SERIE B — INFORMATIK

Generalized Guarding and Partitioning for Rectilinear Polygons

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Abstract

A T_k–guard G in a rectilinear polygon P is a tree of diameter k completely contained in P. The guard G is said to cover a point x if x is visible (or rectangularly visible) from some point contained in G. We investigate the function $r(n, h, k)$, which is the largest number of T_k–guards necessary to cover any rectilinear polygon with h holes and n vertices. The aim of this paper is to prove new lower and upper bounds on parts of this function.
In particular, we show the following bounds:

1. $r(n, 0, k) \leq \left\lfloor \frac{n}{k+4} \right\rfloor$, with equality for even k
2. $r(n, h, 1) = \left\lfloor \frac{3n+4h+4}{16} \right\rfloor$
3. $r(n, h, 2) \leq \left\lfloor \frac{n}{6} \right\rfloor$.

These bounds, along with other lower bounds that we establish, suggest that the presence of holes reduces the number of guards required, if $k > 1$. In the course of proving the upper bounds, new results on partitioning are obtained.

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1 Introduction

Given two points x and y in a rectilinear polygon P, the points x and y are called *rectangularly visible*, denoted $x \square y$, if the smallest aligned rectangle $R(x, y)$ spanned by x and y is contained in P [8]. This is a more restrictive notion than the usual visibility, where one only requires that the line segment (x, y) is contained in P.

In this paper we study the following (rectangular) visibility problem: Let P be a rectilinear polygon with h holes on n vertices. How can one cover P by T_k–guards? Here, a T_k–guard in P is a tree G that has graph-theoretic diameter k and is rectilinearly embedded in P. The region $V(G)$ covered by such a guard is the set of all points rectangularly visible to G: $V(G) = \{x \in P \mid \exists y \in G \text{ such that } x \square y\}$. A collection $\{G_i\}, i \in I$ of T_k–guards covers P if $\bigcup_{i \in I} V(G_i) = P$.

Let us define the following functions:

$$
\begin{aligned}
r(P, k) &= min\{p \mid \exists \text{ a set of } p\ T_k\text{–guards} \\
&\qquad \text{that cover } P\} \\
r(n, h, k) &= max\{r(P, k) \mid P \text{ is a rectilinear polygon} \\
&\qquad \text{with } n \text{ vertices and } h \text{ holes}\}
\end{aligned}
$$

Further, let $g(n, h, k)$ be the function analogous to $r(n, h, k)$ defined for general polygons with the usual visibility notion. The first result concerning these functions is Chvátal's classical Art Gallery Theorem, which in our notation reads $g(n, 0, 0) = \left\lfloor \frac{n}{3} \right\rfloor$. After this result, many combinatorial and algorithmic variations of this problem have been studied; most of these variations can be found in [10] and [11]. For general polygons, it is known that $g(n, 0, k) = \left\lfloor \frac{n}{k+3} \right\rfloor$ [12] and $g(n, h, 0) = \left\lfloor \frac{n+h}{3} \right\rfloor$ [6], [2]. Throughout this paper we use the following non-standard convention: $\left\lfloor \frac{n}{m} \right\rfloor$ is the set to be 1 for $0 < n < m$.

In rectilinear polygons the situation is quite different. For instance, for point guards (T_0–guards), it is known that $r(n, h, 0) = \left\lfloor \frac{n}{4} \right\rfloor$ [7], [4]. This is unusual in that the number of holes does not affect the maximum number of guards required. However, for line guards (T_1–guards) holes make the problem harder: it is known that $r(n, h, 1) \geq \left\lfloor \frac{3n+4h+4}{16} \right\rfloor$ [14]. This bound is tight for $h = 0$ (i.e., $r(n, 0, 1) = \left\lfloor \frac{3n+4}{16} \right\rfloor$) [1]. So what is the correct bound for line guards, and what about general T_k–guards? This paper answers the first question and begins to address the second. We begin with some definitions and coventions.

We use the term (n, h)*–polygon* to denote a rectilinear polygon with h holes and a total of n vertices. Such a polygon is said to be in *general position* if no two reflex vertices can be joined by a horizontal or vertical line segment lying in the interior of the polygon. A short case analysis shows that by perturbing the vertices of a polygon P that is not in general position, we can obtain a polygon P' in general position such that a covering of P' by T_k–guards implies a covering of P by T_k–guards. We henceforth restrict our attention to polygons in general position.

The *rectangular decomposition* of an (n, h)–polygon P is a partition of P into rectangles by extending a horizontal chord into the polygon from every reflex vertex

Figure 1: Rectangular decomposition and R-graph

(see Figure 1). The number of rectangles in this decomposition is $\frac{n-2}{2} + h$ (if the polygon were not in general position this number would be smaller). We define the R–graph of P, denoted $\mathbf{R}(P)$ (or simply $\mathbf{R}$ when P is understood), as a directed graph where each vertex corresponds to a rectangle of the rectangular decomposition of P, and an arc is directed from node A to node B iff they correspond to adjacent rectangles and the chord separating these rectangles forms an entire side of B. The direction of these arcs gives us some visibility information. R–graphs are similar to the H–graphs of O'Rourke [9]. The undirected version of $\mathbf{R}$ is denoted $\tilde{\mathbf{R}}$.

For any pair of neighboring rectangles in a rectangle decomposition there is one vertical polygon edge which is a vertical boundary for both. Depending whether this edge is the left (or right) boundary of both rectangles we will call the rectangles (or their corresponding nodes in $\mathbf{R}(P)$) left (or right) neighbors. The remaining terminology about rectangle decompositions should be self-explanatory (compare with Figure 1):

lower neighbor	(B is a lower neighbor of A),
upper neighbor	(C is an upper neighbor of A),
indegree	($indeg(C) = 1$),
outdegree	($outdeg(A) = 3$),
degree	($deg(D) = indeg(D) + outdeg(D) = 3$).

We note that the property of being a left neighbor is symmetric, in contrast to the property of being a lower neighbor.

The rest of the paper is organized as follows. The next section provides constructions which establish a lower bound for every value of $r(n, h, k)$. The third section contains a proof that $r(n, 0, k) \leq \left\lfloor \frac{n}{k+4} \right\rfloor$, and that equality holds for even k. One feature of our proof is that it provides a procedure for partitioning a simply-connected orthogonal polygon into at most $\left\lfloor \frac{n}{k+4} \right\rfloor$ polygons of size at most $2k + 6$; this generalizes results in [9], [3] for $k = 0$. The fourth section shows that the lower bound

Figure 2: Lower bounds for polygons with no holes

Figure 3: Lower bounds for even k

for line guards is tight and that $r(n,h,2) \leq \left\lfloor \frac{n}{6} \right\rfloor$. The last section is a summary and discussion of future directions.

2 Lower bounds on $r(n,h,k)$

In this section, we establish the following lower bounds on $r(n,h,k)$:

$$r(n,h,k) \geq \begin{cases} \left\lfloor \frac{n-2h}{k+4} \right\rfloor & \text{even } k \\ \left\lfloor \frac{3n+(7-3k)h+4}{3k+13} \right\rfloor & k = 1,3 \\ \left\lfloor \frac{3(n-2h)+4}{3k+13} \right\rfloor & \text{odd } k \geq 5 \end{cases}$$

These bounds are valid only for certain relationships of n/h, and k, as detailed later.

We begin with the $\left\lfloor \frac{n-2h}{k+4} \right\rfloor$ bound for even k. This bound is valid for $\frac{n}{h} \geq k+6$; this condition may be thought of as "having enough vertices per hole to make it interesting." Note that $\frac{n-4}{h}$ must be at least four, because each hole must have at least four vertices. Also, it is already known that $r(n,h,0) = \left\lfloor \frac{n}{4} \right\rfloor$ for $k = 0$ [4], so we need only consider $k \geq 2$.

Figure 2 shows examples of infinite polygon classes that establish a lower bound of $\left\lfloor \frac{n}{k+4} \right\rfloor$ for $h = 0$. The figure shows examples for $k = 2$, $k = 4$, and $k = 6$; these examples consist of $\frac{n}{k+4}$ spiral arms joined in a row; one guard is needed for each arm. Examples for larger k are made by increasing the number of turns on each spiral arm (one more turn per each increase of two in k). Examples for larger

Figure 4: The 1–pinwheel and the 3–pinwheel

n are made by joining more arms to the polygon. Holes made be added to these examples in the following manner: find a spiral arm that does not contain a hole (here we use the property that $\frac{n}{h} \geq k+4$), shorten that spiral by one turn, and add a rectangle in its end. This operation increases n by two and h by one, leaving the numerator (of $\left\lfloor \frac{n-2h}{k+4} \right\rfloor$) unchanged, and ensures that each arm still requires its own guard. Examples of this construction are shown in Figure 3 for $n = 36, h = 3, k = 2$ and $n = 34, h = 2, k = 6$. The class of polygons thus described establishes the $\left\lfloor \frac{n-2h}{k+4} \right\rfloor$ lower bound.

It remains to show lower bounds for odd k. Note that all both of the bounds that we wish to show (one for $k = 1$ and 3, and another for $k \geq 5$ both simplify to $\left\lfloor \frac{3n+4}{3k+13} \right\rfloor$ for $h = 0$. We first establish this bound, and describe the general construction method for odd k.

Let the term *t–pinwheel* denote the $(8t+12, 0)$–polygon formed by connecting four spiral arms of t turns in "pinwheel fashion", as illustrated in Figure 4 for $t = 1$ and $t = 3$. We will construct larger polygons from pinwheels by an operation that we call *grafting*. Grafting consists of clipping one of the spiral arms from a pinwheel, and attaching this fragment to another polygon at the first turn of one of its spiral arms (with the restriction that this spiral arm has not been grafted to before). A polygon which is formed by successively grafting only t–pinwheels to a t–pinwheel is called a *t–growth*. Figure 5 shows two 3–growths, the first the result of one grafting operation and the second the result of two.

In any t–pinwheel or t–growth, the vertices at the end of each spiral arm (one for each arm) form an independent set with respect to paths of length $2t+1$ inside the polygon. Thus, no T_{2t-1}–guard can see two of these vertices. To get lower bound examples for odd k and $h = 0$, we set $k = 2t-1$ ($t = \frac{k+1}{2}$). Any $(\frac{k+1}{2})$–growth resulting from j graftings has $3j+4$ spiral arms (thus requiring $3j+4$ T_k–guards) and $n = (8t+12) + j(6t+10) = (4k+16) + j(3k+13)$ vertices. These growths thus give the desired $\left\lfloor \frac{3n+4}{3k+13} \right\rfloor$ lower bound.

To establish the general $\left\lfloor \frac{3(n-2h)+4}{3k+13} \right\rfloor$ bound, we start with the (holeless) $(\frac{k+1}{2})$–growth and add holes in the same fashion that we did for the even–k examples: find an empty spiral arm, shorten it by one turn, and insert a rectangle. Once again

Figure 5: 3–growths

we have increased n by two and h by one without changing the number of guards required. An example of this construction is shown in Figure 6 for $n = 100, h = 4, k = 5$ (requiring 10 T_5–guards). This establishes the bound if the "enough vertices per hole" condition of $\frac{n}{h} > k + 6\frac{1}{3}$ is satisfied.

Figure 6: A 3-growth with holes added

For $k = 3$, we wish to show a lower bound of $\left\lfloor \frac{3n-2h+4}{22} \right\rfloor$. We start, as expected, with 2-growths, but to add a hole we *increase* the number of turns on a spiral arm by one, and insert an L-shaped hole that sits inside this turn (see Figure 7 for an example). This process adds 8 vertices and 1 hole ($3\Delta n - 2\Delta h = 22$) but the polygon now requires one extra guard, which bears out the formula. This hole insertion may be carried out as long as $\frac{n}{h} > 19\frac{1}{3}$.

Figure 7: Example for $k = 3$

For $k = 1$, the bound of $\left\lfloor \frac{3n+4h+4}{16} \right\rfloor$ is established by starting with 1-growths and adding rectangular holes in the ends of empty spiral arms [14]. Each hole insertion adds 1 hole and 4 vertices, and necessitates 1 extra guard. This construction is valid for $\frac{n}{h} > 9\frac{1}{3}$.

3 Upper bound on $r(n, 0, k)$

In this section, we prove the following upper bound:

Theorem 1 $r(n, 0, k) \leq \left\lfloor \frac{n}{k+4} \right\rfloor$

We actually prove a stronger statement

Theorem 2 *Any $(n, 0)$–polygon in general position can be partitioned into $\left\lfloor \frac{n}{k+4} \right\rfloor$ simply-connected rectilinear polygons of at most $2k + 6$ vertices.*

We recall once more that if $n < k + 4$ then we have to count one for $\left\lfloor \frac{n}{k+4} \right\rfloor$ rather than zero. The following lemma and Theorem 2 imply Theorem 1.

Lemma 3 *Any simply-connected rectilinear polygon of at most $2k + 6$ vertices can be covered by one T_k–guard.*

Lemma 3 can be proved easily by induction on k.

Now it is sufficient to give a proof of Theorem 2 for a polygon P with $n \geq 2k + 8$ vertices.

We let the term *cut* denote either a chord of the horizontal or vertical rectangular decomposition of P or the L-shaped union of two line segments joining two reflex vertices. We prove Theorem 2 inductively, using cuts to subdivide the polygon P. A cut subdivides P into two rectilinear subpolygons of n_1 and n_2 vertices such that $n_1 + n_2 = n + 2$; we refer to such a cut as a (n_1, n_2)–cut. Such a cut will be called good if $\lfloor \frac{n_1}{k+4} \rfloor + \lfloor \frac{n_2}{k+4} \rfloor \leq \lfloor \frac{n}{k+4} \rfloor$, i.e. if the inductive argument can be applied.

Lemma 4 *Let n, n_1, n_2 be even numbers with $n \geq 2k + 8$ and $n_1 + n_2 = n + 2$. An (n_1, n_2)–cut of an $(n, 0)$–polygon is good if one of the following conditions holds:*

(i). $n_1 \leq 2k + 6$ and $n_2 \leq 2k + 6$

(ii). $n_1 \geq k + 4$ and $n_2 \geq k + 4$ and $n_1 \not\equiv 0$ or 1 $(mod\ k + 4)$

(iii). $n_1 \geq k + 4$ and $n_2 \geq k + 4$ and $n_2 \not\equiv 0$ or 1 $(mod\ k + 4)$

(iv). $n_1 \equiv n_2 \equiv 1$ $(mod\ k + 4)$

Proof: (i): $\lfloor \frac{n_1}{k+4} \rfloor + \lfloor \frac{n_2}{k+4} \rfloor = 1 + 1 = \lfloor \frac{2k+8}{k+4} \rfloor \leq \lfloor \frac{n}{k+4} \rfloor$.
(ii),(iii),(iv): Let α_i be the residue n_i (mod $k + 4$). Then in all cases we have $\alpha_1 + \alpha_2 \geq 2$. Moreover $k + 4 \leq n_1$ and $k + 4 \leq n_2$ holds in case (ii) and (iii) by assumption and in case (iv) because otherwise n_1, resp. n_2 (as the number of vertices of P_1, resp. P_2) would be 1. Thus we get
$\lfloor \frac{n_1}{k+4} \rfloor + \lfloor \frac{n_2}{k+4} \rfloor = \lfloor \frac{n_1-\alpha_1}{k+4} \rfloor + \lfloor \frac{n_2-\alpha_2}{k+4} \rfloor \leq \lfloor \frac{n_1+n_2-\alpha_1-\alpha_2}{k+4} \rfloor = \lfloor \frac{n+2-\alpha_1-\alpha_2}{k+4} \rfloor \leq \lfloor \frac{n}{k+4} \rfloor$ □

Corollary *Let n, n_1, n_2 be even numbers with $n_1 + n_2 = n + 2$, $n_1 \geq k + 4$ and $n_2 - 2 \geq k + 4$. If an $(n, 0)$–polygon has an (n_1, n_2)–cut and an $(n_1 + 2, n_2 - 2)$–cut then at least one of them is a good cut.* □

Usually we will apply this corollary in a situation where the region between the two cuts is a rectangle. We use the term *consecutive cuts* to refer to such a pair of cuts.

Proof of Theorem 2. As P is an $(n, 0)$–polygon, the R–graph $\mathbf{R}(P)$ is a tree with $r = \frac{n-2}{2}$ nodes, and therefore it has a node R such that after deleting it, the size of any connected component is at most $\frac{r}{2} = \frac{n-2}{4}$. In terms of the polygon this means that $deg(R)$ horizontal cuts partition the polygon into $deg(R) + 1$ parts: the rectangle R and polygons $P_1, \ldots, P_{deg(R)}$ with $n_1, \ldots, n_{deg(R)}$ vertices such that each n_i is at most $2 \cdot \frac{r}{2} + 2 = \frac{n+2}{2}$. Since any cut creates two new vertices we have $\sum_{i=1}^{deg(R)} n_i = n + 2 \cdot deg(R) - 4$. Transforming this equality as follows

$$\begin{array}{rcll} -n_i & = & -n + \sum_{j \in \{1,\ldots deg(R)\} \setminus \{i\}} n_j + 4 - 2 \cdot deg(R) & \text{and combining it with} \\ 2n_i & \leq & n + 2 & \text{we obtain} \\ n_i & \leq & \sum_{j \in \{1,\ldots deg(R)\} \setminus \{i\}} n_j + 6 - 2 \cdot deg(R) & \text{for any } i \in \{1, \ldots, deg(R)\}. \end{array}$$

Now, we have the three possibilities: R has 2, 3 or 4 neighbors.

Figure 8: Illustration of Case B

Case A: Suppose that $deg(R) = 2$ and assume w.l.o.g. $n_1 \leq n_2$.
Considering the two cuts individually we have an (n_1, n_2+2)–cut and an (n_1+2, n_2)–cut. If moreover $n_1 \geq k+4$ then by the corollary at least one of the cuts is good. Otherwise, if $n_1 < k+4$ then by the inequality derived above we get $n_2 \leq n_1 + 6 - 2 \cdot 2 < k + 4 + 2 \leq 2k + 6$. Thus, the $(n_1 + 2, n_2)$–cut will be good by Lemma 4 (i).

Case B: Suppose that $deg(R) = 3$ and assume w.l.o.g. (by symmetry) that P_1 (resp. P_2 and P_3) meets R via a left upper (resp left lower and right upper) neighboring rectangle.
By the above discussion, we know that $n_1 + n_2 + n_3 = n + 2$ and $n_i \leq n_j + n_k$ for any permutation (i, j, k). Clearly, we have an $(n_1, n_2 + n_3)$–cut, an $(n_2, n_1 + n_3)$–cut and an $(n_3, n_1 + n_2)$–cut, but, there is also a fourth $(n_3 + 2, n_1 + n_2 - 2)$–cut which starts vertically from A down to the horizontal edge thru C or its extension (see Figure 8 for illustration of the typical situations).

Subcase B.1: Suppose that $n_3 \geq k + 4$.
If moreover $n_1 + n_2 - 2 \geq k + 4$ then by the corollary the third or the fourth cut will be good. Otherwise, if $n_1 + n_2 - 2 < k + 4$ then we have $n_3 \leq n_1 + n_2 < k + 6 \leq 2k + 6$ and hence the fourth cut is good by Lemma 4 (i).

Subcase B.2: Suppose that $n_3 < k+4$ and one of the following seven conditions holds:
a) $n_1 < k + 4$; then $n_1 + n_3 \leq 2k + 6$ and $n_2 \leq n_1 + n_3 \leq 2k + 6$. Thus the first cut is good by Lemma 4 (i).
b) $n_2 < k + 4$; then analogously the second cut is good.
c) $n_1 \geq k + 4$ and $n_2 \geq k + 4$ and $n_1 \equiv 0 \pmod{k+4}$; then $(n_1 + n_3) \not\equiv 0$ or $1 \pmod{k+4}$ and the second cut will be good by Lemma 4 (iii).

d) $n_1 \geq k+4$ and $n_2 \geq k+4$ and $n_2 \equiv 0 \pmod{k+4}$; then $(n_2 + n_3) \not\equiv 0$ or $1 \pmod{k+4}$ and the first cut will be good by Lemma 4 (iii).
e) $n_1 \geq k+4$ and $n_2 \geq k+4$ and $n_1 \not\equiv 0$ or $1 \pmod{k+4}$; then the first cut will be good by Lemma 4 (ii).
f) $n_1 \geq k+4$ and $n_2 \geq k+4$ and $n_2 \not\equiv 0$ or $1 \pmod{k+4}$; then analogously the second cut will be good.
g) $n_1 \geq k+4$ and $n_2 \geq k+4$ and $n_1 \equiv n_2 \equiv 1 \pmod{k+4}$ and $n_3 < k+3$; then the first cut will be good by Lemma 4 (iii).

Subcase B.3: Suppose none of the above holds, this means we have $n_1 \equiv n_2 \equiv 1 \pmod{k+4}, n_3 = k+3$
We will find in each possible configuration either a cut with one resulting subpolygon of size $k+7$ or a pair of consecutive cuts.
We call two reflex vertices opposite to each other if they rectangularly see each other and the edges incident to them (considered as rays emanating from these vertices) represent all 4 main compass directions.
Observe that in the case of two opposite reflex vertices, as well as in the case of two neighboring reflex vertices which both rectangularly see a third reflex vertex, one finds consecutive cuts.

Subcase B.3.1: C is right of B
This is either the left or the right configuration shown in Figure 8. We consider the highest reflex vertex D below the horizontal line thru C such that D is visible both from A and B. If there are two such vertices take, say, the left one. Given there is no such vertex the vertical line extensions thru A and B define consecutive cuts. But if we have a vertex D we also have consecutive cuts by the above observation. Note that in all these cuts the subpolygons containing P_3 have size $\geq k+5$ and the remaining parts have size $\geq k+5$ as well, since each contains P_1 completely. Thus, based on the corollary at least one of the cuts is good.

Subcase B.3.2: C is left of A.
If C rectangularly sees the upper neighbor of A, then we connect C with this neighbor (even if it is convex) by an L–shaped cut and obtain a subpolygon containing P_3 of size $k+7$. Otherwise there must be a reflex vertex in P_1 which is opposite to A and we are done.

Subcase B.3.3: C is right of A and left of B.
In this case we can apply the same argument as in subcase B.3.2 to P_2 with the roles of A and C exchanged.

Case C: Suppose that $deg(R) = 4$ and assume w.l.o.g. that P_1 and P_2 (resp. P_3 and P_4) are left (resp. right) neighbors of R. Since $\sum_{i=1}^{4} n_i = n+4$ at least one of the subsums $n_1 + n_2$ or $n_3 + n_4$ is less than or equal to $\frac{n+4}{2}$. By symmetry, we can assume that this holds for the subsum $n_3 + n_4$. Then there is an L–shaped cut such that the polygon P_3' on the right side of this cut has $n_3 + n_4 - 2$ vertices and consists of P_3, P_4 and a portion of R. Now the analysis of Case B can be applied, with P_3' taking the place of P_3 in that analysis. □

4 Upper bounds on $r(n, h, 1)$ and $r(n, h, 2)$

In this section we will prove the following result.

Theorem 5 $\lfloor \frac{3n+4h+4}{16} \rfloor$ *T_1–guards are always sufficient to cover any rectilinear (n, h)-polygon.*

In fact we prove that these guards can be chosen to be polygon edges or edge extensions. Moreover, in the whole section we will deal with the stronger definition of orthogonal visibility: a point x in a polygon P is othogonaly visible from a T_k–guard ($k \geq 1$) G if there is a line of G such that the perpendicular from X to this line is contained in P.

Lemma 6 *Let R_1 and R_2 be adjacent rectangles in $\mathbf{R}$ separated by the extension of some horizontal polygon edge e. Then the following holds:*

(i). If R_2 is an upper (resp. lower) neighbor of R_1 and the arc connecting them is directed from R_2 to R_1 then R_2 is the only upper (resp. lower) neighbor of R_1. Consequently, if indeg$(R_1) = 2$ then outdeg$(R_1) = 0$.

(ii). If G is a T_1–guard on the edge e and its extension then G can watch any rectangle R which can be reached by a directed path in $\mathbf{R}$ starting from R_1 or R_2.

Proof: (i). This follows from the assumption about the general position. (ii). We observe that according to (i) any directed path in $\mathbf{R}$ is also strictly directed in the geometrical sense (either upwards or downwards). Furthermore on a directed path the rectangles get more and more narrow. □

Lemma 7 *If $R_0 R_1 \ldots R_m$ is a directed path in $\mathbf{R}$ and R_{m+1} is another rectangle with an arc directed to R_m then there is a vertical T_1–guard covering all rectangles R_i $(0 \leq i \leq m+1)$.*

Proof: Note that R_m and R_{m+1} have a vertical polygon edge e in common. Since the path from R_0 to R_m is strictly directed in the geometrical sense with the rectangles becoming more and more narrow, e can be extended to R_0. □

We define the *frame* of $\mathbf{R}$ to be the largest subgraph $\mathbf{F}$ such that for every vertex R in F, $deg_{\mathbf{F}}(R) \geq 2$. If there isn't any nonempty subgraph $\mathbf{F}$ fulfilling the above condition (i.e. if $\mathbf{R}$ is a tree) then we define some arbitrary fixed leaf of $\mathbf{R}$ to be the frame. Thus, $\mathbf{R}$ consists of its frame and some attached trees. Denote by $\mathbf{T}$ the set $\mathbf{R} \setminus \mathbf{F}$ of non–frame nodes. For any $R \in \mathbf{T}$ there is a unique path $p(R)$ in $\tilde{\mathbf{R}}$ connecting it to the frame. A node $R \in \mathbf{T}$ with degree ≥ 3 is called a *primary branch* if for any $R' \in \mathbf{T}$ such that $R \in p(R')$, R is the first node of degree ≥ 3 on $p(R')$.

Let $R_0 \in \mathbf{T}$ be a leaf and $p(R_0) = R_0 R_1 \ldots R_m$ with $R_m \in \mathbf{F}$. We define the *branching distance* of R_0 to be the minimal number l $(1 \leq l \leq m)$ such that $deg(R_l) \geq 3$, or m if there is no such number.

Let $G_1, \ldots, G_l$ be a family of T_1–guards in an (n,h)–polygon P and D a rectilinear region covered by them (called a district of the guards). Usually, D will be smaller than the maximal possible region covered by $G_1, \ldots, G_l$. Deleting D from P we obtain a number (say c') of connected regions which are $(n_1, h_1), \ldots, (n_{c'}, h_{c'})$–polygons denoted by $P_1, \ldots, P_{c'}$.
The deletion of D will be called a *reduction* if $l + \sum_{i=1}^{c'} \left\lfloor \frac{3n_i+4h_i+4}{16} \right\rfloor \leq \left\lfloor \frac{3n+4h+4}{16} \right\rfloor$, i.e. if the deletion allows to apply induction. Note, that this definition also makes sense if D is the whole polygon: then we have $c' = 0$, the sum over an empty set is also 0 and we get $l \leq \left\lfloor \frac{3n+4h+4}{16} \right\rfloor$. In the proof we will show that in most situations one can find a reduction by a district of a single guard (i.e. $l = 1$). There will be only one special geometrical configuration where a reduction by a district of two guards is necessary.

The following measures *gain* and *gain*$^+$ will help to formulate sufficient conditions for a district to cause a reduction. Using the notations above we define

$$gain(D) \ := \ 3(n-n') + 4(h-h') + 4(1-c')$$

where $n' = \sum_{i=1}^{c'} n_i$, $h' = \sum_{i=1}^{c'} h_i$. Furthermore let α_i be the residue $3n_i + 4h_i + 4$ (mod 16) for any $1 \leq i \leq c'$. Then we define

$$gain^+(D) \ := \ 3(n-n') + 4(h-h') + 4(1-c') + \sum_{i=1}^{c'} \alpha_i$$

.

Lemma 8 *Let D be a district of a family of T_1–guards $G_1, \ldots, G_l$. in a polygon P. If $gain^+_P(D) \geq l \cdot 16$ then the deletion of D is a reduction.*

Proof: We will make use of the fact that $\left\lfloor \frac{3n_i+4h_i+4}{16} \right\rfloor = \left\lfloor \frac{3n_i+4h_i+4-\alpha_i}{16} \right\rfloor$.

$$l + \sum_{i=1}^{c'} \left\lfloor \frac{3n_i + 4h_i + 4}{16} \right\rfloor \ \leq \ \left\lfloor \frac{16l}{16} \right\rfloor + \sum_{i=1}^{c'} \left\lfloor \frac{3n_i + 4h_i + 4 - \alpha_i}{16} \right\rfloor$$

$$
\begin{aligned}
&\leq \left\lfloor \frac{16l + 3n' + 4h' + 4c' - \sum_{i=1}^{c'} \alpha_i}{16} \right\rfloor \\
&\leq \left\lfloor \frac{gain_P^+(D) + 3n' + 4h' + 4c' - \sum_{i=1}^{c'} \alpha_i}{16} \right\rfloor \\
&\leq \left\lfloor \frac{3n + 4h + 4}{16} \right\rfloor
\end{aligned}
$$

$\square$

It will be very helpful to represent $gain(D)$ using the number $r = \frac{n}{2} + h - 1$ of nodes in $\mathbf{R}(P)$. Thus $n = 2(r - h + 1)$ and $n' = 2(r' - h' + c')$ where r' is the total number of nodes in the graphs $\mathbf{R}(P_i)$, $1 \leq i \leq c'$ and we get

$$gain(D) = 6(r - r') - 2(h - h') + 10(1 - c').$$

The triple $(\delta_r, \delta_h, \delta_c)$, where $\delta_r = r - r'$, $\delta_h = h - h'$, $\delta_c = 1 - c'$, will be called the *type* of D.

Lemma 9 (Expansion Lemma) *Let G be a horizontal T_1–guard in a polygon P and D a district of G. Let P_1 be a polygon representing a connected component of $P \setminus D$, and e be a horizontal edge that bounds P_1 from above and is shared between P_1 and D. Let R be the rectangle of P_1 that contains e. Let $\overline{D}$ be the expansion of D by R and all rectangles reachable from R on directed paths in $\mathbf{R}(P_1)$. If the edge e is (orthogonally) visible from G (see Figure 9, where G runs across the top of the figure), then $\overline{D}$ is also a district of G and the following holds:*

(i). $gain(\overline{D}) \geq gain(D) + 6$

(ii). if $indeg_{P_1}(R) = 0$ then $gain(\overline{D}) \geq gain(D) + 8$.

Proof: Since G covers the whole horizontal width of R, it follows from Lemma 6 (ii) that any rectangle reachable on a directed path in $\mathbf{R}(P_1)$ from R will be covered by G. Let $\mathbf{S}$ be the subtree of $\mathbf{R}(P_1)$ formed by R and all nodes reachable from there on a directed path. Let B denote the set of rectangles in $\mathbf{S}$ that have two lower neighbors and $b = |B|$. The tree $\mathbf{S}$ has at least $2b + 1$ nodes. If we add by breadth first search the rectangles of $\mathbf{S}$ to D starting with R, then for each rectangle from B either the number of connected components of the remaining polygon increases by 1 (say, b_1 times) or the number of holes decreases by 1 ($b_2 = b - b_1$ times). In contrast, adding a rectangle which has no two lower neighbors neither changes δ_h nor increases the number of connected components. So we have

$$gain(\overline{D}) \geq gain(D) + 6(2b+1) - 10b_1 - 2b_2 \geq gain(D) + (12-10)b + 6 \geq gain(D) + 6$$

Figure 9: Illustrating Lemma 9

Now, suppose that $indeg_{P_1}(R) = 0$. We consider the three possibilities $outdeg_{P_1}(R) = 0,\ 1$ or 2.

If $outdeg_{P_1}(R) = 0$ then P_1 consists of R only and adding R to D we reduce the number of connected components of $P \setminus D$ by one, giving $gain(\overline{D}) = gain(D) + 10$. If $outdeg_{P_1}(R) = 1$ then let R' be this unique neighbor of R in P_1. Adding R to D we get a district D' with $gain(D') = gain(D) + 6$ and, moreover, we can apply this lemma once more to D' and the rectangle R' in $P \setminus D'$. Thus we get $gain(\overline{D}) \geq gain(D') + 6 = gain(D) + 12$
Finally, if $outdeg_{P_1}(R) = 2$ then $R \in B$ and thus $b \geq 1$. Our claim follows immediately from the inequality in the first part of the proof. $\square$

The proof of the theorem now follows from the next three lemmata which show that each non-trivial polygon is reducible.

Lemma 10 *If $R_0 \in \mathbf{T}$ is a leaf with branching distance ≥ 3 then there is some reduction with R in the reduction district.*

Proof: Let R_0, R_1, R_2 be the first three rectangles on the path $p(R_0)$. Since $deg(R_1) = deg(R_2) = 2$, the deletion of the region $D = R_0 \cup R_1 \cup R_2$ neither disconnects the remaining polygon nor changes the number of holes and we get $gain(D) = 6 \cdot 3 = 18$. Hence, it is sufficient to show that there is a guard G covering D. Let us consider the directed versions of the edges $\{R_0, R_1\}$ and $\{R_1, R_2\}$.

- If both arcs are directed from R_1 to R_0 and R_2 then a guard placed on a horizontal boundary of R_1 covers D by Lemma 6 (ii).

- If the two arcs form a directed path then a guard on a horizontal boundary of the first rectangle of the directed path will cover D by Lemma 6 (ii).
- If both edges are directed towards R_1 then there is a vertical guard covering D by Lemma 7.

$\square$

Lemma 11 *If all leaves in* **R** *have branching distance* < 3 *and* R *is a primary branching then there is a reduction such that* R *or a part of* R *is in the reduction district.*

Proof: The proof of this lemma is much more complicated than the proof of the preceeding lemma. It requires a rather long case inspection and several tricky arguments. However this is not surprising because both lemmata together yield a new proof for simply connected polygons (cf. [1]).

Let R be a primary branching with neighbors R_1, R_2, R_3 (and possibly R_4, if $deg(R) = 4$) in $\tilde{\mathbf{R}}$. W.l.o.g. we can assume that R_1 is the (unique) neighbor of R on the path $p(R)$ and, moreover, that R_1 is a left lower neighbor of R. By the assumption there are leaves L_2, L_3 (and possibly L_4) such that for any $i \geq 2$ we have either $L_i = R_i$ or L_i is a neighbor of R_i and $deg(R_i) = 2$. Let $\mathbf{N}$ be the set of rectangles consisting of $R_2, R_3, (R_4 \; if \; deg(R) = 4)$ and the leaves $L_2, L_3, (L_4)$ provided they do not coincide with some R_i. We have to distinguish the following cases:

Case A: Suppose that for all rectangles in $\mathbf{N}$ there is a directed path from R to them.
Then we choose a horizontal boundary of R for placing the guard and by Lemma 6 (ii) this guard covers a district D consisting of R and all rectangles from $\mathbf{N}$. Clearly, the type of this district is $(\delta_r, 0, 0)$ and $\delta_r \geq 3$. This implies $gain(D) \geq 18$ and we are done.

Case B: Suppose that for some $i_0 \geq 2$ there is an arc $R_{i_0} \to R$ in **R** ,i.e. R_{i_0} is wider than R.
W.l.o.g. we may assume that $i_0 = 2$. Furthermore we can assume that R_2 is an upper neighbor of R, because otherwise by Lemma 6 (i) R_2 would be the only lower neighbor of R contradicting that R_1 is also a lower neighbor.

Subcase B.1: Suppose that $L_2 = R_2$.
Since $deg(R) \geq 3$ and since there is only one upper neighbor, R_3 has to be a right lower neighbor. Dependently on whether $L_3 \neq R_3$ or $L_3 = R_3$, we place a guard on the extended common vertical edge of L_3 and R_3 or on the extended common vertical edge of R and R_3 and define a district D consisting of R, R_2, R_3 and L_3. Thus, the type of D is $(4, 0, 0)$ or $(3, 0, 0)$ and we are done.

Figure 10: Illustration of subcase B.2

Subcase B.2: Suppose that $L_2 \neq R_2$.
Placing a guard G on the extended horizontal edge which separates L_2 from R_2 we define a district D consisting of these two rectangles, see Figure 10. Since $gain(D) = 12$ does not suffice, we apply the expansion lemma. Indeed, the whole upper boundary of R is orthogonaly visible from G. Hence adding to D the rectangle R and all rectangles reachable from R via a directed path in $\mathbf{R}$ we get a new district $\overline{D}$ with $gain(\overline{D}) \geq 12 + 6 > 16$ and we are done.

Case C: Suppose that neither case A nor case B are valid, i.e. for any $i \geq 2$ there is an arc from R to R_i in $\mathbf{R}$ and there is some $i_0 \geq 2$ such that $L_{i_0} \neq R_{i_0}$ and the arc between them is directed from L_{i_0} to R_{i_0}. Again, w.l.o.g. we assume $i_0 = 2$. Let e be the common vertical polygon edge of R and R_2 and A the lower (resp. upper) polygon vertex of this edge if R_2 is an upper (resp. lower) neighbor of R. We place a vertical guard G on the full extension $\overline{e}$ of e and define a district D dependently on whether A is a reflex vertex or not.

Subcase C.1: Suppose that A is not a reflex vertex.
Then in a first step we define a district D of type $(2,0,0)$ consisting of L_2, R_2 and the remaining segment (i.e. below R_2) of the edge e, see Figure 11 – the left picture. Denoting this segment by e', it is an edge of the polygon $P' = P \setminus (L_2 \cup R_2)$. Let φ be the rotation of the plane by $90°$ such that $e'' = \varphi(e')$ is a top edge in the rotated polygon $P'' = \varphi(P')$, see Figure 11 – the right picture.

Now, we consider the horizontal rectangular decomposition of P'' (i.e. the rotation of the vertical rectangular decomposition of P') and denote by S the rectangle containing e''. Restricting the guard G to P', resp. via rotation to P'', it is placed on the top edge e'' of S. So we can apply the expansion lemma in this situation and we get a district $\overline{D}$ with $gain(\overline{D}) \geq gain(D) + 6 = 18$.

The trick of first cutting out a district of small gain, then rotating the polygon and applying the expansion lemma will be used several more times. Since in con-

Figure 11: Illustration of subcase C.1

trast to the original expansion lemma, we expand here the district in a horizontal direction, we will refer to this trick as the horizontal expansion lemma.

Subcase C.2: Suppose that A is a reflex vertex.
We consider the horizontal polygon edge f which determines the upper boundary of the rectangle R and denote the right polygon vertex on this edge by B, see Figure 12. Let S be the rectilinear rectangle spanned by A and B (in general, S is not a rectangle of the rectangular decomposition).

Subcase C.2.1: Suppose that $S \subseteq P$, i.e. there are no vertices or edges of P in the interior of S.
We define a district D consisting of L_2, R_2 and S. Clearly, this district is covered by G. Since general position was assumed, one can be sure that the deletion of G neither disconnects the remaining region $P' = P \setminus D$ nor changes the number of holes and, furthermore, there is a cut separating the (8,0)–polygon D from the (n',h')–polygon P'. This implies $n' + 8 = n + 2$ or equivalently $\delta_n = 6$ and consequently $gain(D) = 3\delta_n + 4\delta_h + 4\delta_c = 18$.

Subcase C.2.2: Suppose that $S \not\subseteq P$.

Subcase C.2.2.1: Suppose that R_2 is a right neighbor of R.
We will show that summing up all current assumptions we will obtain the following unique situation:
R has two right neighbors R_2 and R_3 both of degree two. Furthermore, we have the following arcs in $\mathbf{R}$: $L_2 \rightarrow R_2 \leftarrow R \rightarrow R_3 \leftarrow L_3$. In fact, if R_2 were the only right

Figure 12: Illustration of subcase C.2.1: $S \subset P$

neighbor of R then either subcase C.1 (A is not a reflex vertex) or subcase C.2.1 ($S \subseteq P$) would apply. Hence, there is a second right neighbor R_3 and since case B is not valid we have an arc $R \to R_3$. Furthermore if R_3 were a leaf or if $R_3 \neq L_3$ and $R_3 \to L_3$ the vertex A would not be reflex and subcase C.1. would be valid. So we obtain the configuration $L_2 \to R_2 \leftarrow R \to R_3 \leftarrow L_3$ and a guard placed on e and its full extension vertically crosses all these rectangles. Thus, defining a district consisting of L_2, R_2, R_3 and L_3 we obtain a reduction of type $(4, 0, 0)$.

Subcase C.2.2.2: Suppose that R_2 is a left neighbor of R.
Since R_1 is a left lower neighbor of R, R_2 must be a left upper neighbor. This subcase is the hardest one. We will analyse it separately as Case E. It will be very useful to exclude several configurations on the right side of R before (Case D). To do this, let $\mathbf{N}'$ be the set of all right neighbors of R (i.e. R_3 and possibly R_4, if $deg(R) = 4$) and of the leaves L_3 (L_4) if they do not coincide with R_3 (R_4).

Case D: Suppose we have all assumptions made in subcase C.2.2.2 and moreover $\mid \mathbf{N}' \mid \geq 2$.
We again examine the cases A, B, and C taking into account the right neighbors only.

Subcase D–A: Suppose that all rectangles in $\mathbf{N}'$ are reachable from R on directed paths. Consider the L–cut starting vertically from the more narrow left neighbor of R to the opposite side of R and then turning to the right side, see Figure 13 where R_2 is more narrow than R_1. This L–cut removes an m–gon D with $m = 2 \cdot \mid \mathbf{N}' \mid \ +4 \geq 8$ that can be covered by a horizontal guard in R. So we get $\delta_n \geq 6$, $\delta_h = \delta_c = 0$ and consequently $gain(D) \geq 18$.

Subcase D–B: If there is a right neighbor R_{i_0} with an arc $R_{i_0} \to R$ in $\mathbf{R}$ then this is a proper subcase of Case B and so we are done.

Figure 13: Illustration of subcase D–A

Subcase D–C: If there is a right neighbor R_{i_0} with arcs $R \rightarrow R_{i_0} \leftarrow L_{i_0}$ we are in the situation of Subcase C.2.2.1.

Case E: Suppose, we have all assumptions made in subcase C.2.2.2 and moreover $\mid \mathbf{N}' \mid < 2$ (the negation of D).
We recall that these assumptions together imply the following configuration: R has a left lower neighbor R_1 (which lies on the unique path connecting R with the frame), a left upper neighbor R_2 with an attached leaf L_2 such that $R \rightarrow R_2 \leftarrow L_2$ and exactly one right neighbor R_3 which is a leaf and we have $R \rightarrow R_3$. Furthermore we know that the lower vertex A of the common vertical edge e of R, R_1 and R_2 is reflex and that the interior of the rectangle S spanned by A and B (the right vertex of the horizontal polygon edge bounding R from above) contains some vertex.
We place a guard onto the full extension $\overline{e}$ of e and define a first district D_1 to consist of the guard position itself plus the rectangles R_2 and L_2. The vertical cut from A (which is part of D_1) causes us to have either $\delta_h = 1$ and $\delta_c = 0$, or $\delta_h = 0$ and $\delta_c = -1$.

Subcase E.1: Suppose that by deleting D_1 we get $\delta_h = 1$ and $\delta_c = 0$.
We have $gain(D_1) = 2 \cdot 6 - 2 = 10$ and in $P \setminus D_1$ and applying the rotated version of Lemma 9 on both sides of of the guard position we obtain a district $\overline{D_1}$ of gain $\geq 10 + 2 \cdot 6 > 16$.

Subcase E.2: Suppose that by deleting D_1 we get $\delta_h = 0$ and $\delta_c = -1$.
We have $gain(D_1) = 2 \cdot 6 - 10 = 2$ and get two polygons P_l and P_r to the left and to the right side of the vertical cut from A. Let R_l (resp. R_r) be the rectangles of the vertical decomposition of P_l (resp. P_r) which contain the vertical cut from A. Note that for both rectangles one can apply the rotated version of Lemma 9, see Figure

Figure 14: Illustration of subcase E.2

14.

Subcase E.2.1: Suppose that in the vertical rectangular decomposition graph of P_r we have $indeg(R_r) \neq 1$.
An application of Lemma 9 (ii) to P_r increases the gain by ≥ 8 and hence we obtain a district $\overline{D_1}$ of gain $\geq 2+8+6=16$.

Subcase E.2.2: Suppose that in the vertical rectangular decomposition graph $\mathbf{R}'$ of P_r we have $indeg(R_r) = 1$.
Applying twice the rotated version of Lemma 9 we get a district $\overline{D_1}$ consisting of R_2, L_2, R_r and R_l. Note that the gain of this district is $2+2\cdot 6 = 14$. The assumption $indeg(R_r) = 1$ implies that if we take a chord in P_r parallel to the guard, and shift it rightwards starting at the guard's location, then the first vertex of P_r that this chord will encounter is a reflex vertex on the upper or lower side of R_r. It is impossible that this vertex is B because of our assumption that the rectangle S contains a polygon vertex. Let C be the highest of all polygon vertices in the interior of S (the left one if there are two highest ones) and let f' be the horizontal edge turning from C to the right, see Figure 15. If R' denotes the rectangle in the vertical rectangular decomposition of P_r that is placed between f and f' then $indeg(R') = 2$, i.e. the right side of R' is either the vertical cut of B and B is a reflex vertex or the vertical cut from the right vertex C' of f' and C' is a reflex vertex, see Figure 16 for all possible configurations.

Note that otherwise we would get a contradiction either to the fact that D is a highest vertex in the interior of S or to the fact that R has exactly one right neighbor R_3 with $R \rightarrow R_3$. Extending R' horizontally to the left (up to R_r) and adding the extended rectangle to $\overline{D_1}$ we get a district D_2 increasing δ_r by 1. Moreover either δ_c decreases by 1 or δ_h increases by 1. In the second case we are done because we

Figure 15: Illustration of subcase E.2.2

Figure 16: The four possible configurations on the right side of R'

get $gain(D_2) = gain(\overline{D_1}) + 6 - 2 = 14 + 6 - 2 > 16$. In the first case we have only $gain(D_2) = gain(\overline{D_1}) + 6 - 10 = 14 + 6 - 10 = 10$. Let P_1, P_2, P_3 be the three $(n_1, h_1)-, (n_2, h_2)-, (n_3, h_3)-$ polygons representing $P \setminus D_2$ where P_1 is the polygon on the right side of R' and P_2 the polygon below the horizontal cut from C. Note that either P_1 is a simple rectangle (Figure 16 (a) and (d)) or R_3 is a leaf in the horizontal rectangular decomposition of P_1 (Figure 16 (c)) or it can be extended (downward) to a leaf R_3' of $\mathbf{R}(P_1)$ (Figure 16 (b)).
For $i \in \{1, 2, 3\}$ let α_i be the residue $3n_i + 4h_i + 4 \quad (mod 16)$.

Subcase E.2.2.1: Suppose that $\alpha_1 \geq 6$.
Then we get $gain^+(D_2) = gain(D_2) + \sum_{i=1}^{3} \alpha_i \geq gain(D_2) + \alpha_1 \geq 16$ and we are done.

Subcase E.2.2.2: Suppose that $\alpha_1 < 6$.
Now we place a second guard horizontally on the edge f and its extension. Note that we have to find a common district of $gain^+$ at least 32. If P_1 is a rectangle we add it to D_2. For the resulting district D_3 we have one rectangle more and one connected component (P_1) less and hence $gain(D_3) = gain(D_2) + 6 + 10 = 26$.
If P_1 is not a rectangle we add to D_2 the leaf R_3 respectively R_3'. The new district D_3 has one rectangle more and the polygon $P_1' = P_1 \setminus R_3$ (respectively $P_1 \setminus R_3'$) has one rectangle or equivalently two vertices less. Hence, the residue α_1' of P_1' is $\alpha_1 - 6$ (mod 16) ≥ 10, and consequently $gain^+(D_3) \geq gain(D_2) + 6 + \alpha_1' \geq 26$.
Finally, we consider the retangle R'' in the horizontal rectangular decomposition of P_2 placed between the vertical cut from A' and the vertical edge from C, see Figure 17. Obviously, R'' is covered by the horizontal guard and Lemma 9 can be applied. Note that this application does not change α_1' and thus for the resulting district D we get $gain^+(D) \geq gain(D_3) + 6 + \alpha_1' \geq 32$. This completes our case inspection. $\square$

We note that applying Lemma 10 and Lemma 11 we can reduce the problem to polygons P such that $\mathbf{R}(P)$ consists only of its frame and leaves or paths of length 2 attached to the frame. In the following we show how to find a place for a reduction in such a polygon.

We need the following definition: An *extremal hole edge* is a polygon edge e on the boundary of a hole such that

1. e connects two reflex vertices and
2. in the partition of P induced by extending e in both directions until it hits the boundary, the region containing e is simply-connected.

We remark that if a polygon has more than one hole, then among all, say, northernmost hole edges there is not necessarily an extremal edge, see Figure 18.

Lemma 12 *If a rectilinear polygon has holes, then it has an extremal hole edge.*

Figure 17: Illustration of subcase E.2.2.2

Figure 18: No northernmost extremal hole edges

Proof: Let us call an edge a reflex edge if it connects two reflex vertices. Clearly, any hole of an (n,h)–polygon P has at least 4 reflex edges. Let E_h denote the set of all horizontal reflex edges of holes in P. We show that E_h contains an extremal edge. First observe that E_h contains a non–empty subset E'_h of reflex cut edges. A horizontal reflex edge is a cut edge if both extensions to the east and the west hit the outer boundary of P. To see that there are such edges one defines the following hole merging procedure. One can merge two holes if an edge extension of a reflex edge of one hits the other hole. In this case we merge the holes by adding this one–sided edge extension as a wall to them. If the extension hits the hole itself one adds to the hole the connected component enclosed by the hole and the one–sided edge extension. We search through the set E_h and apply the procedure whenever it is possible. Remark that this procedure does not create new reflex edges and we are eventually left with a polygon P' which has at least one hole. The set of horizontal reflex edges in P' corresponds exactly to those reflex edges in E'_h. Now to find the extremal edge in P it is clearly sufficient to show the following fact:

Given a polygon Q with a distinguished horizontal edge e on the outer boundary and the property that all horizontal reflex edges are cut edges, there is always an extremal horizontal edge e' such that in the partition of Q induced by e' the simply connected part $Q_{e'}$ containing e' does not contain e.

This can be proved by induction on the number h of holes. It is true for $h = 1$ since the hole has at least 2 extremal edges. If we have more than one hole take any horizontal reflex edge e" and consider $Q_{e"}$. There are two cases to distinguish. Firstly, suppose $Q_{e"}$ is simply connected. Then if $Q_{e"}$ does not contain e we are done, otherwise either there is another horizontal reflex cut edge of the same hole which is extremal or choose any one of these edges, say d, and apply the induction hypothesis to Q_d with the extension of d being the new distinguished boundary edge. Given that $Q_{e"}$ is not simply connected we can apply the induction hypothesis to it with the extension of e' being the new distinguished boundary edge if $e \notin Q_{e"}$. $\square$

Lemma 13 *Let P be a polygon to which Lemma 10 and Lemma 11 cannot be applied. W.l.o.g. let e be a horizontal extremal hole edge bounding the hole from above and let $R \in \mathbf{R}$ be the rectangle having e on its boundary. Then there is a reduction such that R or a rectangular part of R is in the district of the reduction.*

Proof: We note that R has two lower neighbors R_l and R_r. If there are also upper neighbors S_1 and S_2 of R then because e is extremal, each of them is either leaf or of degree two and adjacent to some leaf L_1 or L_2. Analogously to the proof of Lemma 13 let $\mathbf{N}$ be the set consisting of all upper neighbors of R and all leaves adjacent to these neighbors. Again we distinguish three cases:

Case A: Suppose that any rectangle of $\mathbf{N}$ is reachable from R on a directed path in $\mathbf{R}$ (note that this condition holds also if $\mathbf{N}$ is empty).
We place a horizontal guard onto the full extension of e. Clearly, it covers a district

D consisting of R and all rectangles of $\mathbf{N}$. Thus, the type of D is $(1+|\mathbf{N}|, 1, 0)$ and its gain is $6 + 6 \cdot |\mathbf{N}| - 2 \geq 4$. Moreover for both R_l and R_r the expansion lemma can be applied, so the expanded district $\overline{D}$ has a gain $\geq 4 + 2 \cdot 6 = 16$.

Case B: Suppose that there is (exactly) one upper neighbor S_1 and an arc $R \leftarrow S_1$.
Placing a horizontal guard onto the upper boundary of S_1 and extending it as far as possible we can cover R and all rectangles of $\mathbf{N}$ and hence we can proceed further as in Case A.

Case C: Suppose that there is (at least) one upper neighbour S_1 adjacent to a leaf L_1 and arcs $R \rightarrow S_1 \leftarrow L_1$.
W.l.o.g. let S_1 be a left neighbor of R. Placing a vertical guard onto the common vertical polygon edge f of R and S_1 and its extension one can cover a district D consisting of L_1, S_1 and that part of R which is bounded by f on the left side and by the extension of the left boundary of R_r on the right side. So after deleting D the remaining part of R forms together with R_r one rectangle in the rectangular decomposition and thus D is of type $(3, 1, 0)$ and one has $gain(D) = 16$. ◻

We close this section proving the $\left\lfloor \frac{n}{6} \right\rfloor$ upper bound for T_2–guards. For technical convenience in the inductive proof we introduce a slight reformulation of the bound. For any (n, h)–polygon P we define a characteristic number $\chi(P)$ as follows:

$$\chi(P) = \begin{cases} 1 & if\ n = 4\ and\ h = 0 \\ 0 & else \end{cases}$$

Theorem 14 *For any (n, h)–polygon P we have $r(P, 2) \leq \left\lfloor \frac{n+2\chi(P)}{6} \right\rfloor$.*

To prove this theorem one goes along similar lines as in the proof of Theorem 5 where in contrast to the above proof the lemmata for reducing simply connected parts becomes rather trivial. For reducing holes the existence of extremal edges is also essential. Roughly speaking one can use the second arm of a T_2–guard to cover one rectangle more.

Since we want to prove another bound than in Theorem 5 we have to change the definitions of reductions, types and of gain. To avoid confusions with Theorem 5 we will use the notations $gain_2$, and $2-reductions$. (Note that the definitions depend on the bound one wants to prove rather than on the guard type, so a more precise notation would be $gain_{\lfloor \frac{n}{6} \rfloor}$ and $\left\lfloor \frac{n}{6} \right\rfloor$-reduction.)

Let G be a T_2–guard in an (n, h)–polygon P covering a district D and let $P_1, \ldots, P_{c'}$ be the $(n_1, h_1), \ldots, (n_{c'}, h_{c'})$–polygons that are the connected components of $P \setminus D$. The deletion of D will be called a $2-reduction$ if $1 + \sum_{i=1}^{c'} \left\lfloor \frac{n_i+2\chi(P_i)}{6} \right\rfloor \leq \left\lfloor \frac{n+2\chi(P)}{6} \right\rfloor$, i.e. if the deletion of D allows us to apply induction.

Define $\delta_n, \delta_r, \delta_h, \delta_c$ as before and $\delta_\chi = \chi(P) - \sum_{i=1}^{c'} \chi(P_i)$, i.e. analogously as $-\delta_c$ describes the increase of the number of connected components after deleting D, $-\delta_\chi$ describes the increase of the number of connected components that are $(4,0)$–polygons. For shortness, such components will be called *rectangle components*. The tuple $(\delta_r, \delta_h, \delta_c, \delta_\chi)$ will be called the $2-type$ of D. Now, we can introduce the $gain_2$ of a district as follows:

$$gain_2(D) = \delta_n + 2\delta_\chi = 2(\delta_r - \delta_h + \delta_c + \delta_\chi)$$

Lemma 15 *Let D be a district of a T_2–guard G in a polygon P. If $gain_2(D) \geq 6$ then the deletion of D is a 2-reduction.*

The proof of this lemma is analogous to the proof of Lemma 8. In contrast, the following analog to Lemma 9 contains some essential differences.

Lemma 16 (Expansion Lemma) *Let G be a horizontal T_1–guard in a polygon P and D a district of G. Let P_1 be one of the connected components of $P \setminus D$ and R, and e be a horizontal edge that bounds P_1 from above and is shared between P_1 and D. Let R be the rectangle of P_1 that contains e. Let $\overline{D}$ be the expansion of D by R and all rectangles reachable from R on directed paths in $\mathbf{R}(P_1)$. If the edge e is (orthogonally) visible from G (see Figure 9, where G runs across the top of the figure), then $\overline{D}$ is also a district of G and the following holds: Either $gain_2(\overline{D}) \geq gain_2(D) + 2$ or $gain_2(\overline{D}) = gain(D)$ and $P_1 \setminus \overline{D}$ consists of $(|\mathbf{S}|+1)/2$ rectangle components.*

Proof: Let B denote the set of rectangles in $\mathbf{S}$ that have two lower neighbors and $b = |B|$. Then $\mathbf{S}$ has at least $2b+1$ nodes. If we add by breadth first search the rectangles of $\mathbf{S}$ to D starting with R then for each rectangle from B either the number of connected components of the remaining polygon increases by 1 (say, b_1 times) or the number of holes decreases by 1 ($b_2 = b - b_1$ times). In contrast, adding a rectangle that does not have two lower neighbors neither changes δ_h nor increases the number of connected components. Thus, after deleting all rectangles of $\mathbf{S}$ from P_1 the number of remaining connected components (and especially the number b_3 of rectangle components) is bounded by $b_1 + 1$. So we have

$$gain(\overline{D}) \geq gain(D) + 2 \cdot (|\mathbf{S}| - b_1 - b_2 - b_3)$$

Note that

$$|\mathbf{S}| - b_1 - b_2 - b_3 \geq 2b + 1 - b_1 - b_2 - (b_1 + 1) = 2b_1 + 2b_2 + 1 - 2b_1 - b_2 - 1 \geq 0$$

and the left side is equal to zero iff $|\mathbf{S}| = 2b+1$, $b_2 = 0$ and $b_3 = b_1 + 1$. This implies $b_3 = (2b_1 + 2)/2 = (2b+2)/2 = (|\mathbf{S}|+1)/2$, which completes the proof. $\square$

Now we will show that for any polygon one can find a 2–reduction. Obviously, the deletion of any district with 2–type $(3,0,0,0)$ is a 2–reduction. The following observations will be very helpful to extend the results for T_1–guards to T_2–guards.

Figure 19: Illustration of the amplification lemma

Lemma 17 **(Amplification Lemma)** *Let D be a district of a T_1–guard G and suppose that in $P \setminus D$ there is a rectangle component R (see Figure 19).*
Then G can be amplified to a T_2–guard G' covering the district $D' = D \cup R$ with $gain_2(D') = gain_2(D) + 6$.

Proof: Since R was obtained by the deletion of D from P there must be a common point A on the boundaries of P, R and D. Let l be the perpendicular from A to G. Because orthogonal covering is always assumed, l is included in P and moreover it is possible to extend l in such a way that it crosses the entire height of R. Clearly, G together with this extended segment forms a T_2–guard orthogonally covering $D \cup R$. By extending D in this way, one more rectangle is covered, there is one less connected component remaining, and one less rectangle component remaining. Collectively, these changes increase the $gain_2$ by 6. $\square$

Let D be a district of a T_1–guard of type $(3, 0, 0)$. If the remaining polygon $P' = P \setminus D$ is not a 4–gon then the 2–type of D is $(3, 0, 0, 0)$ and hence the deletion of D is also a 2–reduction. Otherwise, if P' is a 4–gon then P must be an 8–gon which clearly can be covered by a T_2–guard and thus P is also 2–reducible in this case.

Let D be a district of a T_1–guard (w.l.o.g. horizontal) of type $(2, 0, 0)$, P_1 a connected component of $P \setminus D$ and $R \in \mathbf{R}(P_1)$ such that the new expansion lemma can be applied. Then we either get $gain_2(\overline{D}) \geq gain_2(D) + 2 \geq 6$ (which implies a 2-reduction) or $gain_2(\overline{D}) = gain_2(D) = 4$ and all $(|\mathbf{S}| + 1)/2$ connected components of $P_1 \setminus \overline{D}$ are 4–gons. In the latter case one can apply the amplification lemma to get a district D^* with $gain_2(D^*) = gain_2(\overline{D}) + 6 = 10$.

Lemma 18 *If $R_0 \in \mathbf{T}$ is a leaf with branching distance ≥ 3 then there is some 2–reduction with R in the reduction district.*

Proof: In this situation one can always find a T_1–guard with a district of type $(3, 0, 0)$ (see proof of Lemma 10), so we are done. ◻

Lemma 19 *If all leaves in* $\mathbf{R}$ *have branching distance* < 3 *and* R *is a primary branching then there is a reduction such that* R *or a part of* R *is in the reduction district.*

Proof: Let us return to the case inspection in the proof of Lemma 11. In case A, B and C.1 there are T_1 guards with districts either of type $(3, 0, 0)$ or of type $(2, 0, 0)$ and such that the new expansion lemma (or its rotated version) can be applied. Taking into account the observations above, we are done with these cases and only case C.2 remains (see Figure 12). As in the proof of Lemma 11, we start with a vertical T_1–guard on the extension $\overline{e}$ of the edge e and a district D consisting of R_2 and L_2 and the guard position. Depending on whether $\overline{e}$ disconnects the polygon or reduces one hole, D has the type $(2, 0, -1)$ or $(2, 1, 0)$. Thus the 2–type of D is either $(2, 0, -1, \delta_\chi)$ where $\delta_\chi \in \{0, -1, -2\}$ or $(2, 1, 0, 0)$. Furthermore, one can apply the new expansion lemma on the right and on the left side of $\overline{e}$. If (before expanding) on one side (resp. on both sides) there is only a rectangle component, i.e. $\delta_\chi = -1$ (resp. $\delta_\chi = -2$) then the expansion on this side (resp. to both sides) removes one (resp. two) rectangle(s) which is also a connected component and especially a connected component being a 4–gon. Thus the extended district has the 2–type $(3, 0, 0, 0)$ (resp. $(4, 0, 1, 0)$) which implies a sufficient $gain_2$ of 6 (resp. 10).

Now we can assume that D is of 2–type $(2, 0, -1, 0)$ or $(2, 1, 0, 0)$ and hence $gain_2(D) = 2$. Applying the new expansion lemma on both sides of $\overline{e}$ we either increase the $gain_2$ twice by 2 (and we are done) or we know that after this step at least on one side there remains a rectangle component. In this case one can apply the amplification lemma increasing the $gain_2$ by 6 and we are done. ◻

The proof of Theorem 14 will be completed by a lemma that shows how to reduce the number of holes.

Lemma 20 *Let* P *be a polygon to which Lemma 18 and Lemma 19 cannot be applied. W.l.o.g. let* e *be a horizontal extremal hole edge bounding a hole from above and let* $R \in \mathbf{R}$ *be the rectangle having* e *on its boundary. Then there is a reduction such that* R *or a rectangular part of* R *is in the district of the reduction.*

Proof: We switch back to the proof of Lemma 11 and note that R has two lower neighbors R_l and R_r. If there are also upper neighbors R_1 and R_2 of R then because e is extremal, each of them is either a leaf or of degree 2 and adjacent to some leaf L_1 or L_2. Let us start assuming that the set $\mathbf{N}$ consisting of all upper neighbors of R and all leaves adjacent to these neighbors is not empty and run trough the case inspection under this additional assumption.

Figure 20: Constuction of the districts $\overline{D'}$ and $\overline{E}$

In **Case A** and **Case B** we have a horizontal guard which first covers a district D consisting of R and all rectangles in $\mathbf{N}$. Hence D is of type $(1+|\mathbf{N}|, 1, 0)$ and of 2–type $(1+|\mathbf{N}|, 1, 0, 0)$. So we get $gain_2(D) \geq 2$ and moreover the new expansion lemma can be applied twice. If both applications increase the $gain_2$ by 2 we are done. Otherwise at least one application causes a rectangle component which can be covered by the amplification lemma, giving a sufficiently large gain.

In **Case C** a vertical guard will be placed onto the full extension $\overline{e}$ of the edge e which covers first a district D consisting of R_1, L_1 and $\overline{e}$. Depending on whether the lower vertex of e is reflex or not, we have $gain_2(D) = 2$ and D can be expanded twice or $gain_2(D) = 4$ with one possible expansion. Again either one gets a sufficient $gain_2$ by the expansion or there remains a rectangle component which will be covered by amplification of the guard.

Finally, we show how to proceed if the set $\mathbf{N}$ is empty. First, we place a horizontal guard onto the upper boundary of R and define a district $D = R$ of 2–type $(1, 1, 0, 0)$ and with $gain_2(D) = 0$. Obviously, two expansions with respect to R_l and R_r are possible. Let $\overline{D}$ be the new district after the expansions, then we have $gain_2(\overline{D}) \geq gain_2(D) = 0$. If there is a rectangle component in $P \setminus \overline{D}$ we can get a sufficiently large $gain_2$ by the amplification lemma. Otherwise both expansions increase the $gain_2$ at least by 2 and we get $gain_2(\overline{D}) \geq gain_2(D) + 2 \cdot 2 = 4$. Note that we are done if one of the expansions adds more than 2 to the $gain_2$, so we can assume that the application of Lemma 16 to R_l (as well as to R_r) increases the gain exactly by 2. As was shown in the proof of Lemma 16 this increase is $\geq 2 \cdot (|\mathbf{S}| - b_1 - b_2 - b_3)$ where $\mathbf{S}$ is the set of all rectangles in $\mathbf{R}$ reachable from R_l on a directed path, $b = b_1 + b_2$ is the number of rectangles in $\mathbf{S}$ with two lower neighbors and b_3 denotes the number of rectangle components in the remaining polygon which is 0 in this case. Since $|\mathbf{S}| \geq b+1$, the only possibility to get exactly 2 for the increase of the $gain_2$ is $b = 0$ and $|\mathbf{S}| = 1$, i.e. R_l has exactly one lower neighbor R'_l with an arc $R'_l \to R_l$, and R_r has exactly one lower neighbor R'_r with an arc $R'_r \to R_r$.

Figure 21: R_0 is a rectangle component in $P \setminus \overline{D'}$ but not in $P \setminus \overline{E}$

Now, we choose the extension $\overline{e}$ of the common vertical edge e of R_l and R_l' for amplifying G. Let D' be the district of the new T_2–guard consisting of $\overline{D}$ and $\overline{e}$ (see Figure 20, the darkly shaded region in the left picture). Again we have to distinguish the two cases whether the lower vertex A of e is reflex or not.

Case 1: A is a reflex vertex.
Then the 2–type of D' is $(3, 2, 0, 0)$ or $(3, 1, -1, 0)$ and thus $gain_2(D') = 2$. Furthermore one can expand D' twice. Let $\overline{D'}$ denote the district obtained in this way.

Subcase 1.1: One of the two expansions increases the $gain_2$ by more than 2 or each expansion increases the $gain_2$ by 2, then obviously $gain_2(\overline{D'}) = 6$.

Subcase 1.2: The application on the left side of $\overline{e}$ does not increase the $gain_2$. Then there is a rectangle component R_0 in $P \setminus \overline{D'}$ on the left side of $\overline{e}$. Consider a vertical T_1 guard H on $\overline{e}$ covering a district E consisting of $\overline{e}$, R and R_l. The district E has $gain_2(E) = 0$, and expanding E on both sides of $\overline{e}$ one gets an extended district $\overline{E}$ with $gain_2(\overline{E}) \geq 0$. Note that the rectangle R_0 is a rectangle component of $P \setminus \overline{E}$ (see Figure 20, the right picture), and we may thus apply amplification to obtain a T_2–guard covering the district $E' = \overline{E} \cup R$ with $gain_2(E') = gain_2(\overline{E}) + 6 \geq 6$.

Subcase 1.3: The application on the right side of $\overline{e}$ does not increase the $gain_2$. Then there is a rectangle component R_0 in $P \setminus \overline{D'}$ on the left side of $\overline{e}$. We will proceed as in Subcase 1.2 and we will be successful if R_0 will be also a rectangle component in $P \setminus \overline{E}$. There is one (and only one) exceptional situation, namely if R_0 is a neighbor of R_r (see Figure 21). Then in $P \setminus \overline{E}$ R_0 and R_r together form a $(6, 0)$–polygon. However, by adding the horizontal arm to H which covers R_0, we also cover R_r and hence the $gain_2$ increases by 6 (we have eliminated two rectangles and one connected component). We note that the exceptional situation described above is the only one because we have $\overline{D'} \setminus \overline{E} = R_r$.

Case 2: A is not a reflex vertex.
Then the 2–type of D' is $(3, 1, 0, 0)$ and thus $gain_2(D') = 4$. Furthermore, it is possible to expand on the right side of $\overline{e}$. One can handle this situation analogously to Case 1, repeating the inspection of the subcases under the pretense that the application of Lemma 16 on the left side of $\overline{e}$ increases the $gain_2$ by exactly 2.

This finishes the proof of this lemma and also the proof of Theorem 14. $\square$

5 Conclusion

We have studied generalized guarding in rectilinear polygons with holes, obtaining general lower bounds and some specific upper bounds. We have found that in the rectilinear world there is a strong difference between odd and even k. Surprisingly, for $k \geq 3$, we have not found lower bounds where increasing h makes polygons require more guards, and we in fact believe that increasing h makes polygons require *less* guards. However, we are unable to establish this, and leave this question unsettled.

We note here that our lower bound constructions give the same bounds even if the usual visibility (rather than rectangle visibility) is used, and the T_k–guards are not rectilinearly embedded; the upper bound arguments (obviously) also hold in this more general situation. The fourth author has previously shown that the even-k upper bound of $r(n, 0, k) \leq \left\lfloor \frac{n}{k+4} \right\rfloor$ holds in this situation [13]; his result is implied by Theorem 1.

There are many questions related to this paper which are yet to be answered. Aside from the usual questions about tight bounds for the generalized guarding problem both for rectilinear and general polygons, we want to mention the following:

- What is the lower bound on $r(n, h, k)$ when $\frac{n}{h}$ is small (lots of rectangular holes)?
- Are there lower bound examples that have a different structure but illustrate the same bounds as our constructions? We conjecture that there are no such examples.
- What are the exact bounds for rectilinear polygons with holes expressed as a function only of n and k? (Wessel showed a lower bound of $\left\lfloor \frac{3n+4}{14} \right\rfloor$ for $k = 1$ [14].)
- To prove Lemma 3, we need only guards that are trees with at most k edges, while the lower bounds hold even for nonrectilinear trees of diameter k. How can one exploit the full power of diameter-k trees to get a better upper bound? What is the situation for guards that are *paths* of diameter (length) k?

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