SERIE B — INFORMATIK

On Mutually Avoiding Sets

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Abstract

Two finite sets of points in the plane are called mutually avoiding if any straight line passing through two points of anyone of these two sets does not intersect the convex hull of the other set. For any integer n, we construct a set of n points in general position in the plane which contains no pair of mutually avoiding sets of size more than $\mathcal{O}(\sqrt{n})$. The given bound is tight up to a constant factor, since Aronov *et al.* [AEGKKPS] showed a polynomial-time algorithm for finding two mutually avoiding sets of size $\Omega(\sqrt{n})$ in any set of n points in general position in the plane.

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1 Introduction

Let A and B be two disjoint finite sets of points in the plane such that their union contains no three points on a line. We say that A *avoids* B if no straight line determined by a pair of points of A intersects the convex hull of B. A and B are called *mutually avoiding* if A avoids B and B avoids A. In this note we investigate the maximum size of a pair of mutually avoiding sets.

Aronov *et al.* [AEGKKPS] showed that any set of n points in general position in the plane (i.e., no three points lie on a line) contains a pair of mutually avoiding sets, both of size at least $\sqrt{n/12}$. Moreover, they gave an algorithm for finding such a pair of sets in time $\mathcal{O}(n \log n)$. In Section 2 we construct, for any integer n, a set of n points in general position in the plane which contains no pair of mutually avoiding sets of size more than $11\sqrt{n}$ each.

Mutually avoiding sets in a $d-$dimensional space are defined similarly. Any set of n points in general position in R^d contains a pair of mutually avoiding sets, both of size at least $\Omega(n^{\frac{1}{d^2-d+1}})$ (see [AEGKKPS]). On the other hand, our method described for the planar case in Section 2 yields a construction of sets of n points in R^d with no pair of mutually avoiding sets of size more than $\mathcal{O}(n^{1-1/d})$.

Now we recall some definitions from [AEGKKPS]. A set of line segments, each joining a pair of the given points, is called *a crossing family* if any two line segments intersect in the interior. Two line segments are called *parallel* if they are two opposite sides of a convex quadrilateral. In the other words, two line segments are parallel if their end-points form two mutually avoiding sets of size 2. It is an easy observation that any pair of avoiding sets of size s can be rebuilt onto s pairwise parallel line segments or onto a crossing family of size s. Aronov *et al.* [AEGKKPS] used this observation and the above result on mutually avoiding sets for finding a crossing family of size $\Omega(\sqrt{n})$ and a set of $\Omega(\sqrt{n})$ pairwise parallel line segments.

The result on pairwise parallel line segments was extended to a higher dimension by Pach. Pach [P] showed that any set of n points in general position in R^d contains at least $\Omega(n^{1/d})$ $d-$dimensional simplices (i.e., $(d+1)-$point subsets) which are pairwise mutually avoiding.

In Section 3 we show a relation between mutually avoiding sets and Erdős' well-known empty-hexagon-problem.

2 Sets with small mutually avoiding subsets

For a finite set P of points in the plane, let $q(P)$ denote the ratio of the maximum distance of any pair of points of P to the minimum distance of any pair of points of P. For example, if P is a square grid $\sqrt{n} \times \sqrt{n}$ then $q(P) = \sqrt{2}(\sqrt{n}-1)$. In this section we show:

Theorem 1 *Let $c > 0$ be a positive constant. Then any set P of n points in the plane satisfying $q(P) \leq c\sqrt{n}$ contains no pair of mutually avoiding sets of size more than $\lceil 2(\sqrt{17}+1)c\sqrt{n}\, \rceil$ each.*

Figure 1: Auxiliary lines and points

One of the basic results about covering says that for any integer $n \geq 2$ there is a set P of n points in the plane with $q(P) < c_0\sqrt{n}$, where $c_0 = \sqrt{2\sqrt{3}/\pi} \approx 1.05$. Such a set P can be found as the triangle grid inside an appropriate disc. If we slightly perturb points of P, we obtain a set in general position still satisfying $q(P) < c_0\sqrt{n}$. According to Theorem 1 this set contains no pair of mutually avoiding sets of size more than $11\sqrt{n}$. (It is obvious for $n \leq 100$. For $n > 100$ we use the estimation $\lceil 2(\sqrt{17}+1)c_0\sqrt{n}\rceil < 2(\sqrt{17}+1)c_0\sqrt{n} + 1 < 11\sqrt{n}$.)

Proof of Theorem 1. Let A and B be two mutually avoiding sets of the set P. Define Cartesian coordinates so that for some positive constant $d \in \langle 0, \frac{1}{2}c\sqrt{n}\rangle$ all points of $A \cup B$ lie in the closed strip between the two vertical lines $p : x = -d$ and $q : x = d$, and one of the sets A and B, say A, has a point on the line p and a point on the line q. Moreover, let the topmost point b_0 of the set B lie on the $x-$axis and let the set A lie "above" the set B (i.e., the set A lies above any straight line connecting two points of B). Since b_0 lies on the $x-$axis, all points of $A \cup B$ lie between the two horizontal lines $r : y = -c\sqrt{n}$ and $s : y = c\sqrt{n}$. Define three points u, v, w as those points in which the line s intersects the line p, the $y-$axis, and the line q, respectively (see Fig. 1).

For each point $b \in B$, let $f(b)$ be that point in which the line segment bv intersects the $x-$axis. Now we show that, for any two points $b, b' \in B$, the distance $|f(b)f(b')|$

Figure 2: Auxiliary points and triangles

between $f(b)$ and $f(b')$ is greater than $\frac{d}{(\sqrt{17}+1)c\sqrt{n}}$.

If the line bb' is horizontal then $|f(b)f(b')| \geq \frac{1}{2}|bb'| \geq \frac{1}{2} > \frac{d}{(\sqrt{17}+1)c\sqrt{n}}$. If the line bb' is not horizontal then it intersects the line s in some point g outside the segment uw (see Fig. 2). Thus $|gv| > d$. Without loss of generality, assume that b is closer to the line s than b'. Let z be that point on $b'v$, for which bz is horizontal.

Now estimate

$$|bz| = \frac{|bb'|}{|gb'|} \cdot |gv| > |bb'| \cdot \frac{|gv|}{|gv| + |vb'|} > 1 \cdot \frac{|gv|}{|gv| + \sqrt{(2c\sqrt{n})^2 + d^2}} >$$

$$> \frac{d}{d + \sqrt{(2c\sqrt{n})^2 + d^2}} \geq \frac{d}{\frac{1}{2}c\sqrt{n} + \sqrt{(2c\sqrt{n})^2 + (\frac{1}{2}c\sqrt{n})^2}} = \frac{d}{(\frac{1}{2} + \sqrt{\frac{17}{4}})c\sqrt{n}}$$

and

$$|f(b)f(b')| \geq \frac{1}{2}|bz| > \frac{d}{(\sqrt{17}+1)c\sqrt{n}}.$$

Since the points $f(b), b \in B$ are placed on a line segment of length $2d$, the size of B is at most $\lceil 2d / \frac{d}{(\sqrt{17}+1)c\sqrt{n}} \rceil = \lceil 2(\sqrt{17}+1)c\sqrt{n} \rceil$. $\square$

3 Relation between mutually avoiding sets and the empty-hexagon problem

Let A be a finite set of points in general position in the plane. A subset S of A of size k is called *convex* if its elements are vertices of a convex k-gon. If S is convex and the interior of the corresponding convex k-gon contains no point of A, then S is called *a k-hole* (or *an empty k-gon*). The classical Erdős-Szekeres Theorem [ES] (1935) says that if the size of A is at least $\binom{2k-4}{k-2}+1$ then A contains a convex subset of size k.

Erdős [E] asked whether the following sharpening of the Erdős-Szekeres theorem is true. Is there a least integer $n(k)$ such that any set of $n(k)$ points in general position in the plane contains a k-hole? He pointed out that $n(4) = 5$ and Harborth [Ha] proved $n(5) = 10$. However, as Horton [Ho] shows, $n(k)$ does not exist for $k \geq 7$. The question about the existence of $n(6)$ (the empty-hexagon-problem) is still open. After a definition we formulate a conjecture which, if true, would imply that the number $n(6)$ exists.

Definition 2 *Let A be a finite set of points in general position in the plane. Let $k \geq 2$, $l \geq 2$. A subset S of A of size $k+l$ is called a (k,l)-set if S is a union of two disjoint sets K and L so that the following three conditions hold:*

(i) $|K| = k, \qquad |L| = l$,

(ii) K and L are mutually avoiding,

(iii) the convex hull of S contains no points of $A - S$.

Conjecture 3 *(Bárány, Valtr) For any two integers $k \geq 2$ and $l \geq 2$, there is an integer $p(k,l)$ such that any set of at least $p(k,l)$ points in general position in the plane contains a (k,l)-set.*

If Conjecture 3 is true for $k = l = 6$ then the number $n(6)$ exists. It follows from the fact that any (6,6)-set contains a 6-hole (it can be shown that either one of the corresponding sets K and L is a 6-hole or there is a 6-hole containing three points of K and three points of L).

Note that all known constructions of large sets with no 6-hole (see [Ho], [V]) satisfy Conjecture 3 already for rather small integers $p(k,l)$.

We cannot even prove that the numbers $p(k,2)$, $k \geq 5$ exist. (Note that $p(2,2) = 5$ and $p(3,2) = 7$ are the minimum values of $p(k,2)$, $k = 2,3$, for which Conjecture 3 holds.) The existence of numbers $p(k,2)$, $k \geq 2$ would imply the following conjecture:

Conjecture 4 *(Bárány, Valtr) For any integer $k > 0$, there is an integer $R(k)$ such that any set of at least $R(k)$ points in general position in the plane contains $k+2$ points $x, y, z_1, z_2, \ldots, z_k$ such that the k sets $\{x, y, z_i\}$, $i = 1, \ldots, k$ are 3-holes (i.e., they form empty triangles).*

Bárány proved that Conjecture 4 holds for $k \leq 10$.

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