

On entanglement evolution across defects in critical chains

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We consider a local quench where two free-fermion half-chains are coupled via a defect. We show that the logarithmic increase of the entanglement entropy is governed by the same effective central charge which appears in the ground-state properties and which is known exactly. For unequal initial filling of the half-chains, we determine the linear increase of the entanglement entropy.

I. INTRODUCTION

The entanglement across defects in critical quantum chains is interesting, because in free-fermion systems it varies continuously with the defect strength [1]. For two half-chains of length L , one finds a logarithmic behaviour of the von Neumann entanglement entropy

$$S = \frac{c_{\text{eff}}}{6} \ln L \quad (1)$$

where $0 \leq c_{\text{eff}} \leq c$ with c (equal to $1/2$ or 1) denoting the central charge of the model. Since e^S , which is the effective number of terms in the Schmidt decomposition, then follows a power law, one can view c_{eff} as a variable critical exponent. It was studied in a number of papers for discrete [1–5] and continuous [6–8] systems and an exact analytical formula was obtained [3], also for the Rényi entropy [7, 10] and for bosons [9, 10]. Technically, the variation is caused by a gap in the single-particle eigenvalue spectrum of the reduced density matrix (RDM) and the relevant physical parameter is the transmission amplitude of the defect.

A similar situation is found if one considers a local quench where one connects two initially separated half-chains by a defect. The entanglement entropy then increases in time as

$$S = \frac{\hat{c}_{\text{eff}}}{3} \ln t \quad (2)$$

as long as $t \ll L$ for finite L and for all times if L is infinite. This was found either purely numerically [2] or by evaluating expressions based on the counting statistics numerically [11–13].

No explicit formulae for \hat{c}_{eff} have been given so far. However, it was observed in the numerics of the transverse Ising model that \hat{c}_{eff} equals c_{eff} , as in the homogeneous case, where the same c appears in (1) and (2) and the formulae follow from conformal considerations [14].

In this note, we look at this problem once again and study the evolution after the quench for an XX (hopping) model via the time-dependent single-particle RDM eigenvalues. We show that those for the defect case are related by a simple exact formula to those for the homogeneous case, if the defect is conformal (i.e. scale-free). The relation is the same as in the static case and thus implies again a gap in the spectrum. This allows to take over the ground-state results and thereby proves the equality of c_{eff} and \hat{c}_{eff} . Since c_{eff} is exactly known, it also provides an analytical formula for \hat{c}_{eff} . One can check that it fully agrees with the expressions and results of the counting statistics.

We also consider the case where the initial filling of the two half-chains is different. Then a somewhat different relation for the spectra exists where the effect of backscattering is manifest. The current which sets in after the quench leads to a steady generation of entanglement between transmitted and reflected parts of the wavefunction and thus to a linear increase of the entropy with time. We give analytical expressions for the coefficient both for conformal and for non-conformal defects.

II. MODEL AND METHOD

We study free fermions hopping on a finite open chain of $2L$ sites. The time evolution for $t > 0$ is governed by the Hamiltonian

$$H' = \frac{1}{2} \sum_{m,n=1}^{2L} H'_{m,n} c_m^\dagger c_n \quad (3)$$

where the nonzero matrix elements are

$$H'_{m,m+1} = H'_{m+1,m} = \begin{cases} -1 & m \neq L \\ -\lambda & m = L \end{cases}, \quad H'_{L,L} = -H'_{L+1,L+1} = \sqrt{1 - \lambda^2} \quad (4)$$

In the middle of the chain there is a defect in the form of a modified bond supplemented with site energies on both sides and characterized by the parameter λ . It will be referred to as the conformal defect and is used to derive all the exact relations. In addition, we will also consider simple bond defects without site energies. The initial system is composed of two disconnected half-chains and the initial state will be specified in the corresponding sections.

In the homogeneous case, $\lambda = 1$, the Hamiltonian (3) is diagonalized by a Fourier transform

and the eigenvectors and eigenvalues are given by

$$\phi_k(m) = \sqrt{\frac{2}{2L+1}} \sin \frac{\pi km}{2L+1}, \quad \Omega_k = -\cos \frac{\pi k}{2L+1} \quad (5)$$

where $k = 1, \dots, 2L$. For the conformal defect the solutions are related to those of the homogeneous chain via

$$\phi'_k(m) = \begin{cases} \alpha_k \phi_k(m) & 1 \leq m \leq L \\ \beta_k \phi_k(m) & L < m \leq 2L \end{cases}, \quad \Omega'_k = \Omega_k \quad (6)$$

and thus the eigenvectors are simply rescaled on the left and right hand side of the defect with the scaling factors

$$\alpha_k^2 = 1 + (-1)^k \sqrt{1 - \lambda^2}, \quad \beta_k^2 = 1 - (-1)^k \sqrt{1 - \lambda^2} \quad (7)$$

Note that, apart from the alternation, α_k and β_k are independent of k and lead to a constant transmission coefficient $T = \lambda^2$. Therefore one has here a lattice realization of the special scale-free defects which can be constructed by gluing together two conformal field theories [10, 15]. The bond defect is non-conformal and leads to a transmission which depends on the wavelength of the incoming particle.

For the entanglement between left and right halves, one needs the RDM for a half-chain which has the form $\rho = e^{-\mathcal{H}}/Z$ with a free-fermion effective Hamiltonian \mathcal{H} , see [16]. In the following, the single-particle eigenvalues of \mathcal{H} will be called $2\omega_l(t)$. They are related via

$$\zeta'_l(t) = \frac{1}{e^{2\omega_l(t)} + 1} \quad (8)$$

to the eigenvalues of the correlation matrix $\mathbf{C}'(t)$ which is the full matrix $\bar{\mathbf{C}}'(t)$ with elements $\langle c_m^\dagger(t) c_n(t) \rangle$ restricted to $1 \leq m, n \leq L$. For $\lambda = 1$, the quench will be called *homogeneous* and the corresponding quantities are denoted by $2\varepsilon_l(t)$, $\zeta_l(t)$ and $\mathbf{C}(t)$, respectively.

Therefore one has to find the time-dependent correlation matrix which can be written in the Heisenberg form

$$\bar{\mathbf{C}}'(t) = e^{i\bar{\mathbf{H}}'t} \bar{\mathbf{C}}(0) e^{-i\bar{\mathbf{H}}'t} \quad (9)$$

where $\bar{\mathbf{H}}'$ denotes the matrix with elements $H'_{m,n}$ and $\bar{\mathbf{C}}(0)$ contains the initial correlations. The entanglement entropy is then given by

$$S(t) = \sum_l \ln(1 + e^{-2\omega_l(t)}) + \sum_l \frac{2\omega_l(t)}{e^{2\omega_l(t)} + 1} = \sum_l H(\zeta'_l(t)) \quad (10)$$

where $H(x) = -x \ln x - (1-x) \ln(1-x)$. The first (second) form proves to be useful for quenches from equal (unequal) fillings.

III. QUENCH FROM EQUAL FILLINGS

In this case, the initial correlation matrix has the block form

$$\bar{\mathbf{C}}(0) = \begin{pmatrix} \mathbf{C}^0 & 0 \\ 0 & \mathbf{C}^0 \end{pmatrix} \quad (11)$$

where the $L \times L$ matrix \mathbf{C}^0 refers to a half-chain in the ground state and has elements

$$C_{mn}^0 = \sum_{k=1}^L \phi_k^0(m) \phi_k^0(n) n_k \quad (12)$$

Here n_k is the occupation number of mode k and the wavefunctions $\phi_k^0(m)$ have the form (5) with $2L$ replaced by L . To evaluate (9) one expresses the exponential operators as

$$\left(e^{\pm i \bar{\mathbf{H}}' t} \right)_{mn} = \sum_{k=1}^{2L} \phi_k'(m) \phi_k'(n) e^{\pm i \Omega_k t} \quad (13)$$

Using (6), defining the overlap matrix

$$B_{kl} = \sum_{j=1}^L \phi_k(j) \phi_l^0(j) = (-1)^{(k+l)} \sum_{j=L+1}^{2L} \phi_k(j) \phi_l^0(j-L) \quad (14)$$

and noting that $\alpha_k \alpha_{k'} + (-1)^{k+k'} \beta_k \beta_{k'} = 2$ for $k - k'$ even and zero otherwise, one arrives at

$$C'_{mn}(t) = 2 \sum_{\substack{k,k'=1 \\ k-k' \text{ even}}}^{2L} \sum_{k''=1}^L \alpha_k \alpha_{k'} B_{kk''} B_{k''k'} \phi_k(m) \phi_{k'}(n) n_{k''} e^{i(\Omega_k - \Omega_{k'})t} \quad (15)$$

The only dependence on the defect strength λ is in the factors $\alpha_k \alpha_{k'}$. The point now is that one can find a simple connection with the homogeneous quench if one considers the matrix $(2\mathbf{C}'(t) - 1)^2$. Then a further overlap matrix appears with elements

$$A_{kl} = \sum_{j=1}^L \phi_k(j) \phi_l(j) = \frac{1}{2(2L+1)} \left[\frac{\sin \frac{\pi}{2}(k-l)}{\sin \frac{\pi(k-l)}{2(2L+1)}} - \frac{\sin \frac{\pi}{2}(k+l)}{\sin \frac{\pi(k+l)}{2(2L+1)}} \right] \quad (16)$$

which, apart from the diagonal ones, vanish for $k - l$ even. The matrix elements of $(2\mathbf{C}'(t) - 1)^2$ are 6-fold sums where each summand is proportional to $\alpha_k \alpha_{k'} \alpha_l \alpha_{l'} A_{k'l}$ with $k - k'$ and $l - l'$ even. For $k' - l$ odd, this gives $\alpha_k \alpha_{k'} \alpha_l \alpha_{l'} = \lambda^2$ and thus a simple prefactor. For $k' = l$, one has $A_{ll} = 1/2$ and a summand

$$\alpha_k \alpha_l^2 \alpha_{l'} B_{kk''} B_{ll''} B_{ll''} B_{l'l''} \phi_k(m) \phi_{l'}(n) (2n_{k''} - 1) (2n_{l''} - 1) e^{i(\Omega_k - \Omega_{l'})t} \quad (17)$$

where $\alpha_k \alpha_l^2 \alpha_{l'} = 2 - \lambda^2 + (-1)^l 2\sqrt{1 - \lambda^2}$ for the allowed indices. The remaining sums can then be evaluated using the identities

$$\sum_{l=1}^{2L} B_{lk''} B_{ll''} = \delta_{k''l''}, \quad \sum_{l=1}^{2L} (-1)^l B_{lk''} B_{ll''} = 0, \quad \sum_{k''=1}^L B_{kk''} B_{l'k''} = A_{kl'} \quad (18)$$

Summing over l therefore eliminates the alternating terms and enforces $k'' = l''$ which leads to $(2n_{k''} - 1)^2 = 1$ independent of k'' and of the filling. The final sum over k gives $(1 - \lambda^2/2)\delta_{mn}$. In this way one arrives at the formula

$$(2\mathbf{C}'(t) - 1)_{mn}^2 = \lambda^2 (2\mathbf{C}(t) - 1)_{mn}^2 + (1 - \lambda^2)\delta_{mn} \quad (19)$$

As a consequence, the single-particle eigenvalues of the corresponding RDMs are related by $\tanh^2 \omega_l(t) = \lambda^2 \tanh^2 \varepsilon_l(t) + 1 - \lambda^2$, or alternatively, writing $\lambda = s$ for the transmission amplitude, by

$$\text{ch } \omega_l(t) = \frac{1}{s} \text{ch } \varepsilon_l(t) \quad (20)$$

This is exactly the same relation one finds for the static defect problem [3] and gives a gap in the ω_l -spectrum. Note that (19) can also be written as $\mathbf{C}'(t)(1 - \mathbf{C}'(t)) = \lambda^2 \mathbf{C}(t)(1 - \mathbf{C}(t))$ and is then identical to the relation for the overlap matrix \mathbf{A} in a (static) continuum system [6–8].

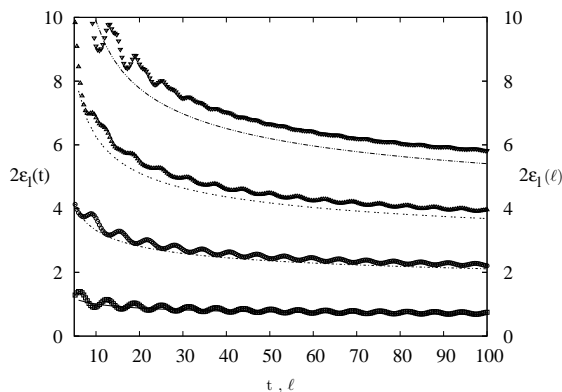


FIG. 1: Low-lying eigenvalues $2\varepsilon_l(t)$ after homogeneous local quench with $2L = 400$ (points) and $2\varepsilon_l(\ell)$ of an interval in an infinite system (lines). The data is shown for even ℓ and for half-filling.

The problem is now reduced to that of the homogeneous quench, but one still needs the $\varepsilon_l(t)$. The known results give a logarithmic behaviour of $S(t)$ with a prefactor $1/3$ that is similar to the scaling of the equilibrium entropy of an interval ℓ in an infinite chain [14]. In the static case, one knows that the low-lying $\varepsilon_l(\ell)$ have a spacing $\pi^2/2 \ln \ell$ for large $\ln \ell$ [17]. This gives a density of

states proportional to $\ln \ell$ and the logarithmic variation of $S(\ell)$. The similarity of the conformal results for the entropy suggests that the $\varepsilon_l(t)$ have an analogous behaviour. This is in fact the case and is shown in Fig. 1. Plotted are the lowest $\varepsilon_l(t)$ in a homogeneous quench together with the lowest $\varepsilon_l(\ell)$ for a segment of length ℓ in an infinite chain. Apart from some oscillations, they coincide closely.

With this observation, one can now follow the same steps as in the static case and write S in (10) as an integral over ε . In this way one finds $\hat{c}_{\text{eff}} = c_{\text{eff}}$, where c_{eff} is $12/\pi^2$ times the integral $I(s)$ given in (26) of [3]. It is depicted on Fig. 2, both as a function of s and $T = s^2$, to allow simple comparison with the numerical results of [11]. The function is nonanalytic around $s = 0$. For the simple bond defect, one knows from the static case that the parameter s is the transmission amplitude at the Fermi level. The relation between ω and ε is then only valid as the levels become dense.

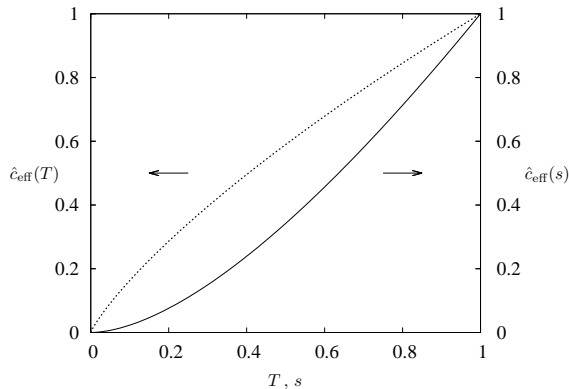


FIG. 2: \hat{c}_{eff} as a function of s (solid line) as well as $T = s^2$ (dashed line).

IV. QUENCH FROM UNEQUAL FILLINGS

If the initial fillings are unequal, a steady particle-current results after connecting the half-chains. In the homogeneous case, the time evolution of the density profile [18], the particle number fluctuations in a half-chain [19] as well as the entanglement entropy [20] have been studied previously. The fluctuations and the entropy both grow logarithmically in time.

We first consider the case where the left hand side of the chain is completely filled and the right hand side is empty. The initial correlation matrix then reads

$$\bar{\mathbf{C}}(0) = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & 0 \end{pmatrix} \quad (21)$$

where $\mathbf{1}$ denotes the $L \times L$ identity matrix. Expanding into eigenvectors one has

$$C'_{mn}(t) = \sum_{k,l=1}^{2L} \alpha_k^2 \alpha_l^2 \phi_k(m) A_{kl} \phi_l(n) e^{i(\Omega_k - \Omega_l)t} = \lambda^2 C_{mn}(t) + (1 - \lambda^2) \delta_{mn} \quad (22)$$

where in the second step we used the property $\alpha_k^2 \alpha_l^2 = \lambda^2$ for all $k - l$ odd, where the overlap matrix \mathbf{A} is nonvanishing. For $k = l$, the alternating piece in α_k^4 gives zero upon summation and the constant piece leads to the second term on the right hand side. Remarkably, one finds the same relation as in Eq. (19) but now for the correlation matrix itself and thus also for its eigenvalues

$$\zeta'_l(t) = \lambda^2 \zeta_l(t) + 1 - \lambda^2 \quad (23)$$

The eigenvalues $\zeta'_l(t)$ and $\zeta_l(t)$ are shown on the left of Fig. 3 and have a simple interpretation. As the half-chains are connected, particles leave the left part of the system and a steady current results. In the pure case, this implies a growing number of $\zeta_l(t) = 0$ eigenvalues. In the presence of a defect, however, there is a probability $R = 1 - T$ for backscattering and the corresponding eigenvalues are $\zeta'_l(t) = 1 - \lambda^2 = R$. It is interesting to compare with the case of a simple bond defect, which is done on the right of Fig. 3. Here such a simple relation as (23) does not hold, since the transmission coefficient depends on the momentum q of the particle as

$$T_q = \frac{\sin^2 q}{\text{ch}^2 \nu - \cos^2 q} \quad (24)$$

where we defined $\lambda = e^{-\nu}$ and $0 \leq q \leq \pi$. Because of this variation, the $\zeta'_l(t)$ curves are not flat.

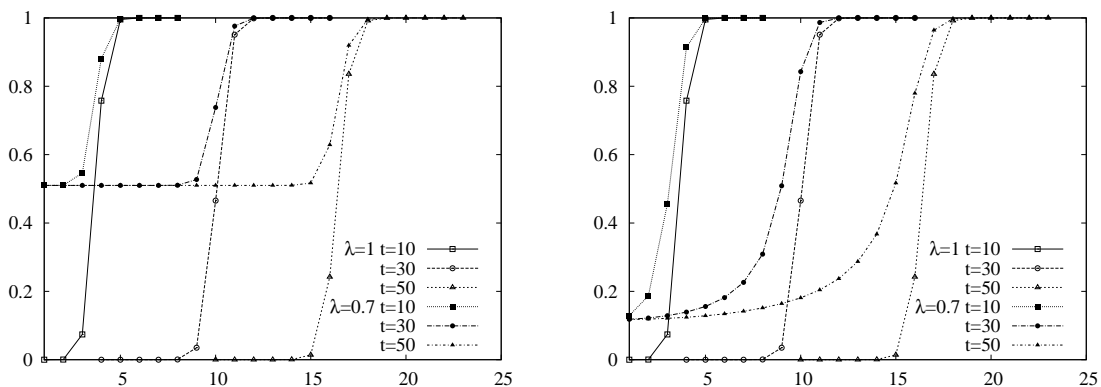


FIG. 3: Eigenvalues $\zeta_l(t)$ (open symbols) and $\zeta'_l(t)$ (filled symbols) for different times and $2L = 200$ sites. Left: conformal defect. Right: bond defect.

The common feature of both defects is that the number of nonzero $\zeta'_l(t)$ eigenvalues, which have a finite contribution to the entropy, grows with time proportionally to the number of transmitted

particles. Because of the steady flux in the center of the chain, this results in a linear contribution to the entanglement. Numerically, this can be observed even for coupling strengths arbitrarily close to $\lambda = 1$, where the entropy becomes logarithmic, $S \sim 1/6 \log t$ [20]. In general, S can be well fitted with the ansatz

$$S(t) = \alpha t + \beta \ln t + \gamma \quad (25)$$

and the coefficients are shown in Fig. 4. In the conformal case, the linear part can be calculated, using the second form of the entropy in Eq. (10), as $H(\lambda^2)t/\pi$. This is just the contribution of a single $\zeta'_i(t)$ on the flat part of the spectrum multiplied by the length of the plateau, which equals the total number t/π of transmitted particles in the homogeneous case [18]. This result, shown by the solid line on the left of the Fig. 4, agrees perfectly with the fitted values of α . Furthermore, it also agrees with the results of [11–13] for the entropy evolution in a quantum point contact in the high-bias regime.

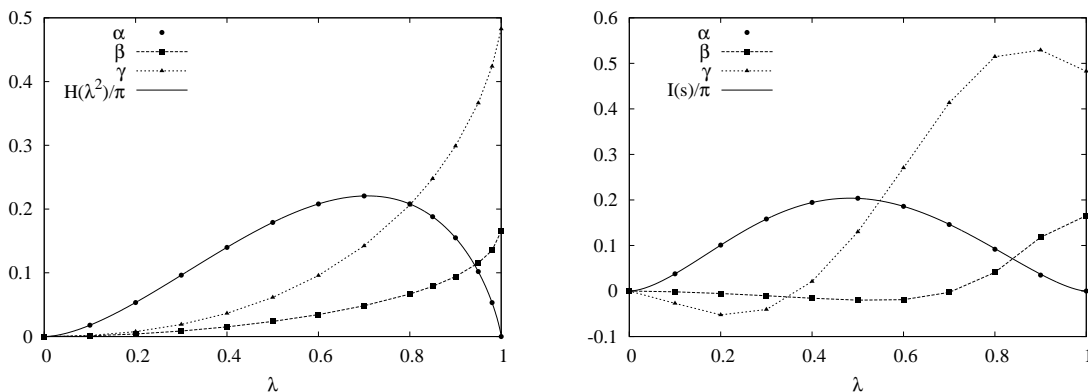


FIG. 4: Fitted coefficients of the ansatz $S(t) = \alpha t + \beta \ln t + \gamma$ together with the analytical formulas obtained for the slope α . Left: conformal defect. Right: bond defect.

In the non-conformal case, the linear growth of the entropy can be obtained by distinguishing the different momentum states and associating a factor $H(T_q)$ with the entanglement which is produced between transmitted and reflected parts of the wavefunction. Thus one writes for $L \rightarrow \infty$

$$\alpha = \int_0^\pi \frac{dq}{2\pi} v_q H(T_q) \quad (26)$$

where $v_q = d\Omega_q/dq = \sin q$ is the velocity of the incoming particle and measures the flux. For the conformal defect where $T = \lambda^2$ one obtains the previous result. For the bond defect, the integral can be carried out by substituting Eq. (24) and using T_q as integration variable. A lengthy

calculation then yields $\alpha = I(s)/\pi$ with

$$I(s) = \ln\left(\frac{1-s^2}{4s^2}\right) - \frac{1}{s} \ln\left(\frac{1-s}{1+s}\right) + \frac{1-s^2}{s} \left[\frac{1}{4} \ln^2\left(\frac{1-s}{1+s}\right) + \text{Li}_2\left(\frac{1-s}{1+s}\right) - \frac{\pi^2}{6} \right] \quad (27)$$

where $\text{Li}_2(s)$ denotes the dilogarithm function and $s = 1/\text{ch } \nu = 2/(\lambda + 1/\lambda)$ is the same amplitude that appears in the equilibrium formulae. This analytical result is again in perfect agreement with the results of the data fits, as shown on the right of Fig. 4.

Note, that the structure of Eq. (26) is very similar to the one found by Fagotti and Calabrese for a *global* quench in the XY model [21]. However, in the present case entanglement is created only *locally* (but steadily) at the defect, in contrast with the global quench where entangled quasi-particle pairs are created everywhere but only at $t = 0$ [22]. It would be interesting to check if this semiclassical picture can be used to calculate other relevant observables as was found recently for the global quench [23, 24].

The above arguments can be generalized to arbitrary initial filling factors n_l and n_r on the left and right. Assuming $n_l > n_r$, the single-particle states with $q < n_r\pi$ will be filled and thus balanced on both sides. In terms of the $\zeta'_l(t)$, the number of eigenvalues on the flat part of the spectrum decreases and one has a more general relation which interpolates between (23) and (20). In the semiclassical picture only wavenumbers $n_r\pi < q < n_l\pi$ will contribute to the current and the integral in Eq. (26) has to be carried out only on this interval. As shown in Fig. 5, this gives again very good agreement with the numerical data, further supporting the semiclassical picture described above. It is possible to evaluate the integral analytically also in this case.

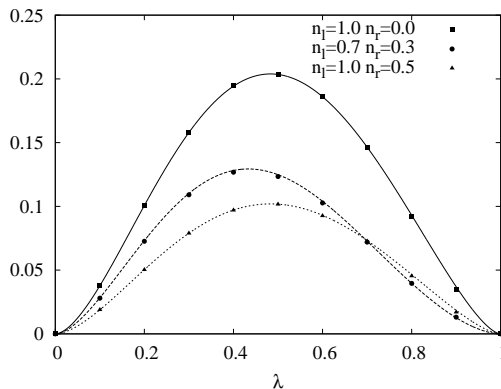


FIG. 5: Slope α of the linear part of the entropy for several filling factors as obtained by fits to the data (points) as well as evaluating the integrals (lines)

V. CONCLUDING REMARKS

There is a close connection of our findings with the work on counting statistics by Klich and Levitov [11]. For the quench with equal fillings, they obtain a density of states for the eigenvalues ζ' which vanishes in an interval around $\zeta' = 1/2$. Thus the spectrum has a gap, and this is just the situation described by (20). Moreover, if one converts their entropy expression (10) into an integral over ε one finds exactly the integral for c_{eff} evaluated in [3]. Finally, the relation between their parameters λ and λ^* is another form of the dispersion relation (20) derived here. Our approach, focussing on the eigenvalues themselves, is more direct and shows the connection with the static defect problem very clearly. The only open point is an analytical derivation of the $\varepsilon_l(t)$ -behaviour, which we inferred from conformal results and numerics.

In this context one should mention that we discussed only the case $t \ll L$. The general CFT formula for the homogeneous quench is [25]

$$S(t) = \frac{c}{3} \ln \left| \frac{2L}{\pi} \sin \frac{\pi v_F t}{2L} \right| + \text{const.} \quad (28)$$

where the Fermi-velocity v_F is set by the filling of the half-chains. This results again from the scaling of the $\varepsilon_l(t)$ with the logarithmic factor and the formula for the defect case is obtained by substituting $c \rightarrow c_{\text{eff}}$.

The relation $\hat{c}_{\text{eff}} = c_{\text{eff}}$ also holds for the corresponding coefficients in the Rényi entropies S_n which are rather simple for S_2 and S_3 [7, 10]. Similarly, the considerations for the biased quench can be adapted to the Rényi functions.

Finally, a quench from unequal fillings also arises in continuum models if one removes a wall which initially confines the particles. A detailed study of the entanglement evolution has appeared recently [26] and can probably be extended to the defect case.

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