Grassmannians, measure partitions and waists of spheres

Dissertation
zur Erlangung des Grades einer
Doktorin der Naturwissenschaften (Dr. rer. nat.)

am Fachbereich Mathematik und Informatik
der Freien Universität Berlin

vorgelegt von

Nevena Palić

Berlin, 2018

Betreuer: Prof. Dr. Günter M. Ziegler

Zweitgutachter: Prof. Dr. Pavle V.M. Blagojević

Drittgutachter: Dr. Roman Karasev

Tag der Disputation: 27.08.2018

Selbstständigkeitserklärung

Hiermit erkläre ich, dass ich diese Arbeit selbstständig verfasst und keine anderen als die angegebenen Hilfsmittel und Quellen verwendet habe. Ich erkläre weiterhin, dass ich die vorliegende Arbeit oder deren Inhalt nicht in einem früheren Promotionsverfahren eingereicht habe.

Nevena Palić
Berlin, 27. September 2018

Acknowledgements

First of all, I am very grateful to my advisors Pavle V.M. Blagojević and Günter M. Ziegler. Thanks for introducing topological combinatorics to me and for sharing a plenty of interesting and very challenging problems. Thanks also for the time, energy and patience invested in my research and our papers. I am especially grateful for all the knowledge they shared with me.

I also want to thank to my further coauthors Roman Karasev and Pavel Galashin for sharing their brilliant ideas with me. Moreover, I am grateful to Rainer Sinn, Steven Karp, Jonathan Kliem, Thomas Lam, Johanna K. Steinmeyer, Marie-Charlotte Brandenburg, Hana Kourimska and Christian Weiland for useful discussions and improvements of the exposition of some parts of this thesis.

Special thanks to everyone who was in the Villa for the past years for providing an inspiring and a pleasant working atmosphere. Furthermore, I would like to thank my family and friends for their unlimited support on my way.

Last but not least, I am grateful to the Berlin Mathematical School and to the Dahlem Research School for the financial and logistical support, and I am particularly grateful to Elke Pose for all the help in the past years.

Contents

Chapter 1

Introduction

In this thesis we apply methods from algebraic topology to problems arising from geometry, combinatorics and functional analysis. The questions are related to Grassmannians, partitions of measures and waists of spheres, whereas the methods include spectral sequences, a cohomological index theory and the equivariant obstruction theory.

In Chapter 2 we consider amplituhedra – images of nonnegative Grassmannians under maps induced by a linear map. They were introduced by physicists Arkani-Hamed & Trnka [5] as objects that conjecturally model the scattering amplitudes of certain quantum field theories. More general, Grassmann polytopes, as introduced by Lam [52], are images of restrictions of the above mentioned maps to closed positroid cells – cells in a CW decomposition of the nonnegative Grassmannian [68, Def. 3.2, Thm. 3.5].

Let $k \geq 1$, $m \geq 0$ and $n \geq k+m$ be integers and let Z be a $(k+m) \times n$ matrix, such that the induced map

$$\widetilde{Z} : \mathrm{G}_k^{\geq 0}(\mathbb{R}^n) \longrightarrow \mathrm{G}_k(\mathbb{R}^{k+m})$$

given by

$$\widetilde{Z}(\mathrm{span}(V)) = \mathrm{span}(VZ^\top),$$

is well defined, where V is a matrix whose row span $\mathrm{span}(V)$ is an element of $\mathrm{G}_k^{\geq 0}(\mathbb{R}^n)$, and $Z^\top$ is the transpose of the matrix Z. The image $\widetilde{Z}(\bar{e})$ of a closed positroid cell $\bar{e}$ in the CW decomposition of $\mathrm{G}_k^{\geq 0}(\mathbb{R}^n)$ is called a *Grassmann polytope*, an it is denoted by $\mathrm{P}_Z(e)$. If in addition all maximal minors of the matrix Z are positive, the image of the map $\widetilde{Z}$ is called an *amplituhedron*, denoted by $\mathcal{A}_{n,k,m}(Z)$, see Definition 2.1.2 for more details.

The topology of amplituhedra and Grassmann polytopes has been known only in a few cases, when they turn out to be homeomorphic to balls. The case $m = 0$ is trivial, whereas when $m = 1$ Karp & Williams [51, Cor. 6.18] have shown that the amplituhedron is homeomorphic to a ball. For $k = 1$ the amplituhedron is a cyclic polytope of dimension m on n vertices [77]. Similarly, for $k = 1$ the Grassmann polytopes are also polytopes. For $n = k + m$ the map $Z\mathbb{R}^n \to \mathbb{R}^{k+m}$ is a linear isomorphism, and consequently the amplituhedron is homeomorphic to the totally nonnegative Grassmannian $\mathrm{G}_k^{\geq 0}(\mathbb{R}^n)$, which was proved to be a ball by Galashin, Karp & Lam [36, Thm. 1.1], and the Grassmann polytope is homeomorphic to the closed positroid cell $\bar{e}$, which is again homeomorphic to a ball, as shown by Rietsch & Williams [72]. Finally, Galashin, Karp & Lam [36, Thm. 1.2] proved that the cyclically symmetric amplituhedra, amplituhedra arising from particularly chosen matrices Z, are homeomorphic to balls whenever m is even.

We show that Grassmann polytopes are contractible for every n, k and m such that $n = k+m+1$, and that amplituhedra are homeomorphic to balls whenever $n = k+m+1$ and m is even. The proofs depend on the classical topological results of Smale [74, Main Thm.] and Whitehead [85, Thm. 1], and on the recent result on cyclically symmetric amplituhedra of Galashin, Karp & Lam [36, Thm. 1.2].

In Chapter 3 and Chapter 4 we study measure partitions. The classical measure partition problems ask whether for a given set of measures in a Euclidean space (for example, volumes of convex bodies) there exists a certain partition of the ambient space that equiparts each of the given measures. One of the first measure partition results is the well-known ham-sandwich theorem, which states that given any d measures in $\mathbb{R}^d$, there exists a hyperplane cut that equiparts each of the measures.

Convex partitions, i.e., partitions into convex subsets, have been studied intensively. For example, Grünbaum asked in 1960 [40, Sec. 4 (v)], motivated by the ham-sandwich theorem, whether any given measure in $\mathbb{R}^d$ can be cut by k affine hyperplanes into 2^k equal pieces. Hadwiger [42] and Ramos [70] asked an even more general question, that has motivated a lot of research on convex partitions, see for example [15].

In Chapter 3 we give conditions under which for any finite collection of functions on the set of convex partitions, there exists a partition of the Euclidean space into convex prisms – products of convex sets of prescribed dimension, such that each of the functions gets equalized. In particular, specifying these functions to be measures, we claim an existence of a convex partition into prisms that equiparts each of the given measures. Similarly, we consider partitions by regular linear fans, and we give conditions under which for any finite collection of functions on the set of convex partitions, there exists a partition of the ambient space by regular linear fans into convex subsets, such that each of the given functions gets equalized on that partition. These two results are proved using theorems about non-existence of certain equivariant maps, which are also provided in Chapter 3. They are, however, proved using the Fadell–Husseini ideal valued cohomological index theory [32].

Chapter 4 is motivated by the conjecture of Holmsen, Kynčl and Valculescu [45, Conj. 3] on partitions of finite colored sets, such that each subset contains points of many colors. We give a few analogous continuous results. Recall that the ham-sandwich theorem implies that for given d measures in $\mathbb{R}^d$, there exists a (convex) partition of $\mathbb{R}^d$ into two half-spaces such that each of them has non-zero measure with respect to each of the d measures. We consider convex partitions of $\mathbb{R}^d$ such that every subset in the partition has positive measure with respect to at least c measures, even when $c > d$. The first result, which gives a sufficient condition on the number of measures in $\mathbb{R}^d$, such that there exists a convex partition with such a property, has an elementary geometric proof. However, the next two results have stronger statements – one gives, in addition, an equipartition of one of the measures, and the other one gives an equipartition of the sum of the measures. It turns out that topological methods are needed in order to prove the equipartition results. We use a novel configuration space/test map scheme – for the first time the test space is the union of an affine arrangement. Thus we show non-existence of equivariant maps from the space of equipartitions into the union of an affine arrangement.

Next, Chapter 5 presents an application of the equivariant obstruction theory on a question from functional analysis. The celebrated waist of the sphere theorem of Gromov

[39, Sec. 1] states that for all integers $n \geq k \geq 1$ and for every continuous map $f : S^n \to \mathbb{R}^k$, there exists a point $z \in \mathbb{R}^k$ such that the n-dimensional volume of the tubular neighborhood $f^{-1}(z) + \varepsilon \subseteq S^n$ is at least as big as the n-dimensional volume of the tubular neighborhood of an equatorial $(n-k)$-sphere $S^{n-k} \subseteq S^n$, for every $\varepsilon > 0$.

We prove that, if f is additionally $\mathbb{Z}_p$-equivariant for some prime p, and if the action of $\mathbb{Z}_p$ on S^n is free and an orientation preserving isometry, we can choose z to be the origin in $\mathbb{R}^k$, i.e., the volume of the tubular neighborhood of the inverse image $f^{-1}(0)$ is at least as big as the volume of the tubular neighborhood of an equatorial $(n-k)$-sphere. The proof follows the ideas of Gromov [39] and Memarian [60], and depends on the theorem which claims that there is no equivariant map between the wreath product of classical configuration spaces and a certain sphere. The non-existence of such a map is proved using equivariant obstruction theory, and the necessary equivariant CW model for the wreath product of configuration spaces is also developed in Chapter 5.

Finally, in Chapter 6 we get back to Grassmannians, where we consider their combinatorial analogues. More precisely, we study oriented matroid Grassmannians, also called MacPhersonians, which were introduced by MacPherson [56], and firstly used by Gel'fand and MacPherson in order to give a combinatorial formula for Pontrjagin classes [38]. An oriented matroid Grassmannian is the order complex of the set of all oriented matroids of given rank and number of elements, ordered by weak maps. Every MacPhersonian is conjectured to be homotopy equivalent to the corresponding Grassmannian, which was proved by Babson [7] for rank 2. For higher rank, this question is still open.

The results of Chapter 6 follow from substantial computations that we run in order to construct MacPhersonians in rank 3 and 4. In particular, since the construction of the MacPhersonian for all but smallest parameters is beyond computational limits, we construct subcomplexes that are fixed under some group action. Then we are allowed to use results of Floyd [34] and Chang & Skjelbred [26] that compare properties of a whole topological space and of its invariant subspace. All examples that we were able to compute support the conjecture.

In order to make this thesis approachable for the reader, we introduce the Fadell–Husseini index theory [32] and the equivariant obstruction theory [30, Sec. II.3] in the Appendix, where we also summarize their properties.

Chapter 2

Nonnegative Grassmannians, Grassmann polytopes and amplituhedra

In this chapter we present results from the paper *Some more amplituhedra are contractible*, which is a joint work with Pavle V.M. Blagojević, Pavel Galashin and Günter M. Ziegler [16].

Introduction and statement of the main result

Introduction

Let n and k be integers such that $n \geq k \geq 1$. If $\mathrm{Mat}_{k,n}$ denotes the space of all real $k \times n$ matrices of rank k, then the real Grassmannian $\mathrm{G}_k(\mathbb{R}^n)$ — the space of all k-dimensional linear subspaces of $\mathbb{R}^n$ — can be defined as the orbit space $\mathrm{G}_k(\mathbb{R}^n) = \mathrm{GL}_k \setminus \mathrm{Mat}_{k,n}$. The totally nonnegative part of the Grassmannian may be defined quite analogously:

Definition 2.1.1 (Postnikov [68, Sec. 3])**.** Let $n \geq k \geq 1$ be integers, let $\mathrm{Mat}_{k,n}^{\geq 0}$ be the space of all real $k \times n$ matrices of rank k all whose maximal minors are nonnegative, and let GL_k^+ denote the group of all real $k \times k$ matrices with positive determinant, which acts freely on $\mathrm{Mat}_{k,n}^{\geq 0}$ by matrix multiplication from the left. The *totally nonnegative Grassmannian* $\mathrm{G}_k^{\geq 0}(\mathbb{R}^n)$ is the orbit space $\mathrm{G}_k^{\geq 0}(\mathbb{R}^n) = \mathrm{GL}_k^+ \setminus \mathrm{Mat}_{k,n}^{\geq 0}$.

The totally nonnegative Grassmannian was introduced and studied by Postnikov in 2006 [68, Sec. 3], building on works by Lusztig [54] and by Fomin & Zelevinsky [35]. Subsequently, the geometric and combinatorial properties of the totally nonnegative Grassmannian were studied intensively. Rietsch & Williams showed that the totally nonnegative Grassmannian is contractible [72, Thm. 1.1]; an earlier argument by Lusztig [55, Sec. 4.4] can also be adapted to prove the same. Galashin, Karp & Lam [36, Thm. 1.1] proved that $\mathrm{G}_k^{\geq 0}(\mathbb{R}^n)$ is indeed homeomorphic to a closed $k(n-k)$-dimensional ball.

In 2014, the physicists Arkani-Hamed & Trnka [5, Sec. 9] introduced the amplituhedra as certain images of the totally nonnegative Grassmannians. They conjectured that their geometry describes scattering amplitudes in some quantum field theories. For a gentle introduction to amplituhedra in physics and mathematics consult [23]. Shortly after, Lam introduced Grassmann polytopes [52], which generalize amplituhedra.

Postnikov [68, Def. 3.2, Thm. 3.5] defined a CW decomposition of the totally nonnegative Grassmannian $\mathrm{G}_k^{\geq 0}(\mathbb{R}^n)$ such that each cell, also called a *positroid cell*, is indexed by the associated matroid – a positroid – of rank k on n elements, see also [69]. Moreover, Rietsch & Williams [72] showed that the closures of positroid cells are contractible and that their boundaries are homotopy equivalent to spheres.

Definition 2.1.2. Let $k \geq 1$, $m \geq 0$ and $n \geq k+m$ be integers, and let Z be a real $(k+m) \times n$ matrix such that the induced map

$$\widetilde{Z} : \mathrm{G}_k^{\geq 0}(\mathbb{R}^n) \longrightarrow \mathrm{G}_k(\mathbb{R}^{k+m})$$

given by

$$\widetilde{Z}(\mathrm{span}(V)) = \mathrm{span}(VZ^\top) \tag{2.1}$$

is well defined. Here $V \in \mathrm{Mat}_{k,n}^{\geq 0}$, span denotes the row span of a matrix, and $Z^\top$ is the transpose of the matrix Z. The image $\widetilde{Z}(\bar{e})$ of a closed positroid cell $\bar{e}$ in the CW decomposition of the nonnegative Grassmannian $\mathrm{G}_k^{\geq 0}(\mathbb{R}^n)$ is called a *Grassmann polytope*, denoted by $\mathrm{P}_Z(e)$. If e is the maximal cell, which for this CW decomposition means $\bar{e} = \mathrm{G}_k^{\geq 0}(\mathbb{R}^n)$, and all $(k+m) \times (k+m)$ minors of the matrix Z are positive, then the Grassmann polytope $\mathrm{P}_Z(e)$ is called an *amplituhedron* and is denoted by $\mathcal{A}_{n,k,m}(Z)$.

The previous definition in particular means that if $v_1, \dots, v_k \in \mathbb{R}^n$ are linearly independent row vectors, then

$$\widetilde{Z}(\mathrm{span}\{v_1, \dots, v_k\}) = \mathrm{span}\{v_1 Z^\top, \dots, v_k Z^\top\}.$$

The map $\widetilde{Z}$ is well defined if $\mathrm{span}(VZ^\top)$ is a k-dimensional subspace of $\mathbb{R}^{k+m}$ for every $V \in \mathrm{Mat}_{k,n}^{\geq 0}$. The fact that the map $\widetilde{Z}$ is well defined when Z is a matrix with positive maximal minors was established by Arkani-Hamed & Trnka in [5] and by Karp in [50, Thm. 4.2]. Lam [52, Prop. 15.2], however, considers a larger class of matrices Z for which the map $\widetilde{Z}$ is still well defined.

The structure of the amplituhedron is known only in a few cases. The case $m = 0$ is trivial, whereas when $m = 1$ Karp & Williams [51, Cor. 6.18] have shown that the amplituhedron is homeomorphic to a ball. For $k = 1$ the amplituhedron is a cyclic polytope of dimension m on n vertices [77], and for $n = k+m$ the map Z is a linear isomorphism, and consequently the amplituhedron is homeomorphic to the totally nonnegative Grassmannian $\mathrm{G}_k^{\geq 0}(\mathbb{R}^n)$, which is a ball by [36, Thm. 1.1]. Finally, Galashin, Karp & Lam [36, Thm. 1.2] proved that the cyclically symmetric amplituhedra, amplituhedra arising from particularly chosen matrices Z, are homeomorphic to balls whenever m is even. The topology of other Grassmann polytopes is unknown.

Main results

Our first result gives a family of contractible Grassmann polytopes.

Theorem 2.1.3. *Let $k \geq 1$ and $m \geq 0$ be integers, and let Z be a real $(k+m) \times (k+m+1)$ matrix such that the map $\widetilde{Z} : \mathrm{G}_k^{\geq 0}(\mathbb{R}^{k+m+1}) \to \mathrm{G}_k(\mathbb{R}^{k+m})$ is well defined. Then the Grassmann polytope $\mathrm{P}_Z(e)$ is contractible for every positroid cell e in the CW decomposition of $\mathrm{G}_k^{\geq 0}(\mathbb{R}^{k+m+1})$.*

The proof of Theorem 2.1.3 relies on classical results of Smale [74, Main Thm.] and Whitehead [85, Thm. 1].

The following is a consequence of Smale's result [74, Main Thm.].

Theorem 2.1.4 (Smale). *Let X and Y be path connected, locally compact, separable metric spaces, and in addition let X be locally contractible. Let $f\colon X \longrightarrow Y$ be a continuous surjective proper map, that is, any inverse image of a compact set is compact. If for every $y \in Y$ the inverse image $f^{-1}(\{y\})$ is contractible, then the induced homomorphism*

$$f_{\#}\colon \pi_i(X) \longrightarrow \pi_i(Y)$$

is an isomorphism for all $i \geq 0$.

Recall that a continuous map $f\colon X \longrightarrow Y$ between topological spaces X and Y is a *weak homotopy equivalence* if the induced map on the path connected components $f_{\#}\colon \pi_0(X) \longrightarrow \pi_0(Y)$ is bijective, and for every point $x_0 \in X$ and for every integer $n \geq 1$ the induced map $f_{\#}\colon \pi_n(X, x_0) \longrightarrow \pi_n(Y, f(x_0))$ is an isomorphism.

Theorem 2.1.5 ([85, Thm. 1]). *Let X and Y be topological spaces that are homotopy equivalent to CW complexes. Then a continuous map $f\colon X \longrightarrow Y$ is a weak homotopy equivalence if and only if it is a homotopy equivalence.*

Since Theorem 2.1.5 requires that spaces have the homotopy type of a CW complex, the following theorem is a necessary ingredient in the proof of Theorem 2.1.3.

Theorem 2.1.6. *Let $k \geq 1$, $m \geq 0$ and $n \geq k+m$ be integers, and let Z be a real $(k+m) \times n$ matrix such that the map $\widetilde{Z}$ is well defined. Then for every positroid cell e in $\mathrm{G}_k^{\geq 0}(\mathbb{R}^n)$, the Grassmann polytope $\mathrm{P}_Z(e)$ is homotopy equivalent to a countable CW complex. Moreover, if $n = k+m+1$, the Grassmann polytope $\mathrm{P}_Z(e)$ is homotopy equivalent to a finite CW complex.*

In order to apply Theorem 2.1.4 to the map $\widetilde{Z}$, we need to understand its fibers. Thus we prove the following theorem.

Theorem 2.1.7. *Let $k \geq 1$ and $m \geq 0$ be integers, and let Z be a real $(k+m) \times (k+m+1)$ matrix such that the map $\widetilde{Z}$ is well defined. Then for every positroid cell e and for every point $y \in \mathrm{P}_Z(e)$, the inverse image $(\widetilde{Z}|_{\bar{e}})^{-1}(\{y\}) = \widetilde{Z}^{-1}(\{y\}) \cap \bar{e}$ under the restriction map $\widetilde{Z}|_{\bar{e}} : \bar{e} \longrightarrow \mathrm{P}_Z(e)$ is contractible.*

The proof of Theorem 2.1.7 is postponed to the next section, whereas the proof of Theorem 2.1.6 is given in Section 2.4. Here we show that Theorem 2.1.7 in combination with Theorem 2.1.4 and Theorem 2.1.5 implies our main result.

Proof of Theorem 2.1.3. Let e be a positroid cell in the CW decomposition of $\mathrm{G}_k^{\geq 0}(\mathbb{R}^{k+m+1})$. We apply Theorem 2.1.4 to the map $\widetilde{Z} : \bar{e} \longrightarrow \mathrm{P}_Z(e)$. The spaces $\bar{e}$ and $\mathrm{P}_Z(e)$, as well as the map $\widetilde{Z}$, satisfy assumptions of Theorem 2.1.4. Furthermore, Theorem 2.1.7 implies that for every $y \in \bar{e}$, the fiber $\widetilde{Z}^{-1}(\{y\})$ is contractible. Thus, from Theorem 2.1.4 we have that the map $\widetilde{Z}$ is a weak homotopy equivalence.

The closed positroid cell $\bar{e}$ is a CW complex. Furthermore, the Grassmann polytope $\mathrm{P}_Z(e)$ is homotopy equivalent to a CW complex, by Theorem 2.1.6. Thus, from Theorem

2.1.5 we conclude that the map $\widetilde{Z}$ is a homotopy equivalence. Hence, the Grassmann polytope $\mathrm{P}_Z(e)$ is homotopy equivalent to the closed positroid cell $\bar{e}$, which is contractible, see [72]. □

In Theorem 2.1.6, we show that Grassmann polytopes are homotopy equivalent to CW complexes, using classical topological results. However, an even stronger result holds.

Theorem 2.1.8. *Every Grassmann polytope is a semialgebraic set. In particular, it admits a triangulation.*

Note that Theorem 2.1.8 claims that every Grassmann polytope $\mathrm{P}_Z(e)$ can be triangulated in a classical sense, thus there exists a simplicial complex T and a homeomorphism $T \to \mathrm{P}_Z(e)$. This is, however, not a triangulation in terms of [52].

In particular, the above theorem gives an implicit answer to [52, Problem 15.9], which asks to describe a Grassmann polytope by inequalities. A related question in the case $m = 2$ was investigated in [4]. We note that a very similar argument to ours was also given by Arkani-Hamed, Bai & Lam in [3, Appendix J].

The proof of Theorem 2.1.8 is given in Section 2.5.

Theorem 2.1.3 in particular implies that all amplituhedra $\mathcal{A}_{k+m+1,k,m}(Z)$ are contractible. Our next result shows that if in addition m is even, they are homeomorphic to balls.

Theorem 2.1.9. *Let $k \geq 1$ be an integer, let $m \geq 0$ be an even integer, and let $Z \in \mathrm{Mat}_{k+m,k+m+1}$ be a matrix with all $(k+m) \times (k+m)$ minors positive. Then the amplituhedron $\mathcal{A}_{k+m+1,k,m}(Z)$ induced by the matrix Z is homeomorphic to a km-dimensional ball.*

The proof of Theorem 2.1.9 is presented in Section 2.3. We remark that the combinatorics of the amplituhedron in the case $n = k + m + 1$ with m even has been recently studied in detail in [37].

Acknowledgement

We are grateful to Rainer Sinn for sharing the knowledge about semialgebraic sets, to Thomas Lam, whose great observations improved the generality of this chapter, and to Steven Karp for helpful comments.

Proof of Theorem 2.1.7

Let $k \geq 1, m \geq 0$ and $n \geq k + m$ be integers and let Z be a real $(k + m) \times n$ matrix such that the map $\widetilde{Z}$ is well defined. Since the action of the group GL_k^+ on $\mathrm{Mat}_{k,n}^{\geq 0}$ is free, there is a fibration

$$\mathrm{GL}_k^+ \longrightarrow \mathrm{Mat}_{k,n}^{\geq 0} \longrightarrow \mathrm{G}_k^{\geq 0}(\mathbb{R}^n). \tag{2.2}$$

The matrix Z, as in Definition 2.1.2, induces a map

$$\begin{aligned} \widehat{Z} : \mathrm{Mat}_{k,n}^{\geq 0} &\longrightarrow \mathrm{Mat}_{k,k+m}, \\ V &\longmapsto VZ^{\top}, \end{aligned}$$

which is again well defined, see for example [52, Prop. 15.2].

Let e be a positroid cell in the CW decomposition of $\mathrm{G}_k^{\geq 0}(\mathbb{R}^n)$, and let $I_e \subseteq \binom{[n]}{k}$ be the family of nonbases (dependent sets) of cardinality k of the matroid that defines the cell e. The maximal minors of a $k \times n$ matrix are indexed by the set $\binom{[n]}{k}$. Denote by $\mathrm{Mat}_{k,n}^{\geq 0}(e)$ the set of all matrices $V \in \mathrm{Mat}_{k,n}^{\geq 0}$ whose minors indexed by elements of I_e are equal to zero. Then every point in $\bar{e} \subseteq \mathrm{G}_k^{\geq 0}(\mathbb{R}^n)$ is represented by a matrix in $\mathrm{Mat}_{k,n}^{\geq 0}(e)$, and the row span of every such matrix lies in $\bar{e}$. In other words, $\bar{e} = \mathrm{GL}_k^+ \setminus \mathrm{Mat}_{k,n}^{\geq 0}(e)$. Thus the restriction of the fibration (2.2) is a fibration

$$\mathrm{GL}_k^+ \longrightarrow \mathrm{Mat}_{k,n}^{\geq 0}(e) \longrightarrow \bar{e}. \tag{2.3}$$

Note that if e is the maximal positroid cell, the set $\mathrm{Mat}_{k,n}^{\geq 0}(e)$ is the whole set $\mathrm{Mat}_{k,n}^{\geq 0}$.

Denote by $\widehat{\mathrm{P}_Z(e)}$ the image of the set $\mathrm{Mat}_{k,n}^{\geq 0}(e)$ under the map $\widehat{Z}$. With a usual abuse of notation, we consider maps $\widehat{Z} : \mathrm{Mat}_{k,n}^{\geq 0}(e) \longrightarrow \widehat{\mathrm{P}_Z(e)}$ and $\widetilde{Z} : \bar{e} \longrightarrow \mathrm{P}_Z(e)$. Then there exists a commutative diagram of spaces and continuous maps

$$\begin{array}{ccc} \mathrm{Mat}_{k,n}^{\geq 0}(e) & \xrightarrow{\widehat{Z}} & \widehat{\mathrm{P}_Z(e)} \\ \downarrow & & \downarrow \\ \bar{e} & \xrightarrow{\widetilde{Z}} & \mathrm{P}_Z(e), \end{array}$$

where vertical maps send any matrix to its row span.

The proof of Theorem 2.1.7 splits into the following two lemmas.

Lemma 2.2.1. *Let $k \geq 1$ and $m \geq 0$ be integers, $n = k + m + 1$, and let Z be a real $(k+m) \times n$ matrix such that the map $\widetilde{Z}$ is well defined. Then for every positroid cell e in the CW decomposition of $\mathrm{G}_k^{\geq 0}(\mathbb{R}^n)$ and for every $W \in \widehat{\mathrm{P}_Z(e)}$, the inverse image $\widehat{Z}^{-1}(\{W\}) \subseteq \mathrm{Mat}_{k,n}^{\geq 0}(e)$ is nonempty and convex.*

Proof. The matrix Z induces a linear map

$$\begin{array}{ccc} \mathbb{R}^n & \longrightarrow & \mathbb{R}^{k+m} \\ v & \longmapsto & vZ^\top, \end{array} \tag{2.4}$$

where $v \in \mathbb{R}^n$ is a row vector. Since $n = k + m + 1$, the kernel of the map (2.4) is 1-dimensional. Fix a generator $a \in \mathbb{R}^n$ of that kernel.

Choose an arbitrary point $W \in \widehat{\mathrm{P}_Z(e)}$, and let U and V be any two points in $\widehat{Z}^{-1}(\{W\})$. Our goal is to show that for every $\lambda \in [0,1]$ the convex combination $(1-\lambda)U + \lambda V$ also belongs to $\widehat{Z}^{-1}(\{W\})$.

Since $UZ^\top = VZ^\top = W$, the rows of the matrix $V - U$ belong to $\ker(Z)$. Consequently, there exists a row vector $x \in \mathbb{R}^k$ such that $V - U = x^\top a$, where a is also considered as a row vector. Thus we have to show that for every $\lambda \in [0,1]$ the convex combination

$$(1-\lambda)U + \lambda V = U + \lambda x^\top a \tag{2.5}$$

belongs to the space $\mathrm{Mat}_{k,n}^{\geq 0}(e)$, this means that every $k \times k$ minor of the matrix (2.5) is nonnegative, and in addition that all the minors of the matrix (2.5) indexed by the nonbases $I_e \subseteq \binom{[n]}{k}$ of the matroid corresponding to e are equal to zero.

A $k \times k$ submatrix of the matrix (2.5) is of the form

$$\begin{pmatrix} u_{1i_1} + \lambda x_1 a_{i_1} & \dots & u_{1i_k} + \lambda x_1 a_{i_k} \\ \vdots & & \vdots \\ u_{ki_1} + \lambda x_k a_{i_1} & \dots & u_{ki_k} + \lambda x_k a_{i_k} \end{pmatrix}, \tag{2.6}$$

where

$$U = \begin{pmatrix} u_{11} & \dots & u_{1n} \\ \vdots & & \vdots \\ u_{k1} & \dots & u_{kn} \end{pmatrix}, \; x = (x_1 \dots x_k), \; a = (a_1 \dots a_n),$$

and $1 \leq i_1 < \dots < i_k \leq n$. The matrix (2.6) can be transformed using row operations into a matrix that contains the variable λ only in one row. Therefore, every $k \times k$ minor of the matrix (2.5) is a polynomial of degree at most 1 in the variable λ. Since it takes nonnegative values for $\lambda = 0$ and $\lambda = 1$, it is also nonnegative for all $\lambda \in [0, 1]$. Thus for every $\lambda \in [0, 1]$, the point $(1 - \lambda)U + \lambda V$ belongs to $\mathrm{Mat}_{k,n}^{\geq 0}$. Similarly, if $\{i_1, \dots, i_k\}$ is a nonbasis of the matroid corresponding to e, then the determinant of the matrix (2.6) is zero for $\lambda = 0$ and $\lambda = 1$, so it is a constant zero-polynomial, meaning that the matrix (2.5) belongs to $\mathrm{Mat}_{k,n}^{\geq 0}(e)$ for every $\lambda \in [0, 1]$. Consequently the set $\widehat{Z}^{-1}(\{W\})$ is convex. □

Lemma 2.2.2. *Let $k \geq 1, m \geq 0$ and $n \geq k + m$ be integers. For every positroid cell e and for every $W \in \widehat{\mathrm{P}_Z(e)}$, the inverse images*

$$\widehat{Z}^{-1}(\{W\}) \subseteq \mathrm{Mat}_{k,n}^{\geq 0}(e) \subseteq \mathrm{Mat}_{k,n}^{\geq 0} \qquad \text{and} \qquad \widetilde{Z}^{-1}(\{\mathrm{span}(W)\}) \subseteq \bar{e} \subseteq \mathrm{G}_k^{\geq 0}(\mathbb{R}^n)$$

are homeomorphic.

Proof. Let $\varphi\colon \widehat{Z}^{-1}(\{W\}) \longrightarrow \widetilde{Z}^{-1}(\{\mathrm{span}(W)\})$ be defined by $\varphi(U) = \mathrm{span}(U)$, where $U \in \widehat{Z}^{-1}(\{W\})$, and span denotes the row span. We prove that φ is a homeomorphism.

Clearly, φ is continuous, so it suffices to find a continuous map

$$\psi : \widetilde{Z}^{-1}(\{\mathrm{span}(W)\}) \longrightarrow \widehat{Z}^{-1}(\{W\}),$$

such that $\varphi \circ \psi$ is the identity map on $\widetilde{Z}^{-1}(\{\mathrm{span}(W)\})$ and $\psi \circ \varphi$ is the identity map on $\widehat{Z}^{-1}(\{W\})$. Let $L \in \widetilde{Z}^{-1}(\{\mathrm{span}(W)\})$. Then there exists a matrix $K \in \mathrm{Mat}_{k,n}^{\geq 0}(e)$ whose rows span the subspace L. Since

$$\mathrm{span}(KZ^\top) = \mathrm{span}(W),$$

there exists a unique $C \in \mathrm{GL}_k$ such that $KZ^\top = CW$. Now define ψ as $\psi(L) = C^{-1}K$. It can be seen using the Cauchy-Binet formula that $\det(C) > 0$. Thus, $C^{-1}K \in \mathrm{Mat}_{k,n}^{\geq 0}(e)$. Even though we have defined the map ψ using an arbitrarily chosen matrix K such that $\mathrm{span}(K) = L$, it can be checked directly that the definition of ψ does not depend on a choice of K.

In order to prove that the map ψ is continuous, we need to show that the choice of a matrix K can be made continuously on $\widetilde{Z}^{-1}(\{\mathrm{span}(W)\})$. The choice of a matrix K is equivalent to the choice of a positively oriented basis for the subspace $L \subseteq \mathbb{R}^n$. Therefore, we need a continuous section of the fiber bundle (2.3) restricted to the set $\widetilde{Z}^{-1}(\{\text{span}(W)\})$. Since the base space $\bar{e}$ is contractible, the fiber bundle (2.3) is trivial. In particular, its restriction on $\widetilde{Z}^{-1}(\{\text{span}(W)\})$ is also trivial, so it admits a continuous section. Therefore, the bases for elements of $\widetilde{Z}^{-1}(\{\text{span}(W)\})$ can be chosen continuously. On the other hand, the matrix C is a solution of the linear system $KZ^\top = CW$, which depends continuously on K, thus it also depends continuously on L.

Lastly,

$$\varphi(\psi(L)) = \varphi(C^{-1}K) = \text{span}(C^{-1}K) = \text{span}(K) = L,$$

holds for every $L \in \widetilde{Z}^{-1}(\{\text{span}(W)\})$, and

$$\psi(\varphi(U)) = \psi(\text{span}(U)) = C^{-1}U,$$

for every $U \in \widehat{Z}^{-1}(\{W\})$, where C is the unique $k \times k$ matrix such that

$$W = \widehat{Z}(U) = UZ^\top = CW,$$

hence C is the identity matrix. □

Finally, Lemma 2.2.1 and Lemma 2.2.2 complete the proof of Theorem 2.1.7.

Proof of Theorem 2.1.9

Let $k \geq 1$, $m \geq 0$ and $n \geq k+m$ be integers, and suppose in addition that m is even. Let $S \in \text{GL}_n$ be given by

$$S(x_1, \dots, x_n) = (x_2, \dots, x_n, (-1)^{k-1}x_1).$$

Denote by $Z_0 \in \text{Mat}_{k+m,n}$ the matrix whose rows are the eigenvectors of the matrix $S + S^\top$ that correspond to the largest $k+m$ eigenvalues. It was shown in [36, Lemma 3.1] that all $(k+m) \times (k+m)$ minors of the matrix Z_0 are positive, thus it defines an amplituhedron $\mathcal{A}_{n,k,m}(Z_0)$, called *cyclically symmetric amplituhedron*. Galashin, Karp & Lam [36, Thm. 1.2] showed that $\mathcal{A}_{n,k,m}(Z_0)$ is homeomorphic to a closed km-dimensional ball whenever the parameter m is even.

We conclude the proof of Theorem 2.1.9 by showing that the amplituhedra $\mathcal{A}_{n,k,m}(Z)$ and $\mathcal{A}_{n,k,m}(Z_0)$ are homeomorphic.

From [50, Cor. 1.12(ii)] we know that entries of every nonzero vector of $\ker(Z_0)$ and of $\ker(Z)$ are nonzero, and they alternate in sign. Since $n = k+m+1$, the kernels of matrices Z and Z_0 are 1-dimensional. Let $a = (a_1, \dots, a_n) \in \mathbb{R}^n$ be a generator of the kernel of Z and let $b = (b_1, \dots, b_n) \in \mathbb{R}^n$ be a generator of the kernel of Z_0[1]. Choose them in such a way that a_1 and b_1 have the same sign. Consequently, for every $1 \leq i \leq n$, the entries a_i and b_i have the same sign. Let D be an $n \times n$ diagonal matrix $D = \text{diag}(\frac{a_1}{b_1}, \dots, \frac{a_n}{b_n})$. The matrix ZD has the same kernel as the matrix Z_0, and since the diagonal entries of the matrix D are positive, all maximal minors of the matrix ZD are positive. The fact that the matrices ZD and Z_0 have the same kernel implies that they have the same row spans, as well. In particular, there exists a matrix $C \in \text{GL}^+_{k+m}$ such that $Z_0 = CZD$.

[1]It follows from the cyclic symmetry of Z_0 that $b_i = (-1)^{i-1}$ for $1 \leq i \leq n$. See [36] for details.

Multiplication by D on the right gives a homeomorphism $\widehat{D} : \mathrm{Mat}_{k,n}^{\geq 0} \longrightarrow \mathrm{Mat}_{k,n}^{\geq 0}$, which induces a homeomorphism $\widetilde{D} : \mathrm{G}_k^{\geq 0}(\mathbb{R}^n) \longrightarrow \mathrm{G}_k^{\geq 0}(\mathbb{R}^n)$. Furthermore, multiplication by $C^\top$ on the right gives a homeomorphism $\widehat{C} : \mathrm{Mat}_{k,k+m} \longrightarrow \mathrm{Mat}_{k,k+m}$, thus the induced map $\widetilde{C} : \mathrm{G}_k(\mathbb{R}^{k+m}) \longrightarrow \mathrm{G}_k(\mathbb{R}^{k+m})$ is also a homeomorphism. Hence, we obtain the commutative diagram of spaces and maps

$$\begin{array}{ccccccc}
\mathrm{Mat}_{k,n}^{\geq 0} & \xrightarrow{\widehat{D}} & \mathrm{Mat}_{k,n}^{\geq 0} & \xrightarrow{\widehat{Z}} & \mathrm{Mat}_{k,k+m} & \xrightarrow{\widehat{C}} & \mathrm{Mat}_{k,k+m} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathrm{G}_k^{\geq 0}(\mathbb{R}^n) & \xrightarrow{\widetilde{D}} & \mathrm{G}_k^{\geq 0}(\mathbb{R}^n) & \xrightarrow{\widetilde{Z}} & \mathrm{G}_k(\mathbb{R}^{k+m}) & \xrightarrow{\widetilde{C}} & \mathrm{G}_k(\mathbb{R}^{k+m}).
\end{array}$$

The image of the composition $\widetilde{C} \circ \widetilde{Z} \circ \widetilde{D}$ of the maps in the lower row of the diagram is the cyclically symmetric amplituhedron $\mathcal{A}_{n,k,m}(Z_0)$ and the image of the map $\widetilde{Z}$ is the amplituhedron $\mathcal{A}_{n,k,m}(Z)$. Since the maps $\widetilde{C}$ and $\widetilde{D}$ are homeomorphisms, these two amplituhedra are homeomorphic. Finally, the fact that the cyclically symmetric amplituhedron $\mathcal{A}_{n,k,m}(Z_0)$ is homeomorphic to a km-dimensional ball [36, Thm. 1.2], when m is even, concludes the argument that every amplituhedron $\mathcal{A}_{n,k,m}(Z)$ is homeomorphic to a km-dimensional ball whenever $n = k + m + 1$ and m is even.

Proof of Theorem 2.1.6

Let e be a positroid cell in the nonnegative Grassmannian $\mathrm{G}_k^{\geq 0}(\mathbb{R}^n)$, and let Z be a matrix that defines the Grassmann polytope $\mathrm{P}_Z(e)$. By [61, Thm. 1], the Grassmann polytope $\mathrm{P}_Z(e)$ has the homotopy type of a countable CW complex if and only if it has the homotopy type of an absolute neighborhood retract (ANR). Furthermore, by [22, p. 240] the space $\mathrm{P}_Z(e)$ is an ANR if it is compact and locally contractible, see also [43, p. 389]. Since the closed positroid cell $\bar{e}$ is compact, the Grassmann polytope $\mathrm{P}_Z(e)$ is also compact. Thus, it remains to show that $\mathrm{P}_Z(e)$ is locally contractible.

Applying the Gram-Schmidt orthogonalization on the fibration (2.3), we obtain a fibration

$$\mathrm{SO}(k) \longrightarrow E_1 \longrightarrow \bar{e}, \tag{2.7}$$

where the total space E_1 is a subspace of the orthonormal Stiefel manifold. Similarly, we obtain a fibration

$$\mathrm{SO}(k) \longrightarrow E_2 \longrightarrow \mathrm{P}_Z(e). \tag{2.8}$$

We also consider a commutative diagram of spaces and continuous maps

$$\begin{array}{ccc}
E_1 & \xrightarrow{\widehat{Z}} & E_2 \\
\downarrow & & \downarrow \\
\bar{e} & \xrightarrow{\widetilde{Z}} & \mathrm{P}_Z(e),
\end{array}$$

where the horizontal maps are induced by the matrix Z, and the vertical maps send any frame to its span.

By [31, p. 81], every Euclidean neighborhood retract (ENR) is locally contractible. On the other hand, if E_2 is an SO(k)-ENR, then the orbit space $\mathrm{P}_Z(e)$ is an ENR, [30, Prop. II.8.9]. Finally, since E_2 is a compact space with a free SO(k)-action, it is an SO(k)-ENR, [47, Thm. 2.1], which completes the argument that the Grassmann polytope $\mathrm{P}_Z(e)$ has a homotopy type of a countable CW complex.

Finally, if $n = k+m+1$ by Theorem 2.1.7 and Theorem 2.1.4, $\mathrm{P}_Z(e)$ is simply connected, so by [61, Prop. 1 + Remark] it is homotopy equivalent to a finite CW complex.

Proof of Theorem 2.1.8

Let e be a positroid cell in the CW decomposition of the nonnegative Grassmannian $\mathrm{G}_k^{\geq 0}(\mathbb{R}^n)$. Set $d = \binom{k+m}{k}$, and consider the Veronese embedding

$$\nu : \mathbb{R}\mathrm{P}^{d-1} \longrightarrow \mathbb{R}^{d\times d}$$

that maps every point $x = (x_1 : \ldots : x_d) \in \mathbb{R}\mathrm{P}^{d-1}$ to the matrix

$$\left(\frac{x_i x_j}{x_1^2 + \cdots + x_d^2}\right)_{ij} \in \mathbb{R}^{d\times d}.$$

The embedding ν maps every linear line $x \in \mathbb{R}^d$ to the matrix of the projection $\mathbb{R}^d \to x$.

Consider also the map

$$\nu : \mathbb{R}^d \longrightarrow \mathbb{R}^{d\times d}$$

given by

$$(x_1, \ldots, x_d) \longmapsto \left(\frac{x_i x_j}{x_1^2 + \cdots + x_d^2}\right)_{ij} \in \mathbb{R}^{d\times d}.$$

Now we obtain the commutative diagram of spaces and maps

$$\begin{array}{ccccccc}
\mathrm{Mat}_{k,n}^{\geq 0} & \xrightarrow{\widehat{Z}} & \mathrm{Mat}_{k,k+m} & \xrightarrow{\gamma} & \mathbb{R}^d \setminus \{0\} & \xrightarrow{\nu} & \mathbb{R}^{d\times d} \\
\downarrow & & \downarrow & & \downarrow \pi & & \downarrow \mathrm{id} \\
\mathrm{G}_k^{\geq 0}(\mathbb{R}^n) & \xrightarrow{\widetilde{Z}} & \mathrm{G}_k(\mathbb{R}^{k+m}) & \xrightarrow{\gamma} & \mathbb{R}\mathrm{P}^{d-1} & \xrightarrow{\nu} & \mathbb{R}^{d\times d},
\end{array}$$

where $\gamma : \mathrm{G}_k(\mathbb{R}^{k+m}) \longrightarrow \mathbb{R}\mathrm{P}^{d-1}$ is the Plücker embedding, $\gamma : \mathrm{Mat}_{k,k+m} \longrightarrow \mathbb{R}^d \setminus \{0\}$ maps every matrix to the tuple of its $k \times k$ minors, and $\pi : \mathbb{R}^d \setminus \{0\} \longrightarrow \mathbb{R}\mathrm{P}^{d-1}$ is the quotient map.

Since the Grassmann polytope $\mathrm{P}_Z(e)$ is embedded into $\mathbb{R}\mathrm{P}^{d-1}$ via γ, and the projective space $\mathbb{R}\mathrm{P}^{d-1}$ is embedded in $\mathbb{R}^{d\times d}$ via ν, we show that $\nu(\gamma(\mathrm{P}_Z(e)))$ is semialgebraic. The commutativity of the diagram above implies that

$$\nu(\gamma(\mathrm{P}_Z(e))) = \nu(\pi(\gamma(\widehat{\mathrm{P}_Z(e)}))) = \nu(\gamma(\widehat{\mathrm{P}_Z(e)})) = \nu(\gamma(\widehat{Z}(\mathrm{Mat}_{k,n}^{\geq 0}(e)))).$$

The set $\mathrm{Mat}_{k,n}^{\geq 0}(e) \subseteq \mathbb{R}^{k\times n}$ is semialgebraic. Since the map $\widehat{Z}$ is multiplication by a matrix, the set $\widehat{\mathrm{P}_Z(e)}$ is also semialgebraic. Furthermore, every coordinate of the map γ is given by a polynomial, thus $\gamma(\widehat{\mathrm{P}_Z(e)}) \subset \mathbb{R}^d \setminus \{0\}$ is semialgebraic, as well. Finally, the map $\nu : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ is a regular rational map, thus it maps semialgebraic sets to semialgebraic sets, see [28, Sec. 2.2.1].

Chapter 3

Equipartitions by prisms and regular fans

Results in this chapter are joint with Pavle V.M. Blagojević.

Introduction

Partitions of measures are classical, well-studied, but still very challenging problems. For example, the ham-sandwich theorem, proved by Banach in 1938, claims that every collection of d measures in $\mathbb{R}^d$ can be simultaneously partitioned into halfs by one hyperplane cut. In 1960 Grünbaum [40, Sect. 4(v)] asked, whether any proper convex body in $\mathbb{R}^d$ can be divided by d affine hyperplanes into 2^d pieces of equal volume. A positive answer in the plane follows by a direct application of the ham-sandwich theorem. Hadwiger answered this question positively in [42] for $d = 3$, whereas Avis [6] gave a negative answer for $d \geq 5$. The case $d = 4$ is still a hard open problem.

Here we consider two naturally related problems – partitions of the Euclidean space into *prisms* on the one hand, and partitions by regular linear hyperplane fans on the other hand.

Definition 3.1.1. An ordered collection of subsets $(C^1, \ldots, C^p)$ of $\mathbb{R}^d$ is a *partition* of $\mathbb{R}^d$ into p subsets if

(1) $\bigcup_{j=1}^{p} C^j = \mathbb{R}^d$,

(2) $\operatorname{int}(C^j) \neq \emptyset$ for every $1 \leq j \leq p$, and

(3) $\operatorname{int}(C^j) \cap \operatorname{int}(C^k) = \emptyset$ for all $1 \leq j < k \leq p$.

A partition $(C^1, \ldots, C^p)$ is called *convex* if all subsets $C^1, \ldots, C^p$ are convex.

Definition 3.1.2. Let $d_1, \ldots, d_k \geq 1$ and $m \geq 0$ be integers. A *prism* in $\mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_k} \times \mathbb{R}^m$ is a product

$$C_1 \times \cdots \times C_k \times \mathbb{R}^m \subseteq \mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_k} \times \mathbb{R}^m,$$

where each $C_i \subseteq \mathbb{R}^{d_i}$ is convex. A *partition of* $\mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_k} \times \mathbb{R}^m$ *into* p^k *prisms* is a family of prisms

$$(C_1^{j_1} \times \cdots \times C_k^{j_k} \times \mathbb{R}^m)_{1 \leq j_1 \leq p, \ldots, 1 \leq j_k \leq p},$$

where $(C_i^1, \ldots, C_i^p)$ is a convex partition of $\mathbb{R}^{d_i}$ into p subsets, for every $1 \leq i \leq k$. Denote by $\operatorname{PP}(d_1, \ldots, d_k; p)$ the set of all partitions of $\mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_k} \times \mathbb{R}^m$ into p^k prisms. In case when $d_1 = \cdots = d_k$, we use a simplified notation $\operatorname{PP}(d, k, p)$.

An example of a prism and a partition into prisms for $k = 2, m = 0, d_1 = 2, d_2 = 1$ and $p = 5$ is given in Figure 3.1 and Figure 3.2. Figure 3.1 shows convex partitions of $\mathbb{R}^2$ and $\mathbb{R}$, and one prism in $\mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R}$ induced by them. Moreover, Figure 3.2 shows a partition of $\mathbb{R}^2 \times \mathbb{R}$ into 25 prisms.

In this chapter we consider only partitions of $\mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_k} \times \mathbb{R}^m$ into prisms, where all spaces $\mathbb{R}^{d_1}, \dots, \mathbb{R}^{d_k}$ are partitioned into exactly p convex subsets. One could, however, apply the same methods on more general partitions, such that the spaces $\mathbb{R}^{d_1}, \dots, \mathbb{R}^{d_k}$ are not necessarily partitioned into the same number of subsets.

Since prisms are convex subsets of $\mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_k} \times \mathbb{R}^m$, every partition into prisms is a convex partition. The space $\mathcal{C}(\mathbb{R}^d, n)$ of all convex partitions of the Euclidean space $\mathbb{R}^d$ into n convex pieces was endowed with a metric by León and Ziegler in [53]. Therefore, we can see $\mathrm{PP}(d_1, \dots, d_k; p)$ as a topological space with the topology inherited from the metric space $\mathcal{C}(\mathbb{R}^{d_1+\cdots+d_k+m}, p^k)$.

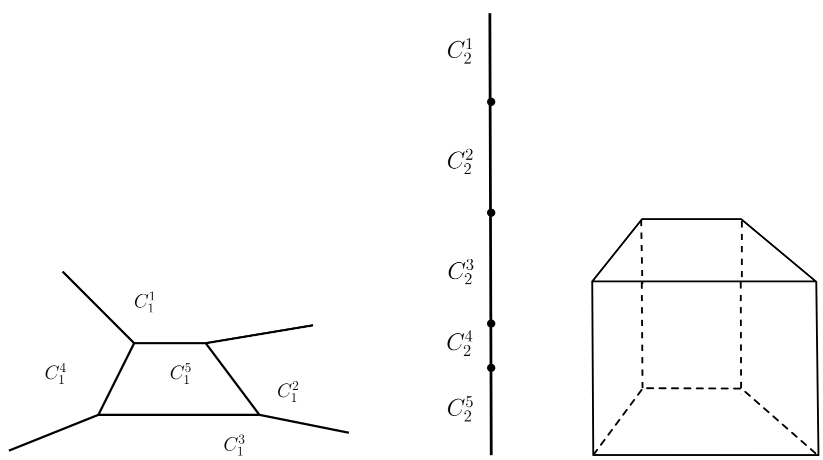


Figure 3.1: A convex partition of $\mathbb{R}^2$ into 5 subsets, a convex partition of $\mathbb{R}$ into 5 subsets, and a prism in $\mathbb{R}^2 \times \mathbb{R}$.

Example 3.1.3. For $k = 1$ every convex partition $(C^1, \dots, C^p)$ of $\mathbb{R}^d$ gives a partition

$$(C^1 \times \mathbb{R}^m, \dots, C^p \times \mathbb{R}^m)$$

of $\mathbb{R}^d \times \mathbb{R}^m$ into p prisms.

Example 3.1.4. The hyperplane arrangement of k coordinate hyperplanes in $\mathbb{R}^k$ partitions $\mathbb{R}^k = \mathbb{R} \times \cdots \times \mathbb{R}$ into 2^k prisms. The convex subsets C_i^1 and C_i^2 that partition $\mathbb{R}$ are the ray of nonnegative and the ray of nonpositive real numbers.

The group $\mathbb{Z}_p$ acts on a partition $(C^1, \dots, C^p)$ of $\mathbb{R}^d$ by cyclically permuting the sets

$$g \cdot (C^1, \dots, C^p) = (C^{g+1}, \dots, C^{g+p}),$$

where the addition of indices is done modulo p.

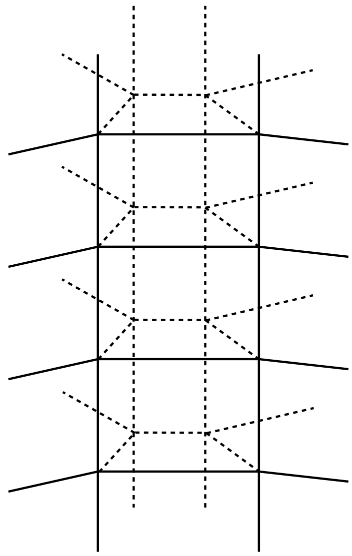


Figure 3.2: A partition of $\mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R}$ into 25 prisms.

Let $\mathbb{Z}_p^k \cong G_1 \times \cdots \times G_k$ be an elementary abelian group, where $G_1 \cong \cdots \cong G_k \cong \mathbb{Z}_p$. Then there is a $\mathbb{Z}_p^k$-action on $\mathrm{PP}(d_1, \ldots, d_k; p)$ defined on every partition $(C_1^{j_1} \times \cdots \times C_k^{j_k} \times \mathbb{R}^m)_{1 \le j_1 \le p, \ldots, 1 \le j_k \le p} \in \mathrm{PP}(d_1, \ldots, d_k; p)$ by

$$(g_1, \ldots, g_k) \cdot (C_1^{j_1} \times \cdots \times C_k^{j_k} \times \mathbb{R}^m) = (C_1^{g_1 + j_1} \times \cdots \times C_k^{g_k + j_k} \times \mathbb{R}^m),$$

for every $(g_1, \ldots, g_k) \in G_1 \times \cdots \times G_k$ and every $1 \le j_1, \ldots, j_k \le p$, where the addition of indices is modulo p.

The symmetric group $\mathfrak{S}_{p^k}$ acts on $\mathbb{R}^{p^k}$ by permuting coordinates. This action induces a $\mathbb{Z}_p^k$-action on $\mathbb{R}^{p^k}$ via the regular embedding $(reg) : \mathbb{Z}_p^k \to \mathfrak{S}_{p^k}$ [1, Ex. 2.7, page 100], which is given by the left translation action of $\mathbb{Z}_p^k$ on itself, so that to each element $g \in \mathbb{Z}_p^k$ we associate permutation $L_g \colon \mathbb{Z}_p^k \to \mathbb{Z}_p^k$ from $\mathfrak{S}_{p^k}$ given by $L_g(x) = g + x$.

Now we are ready to state the first result of this chapter.

Theorem 3.1.5. *Let $d \ge 1$, $k \ge 1$, $r \ge 1$ and $m \ge 0$ be integers, and let p be a prime such that $d > rp^{k-1}$ with at least one of the following conditions satisfied*
(a) *$p = 2$, or*
(b) *4 divides $p - 1$, or*
(c) *k is even, or*
(d) *r is even.*
For every collection $F_1, \ldots, F_r : \mathrm{PP}(d, k, p) \to \mathbb{R}^{p^k}$ of continuous $\mathbb{Z}_p^k$-equivariant maps, and for every absolutely continuous probability measure μ on $\mathbb{R}^d$, there exists a partition $(C_1^{j_1} \times \cdots \times C_k^{j_k} \times \mathbb{R}^m)_{1 \le j_1 \le p, \ldots, 1 \le j_k \le p} \in \mathrm{PP}(d, k, p)$ of $\mathbb{R}^d \times \cdots \times \mathbb{R}^d \times \mathbb{R}^m$ into p^k prisms such that

$$\mu(C_i^j) = \frac{1}{p},$$

for every $1 \leq i \leq k$, $1 \leq j \leq p$, *and such that all functions* $F_1, \dots, F_r$ *equalize on the partition* $(C_1^{j_1} \times \cdots \times C_k^{j_k} \times \mathbb{R}^m)_{1 \leq j_1 \leq p, \dots, 1 \leq j_k \leq p}$, *this means that each function takes the same value on all prisms in the partition.*

If we choose functions $F_1, \dots, F_r$ to be finite absolutely continuous probability measures on $\mathbb{R}^{kd+m}$, then Theorem 3.1.5 becomes a measure partition result saying that for every collection of finite absolutely continuous measures $\mu_1, \dots, \mu_r$ on $\mathbb{R}^{kd+m}$, there exists a partition $(C_1^{j_1} \times \cdots \times C_k^{j_k} \times \mathbb{R}^m)_{1 \leq j_1 \leq p, \dots, 1 \leq j_k \leq p} \in \mathrm{PP}(d, k, p)$ of $\mathbb{R}^d \times \cdots \times \mathbb{R}^d \times \mathbb{R}^m$ into p^k prisms such that

$$\mu_s(C_1^{j_1} \times \cdots \times C_k^{j_k} \times \mathbb{R}^m) = \frac{1}{p^k} \mu_s(\mathbb{R}^{kd+m}),$$

for every prism $C_1^{j_1} \times \cdots \times C_k^{j_k} \times \mathbb{R}^m$ in the partition $(C_1^{j_1} \times \cdots \times C_k^{j_k} \times \mathbb{R}^m)_{1 \leq j_1 \leq p, \dots, 1 \leq j_k \leq p}$ and for every $1 \leq s \leq r$.

The case $k = 1$, $m = 0$ and $r = d$ of Theorem 3.1.5, with F_1 being a finite absolutely continuous probability measure on $\mathbb{R}^d$ (volume, for example), is the problem posed by Nandakumar and Ramana Rao [65], that has been answered positively whenever the number of subsets in the partition of $\mathbb{R}^d$ is a prime power, see [49] and [21, Thm. 1.3].

A *closed half-hyperplane* in $\mathbb{R}^d$ is the set

$$H_v = \{x \in H \mid \langle x, v \rangle \geq 0\},$$

for some hyperplane $H \subset \mathbb{R}^d$ and some vector $v \in H$. Its boundary is the $(d-2)$-dimensional linear space

$$\{x \in H \mid \langle x, v \rangle = 0\}.$$

Our second result considers partitions of the Euclidean space by regular linear p-fans.

Definition 3.1.6. Let $p \geq 2$ be an integer. A *p-fan* in $\mathbb{R}^d$ is the union of p closed half-hyperplanes in $\mathbb{R}^d$ with a common boundary. A p-fan is called *regular* if the angle between any two successive half-hyperplanes is $\frac{2\pi}{p}$, and it is called *linear* if the origin is contained in the common boundary of the half-hyperplanes.

Note that a regular 2-fan in $\mathbb{R}^d$ is just a hyperplane in $\mathbb{R}^d$.

For every $p \geq 2$ a regular p-fan Q partitions $\mathbb{R}^d$ into p convex pieces $C^1, \dots, C^p$. A collection of k regular fans $(Q_1, \dots, Q_k)$ partitions $\mathbb{R}^d$ into p^k convex subsets, some of them being possibly empty, as follows: If a p-fan Q_i partitions $\mathbb{R}^d$ into p convex subsets $C_i^1, \dots, C_i^p$, for every $1 \leq i \leq k$, then the collection of subsets $(C_1^{j_1} \cap \cdots \cap C_k^{j_k})_{1 \leq j_1 \leq p, \dots, 1 \leq j_k \leq p}$ is a convex partition of $\mathbb{R}^d$ into p^k subsets.

Definition 3.1.7. Denote by $\mathrm{LFP}(d, k, p)$ the set of all partitions of the Euclidean space $\mathbb{R}^d$ by k regular linear p-fans.

Similarly as before, the space $\mathrm{LFP}(d, k, p)$ is a subspace of the metric space $\mathcal{C}(\mathbb{R}^d, p^k)$ of all convex partitions of $\mathbb{R}^d$ into p^k pieces, thus it is equipped with the inherited topology.

There is a $\mathbb{Z}_p^k$-action on the space $\mathrm{LFP}(d, k, p)$. Let again $\mathbb{Z}_p^k \cong G_1 \times \cdots \times G_k$ be an elementary abelian p-group, where $G_1 \cong \cdots \cong G_k \cong \mathbb{Z}_p$. Then the action is given on every partition $(C_1^{j_1} \cap \cdots \cap C_k^{j_k})_{1 \leq j_1 \leq p, \dots, 1 \leq j_k \leq p} \in \mathrm{LFP}(d, k, p)$ by

$$(g_1, \dots, g_k) \cdot (C_1^{j_1} \cap \cdots \cap C_k^{j_k}) = (C_1^{g_1 + j_1} \cap \cdots \cap C_k^{g_k + j_k}),$$

for every $(g_1, \dots, g_k) \in \mathbb{Z}_p^k$ and for every $1 \leq j_1, \dots, j_k \leq p$, where the addition of indices is modulo p.

Since a hyperplane is a regular 2-fan, one can consider partitions by regular q-fans as a generalization of the Grünbaum–Hadwiger–Ramos mass partition problem, see for example the results of Simon [73] in the complex space. Our next result gives equipartitons of the real Euclidean space by regular linear fans.

Theorem 3.1.8. *Let $d, k, r \geq 1$ and $m \geq 0$ be integers and let $p \geq 2$ be a prime such that $d > \frac{rp^{k-1}(p-1)}{2}$. Additionally, assume that at least one of the following conditions is satisfied*
(a) $p = 2$, *or*
(b) 4 *divides* $p - 1$, *or*
(c) k *is even, or*
(d) r *is even.*
Then for every collection $F_1, \dots, F_r : \mathrm{LFP}(2d, k, p) \longrightarrow \mathbb{R}^{p^k}$ of continuous $\mathbb{Z}_p^k$-equivariant maps, there exists a partition $(C_1^{j_1} \cap \dots \cap C_k^{j_k})_{1 \leq j_1 \leq p, \dots, 1 \leq j_k \leq p} \in \mathrm{LFP}(2d, k, p)$ of $\mathbb{R}^{2d}$ by k regular linear p-fans into p^k convex subsets such that each of the functions $F_1, \dots, F_r$ equalizes on the partition $(C_1^{j_1} \cap \dots \cap C_k^{j_k})_{1 \leq j_1 \leq p, \dots, 1 \leq j_k \leq p}$, that is, every F_j takes the same value on each convex subset in the partition.

Again, choosing the functions $F_1, \dots, F_r$ to be finite absolutely continuous measures, Theorem 3.1.8 translates into a measure partition result: Given any collection $\mu_1, \dots, \mu_r$ of finite absolutely continuous measures in $\mathbb{R}^{2d}$, there is a partition
$(C_1^{j_1} \cap \dots \cap C_k^{j_k})_{1 \leq j_1 \leq p, \dots, 1 \leq j_k \leq p} \in \mathrm{LFP}(2d, k, p)$ of $\mathbb{R}^{2d}$ by k regular linear p-fans into p^k convex subsets such that

$$\mu_s(C_1^{j_1} \cap \dots \cap C_k^{j_k}) = \frac{1}{p^k} \mu_s(\mathbb{R}^{2d}),$$

for every $1 \leq j_1, \dots, j_k \leq p$ and every $1 \leq s \leq r$.

Solutions to partition problems have often been obtained using tools from algebraic topology. Here we first develop configuration space/test map (CS/TM) schemes in Section 3.2, in order to move from a partition problem to a question of non-existence of equivariant maps, which is then answered in Section 3.4 using the ideal-valued index theory of Fadell and Husseini [32].

From convex partitions to equivariant topology

In this section we develop CS/TM schemes in order to prove Theorems 3.1.5 and 3.1.8. They will lead to statements about non-existence of equivariant maps, Theorem 3.2.3 and Theorem 3.2.4, which will be proved in Section 3.4.

Convex partitions into prisms

In order to partition $\mathbb{R}^{kd+m}$ into p^k prisms, it is necessary to partition each copy of $\mathbb{R}^d$ into p convex sets. Following [21], denote by $\mathrm{EMP}(\mu, p)$ the set of convex partitions of $\mathbb{R}^d$ into p subsets that equipart the measure μ on $\mathbb{R}^d$.

Definition 3.2.1. Let X be a topological space and let $p \geq 1$ be an integer. A *configuration space* is the set of all p-tuples of pairwise disjoint points in X:

$$\mathrm{Conf}(X,p) = \{(x_1,\dots,x_p) \in X^p \mid x_i \neq x_j \text{ for every } i \neq j\}.$$

Blagojević and Ziegler [21, Sec. 2] showed the existence of an $\mathfrak{S}_p$-equivariant map

$$\mathrm{Conf}(\mathbb{R}^d,p) \longrightarrow \mathrm{EMP}(\mu,p),$$

thus every point in the configuration space $\mathrm{Conf}(\mu,p)$ defines a convex equipartition of the measure μ in $\mathbb{R}^d$.

Additionally, we need the following definition.

Definition 3.2.2. Let $d \geq 2$. Denote by W_d the orthogonal complement of the diagonal in $\mathbb{R}^d$:

$$W_d = \{(x_1,\dots,x_d) \in \mathbb{R}^d \mid x_1 + \cdots + x_d = 0\}.$$

It is a $(d-1)$-dimensional linear subspace of $\mathbb{R}^d$.

Now we are ready to develop the CS/TM scheme for equipartitions by prisms.

Since every point in $\mathrm{Conf}(\mathbb{R}^d,p)$ defines a partition of $\mathbb{R}^d$ into p convex sets that are in addition equipartitons of the measure μ, the product $\mathrm{Conf}(\mathbb{R}^d,p)^{\times k}$ can be embedded into $\mathrm{PP}(d,k,p)$, and it represents needed partitions. Moreover, the space $\mathrm{Conf}(\mathbb{R}^d,p)^{\times k}$ inherits the $\mathbb{Z}_p^k$-action. The group $\mathbb{Z}_p^k$ acts on $\mathrm{Conf}(\mathbb{R}^d,p)^{\times k}$ in such a way that each copy of $\mathbb{Z}_p$ cyclically permutes the p points in $\mathbb{R}^d$ that define an element of the configuration space $\mathrm{Conf}(\mathbb{R}^d,p)$.

Define functions $G_1,\dots,G_r : \mathrm{PP}(d,k,p) \longrightarrow W_{p^k} \subset \mathbb{R}^{p^k}$ as

$$\begin{aligned} &G_s((C_1^{j_1}\times\cdots\times C_k^{j_k}\times\mathbb{R}^m)_{1\leq j_1\leq p,\dots,1\leq j_k\leq p}) := \\ &\quad\Big(F_s(C_1^{a_1}\times\cdots\times C_k^{a_k}\times\mathbb{R}^m) - \frac{1}{p^k}\sum_{\substack{1\leq i\leq k\\ 1\leq j_i\leq p}} F_s(C_1^{j_1}\times\cdots\times C_k^{j_k}\times\mathbb{R}^m)\Big)_{1\leq a_1\leq p,\dots,1\leq a_k\leq p},\end{aligned}$$

for every $1 \leq s \leq r$ and every partition $(C_1^{j_1}\times\cdots\times C_k^{j_k}\times\mathbb{R}^m)_{1\leq j_1\leq p,\dots,1\leq j_k\leq p}$ of $\mathbb{R}^{kd+m}$ into p^k prisms. In other words, for a partition

$$P = (C_1^{j_1}\times\cdots\times C_k^{j_k}\times\mathbb{R}^m)_{1\leq j_1\leq p,\dots,1\leq j_k\leq p} \in \mathrm{PP}(d,k,p),$$

the value of the coordinate $(a_1,\dots,a_k)$ of G_s equals the value of F_s on the prism $C_1^{a_1}\times\cdots\times C_k^{a_k}\times\mathbb{R}^m$ with the average value of F_s on the whole partition P subtracted. The maps $G_1,\dots,G_r$ are $\mathbb{Z}_p^k$-equivariant by construction. Moreover, $G_s(P) = 0 \in W_{p^k}$ if and only if the value of the function F_s is the same on every prism of the partition P, for every $1 \leq s \leq r$.

Assume now that the statement of Theorem 3.1.5 does not hold. More precisely, assume that for every partition $P \in \mathrm{PP}(d,k,p)$, at least one of the functions $F_1,\dots,F_r$ does not equalize on P. This, in particular, means that the image of the function

$$G = (G_1,\dots,G_r) : \mathrm{PP}(d,k,p) \longrightarrow W_{p^k}^{\oplus r}$$

does not hit the origin $0 \in W_{p^k}^{\oplus r}$. After restricting the domain to $\mathrm{Conf}(\mathbb{R}^d, p)^{\times k}$ and composing the map G with the projection to the unit sphere in the linear space $W_{p^k}^{\oplus r}$, we obtain a $\mathbb{Z}_p^k$-equivariant map

$$\mathrm{Conf}(\mathbb{R}^d, p)^{\times k} \longrightarrow S(W_{p^k}^{\oplus r}). \tag{3.1}$$

However, we will show in Theorem 3.2.3 that such a map cannot exist. Thus, the proof of Theorem 3.1.5 follows from the CS/TM scheme above and from Theorem 3.2.3.

Theorem 3.2.3. *Let $d \geq 2$, $k \geq 1$ and $r \geq 1$ be integers and let p be a prime. If $d > rp^{k-1}$ and one of the following conditions is satisfied*
(a) *$p = 2$, or*
(b) *4 divides $p-1$, or*
(c) *k is even, or*
(d) *r is even,*
then there is no $\mathbb{Z}_p^k$-equivariant map $\mathrm{Conf}(\mathbb{R}^d, p)^{\times k} \longrightarrow S(W_{p^k}^{\oplus r})$.

The proof of Theorem 3.2.3 is be given in Section 3.4.

Convex partitions by regular linear fans

In this section we develop a CS/TM scheme that leads to a proof of Theorem 3.1.8.

Let us first consider the case when $p \geq 3$ is a prime. Every point in the sphere S^{2d-1} defines a regular linear p-fan in $\mathbb{R}^{2d}$. Indeed, each point $q \in S^{2d-1} \subset \mathbb{R}^{2d}$ is a normal vector of an oriented hyperplane H_q in $\mathbb{R}^{2d}$. Since we can see S^{2d-1} as the join $(S^1)^{*d}$ of d copies of the unit circle, the group $\mathbb{Z}_p$ acts on S^{2d-1} by simultaneously acting on each copy of S^1 by rotations. Denote by $q_1 = q, q_2, \ldots, q_p$ the points of S^{2d-1} that are in the orbit of q, where $q_{i+1} = g \cdot q_i$, for every $i \in \mathbb{Z}_p$, where $g = 1$ is the generator of $\mathbb{Z}_p$ seen as an additive group. The points $q_1, \ldots, q_p$ define p linear hyperplanes $H_{q_1}, \ldots, H_{q_p}$ in $\mathbb{R}^d$ whose intersection is a linear $(d-2)$-dimensional subspace V of $\mathbb{R}^d$. The union of these hyperplanes can be seen as

$$\bigcup_{i=1}^{p} H_{q_i} = U \times V \subset \mathbb{R}^2 \times V \cong \mathbb{R}^{2d},$$

where U is the union of p lines in $V^{\perp}$, the orthogonal complement of V. Denote by ℓ_i the line $H_{q_i} \cap V^{\perp} \subset U$, for every $1 \leq i \leq p$. It is not hard to see that the points $q_1, \ldots, q_p$ lie on a circle in $V^{\perp}$, and that the vector defined by q_i is orthogonal to the line ℓ_i, for every $1 \leq i \leq p$. Consequently, the points $q_1, \ldots, q_p$ define orientations of lines $\ell_1, \ldots, \ell_p$, and the angle between any two consecutive lines is exactly $\frac{\pi}{p}$. An example for $p = 3$ is shown in Figure 3.3(a). Moreover, the order of points $q_1, \ldots q_p$ on the unit circle in $V^{\perp}$ defines an orientation of $V^{\perp}$. If we consider each line ℓ_i as a union of two half-lines emanating from the origin, then the line arrangement $(\ell_1, \ldots, \ell_p)$ defines two regular linear p-fans in $V^{\perp}$, one of them having the positive and one of them having the negative orientation, as shown in Figure 3.3(b). Choose the one with the positive orientation and denote by $\ell_1^+, \ldots, \ell_p^+$ its half-lines, Figure 3.3(c). Then $\ell_i^+ \times V \subset H_{q_i}$ is a half-hyperplane and the half-hyperplanes

$\ell_1^+ \times V, \dots, \ell_p^+ \times V$ define a regular linear p-fan in $\mathbb{R}^{2d}$. Furthermore, this construction is $\mathbb{Z}_p$-invariant.

Using the above construction, we obtain a $\mathbb{Z}_p^k$-equivariant map

$$(S^{2d-1})^k \longrightarrow \mathrm{LFP}(2d,k,p) \tag{3.2}$$

from the product of spheres into the space of all partitions of $\mathbb{R}^{2d}$ by k regular linear p-fans.

We are now ready to develop the CS/TM scheme needed for proving Theorem 3.1.8 in the case when $p \geq 3$.

Let $F_1, \dots, F_r : \mathrm{LFP}(2d,k,p) \longrightarrow \mathbb{R}^{p^k}$ be continuous $\mathbb{Z}_p^k$-equivariant maps. Define maps $G_1, \dots, G_r : \mathrm{LFP}(2d,k,p) \longrightarrow W_{p^k} \subset \mathbb{R}^{p^k}$ as

$$G_s\big((C_1^{j_1} \cap \dots \cap C_k^{j_k})_{1\leq j_1\leq p,\dots,1\leq j_k\leq p}\big) = \big(F_s(C_1^{a_1} \cap \dots \cap C_k^{a_k}) - \frac{1}{p^k} \sum_{\substack{1\leq i\leq k\\ 1\leq j_i\leq p}} F_s(C_1^{j_1} \cap \dots \cap C_k^{j_k})\big)_{1\leq a_1\leq p,\dots,1\leq a_k\leq p},$$

for every partition $(C_1^{j_1} \cap \dots \cap C_k^{j_k})_{1\leq j_1\leq p,\dots,1\leq j_k\leq p} \in \mathrm{LFP}(2d,k,p)$ and for every $1 \leq s \leq r$. In other words, the map G_s is obtained from the map F_s by subtracting coordinatewise the average of F_s on all pieces of the partition. Furthermore, define a map

$$G = (G_1, \dots, G_r) : \mathrm{LFP}(2d,k,p) \to W_{p^k}^{\oplus r}.$$

Since the maps $G_1, \dots, G_r$ are $\mathbb{Z}_p^k$-equivariant, the map G is $\mathbb{Z}_p^k$-equivariant, as well. Note that the image of a partition under G is zero in $W_{p^k}^{\oplus r}$ if and only if all functions $F_1, \dots, F_r$ equalize on that partition.

Assume now that the statement of Theorem 3.1.8 is false, thus assume that there are functions $F_1, \dots, F_r$ such that there is no partition in $\mathrm{LFP}(2d,k,p)$ on which all of them equalize. For such functions $F_1, \dots, F_r$, we obtain a $\mathbb{Z}_p^k$-equivariant function $G : \mathrm{LFP}(2d,k,p) \to W_{p^k}^{\oplus r} \setminus \{0\}$. After composing with a retraction to the unit sphere in $W_{p^k}^{\oplus r}$ on the right, and with the map (3.2) on the left, we obtain a $\mathbb{Z}_p^k$-equivariant map

$$(S^{2d-1})^k \longrightarrow S(W_{p^k}^{\oplus r}). \tag{3.3}$$

Let us for the sake of completeness consider the case when $p = 2$. Every regular linear convex 2-fan is a linear hyperplane. Since every point of the sphere S^{2d-1} defines an oriented hyperplane in $\mathbb{R}^d$, we can identify the space $\mathrm{LFP}(2d,k,2)$ with $(S^{2d-1})^{\times k}$. Similarly as above, assuming that the statement of Theorem 3.1.8 is false, we obtain a $\mathbb{Z}_2^k$-equivariant map

$$(S^{2d-1})^{\times k} \longrightarrow S(W_{p^k}^{\oplus r}). \tag{3.4}$$

Non-existence of maps (3.3) and (3.4) will be shown in Theorem 3.2.4, which yields a contradiction. Therefore, the above CS/TM scheme together with Theorem 3.2.4 yields the proof of Theorem 3.1.8.

Theorem 3.2.4. *Let $d \geq 2$, $k \geq 1$ and $r \geq 1$ be integers and let p be a prime. If $d > \frac{rp^{k-1}(p-1)}{2}$ and one of the following conditions is satisfied*

(a) $p = 2$, *or*
(b) 4 *divides* $p-1$, *or*
(c) k *is even, or*
(d) r *is even,*
then there is no $\mathbb{Z}_p^k$*-equivariant map* $(S^{2d-1})^{\times k} \to_{\mathbb{Z}_p^k} S(W_{p^k}^{\oplus r})$.

The proof of Theorem 3.2.4 is postponed to Section 3.4.

Remark 3.2.5. The choice of the sphere S^{2d-1} as a configuration space is not optimal. Regular linear fans in $\mathbb{R}^{2d}$ can as well be parametrized by the Stiefel manifold $V_2(\mathbb{R}^{2d})$, whose cohomological index can be computed from the work of Makeev [57], and which turns out to give more optimal results. We thank the referee Roman Karasev, and invite the reader to see the subsequent paper for more details.

The Fadell–Husseini index of the sphere $S(W_{p^k}^{\oplus r})$

In order to prove Theorem 3.2.3 and Theorem 3.2.4, we use the Fadell-Husseini ideal valued index theory, see the original paper by Fadell and Husseini [32] and Appendix B of this thesis for more details.

Let Y be a space with an action of a finite group G, and let R be a commutative ring with unit. The *Fadell–Husseini index* of Y with respect to the group G and coefficients R is the kernel ideal of the map in equivariant cohomology induced by the G-equivariant map $p_Y\colon Y \to \mathrm{pt}$

$$\begin{aligned}\mathrm{Index}_G(Y;R) &= \ker\big(p_Y^*\colon H_G^*(\mathrm{pt},R) \longrightarrow H_G^*(Y,R)\big)\\ &= \ker\big(H^*(\mathrm{B}G,R) \longrightarrow H^*(\mathrm{E}G\times_G Y,R)\big).\end{aligned}$$

The rest of this section will be devoted to estimations of the Fadell–Husseini index of the sphere $S(W_{p^k}^{\oplus r})$.

For the purposes of this chapter, the group G will be $\mathbb{Z}_p^k$ and the ring R will be the finite field $\mathbb{F}_p$, for some prime p and some integer $k \geq 1$.

Note that $\mathrm{Index}_{\mathbb{Z}_p^k}(S(W_{p^k}^{\oplus r});\mathbb{F}_p)$ is an ideal in $H^*(\mathbb{Z}_p^k;\mathbb{F}_p)$, the cohomology of the group $\mathbb{Z}_p^k$ with coefficients in the field $\mathbb{F}_p$, that is given by:

$$\begin{aligned}H^*(\mathbb{Z}_2^k;\mathbb{F}_2) &= \mathbb{F}_2[t_1,\dots,t_k], && \deg(t_i)=1,\\ H^*(\mathbb{Z}_p^k;\mathbb{F}_p) &= \mathbb{F}_p[t_1,\dots,t_k]\otimes\Lambda[e_1,\dots,e_k], && \deg(e_i)=1,\ \deg(t_i)=2,\ \text{if } p \text{ is odd,}\end{aligned}$$

where $\Lambda[e_1,\dots,e_k]$ denotes the exterior algebra generated by the elements $e_1,\dots,e_k$.

The next lemma follows from the work of Mann & Milgram [58], see also [18, Sect. 7.2.3].

Lemma 3.3.1 ([58], [18, Sect. 7.2.3])**.** *Let p be a prime. The Fadell-Husseini index* $\mathrm{Index}_{\mathbb{Z}_p^k}(S(W_{p^k}^{\oplus r});\mathbb{F}_p)$ *of the sphere* $S(W_{p^k}^{\oplus r})$ *with respect to the group* $\mathbb{Z}_p^k$ *is generated by the polynomial* ζ^r*, where*

$$\zeta = \begin{cases} \displaystyle\prod_{(\alpha_1,\dots,\alpha_k)\in\mathbb{F}_2^k\setminus\{0\}} (\alpha_1 t_1 + \cdots + \alpha_k t_k), & \text{for } p = 2,\\[2ex] \left(\displaystyle\prod_{(\alpha_1,\dots,\alpha_k)\in\mathbb{F}_p^k\setminus\{0\}} (\alpha_1 t_1 + \cdots + \alpha_k t_k)\right)^{\frac{1}{2}}, & \text{for } p \geq 3.\end{cases}$$

The aim of the following lemma is to understand the polynomial ζ^r.

Lemma 3.3.2. *Let $k \geq 1$ and $r \geq 1$ be integers and let p be a prime. If*
(a) $p = 2$, or
(b) 4 divides $p-1$, or
(c) k is even, or
(d) r is even,
then the generator ζ^r of the Fadell-Husseini index $\mathrm{Index}_{\mathbb{Z}_p^k}(S(W_{p^k}^{\oplus r});\mathbb{F}_p)$ *contains the monomial*

$$t_1^r t_2^{r\frac{p(p-1)}{2}} \dots t_k^{r\frac{p^{k-1}(p-1)}{2}}$$

with a non-zero coefficient.

Proof. First let $p = 2$. A direct computation shows that the polynomial

$$\zeta^r = \prod_{(\alpha_1,\dots,\alpha_k)\in\mathbb{F}_2^k\backslash\{0\}} (\alpha_1 t_1 + \cdots + \alpha_k t_k)^r$$

contains the monomial $t_1^r t_2^{r\cdot 2} \dots t_k^{r 2^{k-1}}$ with a coefficient 1.

Now let $p \geq 3$ be a prime. Let us first understand the polynomial

$$\zeta = \left(\prod_{(\alpha_1,\dots,\alpha_k)\in\mathbb{F}_p^k\backslash\{0\}} (\alpha_1 t_1 + \cdots + \alpha_k t_k) \right)^{\frac{1}{2}}.$$

Consider the set

$$T_p^k = \{(\alpha_1,\dots,\alpha_k) \in \mathbb{F}_p^k\backslash\{0\} \mid 0 \neq \alpha_i < p/2 \text{ and } \alpha_{i+1} = \cdots = \alpha_k = 0 \text{ for some } 1 \leq i \leq k\}.$$

For $(\alpha_1,\dots,\alpha_k) \in \mathbb{F}_p^k \setminus \{0\}$ the following holds:

$$(\alpha_1,\dots,\alpha_k) \in T_p^k \text{ if and only if } (p-\alpha_1,\dots,p-\alpha_k) \notin T_p^k.$$

Now, we can write ζ as

$$\begin{aligned}
\zeta &= \left(\prod_{(\alpha_1,\dots,\alpha_k)\in T_p^k} (\alpha_1 t_1 + \cdots + \alpha_k t_k)((p-\alpha_1)t_1 + \cdots + (p-\alpha_k)t_k) \right)^{\frac{1}{2}} \\
&= \left(\prod_{(\alpha_1,\dots,\alpha_k)\in T_p^k} -(\alpha_1 t_1 + \cdots + \alpha_k t_k)^2 \right)^{\frac{1}{2}} \\
&= \left((-1)^{\frac{p^k-1}{2}} \prod_{(\alpha_1,\dots,\alpha_k)\in T_p^k} (\alpha_1 t_1 + \cdots + \alpha_k t_k)^2 \right)^{\frac{1}{2}},
\end{aligned}$$

where the computations are modulo p.

Therefore, the Fadell-Husseini index of the sphere $S(W_{p^k}^{\oplus r})$ is generated by

$$\zeta^r = \left((-1)^{\frac{p^k-1}{2}} \prod_{(\alpha_1,\dots,\alpha_k)\in T_p^k} (\alpha_1 t_1 + \cdots + \alpha_k t_k)^2\right)^{\frac{r}{2}}.$$

If $\frac{r(p^k-1)}{4}$ is an integer, i.e., r is even or k is even or $4 \mid p-1$, then $\zeta^r = \xi$ or $\zeta^r = -\xi$, where

$$\xi = \prod_{(\alpha_1,\dots,\alpha_k)\in T_p^k} (\alpha_1 t_1 + \cdots + \alpha_k t_k)^r.$$

The polynomial ξ contains the monomial $t_1^{r\frac{p-1}{2}} t_2^{r\frac{p(p-1)}{2}} \dots t_k^{r\frac{p^{k-1}(p-1)}{2}}$ with a coefficient that is a power of $\left(\frac{p-1}{2}\right)!$, thus non-zero in $\mathbb{F}_p$. Therefore, the coefficient of the monomial $t_1^{r\frac{p-1}{2}} t_2^{r\frac{p(p-1)}{2}} \dots t_k^{r\frac{p^{k-1}(p-1)}{2}}$ in the polynomial ζ^r is not zero. □

Remark 3.3.3. In the cases $p = 2$ and $p = 3$ we understand the polynomial ζ^r completely. It equals

$$\zeta^r = \sum_\sigma \pm t_{\sigma(1)}^r t_{\sigma(2)}^{r\cdot p} \cdots t_{\sigma(k)}^{rp^{k-1}},$$

where the summation is over all permutations σ of the set $\{1,\dots,k\}$.

The following lemma obtains better bounds for the index $\mathrm{Index}_{\mathbb{Z}_p^k}(S(W_{p^k}^{\oplus r});\mathbb{F}_p)$ when $k = 2$ and $r = 1$.

Lemma 3.3.4. *Let $p \geq 3$ be a prime.*

(1) *If $4 \mid p-1$, then the coefficient of the monomial $t_1^{\frac{p^2-1}{4}} t_2^{\frac{p^2-1}{4}}$ in the polynomial ζ, the generator of the index $\mathrm{Index}_{\mathbb{Z}_p^2}(S(W_{p^2});\mathbb{F}_p)$, is not equal to zero.*

(2) *If $4 \mid p+1$, then the coefficient of the monomial $t_1^{\frac{(p-1)^2}{4}} t_2^{\frac{(p-1)(p+3)}{4}}$ in the polynomial ζ, the generator of the index $\mathrm{Index}_{\mathbb{Z}_p^2}(S(W_{p^2});\mathbb{F}_p)$, is not equal to zero.*

Proof. Following the proof of Lemma 3.3.2, we note that $\zeta = \xi$ or $\zeta = -\xi$, where

$$\xi = \prod_{(\alpha_1,\alpha_2)\in T_p^2} (\alpha_1 t_1 + \alpha_2 t_2) = \left(\left(\frac{p-1}{2}\right)!\right)^2 t_1^{\frac{p-1}{2}} t_2^{\frac{p-1}{2}} (t_2^{p-1} - t_1^{p-1})^{\frac{p-1}{2}}.$$

If $4 \mid p-1$, the monomial $t_1^{\frac{p^2-1}{4}} t_2^{\frac{p^2-1}{4}}$ has a non-zero coefficient in ξ. Similarly, if $4 \mid p+1$, the coefficient of the monomial $t_1^{\frac{(p-1)^2}{4}} t_2^{\frac{(p-1)(p+3)}{4}}$ in ξ is not zero. □

Proofs

As it has already been mentioned , we prove Theorem 3.2.3 and Theorem 3.2.4 using the Fadell–Husseini index theory [32]. The key ingredient turns out to be its monotonicity property.

Proposition 3.4.1 (Monotonicity of the Fadell–Husseini index, [32, Sect. 2])**.** *Let G be a finite group, R be a commutative ring with unit, and let X and Y be two spaces with a G-action. If there is a G-equivariant map $f : X \to Y$, then*

$$\mathrm{Index}_G(Y; R) \subseteq \mathrm{Index}_G(X; R).$$

In order to prove Theorem 3.2.3, we need to compare the Fadell–Husseini index of the product of configuration spaces to the Fadell–Husseini index of the sphere $S(W_{p^k}^{\oplus r})$. Blagojević, Lück and Ziegler computed the Fadell–Husseini index of the configuration space [18, Thm. 6.1], which together with the product formula [32, Cor. 3.2] yields the following.

Lemma 3.4.2 ([18, Thm. 6.1],[32, Cor. 3.2])**.** *Let p be a prime and let $k \geq 1$ be an integer. Then*

$$\mathrm{Index}_{\mathbb{Z}_p^k}(\mathrm{Conf}(\mathbb{R}^d, p)^{\times k}; \mathbb{F}_p) = \begin{cases} \langle t_1^d, \dots, t_k^d \rangle, & \text{if } p = 2 \\ \langle e_1 t_1^{\frac{(d-1)(p-1)}{2}}, \dots, e_k t_k^{\frac{(d-1)(p-1)}{2}}, \\ \quad t_1^{\frac{(d-1)(p-1)}{2}+1}, \dots, t_k^{\frac{(d-1)(p-1)}{2}+1} \rangle, & \text{if } p \geq 3, \end{cases}$$

where $t_1, \dots, t_k, e_1, \dots, e_k$ are generators of $H^*(\mathbb{Z}_p^k; \mathbb{F}_p)$.

Now we have assembled all ingredients needed for the proof of Theorem 3.2.3.

Proof of Theorem 3.2.3. If there is a $\mathbb{Z}_p^k$-equivariant map $\mathrm{Conf}(\mathbb{R}^d, p)^{\times k} \to S(W_{p^k}^{\oplus r})$, then by Proposition 3.4.1, the index of the sphere $S(W_{p^k}^{\oplus r})$ is a subideal of the index of the product of configuration spaces $\mathrm{Conf}(\mathbb{R}^d, p)^{\times k}$. Therefore, it suffices to show

$$\mathrm{Index}_{\mathbb{Z}_p^k}(S(W_{p^k}^{\oplus r}); \mathbb{F}_p) \not\subseteq \mathrm{Index}_{\mathbb{Z}_p^k}(\mathrm{Conf}(\mathbb{R}^d, p)^{\times k}; \mathbb{F}_p).$$

The ideal $\mathrm{Index}_{\mathbb{Z}_p^k}(S(W_{p^k}^{\oplus r}); \mathbb{F}_p)$ contains the polynomial ζ^r, and if the integers p, k and r satisfy the conditions of the theorem, then ζ^r contains the monomial $t_1^{r\frac{p-1}{2}} t_2^{r\frac{p(p-1)}{2}} \dots t_k^{r\frac{p^{k-1}(p-1)}{2}}$, by Lemma 3.3.2. If $d > rp^{k-1}$, then by Lemma 3.4.2 the polynomial ζ^r does not belong to the ideal $\mathrm{Index}_{\mathbb{Z}_p^k}(\mathrm{Conf}(\mathbb{R}^d, p)^{\times k}; \mathbb{F}_p)$, which concludes the proof. □

Similarly as above, in order to prove Theorem 3.2.4, we compare the index of the product of spheres $(S^{2d-1})^{\times k}$ to the index of the sphere $S(W_{p^k}^{\oplus r})$.

Lemma 3.4.3 ([32])**.** *Let $p \geq 2$ be a prime and let $k \geq 1$ be an integer. Then*

$$\mathrm{Index}_{\mathbb{Z}_p^k}((S^{2d-1})^{\times k}; \mathbb{F}_p) = \begin{cases} \langle t_1^{2d}, \dots, t_k^{2d} \rangle, & \text{if } p = 2, \\ \langle t_1^d, \dots, t_k^d \rangle, & \text{if } p \geq 3, \end{cases}$$

where $t_1, \dots, t_k$ are generators of $H^*(\mathbb{Z}_p^k; \mathbb{F}_p)$.

Proof of Theorem 3.2.4. Similarly as in the proof of Theorem 3.2.3, it suffices to show that

$$\mathrm{Index}_{\mathbb{Z}_p^k}(S(W_{p^k}^{\oplus r}); \mathbb{F}_p) \not\subseteq \mathrm{Index}_{\mathbb{Z}_p^k}((S^{2d-1})^{\times k}; \mathbb{F}_p),$$

which follows from Lemma 3.3.1, Lemma 3.3.2 and Lemma 3.4.3. □

Remark 3.4.4. If $k = 2$ and $r = 1$, one can relax bounds on d in Theorem 3.2.3 and Theorem 3.2.4, and consequently in Theorem 3.1.5 and Theorem 3.1.8, using Lemma 3.3.4.

In particular, the statements of Theorem 3.1.5 and Theorem 3.2.3 are true under the following conditions: $k = 2, r = 1, p \geq 3$ is a prime, and
(a) $d \geq \frac{p+5}{2}$, or
(b) $4 \mid p-1$ and $d \geq \frac{p+3}{2}$.

On the other hand, the statements of Theorem 3.1.8 and Theorem 3.2.4 are true under the following conditions: $k = 2, r = 1, p \geq 3$ is a prime, and
(a) $d > \frac{(p-1)(p+3)}{4}$, or
(b) $4 \mid p-1$ and $d > \frac{p^2-1}{4}$.

Concluding remarks

Remark 3.5.1. One could talk about non-existence of $\mathbb{Z}_p^k$-equivariant maps $\mathrm{Conf}(\mathbb{R}^{d_1}, p) \times \cdots \times \mathrm{Conf}(\mathbb{R}^{d_k}, p) \to S(W_{p^k}^{\oplus r})$ and $S^{2d_1-1} \times \cdots \times S^{2d_k-1} \to S(W_{p^k}^{\oplus r})$, where $d_1, \ldots, d_k$ are not necessarily the same integers. The Fadell–Husseini index of the product of configuration spaces is:

$$\mathrm{Index}_{\mathbb{Z}_p^k}(\mathrm{Conf}(\mathbb{R}^{d_1}, p) \times \cdots \times \mathrm{Conf}(\mathbb{R}^{d_k}, p); \mathbb{F}_p) = \\ \begin{cases} \langle t_1^{d_1}, \ldots, t_k^{d_k} \rangle, & \text{if } p = 2 \\ \langle e_1 t_1^{\frac{(d_1-1)(p-1)}{2}}, \ldots, e_k t_k^{\frac{(d_k-1)(p-1)}{2}}, \\ \quad t_1^{\frac{(d_1-1)(p-1)}{2}+1}, \ldots, t_k^{\frac{(d_k-1)(p-1)}{2}+1} \rangle, & \text{if } p \geq 3, \end{cases}$$

and the Fadell–Husseini index of the product of spheres is:

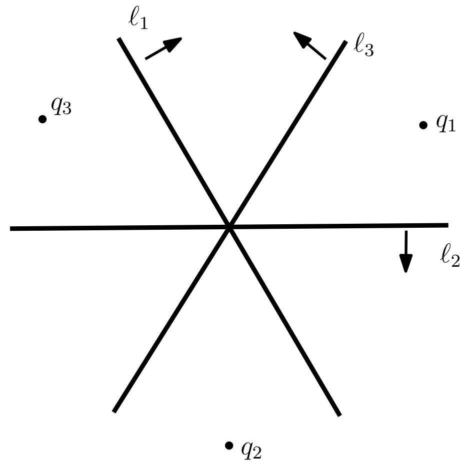
$$\mathrm{Index}_{\mathbb{Z}_p^k}(S^{2d_1-1} \times \cdots \times S^{2d_k-1}; \mathbb{F}_p) = \begin{cases} \langle t_1^{2d_1}, \ldots, t_k^{2d_k} \rangle, & \text{if } p = 2 \\ \langle t_1^{d_1}, \ldots, t_k^{d_k} \rangle, & \text{if } p \geq 3. \end{cases}$$

It suffices to show that the polynomial ζ^r (or any of its monomials) is not an element of the index of the domain of the map (the product of configuration spaces or the product of spheres). For concrete parameters, these polynomials can be computed using, for example, a computer algebra software.

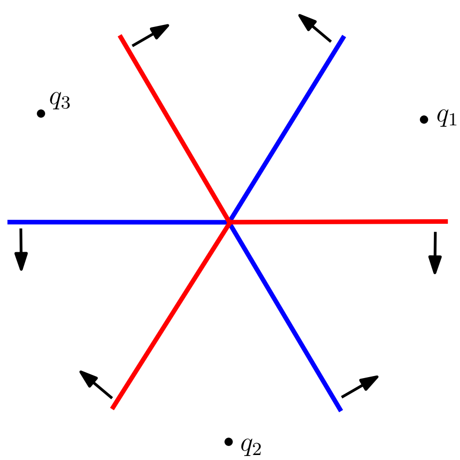
Remark 3.5.2. In their study of Turán numbers of bipartite graphs [13], Blagojević, Bukh and Karasev consider functions $f : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ that are constant on some p-by-p grid. The following is a generalization of [13, Thm. 2].

Corollary 3.5.3. *Let p be a prime and d, k and r be positive integers. If $d > rp^{k-1}$ and one of the following conditions is satisfied*
(a) $p = 2$*, or*
(b) 4 *divides* $p-1$*, or*
(c) k *is even, or*
(d) r *is even,*

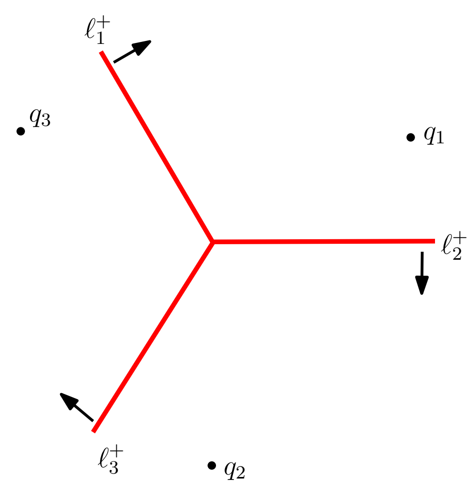
then for every collection of r *continuous function* $f_1, \dots, f_r : (\mathbb{R}^d)^{\times k} \to \mathbb{R}$*, there is a* $\underbrace{p \times \cdots \times p}_{k}$ *grid on which all these functions are constant, i.e. there are sets* $X_1, \dots, X_k \subset \mathbb{R}^d$ *such that* $|X_1| = \cdots = |X_k| = p$ *and the functions* $f_1, \dots, f_r$ *are constant on* $X_1 \times \cdots \times X_k$*.*



(a) The intersection of the oriented hyperplanes H_1, H_2 and H_3 with the space $V^\perp$.



(b) Two 3-fans defined by the line arrangement U.



(c) The regular fan in $V^\perp$ determined by the points q_1, q_2, q_3.

Figure 3.3: The 3-fan in $V^\perp$ determined by the points q_1, q_2 and q_3.

Chapter 4

Cutting a part from many measures

This chapter is based on the paper with the same title [19], and it is a joint work with Pavle V.M. Blagojević and Günter M. Ziegler.

Introduction and statement of the main results

The classical measure partition problems ask whether for a given collection of measures, the ambient Euclidean space can be partitioned in a prescribed way so that each of the given measures gets cut into equal pieces.

The first example of such a result is the well known ham-sandwich theorem, conjectured by Steinhaus and later proved by Banach. It claims that given d measures in $\mathbb{R}^d$, one can cut $\mathbb{R}^d$ by an affine hyperplane into two pieces so that each of the measures is cut into halves. Motivated by the ham sandwich theorem, Grünbaum posed a more general hyperplane measure partition problem in 1960 [40, Sec. 4 (v)]. He asked whether any given measure in the Euclidean space $\mathbb{R}^d$ can be cut by k affine hyperplanes into 2^k equal pieces. An even more general problem was proposed and considered by Hadwiger [42] and Ramos [70]: Determine the minimal dimension d such that for every collection of j measures on $\mathbb{R}^d$ there exists an arrangement of k affine hyperplanes in $\mathbb{R}^d$ that cut all measures into 2^k equal pieces. For a survey on the Grünbaum–Hadwiger–Ramos hyperplane measure partition problem consult [15].

Furthermore, in 2001 Bárány and Matoušek [8] considered partitions of measures on the sphere S^2 by fans with the requirement that each angle of the fan contains a prescribed proportion of every measure.

Measure partition results can also be stated discretely – given a collection of finite sets in $\mathbb{R}^d$, can the ambient space be partitioned in a prescribed way so that each of the given sets gets cut into subsets of the same cardinality. It is not a rare case that a discrete result follows from a continuous one. For example, the discrete ham-sandwich theorem [59, Thm. 3.1.2] is a corollary of the ham-sandwich theorem. It states that given any d finite sets $A_1, \dots, A_d$ in $\mathbb{R}^d$, there exists an affine hyperplane in $\mathbb{R}^d$ which cuts each set A_i into two subsets of cardinality at most $\lfloor \frac{1}{2}|A_i| \rfloor$.

In this chapter we prove a continuous result that is motivated by a discrete conjecture of Holmsen, Kynčl & Valculescu [45, Con. 3]. We consider many measures in the Euclidean space, and instead of searching for equiparting convex partitions (that in general do not exist for a large number of measures), we look for convex partitions that in each piece capture a positive amount from a (large) prescribed number of the given measures.

Definition 4.1.1. Let $d \geq 1$ and $n \geq 1$ be integers. An ordered collection of closed subsets $(C_1, \dots, C_n)$ of $\mathbb{R}^d$ is called a *partition* of $\mathbb{R}^d$ if
(1) $\bigcup_{i=1}^{n} C_i = \mathbb{R}^d$,
(2) $\operatorname{int}(C_i) \neq \emptyset$ for every $1 \leq i \leq n$, and
(3) $\operatorname{int}(C_i) \cap \operatorname{int}(C_j) = \emptyset$ for all $1 \leq i < j \leq n$.
A partition $(C_1, \dots, C_n)$ is called *convex* if all subsets $C_1, \dots, C_n$ are convex.

Let $m \geq 1$, $n \geq 1$, $c \geq 1$ and $d \geq 1$ be integers, and let $\mathcal{M} = (\mu_1, \dots, \mu_m)$ be a collection of m finite absolutely continuous measures in $\mathbb{R}^d$. Moreover, assume that $\mu_j(\mathbb{R}^d) > 0$, for every $1 \leq j \leq m$. We are interested in the existence of a convex partition $(C_1, \dots, C_n)$ of $\mathbb{R}^d$ with the property that each set C_i contains a positive amount of at least c of the measures, that is

$$\#\{j : 1 \leq j \leq m, \, \mu_j(C_i) > 0\} \geq c,$$

for every $1 \leq i \leq n$. In the case when the measures are given by finite point sets, we say that a point set $X \subseteq \mathbb{R}^d$ is in general position if no $d + 1$ points from X lie in an affine hyperplane in $\mathbb{R}^d$. For the point set measures in general position Holmsen, Kynčl and Valculescu proposed the following natural conjecture [45, Con. 3].

Conjecture 4.1.2 (Holmsen, Kynčl, Valculescu, 2017)**.** *Let $d \geq 2$, $\ell \geq 2$, $m \geq 2$ and $n \geq 1$ be integers with $m \geq d$ and $\ell \geq d$. Consider a set $X \subseteq \mathbb{R}^d$ of ℓn points in general position that is colored with at least m different colors. If there exists a partition of the set X into n subsets of size ℓ such that each subset contains points colored by at least d colors, then there exists such a partition of X that in addition has the property that the convex hulls of the n subsets are pairwise disjoint.*

The conjecture was settled for $d = 2$ in the same paper by Holmsen, Kynčl & Valculescu [45]. On the other hand, if instead of finite collections of points one considers finite positive absolutely continuous measures in $\mathbb{R}^d$, Soberón [75] gave a positive answer on splitting d measures in $\mathbb{R}^d$ into convex pieces such that each piece has positive measure with respect to each of the measures. Moreover, he proved existence of convex partitions that equipart all measures. A discretization of Soberón's result by Blagojević, Rote, Steinmeyer and Ziegler [20] gave a positive answer to Conjecture 4.1.2 in the case when $m = d$. In addition, they were able to show that the set X can be partitioned into n subsets in such a way that all color classes are equipartitioned simultaneously.

In this chapter we prove three continuous results of a similar flavor, trying to come closer to a positive answer to Conjecture 4.1.2 in the case when $m \geq d$. The first of the three results is the following.

Theorem 4.1.3. *Let $d \geq 2$, $m \geq 2$, $n \geq 2$, and $c > d$ be integers. If $m \geq n(c-d)+d$, then for every collection $\mathcal{M} = (\mu_1, \dots, \mu_m)$ of m positive finite absolutely continuous measures on $\mathbb{R}^d$, there exists a partition of $\mathbb{R}^d$ into n convex subsets $(C_1, \dots, C_n)$ such that each of the subsets has positive measure with respect to at least c of the measures $\mu_1, \dots, \mu_m$. In other words,*

$$\#\{j : 1 \leq j \leq m, \, \mu_j(C_i) > 0\} \geq c$$

for every $1 \leq i \leq n$.

The following two theorems have stronger statements – in Theorem 4.1.4 we additionally show that one of the measures can be equipartitioned without changing the bound on m, and in Theorem 4.1.5 we show that the sum of all the measures can be equipartitioned if we allow the number m of measures to increase.

Theorem 4.1.4. *Let $d \geq 2$, $m \geq 2$, and $c \geq 2$ be integers, and let $n = p^k$ be a prime power. If*

$$m \geq n(c-d) + \tfrac{dn}{p} - \tfrac{n}{p} + 1,$$

then for every collection $\mathcal{M} = (\mu_1, \ldots, \mu_m)$ of m positive finite absolutely continuous measures on $\mathbb{R}^d$, there exists a partition of $\mathbb{R}^d$ into n convex subsets $(C_1, \ldots, C_n)$ that equiparts the measure μ_m with the additional property that each of the subsets has positive measure with respect to at least c of the measures $\mu_1, \ldots, \mu_m$. In other words,

$$\mu_m(C_1) = \cdots = \mu_m(C_n) = \frac{1}{n}\mu_m(\mathbb{R}^d),$$

and

$$\#\{j : 1 \leq j \leq m,\ \mu_j(C_i) > 0\} \geq c$$

for every $1 \leq i \leq n$.

Theorem 4.1.5. *Let $d \geq 2$, $m \geq 2$, and $c \geq 2$ be integers, and let $n = p^k$ be a prime power. If*
(a) $n(c-1) \geq m$ *and* $\max\{m, n\} \geq n(c-d) + \tfrac{dn}{p} - \tfrac{n}{p} + n$, *or*
(b) $n(c-1) < m$,
then for every collection $\mathcal{M} = (\mu_1, \ldots, \mu_m)$ of m positive finite absolutely continuous measures on $\mathbb{R}^d$, there exists a partition of $\mathbb{R}^d$ into n convex subsets $(C_1, \ldots, C_n)$ that equiparts the sum of the measures $\mu = \mu_1 + \cdots + \mu_m$ with the additional property that each of the subsets has positive measure with respect to at least c of the measures $\mu_1, \ldots, \mu_m$. In other words,

$$\mu(C_1) = \cdots = \mu(C_n) = \frac{1}{n}\mu(\mathbb{R}^d) = \frac{1}{n}\sum_{j=1}^{m}\mu_j(\mathbb{R}^d),$$

and

$$\#\{j : 1 \leq j \leq m,\ \mu_j(C_i) > 0\} \geq c$$

for every $1 \leq i \leq n$.

Previous solutions for measure partition problems relied on a variety of advanced methods from equivariant topology. Different configuration space/test map schemes (CS/TM schemes) related partition problems with the questions of *non*-existence of appropriately constructed equivariant maps from configuration spaces into a suitable test spaces. For example, in the proof of the ham-sandwich theorem a sphere with the antipodal action appears as a test space. The test space in the Grünbaum–Hadwiger–Ramos hyperplane measure partition problem is again a sphere, but with an action of the sign permutation group, while the test space in the Bárány and Matoušek fan partition problem is a complement of an arrangement of linear subspaces equipped with an action of the Dihedral or generalized quaternion group. In this chapter the proof of Theorem 4.1.3 is elementary and it does not use any topology. However, the proofs of Theorem 4.1.4 and Theorem 4.1.5

rely on a novel CS/TM scheme presented in Theorem 4.2.1 and Theorem 4.2.2: For the first time the test space is the union of an arrangement of affine subspaces, equipped in this case with an action of a symmetric group.

In the subsequent work [17] we will make slight modifications of the CS/TM schemes used in the proofs of Theorems 4.1.4 and 4.1.5 and obtain stronger results – not only that one can guarantee each subset to cut a part from many measures, but one also gets that each subset cuts a large part from many measures.

The rest of the chapter is organized as follows. The proofs of Theorem 4.1.4 and Theorem 4.1.5 run in parallel and follow CS/TM schemes that are given in Section 4.2. The topological results about non-existence of equivariant maps are proved in Section 4.3. Finally, the proofs of Theorem 4.1.3, Theorem 4.1.4 and Theorem 4.1.5 are given in Section 4.4. Note that the proof of Theorem 4.1.3 can be read independently of the previous sections.

Existence of a partition from non-existence of a map

In this section we develop CS/TM schemes that relate the existence of convex partitions from Theorems 4.1.4 and 4.1.5 with the non-existence of particular equivariant maps. These two CS/TM schemes are very similar to each other.

Existence of an equipartition of one measure from non-existence of a map

Let $d \geq 2$, $m \geq 2$, $n \geq 1$, and $c \geq 2$ be integers, and let $\mu_1, \dots, \mu_m$ be finite absolutely continuous measures on $\mathbb{R}^d$. Throughout the chapter we assume that $m \geq c$, since it is a requirement that naturally comes from the mass partition problem. Following notation from [21], let $\mathrm{EMP}(\mu_m, n)$ denote the space of all convex partitions of $\mathbb{R}^d$ into n convex pieces $(C_1, \dots, C_n)$ that equipart the measure μ_m, as studied in [53], that is

$$\mu_m(C_1) = \cdots = \mu_m(C_n) = \tfrac{1}{n}\mu_m(\mathbb{R}^d).$$

Now define a continuous map $f\colon \mathrm{EMP}(\mu_m, n) \longrightarrow \mathbb{R}^{(m-1)\times n} \cong (\mathbb{R}^{m-1})^n$ as

$$(C_1, \dots, C_n) \longmapsto \begin{pmatrix} \mu_1(C_1) & \mu_1(C_2) & \dots & \mu_1(C_n) \\ \mu_2(C_1) & \mu_2(C_2) & \dots & \mu_2(C_n) \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{m-1}(C_1) & \mu_{m-1}(C_2) & \dots & \mu_{m-1}(C_n) \end{pmatrix}.$$

The symmetric group $\mathfrak{S}_n$ acts on $\mathrm{EMP}(\mu_m, n)$ and $(\mathbb{R}^{m-1})^n$ as follows

$$\pi \cdot (C_1, \dots, C_n) = (C_{\pi(1)}, \dots, C_{\pi(n)}) \qquad \text{and} \qquad \pi \cdot (Y_1, \dots, Y_n) = (Y_{\pi(1)}, \dots, Y_{\pi(n)}),$$

where $(C_1, \dots, C_n) \in \mathrm{EMP}(\mu_m, n)$, $(Y_1, \dots, Y_n) \in (\mathbb{R}^{m-1})^n$, and $\pi \in \mathfrak{S}_n$. These actions are introduced in such a way that the map f becomes an $\mathfrak{S}_n$-equivariant map. The image of the map f is a subset of an affine set $V \subset \mathbb{R}^{(m-1)\times n} \cong (\mathbb{R}^{m-1})^n$ given by

$$V = \Big\{(y_{jk}) \in \mathbb{R}^{(m-1)\times n} \ : \ \sum_{k=1}^{n} y_{jk} = \mu_j(\mathbb{R}^d) \text{ for every } 1 \leq j \leq m-1\Big\} \cong \mathbb{R}^{(m-1)\times(n-1)}.$$

Consequently, we can assume that $f\colon \mathrm{EMP}(\mu_m, n) \longrightarrow V \subseteq \mathbb{R}^{(m-1)\times n}$.

Let $1 \leq i \leq n$, and let $I \subseteq [m-1]$ be a subset of cardinality $|I| = m - c + 1$, where $[m-1]$ denotes the set of integers $\{1, 2, \dots, m-1\}$. Consider the subspace $L_{I,i}$ of V given by

$$L_{I,i} := \{(y_{jk}) \in V : y_{ri} = 0 \text{ for every } r \in I\},$$

and the associated arrangement

$$\mathcal{A} = \mathcal{A}(m, n, c) = \{L_{I,i} : 1 \leq i \leq n,\ I \subseteq [m-1],\ |I| = m - c + 1\}. \tag{4.1}$$

The arrangement $\mathcal{A}$ is an $\mathfrak{S}_n$-invariant affine arrangement in $\mathbb{R}^{(m-1)\times n}$, meaning that $\pi \cdot L_{I,i} \in \mathcal{A}$ for every $\pi \in \mathfrak{S}_n$. Now we explain the key property of the arrangement $\mathcal{A}$. Let $(C_1, \dots, C_n)$ be a partition of $\mathbb{R}^d$ with a property that at least one of the subsets $C_1, \dots, C_n$ has positive measure with respect to at most $c - 1$ of the measures $\mu_1, \dots, \mu_m$, which means that $(C_1, \dots, C_n)$ is not a partition we are searching for. Since, by construction $\mu_m(C_i) > 0$ for every $1 \leq i \leq n$, it follows that at least one of the subsets $C_1, \dots, C_n$ has positive measure with respect to at most $c - 2$ of the measures $\mu_1, \dots, \mu_{m-1}$. Then there is a column of the matrix $f(C_1, \dots, C_n) \in V \subseteq \mathbb{R}^{(m-1)\times n}$ with at most $c-2$ positive coordinates. In other words, there is a column of the matrix $f(C_1, \dots, C_n)$ with at least $m - c + 1$ zeros, and consequently the matrix $f(C_1, \dots, C_n)$ is an element of the union $\bigcup \mathcal{A} = \bigcup_{L_{I,i} \in \mathcal{A}} L_{I,i}$ of the arrangement $\mathcal{A}$.

Let us now assume that for integers $d \geq 2$, $m \geq 2$, $n \geq 1$, and $c \geq 1$, there exist measures $\mu_1, \dots, \mu_m$ in $\mathbb{R}^d$ such that in every convex partition $(C_1, \dots, C_n)$ of $\mathbb{R}^d$ that equiparts μ_m there is at least one set C_k that does not have positive measure with respect to at least c measures, or equivalently it has measure zero with respect to at least $m-c+1$ of the measures $\mu_1, \dots, \mu_m$. Consequently, $f(C_1, \dots, C_n) \in \bigcup \mathcal{A}$ for every convex partition $(C_1, \dots, C_n)$ of $\mathbb{R}^d$ that equiparts the measure μ_m. In particular, this means that the $\mathfrak{S}_n$-equivariant map f factors as follows

$$\begin{array}{ccc} \mathrm{EMP}(\mu_m, n) & \xrightarrow{\quad f \quad} & V \\ & \searrow^{\widetilde{f}} \quad \nearrow_{i} & \\ & \bigcup \mathcal{A}(m, n, c), & \end{array}$$

where $i\colon \bigcup \mathcal{A} \longrightarrow V$ is the inclusion and $\widetilde{f}\colon \mathrm{EMP}(\mu_m, n) \longrightarrow \bigcup \mathcal{A}$ is an $\mathfrak{S}_n$-equivariant map obtained from f by restricting the codomain. Thus, we have proved the following theorem.

Theorem 4.2.1. *Let $d \geq 2$, $m \geq 2$, $n \geq 1$, and $c \geq 2$ be integers, and let $\mu_1, \dots \mu_m$ be positive finite absolutely continuous measures in $\mathbb{R}^d$ for every $1 \leq j \leq m$. If there is no $\mathfrak{S}_n$-equivariant map*

$$\mathrm{EMP}(\mu_m, n) \longrightarrow \bigcup \mathcal{A}(m, n, c),$$

then there exists a convex partition $(C_1, \dots, C_n)$ of $\mathbb{R}^d$ that equiparts the measure μ_m with the additional property that each of the subsets C_i has positive measure with respect to at least c of the measures $\mu_1, \dots, \mu_m$, that is

$$\mu_m(C_1) = \cdots = \mu_m(C_n) = \frac{1}{n}\mu_m(\mathbb{R}^d),$$

and

$$\#\{j : 1 \leq j \leq m,\, \mu_j(C_i) > 0\} \geq c$$

for every $1 \leq i \leq n$.

Existence of an equipartition of the sum of measures from non-existence of a map

As we have already mentioned, the CS/TM scheme needed for proving Theorem 4.1.5 is very similar to the one presented in Section 4.2.1. Nevertheless, it will be separately developed here.

Let $d \geq 2$, $m \geq 2$, $n \geq 1$, and $c \geq 2$ be integers, and let $\mu_1, \ldots, \mu_m$ be finite absolutely continuous measures on $\mathbb{R}^d$. Denote by μ the sum of the measures $\mu_1, \ldots, \mu_m$, i.e.,

$$\mu = \sum_{j=1}^{m} \mu_j.$$

Similarly as in Section 4.2.1, we define a continuous map $\widetilde{f}\colon \mathrm{EMP}(\mu, n) \longrightarrow \mathbb{R}^{m\times n}$ as

$$(C_1, \ldots, C_n) \longmapsto \begin{pmatrix} \mu_1(C_1) & \mu_1(C_2) & \ldots & \mu_1(C_n) \\ \mu_2(C_1) & \mu_2(C_2) & \ldots & \mu_2(C_n) \\ \vdots & \vdots & \ddots & \vdots \\ \mu_m(C_1) & \mu_m(C_2) & \ldots & \mu_m(C_n) \end{pmatrix},$$

where the domain of the map $\widetilde{f}$ is the space of all convex partitions of $\mathbb{R}^d$ that equipart the measure μ. The map $\widetilde{f}$ is $\mathfrak{S}_n$-equivariant by construction.

The image of the map $\widetilde{f}$ is a subset of an affine set $\widetilde{V} \subset \mathbb{R}^{m\times n}$ given by

$$\widetilde{V} = \Big\{ (y_{jk}) \in \mathbb{R}^{m\times n} \ : \ \begin{array}{ll} \sum_{k=1}^{n} y_{jk} = \mu_j(\mathbb{R}^d) & \text{for every } 1 \leq j \leq m, \text{ and} \\ \sum_{j=1}^{m} y_{jk} = \frac{1}{n}\mu(\mathbb{R}^d) & \text{for every } 1 \leq k \leq n \Big\}. \end{array}$$

Now we define an affine arrangement that resembles the arrangement $\mathcal{A}$ from Section 4.2.1. Let $1 \leq i \leq n$, and let $I \subseteq [m]$ be a subset of cardinality $|I| = m - c + 1$. Consider the subspace $\widetilde{L}_{I,i}$ of $\widetilde{V}$ given by

$$\widetilde{L}_{I,i} := \{(y_{jk}) \in \widetilde{V} : y_{ri} = 0 \text{ for every } r \in I\},$$

and the associated $\mathfrak{S}_n$-invariant arrangement

$$\widetilde{\mathcal{A}} = \widetilde{\mathcal{A}}(m, n, c) = \Big\{ \widetilde{L}_{I,i} : 1 \leq i \leq n,\ I \subseteq [m],\ |I| = m - c + 1 \Big\}. \tag{4.2}$$

Following the steps from Section 4.2.1, we study the key property of the arrangement $\widetilde{\mathcal{A}}$. Let $(C_1, \ldots, C_n)$ be a convex partition of $\mathbb{R}^d$ that does not satisfy the property asked in Theorem 4.1.5. More precisely, assume that for some i the subset C_i has positive measure with respect to at most $c - 1$ of the measures $\mu_1, \ldots, \mu_m$. This means that the i-th column of the matrix $\widetilde{f}(C_1, \ldots, C_n) \in \mathbb{R}^{m\times n}$ has at least $m - c + 1$ zeros. In other words, $\widetilde{f}(C_1, \ldots, C_n) \in \bigcup \widetilde{\mathcal{A}}$. Therefore, we have obtained the following theorem.

Theorem 4.2.2. *Let $d \geq 2$, $m \geq 2$, $n \geq 1$, and $c \geq 2$ be integers, and let $\mu_1, \dots \mu_m$ be positive finite absolutely continuous measures in $\mathbb{R}^d$ for every $1 \leq j \leq m$. If there is no $\mathfrak{S}_n$-equivariant map*

$$\mathrm{EMP}(\mu, n) \longrightarrow \bigcup \widetilde{\mathcal{A}}(m, n, c),$$

then there exists a convex partition $(C_1, \dots, C_n)$ of $\mathbb{R}^d$ that equiparts the measure $\mu = \mu_1 + \dots + \mu_m$ with the additional property that each of the subsets C_i has positive measure with respect to at least c of the measures $\mu_1, \dots, \mu_m$, that is

$$\mu(C_1) = \dots = \mu(C_n) = \frac{1}{n}\mu(\mathbb{R}^d),$$

and

$$\#\{j : 1 \leq j \leq m,\ \mu_j(C_i) > 0\} \geq c$$

for every $1 \leq i \leq n$.

Non-existence of the equivariant maps

This section is devoted to the proof of (non)-existence of equivariant maps from the space of regular convex partitions to appropriate affine arrangements. In section 4.3.1 we consider $\mathfrak{S}_n$-equivariant maps $\mathrm{EMP}(\mu_m, n) \longrightarrow \bigcup \mathcal{A}(m, n, c)$, whereas in Section 4.3.2 $\mathfrak{S}_n$-equivariant maps $\mathrm{EMP}(\mu, n) \longrightarrow \bigcup \widetilde{\mathcal{A}}(m, n, c)$ will be considered for different values of parameters d, m, n and c.

Non-existence of the equivariant map $\mathrm{EMP}(\mu_m, n) \longrightarrow \bigcup \mathcal{A}(m, n, c)$

In order to understand the (non-)existence of an $\mathfrak{S}_n$-equivariant map

$$\mathrm{EMP}(\mu_m, n) \longrightarrow \bigcup \mathcal{A}(m, n, c),$$

we first construct various equivariant maps and prove a few auxiliary lemmas. In the following we use particular tools from the theory of homotopy colimits; for further details on these methods consult for example [24], [84], or [78].

Let X be a topological space and let $n \geq 1$ be an integer. The *ordered configuration space* $\mathrm{Conf}(X, n)$ of n ordered pairwise distinct points of X is the space

$$\mathrm{Conf}(X, n) = \{(x_1, \dots, x_n) \in X^n \mid x_i \neq x_j \text{ for all } 1 \leq i < j \leq n\}.$$

It was shown in [21, Sec. 2] that a subspace of $\mathrm{EMP}(\mu_m, n)$ consisting only of regular convex partitions can be parametrized by the configuration space $\mathrm{Conf}(\mathbb{R}^d, n)$. In particular, we have the following lemma.

Lemma 4.3.1. *There exists an $\mathfrak{S}_n$-equivariant map*

$$\alpha\colon\ \mathrm{Conf}(\mathbb{R}^d, n) \longrightarrow \mathrm{EMP}(\mu_m, n).$$

Let $P = P(\mathcal{A})$ denote the intersection poset of the arrangement $\mathcal{A} = \mathcal{A}(m, n, c)$, ordered by the reverse inclusion. The elements of the poset P are non-empty intersections of subspaces in $\mathcal{A}$, thus they are of the form

$$p_\Lambda = \bigcap_{(i,I)\in\Lambda} L_{I,i} = \{(y_{jk}) \in V \subseteq \mathbb{R}^{(m-1)\times n} \ : \ y_{ji} = 0, \text{ for all } 1 \le i \le n \text{ and } j \in I_i\},$$

where $\Lambda \subset [n] \times \binom{[m-1]}{m-c+1}$ and $I_i = \bigcup_{(i,I)\in\Lambda} I$. Alternatively, each poset element p_Λ can be presented as an $(m-1) \times n$ matrix (a_{jk}), where $a_{jk} = 0$ if and only if $j \in I_k$. In other words, a coordinate a_{jk} in the matrix presentation of p_Λ equals zero if and only if $y_{jk} = 0$ for every element $(y_{jk}) \in p_\Lambda$. An example of the poset $P(\mathcal{A})$ for parameters $n = 2, m = 3$ and $c = 3$ is shown in Figure 4.1.

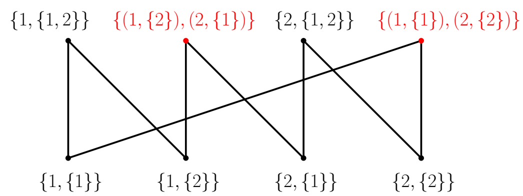


Figure 4.1: Hasse diagram of the poset $P(\mathcal{A}(3, 2, 3))$.

Let $\mathcal{C}$ be the P-diagram that corresponds to the arrangement $\mathcal{A} = \mathcal{A}(m, n, c)$, that is $\mathcal{C}(p_\Lambda) = p_\Lambda$ and $\mathcal{C}(p_{\Lambda'} \supseteq p_{\Lambda''})\colon p_{\Lambda''} \longrightarrow p_{\Lambda'}$ is the inclusion. The Equivariant Projection Lemma [78, Lem. 2.1] implies the following.

Lemma 4.3.2. *The projection map*

$$\operatorname{hocolim}_{P(\mathcal{A})} \mathcal{C} \longrightarrow \operatorname{colim}_{P(\mathcal{A})} \mathcal{C} = \bigcup \mathcal{A}$$

is an $\mathfrak{S}_n$-equivariant homotopy equivalence. In particular, there exists an $\mathfrak{S}_n$-equivariant map

$$\beta\colon \bigcup \mathcal{A} \longrightarrow \operatorname{hocolim}_{P(\mathcal{A})} \mathcal{C}.$$

Now, let Q be the face poset of the $(n-1)$-dimensional simplex, or equivalently a Boolean poset with 2^n elements. Define the monotone map $\varphi\colon P \longrightarrow Q$ by

$$\varphi(p_\Lambda) = \{i \in [n] : (i, I) \in \Lambda \text{ for some } I \subseteq [m-1]\}.$$

Thus, φ maps an element p_Λ to the set of indices of its columns that contain zeros. It is important to notice that φ does not have to be surjective, and therefore we set
$Q' = \varphi(P) \subseteq Q$.

Next we consider the homotopy pushdown $\mathcal{D}$ of the diagram $\mathcal{C}$ along the map φ over Q'. This means that for $q \in Q'$

$$\mathcal{D}(q) = \operatorname{hocolim}_{\varphi^{-1}(Q'_{\ge q})} \mathcal{C}|_{\varphi^{-1}(Q'_{\ge q})} \simeq \Delta(\varphi^{-1}(Q'_{\ge q})),$$

and for every $q \ge r$ in Q' the map $\mathcal{D}(q \ge r)\colon \mathcal{D}(q) \longrightarrow \mathcal{D}(r)$ is the corresponding inclusion. The next result follows from the Homotopy Pushdown Lemma [84, Prop. 3.12] adapted to equivariant setting.

Lemma 4.3.3. *There is an $\mathfrak{S}_n$-equivariant homotopy equivalence*

$$\operatorname{hocolim}_{Q'} \mathcal{D} \longrightarrow \operatorname{hocolim}_{P(\mathcal{A})} \mathcal{C}.$$

In particular, there exists an $\mathfrak{S}_n$-equivariant map

$$\gamma\colon \operatorname{hocolim}_{P(\mathcal{A})} \mathcal{C} \longrightarrow \operatorname{hocolim}_{Q'} \mathcal{D}.$$

We introduce another Q'-diagram $\mathcal{E}$ by setting for $q \in Q'$ that

$$\mathcal{E}(q) = \begin{cases} \Delta(\varphi^{-1}(\hat{1})), & \text{if } q = \hat{1} \in Q' \text{ is the maximal element of } Q, \\ \text{pt}, & \text{otherwise,} \end{cases}$$

and for every $q \geq r$ in Q' the map $\mathcal{E}(q \geq r)$ to be the constant map. In addition, we define a morphism of diagrams $(\Psi, \psi)\colon \mathcal{D} \longrightarrow \mathcal{E}$, where $\psi\colon Q' \longrightarrow Q'$ is the identity map, and $\Psi(q)\colon \mathcal{D}(q) \longrightarrow \mathcal{E}(q)$ is the identity map when q is the maximal element, and constant map otherwise. The morphism (Ψ, ψ) of diagrams induces an $\mathfrak{S}_n$-equivariant map between associated homotopy colimits. Thus, we have established the following.

Lemma 4.3.4. *There exists an $\mathfrak{S}_n$-equivariant map*

$$\delta\colon \operatorname{hocolim}_{Q'} \mathcal{D} \longrightarrow \operatorname{hocolim}_{Q'} \mathcal{E}.$$

In the final lemma we describe the $\operatorname{hocolim}_{Q'} \mathcal{E}$ up to an $\mathfrak{S}_n$-equivariant homotopy. First note that if $q, r \in Q$ are such that $q \geq r$ and $q \in Q'$, then $r \in Q'$. In particular, if $\hat{1} \in Q'$, then $Q' = Q$, where $\hat{1}$ is the maximal element of Q.

Lemma 4.3.5.

(i) *If $\hat{1} \in Q'$, that is $Q' = Q$, then there exists an $\mathfrak{S}_n$-equivariant homotopy equivalence*

$$\operatorname{hocolim}_{Q'} \mathcal{E} \simeq \Delta(Q' \backslash \hat{1}) * \Delta(\varphi^{-1}(\hat{1}))$$

where $\hat{1}$ is the maximal element of Q, and $\dim\left(\Delta(\varphi^{-1}(\hat{1}))\right) = nc - m - 2n + 1$. In particular, there exists an $\mathfrak{S}_n$-equivariant map

$$\eta\colon \operatorname{hocolim}_{Q'} \mathcal{E} \longrightarrow \Delta(Q' \backslash \hat{1}) * \Delta(\varphi^{-1}(\hat{1})).$$

(ii) *If $\hat{1} \notin Q'$ then there exists an $\mathfrak{S}_n$-equivariant homotopy equivalence*

$$\operatorname{hocolim}_{Q'} \mathcal{E} \simeq \Delta(Q'),$$

where $\dim(\Delta(Q')) \leq n - 2$. In particular, there exists an $\mathfrak{S}_n$-equivariant map

$$\eta\colon \operatorname{hocolim}_{Q'} \mathcal{E} \longrightarrow \Delta(Q').$$

Proof. (i) Let us first consider the case when $\hat{1} \in Q'$. Then, since all the maps of the diagram $\mathcal{E}$ are constant maps, the Wedge Lemma [84, Lem. 4.9] yields a homotopy equivalence

$$\operatorname{hocolim}_{Q'} \mathcal{E} \simeq \bigvee_{q \in Q'} (\Delta(Q'_{<q}) * \mathcal{E}(q)) \vee \Delta(Q') \simeq \Delta(Q' \backslash \hat{1}) * \Delta(\varphi^{-1}(\hat{1})).$$

Furthermore, since for $q \neq \hat{1}$ all the spaces $\mathcal{E}(q)$ are points, this homotopy equivalence is an $\mathfrak{S}_n$-equivariant homotopy equivalence.

The poset $\varphi^{-1}(\hat{1})$ consists of all points $p_\Lambda \in P$ that correspond to matrices which have zeros in all columns. Since it is a subposet of $P(\mathcal{A})$, every element of $\varphi^{-1}(\hat{1})$ must contain at least $m-c+1$ zeros in each column and at most $n-1$ zeros in each row. The partial order is given by

$$p_\Lambda \leq p_{\Lambda'} \iff (\forall j \in [m-1])(\forall k \in [n])\ a_{jk} = 0 \Rightarrow a'_{jk} = 0,$$

where $p_\Lambda = (a_{jk})$ and $p_{\Lambda'} = (a'_{jk})$. Maximal chains in the poset $\varphi^{-1}(\hat{1})$ can be obtained by removing zeros from a maximal element p_Λ one by one, taking care that there must be at least $m-c+1$ zeros in each column. Maximal elements of $\varphi^{-1}(\hat{1})$ have exactly one non-zero element in each row, thus $(m-1)(n-1)$ zeros. Since $\hat{1} \in Q'$ minimal elements, however, have $m-c+1$ zeros in each column, thus $n(m-c+1)$ zeros. Therefore, the length of a maximal chain in $\varphi^{-1}(\hat{1})$, and consequently the dimension of its order complex, is $nc-m-2n+1$. In particular, we obtained that when $\hat{1} \in Q'$ then $nc-m-2n+1 \geq 0$, or equivalently $n(c-2)+1 \geq m$.

(ii) Let $\hat{1} \notin Q'$. Then it is not hard to see that $n(c-2)+1 < m$. Again, the Wedge Lemma [84, Lem. 4.9] yields a homotopy equivalence

$$\operatorname{hocolim}_{Q'} \mathcal{E} \simeq \bigvee_{q \in Q'} (\Delta(Q'_{<q}) * \mathcal{E}(q)) \vee \Delta(Q') \simeq \Delta(Q'),$$

since now all the spaces $\mathcal{E}(q)$ are points for $q \in Q'$.

From the assumption $\hat{1} \notin Q'$ we get that $Q' \subseteq Q \backslash \hat{1}$ and consequently $\Delta(Q') \subseteq \Delta(Q \backslash \hat{1})$. On the other hand $\Delta(Q \backslash \hat{1})$ is homeomorphic with the boundary of an $(n-1)$-dimensional simplex and so $\dim(\Delta(Q')) \leq n-2$. □

In the example for parameters $n=2, m=3, c=3$, the poset $\varphi^{-1}(\hat{1})$ consists of two points and no relations between them, as shown in red in Figure 4.1.

Now we have assembled all the ingredients of the proof of the central result about the non-existence of an $\mathfrak{S}_n$-equivariant map $\mathrm{EMP}(\mu_m, n) \longrightarrow \bigcup \mathcal{A}$.

Theorem 4.3.6. *Let $d \geq 2$, $m \geq 2$, and $c \geq 2$ be integers, and let $n = p^k$ be a prime power. If $m \geq n(c-d) + \frac{dn}{p} - \frac{n}{p} + 1$, then there is no $\mathfrak{S}_n$-equivariant map*

$$\mathrm{EMP}(\mu_m, n) \longrightarrow \bigcup \mathcal{A}(m,n,c), \tag{4.3}$$

where μ_m is a finite absolutely continuous measure on $\mathbb{R}^d$, and the affine arrangement $\mathcal{A}(m,n,c)$ is as defined in line (4.1).

Proof. Let $n = p^k$ be a prime power. Denote by $G \cong (\mathbb{Z}/p)^k$ a subgroup of the symmetric group $\mathfrak{S}_n$ given by the regular embedding (reg): $G \longrightarrow \mathfrak{S}_n$, for more details see for example [1, Ex. III.2.7].

In order to prove the non-existence of an $\mathfrak{S}_n$-equivariant map (4.3), we proceed by contradiction. Let $f\colon \mathrm{EMP}(\mu_m, n) \longrightarrow \bigcup \mathcal{A}(m,n,c)$ be an $\mathfrak{S}_n$-equivariant map. Then from Lemmas 4.3.1, 4.3.2, 4.3.3, 4.3.4 and 4.3.5 we get the following composition of $\mathfrak{S}_n$-equivariant maps

$$\begin{array}{ccccccccc} \mathrm{EMP}(\mu_m,n) & \xrightarrow{f} & \bigcup\mathcal{A} & \xrightarrow{\beta} & \mathrm{hocolim}_{P(\mathcal{A})}\,\mathcal{C} & \xrightarrow{\gamma} & \mathrm{hocolim}_{Q'}\,\mathcal{D} & \xrightarrow{\delta} & \mathrm{hocolim}_{Q'}\,\mathcal{E} \\ \alpha\uparrow & & & & & & & & \downarrow\eta \\ \mathrm{Conf}(\mathbb{R}^d,n) & & & & \xrightarrow{g=\eta\circ\delta\circ\gamma\circ\beta\circ f\circ\alpha} & & & & X \end{array}$$

where

$$X=\begin{cases}\Delta(Q'\backslash\hat{1})*\Delta(\varphi^{-1}(\hat{1})), & \text{if } \hat{1}\in Q',\\ \Delta(Q'), & \text{if } \hat{1}\notin Q'.\end{cases}$$

We will reach contradiction with the assumption that the map f exists by proving that, under the assumption of the theorem, the map g cannot exist. More precisely, we will prove that there cannot exist a G-equivariant map

$$\mathrm{Conf}(\mathbb{R}^d,n)\longrightarrow X. \tag{4.4}$$

Our argument starts with the $\mathfrak{S}_n$ and also G-equivariant map $g\colon \mathrm{Conf}(\mathbb{R}^d,n)\longrightarrow X$. The map g induces a morphism between Borel construction fibrations:

$$\begin{array}{ccc} EG\times_G \mathrm{Conf}(\mathbb{R}^d,n) & \xrightarrow{\mathrm{id}\times_G g} & EG\times_G X \\ \lambda\downarrow & & \downarrow\rho \\ BG & \xrightarrow{\mathrm{id}} & BG, \end{array}$$

that induces a morphism between corresponding Serre spectral sequences

$$E_*^{*,*}(g)\colon E_*^{*,*}(\rho)\longrightarrow E_*^{*,*}(\lambda).$$

The crucial property of the morphism $E_*^{*,*}(g)$ we use is that $E_2^{*,0}(g)=\mathrm{id}$. A contradiction with the assumption that there is a map g is going to be obtained from analysis of the morphism $E_*^{*,*}(g)$. For that we first describe the spectral sequences $E_*^{*,*}(\lambda)$ and $E_*^{*,*}(\rho)$.

The Serre spectral sequence of the fibration

$$\mathrm{Conf}(\mathbb{R}^d,n)\longrightarrow EG\times_G \mathrm{Conf}(\mathbb{R}^d,n)\longrightarrow BG$$

has the E_2-term given by

$$E_2^{i,j}(\lambda)=H^i(BG;\mathcal{H}^j(\mathrm{Conf}(\mathbb{R}^d,n);\mathbb{F}_p))\cong H^i(G;H^j(\mathrm{Conf}(\mathbb{R}^d,n);\mathbb{F}_p)).$$

Here $H^i(BG;\mathcal{H}^j(Y;\mathbb{F}_p))$ denotes the cohomology of BG with local coefficients in $H^j(Y;\mathbb{F}_p)$ determined by the action of the fundamental group of the base space $\pi_1(BG)\cong G$. The second description uses the fact that cohomology of the classifying space BG of the group G is by definition the cohomology of the group G with coefficients in the G-module $H^j(\mathrm{Conf}(\mathbb{R}^d,n);\mathbb{F}_p)$. For more details on the cohomology with local coefficients consult for example [44, Sec. 3.H]. The spectral sequence $E_*^{*,*}(\lambda)$ was completely determined in the case $k=1$, i.e., $n=p$ a prime, by Cohen [27, Thm. 8.2] and recently in [18, Thm. 6.1]. A partial description of $E_*^{*,*}(\lambda)$ in the case $k \geq 2$ was given in [18, Thm. 6.3 and Thm. 7.1]. In particular, for $k=1$

$$E_2^{*,*}(\lambda) \cong E_3^{*,*}(\lambda) \cong \cdots \cong E_{(d-1)(n-1)+1}^{*,*}(\lambda) \qquad \text{and} \qquad E_{(d-1)(n-1)+2}^{*,*}(\lambda) \cong \cdots \cong E_\infty^{*,*}(\lambda), \tag{4.5}$$

while for $k \geq 2$

$$E_2^{*,*}(\lambda) \cong E_3^{*,*}(\lambda) \cong \cdots \cong E_{(d-1)(n-\frac{n}{p})+1}^{*,*}(\lambda). \tag{4.6}$$

In the second step we consider the Serre spectral sequence of the fibration

$$X \longrightarrow EG \times_G X \longrightarrow BG$$

whose E_2-term is given by

$$E_2^{i,j}(\rho) = H^i(BG; \mathcal{H}^j(X; \mathbb{F}_p)) \cong H^i(G; H^j(X; \mathbb{F}_p)).$$

We conclude the proof by considering two separate cases.

(a) Let $\hat{1} \in Q'$, or equivalently $nc - m - 2n + 1 \geq 0$. Then the simplicial complex $X = \Delta(Q' \backslash \hat{1}) * \Delta(\varphi^{-1}(\hat{1}))$ is at most $(nc-m-n)$-dimensional, implying that $E_2^{i,j}(\rho) = 0$ for all $j \geq nc - m - n + 1$. Consequently,

$$E_{nc-m-n+2}^{i,j}(\rho) \cong E_{nc-m-n+3}^{i,j}(\rho) \cong \cdots \cong E_\infty^{i,j}(\rho). \tag{4.7}$$

Next, since the path-connected simplicial complex X does not have fixed points with respect to the action of the elementary abelian group G, a consequence of the localization theorem [46, Cor. 1, p. 45] implies that $H^*(G; \mathbb{F}_p) \cong E_2^{*,0}(\rho) \ncong E_\infty^{*,0}(\rho)$. Having in mind (4.7), we conclude that

$$H^*(G; \mathbb{F}_p) \cong E_2^{*,0}(\rho) \ncong E_{nc-m-n+2}^{*,0}(\rho).$$

For our proof, without loss of generality, we can assume that

$$H^*(G; \mathbb{F}_p) \cong E_2^{*,0}(\rho) \cong E_{nc-m-n+1}^{*,0}(\rho) \ncong E_{nc-m-n+2}^{*,0}(\rho). \tag{4.8}$$

Now, from the assumption on m, we deduce that for $k=1$

$$(d-1)(n-1) + 1 \geq nc - m - n + 2,$$

and for $k \geq 2$

$$(d-1)(n - \tfrac{n}{p}) + 1 \geq nc - m - n + 2.$$

Hence the fact that $E_2^{*,0}(g) = \mathrm{id}$, in combination with relations (4.5), (4.6) and (4.8), yields a contradiction: the map $E_{nc-m-n+2}^{*,0}(g)$ sends zero to a non-zero element. This concludes the proof of the theorem in the case when $nc - 2n + 1 \geq m$.

(b) Let $\hat{1} \notin Q'$, or equivalently $nc - m - 2n + 1 < 0$. The simplicial complex $X = \Delta(Q')$ is at most $(n-2)$-dimensional. Hence, $E_2^{i,j}(\rho) = 0$ for all $j \geq n-1$, and

$$E_n^{i,j}(\rho) \cong E_{n+1}^{i,j}(\rho) \cong \cdots \cong E_\infty^{i,j}(\rho). \tag{4.9}$$

The simplicial complex X is path-connected and without fixed points with respect to the action of the elementary abelian group G. Consequence of the localization theorem [46, Cor. 1, p. 45] implies that $H^*(G;\mathbb{F}_p) \cong E_2^{*,0}(\rho) \ncong E_\infty^{*,0}(\rho)$. From (4.9) we have that

$$H^*(G;\mathbb{F}_p) \cong E_2^{*,0}(\rho) \ncong E_n^{*,0}(\rho).$$

For our proof, without loss of generality, we can assume that

$$H^*(G;\mathbb{F}_p) \cong E_2^{*,0}(\rho) \cong E_{n-1}^{*,0}(\rho) \ncong E_n^{*,0}(\rho). \tag{4.10}$$

Now, we need that for $k = 1$

$$(d-1)(n-1)+1 \geq n,$$

and for $k \geq 2$

$$(d-1)(n-\tfrac{n}{p})+1 \geq n$$

is satisfied. Indeed, these conditions are satisfied for $d \geq 2, p \geq 2$ and $n = p^k$. Thus, the fact that $E_2^{*,0}(g) = \mathrm{id}$ with (4.5), (4.6) and (4.10) gives a contradiction: the map $E_n^{*,0}(g)$ sends zero to a non-zero element. This concludes the proof of the theorem in the case when $nc - 2n + 1 < m$. □

The previous proof can also be phrased in the language of the iterated index theory introduced by Volovikov in [82].

Non-existence of the equivariant map $\mathrm{EMP}(\mu,n) \longrightarrow \bigcup \widetilde{\mathcal{A}}(m,n,c)$

Motivated by Theorem 4.2.2, in this section we study the (non-)existence of an $\mathfrak{S}_n$-equivariant map

$$\mathrm{EMP}(\mu,n) \longrightarrow \bigcup \widetilde{\mathcal{A}}(m,n,c)$$

for different values of parameters d, m, n and c. Following the structure of Section 4.3.1, we first prove a few auxiliary lemmas in order to arrive to the topological result, Theorem 4.2.2, at the end of this section.

Recalling that a subspace of $\mathrm{EMP}(\mu,n)$ consisting only of regular convex partitions can be identified with the configuration space $\mathrm{Conf}(\mathbb{R}^d,n)$, see [21, Sec. 2] for more details, we obtain the following lemma.

Lemma 4.3.7. *There exists an $\mathfrak{S}_n$-equivariant map*

$$\widetilde{\alpha}\colon \mathrm{Conf}(\mathbb{R}^d,n) \longrightarrow \mathrm{EMP}(\mu,n).$$

Denote by $\widetilde{P} = P(\widetilde{\mathcal{A}})$ the intersection poset of the affine arrangement $\widetilde{\mathcal{A}}$. Its elements are given by

$$\widetilde{p}_\Lambda = \bigcap_{(i,I)\in\Lambda} \widetilde{L}_{I,i} = \{(y_{jk}) \in \widetilde{V} \subseteq \mathbb{R}^{m\times n} : y_{ji} = 0, \text{ for all } 1 \leq i \leq n \text{ and } j \in I_i\},$$

where $\Lambda \subseteq [n] \times \binom{[m]}{m-c+1}$ and $I_i = \bigcup_{(i,I)\in\Lambda} I$. An element $\widetilde{p}_\Lambda$ can also be seen as an $m \times n$ matrix (a_{jk}), where $a_{jk} = 0$ if and only if $j \in I_k$.

Next we consider a $\widetilde{P}$-diagram $\widetilde{\mathcal{C}}$ determined by the arrangement $\widetilde{\mathcal{A}} = \widetilde{\mathcal{A}}(m,n,c)$. More precisely, we define $\widetilde{\mathcal{C}}(\widetilde{p}_\Lambda) = \widetilde{p}_\Lambda$ and $\widetilde{\mathcal{C}}(\widetilde{p}_{\Lambda'} \supseteq \widetilde{p}_{\Lambda''})\colon \widetilde{p}_{\Lambda''} \longrightarrow \widetilde{p}_{\Lambda'}$ to be the inclusion. The Equivariant Projection Lemma [78, Lem. 2.1] implies the following.

Lemma 4.3.8. *The projection map*

$$\operatorname{hocolim}_{\widetilde{P}} \widetilde{\mathcal{C}} \longrightarrow \operatorname{colim}_{\widetilde{P}} \widetilde{\mathcal{C}} = \bigcup \widetilde{\mathcal{A}}$$

is an $\mathfrak{S}_n$*-equivariant homotopy equivalence. In particular, there exists an* $\mathfrak{S}_n$*-equivariant map*

$$\widetilde{\beta}\colon \bigcup \widetilde{\mathcal{A}} \longrightarrow \operatorname{hocolim}_{\widetilde{P}} \widetilde{\mathcal{C}}.$$

Recall that Q denotes the face poset of an $(n-1)$-dimensional simplex, and define a map $\widetilde{\varphi} : \widetilde{P} \to Q$ by

$$\widetilde{\varphi}(\widetilde{p}_\Lambda) = \{i \in [n] : (i, I) \in \Lambda \text{ for some } I \subset [m]\}.$$

Additionally, denote the poset $\widetilde{\varphi}(\widetilde{P}) \subseteq Q$ by $\widetilde{Q}'$. Note that if $q, r \in Q$ are such that $q \in \widetilde{Q}'$ and $r \leq q$, then r is also an element of $\widetilde{Q}'$. In particular, if $q = \hat{1}$ is the maximal element of Q and $q \in \widetilde{Q}'$, then $\widetilde{Q}' = Q$.

Let $\widetilde{\mathcal{D}}$ be the homotopy pushdown of the diagram $\widetilde{\mathcal{C}}$ along the map $\widetilde{\varphi}$ over $\widetilde{Q}'$. This means that

$$\widetilde{\mathcal{D}}(q) = \operatorname{hocolim}_{\widetilde{\varphi}^{-1}(\widetilde{Q}'_{\geq q})} \widetilde{\mathcal{C}}|_{\widetilde{\varphi}^{-1}(\widetilde{Q}'_{\geq q})} \simeq \Delta(\widetilde{\varphi}^{-1}(\widetilde{Q}'_{\geq q}))$$

for $q \in \widetilde{Q}'$, and the map $\widetilde{\mathcal{D}}(q \geq r)\colon \widetilde{\mathcal{D}}(q) \longrightarrow \widetilde{\mathcal{D}}(r)$ is the corresponding inclusion for every $q \geq r$ in $\widetilde{Q}'$. Once more, the Homotopy Pushdown Lemma [84, Prop. 3.12] adapted to equivariant setting yields the following fact.

Lemma 4.3.9. *There is an* $\mathfrak{S}_n$*-equivariant homotopy equivalence*

$$\operatorname{hocolim}_{\widetilde{Q}'} \widetilde{\mathcal{D}} \longrightarrow \operatorname{hocolim}_{\widetilde{P}} \widetilde{\mathcal{C}}.$$

In particular, there exists an $\mathfrak{S}_n$*-equivariant map*

$$\widetilde{\gamma}\colon \operatorname{hocolim}_{\widetilde{P}} \widetilde{\mathcal{C}} \longrightarrow \operatorname{hocolim}_{\widetilde{Q}'} \widetilde{\mathcal{D}}.$$

Finally, we consider another $\widetilde{Q}'$-diagram $\widetilde{\mathcal{E}}$ by setting for $q \in \widetilde{Q}'$ that

$$\widetilde{\mathcal{E}}(q) = \begin{cases} \Delta(\widetilde{\varphi}^{-1}(\hat{1})), & \text{if } q = \hat{1} \in \widetilde{Q}' \text{ is the maximal element of } Q, \\ \text{pt}, & \text{otherwise,} \end{cases}$$

and the map $\widetilde{\mathcal{E}}(q \geq r)$ to be the constant map for every $q \geq r$ in $\widetilde{Q}'$. Similarly as we have done it in Section 4.3.1, we define a morphism of diagrams $(\widetilde{\Psi}, \widetilde{\psi})\colon \widetilde{\mathcal{D}} \longrightarrow \widetilde{\mathcal{E}}$, where $\widetilde{\psi}\colon \widetilde{Q}' \longrightarrow \widetilde{Q}'$ is the identity map, and $\widetilde{\Psi}(q)\colon \widetilde{\mathcal{D}}(q) \longrightarrow \widetilde{\mathcal{E}}(q)$ is the identity map when $q = \hat{1}$ is the maximal element in Q, and constant map otherwise. Since the morphism $(\widetilde{\Psi}, \widetilde{\psi})$ of diagrams induces an $\mathfrak{S}_n$-equivariant map between associated homotopy colimits, we have established the following.

Lemma 4.3.10. *There exists an* $\mathfrak{S}_n$*-equivariant map*

$$\widetilde{\delta}\colon \operatorname{hocolim}_{\widetilde{Q}'} \widetilde{\mathcal{D}} \longrightarrow \operatorname{hocolim}_{\widetilde{Q}'} \widetilde{\mathcal{E}}.$$

Just like in Section 4.3.1, the final lemma will describe the $\operatorname{hocolim}_{\widetilde{Q}'}\widetilde{\mathcal{E}}$ up to an $\mathfrak{S}_n$-equivariant homotopy.

Lemma 4.3.11.
(i) *If $\hat{1}\in\widetilde{Q}'$, that is if $\widetilde{Q}'=Q$, then there exists an $\mathfrak{S}_n$-equivariant homotopy equivalence*

$$\operatorname{hocolim}_Q\widetilde{\mathcal{E}}\simeq\Delta(Q\backslash\hat{1})*\Delta(\widetilde{\varphi}^{-1}(\hat{1}))$$

where $\hat{1}$ is the maximal element of Q, and $\dim\left(\Delta(\widetilde{\varphi}^{-1}(\hat{1}))\right)=nc-n-\max\{m,n\}$. *In particular, there exists an $\mathfrak{S}_n$-equivariant map*

$$\widetilde{\eta}\colon\ \operatorname{hocolim}_Q\widetilde{\mathcal{E}}\longrightarrow\Delta(Q\backslash\hat{1})*\Delta(\widetilde{\varphi}^{-1}(\hat{1})).$$

(ii) *If $\hat{1}\notin\widetilde{Q}'$ then there exists an $\mathfrak{S}_n$-equivariant homotopy equivalence*

$$\operatorname{hocolim}_{\widetilde{Q}'}\widetilde{\mathcal{E}}\simeq\Delta(\widetilde{Q}'),$$

where $\dim(\Delta(\widetilde{Q}'))\leq n-2$. *In particular, there exists an $\mathfrak{S}_n$-equivariant map*

$$\widetilde{\eta}\colon\ \operatorname{hocolim}_{\widetilde{Q}'}\widetilde{\mathcal{E}}\longrightarrow\Delta(\widetilde{Q}').$$

Proof. The proof of the claim (ii) is identical to the proof of the second part of Lemma 4.3.5. For the claim (i) it suffices to compute the dimension of the simplicial complex $\Delta(\widetilde{\varphi}^{-1}(\hat{1}))$, since the rest of the proof follows the lines of the proof of the first part of Lemma 4.3.5.

The elements of the poset $\widetilde{\varphi}^{-1}(\hat{1})$ are presented by matrices $\widetilde{p}_\Lambda=(a_{jk})$ that contain zeros in every column. The partial order is given by

$$\widetilde{p}_\Lambda\leq\widetilde{p}_{\Lambda'}\iff(\forall j\in[m])(\forall k\in[n])\ a_{jk}=0\Rightarrow a'_{jk}=0,$$

where $\widetilde{p}_\Lambda=(a_{jk})$ and $\widetilde{p}_{\Lambda'}=(a'_{jk})$ are elements of the poset $\widetilde{\varphi}^{-1}(\hat{1})\subseteq\widetilde{P}$. Maximal chains in $\widetilde{\varphi}^{-1}(\hat{1})$ can be obtained by removing zeros one by one from a matrix that represents a maximal element, taking care of the fact that every column has to contain at least $m-c+1$ zeros. The maximal elements are presented by matrices that have at most $n-1$ zeros in each row, and at most $m-1$ zeros in each column. Thus, maximal elements are presented by matrices with $mn-\max(m,n)$ zeros. The minimal elements, on the other hand, are presented by matrices that contain $n(m-c+1)$ zeros. Therefore, the dimension of $\Delta(\widetilde{\varphi}^{-1}(\hat{1}))$ is $nc-n-\max\{m,n\}\geq 0$. Since $c\geq 2$, this implies that $n(c-1)\geq m$. □

Now we are ready to prove the central result about the non-existence of an $\mathfrak{S}_n$-equivariant map $\mathrm{EMP}(\mu,n)\longrightarrow\bigcup\widetilde{\mathcal{A}}$.

Theorem 4.3.12. *Let $d\geq 2$, $m\geq 2$, and $c\geq 2$ be integers, and let $n=p^k$ be a prime power. If*
(a) *$n(c-1)\geq m$ and $\max\{m,n\}\geq n(c-d)+\frac{dn}{p}-\frac{n}{p}+n$, or*
(b) *$n(c-1)<m$,*

then there is no $\mathfrak{S}_n$-equivariant map

$$\mathrm{EMP}(\mu, n) \longrightarrow \bigcup \widetilde{\mathcal{A}}(m, n, c), \tag{4.11}$$

where $\mu = \mu_1 + \cdots + \mu_m$ is the sum of m finite absolutely continuous measures on $\mathbb{R}^d$, and the affine arrangement $\widetilde{\mathcal{A}}(m, n, c)$ is as defined in line (4.2)*.*

Proof. It is not surprising that this proof will follow the lines of the proof of Theorem 4.3.6. Let $n = p^k$ be a prime power and denote by $G \cong (\mathbb{Z}/p)^k$ a subgroup of the symmetric group $\mathfrak{S}_n$ given by the regular embedding $(\mathrm{reg})\colon G \longrightarrow \mathfrak{S}_n$.

The proof will proceed by contradiction. Therefore, assume that

$$\widetilde{f}\colon\ \mathrm{EMP}(\mu_m, n) \longrightarrow \bigcup \widetilde{\mathcal{A}}(m, n, c)$$

is an $\mathfrak{S}_n$-equivariant map. From Lemmas 4.3.7, 4.3.8, 4.3.9, 4.3.10 and 4.3.11 we again get a composition of $\mathfrak{S}_n$-equivariant maps

$$\begin{array}{ccccccccc}
\mathrm{EMP}(\mu, n) & \xrightarrow{\widetilde{f}} & \bigcup \widetilde{A} & \xrightarrow{\widetilde{\beta}} & \mathrm{hocolim}_{\widetilde{P}}\, \widetilde{\mathcal{C}} & \xrightarrow{\widetilde{\gamma}} & \mathrm{hocolim}_{\widetilde{Q}'}\, \widetilde{\mathcal{D}} & \xrightarrow{\widetilde{\delta}} & \mathrm{hocolim}_{\widetilde{Q}'}\, \widetilde{\mathcal{E}} \\
\uparrow{\scriptstyle \widetilde{\alpha}} & & & & & & & & \downarrow{\scriptstyle \widetilde{\eta}} \\
\mathrm{Conf}(\mathbb{R}^d, n) & & & & \overset{\widetilde{g} = \widetilde{\eta} \circ \widetilde{\delta} \circ \widetilde{\gamma} \circ \widetilde{\beta} \circ \widetilde{f} \circ \widetilde{\alpha}}{\dashrightarrow} & & & & \widetilde{X}
\end{array}$$

where

$$\widetilde{X} = \begin{cases} \Delta(Q \backslash \hat{1}) * \Delta(\widetilde{\varphi}^{-1}(\hat{1})), & \text{if } \hat{1} \in \widetilde{Q}', \\ \Delta(\widetilde{Q}'), & \text{if } \hat{1} \notin \widetilde{Q}'. \end{cases}$$

It suffices to show that the map $\widetilde{g}$ cannot exist, since that would contradict the existence of the map $\widetilde{f}$. Actually, we will prove here that there is no G-equivariant map

$$\mathrm{Conf}(\mathbb{R}^d, n) \longrightarrow \widetilde{X}. \tag{4.12}$$

We start by considering the $\mathfrak{S}_n$ and also G-equivariant map $\widetilde{g}\colon \mathrm{Conf}(\mathbb{R}^d, n) \longrightarrow \widetilde{X}$. It induces a morphism between Borel construction fibrations:

$$\begin{array}{ccc}
EG \times_G \mathrm{Conf}(\mathbb{R}^d, n) & \xrightarrow{\mathrm{id} \times_G \widetilde{g}} & EG \times_G \widetilde{X} \\
{\scriptstyle \lambda}\downarrow & & \downarrow{\scriptstyle \widetilde{\rho}} \\
BG & \xrightarrow{\mathrm{id}} & BG,
\end{array}$$

that induces a morphism between corresponding Serre spectral sequences

$$E_*^{*,*}(\widetilde{g})\colon E_*^{*,*}(\widetilde{\rho}) \longrightarrow E_*^{*,*}(\lambda).$$

Like in the proof of Theorem 4.3.6, we use the fact that $E_2^{*,0}(\widetilde{g}) = \mathrm{id}$. Next we analyse the morphism $E_*^{*,*}(\widetilde{g})$. Since the spectral sequence $E_*^{*,*}(\lambda)$ was already described in the proof of Theorem 4.3.6, we concentrate here on the spectral sequence $E_*^{*,*}(\widetilde{\rho})$.

The Serre spectral sequence of the fibration

$$\widetilde{X} \longrightarrow EG \times_G \widetilde{X} \longrightarrow BG$$

has the E_2-term given by

$$E_2^{i,j}(\widetilde{\rho}) = H^i(BG; \mathcal{H}^j(\widetilde{X}; \mathbb{F}_p)) \cong H^i(G; H^j(\widetilde{X}; \mathbb{F}_p)).$$

In order to conclude the proof, we consider two separate cases depending on the values of m and $n(c-1)$.

(a) Let $\hat{1} \in Q'$ and let m satisfy the condition of the theorem. Then $n(c-1) \geq m$, so the simplicial complex $\widetilde{X} = \Delta(Q \backslash \hat{1}) * \Delta(\widetilde{\varphi}^{-1}(\hat{1}))$ is at most $(nc - \max\{m,n\} - 1)$-dimensional. This implies that $E_2^{i,j}(\widetilde{\rho}) = 0$ for all $j \geq nc - \max\{m,n\}$, and consequently,

$$E_{nc-\max\{m,n\}+1}^{i,j}(\widetilde{\rho}) \cong E_{nc-\max\{m,n\}+2}^{i,j}(\widetilde{\rho}) \cong \cdots \cong E_\infty^{i,j}(\widetilde{\rho}). \tag{4.13}$$

Once more a consequence of the localization theorem [46, Cor. 1, p. 45] implies that $H^*(G;\mathbb{F}_p) \cong E_2^{*,0}(\widetilde{\rho}) \ncong E_\infty^{*,0}(\widetilde{\rho})$, since the path-connected simplicial complex $\widetilde{X}$ does not have fixed points with respect to the action of the elementary abelian group G. Having in mind (4.13) we conclude that

$$H^*(G;\mathbb{F}_p) \cong E_2^{*,0}(\widetilde{\rho}) \ncong E_{nc-\max\{m,n\}+1}^{*,0}(\widetilde{\rho}).$$

For our proof, without loss of generality, we can assume that

$$H^*(G;\mathbb{F}_p) \cong E_2^{*,0}(\widetilde{\rho}) \cong E_{nc-\max\{m,n\}-2}^{*,0}(\widetilde{\rho}) \ncong E_{nc-\max\{m,n\}-1}^{*,0}(\widetilde{\rho}). \tag{4.14}$$

Now, the assumption on m and n, means for $k = 1$

$$(d-1)(n-1) + 1 \geq nc - \max\{m,n\} + 1,$$

and for $k \geq 2$

$$(d-1)(n - \tfrac{n}{p}) + 1 \geq nc - \max\{m,n\} + 1.$$

Therefore, the relations (4.5), (4.6) and (4.14), together with the fact that $E_2^{*,0}(\widetilde{g}) = \mathrm{id}$, yield a contradiction: the map $E_{nc-\max\{m,n\}+1}^{*,0}(\widetilde{g})$ sends zero to a non-zero element. This concludes the proof of the theorem in the case when $n(c-1) \geq m$.

(b) Let $\hat{1} \notin \widetilde{Q}'$, or equivalently $n(c-1) < m$. The simplicial complex $\widetilde{X} = \Delta(\widetilde{Q}')$ is at most $(n-2)$-dimensional, by Lemma 4.3.11, which implies that $E_2^{i,j}(\widetilde{\rho}) = 0$ for all $j \geq n-1$. Consequently

$$E_n^{i,j}(\widetilde{\rho}) \cong E_{n+1}^{i,j}(\widetilde{\rho}) \cong \cdots \cong E_\infty^{i,j}(\widetilde{\rho}). \tag{4.15}$$

For the same reason as above, we have $H^*(G;\mathbb{F}_p) \cong E_2^{*,0}(\widetilde{\rho}) \ncong E_\infty^{*,0}(\widetilde{\rho})$. This combined with (4.15) yields

$$H^*(G;\mathbb{F}_p) \cong E_2^{*,0}(\widetilde{\rho}) \ncong E_n^{*,0}(\widetilde{\rho}).$$

Again, without loss of generality, we can assume that

$$H^*(G;\mathbb{F}_p) \cong E_2^{*,0}(\widetilde{\rho}) \cong E_{n-1}^{*,0}(\widetilde{\rho}) \ncong E_n^{*,0}(\widetilde{\rho}). \tag{4.16}$$

In order to complete the proof we need that for $k = 1$

$$(d-1)(n-1)+1 \geq n,$$

and for $k \geq 2$

$$(d-1)(n-\tfrac{n}{p})+1 \geq n.$$

Indeed, both of these inequalities are satisfied, thus the fact that $E_2^{*,0}(\widetilde{g}) = \mathrm{id}$ with (4.5), (4.6) and (4.16) gives a contradiction: the map $E_n^{*,0}(\widetilde{g})$ sends zero to a non-zero element. This concludes the proof of the theorem in the case when $n(c-1) < m$. □

Proofs

Finally, in this section proofs of Theorems 4.1.3, 4.1.4 and 4.1.5 will be presented. The proof of Theorem 4.1.3 is completely geometric and it does not involve any topological methods. In particular, it is independent from Sections 4.2 and 4.3. On the other hand, the proofs of Theorem 4.1.4 and Theorem 4.1.5 heavily depend on the topological results from the previous sections.

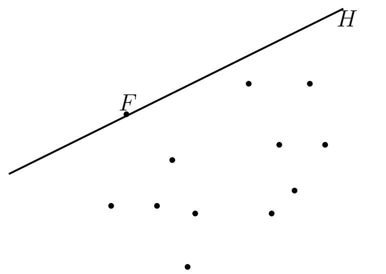
Proof of Theorem 4.1.3

Let d, m, n and c be integers that satisfy assumptions of the theorem. Since the measures $\mu_1, \dots, \mu_m$ are positive and absolutely continuous, the interiors of their supports are non-empty. For every $1 \leq j \leq m$, choose a point $v_j \in \mathrm{int}(\mathrm{supp}(\mu_j))$ in the interior of the support of the measure m_j, and denote by V the set $\{v_1, \dots, v_m\}$. Perturb the points $v_1, \dots, v_m$ if necessary, so that they are in general position, i.e., no $d+1$ of them lie in the same affine hyperplane. The set $P = \mathrm{conv}(V)$ is a d-dimensional polytope in $\mathbb{R}^d$. Choose any $(d-2)$-dimensional face F of the polytope P. Since the points in V are in general position, the face F is a simplex, thus it has $d-1$ vertices. First we want to find an affine hyperplane in $\mathbb{R}^d$ that contains F and such that there are exactly $c-d$ points of V on one of its sides. This cuts $\mathbb{R}^d$ into two half-spaces, one of which has positive measure with respect to at least c of the measures $\mu_1, \dots, \mu_m$, because it intersects interiors of supports of at least c measures. Such a hyperplane exists. Indeed, since F is a face of $P = \mathrm{conv}(V)$, there is a supporting hyperplane H for F, that is a hyperplane that contains the face F such that all other points of P lie on one side of its sides, Figure 4.2(a). Rotate the hyperplane H around the $(d-2)$-dimensional subspace of $\mathbb{R}^d$ spanned by the face F to the position in which there are $c-d$ points on one of its sides and H contains another point of V, see Figure 4.2(b). Denote this point by w, and denote the open half-space that contains $c-d$ points of V by H^+, and by H^- denote the other open half-space determined by H .

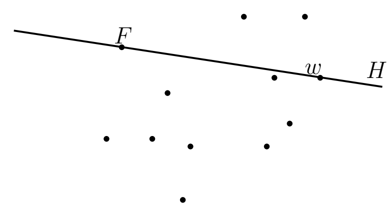
Let V^- be the set $V \cap H^-$, whose cardinality is $m-c$. Consider all half-hyperplanes that contain F in the boundary and a point of V^- in the relative interior. Since the set V is in general position, there are exactly $m-c$ such half-hyperplanes. Label them $K_1, \dots, K_{m-c}$ in order, starting from the half-hyperplane that forms the smallest angle with the half-hyperplane containing F in its boundary and w in its relative interior, as shown in Figure 4.2(c). The union

$$H \cup \{K_{c-2}, K_{2(c-2)}, \dots, K_{(n-2)(c-2)}\}$$

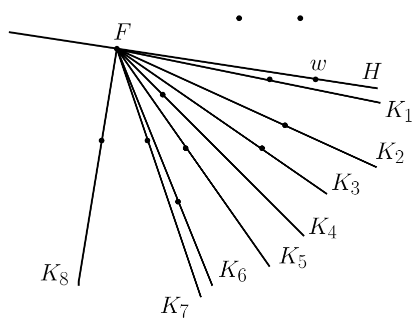
is an n-fan and every region defined by it intersects interiors of supports of at least c of the measures $\mu_1, \dots, \mu_m$. An example for $d = 2$, $n = 5$ and $c = 4$ is shown in Figure 4.2(d).



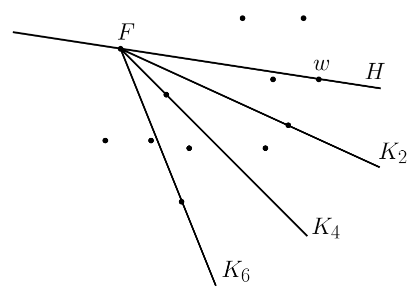
(a) The face F and the hyperplane H.



(b) The face F, the point w and the final position of the hyperplane H.



(c) Labellings of the half-hyperplanes in H^-.



(d) A 5-fan that cuts $\mathbb{R}^2$ into convex pieces so that each piece has positive measure with respect to at least 4 measures.

Figure 4.2: An example of a fan partition of $\mathbb{R}^2$ for $n = 5$ and $c = 4$.

Remark 4.4.1. As a consequence of the previous proof, there is a convex partition $(C_1, \dots, C_n)$ of $\mathbb{R}^d$, such that each piece C_i has positive measure with respect to at least c of the measures $\mu_1, \dots, \mu_m$, and additionally all pieces $C_1, \dots, C_n$ have positive measure with respect to $d-1$ measures $\mu_{j_1}, \dots, \mu_{j_{d-1}}$, where $F = \operatorname{conv}\{v_{j_1}, \dots, v_{j_{d-1}}\}$ and $v_{j_k} \in \operatorname{relint}(\operatorname{supp}(\mu_{j_k}))$, for every $1 \leq k \leq d-1$. In contrast to the statement of Theorem 4.1.4, we cannot guarantee an equipartition, and we cannot choose which measure will be contained in each piece.

Proof of Theorem 4.1.4

Let $d \geq 2$, $m \geq 2$, and $c \geq 2$ be integers, and let $n \geq 2$ be a prime power. Under the assumptions of the theorem on m, Theorem 4.3.6 yields the non-existence of an $\mathfrak{S}_n$-equivariant map

$$\mathrm{EMP}(\mu_m, p) \longrightarrow \bigcup \mathcal{A}(m,n,c).$$

Consequently, Theorem 4.2.1 implies that for every collection of m measures $\mu_1, \dots, \mu_m$ in $\mathbb{R}^d$ there exists a convex partition $(C_1, \dots, C_n)$ of $\mathbb{R}^d$ with the property that each of the subsets C_i has positive measure with respect to at least c of the measures $\mu_1, \dots, \mu_m$. In other words,

$$\#\{j : 1 \leq j \leq m,\, \mu_j(C_i) > 0\} \geq c$$

for every $1 \leq i \leq n$.

Remark 4.4.2. In order to prove the non-existence of the G-equivariant map $f : \mathrm{EMP}(\mu_m, n) \longrightarrow \bigcup \mathcal{A}$, one could directly try to show that there is no G-equivariant map $\mathrm{Conf}(\mathbb{R}^d, n) \longrightarrow \bigcup \mathcal{A} = \mathrm{colim}_{P(\mathcal{A})}\, \mathcal{C}$. However, since the dimension of the order complex of $P(\mathcal{A})$ is

$$\dim(\Delta(P(\mathcal{A}))) = nc - n - c,$$

this method proves Theorem 4.1.4 only for $c \leq d$, which follows directly from the result of Soberón [75].

Proof of Theorem 4.1.5

Proof. Let $d \geq 2$ and $c \geq 2$ be integers, let $n \geq 2$ be a prime power and let $m \geq 2$ be an integer that satisfies the conditions of the theorem. Theorem 4.3.12 yields the non-existence of an $\mathfrak{S}_n$-equivariant map

$$\mathrm{EMP}(\mu, p) \longrightarrow \bigcup \widetilde{\mathcal{A}}(m,n,c),$$

and Theorem 4.2.2 implies that for every collection of m measures $\mu_1, \dots, \mu_m$ in $\mathbb{R}^d$ there exists a convex partition $(C_1, \dots, C_n)$ of $\mathbb{R}^d$ with the property that each of the subsets C_i has positive measure with respect to at least c of the measures $\mu_1, \dots, \mu_m$. In other words,

$$\#\{j : 1 \leq j \leq m,\, \mu_j(C_i) > 0\} \geq c$$

for every $1 \leq i \leq n$. □

Chapter 5

Waists of spheres

The results of this chapter are obtained jointly with Pavle V.M. Blagojević and Roman Karasev.

Introduction

Let $S^n \subset \mathbb{R}^{n+1}$ be the unit n-dimensional sphere centered at the origin. For $X \subseteq S^n$ and $\varepsilon > 0$ denote by $X + \varepsilon$ the ε-tubular neighborhood of X in S^n, i.e.,

$$X + \varepsilon = \bigcup_{x \in X} B(x, \varepsilon),$$

where $B(x, \varepsilon)$ denotes here the intersection of S^n with the open Euclidean ball in $\mathbb{R}^{n+1}$ of radius ε centered at x. Moreover, let vol denote the n-dimensional spherical volume. The following is the celebrated Gromov's waist of the sphere theorem.

Theorem 5.1.1 ([39, Sec. 1])**.** *Let $n \geq k \geq 1$ be integers, and let $f : S^n \longrightarrow \mathbb{R}^k$ be a continuous map. Then there exists $z \in \mathbb{R}^k$ such that*

$$\mathrm{vol}(f^{-1}(z) + \varepsilon) \geq \mathrm{vol}(S^{n-k} + \varepsilon),$$

for every $\varepsilon > 0$, where S^{n-k} denotes the equatorial $(n-k)$-dimensional sphere in S^n.

See, for example, the essay of Guth [41] for a nice exposition about Theorem 5.1.1, its history and relevance.

If the map f in the previous theorem is the restriction of the projection $\mathbb{R}^{n+1} \to \mathbb{R}^k$ to the first k coordinates, then $f^{-1}(0) = S^{n-k}$. Thus, informally said, the waist of the sphere theorem claims that every continuous map $f : S^n \to \mathbb{R}^k$ has a fiber that is at least as big as the largest fiber of the projection.

However, Theorem 5.1.1 does not specify which fiber of the map f is such that its tubular neighborhood has a large volume. Our main result puts some restrictions on the map f in order to claim that the tubular neighborhood of $f^{-1}(0)$ has a large volume.

Let p be a prime and let $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$ denote the cyclic group on p elements. If the action of $\mathbb{Z}_p$ on S^n is an isometry that preserves orientation, then $\mathbb{Z}_p$ also acts on $\mathrm{SO}(n+1)$, where $\mathrm{SO}(n+1)$ is the special orthogonal group. Denote by $\mathrm{Index}_{\mathbb{Z}_p}(\mathrm{SO}(n+1); \mathbb{F}_p) \subseteq H^*_{\mathbb{Z}_p}(\mathrm{B}\,\mathbb{Z}_p; \mathbb{F}_p)$

its Fadell-Husseini index. For details about the Fadell-Husseini index see the original paper of Fadell and Husseini [32] and Appendix B.

For a $\mathbb{Z}_p$-representation R, let $e(R)$ denote the Euler class of the fiber bundle

$$R \longrightarrow \mathrm{E}\,\mathbb{Z}_p \times_{\mathbb{Z}_p} R \longrightarrow \mathrm{B}\,\mathbb{Z}_p. \tag{5.1}$$

Theorem 5.1.2. *Let $n > k \geq 1$ be integers and let p be a prime. Suppose that the action of $\mathbb{Z}_p$ on S^n is free and that it is an orientation-preserving isometry. Moreover, let R be a $\mathbb{Z}_p$-representation of dimension k such that*

$$e(R) \notin \mathrm{Index}_{\mathbb{Z}_p}(\mathrm{SO}(n+1); \mathbb{F}_p).$$

If $f : S^n \longrightarrow R$ is a $\mathbb{Z}_p$-equivariant map, then

$$\mathrm{vol}(f^{-1}(0) + \varepsilon) \geq \mathrm{vol}(S^{n-k} + \varepsilon),$$

for every $\varepsilon > 0$.

Example 5.1.3. Here we give an example of a free $\mathbb{Z}_p$-action on S^n that preserves orientation.

Assume that $n = 2m + 1$ is odd and see the sphere S^n as the join of m copies of S^1. If S^1 is embedded in the complex plane $\mathbb{C}$, then the group $\mathbb{Z}_p$ acts on S^1 by multiplication with the p-th roots of 1. This induces an action on the whole S^n given by

$$g \cdot (\lambda_1 x_1 \oplus \cdots \oplus \lambda_m x_m) = \lambda_1 (g \cdot x_1) \oplus \cdots \oplus \lambda_m (g \cdot x_m),$$

for every $g \in \mathbb{Z}_p$, $x_1, \ldots, x_m \in S^1$ and $\lambda_1, \ldots, \lambda_m \geq 0$, such that $\lambda_1 + \cdots + \lambda_m = 1$. This $\mathbb{Z}_p$-action is free, preserves orientation and is an isometry on S^n.

In order to prove Theorem 5.1.1, Gromov [39] used partitions of the sphere S^n into 2^i pieces that are "flat" [39, Thm. 4.4.A], which were parametrized by the wreath product of spheres. Existence of such a partition was shown using topological methods – non-vanishing of the top Stiefel-Whitney class of a vector bundle [39, Lemma 5.1].

We partition the sphere into p^i "flat" pieces, and parametrize such partitions by the wreath product of configuration spaces. An existence of such a partition is proved in Theorem 5.2.5. The key ingredient in the proof of Theorem 5.2.5 is Theorem 5.3.10, which claims that there is no equivariant map from the wreath product of configuration spaces to a certain sphere. Since the proof of Theorem 5.3.10 depends on the equivariant obstruction theory [30, Sec. II.3], we develop an invariant CW model for the wreath product of configuration spaces.

Remark 5.1.4. In [39] beside Theorem 5.1.1, Gromov proved some results on waists for Gaussian measures. Theorem 5.3.10 can be used to prove analogues of results in [39, Sec. 3], too.

The proof of Theorem 5.1.2 is given in the next section. However, it uses Theorem 5.2.5, whose proof is presented in the last section. Before that, we develop an invariant CW model for the wreath product of configuration spaces in Section 5.3, where we also state the main topological ingredient, Theorem 5.3.10. The proof of Theorem 5.3.10 is, however, given in Section 5.4.

Proof of Theorem 5.1.2

The proof of the main result, Theorem 5.1.2, resembles the Gromov's proof of Theorem 5.1.1, and we build on the Memarian's work [60], who gave a detailed proof of Theorem 5.1.1 based on [39].

Gromov [39, Sec. 5] used iterated partitions of the sphere S^n, where in each step every existing subset was cut into two. Such partitions were encoded by a full rooted tree T of degree 2, and the group of automorphisms of T acted on the space of partitions. Moreover, the space of partitions was identified with a Cartesian product of spheres.

Here we partition S^n iteratively, so that in each step every existing set gets cut into p subsets. Such partitions are parametrized by the product of the special orthogonal group and the wreath product of configuration spaces $\mathrm{Conf}(\mathbb{R}^d, p)$ equipped with an action of the product of the cyclic group $\mathbb{Z}_p$ and the wreath product of copies of $\mathbb{Z}_p$, for an appropriate integer d.

A subset $S \subseteq S^n$ is said to be *convex* if it lies in a hemisphere in S^n and if the cone over it $\mathrm{cone}(S) \subset \mathbb{R}^{n+1}$ is convex.

Definition 5.2.1. Let $n \geq 1$ and $r \geq 1$ be integers. An ordered collection of open subsets $(C_1, \dots, C_r)$ of S^n is called a *partition* of S^n if

(1) $\bigcup_{i=1}^r \overline{C_i} = S^n$, where $\overline{C_i}$ denotes the closure of the set C_i,

(2) $C_i \neq \emptyset$ for every $1 \leq i \leq r$, and

(3) $C_i \cap C_j = \emptyset$ for all $1 \leq i < j \leq r$.

A partition $(C_1, \dots, C_r)$ is called *convex* if all subsets $C_1, \dots, C_r$ are convex.

Remark 5.2.2. In Chapter 3 and Chapter 4 we used partitions of the Euclidean space into closed subsets. Here we work with open subsets of the sphere in order to use results from [60].

We also define the wreath product of the copies of the group $\mathbb{Z}_p$.

Definition 5.2.3. Let $p \geq 2$ be a prime, $\ell \geq 1$ an integer, and denote by $\mathbb{Z}_p$ the finite cyclic group on p elements. Define the *wreath product* $\mathbb{Z}_p^{\wr \ell}$ as follows:

(1) $\mathbb{Z}_p^{\wr 1} = \mathbb{Z}_p$,

(2) $\mathbb{Z}_p^{\wr \ell} = \mathbb{Z}_p^{\wr(\ell-1)} \wr \mathbb{Z}_p = (\mathbb{Z}_p^{\wr(\ell-1)})^{\times p} \rtimes \mathbb{Z}_p$, for $\ell \geq 2$.

Furthermore, for an integer $\ell \geq 1$ define an embedding $\alpha_{p,\ell} : \mathbb{Z}_p^{\wr \ell} \longrightarrow \mathfrak{S}_{p^\ell}$ by:

(1) $\alpha_{p,1} : \mathbb{Z}_p \longrightarrow \mathfrak{S}_p$ maps elements of $\mathbb{Z}_p$ to cyclic permutations, i.e., $\alpha_{p,1}(g)(i) = g + i$, for every $i \in \mathbb{Z}_p$ and for every $g \in \mathbb{Z}_p$.

(2) For $\ell \geq 2$,

$$\alpha_{p,\ell}(g_1, \dots, g_p, h)(i) = \begin{cases} \alpha_{p,\ell-1}(g_{h(1)})(i), & \text{if } 1 \leq i \leq p^{\ell-1} \\ \quad \vdots & \\ \alpha_{p,\ell-1}(g_{h(p)})(i), & \text{if } p^\ell - p^{\ell-1} + 1 \leq i \leq p^\ell, \end{cases}$$

for every $1 \leq i \leq p^\ell$ and for every $(g_1, \dots, g_p, h) \in (\mathbb{Z}_p^{\wr(\ell-1)})^{\times p} \rtimes \mathbb{Z}_p = \mathbb{Z}_p^{\wr \ell}$.

Denote by $[n]_j$ the set of integers $[n]_j = \{(j-1)n+1, \dots, jn\}$, and write $[p^\ell] = \{1, \dots, p^\ell\}$ as

$$[p^\ell] = [p^{\ell-1}]_1 \cup \cdots \cup [p^{\ell-1}]_p.$$

Then for every $(g_1, \dots, g_p, h) \in \mathbb{Z}_p^{\wr \ell}$ and for every $1 \le j \le p$, $g_j \in \mathbb{Z}_p^{\wr(\ell-1)} \subseteq \mathfrak{S}_{p^{\ell-1}}$ permutes elements of $[p^{\ell-1}]_j$, and $h \in \mathbb{Z}_p$ permutes the blocks $[p^{\ell-1}]_1, \dots, [p^{\ell-1}]_p$. Thus, the group $\mathbb{Z}_p^{\wr \ell}$ is a Sylow p-subgroup of the symmetric group $\mathfrak{S}_{p^\ell}$.

Let us now have a look at partitions of S^n that are invariant under the action of the group $\mathbb{Z}_p \times \mathbb{Z}_p^{\wr \ell}$ for some integer ℓ. As expected, they are defined iteratively. Let $P_0 = (C_1^1, \dots, C_p^1)$ be a convex partition of S^n into p subsets, such that $g \cdot C_i = C_{g+i}$ for every $g \in \mathbb{Z}_p$, where the addition is in $\mathbb{Z}_p$. Such partitions are called $\mathbb{Z}_p$-invariant. Now let $\ell \ge 1$. If $P_{\ell-1} = (C_1^{\ell-1}, \dots, C_{p^\ell}^{\ell-1})$ is a $(\mathbb{Z}_p \times \mathbb{Z}_p^{\wr(\ell-1)})$-invariant partition (where we assume that $\mathbb{Z}_p^{\wr 0}$ is a trivial group), define a partition

$$P_\ell = (C_1^\ell, \dots, C_{p^{\ell+1}}^\ell)$$

by partitioning each set $C_i^{\ell-1}$ into p sets $C_{pi-p+1}^\ell, \dots, C_{pi}^\ell$ in a $\mathbb{Z}_p$-equivariant manner, i.e.,

$$g \cdot C_{pi-j}^\ell = \begin{cases} C_{pi-j+g}^\ell, & \text{if } g \le j, \\ C_{pi-j+g+p}^\ell, & \text{if } g > j, \end{cases}$$

for every $g \in \mathbb{Z}_p$. In combination with the $(\mathbb{Z}_p \times \mathbb{Z}_p^{\wr(\ell-1)})$-action on $P_{\ell-1}$, we obtain a $(\mathbb{Z}_p \times \mathbb{Z}_p^{\wr \ell})$-action on P_ℓ. Note that the partition P_ℓ consists of $p^{\ell+1}$ open convex sets.

Denote by $\mathcal{CO}(S^n)$ the family of open convex subsets of S^n, equipped with a topology induced by the Hausdorff distance. A *center map* is any continuous map

$$c : \mathcal{CO}(S^n) \longrightarrow S^n.$$

Definition 5.2.4 ([60, Def. 3.1])**.** Let $n \ge k \ge 0$ be integers and let $\varepsilon > 0$. A set $S \in \mathcal{CO}(S^n)$ is called a (k, ε)*-pancake* if there exists a convex set $S' \subseteq S^n$ of dimension k such that

$$\operatorname{dist}(x, S') \le \varepsilon$$

for every $x \in S$.

Recall that for a group G and a G-representation V, we denote by $e(V)$ the Euler class of the fiber bundle

$$V \longrightarrow \mathrm{E}\, G \times_G V \longrightarrow \mathrm{B}\, G.$$

Moreover, $\operatorname{Index}_G(X; \mathbb{F})$ denotes the Fadell-Husseini index [32] of the space X with respect to the group G and coefficients $\mathbb{F}$. The following is an adaptation of [60, Thm. 4].

Theorem 5.2.5. *Let $n > k \ge 1$ be integers, let p be a prime, let R be a k-dimensional representation of $\mathbb{Z}_p$, such that $e(R) \notin \operatorname{Index}_{\mathbb{Z}_p}(\mathrm{SO}(n+1); \mathbb{F}_p)$, let $\mathbb{Z}_p$ act isometrically, orientation-preservingly and freely on S^n, and let $c : \mathcal{CO}(S^n) \to S^n$ be a $\mathbb{Z}_p$-equivariant center map. If $f : S^n \to R$ is a $\mathbb{Z}_p$-equivariant map, then for every $\varepsilon > 0$ there exists an integer i_ε, such that for every $i \ge i_\varepsilon$ there exists a $(\mathbb{Z}_p \times \mathbb{Z}_p^{\wr i})$-invariant partition Π of S^n into p^{i+1} pieces, such that*

(1) *Every convex subset $S \in \Pi$ is a (k, ε)-pancake,*

(2) *$f(c(S)) = 0$ for every $S \in \Pi$, and*

(3) *all convex subsets $S \in \Pi$ have the same volume.*

The proof of Theorem 5.2.5 is postponed to Section 5.5.

Let us now recall some notation from [60] needed for the proof of Theorem 5.1.2. Following [60, Def. 5.1], consider a set of probability measures on S^n

$$\mathcal{MC}^n = \{\mu_S = \frac{\mathrm{vol}\,|_S}{\mathrm{vol}(S)} \mid S \in \mathcal{CO}(S^n)\}.$$

The vague closure (thus the closure in the vague topology) of $\mathcal{MC}^n$ in the space of probability measures on S^n is called the space of *convexly derived probability measures* on S^n, and it is denoted by $\mathcal{MC}$. A subspace of $\mathcal{MC}$ consisting of measures whose support has dimension k is denoted by $\mathcal{MC}^k$, and $\mathcal{MC}^{\leq k} = \bigcup_{\ell=0}^{k} \mathcal{MC}^\ell$. Moreover, to every finite convex partition Π of S^n, we associate an atomic probability measure $m(\Pi)$ defined by

$$m(\Pi) = \sum_{S \in \Pi} \frac{\mathrm{vol}(S)}{\mathrm{vol}(S^n)} \delta_{\mu_S},$$

where δ_{μ_S} denotes the Dirac measure associated to the point $\mu_S \in \mathcal{MC}^n$, see [60, Def. 5.2]. Denote by $\mathcal{CP}$ the vague closure of the set

$$\{m(\Pi) \mid \Pi \text{ is a finite convex partition of } S^n\}$$

in the space of probability measures on $\mathcal{MC}$.

Definition [60, Def. 5.3] gives certain center maps, to which we will apply Theorem 5.2.5. For a measure $\mu \in \mathcal{CP}$ and for $r > 0$, consider a function $v_{r,\mu} : S^n \to \mathbb{R}$ given by

$$v_{r,\mu}(x) = \mu(B(x, r)),$$

where $B(x, r) \subseteq S^n$ is the intersection of S^n with a Euclidean ball in $\mathbb{R}^{n+1}$ of radius r centered at x. Denote by $M_r(\mu) \subseteq \mathrm{supp}(\mu)$ the set of maximal points of the function $v_{r,\mu}$ on the support of the measure μ. Then for every $S \in \mathcal{CO}(S^n)$, the center $c_r(S)$ is defined to be the barycenter of $\mathrm{conv}(M_r(\mu_S))$. Note that such a defined center map is $\mathbb{Z}_p$-equivariant.

Analogously to [60, Def. 5.4], we define a set of convex partitions in $\mathcal{CP}$ that are *r-adapted* to a continuous map $f : S^n \to R$ as

$$\mathcal{F}_r^0(f) = \{\Pi' \in \mathcal{CP} \mid 0 \in f(\mathrm{conv}(M_r(\mu)) \text{ for every } \mu \in \mathrm{supp}(\Pi')\}.$$

Proof of Theorem 5.1.2. Here we follow the proof of [60, Thm. 1].

Let us first assume that $f : S^n \to R$ is a smooth generic $\mathbb{Z}_p$-equivariant map. Then $f^{-1}(0)$ is $(n-k)$-dimensional, so there exists a constant W, such that

$$\mathrm{vol}(f^{-1}(0) + r) \leq W r^k, \tag{5.2}$$

for every $r > 0$.

For every $r > 0$ and for the center map c_r as defined above, Theorem 5.2.5 yields partitions Π of S^n, such that $m(\Pi) \in \mathcal{F}_r^0(f)$, by Property (2). Property (1), however, means that supports of subsets in Π can be chosen to be contained in arbitrarily thin neighborhoods of at most k-dimensional subsets of S^n. Thus, since the space $\mathcal{F}_r^0(f)$ is closed in $\mathcal{CP}$, see [60, Cor. 5.10], it follows that for every $r > 0$, there exists a partition $\Pi_r \in \mathcal{MC}^{\leq k}$, such that $m(\Pi_r) \in \mathcal{F}_r^0(f)$.

An analogue of [60, Prop. 6] implies that there exists a constant $c > 0$, such that for every $r > 0$ small enough and for every $\Pi \in \mathcal{MC}^{\leq k}$ with $m(\Pi) \in \mathcal{F}_r^0(f)$, the following holds

$$\operatorname{vol}(f^{-1}(0) + \frac{r}{c}) \geq c \sum_{\ell=0}^{k} (\operatorname{vol}(S^{n-\ell} + r) m(\Pi)(\mathcal{MC}^{\ell})). \tag{5.3}$$

Recall that $m(\Pi)$ is a probability measure on $\mathcal{MC}$, and that $\mathcal{MC}^\ell$ is a subspace of $\mathcal{MC}$.

Moreover, an analogue of [60, Prop. 5] claims that for every $\varepsilon > 0$ and for every sequence of reals $r_i > 0$, such that $r_i \to 0$ as $i \to \infty$

$$\operatorname{vol}(f^{-1}(0) + \varepsilon) \geq \operatorname{vol}(S^{n-k} + \varepsilon) \limsup_{i \to \infty} m(\Pi_{r_i})(\mathcal{MC}^k) \tag{5.4}$$

holds.

Inequalities (5.2) and (5.3) imply that

$$\sum_{\ell=0}^{k} (\operatorname{vol}(S^{n-\ell} + r) m(\Pi_r)(\mathcal{MC}^\ell)) \leq \frac{W}{c^{k+1}} r^k,$$

for every $r > 0$ small enough. Letting r to tend to zero, we obtain that $m(\Pi_r)(\mathcal{MC}^\ell)$ tends to zero for every $\ell < k$, and consequently that $m(\Pi_r)(\mathcal{MC}^k)$ tends to 1 as $r \to 0$. In particular, $\limsup_{i \to \infty} m(\Pi_{r_i})(\mathcal{MC}^k) = 1$, which together with (5.4) gives

$$\operatorname{vol}(f^{-1}(0) + \varepsilon) \geq \operatorname{vol}(S^{n-k} + \varepsilon),$$

for every $\varepsilon > 0$.

Assume now that $f : S^n \to R$ is a continuous $\mathbb{Z}_p$-equivariant map. Then it is a uniform limit of smooth generic maps f_j. Moreover, we can choose these maps to be $\mathbb{Z}_p$-equivariant. As we have just shown,

$$\operatorname{vol}(f_j^{-1}(0) + \varepsilon) \geq \operatorname{vol}(S^{n-k} + \varepsilon),$$

holds for every integer j and for every $\varepsilon > 0$.

Let $\delta_j = \| f_j - f \|_\infty$. Then $\delta_j \to 0$ as $j \to \infty$ and $f_j^{-1}(0) \subseteq f^{-1}(B(0, \delta_j))$. Hence,

$$\operatorname{vol}(f^{-1}(B(0, \delta_j)) + \varepsilon) \geq \operatorname{vol}(f_j^{-1}(0) + \varepsilon) \geq \operatorname{vol}(S^{n-k} + \varepsilon),$$

for every integer j and for every $\varepsilon > 0$. Since the volume is a continuous function and

$$f^{-1}(0) + \varepsilon = \bigcap_{j=1}^{\infty} (f^{-1}(B(0, \delta_j)) + \varepsilon),$$

it follows that

$$\operatorname{vol}(f^{-1}(0) + \varepsilon) \geq \operatorname{vol}(S^{n-k} + \varepsilon),$$

for every $\varepsilon > 0$. □

A CW model for the partition space

In Theorem 5.2.5, we aim for $(\mathbb{Z}_p \times \mathbb{Z}_p^{\wr i})$-invariant partitions of the sphere S^n. The space of such partitions will be parametrized by the Cartesian product of the special orthogonal group with the wreath product of configuration spaces. Since we want to apply the equivariant obstruction theory according to tom Dieck [30, Sec. II.3], we need a $\mathbb{Z}_p^{\wr i}$-invariant CW model for the wreath product of configuration spaces. Therefore, we first recapitulate the CW model for the configuration space developed by Blagojević and Ziegler [21, Sect. 3], and later develop a CW model for the wreath product of configuration spaces.

A CW model for the configuration space

Let $d \geq 1$ and $p \geq 2$ be integers. Now we describe a finite CW complex that is an equivariant deformation retract of the configuration space $\mathrm{Conf}(\mathbb{R}^d, p)$, as introduced in [21, Sect. 3].

Definition 5.3.1. Let X be a topological space and let p be an integer. The *(ordered) configuration space* $\mathrm{Conf}(X,p)$ is the set

$$\mathrm{Conf}(X,p) = \{(x_1,\dots,x_p) \in X^p \mid x_i \neq x_j \text{ for every } 1 \leq i < j \leq p\}$$

of all p-tuples of pairwise distinct points in the topological space X.

Denote by $\mathrm{F}(d,p)$ the set

$$\mathrm{F}(d,p) = \{(\sigma, \boldsymbol{i}) \mid \sigma \in \mathfrak{S}_p \text{ is a permutation, and } \boldsymbol{i} \in \{1,\dots,d\}^{p-1}\}.$$

We will also write the pair $(\sigma, \boldsymbol{i})$ as $(\sigma_1 {<_{i_1}} \sigma_2 {<_{i_2}} \cdots {<_{i_{p-1}}} \sigma_p)$, where $\sigma = \sigma_1\sigma_2\dots\sigma_p \in \mathfrak{S}_p$ is a permutation and $\boldsymbol{i} = (i_1,\dots,i_{p-1}) \in \{1,\dots,d\}^{p-1}$. Moreover, let us define a partial order on the set $\mathrm{F}(d,p)$.

Definition 5.3.2. Let $d \geq 1$ and $p \geq 2$ be integers. For $(\sigma, \boldsymbol{i}) = (\sigma_1 {<_{i_1}} \sigma_2 {<_{i_2}} \cdots {<_{i_{p-1}}} \sigma_p)$ and $(\sigma', \boldsymbol{i}') = (\sigma'_1 {<_{i'_1}} \sigma'_2 {<_{i'_2}} \cdots {<_{i'_{p-1}}} \sigma'_p)$ elements of $\mathrm{F}(d,p)$, we say that

$$(\sigma, \boldsymbol{i}) \preceq (\sigma', \boldsymbol{i}')$$

holds if and only if:

whenever $\dots\sigma_k\dots {<_{i'_s}} \dots\sigma_\ell\dots$ appear in this order in $(\sigma', \boldsymbol{i}')$, then
either $\dots\sigma_k\dots {<_{i_s}} \dots\sigma_\ell\dots$ appear in this order in $(\sigma, \boldsymbol{i})$ with $i_s \leq i'_s$,
or $\dots\sigma_\ell\dots {<_{i_s}} \dots\sigma_k\dots$ appear in this order in $(\sigma, \boldsymbol{i})$ with $i_s < i'_s$.

A CW complex is called *regular* if every every attaching map of cells is a homeomorphism on the closed cell, i.e., it does not make identifications on the boundary.

Let $r \geq 1$ be an integer. Denote by W_r the orthogonal complement of the one-dimensional diagonal in $\mathbb{R}^r$:

$$W_r = \{(x_1,\dots,x_r) \in \mathbb{R}^r \mid x_1 + \cdots + x_r = 0\}.$$

It is an $(r-1)$-dimensional linear subspace of $\mathbb{R}^r$.

The following theorem describes the CW model for the configuration space $\mathrm{Conf}(\mathbb{R}^d, p)$.

Theorem 5.3.3 ([21, Thm. 3.13])**.** *Let $d \geq 1$ and $p \geq 2$ be integers. Then there is a finite regular CW complex $\mathcal{F}(d,p)$ whose cells are indexed by strings in* $\mathrm{F}(d,p)$*, and the inclusion relations between the cells are given by:*

The cell $\check{c}(\sigma, \boldsymbol{i})$ associated to $(\sigma, \boldsymbol{i})$ lies in the boundary of the cell $\check{c}(\sigma', \boldsymbol{i}')$ associated to $(\sigma', \boldsymbol{i}')$ if and only if $(\sigma, \boldsymbol{i}) \preceq (\sigma', \boldsymbol{i}')$ in terms of Definition 5.3.2.

The dimension of the CW complex $\mathcal{F}(d,p)$ is $(d-1)(p-1)$, and the dimension of a cell $\check{c}(\sigma, \boldsymbol{i})$ is $(i_1 + \cdots + i_{p-1}) - (p-1)$, where $\boldsymbol{i} = (i_1, \ldots, i_{p-1})$.

The barycentric subdivision $\operatorname{sd}\mathcal{F}(d,p)$ *of the CW complex $\mathcal{F}(d,p)$ has a geometric realization in $W_p^{\oplus d} \subset \mathbb{R}^{d \times p}$, which is an $\mathfrak{S}_p$-equivariant strong deformation retract of the configuration space* $\operatorname{Conf}(\mathbb{R}^d, p)$*.*

The group $\mathfrak{S}_p$ acts on the poset $\mathrm{F}(d,p)$ *by $\pi \cdot (\sigma, \boldsymbol{i}) = (\pi\sigma, \boldsymbol{i})$, which induces an action on the barycentric subdivision* $\operatorname{sd}\mathcal{F}(d,p)$ *given on vertices by $\pi \cdot v(\sigma, \boldsymbol{i}) = v(\pi\sigma, \boldsymbol{i})$, where $\{v(\sigma, \boldsymbol{i}) \mid (\sigma, \boldsymbol{i}) \in \mathrm{F}(d,p)\}$ is the set of vertices of* $\operatorname{sd}\mathcal{F}(d,p)$*.*

Every cell of the CW complex $\mathcal{F}(d,p)$ is given by its combinatorial data $(\sigma, \boldsymbol{i}) = (\sigma_1 <_{i_1} \sigma_2 <_{i_2} \cdots <_{i_{p-1}} \sigma_p)$, and it can be graphically presented by p points $x_{\sigma_1}, \ldots, x_{\sigma_p}$ in $\mathbb{R}^d$, such that the first $i_k - 1$ coordinates of the points σ_{i_k} and $\sigma_{i_{k+1}}$ are the same, and the i_k-th coordinate of the point σ_k is smaller than the i_k-th coordinate of the point σ_{k+1}, for every $1 \leq k \leq p-1$. For further details, consult [21, Sect. 3].

Example 5.3.4. Vertices of the CW complex $\mathcal{F}(d,p)$ correspond to strings $(\sigma, \boldsymbol{i}) \in \mathrm{F}(d,p)$, such that $i_1 = \cdots = i_{p-1} = 1$. There are exactly $p!$ of them, one for each permutation $\sigma \in \mathfrak{S}_p$. For a permutation $\sigma = \sigma_1 \ldots \sigma_p \in \mathfrak{S}_p$, the combinatorial data for the zero-dimensional cell $\check{c}(\sigma, \boldsymbol{i})$ is given by

$$(\sigma_1 <_1 \sigma_2 <_1 \cdots <_1 \sigma_p),$$

and it can be graphically represented by p points $x_1, \ldots, x_p$ in $\mathbb{R}^d$ such that $\pi_1(x_{\sigma_1}) < \cdots < \pi_1(x_{\sigma_p})$, where $\pi_1 : \mathbb{R}^d \to \mathbb{R}$ is the projection to the first coordinate. A vertex that corresponds to the combinatorial data $(6 <_1 5 <_1 2 <_1 7 <_1 1 <_1 4 <_1 3 <_1 8)$ in $\mathcal{F}(2,8)$ is depicted in Figure 5.1 on the left.

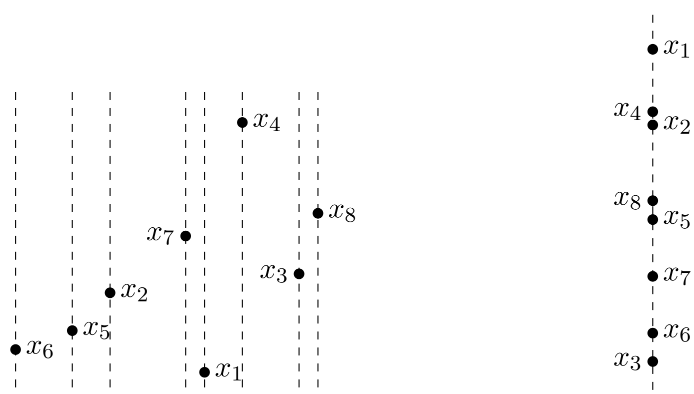


Figure 5.1: Graphical presentation of cells in $\mathcal{F}(2,8)$.

Maximal cells of the CW complex $\mathcal{F}(d,p)$, on the other hand, correspond to strings $(\sigma, \boldsymbol{i}) \in \mathrm{F}(d,p)$, such that $i_1 = \cdots = i_{p-1} = d$. There are also exactly $p!$ of them, one for each permutation $\sigma \in \mathfrak{S}_p$. Similarly as above, for a permutation $\sigma = \sigma_1 \dots \sigma_p \in \mathfrak{S}_p$, the combinatorial data for the maximal cell $\check{c}(\sigma, \boldsymbol{i})$ is given by

$$(\sigma_1 <_d \sigma_2 <_d \cdots <_d \sigma_p),$$

and it can be graphically presented by p points $x_1, \dots, x_p \in \mathbb{R}^d$, such that $\pi_j(x_1) = \cdots = \pi_j(x_p)$ for every $1 \le j \le d-1$, and $\pi_d(x_{\sigma_1}) < \cdots < \pi_d(x_{\sigma_p})$, where $\pi_k : \mathbb{R}^d \to \mathbb{R}$ denotes the projection to the k-th coordinate, for every $1 \le k \le d$. An example for combinatorial data $(3<_2 6<_2 7<_2 5<_2 8<_2 2<_2 4<_2 1)$ is shown in Figure 5.1 on the right.

Example 5.3.5. Let $d \ge 1$ and $p \ge 2$ be integers, and pick a maximal cell $\check{c}(\sigma', \boldsymbol{i}')$ of $\mathcal{F}(d,p)$. Let us examine the boundary of the cell $\check{c}(\sigma', \boldsymbol{i}')$.

Since $\check{c}(\sigma', \boldsymbol{i}')$ is a maximal cell, we have that $i'_1 = \cdots = i'_{p-1} = d$. According to Theorem 5.3.3, a cell $\check{c}(\sigma, \boldsymbol{i})$ lies in the boundary of the cell $\check{c}(\sigma', \boldsymbol{i}')$ if and only if $\dots \sigma_k \dots <_d \dots \sigma_\ell \dots$ appear in this order in $(\sigma', \boldsymbol{i}')$ whenever they appear the same way in $(\sigma, \boldsymbol{i})$. In Figure 5.2, a few cells in the boundary of the maximal cell $\check{c}(3<_2 6<_2 7<_2 5<_2 8<_2 2<_2 4<_2 1)$, which is shown in Figure 5.1, are depicted.

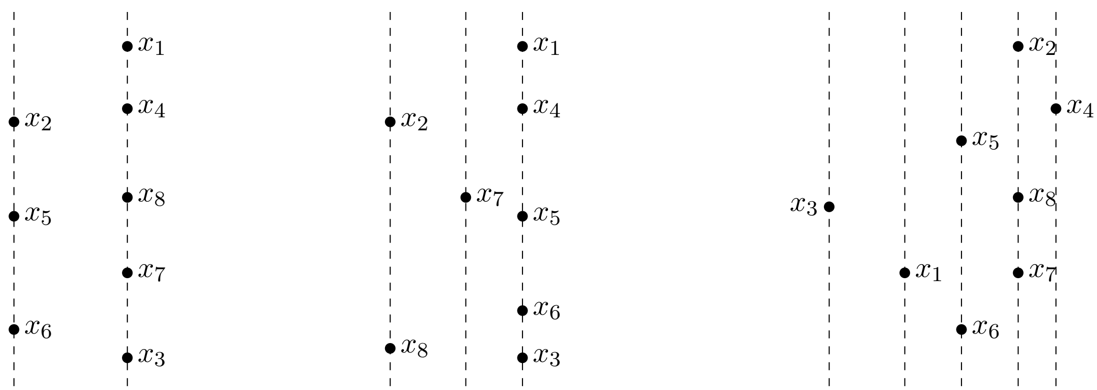


Figure 5.2: Some of the cells in the boundary of the maximal cell $\check{c}(3<_2 6<_2 7<_2 5<_2 8<_2 2<_2 4<_2 1)$.

Let us now concentrate on cells of codimension one, that is, of dimension $(d-1)(p-1)-1$. They are given by strings $(\sigma, \boldsymbol{i}) \in \mathrm{F}(d,p)$, such that $i_1 + \cdots + i_{p-1} = d(p-1) - 1$. Thus, their combinatorial data is

$$(\sigma_1 <_d \sigma_2 <_d \cdots <_d \sigma_k <_{d-1} \sigma_{k+1} <_d \cdots <_d \sigma_p),$$

for some permutation $\sigma \in \mathfrak{S}_p$ and some $1 \le k \le p-1$. In particular, all codimension one faces in the boundary of a maximal cell $\check{c}(\sigma', \boldsymbol{i}')$ can be obtained by splitting the set of integers $[p] = \{1, \dots, p\}$ into two nonempty sets – those that are on the left-hand side of the symbol $<_{d-1}$ and those that are on the right-hand side of the symbol $<_{d-1}$ in the combinatorial presentation $(\sigma_1 <_d \sigma_2 <_d \cdots <_d \sigma_k <_{d-1} \sigma_{k+1} <_d \cdots <_d \sigma_p)$. The order of the integers $\sigma_1, \dots, \sigma_k$ and $\sigma_{k+1}, \dots, \sigma_p$ is the same as their order in $(\sigma', \boldsymbol{i}')$.

For computing the equivariant obstruction cocycle in [21, Sec. 4.1], one needs the following property of the geometric realization of $\operatorname{sd} \mathcal{F}(d,p)$.

Lemma 5.3.6 ([21, Lemma 4.1]). *Let $d \geq 1$ and $p \geq 2$ be integers and denote by $g : W_p^{\oplus d} \to W_p^{\oplus(d-1)}$ the $\mathfrak{S}_p$-equivariant linear projection map obtained by deleting the last row from any matrix $(y_1, \dots, y_p) \in W_p^{\oplus d}$ of row sum 0.*

Then the map g maps all maximal cells of $\mathcal{F}(d,p) \subset W_p^{\oplus d}$ homeomorphically onto the same star-shaped neighborhood $\operatorname{sd}\mathcal{B}(d-1,p)$ of 0 in $W_p^{\oplus(d-1)}$.

The symmetric group acts on $\operatorname{sd}\mathcal{B}(d-1,p)$ by homeomorphisms that reverse orientation according to sgn^{d-1}. Therefore, the maximal cells and the codimension one cells in $\mathcal{F}(d,p)$ can be oriented in such a way that the $\mathfrak{S}_p$-action changes the orientations according to sgn^{d-1}, and the boundary of every maximal cell is the sum of all codimension one cells in its boundary with $+1$ coefficients.

A CW model for the wreath product of configuration spaces

In this section we follow the presentation of [17, Sec. 2], where Ptolemaic epicycles embeddings are introduced. These are embeddings of products of spheres into a configuration space that are invariant under the action of a Sylow 2-subgroup of a symmetric group. We introduce embeddings of products of configuration spaces into a (larger) configuration space that are invariant under the action of a Sylow p-subgroup of the symmetric group.

Definition 5.3.7. For integers $d \geq 1, p \geq 2$ and $\ell \geq 1$ define the space $\mathrm{C}_\ell(d,p)$ by

(1) $\mathrm{C}_1(d,p) = \mathrm{Conf}(\mathbb{R}^d, p)$, and

(2) $\mathrm{C}_\ell(d,p) = (\mathrm{C}_{\ell-1}(d,p))^{\times p} \times \mathrm{C}_1(d,p)$, for $\ell \geq 2$.

Moreover, the left action of the group $\mathbb{Z}_p^{\wr \ell}$ is defined as follows:

(1) For $\ell = 1$, the group $\mathbb{Z}_p^{\wr 1} = \mathbb{Z}_p$ cyclically permutes the elements of the p-tuples in $\mathrm{C}_1(d,p) = \mathrm{Conf}(\mathbb{R}^d, p)$:

$$g \cdot (x_1, \dots, x_p) = (x_{g+1}, \dots, x_{g+p}),$$

for every $g \in \mathbb{Z}_p$ and for every $(x_1, \dots, x_p) \in \mathrm{Conf}(\mathbb{R}^d, p)$, where the addition of indices is in $\mathbb{Z}_p$.

(2) For $\ell \geq 2$,

$$(g_1, \dots, g_p, h) \cdot (X_1, \dots, X_p, Y) = (g_{h+1} \cdot X_{h+1}, \dots, g_{h+p} \cdot X_{h+p}, h \cdot Y),$$

for every $(g_1, \dots, g_p, h) \in \mathbb{Z}_p^{\wr(\ell-1)} \wr \mathbb{Z}_p = \mathbb{Z}_p^{\wr \ell}$ and for every $(X_1, \dots, X_p, Y) \in (\mathrm{C}_{\ell-1}(d,p))^{\times p} \times \mathrm{C}_1(d,p) = \mathrm{C}_\ell(d,p)$, where the addition of indices is in $\mathbb{Z}_p$.

By induction one can see that the space $\mathrm{C}_\ell(d,p)$ is homeomorphic to the product of configuration spaces $\mathrm{Conf}(\mathbb{R}^d, p)^{\times(\frac{p^\ell - 1}{p-1})}$.

The space $\mathrm{C}_\ell(d,p)$ can be embedded in the configuration space $\mathrm{Conf}(\mathbb{R}^d, p^\ell)$ via a *p-adic Ptolemaic epicycles embedding*, that resembles the Ptolemaic epicycles embedding from [17, Sec. 2]. The embedding $\iota_\ell^{d,p} : \mathrm{C}_\ell(d,p) \longrightarrow \mathrm{Conf}(\mathbb{R}^d, p^\ell)$ is given by:

(1) For $\ell = 1$, the map $\iota_1^{d,p} : \mathrm{C}_1(d,p) \longrightarrow \mathrm{Conf}(\mathbb{R}^d, p)$ is the identity.

(2) For $\ell \geq 2$, let $(X_1, \dots, X_p, Y) \in \mathrm{C}_\ell(d,p) = (\mathrm{C}_{\ell-1}(d,p))^{\times p} \times \mathrm{C}_1(d,p)$. Then $Y = (y_1, \dots, y_p) \in \mathrm{C}_1(d,p) = \mathrm{Conf}(\mathbb{R}^d, p)$ defines p pairwise distinct points in $\mathbb{R}^d$. Choose pairwise disjoint open balls $B_1, \dots, B_p \subset \mathbb{R}^d$, such that B_j is centered at y_j for every $1 \leq j \leq p$. Then there are homeomorphisms $\varphi_j : \mathbb{R}^d \longrightarrow B_j$, for every $1 \leq j \leq p$. Denote by $(x_1^j, \dots, x_{p^{\ell-1}}^j)$ the image $\iota_{\ell-1}^{d,p}(X_j) \in \mathrm{Conf}(\mathbb{R}^d, p^{\ell-1})$. Then

$$\iota_\ell^{d,p}(X_1, \dots, X_p, Y) = (\varphi_1(x_1^1), \dots, \varphi_1(x_{p^{\ell-1}}^1), \dots, \varphi_p(x_1^p), \dots, \varphi_p(x_{p^{\ell-1}}^p)).$$

The group $\mathfrak{S}_{p^\ell}$ acts on $\mathrm{Conf}(\mathbb{R}^d, p^\ell)$ by

$$\tau \cdot (x_1, \dots, x_{p^\ell}) = (x_{\tau(1)}, \dots, x_{\tau(p^\ell)}),$$

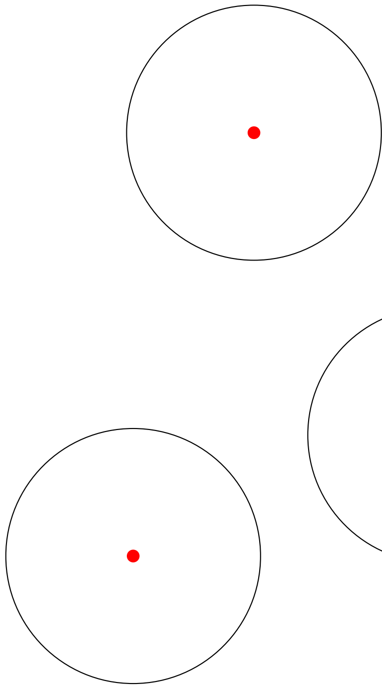
for every permutation $\tau \in \mathfrak{S}_{p^\ell}$. Since the group $\mathbb{Z}_p^{\wr \ell}$ is a subgroup of $\mathfrak{S}_{p^\ell}$, there is a $\mathbb{Z}_p^{\wr \ell}$-action on $\mathrm{Conf}(\mathbb{R}^d, p^\ell)$, which turns the embedding $\iota_\ell^{d,p}$ into a $\mathbb{Z}_p^{\wr \ell}$-equivariant map.

Example 5.3.8. Figure 5.3 shows the image of a point in $\mathrm{C}_3(2,3)$ under the embedding $\iota_3^{2,3}$. More precisely, in Figure 5.3(a) a point $(x_1, x_2, x_3) \in \mathrm{C}_1(2,3) = \mathrm{Conf}(\mathbb{R}^2, 3)$ is shown together with pairwise disjoint balls centered at points x_1, x_2 and x_3. We identify these balls with $\mathbb{R}^2$ via homeomorphisms, and we choose an element of each copy of $\mathrm{Conf}(\mathbb{R}^2, 3)$. These are triples of pairwise disjoint points in $\mathbb{R}^2$, as shown in blue in Figure 5.3(b). Figure 5.3(b) shows pairwise disjoint balls in $\mathbb{R}^2$ centered at these blue points, too. Finally, we identify each of these nine balls with $\mathbb{R}^2$, and choose an element in each copy of $\mathrm{Conf}(\mathbb{R}^2, 3)$. This way, we obtain an element of $\mathrm{C}_3(2,3)$, which is presented by 27 pairwise distinct points in $\mathbb{R}^2$, as in Figure 5.3(c). In other words, an element of $\mathrm{C}_3(2,3)$ is presented by an element of $\mathrm{Conf}(\mathbb{R}^2, 27)$.

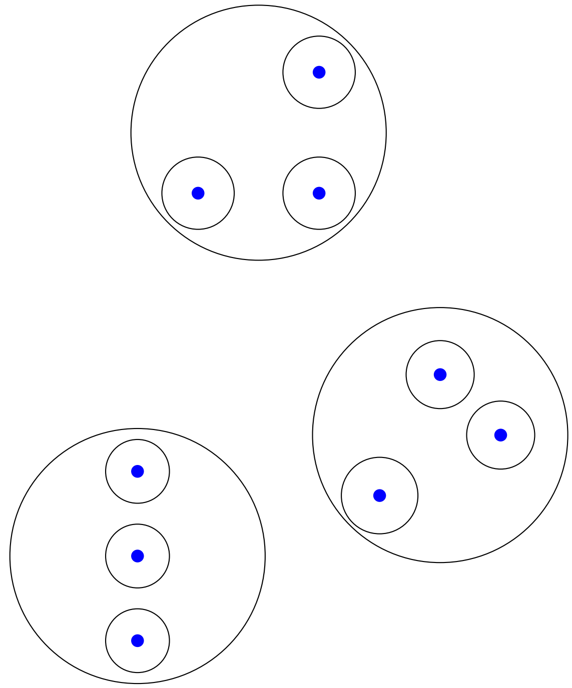
The action of the wreath product $\mathbb{Z}_p \wr \mathbb{Z}_p \wr \mathbb{Z}_p$ on $\mathrm{C}_3(2,3)$ can be seen as follows. Let G_1, G_2 and G_3 be groups that are isomorphic to $\mathbb{Z}_p$. Then we can see the group $\mathbb{Z}_p \wr \mathbb{Z}_p \wr \mathbb{Z}_p$ as

$$\mathbb{Z}_p \wr \mathbb{Z}_p \wr \mathbb{Z}_p \cong (G_3^{\times 3} \rtimes G_2)^{\times 3} \rtimes G_1.$$

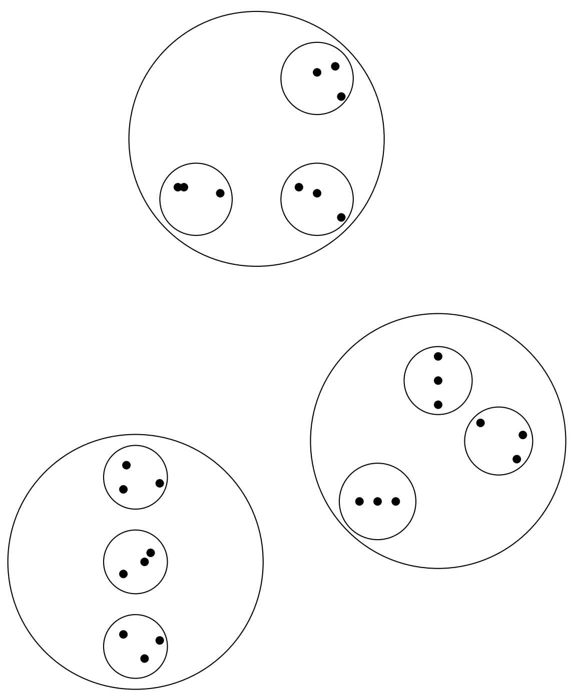
The group G_1 cyclically permutes the red points (and the balls centered at these points). Each copy of G_2 cyclically permutes the three blue points within one big circle, and each copy of the group G_3 cyclically permutes the three black points within one small circle.



(a) Step 1



(b) Step 2



(c) Step 3

Figure 5.3: An embedding $\iota_3^{2,3} : \mathrm{C}_3(2,3) \hookrightarrow \mathrm{Conf}(\mathbb{R}^2, 27)$.

Let us define a $\mathbb{Z}_p^{\wr\ell}$-action on the Euclidean space $\mathbb{R}^{p^\ell}$.

Definition 5.3.9. Let p and ℓ be positive integers. The left action of the group $\mathbb{Z}_p^{\wr\ell}$ on the space $\mathbb{R}^{p^\ell}$ is defined by:

(a) For $\ell = 1$,

$$g \cdot (x_1, \dots, x_p) = (x_{g+1}, \dots, x_{g+p}),$$

where the addition of indices is in $\mathbb{Z}_p$.

(b) For $\ell \geq 2$, see elements of $\mathbb{R}^{p^\ell}$ as real $p \times p^{\ell-1}$ matrices. Then

$$(g_1, \dots, g_p, h) \cdot \begin{pmatrix} v_1 \\ \vdots \\ v_p \end{pmatrix} = \begin{pmatrix} g_{h+1} \cdot v_{h+1} \\ \vdots \\ g_{h+p} \cdot v_{h+p} \end{pmatrix},$$

for every $(g_1, \dots, g_p, h) \in \mathbb{Z}_p^{\wr(\ell-1)} \wr \mathbb{Z}_p = \mathbb{Z}_p^{\wr\ell}$ and for every $\begin{pmatrix} v_1 \\ \vdots \\ v_p \end{pmatrix} \in \mathbb{R}^{p \times p^{\ell-1}} \cong \mathbb{R}^{p^\ell}$, where the addition of indices is in $\mathbb{Z}_p$.

Recall that W_r denotes the space $W_r = \{(x_1, \dots, x_r) \in \mathbb{R}^r \mid x_1 + \dots + x_r = 0\}$, for an integer $r \geq 1$. The subspace $W_{p^\ell} \subset \mathbb{R}^{p^\ell}$ inherits the $\mathbb{Z}_p^{\wr\ell}$-action from Definition 5.3.9. Furthermore, an action of $\mathbb{Z}_p^{\wr\ell}$ can be defined componentwise on $W_{p^\ell}^{\oplus(d-1)}$ by

$$g \cdot (X_1, \dots, X_{d-1}) = (g \cdot X_1, \dots, g \cdot X_{d-1}),$$

for every $g \in \mathbb{Z}_p^{\wr\ell}$ and for every $X_1, \dots, X_{d-1} \in W_{p^\ell}$. Moreover, the unit sphere $S(W_{p^\ell}^{\oplus(d-1)})$ in the linear space $W_{p^\ell}^{\oplus(d-1)}$ inherits the $\mathbb{Z}_p^{\wr\ell}$-action from the ambient space.

We can now state the key topological ingredient needed for the proof of Theorem 5.2.5, whose proof is postponed to Section 5.4.

Theorem 5.3.10. *Let d and ℓ be positive integers and let $p \geq 2$ be a prime. Then there is no $\mathbb{Z}_p^{\wr\ell}$-equivariant map*

$$\mathrm{C}_\ell(d,p) \longrightarrow S(W_{p^\ell}^{\oplus(d-1)}).$$

Remark 5.3.11. By [21, Thm. 1.2], we know that an $\mathfrak{S}_n$-equivariant map $\mathrm{Conf}(\mathbb{R}^d, n) \longrightarrow S(W_n^{\oplus(d-1)})$ exists if and only if n is not a prime power. From the above constructions follows that the composition

$$\mathrm{C}_\ell(d,p) \xrightarrow{\iota_\ell^{d,p}} \mathrm{Conf}(\mathbb{R}^d, p^\ell) \longrightarrow S(W_{p^\ell}^{\oplus(d-1)})$$

of $\mathbb{Z}_p^{\wr\ell}$-equivariant maps exists whenever p is not a prime. Combining it with the statement of Theorem 5.3.10, we obtain that a $\mathbb{Z}_p^{\wr\ell}$-equivariant map $\mathrm{C}_\ell(d,p) \longrightarrow S(W_{p^\ell}^{\oplus(d-1)})$ exists if and only if p is a prime.

Since the proof of Theorem 5.3.10 depends on the equivariant obstruction theory, we need to develop an invariant CW model for the wreath product of configuration spaces.

For integers $d, \ell \geq 1$ and $p \geq 2$, denote by $\mathcal{C}_\ell(d,p)$ the CW complex

(1) $\mathcal{C}_1(d,p) = \mathcal{F}(d,p)$, and

(2) $\mathcal{C}_\ell(d,p) = (\mathcal{C}_{\ell-1}(d,p))^{\times p} \times \mathcal{F}(d,p)$ for $\ell \geq 2$.

In particular, the CW complex $\mathcal{C}_\ell(d,p)$ is the product $\mathcal{C}_\ell(d,p) = \mathcal{F}(d,p)^{\times(\frac{p^\ell-1}{p-1})}$ of $\frac{p^\ell-1}{p-1}$ copies of the CW complex $\mathcal{F}(d,p)$ introduced in Theorem 5.3.3. Thus, cells in $\mathcal{C}_\ell(d,p)$ are products of $\frac{p^\ell-1}{p-1}$ cells in the CW complex $\mathcal{F}(d,p)$. The cell complex $\mathcal{C}_\ell(d,p)$ is given a $\mathbb{Z}_p^{\wr\ell}$-action:

(1) For $\ell = 1$, the group $\mathbb{Z}_p^{\wr 1} = \mathbb{Z}_p$ acts on $\mathcal{C}_1(d,p) = \mathcal{F}(d,p)$ as described in Theorem 5.3.3.

(2) For $\ell \geq 2$, the group $\mathbb{Z}_p^{\wr\ell} = (\mathbb{Z}_p^{\wr(\ell-1)})^{\times p} \rtimes \mathbb{Z}_p$ acts on $\mathcal{C}_\ell(d,p) = (\mathcal{C}_{\ell-1}(d,p))^{\times p} \times \mathcal{C}_1(d,p)$ by
$$(g_1, \dots, g_p, h) \cdot (X_1, \dots, X_p, Y) = (g_{h+1} \cdot X_{h+1}, \dots, g_{h+p} \cdot X_{h+p}, h \cdot Y),$$
for every $g_1, \dots, g_p \in \mathbb{Z}_p^{\wr(\ell-1)}$, $h \in \mathbb{Z}_p$, $X_1, \dots, X_p \in \mathcal{C}_\ell(d,p)$, and $Y \in \mathcal{C}_1(d,p)$.

Remark 5.3.12. There is a commutative diagram of $\mathbb{Z}_p^{\wr\ell}$-equivariant maps.

$$\begin{array}{ccccccc}
\mathrm{Conf}(\mathbb{R}^d, p^\ell) & \xrightarrow{\ r\ } & \mathcal{F}(d,p^\ell) & \xhookrightarrow{\ i\ } & W_{p^\ell}^{\oplus d} & \xhookrightarrow{\ g\ } & W_{p^\ell}^{\oplus(d-1)} \\
\Big\uparrow{\scriptstyle \iota_\ell^{d,p}} & & & & & & \Big\uparrow{\scriptstyle \varphi} \\
\mathrm{C}_\ell(d,p) & \xrightarrow{\ r'\ } & \mathcal{C}_\ell(d,p) & \xhookrightarrow{\ i'\ } & (W_p^{\oplus d})^{\times(\frac{p^\ell-1}{p-1})} & \xrightarrow{\ g'\ } & (W_p^{\oplus(d-1)})^{\times(\frac{p^\ell-1}{p-1})}
\end{array}$$

Here

- r is the deformation retraction from Theorem 5.3.3, and r' is the product of such deformation retractions,
- i is the embedding from Theorem 5.3.3, and i' is the product of such embeddings,
- g is as defined in Lemma 5.3.6, and g' is the product of such maps, and
- φ is an isomorphism.

Example 5.3.13. Let us have a look at some cells in the CW complex $\mathcal{C}_3(2,3)$, which we can see as
$$\mathcal{C}_3(2,3) = (X_3^{\times 3} \times X_2)^{\times 3} \times X_1 \cong X_1 \times X_2^3 \times X_3^9,$$
where X_1, X_2 and X_3 are CW complexes isomorphic to $\mathcal{F}(2,3)$.

The vertices (zero-dimensional cells) of the CW complex $\mathcal{C}_3(2,3)$ are products of vertices in $\mathcal{F}(2,3)$. Let e_1^1 be a vertex in X_1, let e_1^2, e_2^2 and e_3^2 be (not necessarily distinct) vertices in X_2, and let $e_1^3, \dots, e_9^3$ be vertices in X_3. Then

$$e = e_1^1 \times (e_1^2 \times e_2^2 \times e_3^2) \times (e_1^3 \times \cdots \times e_9^3)$$

is a vertex in $\mathcal{C}_3(2,3)$. Every vertex e_i^j is determined by its combinatorial data

$$e_i^j = \check{c}(\sigma_i^j, \boldsymbol{i}),$$

where $\sigma_i^j \in \mathfrak{S}_3$ is a permutation, and $\boldsymbol{i} = (1,1,1)$. Thus, the 13 permutations σ_i^j, for $1 \leq j \leq 3$, $1 \leq i \leq 3^{j-1}$, determine the vertex e in $\mathcal{C}_3(2,3)$. Consequently, the CW complex $\mathcal{C}_3(2,3)$ has 6^{13} vertices.

Assume that $e_1^1 = \check{c}(3{<_1}1{<_1}2)$. Then we can present the vertex e_1^1 graphically, similarly as in Example 5.3.4. However, instead of dots, we use circles for points, see Figure 5.4(a). The vertices $e_1^2 = \check{c}(3{<_1}2{<_1}1), e_2^2 = \check{c}(1{<_1}2{<_1}3)$ and $e_3^2 = \check{c}(2{<_1}1{<_1}3)$ we draw inside of these circles, as in Figure 5.4(b). Finally, in Figure 5.4(c), we show all vertices e_i^j, thus this is a graphical presentation of the vertex $e \in \mathcal{C}_3(2,3)$.

Similarly, maximal cells (or facets) of the CW complex $\mathcal{C}_3(2,3)$ are products of maximal cells of $\mathcal{F}(2,3)$. Therefore, every choice of permutations $\sigma_i^j \in \mathfrak{S}_3$, for every $1 \leq j \leq 3$ and $1 \leq i \leq 3^{j-1}$, is the combinatorial data for a maximal cell in $\mathcal{C}_3(2,3)$. Figure 5.5(a) graphically presents a maximal cell of $\mathcal{C}_3(2,3)$.

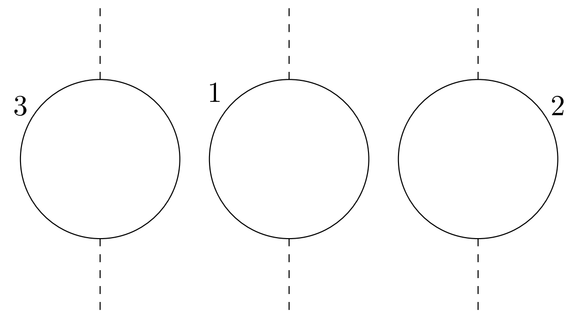
The boundary ∂e of a maximal cell

$$e = e_1^1 \times (e_1^2 \times e_2^2 \times e_3^2) \times (e_1^3 \times \cdots \times e_9^3)$$

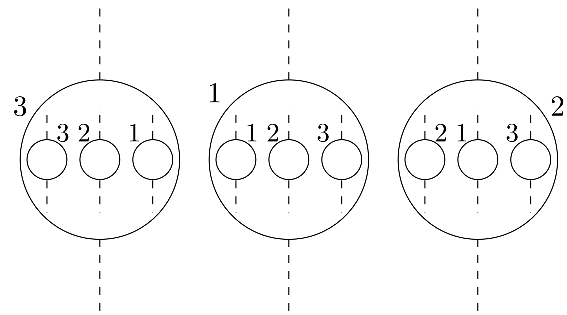
is the union of cells

$$\tau = \tau_1^1 \times (\tau_1^2 \times \tau_2^2 \times \tau_3^2) \times (\tau_1^3 \times \cdots \times \tau_9^3),$$

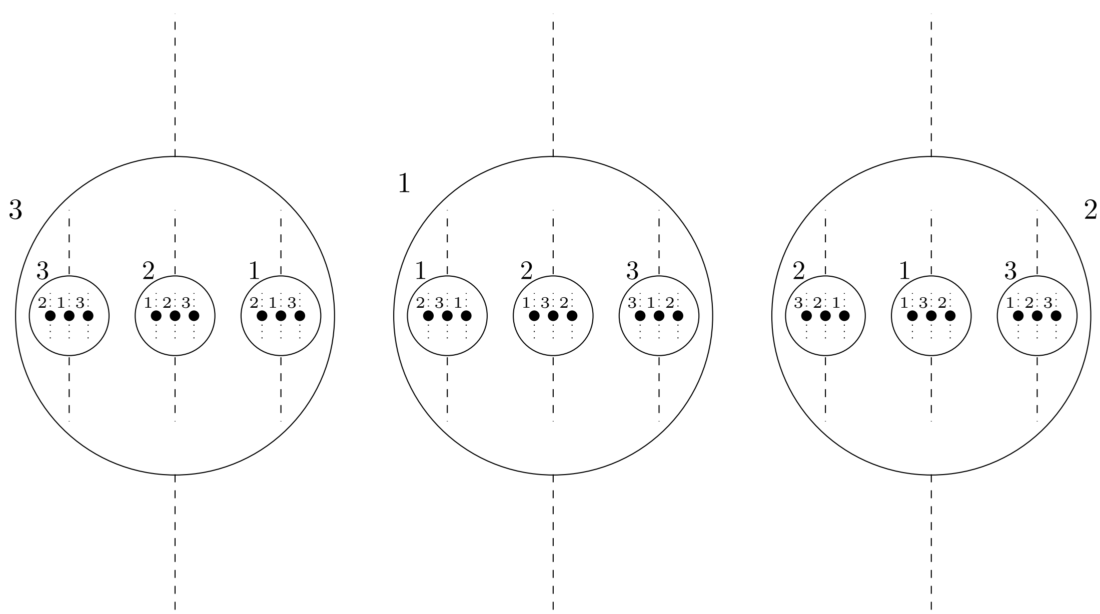
where $\tau_i^j = e_i^j$ or τ_i^j is a cell in the boundary of the cell $e_i^j \in \mathcal{F}(2,3)$ for every i and j. Two codimension one cells in the boundary of the cell e are shown in Figure 5.5.



(a) Vertex e_1^1.

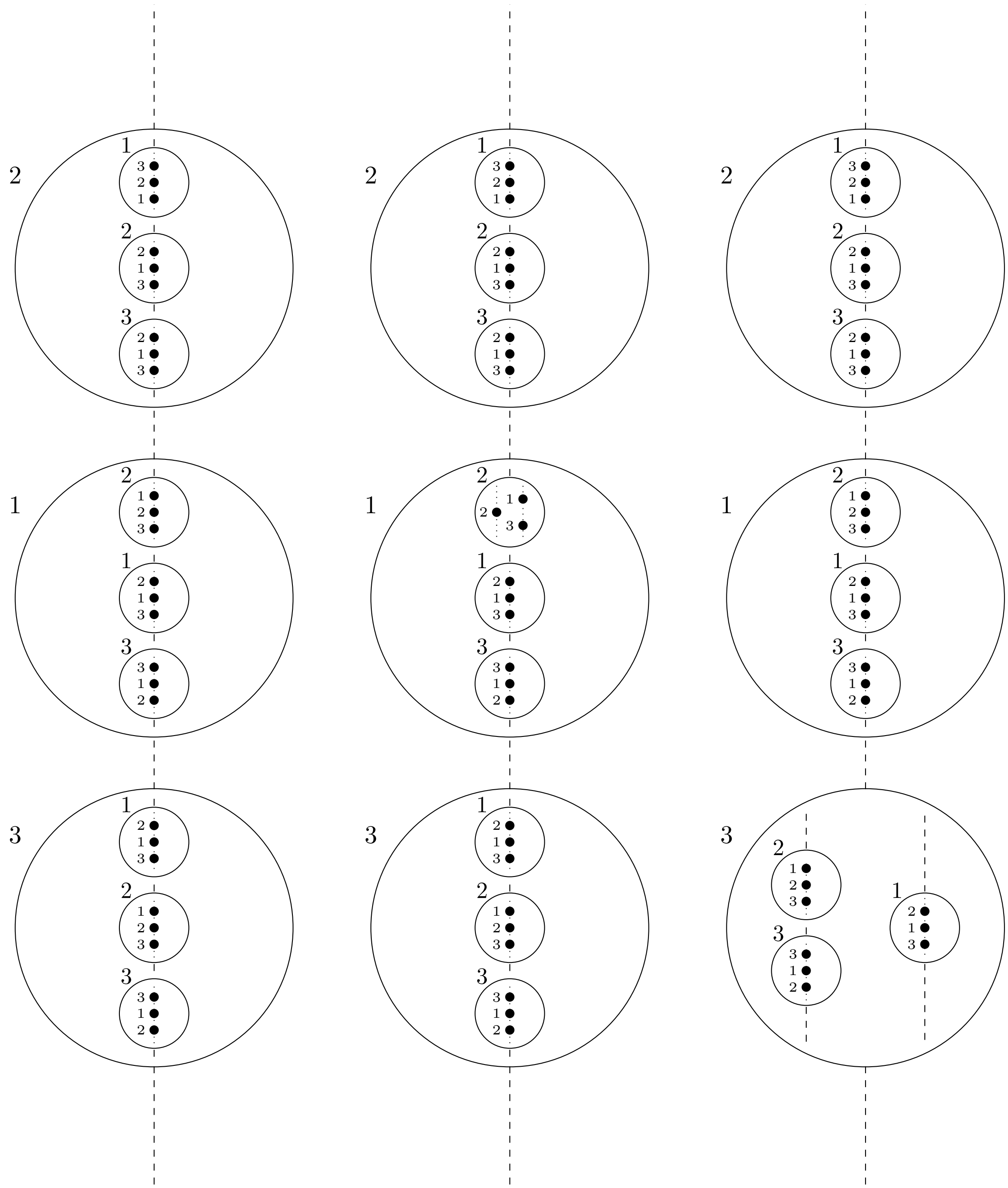


(b) Vertices e_1^1, e_1^2, e_2^2 and e_3^2.



(c) Vertices $e_1^1, e_1^2, e_2^2, e_3^2, e_1^3, \ldots, e_9^3$.

Figure 5.4: A vertex in $\mathcal{C}_3(2,3)$.



(a) A facet $e \in \mathcal{C}_3(2,3)$.

(b) A codimension one cell in ∂e.

(c) Another codimension one cell in ∂e.

Figure 5.5: A maximal cell in $\mathcal{C}_3(2,3)$ and some codimension one cells in its boundary.

Proof of Theorem 5.3.10

Since the map $r' : \mathrm{C}_\ell(d,p) \longrightarrow \mathcal{C}_\ell(d,p)$ is a $\mathbb{Z}_p^{\wr\ell}$-equivariant homotopy equivalence, we proceed by proving the following statement.

Proposition 5.4.1. *Let d and ℓ be positive integers and let $p \geq 2$ be a prime. Then there is no $\mathbb{Z}_p^{\wr\ell}$-equivariant map*

$$\mathcal{C}_\ell(d,p) \longrightarrow S(W_{p^\ell}^{\oplus(d-1)}).$$

The proof of Proposition 5.4.1 uses the equivariant obstruction theory, as described by tom Dieck [30, Sec. II.3]. See also Appendix A for details. Since

- $\mathcal{C}_\ell(d,p)$ is a CW complex of dimension $M = (d-1)(p^\ell - 1)$,
- the action of $\mathbb{Z}_p^{\wr\ell}$ on $\mathcal{C}_\ell(d,p)$ is free when p is a prime,
- $S(W_{p^\ell}^{\oplus(d-1)})$ is an $(M-1)$-dimensional space with a $\mathbb{Z}_p^{\wr\ell}$-action, and
- $S(W_{p^\ell}^{\oplus(d-1)})$ is $(M-1)$-simple and $(M-2)$-connected,

a $\mathbb{Z}_p^{\wr\ell}$-equivariant map $\mathcal{C}_\ell(d,p) \longrightarrow S(W_{p^\ell}^{\oplus(d-1)})$ exists if and only if the primary obstruction

$$\mathfrak{o} = [c_f] \in H^M_{\mathbb{Z}_p^{\wr\ell}}(\mathcal{C}_\ell(d,p); \pi_{M-1}(S(W_{p^\ell}^{\oplus(d-1)})))$$

vanishes, where c_f denotes the obstruction cocycle associated with an equivariant map $f : \mathcal{C}_\ell(d,p) \longrightarrow W_{p^\ell}^{\oplus(d-1)}$ in a general position, see [14, Def. 1.5]. The values of the obstruction cocycle on maximal cells $\check{c} \in \mathcal{C}_\ell(d,p)$ are given by the degrees

$$c_f(\check{c}) = \deg(r \circ f : \partial\check{c} \longrightarrow W_{p^\ell}^{\oplus(d-1)} \setminus \{0\} \longrightarrow S(W_{p^\ell}^{\oplus(d-1)})),$$

where r is the radial projection.

The coefficient $\mathbb{Z}_p^{\wr\ell}$-module $\pi_{M-1}(S(W_{p^\ell}^{\oplus(d-1)}))$ is, via Hurewicz isomorphism, isomorphic with a homology group

$$\pi_{M-1}(S(W_{p^\ell}^{\oplus(d-1)})) \cong H_{M-1}(S(W_{p^\ell}^{\oplus(d-1)}); \mathbb{Z}) =: \mathcal{Z}.$$

The module $\mathcal{Z}$ is as an Abelian group isomorphic to $\mathbb{Z}$. If we see $\mathbb{Z}_p^{\wr\ell}$ as a subgroup of the symmetric group $\mathfrak{S}_{p^\ell}$, then the action of $\mathbb{Z}_p^{\wr\ell}$ on $\mathcal{Z}$ is given by

$$\tau \cdot \xi = (\operatorname{sgn} \tau)^{d-1} \xi,$$

for every $\tau \in \mathbb{Z}_p^{\wr\ell} \subseteq \mathfrak{S}_{p^\ell}$ and for every $\xi \in \mathcal{Z}$, where sgn denotes the sign of the permutation. Note that this action is trivial when p is odd. It is also trivial when $p = 2$ and $\ell > 1$.

In order to compute the primary obstruction $\mathfrak{o}$, we need a $\mathbb{Z}_p^{\wr\ell}$-equivariant map f in general position.

Lemma 5.4.2. *Let $d \geq 1, \ell \geq 1$ and $p \geq 2$ be integers. Then there exists a $\mathbb{Z}_p^{\wr\ell}$-equivariant map $f : \mathcal{C}_\ell(d,p) \longrightarrow W_{p^\ell}^{\oplus(d-1)}$ in general position.*

Moreover, the cells of the CW complex $\mathcal{C}_\ell(d,p)$ can be oriented in such a way that the equivariant obstruction cocycle c_f takes value $+1$ on every oriented maximal cell of $\mathcal{C}_\ell(d,p)$.

Proof. By Lemma 5.3.6, the projection $g : W_p^{\oplus d} \to W_p^{\oplus(d-1)}$ is an $\mathfrak{S}_p$-equivariant map that homeomorphically maps every maximal cell of $\mathcal{F}(d,p)$ to the neighborhood $\operatorname{sd}\mathcal{B}(d-1,p)$ of 0 in $W_p^{\oplus(d-1)}$. Thus, the product map

$$g^{\times(\frac{p^\ell-1}{p-1})} : (W_p^{\oplus d})^{\times(\frac{p^\ell-1}{p-1})} \longrightarrow (W_p^{\oplus(d-1)})^{\times(\frac{p^\ell-1}{p-1})},$$

is a $\mathbb{Z}_p^{\wr\ell}$-equivariant map that maps maximal cells of $\mathcal{C}_\ell(d,p)$ homeomorphically onto a neighborhood $(\operatorname{sd}\mathcal{B}(d-1,p))^{\times(\frac{p^\ell-1}{p-1})}$ of 0 in $(W_p^{\oplus(d-1)})^{\times(\frac{p^\ell-1}{p-1})}$. Since the vector spaces $(W_p^{\oplus(d-1)})^{\times(\frac{p^\ell-1}{p-1})}$ and $W_{p^\ell}^{\oplus(d-1)}$ are $\mathbb{Z}_p^{\wr\ell}$-isomorphic, we obtain a $\mathbb{Z}_p^{\wr\ell}$-equivariant map

$$f : \mathcal{C}_\ell(d,p) \longrightarrow W_{p^\ell}^{\oplus(d-1)}$$

that maps every maximal cell of $\mathcal{C}_\ell(d,p)$ to a neighborhood of 0 in $W_{p^\ell}^{\oplus(d-1)}$. In particular, there are only finitely many zeros of the map f and all of them lie in maximal cells of $\mathcal{C}_\ell(d,p)$.

The space $\operatorname{sd}\mathcal{B}(d-1,p)$ is the barycentric subdivision of a cellular ball of dimension $(d-1)(p-1)$ in $W_p^{\oplus(d-1)}$ with one maximal-dimensional cell (see [21, Sec. 3]). Therefore, we can interpret the image of any maximal cell of $\mathcal{C}_\ell(d,p)$ under the map f as a cellular $(d-1)(p^\ell-1)$-dimensional ball $\mathcal{B}(d-1,p^\ell)$ in $W_{p^\ell}^{\oplus(d-1)}$ with one maximal cell. Following the proof of [21, Lemma 4.1], fix an orientation of the maximal cell of $\mathcal{B}(d-1,p^\ell)$, and orient all codimension one cells in its boundary in such a way that they appear in the boundary of the maximal cell with coefficients $+1$. Now orient all cells of $\mathcal{C}_\ell(d,p)$ so that the restriction of f on every cell is an orientation preserving homeomorphism. With such an orientation, the value of the equivariant obstruction cocycle c_f equals $+1$ on every cell $\check{c} \in \mathcal{C}_\ell(d,p)$. □

Let us fix the orientation of cells in $\mathcal{C}_\ell(d,p)$ as described in Lemma 5.4.2. The next lemma completes the proof of Proposition 5.4.1, which implies Theorem 5.3.10.

Lemma 5.4.3. *Let $d \geq 1$ and $\ell \geq 1$ be integers and let $p \geq 2$ be a prime. Then the primary obstruction $\mathfrak{o} = [c_f]$ does not vanish.*

Proof. The primary obstruction $\mathfrak{o} = [c_f]$ is an element of the equivariant cohomology group $H^M_{\mathbb{Z}_p^{\wr\ell}}(\mathcal{C}_\ell(d,p);\mathcal{Z})$, where $M = (d-1)(p^\ell-1)$ and $\mathcal{Z} = \pi_{M-1}(S(W_{p^\ell}^{\oplus(d-1)}))$. It vanishes if and only if the equivariant cocycle c_f is an equivariant coboundary.

Maximal cells in $\mathcal{C}_\ell(d,p) = (\mathcal{F}(d,p))^{\frac{p^\ell-1}{p-1}} = (\mathcal{F}(d,p))^{1+p+\cdots+p^{\ell-1}}$ are products

$$e_1^1 \times (e_1^2 \times \cdots \times e_p^2) \times \cdots \times (e_1^\ell \times \cdots \times e_{p^{\ell-1}}^\ell),$$

where e_i^j is a maximal cell in $\mathcal{F}(d,p)$ for every $1 \le j \le \ell$ and every $1 \le i \le p^{j-1}$. Recall that every such cell is given by its combinatorial data

$$e_i^j = \check{c}(\sigma_i^j, \boldsymbol{i}),$$

where $\sigma_i^j \in \mathfrak{S}_p$ and $\boldsymbol{i} = (d, \dots, d)$.

Consider the subfamily of all maximal cells in $\mathcal{C}_\ell(d,p)$

$$\mathcal{A} = \{e_1^1 \times (e_1^2 \times \cdots \times e_p^2) \times \cdots \times (e_1^\ell \times \cdots \times e_{p^{\ell-1}}^\ell) \mid \\ (\sigma_i^j)_1 = 1 \text{ for every } 1 \le j \le \ell \text{ and every } 1 \le i \le p^{j-1}\}.$$

There are exactly $((p-1)!)^{\frac{p^\ell-1}{p-1}}$ cells in $\mathcal{A}$, thus

$$c_f(\bigcup \mathcal{A}) = ((p-1)!)^{\frac{p^\ell-1}{p-1}}, \tag{5.5}$$

by Lemma 5.4.2.

The proof continues by contradiction. Therefore, assume that c_f is an equivariant coboundary, i.e., $c_f = \delta c'$ for some equivariant cocycle $c' \in H^{M-1}_{\mathbb{Z}_p^{\wr\ell}}(\mathcal{C}_\ell(d,p); \mathcal{Z})$. Then

$$c_f(\bigcup \mathcal{A}) = \delta c'(\bigcup \mathcal{A}) = c'(\partial(\bigcup \mathcal{A})) = \sum_{\check{c}\in\mathcal{A}} c'(\partial \check{c}).$$

Since the cells of $\mathcal{C}_\ell(d,p)$ are oriented in such a way that the boundary of every maximal cell is a sum of codimension one cells in its boundary with coefficients $+1$, we obtain that

$$c_f(\bigcup \mathcal{A}) = \sum_{\check{c}\in\mathcal{A}} \sum_{\substack{\tau\in\partial\check{c} \\ \dim(\tau)=M-1}} c'(\tau). \tag{5.6}$$

Since c' is a $\mathbb{Z}_p^{\wr\ell}$-equivariant cocycle, its value is the same on every cell in one orbit of the group $\mathbb{Z}_p^{\wr\ell}$. Thus we proceed by examining the orbits of codimension one cells in the boundaries of cells in $\mathcal{A}$.

The boundary of a cell $\check{c} = e_1^1 \times (e_1^2 \times \cdots \times e_p^2) \times \cdots \times (e_1^\ell \times \cdots \times e_{p^{\ell-1}}^\ell)$ is the union

$$\bigcup_{j=1}^{\ell} \bigcup_{i=1}^{p^{j-1}} e_1^1 \times (e_1^2 \times \cdots \times e_p^2) \times \cdots \times (e_1^j \times \cdots \times e_{i-1}^j \times \partial e_i^j \times e_{i+1}^j \times \cdots \times e_{p^{j-1}}^j) \times \cdots \times (e_1^\ell \times \cdots \times e_{p^{\ell-1}}^\ell),$$

and the boundary of a single cell e_i^j in $\mathcal{F}(d,p)$ is understood in Example 5.3.5. Pick one cell

$$\tau = e_1^1 \times (e_1^2 \times \cdots \times e_p^2) \times \cdots \times (e_1^j \times \cdots \times e_{i-1}^j \times f_i^j \times e_{i+1}^j \times \cdots \times e_{p^{j-1}}^j) \times \cdots \times (e_1^\ell \times \cdots \times e_{p^{\ell-1}}^\ell)$$

from the boundary $\partial\check{c}$, where f_i^j is a $((d-1)(p-1)-1)$-dimensional cell in the boundary of e_i^j. Then the cell f_i^j corresponds to the combinatorial data
$(\varphi_1 {<_d} \varphi_2 {<_d} \cdots {<_d} \varphi_\ell {<_{d-1}} \varphi_{\ell+1} {<_d} \cdots {<_d} \varphi_p)$, for some permutation $\varphi = \varphi_1 \dots \varphi_p \in \mathfrak{S}_p$ and some $1 \le \ell \le p-1$.

Denote by $\mathcal{B}$ the set of all maximal cells $e = \check{c}(\sigma, \boldsymbol{i}) \in \mathcal{F}(d,p)$ with $\sigma_1 = 1$. Since f_i^j lies in the boundary of the cell $e_i^j \in \mathcal{B}$, either $\varphi_1 = 1$ or $\varphi_{\ell+1} = 1$ holds. Wlog., suppose that $\varphi_1 = 1$. The orbit of the cell f_i^j under the action of $\mathbb{Z}_p$ on $\mathcal{F}(d,p)$ contains exactly two cells that lie in the boundary of a maximal cell in $\mathcal{B}$ – the cell f_i^j and the cell g_i^j given by the combinatorial data

$$(\varphi_1 - \varphi_{\ell+1} + 1 <_d \cdots <_d \varphi_\ell - \varphi_{\ell+1} + 1 <_{d-1} 1 <_d \varphi_{\ell+2} - \varphi_{\ell+1} + 1 <_d \cdots <_d \varphi_p - \varphi_{\ell+1} + 1),$$

where the addition and subtraction are in $\mathbb{Z}_p$. Consequently, there are only two cells in the orbit of τ under the action of $\mathbb{Z}_p^{\wr \ell}$ on $\mathcal{C}_\ell(d,p)$ that are in the boundary of a cell in $\mathcal{A}$. Denote the other cell by ν.

The cell f_i^j is in the boundary of exactly $\binom{p-1}{\ell-1}$ cells in $\mathcal{B}$, and the cell g_i^j is in the boundary of exactly $\binom{p-1}{\ell}$ cells in $\mathcal{B}$. Hence, the cell τ lies in the boundary of exactly $\binom{p-1}{\ell-1}$ cells in $\mathcal{A}$, and ν lies in the boundary of exactly $\binom{p-1}{\ell}$ cells in $\mathcal{A}$. Thus, the summand $c'(\tau)$ appears $\binom{p-1}{\ell-1} + \binom{p-1}{\ell} = \binom{p}{\ell}$ times in the sum (5.6). Since $1 \le \ell \le p-1$, the number $\binom{p}{\ell}$ is divisible by p. This concludes the argument that the number $c_f(\bigcup \mathcal{A})$ is divisible by p, which is in contradiction with the equation (5.5). $\square$

Proof of Theorem 5.2.5

Finally, in this section we give a proof of Theorem 5.2.5, which completes the proof of the main result.

Let $\varepsilon > 0$ be fixed. First assume that $n > k+1$. We apply the following lemmas from [60]. Recall that $B(x,\nu)$ denotes the closed ball in $\mathbb{R}^{n+1}$ of radius ν centered at x.

Lemma 5.5.1 ([60, Lemma 3.2])**.** *For every $\varepsilon > 0$, there exists $\nu > 0$, such that every convex set $C \subseteq S^n$ that does not contain any ball of the form $B(x,\nu) \cap S^{k+1}$ is a (k,ε)-pancake, where $x \in S^n$ and $S^{k+1} \subseteq S^n$ is any equatorial $(k+1)$-sphere.*

Lemma 5.5.2 ([60, Lemma 3.1])**.** *For every $\nu > 0$, there exists an integer N and a sequence $L_1, \dots, L_N \subseteq \mathbb{R}^n$ of linear $(n-k-1)$-dimensional subspaces, such that for every $x \in S^n$ and for every equatorial $(k+1)$-sphere $S^{k+1} \subseteq S^n$, there exists $1 \le j \le N$ such that*

$$(B(x,\nu) \cap S^{k+1}) \cap L_j \neq \emptyset.$$

Let $\nu > 0$ be the value determined in Lemma 5.5.1, and let $L_1, \dots, L_N$ be the subspaces from Lemma 5.5.2. Moreover, fix a point $x_0 \in S^n$. Set $i_\varepsilon = N$. Since $\mathbb{Z}_p$ acts isometrically on S^n, it also acts on $\mathrm{SO}(n+1)$. Choose an arbitrary $h \in \mathrm{SO}(n+1)$. The set of points $\{g \cdot x_0 \mid g \in \mathbb{Z}_p\}$ defines a partition of S^n into Voronoi cells

$$V_g^h = \{x \in S^n \mid \mathrm{dist}(x, ghx_0) \le \mathrm{dist}(x, g'hx_0) \text{ for every } g' \in \mathbb{Z}_p\},$$

for every $g \in \mathbb{Z}_p$, where dist denotes the Euclidean distance in $\mathbb{R}^{n+1}$. This partition is $\mathbb{Z}_p$-invariant, thus $g' \cdot V_g^h = V_{g'+g}^h$ for every $g, g' \in \mathbb{Z}_p$.

We further want to partition each Voronoi cell into p subsets. Let us first partition the linear subspace $hL_1^\perp = h \cdot L_1^\perp$. For linear maps $a_1, \dots, a_p : hL_1^\perp \to \mathbb{R}$ define a partition $(C_1, \dots, C_p)$ of $hL_1^\perp$ into convex cones

$$C_\ell = \{x \in hL_i^\perp \mid a_\ell(x) \ge a_j(x) \text{ for every } 1 \le j \le p\}.$$

Then $(C_1 \oplus hL_1, \dots, C_p \oplus hL_1)$ is a convex partition of $\mathbb{R}^{n+1}$, and $((C_1 \oplus hL_1) \cap V_{g_0}^h, \dots, (C_p \oplus hL_1) \cap V_{g_0}^h)$ is a convex partition of the Voronoi cell $V_{g_0}^h$, for a fixed element $g_0 \in \mathbb{Z}_p$. Since the linear space $hL_1^\perp$ is $(k+2)$-dimensional, the set of linear maps $hL_1^\perp \to \mathbb{R}$ is parametrized by the configuration space $\mathrm{Conf}(\mathbb{R}^{k+2}, p)$. Partitions of the other Voronoi cells V_g^h are obtained from the partition of $V_{g_0}^h$ by the $\mathbb{Z}_p$-action. Thus, the space $\mathrm{SO}(n+1) \times \mathrm{Conf}(\mathbb{R}^{k+2}, p)$ parametrizes a space of certain convex partitions of S^n into p^2 subsets. Note that $\mathrm{Conf}(\mathbb{R}^{k+2}, p)$ is the space $\mathrm{C}_1(k+2, p)$ as introduced in Definition 5.3.7, and that the induced partitions are $(\mathbb{Z}_p \times \mathbb{Z}_p)$-invariant.

Similarly, the space $\mathrm{C}_2(k+2, p) = \mathrm{Conf}(\mathbb{R}^{k+2}, p)^{\times p} \times \mathrm{Conf}(\mathbb{R}^{k+2}, p)$ parametrizes certain partitions of S^n into p^3 subsets. Indeed, let P be a convex partition of S^n that corresponds to a point $(h, X) \in \mathrm{SO}(n+1) \times \mathrm{Conf}(\mathbb{R}^{k+2}, p)$. Assume that $S_1, \dots, S_p$ are the sets in P such that $S_1, \dots, S_p \subset V_{g_0}^h$. The same way as above, the p copies of $\mathrm{Conf}(\mathbb{R}^{k+2}, p)$ parametrize convex partitions of the sets $S_1, \dots, S_p$ defined by linear maps $hL_2^\perp \to \mathbb{R}$. Indeed, for every $1 \le j \le p$, the j-th copy of $\mathrm{Conf}(\mathbb{R}^{k+2}, p)$ parametrizes a partition of $hL_2^\perp$ into p convex cones, which induces a partition of S_j into p convex sets. Under the $\mathbb{Z}_p$-action, these partitions give partitions of all other sets in P, too. Therefore, the Cartesian product $\mathrm{SO}(n+1) \times \mathrm{C}_2(k+2, p)$ parametrizes certain convex partitions of S^n into p^3 sets, that are $(\mathbb{Z}_p \times \mathbb{Z}_p^{\wr 2})$-invariant.

Proceed iteratively, so that in the i-th step every existing set in the partition of $V_{g_0}^h$ gets partitioned into p convex subsets defined by linear maps $hL_i^\perp \to \mathbb{R}$, and transfer this partition to the remaining Voronoi cells V_g^h using the $\mathbb{Z}_p$-action on S^n. After N steps, we obtain partitions of S^n into p^{N+1} convex subsets that are parametrized by the space $\mathrm{SO}(n+1) \times \mathrm{C}_N(k+2, p)$, and that are $(\mathbb{Z}_p \times \mathbb{Z}_p^{\wr N})$-invariant. We can assume that N is large enough (i.e. $N = i$), since otherwise we can add arbitrary linear $(n-k-1)$-dimensional subspaces $L_j \subseteq \mathbb{R}^{n+1}$. By the choice of the linear subspaces $L_1, \dots, L_N$, Lemma 5.5.2 and 5.5.1 imply that every convex subset in the partition is a (k, ε)-pancake.

Consider a map $\mathrm{SO}(n+1) \times \mathrm{C}_N(k+2, p) \longrightarrow R \oplus (\mathbb{R}^{k+1})^{p^{N+1}}$ that maps every partition $(S_1, \dots, S_{p^{N+1}})$ to

$$(f(c(S_1)), \mathrm{vol}(S_1), \mathrm{vol}(S_1) f(c(S_1)), \dots, \mathrm{vol}(S_{p^N}), \mathrm{vol}(S_{p^N}) f(c(S_{p^N}))),$$

where $(S_1, dots, S_{p^N})$ is a partition of the Voronoi cell $V_{g_0}^h$. Composing it with the product of the identity map $R \to R$ and the projection $(\mathbb{R}^{k+1})^{p^N} \to W_{p^N}^{\oplus(k+1)}$ to the complement of the $(k+1)$-dimensional diagonal in $(\mathbb{R}^{k+1})^{p^N}$, we obtain a test map

$$F : \mathrm{SO}(n+1) \times \mathrm{C}_N(k+2, p) \longrightarrow R \oplus W_{p^N}^{\oplus(k+1)},$$

which is $(\mathbb{Z}_p \times \mathbb{Z}_p^{\wr N})$-equivariant.

Let $(S_1, \dots, S_{p^{N+1}})$ be a partition of S^n such that $F(S_1, \dots, S_{p^{N+1}}) = 0$, which exists by Proposition 5.5.3, that we will state and prove later. Then

$$\begin{aligned} &f(c(S_1)) = 0 \\ &\mathrm{vol}(S_1) = \cdots = \mathrm{vol}(S_{p^N}) \\ &\mathrm{vol}(S_1) f(c(S_1)) = \cdots = \mathrm{vol}(S_{p^N}) f(c(S_{p^N})). \end{aligned}$$

Since the action of $\mathbb{Z}_p$ is an isometry on S^n, it follows that

$$\begin{aligned}\mathrm{vol}(S_1) &= \cdots = \mathrm{vol}(S_{p^{N+1}})\\ \mathrm{vol}(S_1)f(c(S_1)) &= \cdots = \mathrm{vol}(S_{p^{N+1}})f(c(S_{p^{N+1}})).\end{aligned}$$

Therefore, $(S_1, \dots, S_{p^{N+1}})$ is a needed partition.

Let us finally consider the case when $n = k+1$. Again, for a fixed $x_0 \in S^n$ and $h \in \mathrm{SO}(n+1)$, we first partition the sphere S^n into p Voronoi cells V_g^h, for $g \in \mathbb{Z}_p$. We then partition the set $V_{g_0}^h$ into p convex sets $(C_1 \cap V_{g_0}^h, \dots, C_p \cap V_{g_0}^h)$, where $(C_1, \dots, C_p)$ is a convex partition of $\mathbb{R}^{n+1}$ obtained from linear maps $a_1, \dots, a_p : \mathbb{R}^{n+1} \to \mathbb{R}$ as

$$C_\ell = \{x \in S^n \mid a_\ell(x) \geq a_j(x) \text{ for every } 1 \leq j \leq p\},$$

for every $1 \leq \ell \leq p$. The $\mathbb{Z}_p$ action on the sphere induces partitions of the remaining Voronoi cells V_g^h, for every $g \in \mathbb{Z}_p$. We proceed iteratively by partitioning each subset of $V_{g_0}^h$ in an existing partition into p convex sets that are obtained from p linear maps $\mathbb{R}^{n+1} \to \mathbb{R}$. After N steps, we obtain a $(\mathbb{Z}_p \times \mathbb{Z}_p^{\wr N})$-invariant partition of the sphere S^n into p^{N+1} subsets. The family of such partitions is parametrized by $\mathrm{SO}(n+1) \times \mathrm{C}_N(n+1,p)$.

Set i_ε large enough, so that $\nu p^{i_\varepsilon} > \mathrm{vol}(S^n)$, where ν is the parameter from Lemma 5.5.1, and assume that $i = N \geq i_\varepsilon$. If all p^{N+1} subsets in a partition of S^n have the same volume, then by Lemma 5.5.1 all of them are (k,ε)-pancakes.

The same as in the case $n > k+1$, we define a test map

$$F : \mathrm{SO}(n+1) \times \mathrm{C}_N(n+1,p) \longrightarrow R \oplus W_{p^N}^{\oplus n}.$$

If $F(S_1, \dots, S_{p^{N+1}}) = 0$ for some partition $\Pi = (S_1, \dots, S_{p^{N+1}})$, then Π is the required partition. Existence of such a partition is shown in Proposition 5.5.3.

Proposition 5.5.3. *Let i, d, n and k be integers, such that $n > k \geq 1$, and let p be a prime. If $\mathbb{Z}_p$ acts freely and isometrically on S^n, and if R is a $\mathbb{Z}_p$-representation of dimension k such that $e(R) \notin \mathrm{Index}_{\mathbb{Z}_p}(\mathrm{SO}(n+1); \mathbb{F}_p)$, then there is no $(\mathbb{Z}_p \times \mathbb{Z}_p^{\wr i})$-equivariant map*

$$\mathrm{SO}(n+1) \times \mathrm{C}_i(d,p) \longrightarrow S(R \oplus W_{p^i}^{\oplus(d-1)}),$$

where $S(R \oplus W_{p^i}^{\oplus(d-1)})$ is the unit sphere in the linear space $R \oplus W_{p^i}^{\oplus(d-1)}$.

Proof. The proof uses the Fadell-Husseini ideal valued index theory [32]. By the monotonicity property of the Fadell-Husseini index, it suffices to show that

$$\mathrm{Index}_G(S(R \oplus W_{p^i}^{\oplus(d-1)}); \mathbb{F}_p) \nsubseteq \mathrm{Index}_G(\mathrm{SO}(n+1) \times \mathrm{C}_i(d,p); \mathbb{F}_p), \tag{5.7}$$

where $G = \mathbb{Z}_p \times \mathbb{Z}_p^{\wr i}$.

By [32, Prop. 3.1], the index on the right-hand side in (5.7) equals

$$\begin{aligned}&\mathrm{Index}_G(\mathrm{SO}(n+1) \times \mathrm{C}_i(d,p); \mathbb{F}_p) =\\ &\qquad \mathrm{Index}_{\mathbb{Z}_p}(\mathrm{SO}(n+1); \mathbb{F}_p) \otimes H^*(\mathrm{B}\,\mathbb{Z}_p^{\wr i}; \mathbb{F}_p) + H^*(\mathrm{B}\,\mathbb{Z}_p; \mathbb{F}_p) \otimes \mathrm{Index}_{\mathbb{Z}_p^{\wr i}}(\mathrm{C}_i(d,p); \mathbb{F}_p).\end{aligned}$$

It follows from the Gysin long exact sequence [62, Thm. 12.2] that the Euler class $e(R)$ generates the index $\mathrm{Index}_{\mathbb{Z}_p}(S(R);\mathbb{F}_p) \subseteq H^*(\mathrm{B}\,\mathbb{Z}_p;\mathbb{F}_p)$. Similarly, the Euler class $e(W_{p^i}^{\oplus(d-1)})$ of the vector bundle

$$W_{p^i}^{\oplus(d-1)} \longrightarrow \mathrm{C}_i(d,p) \times_{\mathbb{Z}_p^{\wr i}} W_{p^i}^{\oplus(d-1)} \longrightarrow \mathrm{C}_i(d,p)$$

generates the index $\mathrm{Index}_{\mathbb{Z}_p^{\wr i}}(S(W_{p^i}^{\oplus(d-1)});\mathbb{F}_p) \subseteq H^*(\mathrm{B}\,\mathbb{Z}_p^{\wr i};\mathbb{F}_p)$. Furthermore,

$$\mathrm{Index}_G(S(R \oplus W_{p^i}^{\oplus(d-1)});\mathbb{F}_p) = \langle e(R) \cdot e(W_{p^i}^{\oplus(d-1)})\rangle,$$

by [83, Prop. 2.10].

Since $e(R) \notin \mathrm{Index}_{\mathbb{Z}_p}(\mathrm{SO}(n+1);\mathbb{F}_p)$ by the assumption of the theorem, and $e(W_{p^i}^{\oplus(d-1)}) \notin \mathrm{Index}_{\mathbb{Z}_p^{\wr i}}(\mathrm{C}_i(d,p);\mathbb{F}_p)$ by Theorem 5.3.10, [18, Lemma 5.2] and [62, Prop. 9.7], (5.7) follows. □

Chapter 6

Oriented matroid Grassmannians

The theoretical results of this chapter are joint with Pavle V.M. Blagojević and Günter M. Ziegler, whereas the computer computations are done independently by the author.

Introduction

The oriented matroid Grassmannians, later also called MacPhersonians, were introduced in 1993 by Robert MacPherson [56] as a combinatorial analogue to real Grassmannians. The MacPhersonian $\mathrm{MacP}(r,n)$ is the order complex of the partially ordered set of all rank r oriented matroids on a labeled set of n elements, ordered by weak maps. It was a crucial ingredient for giving a combinatorial formula for Pontrjagin classes by Gel'fand and MacPherson [38]. Moreover, MacPherson constructed a canonical map $\mu : \mathrm{G}_r(\mathbb{R}^n) \to \mathrm{MacP}(r,n)$ [56, Prop. 3.2] from the Grassmannian to the oriented matroid Grassmannian. In 2003 Biss [9] published a proof that the map μ is a homotopy equivalence, which would completely determine the topology of the MacPhersonian. However, as Mnëv pointed out in 2007 [63], the paper [9] contains a mistake, see also [10]. Therefore, the following question is still open.

Conjecture 6.1.1. *The map* $\mu : \mathrm{G}_r(\mathbb{R}^n) \longrightarrow \mathrm{MacP}(r,n)$ *is a homotopy equivalence.*

Not much is known about the topology of MacPhersonians. Babson [7] confirmed Conjecture 6.1.1 for $r = 2$. MacPherson [56, Sec. 3.3] claims that a rank MacPhersonian and the corresponding Grassmannian are even homeomorphic, although no proof has been provided yet, consult [67] for details. For $r \geq 3$, the question 6.1.1 is widely open. Anderson and Davis [2, Thm. A] proved that the induced map in cohomology with mod 2 coefficients

$$\mu^* : H^*(\mathrm{MacP}(r,\infty);\mathbb{Z}_2) \longrightarrow H^*(\mathrm{G}_r(\mathbb{R}^\infty);\mathbb{Z}_2)$$

is surjective. The same can be proved for mod 3 coefficients, but for primes $p \geq 5$ one needs different methods, see [76, Thm. 17].

Our approach to Conjecture 6.1.1 is computational. We wrote a code in the programming language C, that constructs MacPhersonians and computes their numerical invariants for small parameters. The main results are summarized in the following two theorems.

Theorem 6.1.2. *The oriented matroid Grassmannian* $\mathrm{MacP}(3,6)$ *is a simplicial complex on* 161 048 *vertices, and the oriented matroid Grassmannian* $\mathrm{MacP}(3,7)$ *is a simplicial* *complex on* 39 339 387 *vertices. Their f-vectors are given in Table 6.1. In particular, the Euler characteristic of* $\mathrm{MacP}(3,6)$ *is* 0*, and the Euler characteristic of* $\mathrm{MacP}(3,7)$ *equals* 3*, which is compatible with Conjecture 6.1.1 that maps* $\mu : \mathrm{G}_3(\mathbb{R}^6) \to \mathrm{MacP}(3,6)$ *and* $\mu : \mathrm{G}_3(\mathbb{R}^7) \to \mathrm{MacP}(3,7)$ *are homotopy equivalences.*

	MacP(3,6)	MacP(3,7)
0	161 048	39 339 387
1	67 506 968	102 912 829 992
2	2 237 230 080	10 573 088 790 768
3	23 453 867 520	280 264 905 278 400
4	114 302 177 280	3 182 159 350 231 680
5	302 970 654 720	19 343 588 635 848 960
6	465 104 977 920	70 610 301 737 848 320
7	413 868 257 280	164 524 030 562 304 000
8	198 394 675 200	251 575 463 004 364 800
9	39 678 935 040	252 698 219 318 845 440
10		161 998 670 765 998 080
11		61 417 168 177 397 760
12		11 422 811 933 245 440
13		536 605 407 313 920

Table 6.1: f-vectors of MacP(3,6) and MacP(3,7).

Theorem 6.1.3. *The Euler characteristic χ of oriented matroid Grassmannians is*

$$\begin{aligned}
\chi(\mathrm{MacP}(3,6)) &= 0, \\
\chi(\mathrm{MacP}(3,7)) &= 3, \\
\chi(\mathrm{MacP}(3,8)) &\equiv 0 \mod 2, \\
\chi(\mathrm{MacP}(3,8)) &\equiv 0 \mod 5, \\
\chi(\mathrm{MacP}(3,8)) &\equiv 0 \mod 7, \\
\chi(\mathrm{MacP}(3,9)) &\equiv 1 \mod 3, \\
\chi(\mathrm{MacP}(3,11)) &\equiv 5 \mod 11, \\
\chi(\mathrm{MacP}(3,13)) &\equiv 6 \mod 13, \\
\chi(\mathrm{MacP}(4,8)) &\equiv 0 \mod 2, \\
\chi(\mathrm{MacP}(4,8)) &\equiv 0 \mod 3, \\
\chi(\mathrm{MacP}(4,9)) &\equiv 0 \mod 3.
\end{aligned}$$

In particular, all computed values are compatible with Conjecture 6.1.1.

The code used for these computations and its output is available online at [66].

Oriented matroids

In this section we introduce oriented matroids, building blocks for MacPhersonians. However, we only state the properties needed for the rest of this chapter and invite the reader to consult [11] for an intensive presentation of the topic.

In order to define oriented matroids, we introduce the following terminology. Let E be a finite set and let $\underline{X} \subseteq E$. A *signed subset* X of E is the set $\underline{X}$ together with a partition (X^+, X^-) of $\underline{X}$ into two disjoint subsets: X^+, called the set of *positive elements* of X, and X^-, called the set of *negative elements* of X. The set $\underline{X} = X^+ \cup X^-$ is called the *support* of X. A signed set can also be seen as a function $X : E \to \{+, -, 0\}$, where $X^+ = X^{-1}(+)$ and $X^- = X^{-1}(-)$.

The *composition* $X \circ Y$ of two signed subsets X and Y of E is the signed subset of E defined by $(X \circ Y)^+ = X^+ \cup (Y^+ \setminus X^-)$ and $(X \circ Y)^- = X^- \cup (Y^- \setminus X^+)$. In other words, the composition of X and Y is the function $X \circ Y : E \to \{+, -, 0\}$ given by

$$X \circ Y(e) = \begin{cases} X(e), & \text{if } X(e) \neq 0 \\ Y(e), & \text{if } X(e) = 0. \end{cases}$$

Let X and Y be signed subsets of E. The *separation set* of X and Y is the set $S(X, Y) = \{e \in E \mid X(e) = -Y(e) \neq 0\}$. The *opposite* of the signed set X is the signed set $-X$, where $(-X)^+ = X^-$ and $(-X)^- = X^+$. In the following definition, $\{+, -, 0\}^E$ denotes the set of all functions $E \to \{+, -, 0\}$ and $0 \in \{+, -, 0\}^E$ denotes the zero-function.

Definition 6.2.1 ([11, Def. 4.1.1])**.** An *oriented matroid* $\mathcal{M}$ is a finite set E together with a set $\mathcal{L} \subseteq \{+, -, 0\}^E$ such that

(1) $0 \in \mathcal{L}$,
(2) if $X \in \mathcal{L}$ then $-X \in \mathcal{L}$,
(3) if $X, Y \in \mathcal{L}$ then $X \circ Y \in \mathcal{L}$,
(4) if $X, Y \in \mathcal{L}$ and $e \in S(X, Y)$ then there is $Z \in \mathcal{L}$ such that $Z(e) = 0$ and $Z(f) = (X \circ Y)(f) = (Y \circ X)(f)$ for all $f \notin S(X, Y)$.

The elements of the set $\mathcal{L}$ are called *covectors* of the oriented matroid $\mathcal{M}$.

The conditions (1)–(4) in the above definition are called covector axioms.

Definition 6.2.2. Let $\mathcal{M}$ be an oriented matroid on the ground set E. A subset I of E is said to be *independent* in $\mathcal{M}$ if for every $e \in I$, there exists a covector X such that $X(e) \neq 0$ and $X(I \setminus \{e\}) = 0$. The *rank* of $\mathcal{M}$, $\operatorname{rank}(\mathcal{M})$, is the maximal cardinality of a set of independent elements of $\mathcal{M}$. Moreover, if the cardinality of I is exactly $\operatorname{rank}(\mathcal{M})$, then I is called a *basis* of $\mathcal{M}$.

There are many equivalent ways of defining an oriented matroid. Here we give another one, which turns out to be more practical for our computations.

Definition 6.2.3 ([11, Def. 3.5.3+Lemma 3.5.4])**.** Let $r \geq 1$ be an integer and let E be a finite set. A *chirotope* of rank r on the set E is a map $\chi : E^r \to \{-1, 0, 1\}$ which satisfies the following properties:

(1) χ is not identically zero,

(2) χ is alternating, that is,

$$\chi(x_{\sigma_1}, x_{\sigma_2}, \ldots, x_{\sigma_r}) = \operatorname{sign}(\sigma)\chi(x_1, x_2, \ldots, x_r),$$

for all $x_1, x_2, \ldots, x_r \in E$ and every permutation σ,

(3) for all $x_1, x_2, \ldots, x_r, y_1, y_2, \ldots, y_r \in E$, such that

$$\chi(x_1, x_2, \ldots, x_r) \cdot \chi(y_1, y_2, \ldots, y_r) \neq 0,$$

there exists $i \in \{1, 2, \ldots, r\}$, such that

$$\chi(y_i, x_2, \ldots, x_r) \cdot \chi(y_1, \ldots, y_{i-1}, x_1, y_{i+1}, \ldots, y_r) = \chi(x_1, x_2, \ldots, x_r) \cdot \chi(y_1, y_2, \ldots, y_r).$$

According to the results of Las Vergnas [11, Thm. 3.5.2] and Lawrence [11, Thm. 3.5.5], every oriented matroid determines exactly two chirotopes χ and $-\chi$, and a chirotope completely determines an oriented matroid. Moreover, if χ is a chirotope, then χ and $-\chi$ determine the same oriented matroid. Thus, we can work with chirotopes, instead of oriented matroids.

In order to construct the MacPhersonian $\operatorname{MacP}(r, n)$, we need to construct all oriented matroids of rank r on the set of elements $E = [n]$. Since chirotopes are alternating, it suffices to store the values of $\chi(x_1, \ldots, x_r)$ for all $x_1 < x_2 < \cdots < x_r$. Thus, we can store a chirotope as a tuple of length $\binom{n}{r}$ with entries $-1, 0, 1$, which, in general, requires less memory (and makes computations easier) than storing the covectors of each oriented matroid.

The following definition introduces weak maps, relations among oriented matroids.

Definition 6.2.4. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two oriented matroids on the same set of elements E. Denote by $\mathcal{L}_1$ the set of covectors of $\mathcal{M}_1$, and by $\mathcal{L}_2$ the set of covectors of $\mathcal{M}_2$. Then we say that there is a *weak map* from $\mathcal{M}_1$ to $\mathcal{M}_2$, and denote it by $\mathcal{M}_1 \rightsquigarrow \mathcal{M}_2$, if

$$\text{for every } Y \in \mathcal{L}_2 \text{ there exists } X \in \mathcal{L}_1, \text{ such that } Y^+ \subseteq X^+ \text{ and } Y^- \subseteq X^-.$$

Let $\chi_1, \chi_2 : E^r \to \{-1, 0, 1\}$ be two chirotopes of the same rank and on the same number of elements. We say that there is a *weak map* $\chi_1 \rightsquigarrow \chi_2$ from χ_1 to χ_2 if

$$\chi_1(e) = \chi_2(e)$$

for every $e \in E$, such that $\chi_2(e) \neq 0$.

The name weak map is somewhat misleading – it is a relation on the set of oriented matroids, and it does not imply that there exists a map between two oriented matroids.

Unwinding the definitions, one can realize that for oriented matroids $\mathcal{M}_1$ and $\mathcal{M}_2$ that define chirotopes χ_1 and χ_2 the following two statements are equivalent

(1) $\mathcal{M}_1 \rightsquigarrow \mathcal{M}_2$

(2) $\chi_1 \rightsquigarrow \chi_2$ or $\chi_1 \rightsquigarrow -\chi_2$,

see [11, Prop. 7.7.5].

The order complex of a partially ordered set P is a simplicial complex whose vertex set is P, and whose simplices correspond to chains in P. Now we recall the definition of the oriented matroid Grassmannian.

Definition 6.2.5. Let $n \geq r \geq 1$ be integers. The *oriented matroid Grassmannian*, also called *MacPhersonian*, is the order complex of the partially ordered set of all oriented matroids of rank r on the set of elements $[n]$, ordered by weak maps.

Explicit constructions

We are interested in understanding $\mathrm{MacP}(r,n)$ for various parameters r and n. In particular, we want to compare certain invariants of the Grassmannian $G(r,n)$ and the MacPhersonian $\mathrm{MacP}(r,n)$. In his thesis Babson [7] proved that $G(2,n)$ and $\mathrm{MacP}(2,n)$ are homotopy equivalent for every integer $n \geq 2$. Hence, the natural step is to look at MacPhersonians of rank 3. Another reason to focus on exactly this rank is that most of unusual properties that an oriented matroid can have appear already in rank 3. For example, there is a rank 3 oriented matroid whose extension space is not homeomorphic to a ball, or whose realization space can be isomorphic to any given semialgebraic set [64]. Under duality, we can use the known facts for $\mathrm{MacP}(n-3,n)$ when $n \leq 5$, in order to understand $\mathrm{MacP}(3,n)$. In particular, for $n \leq 5$, $G(3,n)$ and $\mathrm{MacP}(3,n)$ are homotopy equivalent.

The parameter $n = 7$ is the smallest one, such that the Grassmannian and the MacPhersonian do not have the same dimension. Additionally, in $\mathrm{MacP}(3,7)$ there are oriented matroids whose extension spaces are not homeomorphic to a ball, see Figure 6.1(a). Therefore, we aimed to construct the MacPhersonian $\mathrm{MacP}(3,7)$.

We have written a computer code that for a given rank r and a number of elements n constructs all oriented matroids of rank r on n elements. We make use of representatives of reorientation classes of oriented matroids found by Finschi [33]. For every uniform oriented matroid $\mathcal{M}$ presented in [33], we find all oriented matroids $\mathcal{M}'$ such that $\mathcal{M} \rightsquigarrow \mathcal{M}'$, i.e., such that $\mathcal{M}$ weakly maps to $\mathcal{M}'$. This is done by checking for each subset of bases of $\mathcal{M}$, whether it satisfies the chirotope axioms [11, Def. 3.5.3 + Lemma 3.5.4]. In the end, we construct all oriented matroids that are obtained from such oriented matroids $\mathcal{M}'$ by permuting and reorienting its elements. As a result, all oriented matroids of rank r on n elements are obtained.

For $r = 2$ these computations confirm the results from [67, Sec. 4.2]. We also run the computations for $r = 3$ and $n \leq 7$. For larger parameters, these computations have not been run yet.

There are exactly 161 048 oriented matroids of rank 3 on 6 elements, and 39 339 387 oriented matroids of rank 3 on 7 elements. The complete list of them is given in [66]. In Table 6.2, one can find the number of oriented matroids with a given number of bases. As expected, there are $20 = \binom{6}{3}$ rank 3 oriented matroids on 6 elements that have only one basis, and $35 = \binom{7}{3}$ oriented matroids of rank 3 on 7 elements that have only one basis. An interesting observation is that there are no oriented matroids of rank 3 on 6 elements with 11 bases.

We have also computed the f-vectors of the simplicial complexes $\mathrm{MacP}(3,6)$ and $\mathrm{MacP}(3,7)$, which are given in Table 6.1. The Euler characteristic of $\mathrm{MacP}(3,6)$ equals 0, and the Euler characteristic of $\mathrm{MacP}(3,7)$ equals 3. These values are the same as the Euler characteristic of the corresponding Grassmannians.

The smallest parameters for which the MacPhersonian and the Grassmannian do not have the same dimension are $r = 3$ and $n = 7$ – the dimension of $\mathrm{MacP}(3,7)$ is 13, and the dimension of $\mathrm{G}_3(\mathbb{R}^7)$ is 12. However, the following proposition implies that examining the homology $H_{13}(\mathrm{MacP}(3,7))$ or the cohomology $H^{13}(\mathrm{MacP}(3,7))$ does not give an evidence of a counterexample to Conjecture 6.1.1.

Proposition 6.3.1. *Simplicial complex* MacP(3, 7) *collapses to a 12-dimensional subcomplex.*

Proof. Every chain

$$c = \mathcal{M}_{13} \rightsquigarrow \mathcal{M}_{12} \rightsquigarrow \cdots \rightsquigarrow \mathcal{M}_0$$

of length 13 in the poset MacP(3, 7) contains an interval

$$[\mathcal{M}_9, \mathcal{M}_{11}] = \mathcal{M}_{11} \rightsquigarrow \mathcal{M}_{10} \rightsquigarrow \mathcal{M}_9$$

(up to reorientation and permutation of elements) as a subchain, where the realizations of oriented matroids $\mathcal{M}_9$, $\mathcal{M}_{10}$ and $\mathcal{M}_{11}$ are shown in Figure 6.1. The interval $[\mathcal{M}_9, \mathcal{M}_{11}]$ in the poset MacP(3, 7) contains exactly three elements, because there is no other oriented matroid $\mathcal{M}$, such that $\mathcal{M}_{11} \rightsquigarrow \mathcal{M} \rightsquigarrow \mathcal{M}_9$. Hence, the chain c is the only chain of length 13 that contains the chain

$$c' = \mathcal{M}_{13} \rightsquigarrow \mathcal{M}_{12} \rightsquigarrow \mathcal{M}_{11} \rightsquigarrow \mathcal{M}_9 \rightsquigarrow \mathcal{M}_8 \rightsquigarrow \cdots \rightsquigarrow \mathcal{M}_0$$

as a subchain. Therefore, the 13-dimensional face of the simplicial complexMacP(3, 7) determined by the chain c has a free face – the 12-dimensional face determined by the chain c', thus it can be collapsed.

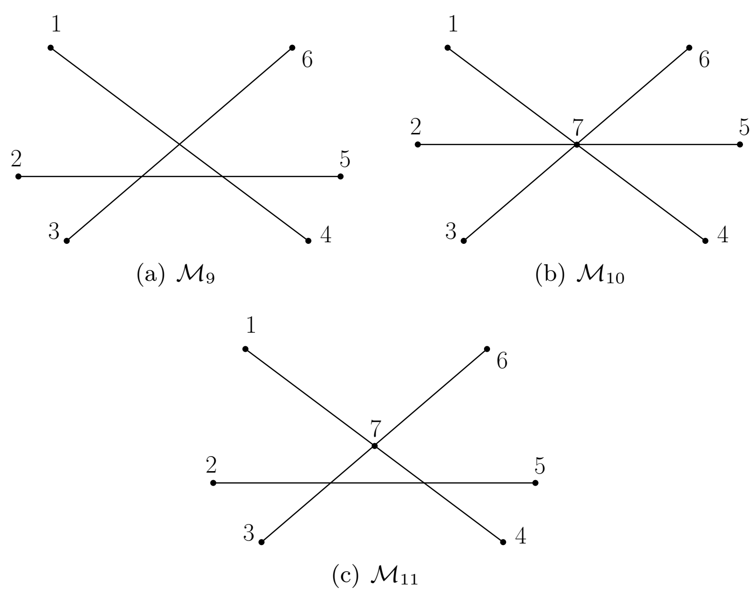


Figure 6.1: A chain of length 13 in MacP(3, 7).

□

Remark 6.3.2. Since the oriented matroids of rank 3 on 8 elements do not show any different behavior from those on 7 elements, the previous proof can be used to show that MacP(3, 8) collapses to a 15-dimensional subcomplex.

Further computations

In this section we would like to compute invariants of further MacPhersonians. However, constructing all oriented matroids and computing the Euler characteristic or any other invariant of the simplicial complex $\mathrm{MacP}(r,n)$ for $r \geq 3$ and $n \geq 8$ is beyond our computational limits. Therefore, we try to understand some properties of MacPhersonians based on their subcomplexes.

For a group G and a G-space X, denote by X^G the set of fixed points in X under the action of G. The following theorem is due to Floyd, and it gives a relation between the Euler characteristic of X and the Euler characteristic of X^G.

Theorem 6.4.1 ([34], see also [25, Thm. III.4.3], [12, p. 267])**.** *Let p be a prime, G be a p-group and let X be a G-space. Then*

$$\chi(X) \equiv \chi(X^G) \mod p.$$

Thus, in order to compute the Euler characteristic of a MacPhersonian (modulo a prime), it suffices to define a group action on the MacPhersonian and to compute the Euler characteristic of the fixed point set, which usually requires less computational power.

The symmetric group $\mathfrak{S}_n$ acts on the vertices of a MacPhersonian $\mathrm{MacP}(r,n)$ by permuting the elements of every oriented matroid. More precisely, for a rank r oriented matroid $\mathcal{M}$ on the set of elements $[n] = \{1,2,\dots,n\}$ with the set of covectors $\mathcal{L}$, the oriented matroid $\mathcal{M}' = \sigma \cdot \mathcal{M}$ is the oriented matroid of rank r on the set of elements $[n]$ with the set of covectors

$$\mathcal{L}' = \{\sigma \cdot X \mid X \in \mathcal{L}\},$$

where

$$(\sigma \cdot X)(e) = X(\sigma_e),$$

for every $X \in \mathcal{L}$, $e \in [n]$ and every $\sigma = \sigma_1 \dots \sigma_n \in \mathfrak{S}_n$. The $\mathfrak{S}_n$-action on the vertices of $\mathrm{MacP}(r,n)$ can be linearly extended, so that we obtain a $\mathfrak{S}_n$-action on the whole simplicial complex $\mathrm{MacP}(r,n)$.

Proposition 6.4.2. *Let $G \subseteq \mathfrak{S}_n$ be a subgroup of the symmetric group, and let $S \subset \mathrm{MacP}(r,n)$ be the set of vertices of $\mathrm{MacP}(r,n)$ that are fixed under the action of G. Then*

$$\mathrm{MacP}(r,n)[S] = \mathrm{MacP}(r,n)^G,$$

where $\mathrm{MacP}(r,n)[S]$ is the induced subcomplex, thus a simplicial complex consisting of all simplices $\Delta \in \mathrm{MacP}(r,n)$, such that the vertex set of Δ is a subset of S.

Proof. Every point of the simplicial complex $\mathrm{MacP}(r,n)$ is of the form

$$x = \lambda_1 \mathcal{M}_1 + \dots + \lambda_k \mathcal{M}_k,$$

for some $k \geq 1$, $\mathcal{M}_1, \dots, \mathcal{M}_k$ vertices of $\mathrm{MacP}(r,n)$, and $\lambda_1, \dots, \lambda_k \geq 0$, such that $\lambda_1 + \dots + \lambda_k = 1$. Wlog., we can assume that $\mathcal{M}_1 \rightsquigarrow \dots \rightsquigarrow \mathcal{M}_k$. Moreover, the G-action is given by

$$\sigma \cdot x = \lambda_1(\sigma \cdot \mathcal{M}_1) + \dots + \lambda_k(\sigma \cdot \mathcal{M}_k),$$

for every $\sigma \in G$.

Clearly, if $\mathcal{M}_1, \dots, \mathcal{M}_k \in S$, then x is fixed under the G-action. On the other hand, assume that $x \in \mathrm{MacP}(r,n)^G$. Then $\{\mathcal{M}_1, \dots, \mathcal{M}_k\} = \{\sigma \cdot \mathcal{M}_1, \dots, \sigma \cdot \mathcal{M}_k\}$ for every $\sigma \in G$. The oriented matroids $\mathcal{M}_i$ and $\sigma \cdot \mathcal{M}_i$ have the same number of bases, for every $1 \le i \le k$, whereas the oriented matroids $\mathcal{M}_i$ and $\mathcal{M}_j$ have a different number of bases, for every $1 \le i < j \le k$. Thus, the oriented matroids $\mathcal{M}_1, \dots, \mathcal{M}_k$ are fixed under G. Consequently, $x \in \mathrm{MacP}(r,n)[S]$. □

Definition 6.4.3. A finite-dimensional connected topological space X is called a *Poincaré duality space* over a ring R if its cohomology ring $H^*(X;R)$ is finitely generated and if there exists an integer d and an element $\nu \in H_d(X;R)$ such that the cap product

$$\cap\, \nu : H^i(X;R) \to H_{d-i}(X;R)$$

is an isomorphism for every $i \ge 0$. We say that the *formal dimension* of X is d.

By the Poincaré duality theorem, every compact oriented manifold is a Poincaré duality space [29, Thm. 18.3.4]. In particular, the Grassmannian $\mathrm{G}_r(\mathbb{R}^n)$ is a Poincaré duality space for every $r \ge 1$ and for every $n \ge 2$ even. A positive answer to Conjecture 6.1.1 would imply that the MacPhersonian $\mathrm{MacP}(r,n)$ is a Poincaré duality space whenever n is even. Thus, we can make use of the following.

Theorem 6.4.4 ([26])**.** *Let p be a prime. If a $\mathbb{Z}_p$-space X is a Poincaré duality space over $\mathbb{F}_p$ of formal dimension d, then each connected component of the fixed point set $X^{\mathbb{Z}_p}$ is a Poincaré duality space over $\mathbb{F}_p$. If $p \neq 2$, then the formal dimension of every connected component of $X^{\mathbb{Z}_p}$ is congruent to $d \mod 2$.*

In order to apply Theorem 6.4.1 and Theorem 6.4.4, it suffices by Proposition 6.4.2 to find fixed vertices of the MacPhersonian under the action of an appropriate group. Following this approach, we consider various group actions on MacPhersonians in the next section.

Since we want to compare numerical properties of MacPhersonians and Grassmannians, it is necessary to know the homology or the cohomology of Grassmannians. The cellular homology groups of Grassmannians $\mathrm{G}_r(\mathbb{R}^n)$ for $r = 3$ and $r = 4$, and for $n \le 13$ are computed using the Schubert CW decomposition of Grassmannians and the formula for differentials given by Jungkind [48, p. 24], see Table 6.4 and Table 6.5.

Results

The complete list of results is available at [66].

$\mathbb{Z}_3$-action on $\mathrm{MacP}(3,9)$

The smallest number of elements n, such that there exists a non-realizable rank 3 oriented matroid on n elements is 9. In particular, the map

$$\mu : \mathrm{G}_3(\mathbb{R}^9) \longrightarrow \mathrm{MacP}(3,9)$$

is not surjective. Therefore, we searched for a group action on $\mathrm{MacP}(3,9)$ that fixes a non-realizable oriented matroid, a vertex of $\mathrm{MacP}(3,9)$ that is not in the image of μ. The group

G generated by the permutation $(1,4,7)(2,5,8)(3,6,9)$ fixes the non-realizable uniform oriented matroid whose corresponding pseudoline arrangement is depicted in Figure 6.2. We are grateful to Jürgen Richter-Gebert for sharing this example with us.

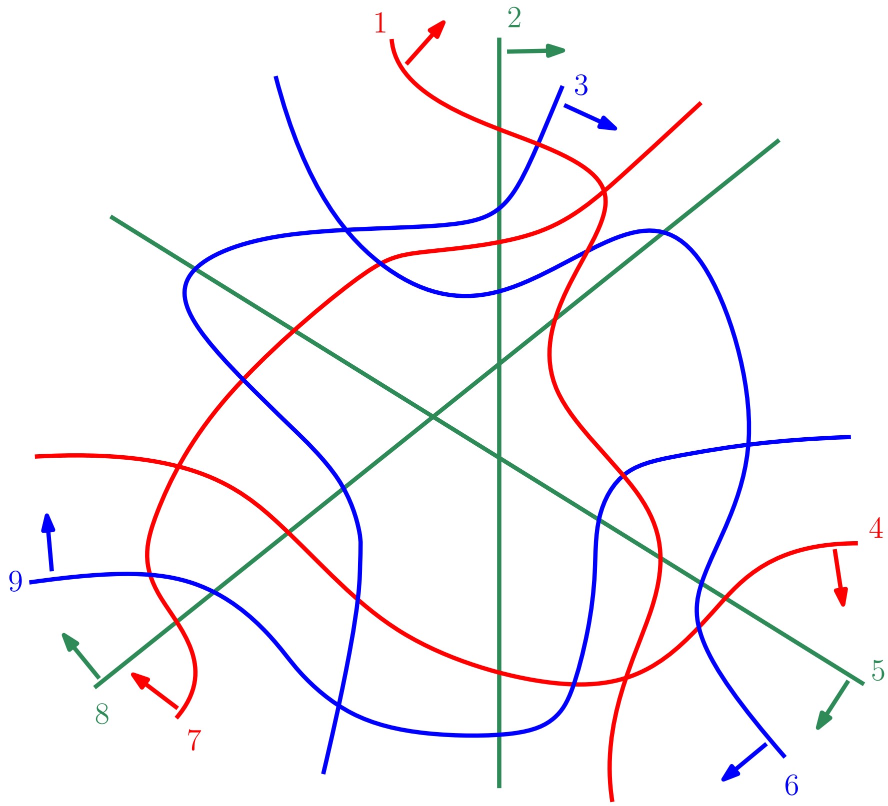


Figure 6.2: A non-realizable uniform oriented matroid of rank 3 on 9 elements, fixed under the action of $G \cong \mathbb{Z}_3$.

In order to construct fixed points under the action of G, we first construct ordinary matroids of rank 3 on 9 elements, and then we orient them. The procedure is explained in Example 6.4.5. The simplicial complex $\mathrm{MacP}(3,9)^G$ is 8-dimensional, and it has two connected components – one of them consists of only one point. The f-vector is

$$f(\mathrm{MacP}(3,9)^G) = (77\,836, 3\,127\,752, 27\,156\,816, 95\,617\,008,\\ 165\,209\,760, 146\,524\,608, 62\,584\,704, 10\,091\,520, 331\,776).$$

In particular, the Euler characteristic of $\mathrm{MacP}(3,9)^G$ equals 4 – it is the same as the Euler characteristic of the Grassmannian $\mathrm{G}_3(\mathbb{R}^9)$.

Example 6.4.5. Here we construct an oriented matroid $\mathcal{M}$ of rank 3 on 9 elements that is fixed under the G-action.

First we make step-by-step an ordinary matroid M that is fixed under the G-action. For each triple (a,b,c) of elements of M, we decide whether it is a basis. Set, for example, $(1,2,3)$ to be a basis in M. Since the matroid M is supposed to be G-invariant, then

$(4,5,6)$ and $(7,8,9)$ are also bases. Furthermore, since $(1,2,3)$ and $(4,5,6)$ are bases, by the bases exchange property of matroids, at least one of the triples $(2,3,4)$, $(2,3,5)$ and $(2,3,6)$ has to be a basis in M. In particular, if for all these triples we have already decided not to be bases, we can discard M, since it cannot be extended to a matroid. Similarly, at least one of the triples $(1,3,4)$, $(1,3,5)$ and $(1,3,6)$, and at least one of the triples $(1,2,4)$, $(1,2,5)$ and $(1,2,6)$ has to be a basis in M. We apply the same procedure to all other pairs of triples $(a,b,c),(d,e,f)$ that have already been determined to be bases.

Finally, we check for every possible reorientation of elements of M, whether it is an oriented matroid. In particular, since we store oriented matroids as chirotopes, we check chirotope axioms [11, Lemma 3.5.4].

The subcomplex of MacP$(3,9)$ fixed under the action of $G \cong \mathbb{Z}_3$ is rather large, and further computations are not trivial. Therefore, we proceed by constructing fixed points in MacP$(3,9)$ under a larger group.

$(\mathbb{Z}_3 \times \mathbb{Z}_3)$-action on MacP$(3,9)$ and MacP$(4,9)$

In this section we consider the transitive action of the group $G \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ generated by the permutations $(1,2,3)(4,5,6)(7,8,9)$ and $(1,4,7)(2,5,8)(3,6,9)$. It fixes only 4 vertices of MacP$(3,9)$. Each of them corresponds to an oriented matroid with 27 bases and the underlying simple oriented matroids have only 3 elements. Their realizations on a 2-dimensional sphere are shown in Figure 6.3.

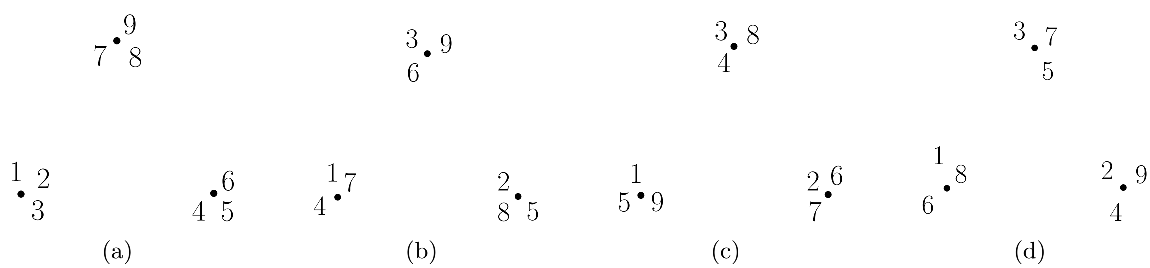


Figure 6.3: Vertices of MacP$(3,9)$ fixed under the action of $G \cong \mathbb{Z}_3 \times \mathbb{Z}_3$.

If we use the lexicographical order of bases for an oriented matroid, i.e., the order

$$(1,2,3) < (1,2,4) < \cdots < (1,2,9) < (1,3,4) \cdots < (7,8,9),$$

the chirotopes that correspond to these four fixed points are listed here.

000000000000000+++0++++++00000000000+++0++++++000
00+++0++++++00000000000000000000000

+00+00+0−00−000000+00+0−000++00+000−00−0000+000−0
+00+00−00000+00+00+0−000++000−0+00+

++000+000−−000−−0000000+0+000+000++000+00−000000+
00000++0000−0−0++0000−0−00+0+0000++

+0+0+00−0000−+0+00000−0000−000+0+000000+0+0−00+00
0−0−0000000+00++0+0−00+0000+00+0−0+

We again obtain that the Euler characteristic of $\mathrm{MacP}(3,9)^G$ is the same as the Euler characteristic of the Grassmannian $\mathrm{G}_3(\mathbb{R}^9)$.

The same group acts on $\mathrm{MacP}(4,9)$, as well. It fixes 6 vertices of $\mathrm{MacP}(4,9)$, that again form a 0-dimensional simplicial complex, whose Euler characteristic is the same as the Euler characteristic of the corresponding Grassmannian. The chirotopes of these fixed points are the following, where the order ob bases is again lexicographical.

000000+−0−+++0−−+0+−+−+0+−−−0++−0−+−00−+0+
−0000−+−+−+−+0+−0−+++0−−+0+−+0+0−−0++−++0−
000−+−00−+0+−0−+−−+0+−+0000000+−+−+−0+−+00

000000+−+0−+−+00−++−+−+−0+−+−00+−−+−0−+0+−
0+−+−+0000−+−0+−+0−+−+00−++−+00−+0+−−+−0−+
+−+00000+0−−0+000+0−−+−+−+0000+−+−+−0+−+00

000000+−−+0+0−++0−+−+−++−0−0+−−0+−+−0+0−−0
+−+−+0−+−+0000+−−+0+0−++0−+−+0−+0+−0000−+0
−+−+−+000−+0+−+−+0−+000−+−0000+−+−+−0+−+00

+++−−−0−++0−++000−0+++0−0−−0+0−0−−0++−−0+0
++−−+00+000−0−++−0000++0++−−0+−−00−0+000−−
+++00+−+00++0+0−00−−00+−−+−+++0−−+0−+++0++

+++−−−−00+++−000++−−0−−00+0−−0−−00−−+0+00+
+00+−−0+00+++−0−+0+−+0+0−0+0−+−0−0++−−+00+
0−000+−+−0−−000−0−−0+−−+00−+++−0+−−0+0−−++

+++−−−++0−00+0++0++0−0+0−−+0−−+00++0+0−−00
−00+−0−−−+0−0−−0++0+00−++00++++00++0+00−0−
0−0+−−−+0−−++0+++0+000++00−+++++00+++−0+++

$\mathbb{Z}_p$-action on $\mathrm{MacP}(3,p)$

For a prime $p \geq 2$, one can define the action on $\mathrm{MacP}(3,p)$ by the group $\mathbb{Z}_p \subseteq \mathfrak{S}_p$ that cyclically permutes the elements of oriented matroids. In other words, let $G \subseteq \mathfrak{S}_p$ be the cyclic group generated by the permutation $(1, 2, 3, \ldots, p)$. Using the same methods as above, we have computed vertices of $\mathrm{MacP}(3, p)$ that are fixed under the G-action.

For $p = 5$, there are 2 fixed points under the action of $\mathbb{Z}_p$ in $\mathrm{MacP}(3, 5)$. Those are the following uniform oriented matroids.

$$
\begin{gathered}
++++++++++\\
+-+--++--+
\end{gathered}
$$

In particular, they form a zero-dimensional simplicial complex, whose Euler characteristic equals 2, the same as $\chi(\mathrm{G}_3(\mathbb{R}^5))$.

There are 3 fixed points in $\mathrm{MacP}(3, 7)$ under the action of $\mathbb{Z}_7$, and all of them are again uniform oriented matroids.

$$
\begin{gathered}
+++++++++++++++++++++++++++++++++++\\
++-+++--+---+++++-++----+++-+--++++\\
+-+-+----+-+--++-+----+--+-+--++--+
\end{gathered}
$$

Therefore, the simplicial complex $\mathrm{MacP}(3, 7)^{\mathbb{Z}_7}$ is zero-dimensional, and its Euler characteristic is 3, the same as $\chi(\mathrm{G}_3(\mathbb{R}^7))$.

Also for $p = 11$, we obtain the expected 5 oriented matroids of rank 3 on 11 elements that are fixed under the action of $\mathbb{Z}_{11}$, and all of them are uniform. We omit listing them here. Finally, the largest prime for which we computed fixed points is $p = 13$. There are 6 uniform oriented matroids of rank 3 on 13 elements, that are fixed under the action of $\mathbb{Z}_{13}$.

$(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$-action on $\mathrm{MacP}(3, 8)$ and $\mathrm{MacP}(4, 8)$

Consider the group $G \subseteq \mathfrak{S}_8$ generated by the permutations $(1, 2)(3, 4)(5, 6)(7, 8)$, $(1, 3)(2, 4)(5, 7)(6, 8)$ and $(1, 5)(2, 6)(3, 7)(4, 8)$. It is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

There are 56 vertices of $\mathrm{MacP}(3, 8)$ that are fixed under the action of G, and they form a 0-dimensional simplicial complex. Each of the fixed points is an oriented matroid with 32 bases. Thus every connected component of the simplicial complex $\mathrm{MacP}(3, 8)^G$ is a point. In particular, the Euler characteristic of $\mathrm{MacP}(3, 8)^G$ is even, as well as $\chi(\mathrm{G}_3(\mathbb{R}^8))$.

Similarly, there are 70 vertices of $\mathrm{MacP}(4, 8)$ that are fixed under the action of G, and they also form a 0-dimensional simplicial complex.

$\mathbb{Z}_p$-action on $\mathrm{MacP}(3, n)$

For n being even and p being a prime, we can try to make use of Theorem 6.4.4.

We defined actions on $\mathrm{MacP}(3, 6)$ by four different groups generated by the permutations

(1) $(1, 2, 3)$,
(2) $(1, 2, 3, 4, 5)$,
(3) $(1, 2, 3)$ and $(4, 5, 6)$, and
(4) $(1, 2, 3)(4, 5, 6)$,

and actions by groups generated by the permutations

(1) $(1, 2, 3, 4, 5)$,
(2) $(1, 2, 3, 4, 5, 6, 7)$

on $\mathrm{MacP}(3, 8)$.

The dimension, Euler characteristic, f-vectors, integer homology and integer cohomology of the fixed point sets are given in Table 6.3. These computations were obtained using the software SageMath [80]. In particular, we can see that none of the computed values contradicts Theorem 6.4.1 nor Theorem 6.4.4.

Action of the Sylow p-subgroups

The Sylow 2-subgroup of $\mathfrak{S}_8$ is the wreath product $\mathbb{Z}_2 \wr \mathbb{Z}_2 \wr \mathbb{Z}_2$, and the Sylow 3-subgroup of $\mathfrak{S}_9$ is the wreath product $\mathbb{Z}_3 \wr \mathbb{Z}_3$.

The action of the wreath product $\mathbb{Z}_2 \wr \mathbb{Z}_2 \wr \mathbb{Z}_2$ on $\mathrm{MacP}(3, 8)$ and on $\mathrm{MacP}(4, 8)$ is fixed-point free.

The action of the group $\mathbb{Z}_3 \wr \mathbb{Z}_3$ has exactly one fixed point in $\mathrm{MacP}(3, 9)$, whereas there is no fixed oriented matroid of rank 4 on 9 elements under the action of the group $\mathbb{Z}_3 \wr \mathbb{Z}_3$. Hence, the action of $\mathbb{Z}_3 \wr \mathbb{Z}_3$ on $\mathrm{MacP}(4, 9)$ is fixed-point free. The chirotope of the rank 3 oriented matroid that is fixed under the action of this group is

$$000000000000000{+}{+}{+}0{+}{+}{+}{+}{+}{+}00000000000{+}{+}{+}0{+}{+}$$
$$ {+}{+}{+}{+}00000{+}{+}{+}0{+}{+}{+}{+}{+}{+}00000000000000000000000.$$

Groups that fix particular oriented matroids

So far, we have defined group actions on MacPhersonians, and searched for fixed points under these group actions. Since the fixed point subsets of a MacPhersonian give some information about the whole MacPhersonian, we consider in this section groups that fix oriented matroids whose inverse images under the map $\mu : \mathrm{G}_r(\mathbb{R}^n) \to \mathrm{MacP}(r, n)$ are not contractible, for example, non-realizable ones or those with disconnected realization spaces.

The first example of a non-realizable oriented matroid one usually meets is the non-Pappus oriented matroid of rank 3 on 9 elements, see Figure 6.4. Its pseudoline arrangement is obtained from the Pappus configuration of 9 lines in the plane, with one line replaced by a (non-straight) pseudoline. The non-Pappus oriented matroid is fixed only under the subgroup of $\mathfrak{S}_9$ of order 2, thus the subcomplex of $\mathrm{MacP}(3, 9)$ of fixed points under that group is too large. Therefore, we have not computed it. Another example of a non-realizable oriented matroid of rank 3 on 9 elements is given in Section 6.4.1.

In $\mathrm{MacP}(4, 8)$ we have considered two non-realizable oriented matroids. First, the oriented matroid RS(8), see [11, Sect. 1.5], is fixed under the action of the group G generated by the permutations $(1, 4)(2, 3)(5, 8)(6, 7)$ and $(1, 7)(2, 8)(3, 5)(4, 6)$, which is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. There are 26 998 fixed points that form 19 connected components - 6 of them are points, 12 of them are Klein bottles on 24 vertices each, and the last one is 7-dimensional on the remaining 26 704 vertices. The f-vector of the simplicial complex of fixed points is

$$f = (26\,998, 494\,160, 2\,800\,992, 7\,552\,384, 11\,153\,664, 9\,289\,728, 4\,091\,904, 737\,280),$$

and the Euler characteristic is 6, the same as the Euler characteristic of the Grassmannian $\mathrm{G}_4(\mathbb{R}^8)$. Note that the Klein bottle is not a Poincaré duality space, but we cannot apply Theorem 6.4.4, because the group that acts is $\mathbb{Z}_2 \times \mathbb{Z}_2$.

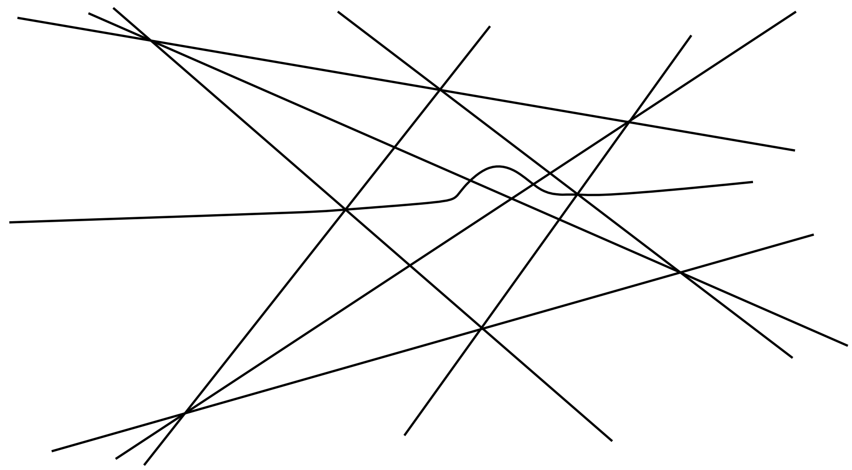


Figure 6.4: A non-Pappus oriented matroid.

The rank 4 oriented matroid on 8 elements EFM(8) (see [11, Exam. 10.4.1]) is fixed under the action of the dihedral group D_3 generated by the permutations $(1,2,3)(4,5,6)$ and $(1,6)(2,5)(3,4)(7,8)$. There are 30 462 fixed points under the action of $\mathbb{Z}_3$ as a subgroup of D_3 forming three connected components, two of which are points. The f-vector of the simplicial complex of fixed points in MacP(4, 8) under the action of $\mathbb{Z}_3 \subseteq D_3$ is

$$f = (30\,462, 2\,137\,512, 24\,865\,392, 112\,512\,576, 255\,113\,088, \\ 317\,613\,312, 218\,631\,168, 76\,796\,928, 10\,420\,224),$$

and the Euler characteristic is also 6. Only 1298 of these points are fixed under the action of the whole dihedral group D_3. They form a 5-dimensional simplicial complex with the f-vector

$$f = (1298, 14\,544, 41\,856, 46\,464, 19\,008, 1152),$$

whose Euler characteristic is 2. There are five connected components - two of the connected components are points, two are circles with 6 vertices each, and the last component is 5-dimensional with 1284 vertices.

The oriented matroid EFM(8) is also fixed under a very specific action of the group $\mathbb{Z}_2$: compose transpositions $(2,3)$ and $(4,5)$ with the reorientation -4568 and oriented matroid duality. The order in the composition does not matter, because all these actions commute. There are 639 584 fixed rank 4 oriented matroids on 8 elements under that group action. It, however, still remains to understand the simplicial complex.

There are a few more interesting oriented matroids that we considered, but which are not suited for our computations. The oriented matroid J(9) [11, Exam. 10.4.4], arises from EFM(8) by a single element extension. It is a uniform oriented matroid of rank 4 on 9 elements. However, it is fixed only under the action of the trivial subgroup of $\mathfrak{S}_9$. Tsukamoto [81] published an example of a rank 3 uniform oriented matroid $\mathcal{M}$ on 13 elements, whose realization space is disconnected. This means that the fiber $\mu^{-1}(\{\mathcal{M}\})$ is disconnected, which raises suspicion that the map μ is not a homotopy equivalence. Unfortunately, this oriented matroid is also fixed only under the trivial subgroup of $\mathfrak{S}_{13}$. Furthermore, the oriented matroid of Suvorov [79], a rank 3 oriented matroid on 14 elements, whose realization space is disconnected, is also fixed only under the trivial subgroup of $\mathfrak{S}_{14}$. Finally, the example of an oriented matroid with a disconnected realization space given by Richter-Gebert [71] is fixed only under a subgroup of $\mathfrak{S}_{14}$ of cardinality 2. Constructing the whole subcomplex of $\mathrm{MacP}(3, 14)$ that is fixed under this group action is beyond our computational limits.

Number of bases	$r = 3$ $n = 6$	$r = 3$ $n = 7$
1	20	35
2	180	420
3	480	1400
4	600	2380
5	1440	5376
6	1680	7560
7	1920	10 080
8	3000	17 640
9	6080	36 960
10	4384	26 656
11	0	13 440
12	3600	44 800
13	5760	72 800
14	8640	107 520
15	11 520	100 800
16	12 480	104 160
17	11 520	94 080
18	37 440	312 480
19	38 400	349 440
20	11 904	118 272
21		168 000
22		147 840
23		282 240
24		221 760
25		403 200
26		564 480
27		1 128 960
28		1 209 600
29		1 787 520
30		2 435 328
31		2 849 280
32		7 042 560
33		10 725 120
34		7 203 840
35		1 743 360
total	161 048	39 339 387

Table 6.2: The number of oriented matroids of rank 3 on 6 or 7 elements, sorted by the number of bases.

MacP(3,6)	$\mathbb{Z}_3$	$\mathbb{Z}_5$	$\mathbb{Z}_3 \times \mathbb{Z}_3$	$\mathbb{Z}_3$ diag.	MacP(3,8)	$\mathbb{Z}_5$	$\mathbb{Z}_7$
dimension	3	1	1	3	dimension	3	1
Euler ch.	0	0	0	0	Euler ch.	0	0
f_0	80	8	8	104	f_0	120	12
f_1	464	8	8	584	f_1	696	12
f_2	768			960	f_2	1152	
f_3	384			460	f_3	576	
H_0	$\mathbb{Z}^2$	$\mathbb{Z}^2$	$\mathbb{Z}^2$	$\mathbb{Z}$	H_0	$\mathbb{Z}^3$	$\mathbb{Z}^3$
H_1	$\mathbb{Z}_2^2$	$\mathbb{Z}^2$	$\mathbb{Z}^2$	$\mathbb{Z}$	H_1	$\mathbb{Z}_2^3$	$\mathbb{Z}^3$
H_2	0			$\mathbb{Z}$	H_2	0	
H_3	$\mathbb{Z}^2$			$\mathbb{Z}$	H_3	$\mathbb{Z}^3$	
H^0	$\mathbb{Z}^2$	$\mathbb{Z}^2$	$\mathbb{Z}^2$	$\mathbb{Z}$	H^0	$\mathbb{Z}^3$	$\mathbb{Z}^3$
H^1	0	$\mathbb{Z}^2$	$\mathbb{Z}^2$	$\mathbb{Z}$	H^1	0	$\mathbb{Z}^3$
H^2	$\mathbb{Z}_2^2$			$\mathbb{Z}$	H^2	$\mathbb{Z}_2^3$	
H^3	$\mathbb{Z}^2$			$\mathbb{Z}$	H^3	$\mathbb{Z}^3$	

Table 6.3: The f-vectors, homology and cohomology groups of order complexes of fixed point sets in MacP(3,6) and MacP(3,8) under group actions.

	$G_3(\mathbb{R}^6)$	$G_3(\mathbb{R}^7)$	$G_3(\mathbb{R}^8)$	$G_3(\mathbb{R}^9)$	$G_3(\mathbb{R}^{10})$	$G_3(\mathbb{R}^{11})$	$G_3(\mathbb{R}^{12})$	$G_3(\mathbb{R}^{13})$
H_0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
H_1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
H_2	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
H_3	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$
H_4	$\mathbb{Z}$	$\mathbb{Z} \times \mathbb{Z}_2$	$\mathbb{Z} \times \mathbb{Z}_2$	$\mathbb{Z} \times \mathbb{Z}_2$	$\mathbb{Z} \times \mathbb{Z}_2$	$\mathbb{Z} \times \mathbb{Z}_2$	$\mathbb{Z} \times \mathbb{Z}_2$	$\mathbb{Z} \times \mathbb{Z}_2$
H_5	$\mathbb{Z} \times \mathbb{Z}_2^2$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^4$
H_6	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^3$
H_7	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z} \times \mathbb{Z}_2^3$	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^5$	$\mathbb{Z}_2^5$	$\mathbb{Z}_2^5$	$\mathbb{Z}_2^5$
H_8	0	$\mathbb{Z} \times \mathbb{Z}_2$	$\mathbb{Z} \times \mathbb{Z}_2^2$	$\mathbb{Z} \times \mathbb{Z}_2^3$	$\mathbb{Z} \times \mathbb{Z}_2^3$	$\mathbb{Z} \times \mathbb{Z}_2^4$	$\mathbb{Z} \times \mathbb{Z}_2^4$	$\mathbb{Z} \times \mathbb{Z}_2^4$
H_9	$\mathbb{Z}$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^5$	$\mathbb{Z} \times \mathbb{Z}_2^6$	$\mathbb{Z}_2^7$	$\mathbb{Z}_2^8$	$\mathbb{Z}_2^8$
H_{10}		0	$\mathbb{Z}_2$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^5$	$\mathbb{Z}_2^5$	$\mathbb{Z}_2^6$
H_{11}		$\mathbb{Z}_2$	$\mathbb{Z} \times \mathbb{Z}_2^2$	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^6$	$\mathbb{Z}_2^7$	$\mathbb{Z} \times \mathbb{Z}_2^8$	$\mathbb{Z}_2^9$
H_{12}		0	$\mathbb{Z}_2$	$\mathbb{Z} \times \mathbb{Z}_2^2$	$\mathbb{Z} \times \mathbb{Z}_2^3$	$\mathbb{Z} \times \mathbb{Z}_2^5$	$\mathbb{Z} \times \mathbb{Z}_2^6$	$\mathbb{Z} \times \mathbb{Z}_2^7$
H_{13}			$\mathbb{Z}_2$	$\mathbb{Z}_2^3$	$\mathbb{Z} \times \mathbb{Z}_2^5$	$\mathbb{Z}_2^7$	$\mathbb{Z}_2^9$	$\mathbb{Z}_2^{10}$
H_{14}			0	$\mathbb{Z}_2$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^5$	$\mathbb{Z}_2^6$	$\mathbb{Z}_2^8$
H_{15}			$\mathbb{Z}$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^6$	$\mathbb{Z} \times \mathbb{Z}_2^8$	$\mathbb{Z}_2^{10}$
H_{16}				0	$\mathbb{Z}_2$	$\mathbb{Z} \times \mathbb{Z}_2^3$	$\mathbb{Z} \times \mathbb{Z}_2^5$	$\mathbb{Z} \times \mathbb{Z}_2^7$
H_{17}				$\mathbb{Z}_2$	$\mathbb{Z} \times \mathbb{Z}_2^2$	$\mathbb{Z}_2^5$	$\mathbb{Z}_2^8$	$\mathbb{Z}_2^{10}$
H_{18}				0	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^7$
H_{19}					$\mathbb{Z}_2$	$\mathbb{Z}_2^3$	$\mathbb{Z} \times \mathbb{Z}_2^5$	$\mathbb{Z}_2^8$
H_{20}					0	$\mathbb{Z}_2$	$\mathbb{Z}_2^3$	$\mathbb{Z} \times \mathbb{Z}_2^5$
H_{21}					$\mathbb{Z}$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^7$
H_{22}						0	$\mathbb{Z}_2$	$\mathbb{Z}_2^3$
H_{23}						$\mathbb{Z}_2$	$\mathbb{Z} \times \mathbb{Z}_2^2$	$\mathbb{Z}_2^5$
H_{24}						0	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$
H_{25}							$\mathbb{Z}_2$	$\mathbb{Z}_2^3$
H_{26}							0	$\mathbb{Z}_2$
H_{27}							$\mathbb{Z}$	$\mathbb{Z}_2^2$
H_{28}								0
H_{29}								$\mathbb{Z}_2$
H_{30}								0

Table 6.4: Integer homology of Grassmannians.

	$G_4(\mathbb{R}^8)$	$G_4(\mathbb{R}^9)$	$G_4(\mathbb{R}^{10})$	$G_4(\mathbb{R}^{11})$	$G_4(\mathbb{R}^{12})$	$G_4(\mathbb{R}^{13})$
H_0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
H_1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
H_2	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
H_3	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$
H_4	$\mathbb{Z} \times \mathbb{Z}_2^2$	$\mathbb{Z} \times \mathbb{Z}_2^2$	$\mathbb{Z} \times \mathbb{Z}_2^2$	$\mathbb{Z} \times \mathbb{Z}_2^2$	$\mathbb{Z} \times \mathbb{Z}_2^2$	$\mathbb{Z} \times \mathbb{Z}_2^2$
H_5	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^4$
H_6	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^5$	$\mathbb{Z}_2^5$	$\mathbb{Z}_2^5$	$\mathbb{Z}_2^5$
H_7	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^5$	$\mathbb{Z}_2^5$	$\mathbb{Z}_2^6$	$\mathbb{Z}_2^6$	$\mathbb{Z}_2^6$
H_8	$\mathbb{Z}^2 \times \mathbb{Z}_2^3$	$\mathbb{Z}^2 \times \mathbb{Z}_2^4$	$\mathbb{Z}^2 \times \mathbb{Z}_2^6$	$\mathbb{Z}^2 \times \mathbb{Z}_2^6$	$\mathbb{Z}^2 \times \mathbb{Z}_2^7$	$\mathbb{Z}^2 \times \mathbb{Z}_2^7$
H_9	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^7$	$\mathbb{Z}_2^8$	$\mathbb{Z}_2^{10}$	$\mathbb{Z}_2^{10}$	$\mathbb{Z}_2^{11}$
H_{10}	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^5$	$\mathbb{Z}_2^8$	$\mathbb{Z}_2^9$	$\mathbb{Z}_2^{11}$	$\mathbb{Z}_2^{11}$
H_{11}	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^6$	$\mathbb{Z}_2^8$	$\mathbb{Z}_2^{11}$	$\mathbb{Z}_2^{12}$	$\mathbb{Z}_2^{14}$
H_{12}	$\mathbb{Z} \times \mathbb{Z}_2^2$	$\mathbb{Z} \times \mathbb{Z}_2^4$	$\mathbb{Z}^2 \times \mathbb{Z}_2^8$	$\mathbb{Z}^2 \times \mathbb{Z}_2^{10}$	$\mathbb{Z}^2 \times \mathbb{Z}_2^{13}$	$\mathbb{Z}^2 \times \mathbb{Z}_2^{14}$
H_{13}	$\mathbb{Z}_2$	$\mathbb{Z}_2^5$	$\mathbb{Z}_2^8$	$\mathbb{Z}_2^{13}$	$\mathbb{Z}_2^{15}$	$\mathbb{Z}_2^{18}$
H_{14}	$\mathbb{Z}_2$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^8$	$\mathbb{Z}_2^{11}$	$\mathbb{Z}_2^{16}$	$\mathbb{Z}_2^{18}$
H_{15}	0	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^6$	$\mathbb{Z}_2^{12}$	$\mathbb{Z}_2^{15}$	$\mathbb{Z}_2^{20}$
H_{16}	$\mathbb{Z}$	$\mathbb{Z} \times \mathbb{Z}_2$	$\mathbb{Z}^2 \times \mathbb{Z}_2^5$	$\mathbb{Z}^2 \times \mathbb{Z}_2^9$	$\mathbb{Z}^3 \times \mathbb{Z}_2^{15}$	$\mathbb{Z}^3 \times \mathbb{Z}_2^{18}$
H_{17}		$\mathbb{Z}_2^2$	$\mathbb{Z}_2^5$	$\mathbb{Z}_2^{11}$	$\mathbb{Z}_2^{16}$	$\mathbb{Z}_2^{23}$
H_{18}		0	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^8$	$\mathbb{Z}_2^{15}$	$\mathbb{Z}_2^{20}$
H_{19}		$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^8$	$\mathbb{Z}_2^{13}$	$\mathbb{Z}_2^{21}$
H_{20}		0	$\mathbb{Z} \times \mathbb{Z}_2^2$	$\mathbb{Z} \times \mathbb{Z}_2^5$	$\mathbb{Z}^2 \times \mathbb{Z}_2^{12}$	$\mathbb{Z}^2 \times \mathbb{Z}_2^{18}$
H_{21}			$\mathbb{Z}_2$	$\mathbb{Z}_2^6$	$\mathbb{Z}_2^{11}$	$\mathbb{Z}_2^{20}$
H_{22}			$\mathbb{Z}_2$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^{10}$	$\mathbb{Z}_2^{16}$
H_{23}			0	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^7$	$\mathbb{Z}_2^{16}$
H_{24}			$\mathbb{Z}$	$\mathbb{Z} \times \mathbb{Z}_2$	$\mathbb{Z}^2 \times \mathbb{Z}_2^6$	$\mathbb{Z}^2 \times \mathbb{Z}_2^{12}$
H_{25}				$\mathbb{Z}_2^2$	$\mathbb{Z}_2^5$	$\mathbb{Z}_2^{13}$
H_{26}				0	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^9$
H_{27}				$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^9$
H_{28}				0	$\mathbb{Z} \times \mathbb{Z}_2^2$	$\mathbb{Z} \times \mathbb{Z}_2^5$
H_{29}					$\mathbb{Z}_2$	$\mathbb{Z}_2^6$
H_{30}					$\mathbb{Z}_2$	$\mathbb{Z}_2^3$
H_{31}					0	$\mathbb{Z}_2^3$
H_{32}					$\mathbb{Z}$	$\mathbb{Z} \times \mathbb{Z}_2$
H_{33}						$\mathbb{Z}_2^2$
H_{34}						0
H_{35}						$\mathbb{Z}_2$
H_{36}						0

Table 6.5: Integer homology of Grassmannians.

Appendix A

Equivariant obstruction theory

This is an exposition on the equivariant obstruction theory, as introduced by tom Dieck [30, Sec. II.3], which also follows [14, Sec. 1]. In this thesis we use the equivariant obstruction theory to give a criterion of existence of equivariant maps. More precisely, if G is a finite group, (X, A) is a relative G-CW complex, such that the action of G on $X \setminus A$ is free, and Y is a G space, the equivariant obstruction theory answers the question, whether a G-equivariant map $f : A \to Y$ can be extended to a G-equivariant map $F : X \to Y$.

Definition A.1. Let X be a CW complex and let G be a finite group. We say that X is a *G-CW complex* or *G-invariant CW complex* if there is an action of G on X such that

(i) For every open cell $e \in X$ and for every $g \in G$, the left translation $g \cdot e$ is also an open cell in X, and

(ii) If $g \cdot e = e$, then the map $x \mapsto g \cdot x$ is the identity on e.

A relative CW complex (X, A) is said to be a *relative G-CW complex* if both X and A are G-CW complexes, where the action of G on A is the restriction of the G-action on X.

Let G be a finite group and let (X, A) be a relative G-CW complex, such that the action of G on $X \setminus A$ is free. Since the action of G is free on every skeleton of $X \setminus A$, it induces a free G-action on the cellular chain complex $C_*(X, A)$, which turns it into a chain complex of free $\mathbb{Z}[G]$-modules.

Let M be a $\mathbb{Z}[G]$-module. The homology of the cochain complex $C^*_G(X, A; M) = \hom_{\mathbb{Z}[G]}(C_*(X, A), M)$ is the equivariant cohomology $H^*_G(X, A; M)$ of (X, A) with coefficients in M.

Let Y be a path connected G-space. Moreover, suppose that Y is *n-simple* for a fixed integer n, i.e., suppose that the fundamental group $\pi_1(Y, y_0)$ acts trivially on $\pi_n(Y, y_0)$ for every $y_0 \in Y$. The G-action on Y induces a G action on the set of free homotopy classes $[S^n, Y]$, and since $\pi_1(Y)$ acts trivially on $\pi_n(Y)$, it induces a G-action on the homotopy group $\pi_n(Y)$, as well. This turns $\pi_n(Y)$ into a $\mathbb{Z}[G]$-module. Therefore, we can talk about equivariant cohomology $H^*_G(X, A; \pi_n(Y))$ for every relative G-CW complex (X, A) with a free G action on $X \setminus A$, and for every n-simple path connected G-space Y.

Let X_k denote the k-th skeleton of X, and let $[X_k, Y]_G$ denote the set of homotopy classes of G-equivariant maps $X_k \to Y$.

Proposition A.2 ([30, Thm. II.3.10])**.** *Let $n \geq 1$ be an integer. Then there exists an exact obstruction sequence*

$$[X_{n+1}, Y]_G \longrightarrow \operatorname{im}([X_n, Y]_G \longrightarrow [X_{n-1}, Y]_G) \xrightarrow{\mathfrak{o}_G^{n+1}} H_G^{n+1}(X, A; \pi_n(Y)),$$

which is natural in (X, A) and Y.

If $f : X_{n-1} \to Y$ is a G-equivariant map that can be continuously and equivariantly extended to a map $X_n \to Y$, then its homotopy class $[f] \in \operatorname{im}([X_n, Y]_G \longrightarrow [X_{n-1}, Y]_G)$ defines an element $\mathfrak{o}_G^{n+1}([f]) \in H_G^{n+1}(X, A; \pi_n(Y))$, called an *obstruction element*. By the exactness of the above sequence, the obstruction element $\mathfrak{o}_G^{n+1}([f])$ vanishes if and only if there is a map in the homotopy class of f that continuously and G-equivariantly extends to X_{n+1}.

The obstruction element can also be introduced on the cochain level. Let $h : X_n \to Y$ be a G-equivariant map, let $e \in C_{n+1}(X, A)$ be a generator, and denote by $\varphi : (D^{n+1}, S^n) \longrightarrow (X_{n+1}, X_n)$ the attaching map for the cell e. Then the composition

$$S^n \xrightarrow{\varphi|_{S^n}} X_n \xrightarrow{h} Y$$

defines the *obstruction cochain* $\mathfrak{o}_G^{n+1}(h) \in C_G^{n+1}(X, A; \pi_n(Y))$ given by

$$\mathfrak{o}_G^{n+1}(h)(e) = [h \circ \varphi] \in [S^n, Y].$$

If $\pi_n(Y) \cong \mathbb{Z}$, then $\mathfrak{o}_G^{n+1}(h)(e) = \deg(h \circ \varphi)$.

Consult [30, p. 115ff.] for the proof that the cohomology class of the obstruction cochain is the obstruction element defined above.

The primary obstruction

Recall that (X, A) is a relative G-CW complex, such that the action of G on $X \setminus A$ is free, and that Y is a path connected, n-simple G-space, for some integer $n \geq 1$. Let us now assume in addition that the space Y is $(n-1)$-*connected*, thus that Y is nonempty, path connected and that $\pi_i(Y) = 0$ for every $1 \leq i \leq n-1$.

Proposition A.3 ([30, Prop. II.3.15])**.**

(i) *For every G-map $f : A \to Y$, there is a G-map $h : X_n \to Y$, such that $h|_A = f$.*

(ii) *Any two G-extensions of f are G-homotopic* rel A *on X_{n-1}.*

(iii) *Let $H : A \times I \to Y$ be a G-homotopy between G-maps $f, g : A \to Y$, and let $k, h : X_n \to Y$ be G-extensions of f and g. Then there exists a G-homotopy $K : X_{n-1} \times I \to Y$ between $k|_{X_{n-1}}$ and $h|_{X_{n-1}}$, such that $K|_{A \times I} = H$.*

Let $f : A \to Y$ be a G-map. Proposition A.3(i) implies that the set $[X_n, Y]_G$ is nonempty, and Proposition A.3(ii) implies that every extension $h : X_n \to Y$ of f defines the same element in $\operatorname{im}([X_n, Y]_G \longrightarrow [X_{n-1}, Y]_G)$. Finally, Proposition A.3(iii) implies that every map $g : A \to Y$ that is G-homotopic to f defines the same element in $\mathrm{im}([X_n, Y]_G \longrightarrow [X_{n-1}, Y]_G)$. Therefore, there is a well-defined map

$$\gamma : [A, Y]_G \longrightarrow H_G^{n+1}(X, A; \pi_n(Y)),$$

given by $\gamma([f]) = \mathfrak{o}_G^{n+1}([h])$, where $h : X_n \to Y$ is a G-extension of f. The element $\gamma([f]) \in H_G^{n+1}(X, A; \pi_n(Y))$ is called *the primary obstruction to extending* f. If $\mathrm{im}([X_n, Y]_G \longrightarrow [X_{n-1}, Y]_G) = \{*\}$, then the element $\mathfrak{o}_G^{n+1}(*) \in H_G^{n+1}(X, A; \pi_n(Y))$ is called *the primary obstruction*, and it does not depend on the choice of the map $X_n \to Y$.

Corollary A.4. *Let X be a free G-CW complex, and let Y be an $(n-1)$-connected and n-simple G-space for some integer $n \geq 1$. Then the primary obstruction element $\mathfrak{o}_G^{n+1}([h]) \in H_G^{n+1}(X; \pi_n(Y))$ does not depend on the choice of a G-map $h : X_n \to Y$.*

Appendix B

The Fadell-Husseini index

Fadell & Husseini [32] developed an ideal-valued cohomological index theory, which has been proven to be very useful in topological combinatorics. Consult the paper [32] for the original exposition and proofs. In this thesis we exploit the monotonicity property of the Fadell-Husseini index, which gives a sufficient condition for non-existence of equivariant maps.

Let X be a topological space, G be a finite group acting on X and $\mathbb{F}$ be a field. Here we recapitulate the Fadell-Husseini index theory only in the generality needed for this thesis.

Definition B.1. The *Borel construction* of X is the orbit space

$$X \times_G \mathrm{E}\,G = (X \times \mathrm{E}\,G)/G,$$

where $\mathrm{E}\,G$ is the total space of the universal bundle over the classifying space $\mathrm{B}\,G$, and the action of G on the product $X \times \mathrm{E}\,G$ is diagonal.

Note that if X is a point, the Borel construction $X \times_G \mathrm{E}\,G$ is the classifying space $\mathrm{B}\,G$.

Definition B.2. The *G-equivariant cohomology* of X is defined as the ordinary cohomology of its Borel construction

$$H^*_G(X; \mathbb{F}) = H^*(X \times_G \mathrm{E}\,G; \mathbb{F}).$$

In particular, $H^*_G(\mathrm{pt}; \mathbb{F}) = H^*(\mathrm{B}\,G; \mathbb{F})$, which is isomorphic to the group cohomology $H^*(G; \mathbb{F})$.

There is a Serre spectral sequence associated to the universal bundle

$$X \longrightarrow X \times_G \mathrm{E}\,G \longrightarrow \mathrm{B}\,G,$$

whose E_2-term is given by

$$E_2^{p,q} = H^p(\mathrm{B}\,G; \mathcal{H}^q(X; \mathbb{F})) \Rightarrow H^{p+q}(X \times_G \mathrm{E}\,G; \mathbb{F}),$$

which is often used for computing the Fadell-Husseini index, see for example [18].

The constant map $p_X : X \to \mathrm{pt}$ is G-equivariant, and it induces a map of G-equivariant cohomology groups

$$p_X^* \colon H^*_G(\mathrm{pt}; \mathbb{F}) \longrightarrow H^*_G(X; \mathbb{F}).$$

Definition B.3. The *Fadell–Husseini index* of the space X with respect to the group G and coefficients $\mathbb{F}$ is the kernel ideal of the map in equivariant cohomology induced by the G-equivariant map $p_X : X \to \mathrm{pt}$:

$$\begin{aligned}\mathrm{Index}_G(X;\mathbb{F}) &= \ker\big(p_X^* : H_G^*(\mathrm{pt};\mathbb{F}) \longrightarrow H_G^*(X;\mathbb{F})\big)\\ &= \ker\big(H^*(\mathrm{B}\,G;\mathbb{F}) \longrightarrow H^*(\mathrm{E}\,G \times_G X;\mathbb{F})\big).\end{aligned}$$

Example B.4. Let $X = S^n$ be a sphere, $G = \mathbb{Z}_2$ act antipodally on S^n, and let $\mathbb{F} = \mathbb{F}_2$ be the field on two elements. Then $\mathrm{E}\,\mathbb{Z}_2 = S^\infty$ and $\mathrm{B}\,\mathbb{Z}_2 = \mathbb{R}\mathrm{P}^\infty$. Moreover, the Borel construction $\mathrm{E}\,\mathbb{Z}_2 \times_{\mathbb{Z}_2} S^n$ is homotopy equivalent to $\mathbb{R}\mathrm{P}^n$. Since $H^*(\mathbb{R}\mathrm{P}^\infty;\mathbb{F}_2) = \mathbb{F}_2[t]$ with $\deg(t) = 1$, and $H^*(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2) = \mathbb{F}_2[t]/\langle t^{n+1}\rangle$, it follows that $\mathrm{Index}_{\mathbb{Z}_2}(S^n;\mathbb{F}_2) = \langle t^{n+1}\rangle$.

Example B.5. If there exists $x_0 \in X$ such that $g \cdot x_0 = x_0$ for every $g \in G$, then $\mathrm{Index}_G(X;\mathbb{F}) = 0$.

Indeed, consider the commutative diagram of G-spaces and G-maps

$$\begin{array}{ccc}\{x_0\} & \longrightarrow & X\\ & \searrow & \downarrow{\scriptstyle p_X}\\ & & \mathrm{pt}\end{array}$$

It induces a commutative diagram in G-equivariant cohomology

$$\begin{array}{ccc}H_G^*(\{x_0\};\mathbb{F}) & \longleftarrow & H_G^*(X;\mathbb{F})\\ & \nwarrow & \uparrow{\scriptstyle p_X^*}\\ & & H_G^*(\mathrm{pt};\mathbb{F})\end{array}$$

where the map $H_G^*(\mathrm{pt};\mathbb{F}) \longrightarrow H_G^*(\{x_0\};\mathbb{F})$ is an isomorphism. Consequently, the map p_X^* is injective, thus $\mathrm{Index}_G(X;\mathbb{F}) = \ker(p_X^*) = 0$.

Properties

The most useful property of the Fadell–Husseini index for our applications is that it yields a necessary condition for the existence of a G-equivariant map $X \to Y$.

Proposition B.6. *Let X and Y be two G-spaces. If there exists a G-equivariant map $X \to Y$, then*

$$\mathrm{Index}_G(Y;\mathbb{F}) \subseteq \mathrm{Index}_G(X;\mathbb{F}).$$

Now we list a few properties that help compute the Fadell-Husseini index.

Proposition B.7. *Let X be a G-space, Y be an H-space and let $\mathbb{F}$ be a field. Then*

$$\mathrm{Index}_{G\times H}(X\times Y;\mathbb{F}) = \mathrm{Index}_G(X;\mathbb{F}) \otimes H^*(\mathrm{B}\,H;\mathbb{F}) + H^*(\mathrm{B}\,G;\mathbb{F}) \otimes \mathrm{Index}_H(Y;\mathbb{F}),$$

where the action of $G \times H$ on $X \times Y$ is diagonal, i.e., $(g,h)\cdot(x,y) = (g\cdot x, h\cdot y)$ for every $g \in G, h \in H, x \in X$ and $y \in Y$.

Suppose in addition that $H^*(\mathrm{B}\,G;\mathbb{F}) = \mathbb{F}[x_1,\dots,x_n]$ and $H^*(\mathrm{B}\,H;\mathbb{F}) = \mathbb{F}[y_1,\dots,y_m]$. If $\mathrm{Index}_G(X;\mathbb{F}) = \langle p_1,\dots,p_i\rangle$ and $\mathrm{Index}_H(Y;\mathbb{F}) = \langle q_1,\dots,q_j\rangle$, then

$$\mathrm{Index}_{G\times H}(X\times Y;\mathbb{F}) = \langle p_1,\dots,p_i,q_1,\dots,q_j\rangle.$$

Example B.8. Let $S^{n_1}, \dots, S^{n_k}$ be spheres with an antipodal $\mathbb{Z}_2$-action. Then the group $\mathbb{Z}_2^k = \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$ acts diagonally on the product $S^{n_1} \times \cdots \times S^{n_k}$. Since $H^*(\mathrm{B}\,\mathbb{Z}_2^k; \mathbb{F}_2) \cong \mathbb{F}_2[t_1, \dots, t_k]$, it follows from Example B.4 that

$$\mathrm{Index}_{\mathbb{Z}_2^k}(S^{n_1} \times \cdots \times S^{n_k}; \mathbb{F}_2) = \langle t_1^{n_1+1}, \dots, t_k^{n_k+1} \rangle.$$

Proposition B.9. *Let X be a G-space, Y be an H-space, and let $\mathbb{F}$ be a field. If the group $G \times H$ acts on the join $X * Y$ by $(g,h) \cdot (\lambda x \oplus (1-\lambda)y) = \lambda(g \cdot x) \oplus (1-\lambda)(h \cdot y)$, then*

$$\mathrm{Index}_{G\times H}(X * Y; \mathbb{F}) \subseteq (\mathrm{Index}_G(X; \mathbb{F}) \otimes H^*(\mathrm{B}\,H; \mathbb{F})) \cap (H^*(\mathrm{B}\,G; \mathbb{F}) \otimes \mathrm{Index}_H(Y; \mathbb{F})),$$

where $\cap$ denotes the intersection of ideals, and

$$\mathrm{Index}_{G\times H}(X * Y; \mathbb{F}) \supseteq (\mathrm{Index}_G(X; \mathbb{F}) \otimes H^*(\mathrm{B}\,H; \mathbb{F})) \cdot (H^*(\mathrm{B}\,G; \mathbb{F}) \otimes \mathrm{Index}_H(Y; \mathbb{F})),$$

where $\cdot$ denotes the product of ideals.

Example B.10. Let $S^{n_1}, \dots, S^{n_k}$ again be spheres with an antipodal $\mathbb{Z}_2$-action. Since $\mathrm{Index}_{\mathbb{Z}_2}(S^{n_i}; \mathbb{F}_2) = \langle t_i^{n_i+1} \rangle \subseteq H^*(\mathrm{B}\,\mathbb{Z}_2; \mathbb{F}_2) = \langle t_i \rangle$, we get that the intersection of ideals and the product of ideals in the previous proposition are the same. Therefore,

$$\mathrm{Index}_{\mathbb{Z}_2^k}(S^{n_1} * \cdots * S^{n_k}; \mathbb{F}_2) = \langle t_1^{n_1+1} \cdots t_k^{n_k+1} \rangle.$$

Proposition B.11. *Let X and Y be two G-spaces. If G acts diagonally on the product $X \times Y$ and on the join $X * Y$, i.e., $g \cdot (x,y) = (g \cdot x, g \cdot y)$, and $g \cdot (\lambda x \oplus (1-\lambda)y) = \lambda(g \cdot x) \oplus (1-\lambda)(g \cdot y)$, then*

$$\mathrm{Index}_G(X \times Y; \mathbb{F}) \supseteq \mathrm{Index}_G(X; \mathbb{F}) \cap \mathrm{Index}_G(Y; \mathbb{F}),$$

and

$$\mathrm{Index}_G(X * Y; \mathbb{F}) \supseteq \mathrm{Index}_G(X; \mathbb{F}) \cdot \mathrm{Index}_G(Y; \mathbb{F}).$$

Proposition B.12. *Let U and V be two G-representations, i.e., real vector spaces with a G-action, and denote by $S(U)$ the unit sphere in U, and by $S(V)$ the unit sphere in V. If* $\mathrm{Index}_G(S(U); \mathbb{F}) = \langle f \rangle$ *and* $\mathrm{Index}_G(S(V); \mathbb{F}) = \langle g \rangle$, *then*

$$\mathrm{Index}_G(S(U \oplus V); \mathbb{F}) = \langle f \cdot g \rangle.$$

Bibliography

[1] Alejandro Adem and James R. Milgram, *Cohomology of Finite Groups*, second ed., Grundlehren der Mathematischen Wissenschaften, vol. 309, Springer-Verlag, Berlin, 2004.

[2] Laura Anderson and James F. Davis, *Mod 2 cohomology of combinatorial Grassmannians*, Sel. math., New ser. **8** (2002), 161–200.

[3] Nima Arkani-Hamed, Yuntao Bai, and Thomas Lam, *Positive geometries and canonical forms*, Preprint, March 2017, 123 pages, arXiv:1703.04541.

[4] Nima Arkani-Hamed, Hugh Thomas, and Jaroslav Trnka, *Unwinding the amplituhedron in binary*, Journal of High Energy Physics **1** (2018), 016.

[5] Nima Arkani-Hamed and Jaroslav Trnka, *The amplituhedron*, Journal of High Energy Physics **2014** (2014).

[6] David Avis, *Non-partitionable point sets*, Inform. Process. Letters **19** (1984), 125–129.

[7] Eric Babson, *A combinatorial flag space*, Ph.D. thesis, Massachusetts Institute of Technology, 1994, http://hdl.handle.net/1721.1/12052.

[8] Imre Bárány and Jiři Matoušek, *Simultaneous partitions of measures by k-fans*, Discrete Comput. Geom. **25** (2001), 317–334.

[9] Daniel K. Biss, *The homotopy type of the matroid Grassmannian*, Annals of Mathematics **158** (2003), 929–952, Erratum: 170:493, 2009.

[10] ______, *Erratum to "The homotopy type of the matroid Grassmannian"*, Annals of Mathematics **170** (2009), 493.

[11] Anders Björner, Michael Las Vergnas, Bernd Sturmfels, Neil White, and Günter M. Ziegler, *Oriented Matroids*, Encyclopedia of Mathematics and its Applications, Cambridge University Press, 1999, second edition.

[12] Anders Björner, Jiří Matoušek, and Günter M. Ziegler, *Using brouwer's fixed point theorem*, A Journey Through Discrete Mathematics: A Tribute to Jiří Matoušek (Martin Loebl, Jaroslav Nešetřil, and Robin Thomas, eds.), Springer International Publishing, Cham, 2017, pp. 221–271.

[13] Pavle V. M. Blagojević, Boris Bukh, and Roman Karasev, *Turán numbers for $k_{s,t}$-free graphs: Topological obstructions and algebraic constructions*, Israel J. Math.F **197** (2013), 199–214.

[14] Pavle V. M. Blagojević and Aleksandra S. Dimitrijević Blagojević, *Using equivariant obstruction theory in combinatorial geometry*, Topology and its Applications **154** (2007), 2635–2655.

[15] Pavle V. M. Blagojević, Florian Frick, Albert Haase, and Günter M. Ziegler, *Topology of the Grünbaum–Hadwiger–Ramos hyperplane mass partition problem*, Preprint, 27 pages, February 2015, arXiv:1502.02975; Transactions Amer. Math. Soc., to appear.

[16] Pavle V. M. Blagojević, Pavel Galashin, Nevena Palić, and Günter M. Ziegler, *Some more amplituhedra are contractible*, Preprint, 7 pages, arXiv:1806.00827, June 2018.

[17] Pavle V. M. Blagojević, Jonathan Kliem, Nevena Palić, Johanna K. Steinmeyer, and Günter M. Ziegler, *Discretizing measure partition results*, in preparation.

[18] Pavle V. M. Blagojević, Wolfgang Lück, and Günter M. Ziegler, *Equivariant topology of configuration spaces*, J. Topol. **8** (2015), 414–456.

[19] Pavle V. M. Blagojević, Nevena Palić, and Günter M. Ziegler, *Cutting a part from many measures*, Preprint, October 2017, 15 pages, arXiv:1710.05118.

[20] Pavle V. M. Blagojević, Günter Rote, Johanna Steinmeyer, and Günter M. Ziegler, *Convex equipartitions of colored point sets*, Discrete Comput. Geometry (2017), Published online, 9 pages; arXiv:1705.03953.

[21] Pavle V. M. Blagojević and Günter M. Ziegler, *Convex equipartitions via equivariant obstruction theory*, Israel J. Math. **200** (2014), 49–77.

[22] Karol Borsuk, *Über eine Klasse von lokal zusammenhängenden Räumen*, Fund. Math. **19** (1932), 220–242.

[23] Jacob Bourjaily and Hugh Thomas, *What is the amplituhedron?*, Notices of the AMS **65** (2018), 167–169.

[24] Aldridge. K. Bousfield and Daniel. M. Kan, *Homotopy Limits, Completions and Localizations*, Lecture Notes in Mathematics, Vol. 304, Springer-Verlag, Berlin-New York, 1972.

[25] Glen E. Bredon, *Introduction to Compact Transformation Groups*, Academic Press, New York-London, 1972, Pure and Applied Mathematics, Vol. 46.

[26] Theodore Chang and Tor Skjelbred, *Group actions on Poincaré duality spaces*, Bull. Amer. Math. Soc. **78** (1972), no. 6, 1024–1026.

[27] Frederick R. Cohen, *The homology of $\mathcal{C}_{n+1}$-spaces, $n \geq 0$*, The Homology of Iterated Loop Spaces, Springer Berlin Heidelberg, Berlin, Heidelberg, 1976, pp. 207–351.

[28] Michel Coste, *An Introduction to Semialgebraic Geometry*, Université de Rennes, 2002, 78 pages, `http://gcomte.perso.math.cnrs.fr/M2/CosteIntroToSemialGeo.pdf`.

[29] Tammo tom Dieck, *Algebraic Topology*, EMS Textbooks in Mathematics, European Mathematical Society (EMS), Zürich, 2008.

[30] Tammo tom Dieck, *Transformation Groups*, Studies in Mathematics, vol. 8, Walter de Gruyter, Berlin, 1987.

[31] Albrecht Dold, *Lectures on Algebraic Topology*, second ed., Grundlehren Math. Wissenschaften, vol. 200, Springer-Verlag, Berlin, New York, 1980.

[32] Edward R. Fadell and Sufian Y. Husseini, *An ideal-valued cohomological index theory with applications to Borsuk-Ulam and Bourgin-Yang theorems*, Ergodic Theory Dynam. Systems **8*** (1988), 73–85.

[33] Lukas Finschi, *Homepage of oriented matroids*, `http://www.om.math.ethz.ch`.

[34] Edwin E. Floyd, *On periodic maps and the Euler characteristics of associated spaces*, Trans. Amer. Math. Soc. **72** (1952), 138–147.

[35] Sergey Fomin and Andrei Zelevinsky, *Double Bruhat cells and total positivity*, J. Amer. Math. Soc. **12** (1999), 335–380.

[36] Pavel Galashin, Steven N. Karp, and Thomas Lam, *The totally nonnegative Grassmannian is a ball*, Preprint, March 2018, 19 pages, arXiv:1707.02010.

[37] Pavel Galashin and Thomas Lam, *Parity duality for the amplituhedron*, Preprint, May 2018, 52 pages, arXiv:1805.00600.

[38] Izrail' M. Gel'fand and Robert D. MacPherson, *A combinatorial formula for the Pontrjagin classes*, Bull. Amer. Math. Soc. **26** (1992), no. 2, 304–309.

[39] Mikhael Gromov, *Isoperimetry of waists and concentration of maps*, Geom. Funct. Anal. **13** (2003), 178–215.

[40] Branko Grünbaum, *Partitions of mass-distributions and of convex bodies by hyperplanes*, Pacific J. Math. **10** (1960), 1257–1261.

[41] Larry Guth, *The waist inequality in Gromov's work*, Preprint, 13 pages, `http://math.mit.edu/~lguth/Exposition/waist.pdf`.

[42] Hugo Hadwiger, *Simultane Vierteilung zweier Körper*, Arch. Math. (Basel) **17** (1966), 274–278.

[43] Olof Hanner, *Some theorems on absolute neighborhood retracts*, Ark. Mat. **1** (1951), 389–408.

[44] Allen Hatcher, *Algebraic Topology*, Cambridge University Press, Cambridge, 2002.

[45] Andreas F. Holmsen, Jan Kynčl, and Claudiu Valculescu, *Near equipartitions of colored point sets*, Comput. Geom. **65** (2017), 35–42.

[46] Wu Yi Hsiang, *Cohomology Theory of Topological Transformation Groups*, Springer-Verlag, New York-Heidelberg, 1975, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 85.

[47] Jan W. Jaworowski, *Extensions of G-maps and Euclidean G-retracts*, Math. Z. **146** (1976), 143–148.

[48] Stefan J. Jungkind, *Some computations of the homology of real Grassmannian manifolds*, Master's thesis, The University of British Columbia, 1979.

[49] Roman Karasev, Alfredo Hubard, and Boris Aronov, *Convex equipartitions: the spicy chicken theorem*, Geom. Dedicata **170** (2014), 263–279.

[50] Steven N. Karp, *Sign variation, the Grassmannian, and total positivity*, J. Combin. Theory Ser. A **145** (2017), 308–339.

[51] Steven N. Karp and Lauren K. Williams, *The $m = 1$ amplituhedron and cyclic hyperplane arrangements*, Sém. Lothar. Combin. **78B** (2017), (Proc. 29th Conference on Formal Power Series and Algebraic Combinatorics, London), Article #20, 12 pages.

[52] Thomas Lam, *Totally nonnegative Grassmannian and Grassmann polytopes*, Current developments in mathematics 2014, Int. Press, Somerville, MA, 2016, pp. 51–152.

[53] Emerson León and Günter M. Ziegler, *Spaces of convex n-partitions*, Preprint, November 2015, 22 pages, arXiv:1511.02904; in: "New Trends in Intuitive Geometry" (G. Ambrus, I. Bárány, K. J. Böröczky, G. Fejes Tóth, J. Pach, eds.), Bolyai Society Mathematical Studies, vol. 27, Springer, Berlin Heidelberg, 2018, to appear.

[54] George Lusztig, *Total positivity in reductive groups*, Lie theory and geometry, Progr. Math., vol. 123, Birkhäuser Boston, Boston, MA, 1994, pp. 531–568.

[55] ______, *Introduction to total positivity*, Positivity in Lie theory: open problems, De Gruyter Exp. Math., vol. 26, de Gruyter, Berlin, 1998, pp. 133–145.

[56] Robert MacPherson, *Combinatorial differential manifolds*, Topological methods in modern mathematics (Houston, TX), Publish or Perish, 1993, pp. 203–221.

[57] Vladimir V. Makeev, *The Knaster problem and almost spherical sections*, Math. of the USSR-Sbornik **66** (1990), 431–438.

[58] Benjamin M. Mann and R. James Milgram, *On the Chern classes of the regular representations of some finite groups*, Proceedings of the Edinburgh Mathematical Society **25** (1982), 259–268.

[59] Jiří Matoušek, *Using the Borsuk-Ulam Theorem*, Springer-Verlag, Berlin-Heidelberg-New York, 2003.

[60] Yashar Memarian, *On Gromov's waist of the sphere theorem*, J. Topol. Anal. **3** (2011), 7–36.

[61] John Milnor, *On spaces having the homotopy type of a* CW*-complex*, Trans. Amer. Math. Soc. **90** (1959), 272–280.

[62] John W. Milnor and James D. Stasheff, *Characteristic Classes*, Princeton University Press, Princeton, N. J.; University of Tokyo Press, Tokyo, 1974, Annals of Mathematics Studies, No. 76.

[63] Nicolai E. Mnëv, *On D.K. Biss' papers "The homotopy type of the matroid Grassmannian" Annals of Mathematics 158 (2003) 929 – 952 and "Oriented matroids, complex manifolds, and a combinatorial model for BU" Advances in Mathematics 179 (2003) 250 – 290*, Preprint, September 2007, 2pages, `arXiv:0709.1291`.

[64] Nicolai. E. Mnëv, *The universality theorems on the classification problem of configuration varieties and convex polytopes varieties*, Topology and geometry—Rohlin Seminar, Lecture Notes in Math., vol. 1346, Springer, Berlin, 1988, pp. 527–543.

[65] R. Nandakumar and N. Ramana Rao, *'Fair' partitions of polygons: An introduction*, Proc. Indian Acad. Sci. (Math. Sci.) **122** (2012), 459–467.

[66] Nevena Palić, *GitHub repository*, `https://github.com/nevenapalic/MacPhersonian`.

[67] ______, *On the rank 2 oriented matroid Grassmannian*, Master's thesis, Freie Universität Berlin, 2016, `https://userwikis.fu-berlin.de/pages/viewpage.action?pageId=580190224&preview=/580190224/614629385/Masterarbeit%20-%20Nevena%20Palic.pdf`.

[68] Alexander Postnikov, *Total positivity, Grassmannians, and networks*, Preprint, September 2006, 85 pages; arXiv:0609764.

[69] Alexander Postnikov, David Speyer, and Lauren Williams, *Matching polytopes, toric geometry, and the totally non-negative Grassmannian*, J. Algebraic Combin. **30** (2009), 173–191.

[70] Edgar A. Ramos, *Equipartition of mass distributions by hyperplanes*, Discrete Comput. Geom. **15** (1996), 147–167.

[71] Jürgen Richter-Gebert, *Two interesting oriented matroids*, Doc. Math. **1** (1996), no. 7, 137–148.

[72] Konstanze Rietsch and Lauren Williams, *Discrete Morse theory for totally non-negative flag varieties*, Adv. Math. **223** (2010), 1855–1884.

[73] Steven Simon, *Measure equipartitions via finite Fourier analysis*, Geom. Dedicata **179** (2015), 217–228.

[74] Stephen Smale, *A Vietoris mapping theorem for homotopy*, Proc. Amer. Math. Soc. **8** (1957), 604–610.

[75] Pablo Soberón, *Balanced convex partitions of measures in* $\mathbb{R}^d$, Mathematika **58** (2012), 71–76.

[76] James Stasheff, *A classification theorem for fibre spaces*, Topology **2** (1963), 239–246.

[77] Bernd Sturmfels, *Totally positive matrices and cyclic polytopes*, Proceedings of the Victoria Conference on Combinatorial Matrix Analysis (Victoria, BC, 1987), vol. 107, 1988, pp. 275–281.

[78] Sheila Sundaram and Volkmar Welker, *Group actions on arrangements of linear subspaces and applications to configuration spaces*, Trans. Amer. Math. Soc. **349** (1997), no. 4, 1389–1420.

[79] P. Suvorov, *Isotopic but not rigidly isotopic plane systems of straight lines*, Topology and Geometry - Rohlin Seminar (Heidelberg) (O. Ya. Viro, ed.), Lecture notes in mathematics, vol. 1346, Springer, 1988, pp. 545–556.

[80] The Sage Developers, *Sagemath, the Sage Mathematics Software System (Version 7.4)*, 2018, `http://www.sagemath.org`.

[81] Yasuyuki Tsukamoto, *New examples of oriented matroids with disconnected realization spaces*, Discrete and Computational Geometry **49** (2013), 287–295.

[82] Aleksej Yu. Volovikov, *On the index of G-spaces*, Mat. Sb. **191** (2000), 3–22.

[83] Rade T. Živaljević, *User's guide to equivariant methods in combinatorics. II*, Publ. Inst. Math. (Beograd) (N.S.) **64(78)** (1998), 107–132, 50th anniversary of the Mathematical Institute, Serbian Academy of Sciences and Arts (Belgrade, 1996).

[84] Volkmar Welker, Günter M. Ziegler, and Rade T. Živaljević, *Homotopy colimits—comparison lemmas for combinatorial applications*, J. Reine Angew. Math. **509** (1999), 117–149.

[85] John H. C. Whitehead, *Combinatorial homotopy I*, Bulletin Amer. Math. Soc. **55** (1949), 213–245.

Abstract

Grassmannians, measure partitions and waists of spheres

In this thesis we apply methods from algebraic topology on questions from geometry, combinatorics and functional analysis.

First we study amplituhedra – images of the totally nonnegative Grassmannians under projections that are induced by linear maps. They were introduced in Physics by Arkani-Hamed & Trnka (*Journal of High Energy Physics*, 2014) as model spaces that should provide a better understanding of the scattering amplitudes of quantum field theories. The topology of the amplituhedra has been known only in a few special cases, where they turned out to be homeomorphic to balls. Amplituhedra are special cases of Grassmann polytopes introduced by Lam (*Current developments in mathematics* 2014, Int. Press). We show that some further amplituhedra are homeomorphic to balls, and that some more Grassmann polytopes and amplituhedra are contractible.

Next we study equipartitions of measures in a Euclidean space by certain families of convex sets. Our first result gives partitions of the ambient space into convex prisms – products of convex sets, that equipart a given set of measures, and our second result gives partitions of the Euclidean space by regular linear fans, that also equipart a given set of measures.

The next result is a continuous analogue of the conjecture of Holmsen, Kynčl and Valculescu (*Computational Geometry*, 2017). For given a large enough family of positive finite absolutely continuous measures in the Euclidean space, we prove that there exists a partition of the ambient space, such that every set in the partition has positive measure with respect to at least c of the given measures, where we allow c to be greater than the dimension of the ambient Euclidean space. Additionally, we obtain an equipartition of one of the measures. The proof relies on a configuration space/test map scheme that translates this problem into a novel question from equivariant topology: We show non-existence of equivariant maps from the ordered configuration space into the union of an affine arrangement.

Furthermore, we prove an extension of the Gromov's theorem on the waists of spheres (*Geometric and Functional Analysis*, 2003). Gromov showed that for every $n > k \geq 1$ and for every continuous map $f : S^n \to \mathbb{R}^k$ from a sphere to a Euclidean space, there exists a point $z \in \mathbb{R}^k$, such that the volume of the tubular neighborhood of the inverse image $f^{-1}(z)$ is at least as large as the volume of the tubular neighborhood of the $(n-k)$-dimensional equatorial sphere. We show that if the map f is $\mathbb{Z}_p$-equivariant for a prime p, and if the action of $\mathbb{Z}_p$ on S^n and $\mathbb{R}^k$ satisfies certain properties, one can choose z in Gromov's theorem to be the origin in $\mathbb{R}^k$.

Finally, we study oriented matroid Grassmannians, also called MacPhersonians. An oriented matroid Grassmannian is the order complex of the set of all oriented matroids of a fixed rank and a fixed number of elements, ordered by weak maps. They were introduced by MacPherson (*Topological Methods in Modern Mathematics*, 1993), and firstly used by Gel'fand and MacPherson to give a combinatorial formula for Pontrjagin classes. For a given rank and a number of elements, the MacPhersonian is conjectured to be homotopy equivalent to the corresponding Grassmannian. We give some computational evidence in rank 3 and 4 that support the conjecture.

Zusammenfassung

Grassmannians, measure partitions and waists of spheres

Diese Arbeit behandelt die Anwendung von Methoden der algebraischen Topologie auf Fragestellungen der Geometrie, Kombinatorik und Funktionalanalysis.

Zunächst betrachten wir Amplitueder. Diese sind Bilder der total-nichtnegativen Grassmann-Mannigfaltigkeiten unter Projektionen, die durch lineare Abbildungen induziert sind. Amplitueder wurden in der Physik von Arkani-Hamed & Trnka (*Journal of High Energy Physics*, 2014) als Modellräume eingeführt, um ein besseres Verständnis der Streuamplituden von Quantenfeldtheorien zu ermöglichen. Die Topologie der Amplitueder war zuvor lediglich in wenigen Spezialfällen bekannt, in welchen sie homöomorph zu Bällen sind. Die Amplitueder sind spezielle Grassmann-Polytope, eingeführt von Lam (*Current Developments in Mathematics* 2014, International Press). Wir zeigen für weitere Amplitueder, dass sie homöomorph zu Bällen sind und für einige weitere Grassmann-Polytope und Amplitueder, dass sie zusammenziehbar sind.

Anschließend betrachten wir Equipartitionen von Maßen im Euklidischen Raum durch bestimmte Familien von konvexen Mengen. Das erste Resultat liefert Partitionen des umgebenden Raumes in konvexe Prismen, Produkte konvexer Mengen, welche eine gegebene Menge von Maßen equipartitionieren. Das zweite Resultat liefert Partitionen des Euklidischen Raumes durch reguläre lineare Fächer, welche ebenso eine gegebene Menge an Maßen equipartitionieren.

Unser nächstes Resultat ist ein stetiges Analogon der Vermutung von Holmsen, Kynčl und Valculescu (*Discrete & Computational Geometry*, 2017). Für jede hinreichend große gegebene Menge an positiven, endlichen, absolut stetigen Maßen im Euklidischen Raum zeigen wir die Existenz einer Partition des umgebenden Raumes, in der jeder Teil der Partition positives Maß bezüglich mindestens c der gegebenen Maße hat. Hierbei darf c die Dimension des umgebenden Euklidischen Raumes übersteigen. Zusätzlich erhalten wir eine Equipartition eines der Maße. Der Beweis beruht auf einem Konfigurationsraum/Testabbildungs-Schema, welches das Problem in eine Fragestellung der äquivarianten Topologie übersetzt: Wir zeigen die Nichtexistenz äquivarianter Abbildungen von dem geordneten Konfigurationsraum in ein affines Arrangement.

Des Weiteren zeigen wir eine Erweiterung von Gromov's "Waist of the sphere"-Satz (*Geometric and Functional Analysis*, 2003). Dieser besagt, dass für alle $n > k \geq 1$ und alle stetigen Abbildungen $f : S^n \to \mathbb{R}^k$ von der Sphäre in den Euklidischen Raum ein Punkt $z \in \mathbb{R}^k$ existiert, sodass das Volumen der tubularen Umgebung des Urbilds $f^{-1}(z)$ mindestens so groß ist wie das Volumen der tubularen Umgebung der $(n-k)$-dimensionalen äquatorialen Sphere. Wir zeigen, dass man, bei $\mathbb{Z}_p$-Äquivarianz der Abbildung f für alle Primzahlen p, in Gromovs Satz ein solches z als Ursprung des $\mathbb{R}^k$ wählen kann, sofern die Wirkung von $\mathbb{Z}_p$ auf S^n und $\mathbb{R}^k$ bestimmte Eigenschaften erfüllt.

Schließlich betrachten wir "oriented matroid Grassmannians", die als "MacPhersonians" bekannt geworden sind. Eine MacPhersonian ist der Ordnungskomplex der Menge aller orientierten Matroiden von festem Rang und fester Anzahl an Elementen, geordnet durch schwache Abbildungen. Sie wurden von MacPherson eingeführt (*Topological Methods in Modern Mathematics*, 1993) und dann von Gelfand und MacPherson benutzt, um eine kombinatorische Formel für Pontrjagin-Klassen anzugeben. Für gegebenen Rang und Anzahl der Elemente wird vermutet, dass die MacPhersonian zu der zugehörigen Grassmann-Mannigfaltigkeit homotopieäquivalent ist. Wir geben computergestützte Hinweise auf die Gültigkeit der Vermutung für Rang 3 und 4.