Lexicographic Fréchet Matchings

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Abstract

The Fréchet distance between two curves is the maximum distance in a simultaneous traversal of the two curves. We refine this notion by not only looking at the maximum distance but at all other distances. Roughly speaking, we want to minimize the time $T(s)$ during which the distance exceeds a threshold s, subject to upper speed constraints. We optimize these times lexicographically, giving more weight to larger distances s. For polygonal curves in general position, this criterion produces a unique monotone matching between the points on the two curves, which is important for applications like morphing, and we can compute this matching in polynomial time.

1 Introduction

The classical Fréchet distance is a bottleneck criterion: it measures the maximum distance in the simultaneous traversal. Since the maximum is generically achieved at a single point, the matching is far from unique. In an attempt to address this issue, Buchin, Buchin, Meulemans, and Speckmann (ESA'12) [2] have introduced *locally correct* Fréchet matchings: The maximum distance of matched points on any two matched *subcurves* should not exceed the Fréchet distance between these subcurves. This criterion still does not yield unique matchings. We will review this and other related work later in Section 6. We propose a strengthening of the Fréchet criterion, which, under general position assumptions, produces unique matchings.

Let $P\colon [0, L_P] \to \mathbb{R}^d$ and $Q\colon [0, L_Q] \to \mathbb{R}^d$ be two curves. Two nondecreasing bijections $\alpha\colon [0, M] \to [0, L_P]$ and $\beta\colon [0, M] \to [0, L_Q]$ define a *simultaneous traversal* $(P(\alpha(t)), Q(\beta(t)))$ $(0 \le t \le M)$ of these curves. The classical Fréchet distance looks for a traversal that minimizes the maximum distance $\max\{\, \|P(\alpha(t)) - Q(\beta(t))\| \mid 0 \le t \le M \,\}$ between matched points.

To refine this notion, we look not only at the maximum, but at the whole *distance function* $f\colon [0, M] \to \mathbb{R}_{\ge 0}$:

$$f(t) = \|P(\alpha(t)) - Q(\beta(t))\|$$

The idea is to minimize this function lexicographically: we not only minimize the largest value, but also the "second-largest" value, subject to the largest being smallest, and so on.

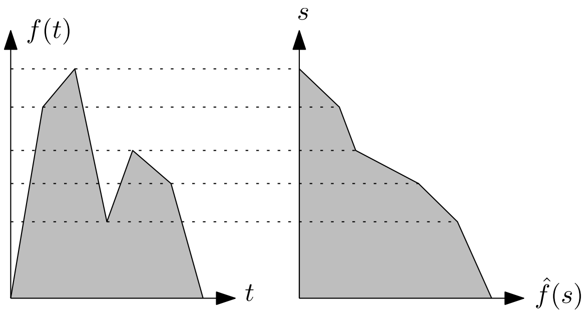

Figure 1: A function f and its profile $\hat f$

In terms of a continuous function f, we formalize this as follows. The *profile function* $\hat f\colon \mathbb{R} \to \mathbb{R}_{\ge 0}$ associated to f measures for each threshold value s the amount of time that $f(t)$ is at least s:

$$\hat f(s) = \mu(\{\, t \mid f(t) \ge s \,\}),$$

where μ denotes the Lebesgue measure.

Figure 1 shows an example. Since the argument variable s corresponds to the dependent variable of f, we have drawn s on the vertical axis, and $\hat f(s)$ to the right. Pictorially, the graph of $\hat f$ can be obtained by shifting the shaded area below the graph of f horizontally to the left until it abuts the y-axis.

We compare two profiles $\hat f$ and $\hat g$ lexicographically, giving most weight to values for the largest arguments s, see Figure 2:

$$\hat f \prec_{\text{lex}} \hat g \iff \exists s_0\colon \big[(\forall s \ge s_0\colon \hat f(s) = \hat g(s)) \wedge \forall \varepsilon > 0\colon \exists s \in (s_0 - \varepsilon, s_0)\colon \hat f(s) < \hat g(s)\big]$$

For piecewise algebraic functions, this comparison is always well-defined. The maximum $s_{\max} = \max_t f(t)$ can be recovered from $\hat f$: $\hat f(s) = 0$ for $s > s_{\max}$, $\hat f(0) = M$, and $\hat f$ is strictly decreasing function between 0 and $s_{\max}$. (This follows from the continuity of f.) Therefore, lexicographic minimization of $\hat f$ subsumes and extends the classical Fréchet objective function of minimizing the maximum of f.

The lexicographic minimization of $\hat f$ requires some normalization in order to make sense. Otherwise we could simply traverse the two curves at a larger speed and accordingly scale down $\hat f$.

We therefore make the following natural assumption.

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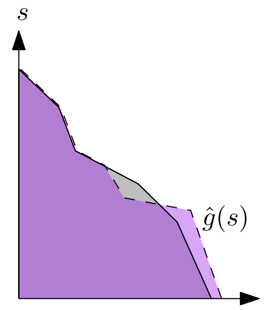

Figure 2: $\hat{g} \prec_{\text{lex}} \hat{f}$ because when the functions first branch apart as s goes down from larger to smaller values, $\hat{g}$ is below $\hat{f}$.

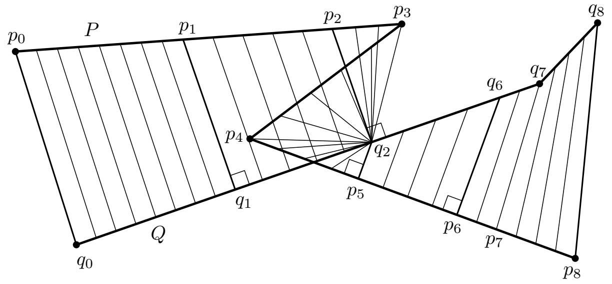

Figure 3: A lexicographic Fréchet matching between two curves P and Q.

Assumption 1 *The speed at which the curves P and Q are traversed by the parametrizations α and β is bounded by* 1.

Before we go into more details about the lexicographic Fréchet matching, let us look at Figure 3 and observe some of its features at a small example. Starting in the endpoints $pq = p_0q_0$, both points p and q advance at full speed. At p_1q_1, the matching edge becomes perpendicular to Q. The point $p \in P$ continues to move at full speed, but q follows as the closest point on Q. At q_2 the point q must remain stationary, because otherwise the distance to p_4 would increase. (The point q_2 is defined by the condition $p_3q_2 = p_4q_2$. This is the classical critical situation for the Fréchet distance.) The point p advances via the vertex p_3 to p_4. It then proceeds to p_5 "as fast as possible" while q remains stationary. Here the distance reaches a local minimum. Let us look at the situation from the other end in reverse. Similar to the beginning, p and q move at full speed till p_6q_6. The point q continues at full speed to p_2, while p follows, keeping the segment pq perpendicular to P until p reaches p_5.

2 Definitions

P and Q are polygonal chains with m and n edges, respectively. We assume that they lie in the plane, but the results generalize easily to arbitrary dimensions.

From now on, we will assume that P and Q are

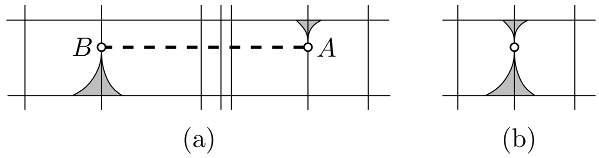

Figure 4: Critical events for the Fréchet distance. (a) A horizontal passage between two cells in a row opens. (b) A direct passage between adjacent cells opens. This is a special case of (a).

given by an *arc-length* parametrization. This is in contrast to the prevalent convention where a unit-length parameter interval is allotted to each segment.

We consider the rectangular parameter area $R = [0, L_P] \times [0, L_q]$, where a point $(x, y) \in R$ represents the point pair $(P(x), Q(y))$ and accordingly defines the *height function*

$$\delta(x, y) := \|P(x) - Q(y)\|_2.$$

A joint parametrization $(\alpha(t), \beta(t))$ corresponds to a continuous curve in R from $(0, 0)$ to the opposite corner (L_P, L_Q) that is monotone in x and y direction, and it leads to the distance function $f(t) = \delta(\alpha(t), \beta(t)) = \|P(\alpha(t)) - Q(\beta(t))\|_2$. We will say that (α, β) induces a *matching* between the points of P and Q although it is not a matching in the usual technical sense of a one-to-one mapping. (It is a *correspondence*.) The speed constraint translates into the condition that the curve is Lipschitz continuous with respect to the L_∞-norm: $\|(\alpha(t_2), \beta(t_2)) - (\alpha(t_1), \beta(t_1))\|_{\max} \le |t_2 - t_1|$.

The free-space diagram F_ε for distance ε is the set $\{\, (x, y) \in R \mid \|P(x) - Q(y)\|_2 \le s \,\}$. The fundamental insight of Alt and Godau [1] is that the Fréchet distance equals the smallest ε for which a monotone path from $(0, 0)$ to (L_P, L_Q) exists in F_ε.

The parameter region R is divided into mn rectangular *cells* according to the partition of P and Q into segments. In each cell, the height function has the form

$$\delta(x, y) = \sqrt{(x - a)^2 + (y - b)^2 + \lambda(x - a)(y - b)}$$

for some parameters a, b, and $-2 \le \lambda \le 2$. Its level sets $\delta(x, y) = \text{const}$ are similar ellipses with common center (a, b) and axes in the $\pm 45°$ directions, possibly degenerating to parallel strips.

3 The Lexicographic Fréchet Matching

We first sketch the idea for constructing a lexicographic Fréchet matching. As ε varies, the free-space diagram grows or shrinks and the monotone connectivity of the free-space diagram changes. It is well-known that the critical values ε, where something happens look like in Figure 4. There are also vertical versions of these events, where the x and y coordinates are interchanged. The Fréchet distance is equal to one of these critical values.

Let us assume that the Fréchet distance ε is determined by a situation as in Figure 4a. We assume that the input is in sufficiently *general position*, and therefore this is the only critical event with this value. This implies that any monotone path must go through points B and A (otherwise the maximum distance would exceed ε). From B to A the path has no choice but to go on a straight line, and it will do so at maximum speed. This allows us to decompose the problem: Find the optimum path from $(0, 0)$ to point B, and from point A to (L_P, L_Q).

Let us concentrate on the second subpath. The maximum distance ε is assumed at point A. To minimize the time during which the distance is close to ε, the distance should decrease as fast as possible.

4 Steepest Descent

As mentioned above, the level sets of the function δ are concentric ellipses, see Figure 5. The points where the ellipses have vertical tangents and δ has a horizontal gradient lie on a line ℓ through the common center, which separates the region where the gradient points downward from the region where it points upward.

The region of R that can be reached from A in some fixed time by a monotone path is a square with lower endpoint at A. We want to go to the smallest possible value in this square. There are several cases. (a) A is above ℓ. Then we see that the path has to move horizontally to the right in order to decrease δ as fast as possible. (b) A is below ℓ. Then the path has to move at 45°, parallel to the dashed axis of the ellipses. (c) A lies on ℓ. Then the path remains on ℓ. We also have the cases that (c′) A is on the line ℓ' where the ellipses have horizontal tangents, and (a′) A is to the right of ℓ'. These cases are analogous to (c) and (a).

When the point moves in the direction of case (a) or (b), it will eventually hit the line ℓ. In this case, it switches to case (c) and continues on ℓ. In Figure 3, such a transition occurs at p_1q_1 and at p_6q_6 (coming from p_7q_7).

Lemma 1 *A steepest-descent path in R is a polygonal path. Before it leaves a cell or arrives in the local minimum (at the center of the ellipses), it makes at most one bend inside the cell.*

As we follow a steepest-descent path and decrease the height ε, we must ensure that we can still reach the target point by a monotone path in the free-space diagram F_ε. Thus, we must watch for events of Figure 4. In addition we must watch for events involving the moving point A, as shown in Figure 6.

The general procedure is as follows. We are looking for an optimal monotone path between two points A and B in R. If $\varepsilon = d(A) > d(B)$, we follow a steepest descent path from A, and we simultaneously decrease ε and watch for events in which the monotone connectivity changes: This could be either an event like in Figure 4, or an event involving the moving point A, like in Figure 6 (or its vertical counterpart), or the event that A and B get the same x or y coordinate. In each case, we identify a point or a horizontal or vertical segment through which the optimum path must pass, and we can partition the problem into smaller subproblems (which may be empty). If A moves to an adjacent cell, we follow a new descent path in this cell.

The case $d(B) > d(A)$ is symmetric. If $d(A) = d(B)$, we simultaneously follow two steepest descent paths from A and B.

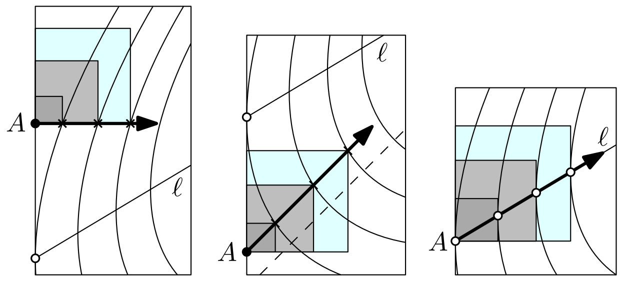

Figure 5: Steepest monotone descent from a point A

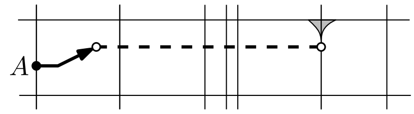

Figure 6: A critical event when point A moves on a steepest-descent path

5 Algorithm

The basic building block of the algorithm is the following. We are given two points $A, B \in R$ and want to find the best monotone path between them. As described above, we have to watch for events that might change the connectivity. When A or B leaves its cell, or when we identify an event that destroys the connectivity, we have made some progress: either the path split into two subpaths at the boundary of a cell or at the optimal path in at least one cell becomes completely known (or both). It follows that such *major events* can happen at most $O(m+n)$ times. Between two major events, A and B are confined to their cell. We have to search through the critical values of all potential events of the types of Figure 4 or Figure 6. This is done by the usual reduction to the *decision problem*, which asks for a given ε whether there is a monotone path between two points whose height never exceeds ε. This question can be decided in $O(mn)$ time. The search with in the critical values of the events of Figure 4 is addressed in the classical Fréchet distance algorithm, and it can be done in $O(mn \log(m+n))$ time [1]. There are only $O(m+n)$ potential events of the type in Figure 6, and a binary search will resolve these events also in $O(mn \log(m+n))$ time. Since we have $O(m+n)$ major phases, we obtain:

Theorem 2 *In the lexicographic Fréchet matching between two polygonal chains of m and n pieces, the simultaneous parametrization has $O(m+n)$ linear pieces. If the input is in general position, it can be computed in $O(mn(m+n)\log(m+n))$ time.*

At least one point always moves at full speed. The other point can either (a) *travel at full speed as well*; (b) *remain stationary*; *or* (c) *follow the first point as the closest point* (*projection*) *on its edge.*

6 Related Work

The nonuniqueness of the Fréchet matching has been noted by many. For example, Helmut Alt [private communication] has considered the path integral of the height function $\delta(x, y)$. Minimization of this functional requires tools from calculus of variations.

As mentioned in the introduction, Buchin et al. [2] have introduced *locally correct* Fréchet matchings to get rid of *some* of the freedom of the Fréchet matching. Their algorithm is similar to our algorithm, as it also partitions the problem into subproblems at critical events. By contrast, we optimize the resulting subproblems more aggressively. Locally correct Fréchet matchings are far from unique. For example, any parametrization where the distance is monotonically increasing throughout, or where the distance has a single local minimum or a single local maximum (which is, however, required to equal the Fréchet distance) is locally correct. On the other hand, it is easy to see that the lexicographic Fréchet matching must be locally correct: replacing a matching between two subcurves by a matching with a smaller maximum would yield a better profile.

Speed limits in connection with the Fréchet distance have also been considered by Maheshwari, Sack, Shahbaz, and Zarrabi-Zadeh [4]. They imposed upper and lower speed limits on each segment of P and Q, and computed the (classical, i.e., bottleneck) Fréchet distance under these constraints. (Without lower speed limits, their problem becomes equivalent to the classical Fréchet distance.)

One can of course look at paths on terrains that do not come from a height function δ resulting from the distance between two paths, and without the monotonicity constraint. The question of minimizing the maximum height on a path from A to B has been addressed by de Berg and van Kreveld [3], besides some other criteria. It would be interesting to extend our lexicographic approach to this setting. On polyhedral terrains, gradient paths are straightforward to find.

7 Degenerate Inputs

Several critical events can occur simultaneously: there could either be a sequence of critical passages that have to be passed in succession, or their might be alternative pathways. I have ignored this possibility so far. Buchin et al. [2] have shown that, for obtaining a locally correct Fréchet matching in this case, a certain set of minimal critical events can be arbitrarily chosen. In our setting, we would have to compare the steepest-descent paths from the different critical points on the basis of “how steeply” they descend. The possiblity of ties would still remain. I have not properly investigated how to handle degeneracies. It would be instructive to look at some examples where the critical section of the Fréchet matching is globally nonunique.

8 Different Normalizations

We can impose an individual speed limit for each edge of P and Q. The algorithm and the analysis can easily accommodate this. Theorem 2 holds without change.

A different assumption would bound the *sum* of the speeds instead of their maximum. This corresponds to the L_1 metric in the parameter plane. Our approach can be modified to deal with this variation. The speed on one path has to be traded against the speed on the other path. This has the effect that the “stop-and-go” case of one point remaining stationary, where the matching is not one-to-one, occurs more often.

One can also consider the Euclidean metric in the parameter plane, which translates to the condition that the *sum of the squared speeds* is bounded by 1. This constraint seems somewhat unnatural for the problem; moreover, it makes the gradient paths nonlinear and may lead to nasty differential equations.

References

[1] H. Alt and M. Godau. Computing the Fréchet distance between two polygonal curves. *Internat. J. Comput. Geom. Appl.*, 5:75–91, 1995.

[2] K. Buchin, M. Buchin, W. Meulemans, and B. Speckmann. Locally correct Fréchet matchings. In *Proc. 20th Ann. European Conf. Algorithms (ESA'12)*, volume 7501 of *Lect. Notes Comput. Sci.*, pages 229–240. Springer-Verlag, 2012. arXiv:1206.6257.

[3] M. de Berg and M. van Kreveld. Trekking in the Alps without freezing or getting tired. *Algorithmica*, 18:306–323, 1997.

[4] A. Maheshwari, J.-R. Sack, K. Shahbaz, and H. Zarrabi-Zadeh. Fréchet distance with speed limits. *Comput. Geom. Theory Appl.*, 44(2):110–120, 2011.