to a classical characterization of interval orders.

Finally, in Section 7 we show that our bijections may help solving optimization and counting problems. The idea is to use dynamic programming such that the dependency graph of the dynamic program is one of the lattices $\mathcal{A}(P)$, $\mathcal{A}_M(P)$, $\mathcal{S}(P)$ or $\mathcal{S}_M(P)$. The time complexity of this approach depends on the size of the lattice. In some cases this size is polynomially bounded and we obtain polynomial algorithms, e.g., maximal interval reductions of an N-free order can be counted in quadratic time.

1.2 Basics

An *order* $P = (V, <_P)$ consists of a finite set V of *elements of* P and the *relations of* P which are pairs (x, y) of elements with $x <_P y$. The relation $<_P$ is transitive and irreflexive. In some cases we will see a reflexive order relation $\leq_P$.

An order Q is an *interval order* if there is a pair (l, r) of functions assigning to each element $x \in V_Q$ real numbers l_x, r_x on the real line so that $l_x < r_x$ for all $x \in V_Q$, i.e., (l_x, r_x) is an open interval, and $x <_Q y$ if and only if $r_x \leq l_y$ for all $x, y \in V_Q$. The pair (l, r) is called a *representation* of the interval order Q. (See Möhring [10] for a good introduction to interval orders.)

Definition 1 *An* interval reduction *of an order P is an interval order Q on the same ground set such that $x <_Q y$ implies $x <_P y$ for all x, y. An interval reduction Q of P is a* maximal interval reduction *if there is no interval reduction R between Q and P, i.e., with $<_Q \subset <_R \subseteq <_P$.*

An interval extension *of P is an interval order Q on the same ground set such that $x <_P y$ implies $x <_Q y$ for all x, y. An interval extension Q of P is a* minimal interval extension *if there is no interval extension R between P and Q, i.e., with $<_P \subseteq <_R \subset <_Q$.*

1.3 Interval Representations

Throughout this paper we work with *integer representations* of interval orders, i.e., with representations in which all interval end-points are non-negative integers. An integer representation (l, r) such that there is a positive integer K with $l(V) \cup r(V) = \{0, 1, \ldots, K\}$ is called a *dense representation.* The number K is the *magnitude of the representation.* The *magnitude* $\mu(Q)$ *of an interval order* Q is defined as the minimal magnitude of an interval representation of Q.

It is well known (see [10, 4]) that any interval order Q has a unique representation of magnitude $\mu(Q)$ this representation is called the *canonical representation.* The canonical representation of Q is closely related to the lattice of maximal antichains $\mathcal{A}_M(Q)$ of Q.

Characterization 1 $\mathcal{A}_M(P)$ *is a chain iff P is an interval order.*

The canonical representation (l, r) of Q can be obtained from the chain $A_1, .., A_k$ of maximal antichains of P by defining $l_x = -1 + \min\{i : x \in A_i\}$ and $r_x = \max\{i : x \in A_i\}$. Therefore, the magnitude of an interval order is just the number of maximal antichains of the order. Another nice characterization of the canonical representation of Q is the following: Given a representation (l, r) of magnitude K such that $l(V) = \{0, 1, \ldots, K-1\}$ and $r(V) = \{1, \ldots, K-1, K\}$ then $K = \mu(Q)$ and (l, r) is the canonical representation of Q.

Given a dense representation of an interval order Q let $D_i = \{x \in V_Q : r_x \leq i\}$. Each D_i for $i = 0, .., K$ is an ideal (downward closed set, i.e., $x \in D$ and $y <_Q x$ implies $y \in D$) of Q, hence we obtain a chain

$$\emptyset = D_0 \subseteq D_1 \subseteq D_2 \subseteq \ldots \subseteq D_K = V_Q \tag{1}$$

of ideals. Symmetrically, $U_i = \{x \in V_Q : i \leq l_x\}$ is a filter (upward closed set) of Q and

$$V_Q = U_0 \supseteq U_1 \supseteq U_2 \supseteq \ldots \supseteq U_K = \emptyset \tag{2}$$

is a chain of filters. Let M_i be the set of elements whose intervals contain $(i-1,i)$. Obviously M_i is an antichain of Q and D_{i-1}, M_i, U_i is a partition of V_Q.

Definition 2 *Let $P = (V, <_P)$ be an order and $Q = (V, <_Q)$ be an interval order on the same ground set. A dense representation of Q is called an* essential representation for P *if D_i is an ideal of P and U_i is a filter of P for all $i = 0,..,K$. An interval order Q is called* essential for P *if Q admits an essential representation for P.*

Note that if Q is an essential interval order for P then Q can be an extension of P, it can be a reduction of P or it can be neither extension nor reduction. Note also that the conditions on the ideals and on the filters in the definition of essential representations are independent of each other. To see this consider $V = \{a, b\}$ and P on V with $a < b$. The interval order Q on V with representation $(l_a, r_a) = (1, 2)$ and $(l_b, r_b) = (0, 2)$ has $D_0 = D_1 = \emptyset$ and $D_2 = V$, hence all D_i are ideals of P but $U_1 = \{a\}$ which is not a filter of P.

For $X \subset V$ let $X^{\downarrow}$ be the smallest ideal containing X, i.e., $X^{\downarrow} = \{y : y \leq_P x \text{ for some } x \in X\}$, and let $X^{\uparrow} = \{y : x \leq_P y \text{ for some } x \in X\}$ be the smallest filter containing X. With this additional notation we can characterize essential representations of interval orders as those with $D_i = D_i^{\downarrow}$ and $U_i = U_i^{\uparrow}$ for all i.

Definition 3 *A dense representation of an interval order Q is called a* strongly-essential representation for P *if the ideals of M_i taken in Q and P equal each other for all i. Formally noted this condition is $M_i^{\downarrow} = D_{i-1} \cup M_i$. Interval order Q is called* strongly-essential for P *if Q admits a strongly-essential representation for P.*

Observe that the definition of strongly-essential is asymmetric. It only imposes conditions on ideals and not on filters.

Lemma 1 *A representation (l, r) of an interval order Q which is strongly-essential for P is also essential for P.*

Proof. Let I be an ideal in P. Let M be the set of maximal elements of I, i.e., $M = \text{MAX}(I)$, then $I \setminus M$ is an ideal and the complement $V \setminus I$ of I is a filter of P.

We first show that D_i is an ideal. $D_i = (D_i \cup M_{i+1}) \setminus M_{i+1} = M_{i+1}^{\downarrow} \setminus M_{i+1}$ which is an ideal by the above fact. Now consider U_i. Since D_{i-1}, M_i, U_i is a partition $U_i = V \setminus (D_{i-1} \cup M_i) = V \setminus M_i^{\downarrow}$ which is a filter by the above. □

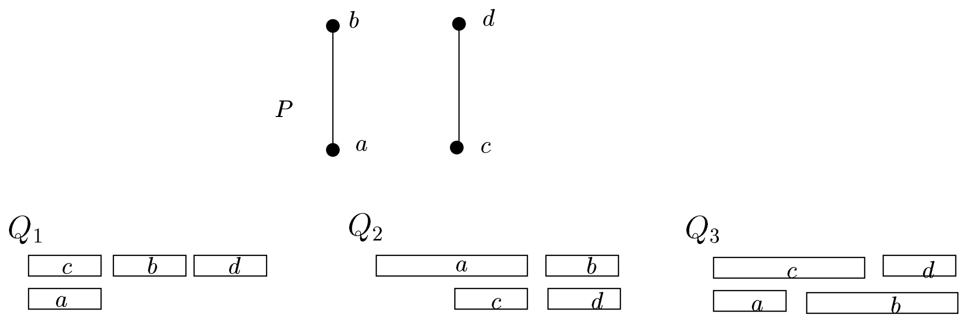


Figure 1: Representations of interval extensions

1.4 Essential Representations for Interval Extensions and Reductions

Now we are able to establish basic connections between essential interval representations and maximality and minimality of interval reductions and extension.

Lemma 2 *Every representation of an interval extension Q of P is essential.*

Proof. Let $x \in D_i$, since Q is an extension of P the set $\{y : y <_P x\}$, i.e., the ideal generated by x is a subset of D_i. Since $D_i^{\downarrow} = \bigcup_{x \in D_i} \{y : y <_P x\}$ and unions of ideals are ideals we obtain $D_i^{\downarrow} = D_i$. The dual argument shows $U_i^{\uparrow} = U_i$. □

With respect to the property *strongly-essential* representations of interval extensions of P behave less nicely. This is exemplified in Example 1.

Example 1 Let $P = II$, the parallel composition of two 2-element chains. Figure 1 displays three represented interval extensions of P. The representation of Q_1 is canonical but Q_1 has no strongly-essential representation. The representation of Q_2 is strongly-essential but not canonical. Note that, because of the asymmetry in the definition of strongly-essential the reflection of the representation Q_2 is not strongly-essential for the dual of P. Q_3 is a minimal extension of P, the representation is canonical and hence, by Lemma 4 strongly-essential.

The next lemma shows that a strongly-essential representation fulfills one side of the defining conditions for canonical representations.

Lemma 3 *If (l, r) is a dense representation of magnitude K and (l, r) is strongly-essential for some P then $l(V) = \{0, \ldots, K-1\}$.*

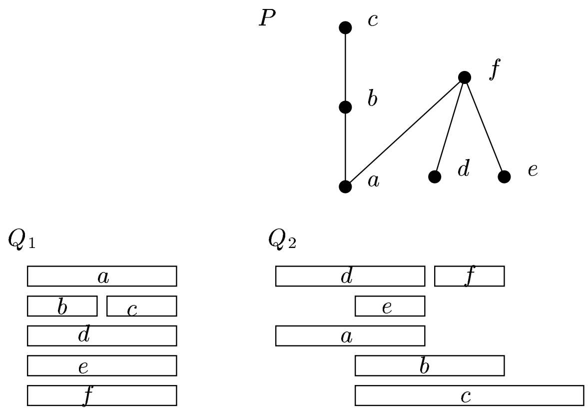


Figure 2: Representations of interval reductions

Proof. If $i \notin l(V)$ then $M_{i+1} \subset M_i$. Let $x \in M_i \setminus M_{i+1}$ since x is incomparable with all elements of M_{i+1} it is not in $M_{i+1}^{\downarrow}$. Since $x \in D_i$ we find that $M_{i+1}^{\downarrow} \neq D_i \cup M_{i+1}$ in contradiction to the definition of strongly-essential. □

Lemma 4 *The canonical representation of a minimal interval extension Q of P is strongly-essential for P.*

Proof. Let (l, r) be the canonical representation of a minimal interval extension Q of P. Since Q is an extension $M_i^{\downarrow} \subseteq D_{i-1} \cup M_i$ for all i. We have to show that they are in fact equal.

Suppose there is $x \in (D_{i-1} \cup M_i) \setminus M_i^{\downarrow}$ such that the right endpoint of the interval of x is $r_x \leq i - 1$ but all y with $x <_P y$ have $l_y \geq i$. Then, redefining $r_x^{new} = i$ gives an interval extension of P. Since (l, r) is canonical the set $U_{i-1} \setminus U_i$ is nonempty and the new representation has fewer relations then the old. This contradicts the minimality of the interval extension Q. □

We now turn to interval reductions and their relation to the notion of essential representations.

Lemma 5 *The canonical representation (l, r) of a maximal interval reduction Q of P is essential for P.*

Proof. Assume, that D_i is not an ideal for some i. Let $y \in D_i^{\downarrow} \setminus D_i$. There is an $x \in D_i$, i.e., $r_x \leq i$, with $y <_P x$. All elements in U_i are greater than x in Q, hence, they are greater that x and y in P. This shows that redefining $r_y^{new} = i < r_y^{old}$ gives an interval reduction of P. Since (l, r) is canonical the set $U_i \setminus U_{i+1}$ is nonempty and the new interval reduction has more relations then the old. This contradicts the maximality of interval reduction Q. The evidence that $U_i^{\uparrow} = U_i$ is obtained by a completely symmetric argument. □

Example 2 Figure 2 shows an order P with three represented interval reductions. The representation of Q_1 is canonical but Q_1 has no essential representation for P. The representation of Q_2 is essential but not canonical. Interchanging r_b and r_c gives a non-essential representation of Q_2. Q_3 is a maximal reduction of P, the representation is canonical and hence, by Lemma 5 essential.

1.5 Interval Orders in the Extension Lattice

Consider interval orders on the ground set V as elements of the *extension lattice* $\text{Ext}(V)$. This lattice has as elements all orders on the ground set V with relations $P <_{\text{Ext}} Q$ iff $<_P \subset <_Q$. The $\hat{0}$ of the lattice is the antichain on V and there is an artificial $\hat{1}$ above the linear orders on V. $\text{Ext}(V)$ is a ranked lattice where the rank of an element P of $\text{Ext}(V)$, i.e., of an order P, is the number of relations of P. We show that the set of interval orders is in a certain sense dense in $\text{Ext}(V)$.

Theorem 1 *Let Q_1 and Q_2 be interval orders with $Q_1 <_{\text{Ext}} Q_2$. If Q_2 has at least two relations more then Q_1 then there is an interval order R in between, i.e., with $Q_1 <_{\text{Ext}} R <_{\text{Ext}} Q_2$.*

The proof is based on the following lemma.

Lemma 6 *Let Q_1 and Q_2 be interval orders with $Q_1 <_{\text{Ext}} Q_2$ and let Q_2 have at least two relations more then Q_1. Then there is a pair x, y of elements with $x < y$ in Q_2 and in the canonical representation (l, r) of Q_1 $r_x = l_y + 1$, hence $x||y$ in Q_1.*

Proof. (Theorem 1) Let (l, r) be the canonical representation of Q_1 and x, y be a pair of elements as in Lemma 6. Define R as the interval order with representation (l', r') where $l'_z = l_z$ for all $z \neq y$ and $l'_y = l_y + 1/2$ and $r'_z = r_z$ for all $z \neq x$ and $r'_x = r_x - 1/2$. The relations of R are the relations of Q_1 together with $x < y$. □

Proof. (Lemma 6) Let $A_1, A_2, \ldots, A_{K_1}$ be the unique chain of maximal antichains in $\mathcal{A}_M(Q_1)$ and let $B_1, B_2, \ldots, B_{K_2}$ be the chain of maximal antichains in $\mathcal{A}_M(Q_2)$. Recall that the canonical representation (l, r) of Q_1 is given by $l_x = -1 + \min\{i : x \in A_i\}$ and $r_x = \max\{i : x \in A_i\}$.

Let i be minimal such that $A_i \neq B_i$. We claim that $B_i \subset A_i$. Otherwise the fact that A_i a maximal antichain of Q_1 would imply that two elements of B_i are comparable in Q_1. Since elements of B_i are an antichain of Q_2 this contradicts $Q_1 <_{\text{Ext}} Q_2$.

If there is a x in $A_i \setminus B_i$ with $x \in B_{i-1}$. Then in Q_2 element x is less then all elements not in $\bigcup_{j<i} B_j = \bigcup_{j<i} A_j$. Since $r_x \geq i$ any element y with $l_y = r_x - 1$ will supplement x to form a pair x, y as claimed.

We now assume that there is no x as in the previous paragraph. Let x be any element in $B_i \setminus B_{i+1}$. Such an x exists since B_i is a maximal antichain. From the definitions we have $x < z$ in Q_2 for all elements not in $\bigcup_{j \leq i} B_j$. Obviously, $r_x \geq i$ in the canonical representation of Q_1. If $r_x > i$ then as in the previous paragraph we may choose any element y with $l_y = r_x - 1$ to form with x a pair x, y as claimed.

If x chosen as before has $r_x = i$ then $x < y$ in Q_2 for all $y \in (A_i \setminus B_i) \setminus B_{i-1} = A_i \setminus B_i$ since we assume that the first case doesn't apply. Again the same assumption implies that such a y exists and has $l_y = i - 1$. With x, y we then have found a pair x, y as claimed. □

2 Chains in the Lattice of Antichains

In this section we characterize two classes of orders corresponding to chains in the lattice of antichains $\mathcal{A}(P)$ of an order P. We begin with a brief review of some basic facts about $\mathcal{A}(P)$.

The lattice $\mathcal{A}(P)$ is most conveniently described as the set of all ideals of P ordered by inclusion. We take this as definition. Since unions and intersections of ideals are ideals the lattice $\mathcal{A}(P)$ is distributive. The fundamental theorem of finite distributive lattices states that for every finite distributive lattice L there is an order P such that L is isomorphic to $\mathcal{A}(P)$ (see [14], Chapter 3).

For our naming of $\mathcal{A}(P)$ as the lattice of antichains of P recall the one-to-one correspondence between ideals and antichains: With an ideal I of P associate the antichain $A_I = \textsc{Max}(I)$ of maximal elements of I. Conversely, with antichain A associate the ideal $A^{\downarrow}$. Figure 3(a) shows an example.

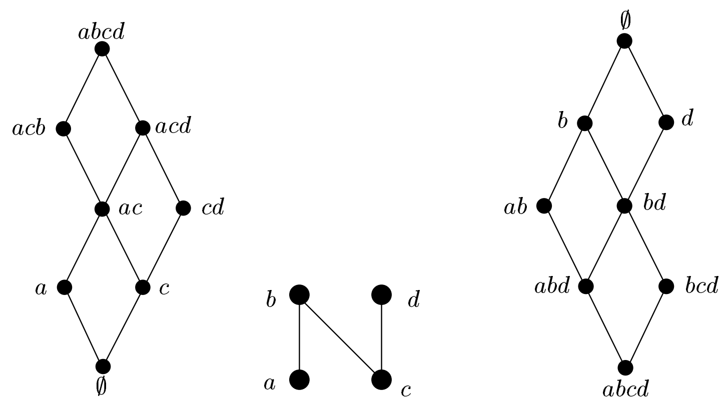
Taking complements in the ground set of P bijectively maps ideals to filters and vice versa, hence, $\mathcal{A}(P)$ is the set of filters of P ordered by reverse inclusion (see Figure 3(c)). Filters have their own 'natural' one-to-one correspondence with antichains. With a filter F of P associate the antichain $A_F = \textsc{Min}(F)$ of minimal elements of F. Conversely, with antichain A associate the filter $A^{\uparrow}$. We have mentioned these 'dual' representations of $\mathcal{A}(P)$ since they will help understand the reasons for the asymmetry in the property strongly-essential.

Minimum and maximum of $\mathcal{A}(P)$ are given by $\hat{0} = \emptyset$ and $\hat{1} = V$. A chain in $\mathcal{A}(P)$ is called a *closed chain* if it contains $\hat{0}$ and $\hat{1}$.

With the next two bijections we characterize two classes of orders corresponding to chains in $\mathcal{A}(P)$.

Bijection 3 *There is a bijection between closed chains in $\mathcal{A}(P)$ and weak-orders which are essential for P.*

Bijection 4 *There is a bijection between chains in $\mathcal{A}(P)$ and strongly-essential representations of interval extensions of induced suborders of P.*



(a) The ideals of N (b) The order N (c) The filters of N

Figure 3: An example for $\mathcal{A}(P)$

We give the two mappings from chains in $\mathcal{A}(P)$ to representations of interval orders right away. The proofs of the theorems are given in subsequent subsections.

Bij. 3 A closed chain $\emptyset = I_0, I_1, \ldots, I_{K-1}, I_K = V$ in $\mathcal{A}(P)$ induces an ordered partition $(I_1, I_2 \setminus I_1, \ldots, I_K \setminus I_{K-1})$ of V. Let $B_i = I_i \setminus I_{i-1}$ be the ith block of this partition. Define the order $W = (V, <_W)$ by $x <_W y$ if and only if $x \in B_i$, $y \in B_j$ and $i < j$. If $*$ denotes serial composition W can be written as $W = B_1 * B_2 * \ldots * B_K$.

In case the chain is a maximal chain W will be a linear extension. In this sense Bijection 3 extends Bijection 1.

Example 3 There are 21 closed chains in $\mathcal{A}(N)$, where N is the order of Figure 3. There is one chain with 2 elements, six with 3, ten with 4 and five with 5 elements, i.e., linear extensions. All of these give rise to different weak orders that are essential for N. Bijection 3 proves that these are indeed all.

Bij. 4 Transform a chain $I_1, I_2, \ldots, I_K$ in $\mathcal{A}(P)$ into $A_1, A_2, \ldots, A_K$ with $A_i = \textsc{Max}(I_i)$. From this chain of antichains define $V_Q = \bigcup_{i=1}^K A_i$. The representation (l, r) of an interval order Q on V_Q is given by $l_x = -1 + \min\{i : x \in A_i\}$ and $r_x = \max\{i : x \in A_i\}$ for all $x \in V_Q$.

In case all antichains A_i in the chain are maximal antichains of P and the chain is maximal with this property, i.e., if the chain is a maximal chain in $\mathcal{A}_M(P)$ the representation (l, r) is the canonical representation of a minimal interval extension of P. In this sense Bijection 4 extends Bijection 2.

2.1 Bijection 3 and Weak-Orders Related to P

Weak-orders can be defined as the interval orders admitting an integer representation with all intervals of unit length. Such a representation for the order W corresponding to the chain in $\mathcal{A}(P)$ is given by $l_x = i-1$ and $r_x = i$ for all $x \in B_i$. This representation is essential since $D_i = I_i$ which is an ideal by definition and $F_i = V \setminus I_i$ which clearly is a filter. Hence W is a weak-order and essential for P.

Let W be a weak-order which is essential for P and let D_i be defined by the canonical representation of W. From canonical we obtain that $D_i \subset D_j$ for $1 \leq i < j \leq K$ and from essential we obtain that D_i is an ideal in P. Hence $D_1, D_2, \dots, D_K$ is a chain in $\mathcal{A}(P)$.

The above considerations show that the image of the mapping is the set of weak-orders W which are essential for P. Since the mapping is obviously injective and all sets are finite it is a bijection. This completes the proof of Bijection 3. □

Let W be a weak-order which is essential for P. Each block B_i is an antichain of W and can thus be seen as a reduction of the suborder induced by P on this set. On the other hand W contains all relations in $B_i \times B_j$ for $i < j$. Since W is essential for P every pair x, y with $x \in B_i$ and $y \in B_j$ is either incomparable in P or $x <_P y$. Therefore, between blocks W behaves as an extension of P.

This observation enables us to characterize weak-order extensions of an order P. A detailed treatment of weak-order extensions has recently been given by Bertet et. al [2]. Therefore, we will confine us to the indication of the main ideas.

An essential weak-order is an extension of P if and only if every 'reduction class' B_i is already an antichain of P. In terms of the chain in $\mathcal{A}(P)$ corresponding to W this condition translates to the condition that $I_i \setminus I_{i-1}$ is an antichain in P. Call a pair (I_{i-1}, I_i) of ideals *legal* if $I_i \setminus I_{i-1}$ is an antichain in P.

Proposition 1 *The weak-order extensions of P correspond bijectively to closed chains in $\mathcal{A}(P)$ such that every pair (I_{i-1}, I_i) of the chain is legal.*

Two weak-orders W_1 and W_2 on the same ground set have $W_1 <_{\text{Ext}} W_2$ if and only if the partition $(B_1, \dots, B_{K_2})$ induced by W_2 is a refinement of the partition $(A_1, \dots, A_{K_1})$ induced by W_1. From this we conclude a characterization of minimal weak-order extensions of P.

Proposition 2 *The minimal weak-order extensions of P correspond bijectively to closed chains in $\mathcal{A}(P)$ such that every pair (I_{i-1}, I_i) but no pair (I_i, I_j) with $j - i > 1$ of the chain is legal.*

The question concerning weak-order reductions suggests itself. However, weak-order reductions don't carry much structure. Let B_i, B_{i+1} be two adjacent blocks of the partition induced by a weak-order reduction W of P then $(B_i^{\downarrow}, B_{i+1}^{\uparrow})$ is a partition of V and $x <_P y$ for all $x \in B_i^{\downarrow}$ and $y \in B_{i+1}^{\uparrow}$. Hence, $(B_i^{\downarrow}, B_{i+1}^{\uparrow})$ is a serial decomposition of P. It follows easily that every order has a unique maximal weak-order reduction W with the blocks of W corresponding to connected components of the cocomparability graph of P. Every other weak-order reduction induces a coarser partition than W.

2.2 Proof of Bijection 4

Recall the mapping Bij. 4 from chains in $\mathcal{A}(P)$ to representations of interval orders: For chain $C = (A_1, A_2, \ldots, A_K)$ and $x \in \bigcup_{i=1}^K A_i$ define $l_x = -1 + \min\{i : x \in A_i\}$ and $r_x = \max\{i : x \in A_i\}$. Let ρ be this mapping, i.e., $\rho(C) = (l, r)$.

The following observations are immediate:

- $\rho(C)$ is a representation of an interval order Q on $V_Q = \bigcup_{i=1}^K A_i$.
- ρ is injective.

Lemma 7 *Let $C = (A_1, A_2, \ldots, A_K)$ be a chain in $\mathcal{A}(P)$. The interval order Q represented by $\rho(C)$ is an interval extension of the order P' induced by P on $\bigcup_{i=1}^K A_i$, moreover, $\rho(C)$ is strongly-essential for P'.*

Proof. Suppose that $(l, r) = \rho(C)$ is not an interval extension of P'. Then $x, y \in \bigcup_{i=1}^K A_i$ with $x <_{P'} y$ but $l_y < r_x$ exist. Let $i = r_x$ and $x \in A_i$ and $y \in A_{l_y+1}$. From $A_{l_y+1} <_{\mathcal{A}} A_i$ it follows that $A_{l_y+1} \subset A_i^{\downarrow}$. This implies that there is a $z \in A_i$ with $y <_P z$. Hence, $x <_P z$ by transitivity. Now $x, z \in A_i$ contradicts that A_i is an antichain.

For strongly-essential we have to show that $M_i^{\downarrow} = D_{i-1} \cup M_i$ for all i, note that here the down-set operator $\downarrow$ has to be taken in P'. From the definition of $\rho(C)$ it follows that $M_i = A_i$. Since C is a chain of antichains in $\mathcal{A}(P')$ the corresponding ideals form a chain by inclusion, i.e., $M_1^{\downarrow} \subset M_2^{\downarrow} \subset \ldots \subset M_K^{\downarrow}$. Since (l, r) represents an extension of P' we have the inclusion $M_i^{\downarrow} \subseteq D_{i-1} \cup M_i$. Any $x \in D_{i-1} \cup M_i$ is an element of M_j for some $j < i$, hence, $x \in M_j^{\downarrow} \subset M_i^{\downarrow}$. This proves equality and the lemma. □

Lemma 8 *Let (l, r) be a representation of an interval extension Q of a suborder P' of P such that (l, r) is strongly-essential for P' then there is a chain C in $\mathcal{A}(P)$ with $(l, r) = \rho(C)$.*

Proof. Define $C = (M_1, M_2, \ldots, M_K)$ with K the magnitude of (l, r). Since Q is an extension of P' each M_i is an antichain in P' and hence in P. It remains to show that C is a chain, i.e., $M_i^{\downarrow} \subset M_j^{\downarrow}$ for $i < j$. This is a direct consequence of strongly-essential: $M_i^{\downarrow} = D_{i-1} \cup M_i \subset D_{j-1} \cup M_j = M_j^{\downarrow}$. To be precise this shows that C is a chain in $\mathcal{A}(P')$, however, as antichains the elements of $\mathcal{A}(P')$ are elements of $\mathcal{A}(P)$ and the chains of $\mathcal{A}(P')$ are chains of $\mathcal{A}(P)$. □

The two lemmas show that the image of ρ is precisely the set of representations of interval extensions of suborders P' of P which are strongly-essential for P'. Since as noted before ρ is injective and all sets are finite ρ is a bijection. This completes the proof of Bijection 4.

Remark 1 In Example 1 and Lemma 3 we have noted the asymmetry in the definition of the property strongly-essential which probably might better be called 'left-strongly-essential'. If we choose to associate the antichain $\textsc{Min}(V \setminus I)$ with ideal I instead of $\textsc{Max}(I)$ the mapping from chains in $\mathcal{A}(P)$ to representations would lead to 'right-strongly-essential' representations.

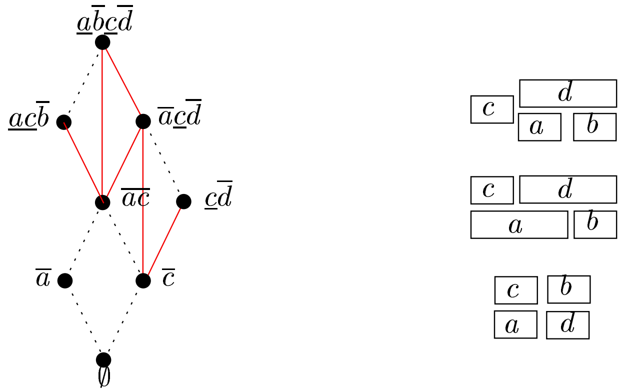
The above bijection may give an abundance of representations for the same interval extension and it gives interval extensions of suborders P' of P. We might, however, be interested in generating every interval extensions of P itself without too much overhead. We now discuss a way for achieving this. The result will be similar in flavor to the results about weak-order extensions (Proposition 1). Again we specify legal edges in the lattice such that the objects we look for correspond to chains with every consecutive pair in the chain being a legal pair.

Let $1 < i < K$ and (l, r) be a canonical representation of an interval order Q of magnitude K. Then there is an x with $r_x = i$ and a y with $l_y = i$. If Q is an extension of P and $C = (A_1, A_2, \ldots, A_K)$ is the chain of antichains in $\mathcal{A}(P)$ with $\rho(C) = (l, r)$ then $y \in A_{i+1} \setminus A_i$ and $x \in A_i \setminus A_{i+1}$. Moreover, if $V_Q = V$ then every element $x \in V$ is in some A_i. It is easily seen that a necessary and sufficient condition for this is that $A_{i+1}^{\downarrow} \setminus A_i^{\downarrow} \subset A_{i+1}$ for $i = 1, \ldots, K-1$.

Call a pair (A, A') of of antichains with $A <_{\mathcal{A}} A'$ *legal* if $A_{i+1} \setminus A_i \neq \emptyset$ and $A_i \setminus A_{i+1} \neq \emptyset$ and $A_{i+1}^{\downarrow} \setminus A_i^{\downarrow} \subset A_{i+1}$. We have seen that $\rho(C)$ is a canonical representation of an extension of P only if every consecutive pair of antichains in C is legal. Together with Bijection 4 we obtain the proposition.

Proposition 3 *The strongly-essential interval extensions of P correspond bijectively to chains in $\mathcal{A}(P)$ such that every pair (A_i, A_{i+1}) of the chain is legal and $\bigcup_i A_i = V$.*

Example 4 Consider again the N as an example, see Fig. 4. $\mathcal{A}(N)$ has 6 legal arcs, but only four of them are used in paths that collect all vertices in V. In Fig. 4(a) the maximal (minimal) elements are overlined (underlined). In fact there are three strongly essential interval extensions of N,



(a) The lattice $\mathcal{A}(N)$, legal arcs are drawn solid.

(b) Representations of strongly-essential extensions of N.

Figure 4: The strongly essential extensions of N

one such extension being N itself. The canonical interval models are given in Fig. 4(b). Here again the asymetry of the definition of strongly essential occurs: the analogous extension to the one in the first row where we just add an arc (b, d) instead of (c, a) is not strongly essential for N.

3 Bijection 2 and Minimal Interval Extensions

In this section we give a new proof of the result for Bijection 2, i.e., the result of Habib et. al. [7].

Let $C = (A_1, A_2, \ldots, A_K)$ be a maximal chain in $\mathcal{A}_M(P)$. Recall the representation $\rho(C) = (l, r)$ given by $l_x + 1 = \min\{i : x \in A_i\}$ and $r_x = \max\{i : x \in A_i\}$, and let $Q(C)$ be the interval order corresponding to this representation. In Lemma 9 we show that $Q(C)$ is a minimal interval extension of P and C is the unique chain in $\mathcal{A}_M(Q)$. Conversely, given a minimal interval extension Q of P Lemma 10 shows that the unique chain in $\mathcal{A}_M(Q)$ is a chain in $\mathcal{A}_M(P)$. Together the two lemmas readily establish the bijection.

Lemma 9 *Let $C = (A_1, A_2, \ldots, A_K)$ be a maximal chain in $\mathcal{A}_M(P)$ the corresponding interval order Q is a minimal interval extension of P. Moreover, C is the unique chain in $\mathcal{A}_M(Q)$.*

Proof. We first show that each element $x \in V$ is contained in at least one antichain of the chain C. Suppose not, since the A_i are maximal x is comparable to at least one element in each A_i. Since $A_1 = \textsc{Min}(P)$ and $A_K = \textsc{Max}(P)$ we find an i such that $a <_P x$ for some $a \in A_i$ and $x <_P b$

for some $b \in A_{i+1}$. let B be a maximal antichain containing x in the order induced by P on $A_i^{\uparrow} \cap A_{i+1}^{\downarrow}$. It is easily shown that B is a maximal antichain of P and $A_i <_{\mathcal{A}_M} B <_{\mathcal{A}_M} A_{i+1}$ contradicting the maximality of the chain.

Since A_i and A_{i+1} are both maximal antichains of P there is an $x \in A_i \setminus A_{i+1}$ and a $y \in A_{i+1} \setminus A_i$. Hence, $r_x = i = l_y$ and the representation is canonical. In general the antichains M_i of the canonical representation of an interval order Q are the maximal antichains of Q and the unique chain in $\mathcal{A}(Q)$ is $M_1, \ldots, M_K$. In our case $M_i = A_i$ for all i and the unique chain in $\mathcal{A}(Q)$ equals C.

It remains to show that Q is a minimal interval extension of P. It is an extension of P since $x <_P y$ implies that all antichains containing x precede all antichains containing y in the chain C. Now suppose that R' is an interval order with $P <_{\text{Ext}} R' <_{\text{Ext}} Q$. We apply Lemma 6 to find incomparable elements x, y with $r_x = i$ and $l_y = i$. Note that A_i is the last antichain in $A_1, A_2, \ldots, A_K$ containing x and A_{i+1} is the first antichain containing y. Let B be a maximal antichain containing x and y in the order induced by P on $A_i^{\uparrow} \cap A_{i+1}^{\downarrow}$. As before B is a maximal antichain of P and $A_i <_{\mathcal{A}_M} B <_{\mathcal{A}_M} A_{i+1}$ contradicting the maximality of the chain. □

Lemma 10 *Let Q be a minimal interval extension of P. The unique chain $M_1, M_2, \ldots, M_K$ of maximal antichains in $\mathcal{A}(Q)$ is a maximal chain $\mathcal{A}(P)$.*

Proof. Since Q is an extension each M_i is an antichain of P. Suppose an antichain M_i is not maximal. Let A be a maximal antichain containing M_i. Let x be an element in $A \setminus M_i$ and suppose that $l_x \geq i$. Among all elements x in $A^{\downarrow}$ with $l_x \geq i$ choose one with l_x minimal. We claim that extending the interval of x to the left until $l_x = i - 1$ still gives an interval extension contradicting the minimality of Q. Suppose the claim is not true then at some point the interval of x will be blocked by the interval of some y with $y <_P x$. Element y is in $A^{\downarrow}$ but not in M_i, hence, with $r_y \geq i$ also $l_y \geq i$ contradicting the choice of x. If all elements $x \in A \setminus M_i$ have $r_x \leq i - 1$ a symmetrical argument gives a contradiction.

The maximal antichains $M_1, M_2, \ldots, M_K$ form a chain in $\mathcal{A}_M(P)$. Otherwise, i.e., if there are $i < j$ with $M_i \not<_{\mathcal{A}_M} M_j$ we would find $x \in M_i^{\downarrow}$ with $x \notin M_j^{\downarrow}$. An argument very similar to the one in the previous paragraph shows that in this case we could move r_x to the right to obtain an interval extension with less relations thus contradicting the minimality of Q.

It remains to show that the chain $M_1, M_2, \ldots, M_K$ in $\mathcal{A}_M(P)$ is a maximal chain. Suppose the chain could be refined by B with $M_i <_{\mathcal{A}_M} B <_{\mathcal{A}_M)} M_{i+1}$. It follows from the maximality of the antichains that there are $x, y \in B$ with $x \in M_i \setminus M_{i+1}$ and $y \in M_{i+1} \setminus M_i$. The interval order corresponding to the refined chain is an extension of P with at least one relation less than Q. Again, this contradicts the minimality of Q. □

4 The Separation Lattice

In this section we define the separation lattices and provide some material about them that will be needed in our discussion of interval reductions. A *separation* of an order $P = (V, <_P)$ is a pair (I, F) of subsets of V such that

(1) I and F are disjoint,

(2) $x <_P y$ for all $x \in I$ and $y \in F$ and

(3) I is an ideal and F is a filter of P.

The name separation was motivated by the observation that $V \setminus (I \cup F)$ is a separator of the cocomparability graph of P whenever (I, F) is a separation of P. This mapping $(I, F) \to V \setminus (I \cup F)$ from separations of P to separators of the cocomparability graph is onto but not one-to-one. In general a separation is not characterized by the supporting set $I \cup F$, e.g.,if $I \cup F$ is a series composition of more then two components it is not.

It is natural to define an order relation on the set $\mathcal{S}(P)$ of separations of P by

$$(I, F) \leq_{\mathcal{S}} (I', F') \iff I \subseteq I' \text{ and } F \supseteq F' \tag{3}$$

Examples of separation lattices are given in Figure 5. From left to right this figure shows $SL(P)$ for the $P = L_3$, the 3-element chain, for $P = N$, see Fig. 3, and for $P = II$, see Fig. 1.

Theorem 2 $\mathcal{S}(P)$ *with the above defined order relation is a distributive lattice.*

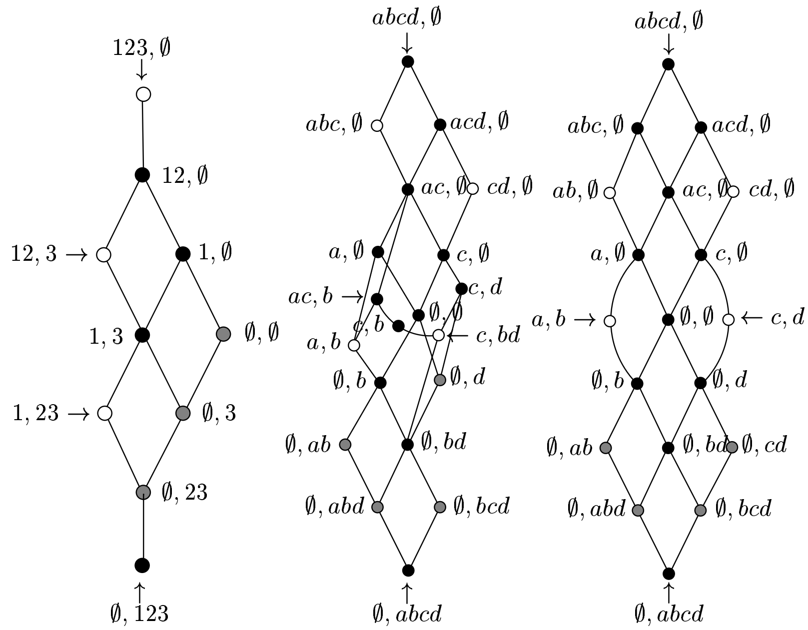
Proof. Let (I, F) and (I', F') be separations. $(I, F) \leq_S (I'', F'')$ and $(I', F') \leq_S (I'', F'')$ require $I \cup I' \subseteq I''$ and $F \cap F' \supseteq F''$, since $(I \cup I', F \cap F')$ is itself a separation it is the join of (I, F) and (I', F'). We denote the join operation by $\sqcup$, i.e., $(I, F) \sqcup (I', F') = (I \cup I', F \cap F')$.

Similarly the meet of two separations exists and is given by $(I, F) \sqcap (I', F') = (I \cap I', F \cup F')$.

From the definitions of join and meet it is obvious that there is an embedding of $\mathcal{S}(P)$ into a product of boolean lattices. Hence, $\mathcal{S}(P)$ is itself distributive. Minimum and maximum of $\mathcal{S}(P)$ are given by $\hat{0} = (\emptyset, V)$ and $\hat{1} = (V, \emptyset)$. □

By the fundamental theorem of finite distributive lattices every such lattice is the lattice of antichains of some order. More precisely, a distributive lattice L is isomorphic to the lattice of antichains $\mathcal{A}(J_L)$ where J_L is the set of join irreducibles of L with the order relation induced by L. (See [14] and note that the element $\hat{0}$ of a lattice is not considered to be join irreducible).

In the next lemma we characterize the join-irreducible elements of $\mathcal{S}(P)$. Let $\mathrm{Pr}[x] = \{y : y \leq x\}$ and $\mathrm{Su}(x) = \{y : x < y\}$.



(a) $\mathcal{S}(L_3)$ (b) $\mathcal{S}(N)$ (c) $\mathcal{S}(I\!I)$

Figure 5: Examples for the Separation Lattice

Lemma 11 *Besides $\hat{0} = (\emptyset, V)$ the join irreducible elements of the separation lattice $\mathcal{S}(P)$ are*

(a) $(\emptyset, V \setminus \mathrm{Pr}[x])$ *for all $x \in V$ together with*

(b) $(\mathrm{Pr}[x], \mathrm{Su}(x))$ *for all $x \in V$.*

Proof. The elements of $\mathcal{S}(P)$ described in (a) and (b) obviously are join irreducible. Moreover, every separation is obtained as join of these elements. This shows that the described separations are the elements of $J_{\mathcal{S}(P)}$. □

Example 5 In Figure 5 the copies of P in $\mathcal{S}(P)$ induced by the separations of type (a) in the lemma is indicated by grey circles. A second copy of P in $\mathcal{S}(P)$ induced by the separations of type (b) in the lemma is indicated by white circles.

Lemma 12 *The order relation of $J_{\mathcal{S}(P)}$ is isomorphic to the order Q on two copies V' and V'' of V with relations:*

$$x' <_Q y' \text{ iff } x <_P y, \quad x'' <_Q y'' \text{ iff } x <_P y, \quad x' <_Q y'' \text{ iff } y \not<_P x.$$

Proof. Associate x' with $(\emptyset, V \setminus \mathrm{Pr}[x])$ and x'' with $(\mathrm{Pr}[x], \mathrm{Su}(x))$. The claimed relations are easily verified. □

It is noticeable that $\mathcal{S}(P)$ contains two almost disjoint copies of $\mathcal{A}(P)$ one with the first component $\emptyset$, corresponding to the interpretation of $\mathcal{A}(P)$ as filters, and one with the second component $\emptyset$, corresponding to the interpretation of $\mathcal{A}(P)$ as ideals, stacked on top of each other at the common element $(\emptyset, \emptyset)$.
We generalize notation and define for any subset $X \subseteq V$

- $\mathrm{Pr}(X) = \{y \in V \mid y <_P x \text{ for all } x \in X\}$,
- $\mathrm{Su}(X) = \{y \in V \mid y >_P x \text{ for all } x \in X\}$.

Note that $\mathrm{Pr}(X)$ and $\mathrm{Su}(X)$ are defined as intersections of predecessor- and successor-sets of elements of X. In contrast $X^{\downarrow}$ and $X^{\uparrow}$ are unions of predecessor- and successor-sets of elements of X. Clearly, $\mathrm{Pr}(X)$ is an ideal and $\mathrm{Su}(X)$ is a filter of P.

The following list collects some properties of the mappings Pr and Su for $X \subseteq Y \subseteq V$.

(0) $\mathrm{Pr}(Y) \subseteq \mathrm{Pr}(X)$ and $\mathrm{Su}(Y) \subseteq \mathrm{Su}(X)$.

(1) $X \subseteq \mathrm{Pr}(\mathrm{Su}(X))$ and $X \subseteq \mathrm{Su}(\mathrm{Pr}(X))$.

(2) $\mathrm{Pr}(\mathrm{Su}(X)) \subseteq \mathrm{Pr}(\mathrm{Su}(Y))$ and $\mathrm{Su}(\mathrm{Pr}(X)) \subseteq \mathrm{Su}(\mathrm{Pr}(Y))$.

(3) $\mathrm{Su}(\mathrm{Pr}(\mathrm{Su}(X))) = \mathrm{Su}(X)$ and $\mathrm{Pr}(\mathrm{Su}(\mathrm{Pr}(X))) = \mathrm{Pr}(X)$.

Properties 1–3 show that $\mathrm{Pr} \circ \mathrm{Su}$ and $\mathrm{Su} \circ \mathrm{Pr}$ are closure operators. Observe further that for any $X \subseteq V$ the pairs

- $\alpha(X) = (\mathrm{Pr}(X), \mathrm{Su}(\mathrm{Pr}(X)))$ and
- $\omega(X) = (\mathrm{Pr}(\mathrm{Su}(X)), \mathrm{Su}(X))$

are separations.

A separation (I, F) is *maximal* if there is no separation (I', F') with $I \subseteq I'$, $F \subseteq F'$ and $I \cup F \subset I' \cup F'$. Let $\mathcal{S}_M(P)$ denotes the set of maximal separations of P with the order relation induced by $\mathcal{S}(P)$. In Figure 5 the elements of $\mathcal{S}_M(P)$ are marked with arrows.

Lemma 13 *A separation (I, F) is maximal if and only if $I = \mathrm{Pr}(F)$ and $F = \mathrm{Su}(I)$.*

Proof. For every ideal I the pair $(I, \mathrm{Su}(I))$ and for every filter F the pair $(\mathrm{Pr}(F), F)$ are separations. Since $I \subseteq \mathrm{Pr}(F)$ and $F \subseteq \mathrm{Su}(I)$ for every separation (I, F) we find that $I = \mathrm{Pr}(F)$ and $F = \mathrm{Su}(I)$ are necessary conditions for the maximality of (I, F).

Combining $I = \mathrm{Pr}(F)$ and $F = \mathrm{Su}(I)$ gives $F = \mathrm{Su}(\mathrm{Pr}(F))$, hence, $(I, F) = (\mathrm{Pr}(F), \mathrm{Su}(\mathrm{Pr}(F)))$ and every separation $(\mathrm{Pr}(F), Y)$ has $Y \subseteq \mathrm{Su}(\mathrm{Pr}(F))$. On the other hand every separation $(Y, \mathrm{Su}(\mathrm{Pr}(F))$ has $Y \subseteq \mathrm{Pr}(\mathrm{Su}(\mathrm{Pr}(F))) = \mathrm{Pr}(F)$. This shows that (I, F) is maximal, i.e., an element of $\mathcal{S}_M(P)$. $\square$

As special case consider the case of an interval order Q. Given the canonical representation of Q the maximal separations are exactly the pairs (D_i, U_i) for $i = 0, .., K$. In particular the order induced by $\mathcal{S}(Q)$ on the maximal separations is a chain. In fact this property charcterizes interval orders.

Characterization 2 *$\mathcal{S}_M(P)$ is a chain iff P is an interval order.*

In general $\mathcal{S}_M(P)$ is only an suborder of $\mathcal{S}(P)$, but not a sublattice, i.e., for $S, S' \in \mathcal{S}_M(P)$ the join $S \sqcup S'$ need not be in $\mathcal{S}_M(P)$. Nevertheless, $\mathcal{S}_M(P)$ is itself a lattice, this will be shown in Theorem 3.

Lemma 14 *For every $X \subseteq V$, $\alpha(X)$ and $\omega(X)$ are maximal separations. Moreover, $\alpha(X)$ is the unique $(<_S)$–maximal separation in $\mathcal{S}_M(P)$ with X contained in the filter. Symmetrically, $\omega(X)$ is the unique $(<_S)$ minimal separation in $\mathcal{S}_M(P)$ with X contained in the ideal.*

Proof. $\alpha(X) = (\mathrm{Pr}(X), \mathrm{Su}(\mathrm{Pr}(X)))$. Since $\mathrm{Pr}(X) = \mathrm{Pr}(\mathrm{Su}(\mathrm{Pr}(X)))$ and trivially $\mathrm{Su}(\mathrm{Pr}(X)) = \mathrm{Su}(\mathrm{Pr}(X))$ Lemma 13 shows that $\alpha(X)$ is in $\mathcal{S}_M(P)$.

Suppose (I, F) is a maximal separation with $X \subseteq F$. It follows that $I = \mathrm{Pr}(F) \subseteq \mathrm{Pr}(X)$. Assuming $\alpha(X) \leq_S (I, F)$ we conclude $I = \mathrm{Pr}(X)$ and hence $(I, F) = \alpha(X)$.

The assertions concerning $\omega(X) = (\mathrm{Pr}(\mathrm{Su}(X)), \mathrm{Su}(X))$ are proved by dual arguments. □

Theorem 3 *$\mathcal{S}_M(P)$ is a lattice with lattice operations*

- *$S \vee S' = \omega(I \cup I')$ and*
- *$S \wedge S' = \alpha(F \cup F')$*

for $S = (I, F)$ and $S' = (I', F')$.

Proof. From Lemma 14 we know that $S \vee S'$ and $S \wedge S'$ are in $\mathcal{S}_M(P)$. We now show that $\vee$ is a supremum operation. The analogous claim for $\wedge$ again follows by a dual argument.

Let $S \vee S' = (I'', F'')$. From the definition of $\omega(I \cup I')$ and the first closure property $I \cup I' \subseteq I''$. On the other hand $F'' = \mathrm{Su}(I \cup I') = \mathrm{Su}(I) \cap \mathrm{Su}(I') = F \cap F'$. Hence, $S \vee S'$ is a common successor of S and S'.

A successor $S''' = (I''', F''')$ of S and S' has $F''' \subseteq F \cap F' = F''$, hence $I''' \supseteq \mathrm{Pr}(F'') = I''$. This shows $S \vee S' = (I'', F'') \leq_S (I''', F''')$. □

We note that the lattice $\mathcal{S}_M(P)$ has already received some attention in the context of concept analysis where it is known as the concept lattice $\mathcal{B}(P, P, <)$. Jutta Mitas [9] pointed out that $\mathcal{B}(P, P, <)$ is isomorphic to $\mathcal{L}(\mathrm{PrSu}(P))$ the Dedekind-MacNeille completion of $\mathrm{PrSu}(P)$, with $\mathrm{PrSu}(P)$ being the inclusion order of the sets $\{\mathrm{Pr}(x) : x \in V\} \cup \{\mathrm{Pr}(\mathrm{Su}(x)) : x \in V\}$.

The order PrSu(P) has been investigated in studies of the interval dimension of an order (e.g., [3, 4, 5, 8]). In particular it has been shown repeatedly that

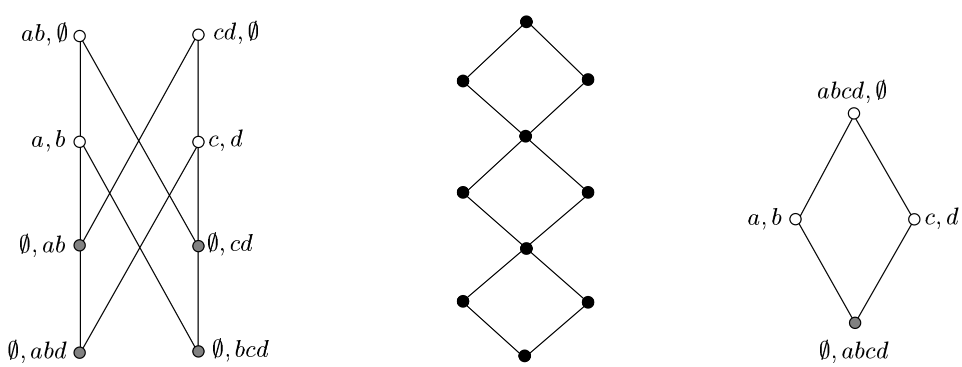
$$\mathrm{Idim}(P) = \dim(\mathrm{PrSu}(P)). \tag{4}$$

Yet another representation of $\mathcal{S}_M(P)$ can be derived from the fact that the concept lattice $\mathcal{B}(G, M, I)$ is isomorphic to the lattice of maximal antichains of the height one poset with minimal elements G, maximal elements M and $g < m$ iff $(g, m) \notin I$. In our case the height one poset in question has a copy V' of V as minimal elements a second copy V'' as maximal elements and relations $x' < y''$ iff $x \not<_P y$. This is cast into the following formula

$$\mathcal{S}_M(P) = \mathcal{A}_M\left(\overline{\mathrm{Bip}(\mathrm{P}^\circ)}\right). \tag{5}$$

Recall the characterization of $J_{\mathcal{S}(P)}$ in Lemma 11 and 12. It follows that $J_{\mathcal{S}(P)}$ can be obtained from $\overline{\mathrm{Bip}(\mathrm{P}^\circ)}$ by adding the relations of P on each of the sets V' and V''. It is therefore natural to ask whether $\mathcal{S}_M(P)$ equals $\mathcal{A}_M(J_{\mathcal{S}(P)})$.

Example 6 Consider the order $P = \mathit{II}$. Fig. 6 shows the order $J_{\mathcal{S}(\mathit{II})}$ induced by the join irreducible elements of $\mathcal{S}(\mathit{II})$, the lattice $\mathcal{A}_M(J_{\mathcal{S}(\mathit{II})})$ of maximal antichains of this order and the lattice $\mathcal{S}_M(\mathit{II})$ of maximal separations of II. The example shows that in general $\mathcal{S}_M(P) \neq \mathcal{A}_M(J_{\mathcal{S}(P)})$.



(a) $J_{\mathcal{S}(\mathit{II})}$ (b) $\mathcal{A}_M(J_{\mathcal{S}(\mathit{II})})$ (c) $\mathcal{S}_M(\mathit{II})$

Figure 6: Objects related to II

4.1 Digression: N-free Orders

In this subsection we give a characterization of $\mathcal{S}_M(P)$ for N-free orders. Readers not particularly interested in N-free orders may skip to the beginning of the next section.

The characterization of $\mathcal{S}_M(P)$ is based on the observation (Theorem 4) that for an N-free order P the root order $\mathrm{Root}(P)$ and the predecessor-successor order $\mathrm{PrSu}(P)$ are almost equal. Let $\mathrm{Root}^*(P)$ be $\mathrm{Root}(P)$ augmented by a global minimal element 0 and a global maximal element 1.

Theorem 4 *For an N-free order* $\mathrm{Root}^*(P) = \mathrm{PrSu}(P)$.

We have already noted that $\mathcal{S}_M(P) = \mathcal{L}(\mathrm{PrSu}(P))$, hence, the theorem implies another characterization of $\mathcal{S}_M(P)$ for N-free orders P

$$\mathcal{S}_M(P) = \mathcal{L}(\mathrm{Root}(P)). \tag{6}$$

Another consequence is an relation involving interval-dimension and dimension that appeared first in [6]. Combining (4) with the theorem we obtain that for N-free orders P

$$\mathrm{Idim}(P) = \dim(\mathrm{Root}(P)). \tag{7}$$

Similar results have been shown involving subdivisions of an order [4]. We step into the proof of Theorem 4 with an easy lemma.

Lemma 15 *Let P be an N-free order. For all $x \in V$ and all immediate successors y, y' of x the predecessor sets of y and y' are equal, i.e,* $\mathrm{Pr}(y) = \mathrm{Pr}(y')$.

Proof. Otherwise there is an a such that a, x, y, y' form an N, a contradiction. □

An immediate consequence of this lemma is that

$$\{\mathrm{Pr}(\mathrm{Su}(x)) : x \in V\} \subseteq \{\mathrm{Pr}(x) : x \in V\} \cup V. \tag{8}$$

Therefore $\mathrm{PrSu}(P)$ is the inclusion order of the predecessor sets with a maximal element 1 adjoint.

Usually one associates with an N-free order the root-digraph which is the unique digraph D with a minimal number of vertices having P as line graph. To stay within the class of orders $\mathrm{Root}(P)$ is defined in [6] as the transitive reduction (minus 0 and 1) of D. The vertices of $\mathrm{Root}(P)$ are characterized as the induced complete bipartite subgraphs of the diagram of P. Two such subgraphs v_1, v_2 are in relation $v_1 < v_2$ if some maximal elements of v_1 are predecessors of some minimal elements of v_2. An element $x \in V$ is contained in at most two such bipartite subgraphs. If x is non-minimal it is contained in $(\textsc{Max}(\mathrm{Pr}(x)), \textsc{Min}(\mathrm{Su}(\mathrm{Pr}(x))))$ and if x is non-maximal it is contained in $(\textsc{Max}(\mathrm{Pr}(\mathrm{Su}(x))), \textsc{Min}(\mathrm{Su}(x)))$. Note that every pair $(\textsc{Max}(\mathrm{Pr}(\mathrm{Su}(x))), \textsc{Min}(\mathrm{Su}(x)))$ can equally well be written as $(\textsc{Max}(\mathrm{Pr}(y)), \textsc{Min}(\mathrm{Su}(\mathrm{Pr}(y))))$ for some y. This shows that projection onto the first component is a bijection between $\mathrm{Root}(P)$ and the non-extremal elements of $\mathrm{PrSu}(P)$. This completes the proof of Theorem 4.

5 Chains in the Separation Lattice

Bijection 5 *There is a bijection between closed chains in $\mathcal{S}(P)$ and dense, essential representations of interval reductions of P.*

Bij. 5 Given a closed chain $C = (\hat{0} = S_0 <_{\mathcal{S}} S_1 <_{\mathcal{S}} \ldots <_{\mathcal{S}} S_K = \hat{1})$ in $\mathcal{S}(P)$ with $S_i = (I_i, F_i)$ let $r_x = \min\{i : x \in I_i\}$ and $l_x = \max\{i : x \in F_i\}$. This defines a representation $(l, r) = \sigma(C)$ of an interval order of magnitude K.

In Lemma 16 we show that $\sigma(C)$ is a dense representation of an essential interval reduction of P. Conversely, Lemma 17 shows that every such representation is the image of a closed chain in $\mathcal{S}(P)$. Since σ is easily seen to be injective this establishes the bijection of the theorem.

Lemma 16 *Let C be a closed chain in $\mathcal{S}(P)$. The representation $\sigma(C)$ is a dense representation of an essential interval reduction of P.*

Proof. Let $(l, r) = \sigma(C)$ if $r_x \leq l_y$ then there is an i such that $r_x \leq i \leq l_y$, hence, $x \in I_i$ and $y \in F_i$ which implies $x <_P y$ since S_i is a separation. Hence, (l, r) represents an interval reduction. Since $S_i \neq S_{i+1}$ the representation is dense.

For essential we have to show that $D_i = \{x : r_x \leq i\}$ is an ideal of P. Since $D_i = I_i$ this is true by the definition of separations. Symmetrically $U_i = \{x : l_x \geq i\} = F_i$ is a filter. □

Lemma 17 *Let (l, r) be a dense representation of an essential interval reduction of P then there is chain C in $\mathcal{S}(P)$ such that $(l, r) = \sigma(C)$.*

Proof. Given a dense representation (l, r) of a reduction of P, let $S_i = (D_i^{\downarrow}, U_i^{\uparrow})$ for $i = 0, \ldots, K$ with K being the magnitude of the representation. We claim that S_i is a separation of P. Since (l, r) is a reduction we have $x <_P y$ for all $x \in D_i$ and $y \in U_i$. The statement for $x \in D_i^{\downarrow}$ and $y \in U_i^{\uparrow}$ follows from transitivity.

The containments $D_i \subseteq D_j$ and $U_i \supseteq U_j$ for all $i < j$ is passed on to $D_i^{\downarrow} \subseteq D_j^{\downarrow}$ and $U_i^{\uparrow} \supseteq U_j^{\uparrow}$. Hence $S_i \leq_{\mathcal{S}} S_j$ for $i < j$ and the sequence forms a chain in $\mathcal{S}(P)$. The first element of the chain is $S_0 = (\emptyset, V) = \hat{0}$ and the last is $S_K = (V, \emptyset) = \hat{1}$, hence, the chain is complete. □

As result of this section we have so far obtained a nice bijection. This bijection, however, gives an abundance of representations for the same interval reduction. We now discuss a possibility of avoiding multiple generation. The result will be similar in flavor to the results about weak-order extensions (Proposition 1) and strongly-essential interval extensions (Proposition 3). Again we specify legal edges in the lattice such that the objects we look for correspond to chains with every consecutive pair in the chain being a legal pair.

Let $0 < i \leq K$ and (l, r) be a canonical representation of an interval order Q of magnitude K. Then there is an x with $r_x = i$ and a y with $l_y = i - 1$. If Q is a reduction of P and $C = (S_0, S_1, \ldots, S_K)$ with $S_i = (I_i, F_i)$ is the chain in $\mathcal{S}(P)$ with $\sigma(C) = (l, r)$ then $x \in I_i \setminus I_{i-1}$ and $y \in F_{i-1} \setminus F_i$. It follows that in the chain corresponding to the canonical representation of an interval reduction any two consecutive separations differ in both components. It is easily seen that the converse is true as well. If in chain C in $\mathcal{S}(P)$ any two consecutive separations differ in both components then $\sigma(C)$ is canonical. Call a pair (S_{i-1}, S_i) of separations *legal* if $I_i \setminus I_{i-1} \neq \emptyset$ and $F_{i-1} \setminus F_i \neq \emptyset$. We summarize these considerations with a proposition.

Proposition 4 *The essential interval reductions of P correspond bijectively to closed chains in $\mathcal{S}(P)$ such that every pair (S_{i-1}, S_i) of the chain is legal.*

Observe that any transitive arc $(S, T) \in \mathcal{S}(P)$ with (S, R) and $(R, T) \in \mathcal{S}(P)$ is legal if any of (S, R) or (R, T) is legal. In particular if we have that (S, R) and $(R, T) \in \mathcal{S}(P)$ are both legal then $(S, T) \in \mathcal{S}(P)$ is legal, too. Thus the legal arcs give rise to a partial order. This partial order has a lot of minimal and maximal elements, in particular all separations where either the ideal or the filter correponds to the empty set. To compute all essential interval reductions we may restrict ourselves to the partial order that is induced by those elements that can be included into some closed chain of legal arcs.

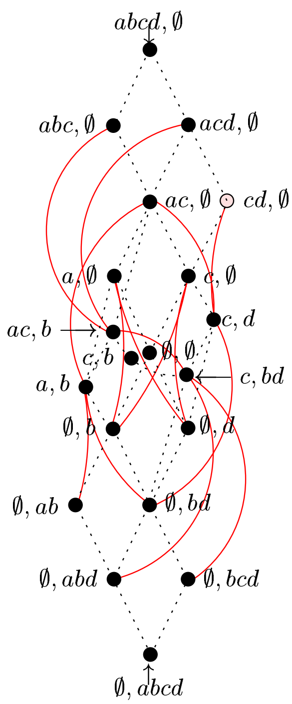
Example 7 Consider again N as an example. Fig. 7(a) shows the illegal arcs of the diagram of $\mathcal{S}(N)$ as dotted lines and those legal arcs that can not be found as transitive legal arcs (see pthe revious paragraph) as solid lines. If we restrict the picture to those elements that can be included into some closed chain of legal arcs we obtain the partial order that is shown in Fig. 7(b). It has one closed chain of length 1, five of length 2 and one of length 3. So in total we obtain seven essential interval reductions for N.

On the other hand N has $2^3 = 8$ reductions out of which only one, namely the II is not an interval order. So N has seven interval reductions and by Proposition 4 all of them must be essential.

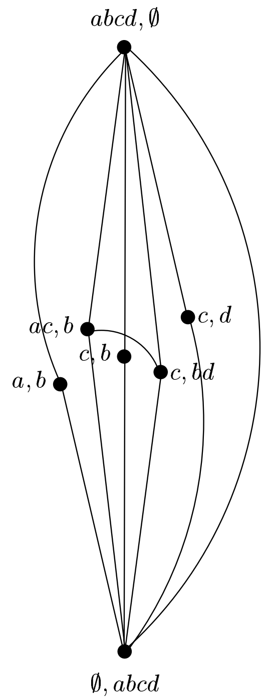
6 Maximal Interval Reductions

In this section we show that the mapping σ from Section 5 specializes to a bijection between maximal chains in $\mathcal{S}_M(P)$ and maximal interval reductions of P.

Bijection 6 *The mapping σ is a bijection between maximal chains in the lattice of maximal separations $\mathcal{S}_M(P)$ and maximal interval reductions of P.*



(a) $\mathcal{S}(N)$ with some but not all of its legal arcs



(b) Legal arcs belonging to closed chains

Figure 7: Legal Arcs in the Separation Lattice

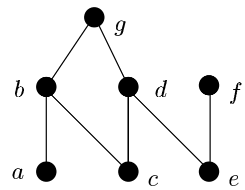
The mapping is the same as in the previous section, i.e., the representation $(l, r) = \sigma(C)$ corresponding to chain $C = (\hat{0}, S_1, \ldots, S_{K-1}, \hat{1})$ with $S_i = (I_i, F_i)$ is given by $r_x = \min\{i : x \in I_i\}$ and $l_x = \max\{i : x \in F_i\}$.

Example 8 Consider the order P given in Fig. 8(a) as an example. $\mathcal{S}_M(P)$ has 7 elements and is shown in Fig. 8(c). It has three maximal chains that correspond to three maximal interval reductions for which interval models (drawn with l_i bottom and r_i top) are given on the right. The diagrams of these reductions can be seen in Fig. 8(b).

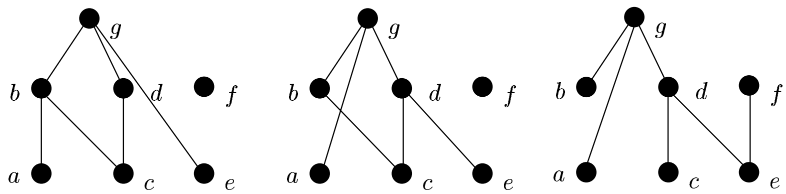
The mapping is obviously injective, Lemma 18 below shows that it maps maximal chains in $\mathcal{S}_M(P)$ to canonical representations of maximal interval reduction and Lemma 19 shows that every such representation is in the image of mapping σ. Together this proves the theorem.

Lemma 18 *Let* $C = (\hat{0} = S_0, S_1, \ldots, S_K = \hat{1})$ *be a maximal chain in* $\mathcal{S}_M(P)$. *Then* $(l, r) = \sigma(C)$ *is the canonical representation of a maximal interval reduction* Q *of* P.

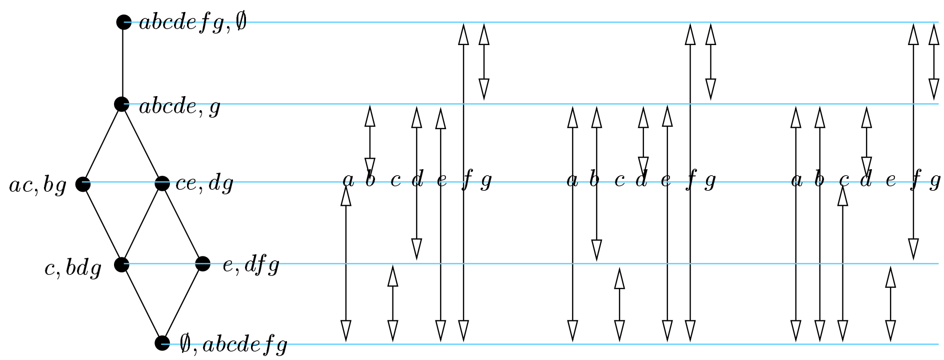
Proof. From Lemma 16 we already know that we obtain a dense and essential representation of an interval reduction of P. We claim that the



(a) The order P



(b) The maximal interval reductions of P



(c) $\mathcal{S}_M(P)$ and interval models for the maximal interval reductions

Figure 8: An order P and its maximal interval reductions

representation is the canonical one. If there were an i with $0 < i \leq K$ but i not a right end-point of an interval then $I_i = I_{i-1}$. From $F_i \subset F_{i-1}$ we immediately see that S_i is not a maximal separation. Symmetric arguments show that every i with $0 \leq i < K$ is a left end-point of an interval, hence, the representation is canonical.

Suppose that Q is not a maximal reduction. It follows that there is an interval reduction Q^* with $Q <_{\text{Ext}} Q^* <_{\text{Ext}} P$. From Lemma 6 we obtain the existence of a pair x, y of elements with $x <_{Q^*} y$, hence, $x <_P y$ and $r_x = i = l_y + 1$. Let $S_i = (I_i, F_i)$ and $S_{i+1} = (I_{i+1}, F_{i+1})$. From $x <_P y$ it follows that $S = (I_i \cup \{x\}, F_{i+1} \cup \{y\})$ also is a separation. Let $S^* = (I^*, F^*)$ be a maximal separation with $I_i \cup \{x\} \subseteq I^*$ and $F_{i+1} \cup \{y\} \subseteq F^*$. It is easy to see that $S_i <_{\mathcal{S}_M} S^* <_{\mathcal{S}_M} S_{i+1}$ in contradiction to the maximality of the chain. $\square$

Lemma 19 *Let Q be a maximal interval reduction of P and let (l, r) be the canonical representation of Q. Then there is a maximal closed chain $C = (S_0, S_1, \ldots, S_K)$ in $\mathcal{S}_M(P)$ with $(l, r) = \sigma(C)$.*

Proof. From Lemma 5 we know that (l, r) is essential. Therefore, $S_i = (D_i, U_i)$ is a separation of P. If S_i were not maximal then either there exists x such that $S' = (D_i \cup \{x\}, U_i)$ is a separation or there exists y such that $S'' = (D_i, U_i \cup \{y\})$ is a separation. In the first case $x <_P y$ for all $y \in U_i$. Redefining $r_y^{new} = i < r_y^{old}$ gives an interval reduction of P. Since (l, r) is canonical the set $U_i \setminus U_{i+1}$ is nonempty and the new interval reduction has more relations then the old. This contradicts the maximality. The argument for the second case is symmetric. Hence the separations S_i are maximal.

It is immediate that $S_0, S_1, \ldots, S_K$ is a complete chain. Assuming that the chain is not maximal there exist a maximal separation $S' = (I', F')$ and an index i such that $S_i <_{\mathcal{S}_M} S' <_{\mathcal{S}_M} S_{i+1}$. It follows that $D_i \subset I' \subset D_{i+1}$ and $U_i \supset F' \supset U_{i+1}$. Choose $x \in I' \setminus D_i$ and $y \in F' \setminus U_{i+1}$ it follows that $r_x = i+1$ and $l_y = i$ and $x <_P y$. Defining $r_x = l_y = i + 1/2$ gives an interval reduction with more relations. This contradicts the maximality of Q. □

7 Optimization and Dynamic Programming

In this final section we show that our bijections may help solving optimization and counting problems. The idea is to use dynamic programming such that the dependency graph of the dynamic program is one of the lattices $\mathcal{A}(P)$, $\mathcal{A}_M(P)$, $\mathcal{S}(P)$ or $\mathcal{S}_M(P)$. We restrain ourselves to detail the case $\mathcal{S}_M$, however, the techniques described here fully apply to the other lattices.

To formalize the idea we introduce the notion of an *upward propagated invariant* of ordered sets. Upward propagated invariants of P are shown to be computable by a dynamic program such that the states of the program naturally correspond to the elements of $\mathcal{S}_M(P)$ and the relations of $\mathcal{S}_M(P)$ capture the restrictions on the order in which the states have to be processed.

Definition 4 *Let h be an invariant of ordered sets. We say that h can be* upward propagated *(in $\mathcal{S}_M$) when for each order P there is a* state *function $s : \mathcal{S}_M(P) \longrightarrow \mathbf{S}$ that has the following properties:*

(i) There is an algorithm that is polynomial in the size of $\mathcal{S}_M(P)$ that given $s(\hat{1})$ outputs $h(P)$.

(ii) There is an algorithm that is polynomial in the size of P that given $s(S_1), \ldots, s(S_k)$ for any separation S with immediate predecessors $S_1, \ldots, S_k$ outputs $s(S)$.

Here (i) ensures that knowing the final state on the maximum element of $\mathcal{S}_M(P)$ it is possible to compute the value of the invariant $h(P)$. (ii) guarantees that in fact with any reasonable search strategy that processes $\mathcal{S}_M(P)$ in a bottom up way $s(\hat{1})$ (and thus $h(P)$) can be computed in time polynomial in the size of $\mathcal{S}_M(P)$.

Clearly that such a notion of *upward propagated* can be generalized to any other lattice (or even directed acyclic graph, see [2]) that is associated to an order P.

Theorem 5 *Any upward propagated invariant $h(P)$ of orders can be computed in time polynomial in the size of $\mathcal{S}_M$.*

Proof. With what is stated above, for a proof it suffices to show that $\mathcal{S}_M(P)$ can be generated in time polynomial in its size.

Therefore, just observe that this lattice can be explored in a DFS, by starting from the minimum element, say. Clearly that, when positioned on a certain separation S, we may easily enumerate all immediate successors of S in $\mathcal{S}_M$. To not visit any separation twice we have to store all separations visited so far in an appropriate data structure. Such a data structure could for example be a dictionary containing the separations encoded as strings of elements. □

In Theorem 6 we give some examples of parameters of P depending on its interval reductions that can be computed by this approach. These parameters may play a role for scheduling problems: Assume that the elements of P correspond to jobs that have to be scheduled on identical maschines subject to two conditions:

(i) Unrelated (uncomparable) jobs have to be active simultaneously to be able, e.g., to communicate some data.

(ii) Fcor two related jobs $x < y$ it is not allowed that y is finished before x is released, i.e., it is not allowed to reverse an order relation.

It is easy to see that feasible schedules for this problem correspond to an interval reductions of P. The minimum width of an interval reduction of P then translates to the minimum number of machines that are needed for a feasible schedule. The *workload* of such a feasible schedule Q is the sum over all processing times, or stated in our context, sum of the interval lengths of all intervals in the canonical representation of Q.

The bijection between maximal interval reductions and chains in $\mathcal{S}_M(P)$ allows to optimize these scheduling parameters in a complexity proportional to the size of $\mathcal{S}_M$. This has interesting consequences for orders where this lattice is small, e.g., for N-free orders.

As an easy corollary of Theorem 5 we obtain the following:

Theorem 6 *The following invariants can be computed in time polynomial in the size of $\mathcal{S}_M$:*

1. *The number of maximal interval reductions of P.*
2. *The minimum width of an interval reduction of P.*
3. *The minimum workload of P.*

Proof. It is easy to see that all three invariants are upward propagated. By Theorem 6, the *number of maximal interval reductions* is just the number of maximal paths in $\mathcal{S}_M$. Define $s(S)$ as the number of maximal paths from $\hat{0}$ to S, this number can locally be computed as sum of the number of maximal paths leading to the immediate predecessors of S.

The *minimum width* of an interval reduction is clearly admitted by a maximal interval reduction. A maximal interval reduction Q corresponds to a maximum chain in $\mathcal{S}_M(P)$ and a maximal antichain A of Q corresponds to a covering relation on that maximum chain: if $S' = (I', F')$ is followed by $S = (I, F)$ in the chain, the antichain is given by $A = I \setminus I'$. Therefore, minimum width is an upward propagated invariant.

The *minimum workload* is the minimum of this value over all immediate predecessors S' of S in $\mathcal{S}_M(P)$ plus the weight of the antichain corresponding to the the covering relation (S', S) as before. □

The following corollary shows that for an important class of orders the problem of counting interval reductions is completely different from the problem of counting interval *extensions*. In [6] it was shown that counting such extension is already #P-complete for the class of N-free orders.

Corollary 20 *The above problems can be solved in quadratic time if P is N-free.*

Proof. As we have seen in the proof of Theorem 4, maximal separations of an N-free order P correspond to maximal bipartites in the transitive reduction or P. It is known that an N-free order P has at most a linear number of such bipartites (see [15, 6]). Therefore, for an N-free order P the size of $\mathcal{S}_M(P)$ is linear and the number of relations of $\mathcal{S}_M(P)$ is at most quadratic in the size of P. We leave it to the reader to supplement the missing algorithmic details to complete the proof. □

References

[1] G. Birkhoff. Lattice theory. 3rd ed. American Mathematical Society Colloquium Publications, 25. Providence, Rhode Island: American Mathematical Society (1979).

[2] K. Bertet, J. Gustedt and M. Morvan. Weak-Order Extensions of an Order. Proceedings of WG97, volume 1198 of LNCS, 65–77 (1997).

[3] O. Cogis. On the Ferrers dimension of a digraph. Discrete Math. 38, 47-52 (1982).

[4] S. Felsner, M. Habib and R.H. Möhring. On the interplay of interval dimension and dimension. SIAM J. on Discr. Math. 7, 32–40 (1994)

[5] S. Felsner. 3-Interval Irreducible Partially Ordered Sets. Order 11, 97–125 (1994)

[6] J. Gustedt and M. Morvan. N-free orders and minimal interval extensions. Order 9, No.3, 291-302 (1992).

[7] M. Habib, M. Morvan, M. Pouzet and J.-X. Rampon. Extensions intervallaires minimales. (Minimal interval extensions). C. R. Acad. Sci., Paris, Ser. I 313, No.13, 893-898 (1991).

[8] L. Langley. A recognition algorithm for orders of interval dimension two. Discrete Appl. Math. 60, 257-266 (1995).

[9] J. Mitas. Interval orders based on arbitrary ordered sets. Discrete Math. 144, 75–95 (1995).

[10] R.H. Möhring. Computationally tractable classes of ordered sets. 105–194 In: Algorithms and Orders, I. Rival (Ed) Reidel Dordrecht (1989)

[11] M. Morvan and L. Nourine. Simplicial elimination schemes, extremal lattices and maximal antichain lattices. Order 13, No.2, 159-173 (1996).

[12] K. Reuter. The jump number and the lattice of maximal antichains. Discrete Math. 88, 289-307 (1991).

[13] R. Stanley. Ordered Structures and Partitions. Memoirs of the AMS 119, (1972).

[14] R. Stanley. Enumerative Combinatorics, Volume 1. The Wadsworth & Brooks/Cole Mathematics Series, Monterey, California, (1986).

[15] M.M. Sysło. A labeling algorithm to recognize a line graph and output its root graph. Inform. Process. Lett. 15, 28–30 (1982).