BUILDING CONCEPTUAL UNDERSTANDINGS OF EQUIVALENCE

by

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DEDICATION

It is with affectionate pleasure to dedicate this work to my husband Cody Rich, for his ever-lasting love and support along this journey. His encouragement gave me strength to continue writing into the long, solitary evenings. His strength to lead our family, drive our children to every sporting event, doctor's appointment, birthday party, and school function is a testimony of true love. Words of gratitude will never express how truly thankful I am to have had Cody by my side through this experience.

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ABSTRACT

The equal sign is prevalent at all levels of mathematics however many students misunderstand the meaning of the equal sign and consider it an operational symbol for the completion of an algorithm (Baroody & Ginsburg, 1983; Rittle-Johnson & Alibali, 1999). Three constructs were studied through the lens of the Developing Mathematical Thinking (Brendefur, 2008), Relational Thinking, Spatial Reasoning and Modes of Representation. A review of literature was conducted to examine the effects of mathematics instruction on the development of students' conceptual understanding of equivalence through the integration of spatial reasoning and relational thinking. The Developing Mathematical Thinking (DMT) curricular resources integrate Bruner's enactive, iconic, and symbolic modes of representations (1966), using tasks designed to strengthen students' spatial reasoning and relational thinking to develop mathematical equivalence. The research question "What is the effect of integrating iconic teaching methodology into mathematics instruction on first grade students' relational thinking and spatial reasoning performance?" was analyzed to determine whether there was a significant difference in pre-and posttest scores for the two groups. Students were found to have a better opportunity to develop conceptual understanding of mathematics in their early years of school when taught with the progression of EIS, relational thinking and spatial reasoning.

Keywords: equivalence, spatial reasoning, relational thinking, mathematical modeling

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LIST OF ABBREVIATIONS

DMT	Developing Mathematical Thinking
PMA-S	Primary Mathematics Assessment Screener
EIS	Enactive, Iconic, Symbolic representation
CRA	Concrete, Representative, Abstract

CHAPTER 1: INTRODUCTION

Background

Equivalence has been one of the primary focal points within the body of literature, more specifically, understanding the functionality of the equal sign (Falkner, Levi, & Carpenter, 1999; Kieran, 1981; McNeil & Alibali, 2005). This understanding is critical in order for students to be able to use multiple representations and relational thinking to recognize patterns and generalize in many areas of study within the field of mathematics. The concept of equivalence is crucial for developing algebraic reasoning (Falkner et al., 1999).

The equal sign is prevalent at all levels of mathematics however many students misunderstand the meaning of the equal sign and consider it an operational symbol for the completion of an algorithm (Baroody & Ginsburg, 1983; Kieran, 1981; Rittle-Johnson & Alibali, 1999). Students are introduced to the equal sign early on in school strictly in its symbolic form with little attention paid to the conceptual meaning or the relational function it plays in algebra. This common misconception of the equal sign has been found to reinforce student’s mathematical tendency to focus on counting and addition in the early years of elementary school (Seo & Ginsburg, 2003).

Misconceptions can be avoided when instruction shifts to allow student’s view of the equal sign as an expressed relation rather than solely as an operator symbol (Carpenter & Levi, 2000). When the equal sign is viewed as merely operational in early elementary school, children can successfully solve simple equations such as $2 + 3 = 5$.

However, this type of thinking leaves middle school students struggling to solve for unknown numbers in algebra courses because they assume the equal sign is misplaced (Kieran, 1981). Instead, if students develop an understanding of the equal sign as an expressed relation, they can begin to develop the understanding of solving for an unknown or variable within an equivalence statement. It is important that students develop the understanding of the equal sign as an expressed relation as early as kindergarten to avoid misconceptions of the equal sign (Knuth, 2006), therefore, teachers need to structure mathematics instruction appropriately (Knuth, Stephens, McNeil, & Alibali, 2006).

Problem Statement

Students are unable to connect their operational knowledge of the equal sign from elementary school mathematics to the relational thinking of the equal sign needed in middle school algebra classes. This disconnected thinking leads to the memorization of rules and meaningless operations with very little conceptual understanding (Herscovics & Linchevski, 1994). Common misconceptions such as these can be long-standing and persist into middle, high school and occasionally college level courses (McNeil & Alibali, 2005). Simple arithmetic problems in elementary school promote operational thinking, oftentimes making it difficult for students to generalize beyond the given problem. Children become entrenched in the operational view of the equal sign, and often times procedures become deeply rooted in students' minds (Chesney & McNeil, 2014). Altering the elementary school curriculum with an emphasis on demonstrating the relational view of the equal sign can build students schema and improve their mathematical performance (McNeil & Alibali, 2005).

Purpose Statement

The purpose of this study was to investigate whether there is a significant difference in first grade students' performance in relational thinking and spatial reasoning when they learn to construct and compare numbers using the progression of enactive, iconic, and symbolic representation.

The teachers in the treatment condition taught mathematics lessons intentionally designed to focus on the integration of enactive, iconic, and symbolic representation to strengthen students' relational thinking and spatial reasoning performance in first grade. The teachers in the comparison group taught the school district's adopted curriculum which with the Common Core Standards for first grade mathematics.

Nature of the study

The study compared relational thinking and spatial reasoning for first grade students whose teacher received professional development to increase use of enactive, iconic, symbolic representation in teaching (EIS group) and those whose teacher received no professional development and taught in a more traditional manner (Traditional group). Both groups were tested using the Primary Mathematics Assessment Screener (PMA-S) in September, prior to the mathematics instruction, and again mid-May after the mathematics instruction; therefore, student performance was also compared across time. Thus, this study used a 2 (EIS group versus Comparison group) x 2 (pretest versus posttest) design. The dependent variable was the students' knowledge of relational thinking and spatial reasoning measured with the PMA-S developed for grades kindergarten through second grade (Brendefur, Strother, & Thiede, 2012) to assess students' knowledge of mathematics with a short, comprehensive and predictive screener.

The screener builds a profile of students' strengths and weaknesses for 6 dimensions: number sense and sequencing, number facts, contextual problems, relational thinking, measurement, spatial reasoning. The Primary Mathematics Assessment Screener (PMA-S) was administered at the beginning of the study in September as a pretest, and again in May as a posttest. The goal of this study was to determine whether student achievement on the PMA-S differed between the EIS and Traditional groups, and whether achievement differed across time.

The larger population of interest for this study is first grade classrooms in Idaho. Within this larger population, the study consisted of first grade classrooms from five school districts. Two of the school districts serve between 15,650 to 26,240 students, and three of the districts serve between 600 to 1725 students. There were over 2600 students with Limited English Proficiency (LEP) comprising approximately 8% of the total districts. In these districts, the student demographics range from 79.3% white, 10.3% Hispanic/Latino, 5.9% Asian, 3.3% Black, 0.9% Native American, and 0.8% Pacific Islander. For this study, the sampling frame will be first grade classrooms in Idaho chosen on the basis of similarly matched demographics related to students who receive free and reduced lunch assistance. The target population was first grade teachers in general education classrooms included in this study.

As noted above, this study used a 2 (Treatment group versus Comparison group) x 2 (pretest versus posttest) design. The dependent variable was the students' knowledge of relational thinking and spatial reasoning measured with the PMA-S. The goal of this study was to determine whether student achievement on the PMA-S differed between the EIS and Traditional groups, and whether achievement differed across time.

Research Question and Hypothesis

To address the primary purpose of this research study, the following research question and hypothesis was investigated:

1. What is the effect of integrating iconic representation through student drawings in conjunction with the enactive, iconic and symbolic teaching methodology into mathematics instruction on first grade students' relational thinking and spatial reasoning performance?

$H1$: There is a positive effect on integrating enactive, iconic and symbolic teaching methodology into mathematics instruction on first grade students' relational thinking and spatial reasoning performance.

H_01: There is not a positive effect on integrating iconic teaching methodology into mathematics instruction on first grade students' relational thinking and spatial reasoning performance.

Theoretical Framework

This study was informed by three different constructs: Relational thinking, spatial reasoning and modes of representation. The Developing Mathematical Thinking (Brendefur, 2008) framework provided a lens for which the three constructs were viewed. The DMT was designed to help teachers critically analyze their mathematical practices to better serve their students. The framework includes five major components: taking student's ideas seriously, pressing students conceptually, encouraging multiple strategies and models, addressing misconceptions, and focusing on structure.

Relational Thinking

Oftentimes, instructional practices are centered around lessons which strengthen an operational view of the equal sign. As students continue to formulate ideas about the equal sign over the course of their elementary years, the ability to reverse the entrenched ideas becomes much more challenging (Chesney & McNeil, 2014). Simple arithmetic problems in elementary school promote operational thinking, making it difficult for students to generalize beyond the given problem. Altering the elementary school curriculum with a relational view of the equal sign can build students schema and improve their mathematical performance (McNeil & Alibali, 2005). Therefore, K-12 reform has included an integration of meaningful lessons designed to enhance algebraic thinking into the primary years of school across all mathematical domains, pressing students to use critical thinking (Kaput, 2000).

Spatial Reasoning

As educators become more aware of the need for relational thinking tasks it is important to recognize the critical role spatial reasoning and mathematical modeling play in the overall development of algebraic thinking and the equal sign. Developing conceptual understanding of the equal sign tends to focus on the symbolic numerical relationships in equations. The National Research Council report (2006) urges educators to recognize the importance of developing spatial reasoning skills with students across all areas of mathematics.

Modes of Representation

Bruner's (1966) modes of representation describe the process of enriching students' understanding by working through enactive, iconic and symbolic (EIS) models.

The enactive stage is critical to developing connections to a task and allows for better recall later on with the symbolic form of the equal sign and equations. Many mathematical concepts rely on an understanding of the equal sign as an expressed relation. It is critical for teachers to expose students to different methods of modeling relationships with multiple representations. Much of the mathematical instruction is limited to the equal sign taught within the confounds of fact fluency in a very abstract, symbolic way. Struggling students need to make the necessary connections to equality and relational thinking from a more visual approach. Visual models support the development of these ideas. Understanding the concepts of the equal sign as an expressed relation is more likely to transfer when visual models are used to support conceptual development. Students will have a better opportunity to generalize and build on existing foundational knowledge of equivalence throughout their mathematical careers.

Mathematical Modeling

Many students have difficulty understanding concepts without being able to first see a visual or pictorial image of an idea in their mind (Arwood & Young, 2000). Auditory learners consist of 5-15% of our general K-12 population, leaving 85-95% of our learners equipped with a visual learning system to acquire new concepts (Arwood et.al, 2009). Mathematics curricula loaded with symbolic representation require students to memorize procedures, denying the student an opportunity to utilize their visual thinking modality in the process of building conceptual understanding. Mathematical modeling is a way to express what a student visualizes, granting access for them to see the hidden meaning behind the mathematical symbols, such as the equal sign and its various meanings (Cai et al., 2014; Gravemeijer, 1999).

Definitions of Terms

The operational definitions for this study were as follows:

Relational Thinking-The ability to recognize the equal sign represents a relationship between both sides of the equation, and that there is a need for balance (Matthews, 2015).

Spatial Reasoning-The capacity to think about objects in 3D, draw conclusions about those objects with limited information, and determine how an object might look when rotated (123test.com).

Visualization-Create an image in the mind, hold it and then transform or manipulate to be different (Ontario Ministry of Education, 2014).

Mental Rotation-The ability to visualize the necessary transformations of numbers within equations (Cheng & Mix, 2014).

Gesturing-Allows students to explain the visual imagery taking place inside one's head as they work to problem solve a specific task (Ehrlich et al., 2006).

Mathematical Modeling-A way to express what a student is visualizing and reveal the meaning behind the mathematical symbols, such as the equal sign (Cai, et.al., 2014).

Enactive, Iconic, Symbolic-Instruction designed to include a progression of representations beginning with the physical manipulatives to a pictorial representation depicting the concrete representation, and finally the symbolic form to support conceptual understandings (Bruner, 1966; Fyfe et al., 2014).

Assumptions, Limitations, and Scope

It is assumed that all teachers in the EIS group implemented the modules with fidelity as was spelled out in the training sessions. It is also assumed the PMA-S was administered to students in both EIS and Traditional groups in September before any mathematics lessons were taught, and mid-May at the conclusion of the mathematics lessons.

The study was limited in random assignment. School district approval to implement a supplemental mathematics resource, collect and analyze students' scores from the PMA-S was attained through district level administrators. Approval then allowed district mathematics curriculum directors to recruit first grade classroom teachers for the EIS and Traditional groups. The participating schools were chosen based on their willingness to participate in the study, leaving two of the schools in the comparison group from a more affluent area, and the other six schools from a lower income area.

In this study, a two by two repeated measures analysis of variance was used to determine the significance of iconic teaching methods on students' relational thinking and spatial reasoning on first grade students. The setting was narrowed to include Twenty-three first grade general education classrooms.

Significance of the Study

The current study provides valuable insight on the integration of the enactive, iconic, and symbolic representation into first grade mathematics lessons. The results and discussion adds to the body of research concerning the effects of spatial reasoning and relational thinking on students' mathematical competency. Educational researchers with a focus on the pedagogy of mathematics in primary school-aged children, in particular, the link between students' relational thinking and spatial reasoning skills on mathematical competency can benefit from the insights of this study.

Summary

Simple arithmetic problems in elementary school promote operational thinking, often times making it difficult for students to generalize beyond the given problem (Chesney & McNeil, 2014). Altering the elementary school curriculum with a relational view of the equal sign can build students schema and improve their mathematical performance (McNeil & Alibali, 2005). Many mathematical concepts rely on an understanding of the equal sign as an expressed relation. It is critical for teachers to expose students to different methods of modeling relationships with multiple representations. Therefore, the general purpose of this study was to investigate whether there was a significant difference in first grade students' performance in relational thinking and spatial reasoning when they had learned to construct and compare numbers using iconic modeling.

Chapter one provided a brief summary of relevant research concerning children's view of the equal sign, relational thinking, and spatial reasoning. The focus of the research is to explore the significance of integrating EIS representation into first grade mathematics' lessons.

Chapter two covers a thorough review of literature of relational thinking, spatial reasoning and mathematical modeling. A theoretical framework describes the lens in which a curriculum integrating enactive, iconic, and symbolic representation effects students' relational thinking and spatial reasoning.

Next, chapter three presents the methodology of the two-way analysis for this study. The design of the study is to determine the significant effects the two independent variables (EIS representation or Traditional instruction) have on students' conceptual understanding of relational thinking and spatial reasoning measured by the Primary Mathematics Assessment (PMA-S).

Chapter four is an analysis of the data collected from the pre-and posttest. A two-way repeated measures analysis of variance was used to assess the effects of both independent variables on the dependent variable from pretest to posttest.

Lastly, chapter five provides conclusions, discussion, rival explanations, and recommendations for further study. The goal of this chapter is to highlight the significant findings of integrating EIS representation into first grade mathematics lessons to improve students' relational thinking and spatial reasoning.

CHAPTER 2: LITERATURE REVIEW

Introduction

The most recent educational reform for mathematics, Common Core State Standards (CCSS) emphasizes the need for a balance between conceptual understanding and procedural knowledge (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). However, a great deal of time is centered on arithmetic facts and fluency with very little context, which leaves students in early elementary school lacking a deep conceptual understanding of the equal sign (Knuth, Stephens, McNeil, & Alibali, 2006). Students are limited to a repeated application of procedural knowledge with very little connections to context (Hunter, 2007). This becomes a major issue as elementary students transition to middle school algebra, requiring less procedural knowledge and a deeper understanding of relationships within an equation (Warren & Cooper, 2005). However, traditional elementary curricula tend not to promote spatial reasoning, relational thinking, or the integration of visual models (Fyfe, McNeil, Son, & Goldstone, 2014; Hattikudur & Alibali, 2010).

The purpose of this literature review is to provide an overview of the ways in which relational thinking, spatial reasoning, and mathematical models influence students' mathematical competency early elementary school mathematics. The literature has been outlined through a theoretical framework to see how integrating enactive, iconic, and symbolic representation effects students' relational thinking and spatial reasoning.

First, relational thinking and the various meanings of the equal sign will be defined. Next, three of the most well-established connections to mathematical competency and spatial reasoning will be explained, in addition to the ways in which visualization and mathematical models allow students to connect their ideas to the abstract symbolic representation of mathematics. Following this section, the similarities and differences between the progressions of conceptual understanding using the concrete, representation, and abstract framework and Bruner’s (1966) enactive, iconic, and symbolic representations will be delineated.

The last section of the literature review will explain the impact textbooks have in the development of mathematical knowledge related to relational thinking, spatial reasoning and students’ misconceptions based off recommendations from the Common Core State Standards (2010). Two curricula, Bridges in Mathematics (Frykholm, 2016) and Developing Mathematical Thinking (Brendefur, 2014) illuminated students’ understanding of the equal sign by incorporating relational thinking, spatial reasoning and mathematical modeling. The goal of this literature review was to gain a deeper understanding of ways to integrate relational thinking and spatial reasoning into mathematics lessons.

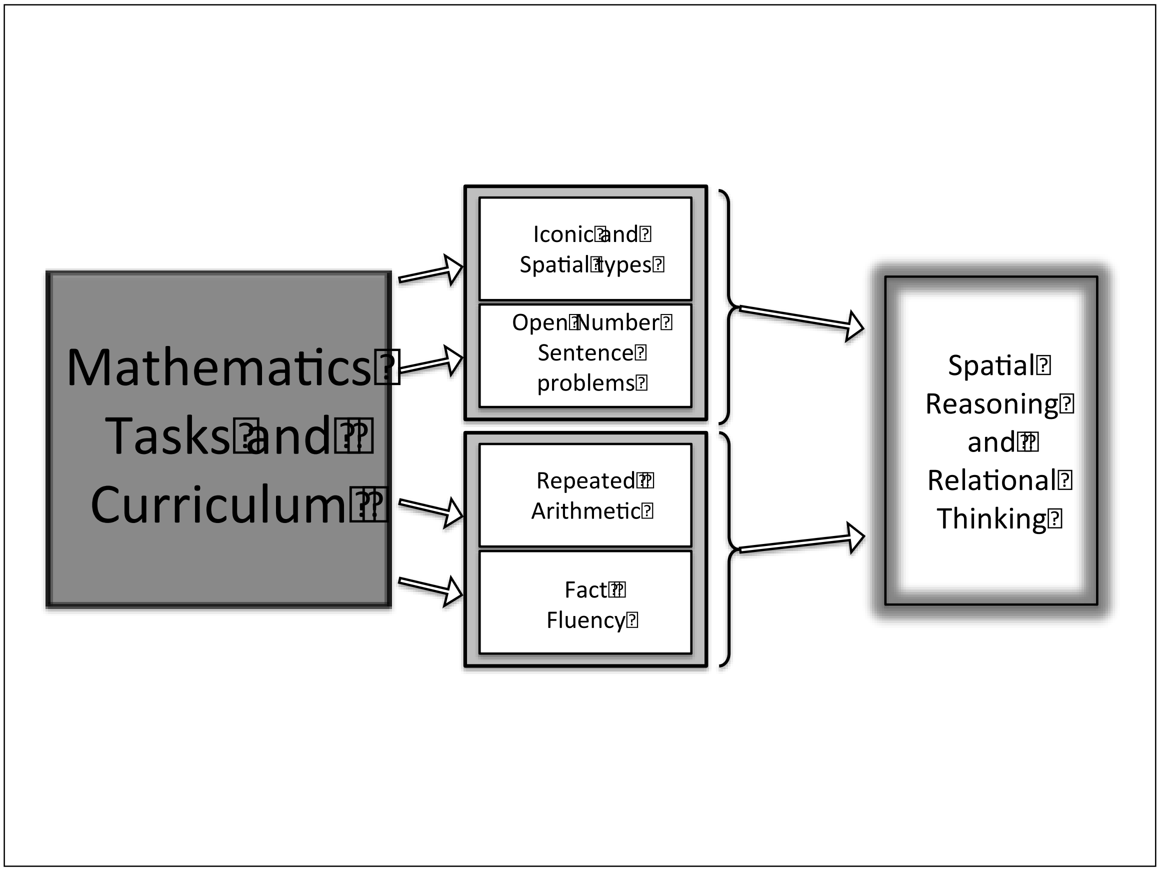

Figure 1. **Framework for developing spatial reasoning and relational thinking.**

Relational Thinking and the Equal Sign

Most elementary students begin to develop their awareness of the equal sign's functionality at an operational level, where the equal sign acts as a symbol to perform a calculation or action (Carraher et al. 2006). When the bulk of instruction is focused on procedures and computing facts many elementary students develop a shallow understanding of the equal sign and consider it an operational symbol (Baroody & Ginsburg, 1983; Rittle-Johnson, Matthews, Taylor, & McEldoon, 2011). For instance, students with an operational view of the equal sign will reject any equations presented outside of the traditional format, a + b = c, and will define the purpose of the equal sign as a cue to perform the calculations on the left side of the equal sign to get an answer (Behr, 1980; Carpenter et al., 2003). However, given more exposure to a variety of equations, students can become more flexible with their thinking and progress to different levels of understanding (Blanton & Kaput, 2005). Mathematics instruction for early elementary classrooms should foster relational thinking by including tasks designed to draw attention to how numbers relate to one another, and develop the flexibility to think of numbers in a variety of ways to establish the idea of equivalence (Cheng & Mix, 2014; A. Stephens, Blanton, Knuth, Isler, & Gardiner, 2015).

Matthews et al (2012) developed a construct map based on the research of Carpenter (2003) and Hunter (2007) to explain the continuum of relational thinking for students' thinking. Figure 2 describes the student thinking associated with each level of understanding.

Comparative Relational

- Students recognize that the equal sign represents a relationship between both sides of the equation, and that there is a need for balance.
- Students are aware of the relationships among the numbers and do not need to perform a calculation to determine equivalence.

Basic Relational

- Students flexibly accept non-traditional equations as correctly written, such as 3+2=4+1.
- Students determine equivalence by performing calculations to both sides of the equation.

Flexible Operational

- Students still consider the equal sign as a symbol for calculation.
- Students begin to recognize equations written in a non-traditional way as acceptable, such as 8=3+5, and 3=3.

Rigid Operational

- Students consider the equal sign as a symbol for calculation.
- Students only consider equations written in the traditional format to be acceptable, such as 3+2=5, and missing term equations such as, 5+__=8.

Figure 2. Continuum of students' understanding for the equal sign (Matthews, et al., 2012).

The first level of student understanding is called Rigid Operational. Students at this level are calculating traditional or missing term equations. Traditional equations written a + b = c, place the equal sign as a function for solving the addition problem a + b to produce an answer. This traditional format instills an operational view of the equal sign (McNeil et al., 2006). With exposure to non-traditional equations such as, a = b + c, students become more flexible in their determination of a correctly written equation. However, their view of the equal sign still remains as a cue for calculation. As students move into the Basic Relational stage, their flexibility to solve equations written with operations on both sides of the equals sign increases. However, it is not until the final stage, Comparative Relational when students consider the number relations on each side of the equal sign to determine equivalency, and their need to calculate diminishes. This level of relational thinking demonstrates students' knowledge about how the equal sign relates to the entire equation, where they are looking for relatable numbers in the equation prior to solving the problem (Jacobs, Franke, Carpenter, Levi, & Battey, 2007). Identifying these relationships in equations and their connections with the numbers is a critical component of mathematical understanding. Developing and applying the knowledge of relational thinking to solve mathematical equivalence problems will increase early algebraic understanding (Byrd, McNeil, Chesney, & Matthews, 2015; Carpenter & Levi, 2000; Molina, Castro, & Ambrose, 2005; Rittle-Johnson et al., 2011)NCTM, 2000). Students who think at the Comparative Relational level have a strong understanding of the equal sign, and a deeper connection to algebraic reasoning (Carpenter et al., 2003; Hunter, 2007).

The natural tendency for students as young as kindergarten is to demonstrate an operational view of the equal sign, however, they do have the capabilities to think relationally if given the opportunity (Baroody & Ginsburg, 1983). Therefore, relational thinking skills should be explicitly taught at an early age to avoid a deep-rooted set of operational skills (McNeil & Alibali, 2005). Relational thinking involves flexible thinking to determine how numbers can be manipulated before answering a problem. Using relational thinking to solve an algebraic equation requires the conceptual understanding that each time a number is manipulated the equation remains equivalent.

Providing students with a progression of non-traditional number sentences focused on numerical relationships and patterns will develop relational thinking. As a starting point for young students reversing the order of the number sentence to begin with the answer such as 3 = 2 + 1 presses students to accept that the answer does not always need to be after the operation (Matthews, 2012; Warren & Cooper, 2005). Next, students develop their understanding of the term equal as they begin to recognize that both sides of the equation compute to the same quantity through exposure to non-traditional equations written with the operations on both sides of the equal sign (Carpenter et al., 2003). Students who possess the conceptual knowledge of equivalence recognize transformations can occur by adding the same number to both sides of the equal sign without changing the structure of the equation. For example, when asked whether the equation 18 + 3 = 16 + 5 is true or false, students who are taught to think about the relationship between 18 and 16, notice that 18 is 2 more than 16, and reason that it must be true because 5 is 2 more than 3. Unfortunately, if students are not taught to look at equations relationally, then the transformations between 18 and 16 simply become proceduralized and learned as memorized rules (Jacobs et al., 2007). This strategy shows a level of relational thinking in which students use number relations to make the problem more manageable. Thinking relationally, therefore, is different from applying a collection of memorized mathematical rules and procedures (Hattikudur & Alibali, 2010). Students who think relationally identify number relations and reason about which transformations make sense in a particular problem (Carraher, Schliemann, Brizuela, & Earnest, 2006).

Providing students with true or false equations can be another way to press students to think about number relationships. Equations such as 14 + 18 = 13 + 17 are more compatible with instructing students to see number relationships because a numerical answer is not required. Engaging students in a discussion of how the numbers relate to each other to determine whether the equation is true or false strengthens their conceptual understandings of equivalence (Carpenter et al., 2003). Students with sufficient conceptual knowledge of how these number properties are applied have the understanding to transfer their procedural knowledge of mathematical equations to algebraic thinking (Stephens et al., 2015). Meaningful discussions about number relationships and the transferability of those ideas helps students make more mathematical generalizations (Bastable & Schifter, 2008).

Students need time to develop relational thinking, with practice designed to explicitly examine the way in which numbers relate, and ways that those relations can generalize to other areas of mathematics (Bastable & Schifter, 2008; Blanton & Kaput, 2005; Carpenter & Levi, 2000; Carraher, Schliemann, Brizuela, & Earnest, 2006). One way to increase conceptual understanding is to increase the exposure of problem solving tasks involving non-traditional equations (Matthews, 2012). It has been shown that students as young as kindergarten and first grade have informal knowledge of number relations, however, the mathematics presented in traditional textbooks do not explicitly draw out these relations, allow time for the relations to organically emerge, or instruct students to determine how the ideas can be generalized (Blanton & Kaput, 2005). Consequently, there is a need for mathematics instruction to incorporate more than just the traditional format of equations into daily lessons, and include ways to represent relational equivalence (Ellis, 2011; Molina et al., 2005). Later, I explain how curricula can be designed to foster relational thinking and develop students' understandings of the equal sign.

Spatial Reasoning and Mathematical Competency

Many researchers have confirmed spatial reasoning skills and mathematical competency to be directly related to each other (Battista 1990; Casey et al. 2015; Reuhkala 2001; Rohde and Thompson 2007; Zhang et al. 2014). Training with specific spatial reasoning tasks will improve students' abilities in the Science, Technology, Engineering, and Mathematics (STEM) fields (Newcombe & Frick, 2010; Uttal et al., 2012). There is a strong link between spatial reasoning ability and geometry where strong visuospatial skills predict how well students will complete 3-D geometry tasks (Clements 2004; Clements & Battista 1992; Pittalis & Christou, 2010). As educators become more aware of the need for relational thinking tasks it is important to recognize the critical role spatial reasoning and mathematical modeling play in the overall development of algebraic thinking and the equal sign. Developing conceptual understanding of the equal sign tends to focus on the symbolic numerical relationships in equations. However, there is evidence to suggest that spatial reasoning also plays an important role in the development of these relationships through a visual lens (Drefs & D'Amour, 2014; Uttal et al., 2012). The National Research Council report (2006) urges educators to recognize the importance of developing spatial reasoning skills with students across all areas of mathematics. The National Governors Association (2010) suggests more spatial reasoning be integrated into the elementary mathematics curriculum to promote relational thinking skills, and mathematical modeling to be a key component for students to explain their thinking when representing algebraic concepts.

It has been found that students with strong spatial reasoning skills generally do well in mathematics (Mix & Cheng, 2012). Mix and Cheng (2012) claim, “The relation between spatial ability and mathematics is so well established that it no longer makes sense to ask whether they are related” (Mix & Cheng, 2012, p. 206). The topic for researchers now is to determine how they are related. Strong spatial reasoning skills and mathematics competency is not limited to one specific mathematical topic or spatial reasoning task (Ontario Ministry of Education, 2014). Spatial reasoning is a critical piece for developing the way students think about equations (Cheng & Mix, 2014). Given the opportunity, students’ spatial reasoning skills can increase when practice is integrated and supported throughout mathematics instruction (B. N. Verdine et al., 2013). By the time students reach kindergarten, their spatial reasoning skills predict their overall mathematical success (Verdine et. al., 2014). Therefore, students’ educational experience in early elementary school should have an intentional focus on improving spatial reasoning skills. The focus of this section is to reveal the connection between spatial reasoning and the ability to conceptualize mathematical symbols such as the equal sign through the use of spatial orientation on a number line, gesture, visualization, and mental rotation.

Studies have shown improvement in certain types of spatial reasoning tasks transferring to other types of mathematical tasks. A crucial component to understanding ordinality—the position of a number in relation to its location on a number line, and magnitude—the size of a number is directly linked with the development of a precise spatial representation of numbers to the symbolic representations of numbers and the visual space of a number (Dehaene, Bossini, & Giraux, 1993). The number line has been shown in cognitive studies to be important for the development of numerical knowledge (Booth & Siegler, 2008; Kucian et al., 2011; Schneider & Grabner, 2009). Siegler and Ramani (2008) report that students who play board games such as Chutes and Ladders increase rote counting skills, number identification, and the conceptual understanding of numerical magnitude. Additionally, activities which include puzzles, video games, and blocks with significant connections to spatial reasoning skills and mathematical competency improves accuracy of symbolically representing a number line (Gunderson et al., 2012; Uttal 2012).

Problem solving tasks regarding orientation, transformations and movement of shapes create an opportunity among students and the teacher to engage in rich, mathematical discourse. As students discuss their thinking, they will use their hands to gesture while attempting to convey their thoughts surrounding the task. Gesturing allows students to explain the visual imagery taking place inside their head as they work to problem solve specific tasks (Ehrlich et al., 2006). Students' gestures represent the movement of the transformation and creates an avenue for their thinking to emerge through the discussion. Alibali and Nathan, (2012) found gestures to be an excellent tool for teaching students how to solve spatial transformation tasks by placing an emphasis on the importance of moving the pieces without the actual physical movement. In essence, they used their hands to gesture what their mind was creating and convey mathematical thinking.

The ability to gesture what the mind is thinking is dependent upon students' ability to visualize mathematical transformations (Ontario Ministry of Education, 2014). The ability to think relationally requires students to visualize how numbers can be manipulated and rearranged in an equation (M. Stephens & Armanto, 2010). Therefore, visualization is a key component across mathematical topics (Ontario Ministry of Education, 2014). Spatial visualization tasks require students to create an image in their mind, hold the image, and then mentally transform or manipulate that image to be different. Some examples of these types of tasks include composing and decomposing pattern blocks to determine a new composed image, imagining transformations and perspectives of a three-dimensional cube, or activities that involve mentally folding a two-dimensional shape to form a new three-dimensional shape. In addition to spatial visualization, mental rotation has also been shown to increase student performance in mathematics (Cheng & Mix, 2014).

Students who are allotted time to practice mental rotation have demonstrated the ability to solve a series of multi-step word problems (Casey et al., 2015). Mental rotation consists of the ability to look at an object, or picture of an object and visualize what it might look like when rotated in 2-d or 3-d space. The most recent study of spatial training with mental rotation was conducted with young students developing number sense, counting sequence, fact fluency, and missing term problems (Cheng & Mix, 2012). Although the other areas showed improvement with the spatial training, missing term problems such as 2 + __ = 6 indicated the most significant effect size. Much like the relational skills needed to find the most efficient way to solve missing term problems, the completion of mental rotation tasks during spatial training helped to strengthen students' ability to visualize the necessary transformations of numbers within equations for simpler computation (Cheng & Mix, 2014).

It is important to note that mental rotation and spatial visualization are both subsets to spatial reasoning, and that much of their characteristics overlap (Ontario Ministry of Education, 2014). Developing both skills is a powerful way to connect back to the bigger idea of conceptual understanding for relational thinking, spatial reasoning, and equivalence (Suh & Moyer, 2007; Oropeza & Cortez, 2015).

Mathematical Modeling

Many students have difficulty understanding concepts without being able to first see a visual or pictorial image of an idea in their mind (Arwood & Young, 2000). Visualization helps students use the pictures or shapes in their mind to recall, understand, make connections, clarify, and remember new information (Arwood & Kaulitz, 2007). Auditory learners consist of 5-15% of our general k-12 population, leaving 85-95% of our learners equipped with a visual learning system to acquire new concepts (Arwood et.al, 2009). Mathematics curricula loaded with symbolic representation require students to memorize procedures, denying the student an opportunity to utilize their visual thinking modality in the process of building conceptual understanding. However, implementing visual representations into daily mathematics lessons can support the learning process to increase conceptual understandings (Arwood, 1991).

Mathematical modeling offers students a visual way to represent their thinking and make the necessary connections to problem-solving situations (Erbas, Kertil, Cetinkaya, Alacaci, & Bas, 2014). In order for visual thinkers to be awarded access to the symbolic representation of mathematics it is necessary for their ideas to be visually connected to the symbols in an equation (Arcavi, 2003). Thus, guiding students to recognize their visualizations as a valid path to express their thinking can deepen their conceptual understanding. Arcavi (2003) explains visualization as a method to see something that is typically unseen, and when a student is able to draw mathematics their conceptual understanding becomes more clear. Mathematical modeling is a way to express what a student visualizes, granting access for them to see the hidden meaning behind the mathematical symbols, such as the equal sign and its various meanings (Cai et al., 2014; Gravemeijer, 1999). Students should represent their understandings of numerical and spatial relations through mathematical modeling to build conceptual understanding (Deliyianni et al., 2009; Deloache, 1991). As conceptual understandings develop, background knowledge increases and students' ability to apply skills across different mathematical domains becomes for fluid (Baroody, et al., 2007; Lowrie & Kay, 2001). Mathematical modeling helps students to make connections between symbolic equations and their visual representation of how the numbers relate to one another. Providing visual models grants students the access to develop appropriate understandings of the equal sign in addition to increased ability to communicate their mathematical thinking. Students' thinking becomes more flexible when viewing symbolic equations and they can shift between both symbolic and visual representation with greater ease (Anderson-Pence et.al., 2014; Arcavi, 2003). Often times, students who practice symbolic problems have little opportunity to develop their conceptual understanding which leads to multiple misconceptions or misapplications to procedures (Alibali, 2012). However, visual models can be used to highlight the misconceptions or errors in student work and can be used to teach the deeper meaning of a problem or concept (Arcavi, 2003; Blum & Borromeo Ferri, 2009; Gellert & Steinbring, 2014; Saenze-Ludlow & Walgamuth, 1998).

Spending time reviewing student errors and misconceptions help students begin to see the structure of equations (Brendefur, 2012).

Strong visualization and spatial reasoning skills contribute greatly to students' ability to organize the structure of equations and understand the function of the equal sign (N. McNeil & Alibali, 2004). Mathematical models can be a way to connect one's visualization to their understandings of the problem (Anderson-Pence et al., 2014). The model connects the visualization into the spatial layout of an equation so students can devise a solution to solve the problem (Van den Heuvel-Panhuizen, 2014). As students visualize the problem, they flexibly decode the context into the spatial layout of an equation (Hegarty, Mayer, & Monk, 1995).

When given the opportunity, students can develop the necessary spatial skills to visualize mathematics. Gesturing assists students to communicate their thinking. Mental rotation and spatial visualization can strengthen students' ability to solve non-traditional equations and develop conceptual understanding of the equal sign. Therefore, promoting relational thinking tasks through spatial reasoning and mathematical modeling early on in students' learning can promote mathematical competency and algebraic thinking.

Enactive, Iconic and Symbolic Representations

Mathematical modeling has shown to be helpful for students to connect abstract symbols to students' thinking. Instructional tasks heavily focused on abstract symbols tend to draw out the use of rote, memorized skill practice which has been shown to compete with the development of the conceptual meaning of the equal sign, relational thinking, and spatial reasoning skills (Koedinger & Nathan, 2004). One way to help students make the connections between the numbers and symbols is to incorporate concrete materials for students to manipulate during their practice and application (Brown, McNeil & Glenberg, 2009). Including concrete manipulatives for mathematical tasks has been shown to improve student understanding and retention of the practiced concept (Schwartz & Martin, 2005). Although the use of concrete materials in isolation does not always guarantee that students will flexibly transfer the concrete representation to the symbolic representation of an equation (McNeil & Jarvin, 2007). Alternatively, instruction designed to include a progression of representations beginning with the physical manipulatives to a pictorial representation depicting the concrete representation, and finally to the symbolic form of an equation can support conceptual understanding (Bruner, 1966; Fyfe et. al., 2014; Gravemeijer, 2003).

According to Bruner (1966), students access their background knowledge of the representations to help make connections when the abstract symbols are isolated from other context. Concrete materials provide an opportunity for students to build background knowledge with images depicting the meaning of the abstract symbols. When new abstract symbols are introduced, students can use their visual background knowledge as a retrieval mechanism to help remind them of the relevant concepts. Bruner’s modes of representations begin with the enactive stage, which includes manipulatives, or concrete, physical objects. The second stage is iconic, which represents any visual representations like diagrams, number lines and graphs. Finally, the third stage is symbolic, which are abstract symbols like equations and algorithms.

Many classrooms utilize concrete objects such as toys, tiles, and blocks to help children understand abstract mathematical concepts (Correa, Perry, Sims, Miller, & Fang, 2008; Laski, Jor'dan, Daoust, & Murray, 2015). The average elementary teacher uses manipulatives nearly every day (Uribe-Flórez & Wilkins, 2010). The use of concrete materials used to demonstrate a mathematical concept or aide in the understanding of a procedure is based on the Piagetian idea that young children's thinking is concrete in nature (Bruner, 1966; Montessori, 1964; Piaget, 1953). However, the effectiveness of such objects has been mixed, where some studies report the benefits and promote learning, others report no benefits, and some report manipulatives as a hindrance or distraction to learning (McNeil & Jarvin, 2007). Concrete manipulatives which contain unnecessary details, can distract the learner from thinking about the actual concept to irrelevant information which then has the potential to limit the transfer of knowledge from the given task to different problems (Belenky & Schalk 2014; Goldstone and Sakamoto 2003; Kaminski et al. 2005; Kaminski et al. 2009). Some claim it can be better to exclude the use of concrete manipulatives to focus students' attention on structure and representational aspects, rather than on surface features (McNeil & Uttal, 2009; Belenky & Schalk, 2014). The goal becomes memorization for increased procedural transferability and generalizability to other situations where a student can be systematically taught the relevant symbolic representation (Kaminski et al. 2009; Son et al. 2008). However, symbolic representations do not always lead to success for every student. Solving problems strictly in symbolic form leads to inefficient solution strategies, entrenchment of operational procedures, and inconsistent errors (Carraher, 1985; Stigler et al. 2010; Koedinger, 2004; McNeil, 2005). Concrete manipulatives need to be more than a tool for learning, rather, they should lead and connect to the abstract, symbolic representation (Kaminski et al. 2009). As a whole, mathematics instruction that isolates the symbolic representations leads students to manipulate symbols without conceptual understanding and a weakened ability to solve problems outside of their procedural understandings (Carbonneau, Marley, & Selig, 2013; Lucas 1966; Steger 1977; Fujimara 2001; Nishida, 2007).

Manipulatives are helpful for students to see the mathematical concept of the abstract symbols and numbers in an equation. However, without the inclusion of iconic representation some students may become too dependent upon the manipulative and struggle to transition flexibly between the concrete and abstract symbols (McNeil & Jarvin, 2007). The following sections will discuss two frameworks used to teach mathematics, which utilize the progression of enactive, iconic, symbolic representation that act as a bridge between the concrete and abstract symbols for understanding mathematical ideas of relational thinking, spatial reasoning and the equal sign.

Concrete to Representational to Abstract (CRA)

When teaching mathematics, the use of concrete objects (concrete), pictorial representations (representational), followed by abstract symbols (abstract) is called the Concrete to Representational to Abstract (CRA) instructional strategy mostly used with students in special education (Witzel, 2005). This approach has found to increase the understanding of abstract mathematical concepts and ideas (Witzel, Mercer & Miller, 2003). The CRA instructional approach is a three-stage process. The first stage allows students to manipulate concrete objects to solve problems. During the concrete phase students see, hear, and move objects to demonstrate what is happening with the numbers as well as the procedures to solve the problems. When implementing instruction in the concrete stage, the teacher demonstrates solving mathematics problems through modeling. When modeling, the teacher shows students what is happening with the numbers as well as the mathematical procedures to solve the problems.

The concrete stage is followed by a pictorial representation of whatever concept was physically manipulated in the concrete stage. The representational level of instruction provides a transition from the concrete to the abstract level. The representational stage acts as a bridge, building the necessary connections between solving problems using objects in the concrete stage to solving problems using numbers in the abstract stage. Students use pictures or drawings to represent a solution to the same concept that was manipulated with objects in the concrete stage (Flores, 2009).

The final stage of CRA requires students to solve mathematical problems abstractly using numbers only (Flores, 2009; Kaffer & Miller, 2011; Hinton et al., 2014). To assist in applying the procedures to abstract equations, students are taught mnemonic devices. These types of devices help the student to remember how to structure or organize a solution to a particular problem, and provide cues for sequential steps if they do not remember a particular fact or procedure. One example of a common mnemonic is DRAW developed by Mercer and Miller (1992) to teach place value and fact fluency to students who were at risk for mathematics failure. The DRAW strategy consists of four steps and the mnemonic “DRAW” to help students remember each step. When using DRAW students are to “D” discover the sign, “R” read the problem, “A” answer or draw a conceptual representation of the problem using lines and tallies, and “W” write the answer and check. Mnemonics such as DRAW are used to help students remember to use each step to solve basic addition, subtraction, and multiplication problems that involve regrouping. Other mnemonics have been developed to teach fluency for problems that involve place value, addition and subtraction of numbers, multiplication, fractions, integers and algebra. The abstract problem is taught using memorization of mathematical procedures through mnemonics until the student learns the procedure automatically (Flores, 2009; Witzel, 2005; Witzel et al., 2003).

Each level of CRA is strategically designed to prepare the student for the next level of learning (Witzel, 2005). An example of CRA can be viewed through a re-grouping lesson for the subtraction problem 32 take away 15, where students must break apart the 32 into two units of ten and twelve units of one to compete the algorithm (Flores, 2010). For the concrete stage, base-ten units of ten and units of one cubes are used to build the visual representation of the subtraction problem. Once the answer is revealed, students notate the units of tens and ones next to the algorithm as a way to show how the regrouping procedure works. Lastly, in the abstract stage students perform the standard procedures for solving the algorithm.

The goal of CRA is to help students successfully perform at the abstract level. The CRA framework utilizes the three stages as tools to aide in accurate computation and provides students with a concrete, visual tool to develop the necessary procedural skills for solving abstract equations (Maccini, Mulcahy, & Wilson, 2007; Miller & Hudson, 2007). The stages of CRA provide students with manipulatives to visually represent their solutions to mathematical problems without posing a hindrance on the learning of the concepts. The next section describes a slightly different approach to modeling equations through a progression of stages.

Concreteness Fading

Fyfe and her colleagues (2014) use the term concreteness fading to explain Bruner's progression of enactive, iconic, and symbolic (EIS) representations where physical representation of a mathematical concept can gradually become more abstract. Concreteness fading progresses from the enactive stage, which includes concrete physical objects; the iconic stage, which includes a picture or visual model; and finally the symbolic stage, which includes an abstract model of the concept. For example, the addition problem of 2 + 3 could first be represented by physical objects such as birds or cubes, next by a visual diagram of the cubes representing the birds, and finally by a number sentence. The goal of concreteness fading is to start with a manipulative to help students make the necessary connections and then gradually move away from the physical objects to the most efficient, iconic and abstract representations.

The equal sign can be represented through balancing objects on a seesaw as a visual representation of equivalence (Mann, 2004). Using visuals such as the seesaw develop the necessary background knowledge and imagery for the concept of equivalence, which better prepares students to think of solutions for missing term problems such as 5 + 6 = __ + 2. Giving students the visual representation of the seesaw prior to the discussion of relational thinking prepares students' background knowledge and assists in the sense making for devising a plan to solve the equation (Mann, 2004).

Lessons structured to begin with an enactive example serve as a visual model to progressively link conceptual understandings to the symbols in a meaningful way (Chesney & McNeil, 2014). Concreteness fading encourages teachers to develop conceptual understandings through the EIS progression and provide students with new concepts connected to their own background knowledge (Fyfe et al., 2014).

McNeil and Fyfe (2012) conducted a study with undergraduates to learn a mathematical concept in one of three conditions: concrete, abstract, or concreteness fading. The concreteness fading progression included a transitional phase connecting both the concrete and abstract without the unimportant details. A transfer test was given to the students immediately after the treatment, 1 week later, and 3 weeks later. Students in the concreteness fading condition showed the best transfer performance all three times. In two additional studies by Fyfe and McNeil (2009), students received instruction on missing term problems. In the concrete treatment, problems were presented using toy bears on a balance scale. In the abstract treatment, problems were presented in symbolic form on paper. Problems from the concreteness fading treatment were presented using the progression of concrete manipulatives, to worksheets using pictures to represent the bears, and lastly with symbolic equations. An example of this progression is in figure two below. Children in the concreteness fading treatment solved more transfer problems correctly than children in the other treatments suggesting the progression from enactive, to the iconic, and then to the symbolic stage play a pivotal role in students' mathematical ability to solve missing term problems.

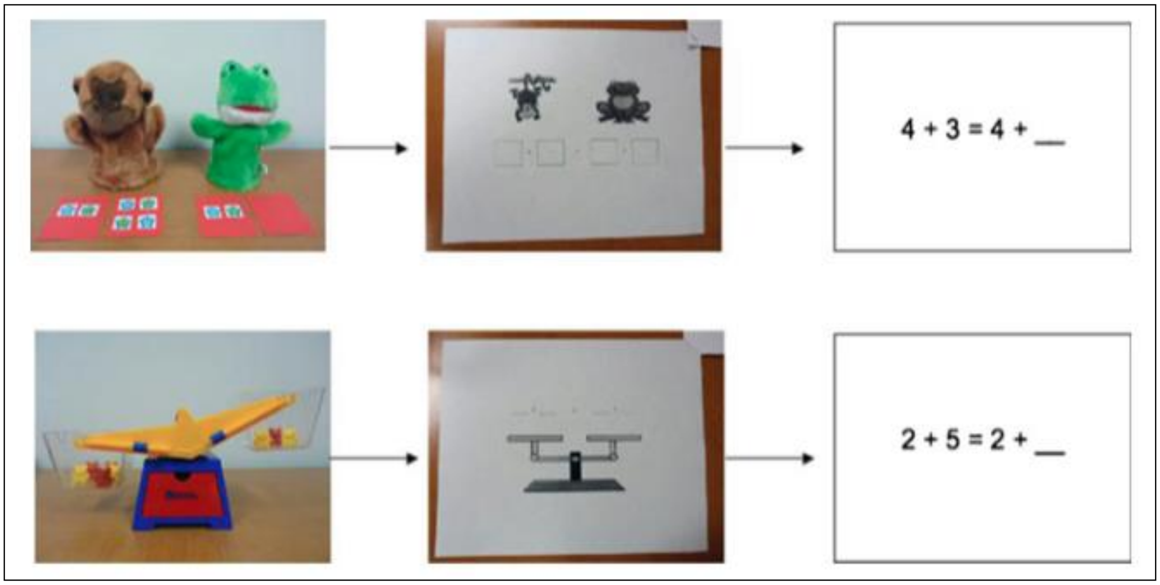

Figure 3. Progression of concreteness fading (Fyfe & McNeil, 2009).

Concreteness fading offers the opportunity for students to visually represent mathematical concepts in a meaningful, connected way. The progression from enactively representing an equation to modeling the problem on paper bridges the gap to the symbolic equation. This type of modeling develops background knowledge and imagery in the student's memory for later retrieval for solving symbolic equations.

Differences Between CRA and Concreteness Fading

Both concreteness fading and CRA are designed to focus instruction on conceptual understanding using Bruner's theory of enactive, iconic and symbolic representation. The CRA approach compartmentalizes the progression into each of the three stages; beginning with direct instruction through teacher demonstration followed by teacher guidance, and student mastery over three lessons. The concreteness fading approach comprises the three modes of representations with more fluidity and flexibility for each concept. The CRA approach relies more heavily on the memorization of the procedures for solving the symbolic representation, whereas the concreteness fading approach tends to focus on the development of conceptual understanding, and progressively connects to the symbolic representation. CRA is presented in discreet stages, leaving the impression that once a concept has been mastered, the stage becomes a distant memory. Concreteness fading has students build a visual representation through the enactive stage to access background knowledge, draw out their thinking with a picture and connect to the symbolic stage. CRA uses a variety of procedural mnemonic devices for particular skills, which are not necessarily generalizable to other mathematical concepts or tasks.

Concrete to representation to abstract (CRA) and concreteness fading have established positive connections to mathematical proficiency (McNeil & Fyfe 2012). CRA offers a visual model for students to follow a set of procedures when solving equations. Concreteness fading prepares students to visually represent mathematical topics more flexibly and fluidly through the progression of Bruner's (1966) enactive, iconic, and symbolic stages. In the next section I will explain how traditional textbooks and curriculum apply these frameworks to relational thinking and spatial reasoning tasks to develop students conceptual understanding of the equal sign (Brendefur, 2015). Following that section, I describe a framework developed to support Bruner's notion of enactive, iconic, and symbolic representations.

Textbooks, Curriculum and Instructional Tasks

Literature presented so far has shown that intentionally promoting relational thinking and spatial reasoning allows students to conceptualize the idea of the equal sign and equivalence. However, disregarding the equal sign as an important symbol for the concept of equivalence and algebraic thinking has been described as one of the biggest obstacles to achieve this level of understanding for many generations (Renwick, 1932; Carpenter et al., 2003). Teacher manuals and material used in classrooms affect the way teacher's present equations and explain the purpose of the equal sign, and many textbooks introduce student's to the equal sign as early as Kindergarten, although the conceptual meanings of the symbolic representation are rarely explicitly taught (Knuth et al., 2005). Curricular material with frequent use of activities having an operational focus develops one way to think, encouraging an operational understanding of the equal sign (Molina et al., 2005). This disconnected thinking leads to the memorization of rules and procedures with very little conceptual understanding, multiple misconceptions, and a difficult transition to algebraic thinking (Ginsburg et al., 2008; Herscovics & Linchevski, 1994).

The textbooks used in classrooms determine how the teacher will focus the lessons, therefore the mathematical topics in textbooks are one of the important ways to influence students' conceptual understandings of the equal sign and development of algebraic thinking (Hattikudur & Alibali, 2010; Knuth et al., 2006; Reys, Reys, & Chavez, 2004; Seo & Ginsburg, 2003). Although the equal sign is present at all levels of mathematics, textbooks using repeated practice of the traditional format $a + b = c$ develops a narrow operational viewpoint (Baroody & Ginsburg, 1983; Rittle-Johnson et al., 2011; B. Rittle-Johnson & Alibali, 1999). Most equations in traditional curriculum are written with the operations to the left of the equal sign and the answer blank to the right (N. M. McNeil, 2007; Seo & Ginsburg, 2003). Much of the curriculum promote operational thinking through repeatedly performing operations in the traditional format of $a + b = c$ (Blanton & Kaput, 2005; Byrd et al., 2015; Falkner, Levi, & Carpenter, 1999) rather than developing algebraic thinking through exposing students to a variety of nontraditional problems such as a = b + c, true/false statements, and missing term problems. Textbooks that overuse the traditional format promote students' operational view developing a rigid and inflexible application of procedures on non-traditional equations (McNeil et al., 2006). For example, in order for students to solve 2 + 3 + 4 = __ + 7, many with an operational view will reorganize the numbers to the left of the equal sign so an answer can be placed to the right (McNeil & Alibali, 2004). Students frequently use the equal sign as a link between steps and simply misapply shortcuts in their procedural work. Students in upper elementary grades predominantly use inappropriate strategies to solve open number sentence problems, and few students apply relational thinking to equivalence problems (Hunter, 2007). Studies by Falkner (1999) and McNeil & Alibali (2004) describe children's understandings of the equal sign as a "do something" function when two numbers are added together to find the answer. Falkner found students viewed the equal sign as an operational symbol rather than an equivalence statement in missing term problems such as 8 + 4 = __ + 5. Students would add the left side of the equation and insert the answer after the equal sign, ignoring the + 5 completely. Similarly, McNeil and Alibali (2004) found that many students added all the numbers together treating the equal sign as an operational symbol and ignored the numbers to the right of the equal sign. Although students can successfully solve simple equations such as 2 + 3 = 5, this type of repeated practice perpetuates the operational view. With continued practice, students' knowledge of operational practice will increase in strength. When students are presented with mathematical problems, their prior knowledge of these patterns is activated which influences how the information is stored to memory. Repeated practice of traditional equations blocks students' ability to learn new ideas and generalize to other mathematical topics, such as missing term problems (N. McNeil, 2008). If a student is given an equivalence problem, their application of the practiced pattern leads to multiple errors, incorrect strategies and misconceptions of the equal sign's purpose (N. M. McNeil & Alibali, 2005).

Textbooks that over emphasize traditional equations hinder students' performance on algebraic thinking. This operational view becomes entrenched in students' minds, and much of their resistance to learning other strategies becomes difficult to overcome. According to Langer (2000), this type of practice promotes mindless learning, a place in the mind where students get stuck in a rhythmic pattern of taking in information. Students who are not taught to think limit their ability to problem solve and obstruct the learning of new information. Therefore, instead of constructing new strategies and developing the flexibility to think of numbers relationally, mindless repeated practice of the traditional format leads students to be content, relying on the strategies they have used many times in the past.

Seo and Ginsburg (2003) examined elementary school curricula and found the equal sign was rarely presented without plus or minus signs, where most number sentences were presented in a traditional format, such as $a + b = c$ or $a – b = c$. The textbooks did not offer teachers much assistance with the lesson implementation nor did they offer support for students' understanding of the equal sign with a relational meaning. McNeil et al. (2006) examined middle school textbooks and found that non-traditional equations such as true/false statements with the operations occurring on both sides of the equal sign hardly ever appeared in any of the textbooks.

Li and Ding (2008) compared United States and Chinese sixth grade students on their ability to solve non-traditional equations. The difference between the two sets of students is strikingly different, with 28% of United States students successfully solving non-traditional equations, to 99% of Chinese students successfully solving the same types of problems. The discrepancy was found to be related to the instructional design of the curriculum structured in the teacher's manuals, which guides teachers' instruction and the attention paid (or not paid) to equivalence problems. Additionally, teachers' manuals are scripted with language suggesting that the equal sign is a signal for an operation to occur (Li et al., 2008). For example, the language for the addition problem 3 + 2 = 5 is stated as "three and three makes six", placing the equal sign as the operator for combining the 3 and 2, rather than a relation between both sides of the equal sign. In contrast, the Chinese teachers' manuals used a variety of language and symbols to represent the relationships between both sides of the equation, such as, greater than and less than. The textbook and student material also included a variety of traditional and non-traditional equations to increase flexibility and improve relational thinking for equivalence problems. U.S. curricula is written with instructional language in the teachers' manual to operationalize the equal sign, contributing to students' difficulty to develop conceptual understanding of the equal sign and its multiple meanings (Li et al., 2008).

Designing curricula for young students to have multiple opportunities for practice solving both traditional and non-traditional equations has shown to successfully develop a deep understanding of equivalence and flexibility to problem solve (McNeil et al. 2012). Understanding how the equal sign functions is pivotal in order for students to flexibly use multiple representations and relational thinking to view numbers and then generalize to other areas of mathematics (Falkner et al., 1999; Kieran, 1981; Rittle-Johnson et al., 2011). Curricula and textbooks often times determine the instructional approach and language used by classroom teachers, which significantly impacts the amount of exposure students have to relational thinking and the multiple meanings of the equal sign (Reys, Lindquist, Lambdin, Smith & Suydam, 2003; Reys, Reys, and Chavez 2004; Schmidt et al., 2005).

Powell (2012) wanted to determine whether eight elementary mathematics textbooks promote operational thinking with traditional equations, or contribute to the development of relational thinking, equivalence and the multiple meanings of the equal sign. The textbooks were commonly found in elementary schools throughout the United States and included the grade bands Kindergarten through fifth grade. Across all of the curricula chosen, the equal sign was only mentioned, at most, eight times throughout each teacher manual for each grade level. Although some of the curricula offered the opportunity for students to practice non-traditional equations, none of them provided a comprehensive set of instruction explicitly developed for conceptual understanding of the equal sign and relational thinking.

While research continues to support the development of relational thinking and spatial reasoning through conceptual understanding of the equal sign, teachers continue to lack the necessary resources to adequately design effective instructional tasks. According to the work of McNeil et al. (2004) Reys et al. (2003) and Canobi (2009) choosing curricula that offers a balance of relational thinking and spatial reasoning tasks is not easy. However, it is important to recognize the changes to math curricula beginning to emerge with the onset of the recommendations for mathematical models, relational thinking, and spatial reasoning suggested by NCTM (2010) and the CCSS (2010). The next section highlights two of those curricular options.

Bridges in Mathematics

Bridges in Mathematics (Frykholm, 2016) is a k-5 curriculum designed to cover all of the Common Core State Standards with the inclusion of the mathematical practices to help students develop conceptual understandings of the mathematics. The primary grades, specifically kindergarten and first grade utilize multiple visual models to deepen students' mathematical understandings, such as the number rack, ten-frame, dominos, unifix cubes, bundles and sticks, and the number line. Bridges in Mathematics supports teachers' instruction to improve mathematical dialog, critical thinking and problem solving. Students are given multiple opportunities to use concrete manipulatives to represent solutions for given tasks. Each of the models are used to visually represent how to compose and decompose numbers, see the relationships between numbers, addition and subtraction, and discover place value through grouping strategies of tens and ones.

As was commonly noted in many textbooks by Seo and Ginsburg (2003) and McNeil (2005), the equal sign is first introduced in the context of addition with the traditional form of a + b = c. For example, in one lesson the teacher poses a problem where students are expected to represent the following context symbolically "Sage has 2 green Popsicles in her left hand and 4 purple Popsicles in her right hand. How many Popsicles does she have in all?" When students observe number sentences such as this, they will most likely interpret the equal sign as a function to perform an action, and reinforce their operational view. See figure 3 for an example of the expected student work scripted in the teacher's guide.

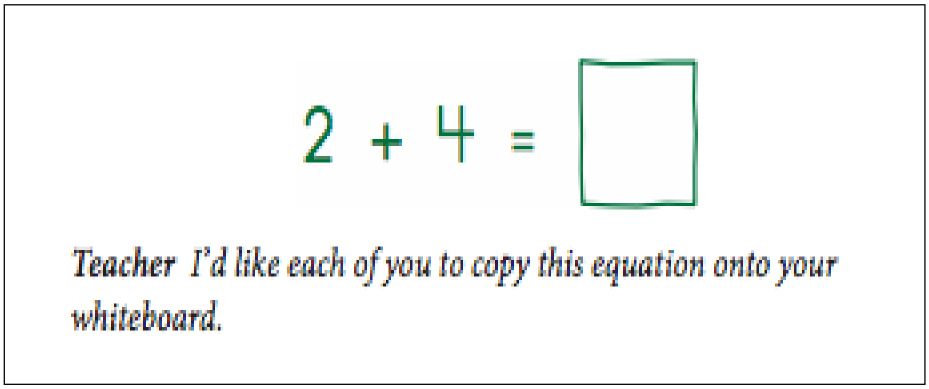

Figure 4. Student work in Bridges for Mathematics teacher's guide.

The authors note the importance of developing algebraic thinking through flexibly solving a variety of equations. To promote algebraic thinking and the notion that the equal sign is more than just a signal to perform an operation, students play fact family games to determine the relationships between quantities. The teacher displays the dot card and tells them the total number of dots, but then only reveals part of the dot card. Students are encouraged to look at the dot patterns and make assumptions of the possible combinations based upon the particular view. Figure 4 shows an example of the game and the suggested dialog of the teacher.

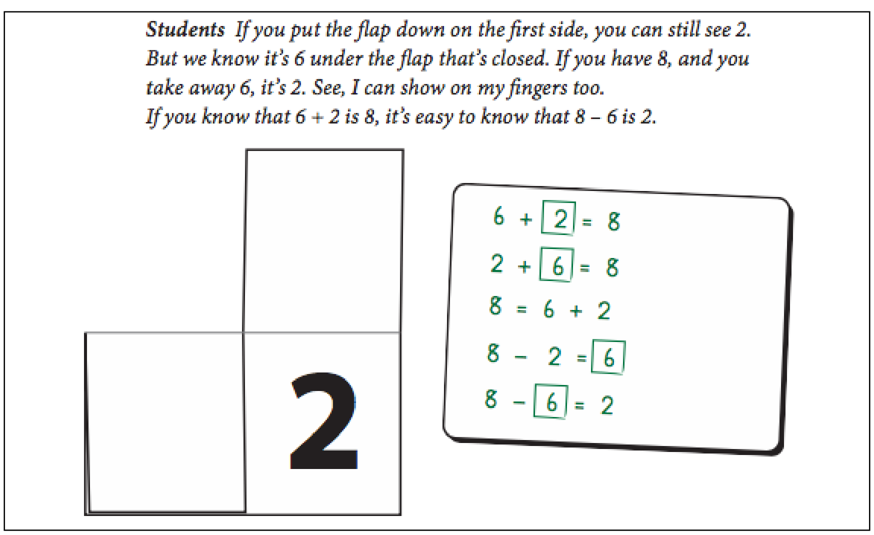

Figure 5. Double-Flap dot card game to encourage algebraic thinking.

Much of the lessons focus on the use of concrete representations paired with the symbolic representations, however the iconic representations are not present. For example, students are asked to represent their thinking using a number rack (refer to figure four for example) when presented with equations. The textbook provides the teacher with a variety of possible student responses to the equation 2 + 4 = ___ using the number rack, and the corresponding equation.

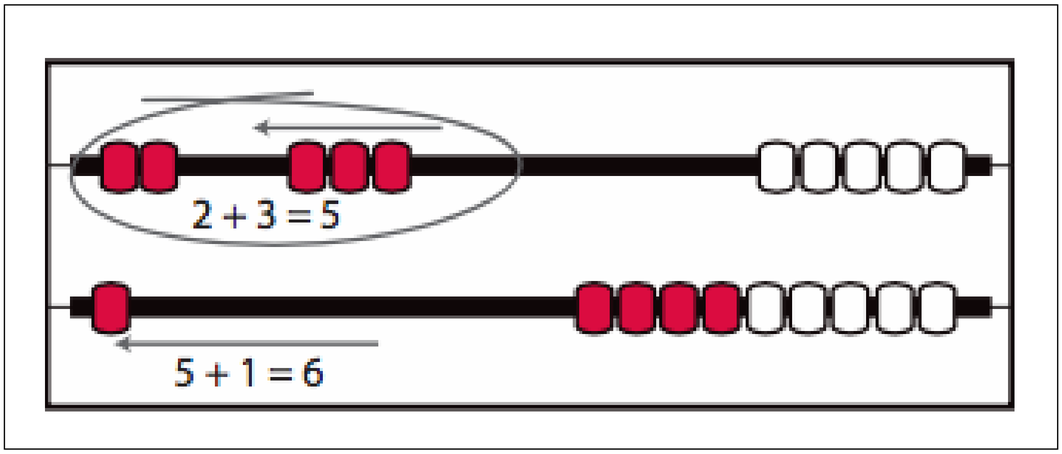

Figure 6. **Number rack student tool in Bridges to Mathematics (2016).**

The kindergarten through second grade curriculum is heavily emphasizes the use of concrete representations as a tool for solving problems. Overall, each lesson strategically provides opportunities for students to engage in the enactive and symbolic modes of representation. However, with a few exceptions where a number line, hundred chart or unifix cubes are presented, the curriculum does not emphasize the use of iconic representation. Although geometry topics are included in the Bridges curriculum, the emphasis remains on identifying, describing and comparing the attributes of two-dimensional shapes. As Cheng and Mix (2014) and Fyfe et.al (2014) revealed through their research, the need to connect relational thinking tasks and spatial reasoning is critical for the development of students' conceptual knowledge of the equal sign.

Curriculum has shown to be a major contributor to students' misconceptions and lack of algebraic knowledge (Seo & Ginsburg, 2003; McNeil, 2014). The following section describes a curriculum developed through a framework designed to promote relational thinking and spatial reasoning using visual models for students to communicate their mathematical ideas.

Developing Mathematical Thinking

Research presented so far supports the claim that curriculum should include ways to promote relational thinking and spatial reasoning through mathematical modeling to develop students' conceptual understandings of the equal sign (Carpenter et al., 2003; McNeil & Alibali 2004; Knuth et al. 2006). Mathematical tasks should include both traditional and non-traditional equations (Molina, 2005; Rittle-Johnson et. al. 2011). The use of mathematical modeling should connect through a progression of concrete, visual representation to an iconic model and then to the formal, abstract symbols of an equation (Fyfe et al., 2015). Spatial reasoning tasks should be integrated throughout the instructional year to increase students' flexibility with the structure of equations and mathematical competency (Ontario Ministry, 2014; Cheng & Mix, 2014). The Developing Mathematical Thinking (DMT) framework offers a comprehensive curriculum designed to encompass all of the necessary components for students to develop conceptual understandings of the equal sign and advance into middle school algebra with less misconceptions and more mathematical competency. The DMT is an alternative to the typical curriculum for teaching mathematics to help teachers develop a different approach to how mathematics is taught (Brendefur, 2015). Much like the Common Core State Standards mathematical practices, the DMT framework consists of five key elements for teachers to reflect upon as they plan, prepare and instruct mathematics lessons. The DMT five key elements (Brendefur, 2008) are as follows: taking student's ideas seriously, encouraging multiple solution strategies and models, pressing students conceptually, addressing misconceptions, and maintaining a focus on the structure of the mathematics. Using students' informal strategies values their thinking and gives the teacher insight as to the level of understanding each student has. Teachers use the five elements of the DMT to develop more efficient strategies and multiple models for solutions to mathematical problems. Students are encouraged to talk with others about their thinking, compare solutions, and make corrections to their errors. One of the most critical components of the framework is to draw attention to the structural components in mathematics, which extend across grade levels and topics. The DMT curriculum is designed to focus students' attention on the structural components of mathematics, which are woven throughout the course of their elementary school years.

One of the ways the DMT curriculum connects student thinking is through the inclusion of Bruner's (1966) enactive, iconic and symbolic models. Each unit is comprised of tasks centered on the EIS framework to develop a strong foundation for the development of conceptual understanding for solving problems (Brendefur, 2012). For example, students in first grade are given a contextual problem about ten children playing in sandbox, where they need to determine whether six of the children are boys, then how many children are girls? Students first demonstrate their thinking using unifix cubes, followed by drawing an iconic bar model to match their unifix cubes model. The symbolic representation of the numbers is then attached with labels. Relational thinking is considered an important component in the DMT curriculum. To highlight the variety of ways to represent the number ten, students are asked to demonstrate the other possible representations for making ten following the EIS progression. Modeling all of the possible combinations for ten emphasizes the idea of equivalence, and using the EIS progression helps all students visually see how the numbers relate to one another. Figure six provides a sample solution for the students to use as a model.

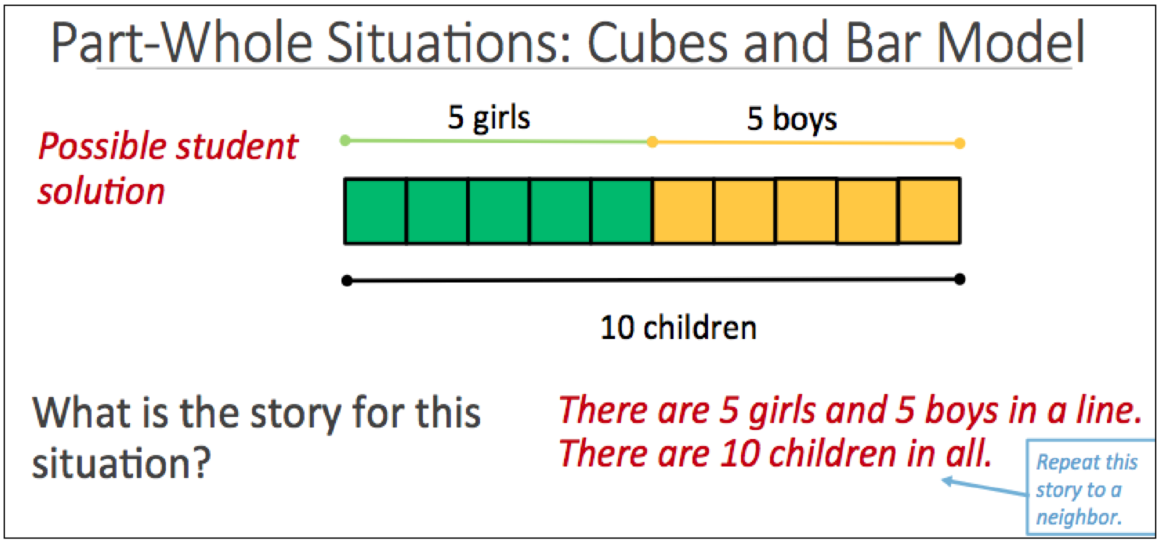

Figure 7. **Sample solution for making ten in Unit 3 of the DMT curriculum.**

As students become fluent with facts within ten, they are introduced to the variety of ways to compose the teen numbers using units of tens and ones. For example, one task is to represent each teen number using units of one. Eventually, students begin to recognize the inefficiency of counting each unit of one. At that point, the teacher introduces a more efficient way of building the teen numbers by using a unit of ten. Over time, students independently build efficient models for larger numbers based off their previous experiences building with units of one. Once again, tasks such as these expose students to relational thinking and highlight the structure of equivalence through the use of mathematical modeling.

The DMT curriculum encourages students to represent solutions to contextual problems, explain their solutions, and then generalize their understandings to other concepts. An example of this is with contextual compare problems presented in unit three where students represent the number of blocks used to build two different towers. The task states that one tower is eight blocks tall, and another tower is six blocks tall. Students are asked to represent both towers using unifix cubes, determine whose tower is tallest and by how much. Next, students draw an iconic representation of the towers, paying attention to the spatial relationship between the number seven and four. The drawing should depict that one tower is taller than the other, and the enactive model is used to determine the difference between the numbers seven and four. Lastly, students connect their understandings of the relationship between the two towers back to the symbolic representation by notating 8 – 6 = 2. As students fluently build models to represent the context, they are then asked to look at a given set of numbers, build the models with unifix cubes to match, draw an iconic representation of the models, and create their own story to match their model. Students work in partners to listen to the story, but then also explain the relationships between the two towers. With this activity, students often times gesture with their hands to explain how many more blocks are in one tower than the other tower.

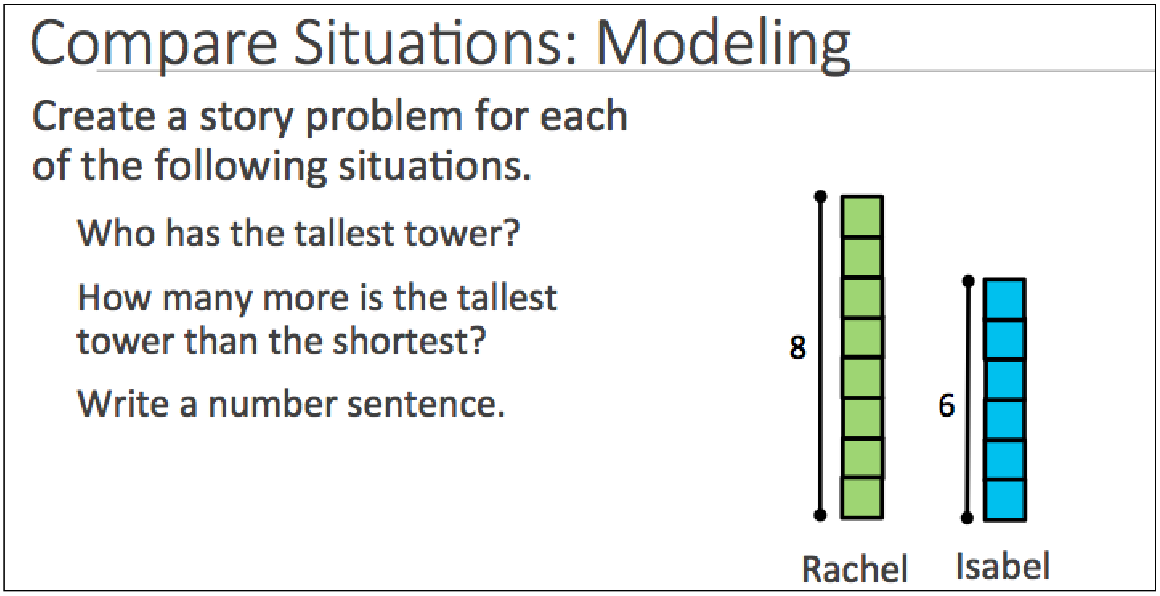

Figure 8. Example of student work mat from Unit 3 (Brendefur, 2016).

Students who understand the concepts from this task are then able to transfer their knowledge to the task in unit four with a spatial reasoning activity using pattern blocks. Each student is given the outline of a figure and a variety of pattern block shapes to fill in the space. The task encourages students to mentally rotate and visualize how different shapes might fit to compose the given figure. Once the figure is covered completely with pattern blocks, students record the specific shapes used on a line plot graph. Then, students trade their line plot graphs with a partner and try to recompose the figure based upon the data on the graph. The understandings of comparing two quantities from the previous unit is needed with this graphing activity as students are asked to compare their total amount of pattern blocks needed to complete the figure with the total amount their partner used. An example of the student work mat can be seen in figure eight.

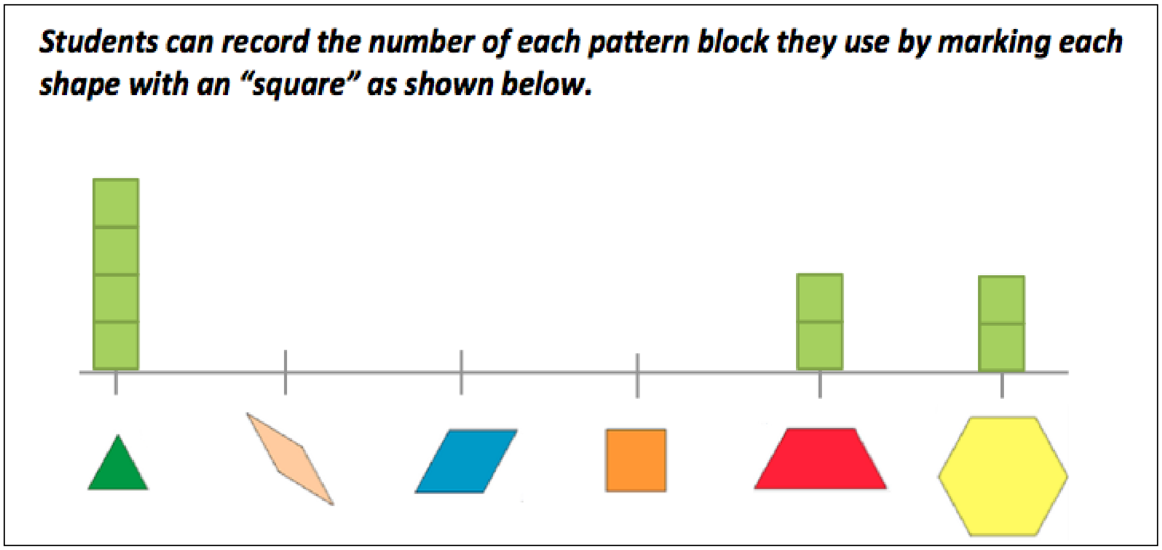

Figure 9. **Student work mat to graph the shapes used to compose a given figure.**

As suggested by National Governors Association for Best Practices (2010), the DMT framework and curriculum intentionally focus on building students' conceptual understandings of mathematical concepts through relational thinking and spatial reasoning tasks. Each task presents students with meaningful problem solving situations where they are encouraged to begin to represent their thinking through enactive mathematical modeling, followed by an iconic representation depicting their thinking, and lastly with a connection to the symbolic representation of the problem. Students are encouraged to communicate their thinking with partners to check for understanding or assess any misconceptions that may arise. The structural components are intentionally highlighted within each lesson to foster deep conceptual understanding and help students generalize their knowledge to other tasks throughout the year. The overlapping of conceptual understandings is woven throughout the complete year to help students build a strong foundation in the mathematical concepts for first grade. Overall, the DMT framework delivers a comprehensive curriculum designed to increase students' relational thinking and spatial reasoning skills, which also encourages students' flexibility with understanding the various meanings of the equal sign.

Conclusion

Most elementary students begin to develop their awareness of the equal sign's functionality at an operational level, where the equal sign acts as a symbol to perform a calculation or action (Carraher et al. 2006). When the bulk of instruction is focused on procedures and computing facts many elementary students develop a shallow understanding of the equal sign and consider it an operational symbol (Baroody & Ginsburg, 1983; Rittle-Johnson et al., 2011). Mathematics instruction for early elementary classrooms should foster relational thinking by including tasks designed to draw attention to how numbers relate to one another, and develop the flexibility to think of numbers in a variety of ways to establish the idea of equivalence (Cheng & Mix, 2014; Stephens et al., 2015). Mathematical tasks should include both traditional and non-traditional equations (Molina, 2005; Rittle-Johnson et. al. 2011).

As educators become more aware of the need for relational thinking tasks it is important to recognize the critical role spatial reasoning and mathematical modeling play in the overall development of algebraic thinking and the equal sign. The National Research Council report (2006) and the National Governors Association Center for Best Practices (NCTM, 2010) suggests more spatial reasoning be integrated into the elementary mathematics curriculum to promote relational thinking skills. Spatial visualization, gesturing, and mental rotation have been shown to increase student performance in mathematics (Cheng & Mix, 2014).

Mathematical modeling gives students a visual representation to explain their mathematical thinking (Erbas et al., 2014). The use of mathematical modeling should connect through a progression of concrete, visual representation to an iconic model and then to the formal, abstract symbols of an equation (Fyfe et al., 2015). Curriculum should support students' conceptual understandings through the integration of relational thinking, spatial reasoning and mathematical models by incorporating Bruner's EIS framework.

The Developing Mathematical Thinking (DMT) framework offers a comprehensive curriculum designed to encompass all of the necessary components for students to develop conceptual understandings of the equal sign and advance into middle school algebra with less misconceptions and more mathematical competency. With the present literature review, I propose a study to investigate the effectiveness of a curriculum that focuses on the EIS framework on first grade students' conceptual understandings of the equal sign, relational thinking and spatial reasoning.

CHAPTER 3: RESEARCH METHOD

Purpose

The purpose of this study was to investigate whether there was a significant difference in first grade students' performance in relational thinking and spatial reasoning when they learn to construct and compare numbers using iconic modeling. The study compared relational thinking and spatial reasoning for first grade students whose teacher received a curriculum designed to increase use of enactive, iconic, symbolic representation in teaching (EIS group) and those whose teacher received district adopted curriculum with a more traditional instructional method (Traditional group). Both groups were tested using the Primary Mathematics Assessment Screener (PMA-S; Brendefur, 2012) in September, prior to the mathematics instruction, and again mid-May after the mathematics instruction; therefore, student performance was also compared across time. Thus, this study used a 2 (EIS group versus Comparison group) x 2 (pretest versus posttest) design.

To address the primary purpose of this research study, the following research question and hypothesis was investigated:

1. What is the effect of integrating iconic representation through student drawings in conjunction with the enactive, iconic and symbolic teaching methodology into mathematics instruction on first grade students' relational thinking and spatial reasoning performance?

$H1$: There is a positive effect on integrating enactive, iconic and symbolic teaching methodology into mathematics instruction on first grade students' relational thinking and spatial reasoning performance.

$H_0 1$: There is not a positive effect on integrating iconic teaching methodology into mathematics instruction on first grade students' relational thinking and spatial reasoning performance.

This chapter discusses the research design, setting and participants, instructional modules, assessment tool, timeline, and analysis.

Research Design and Approach

As noted above, this study used a 2 (EIS group versus Traditional group) x 2 (pretest versus posttest) design. The dependent variable was the students' understanding of relational thinking and spatial reasoning measured with the PMA-S. The goal of this study was to determine whether student achievement on the PMA-S differed between the EIS and Traditional groups, and whether achievement differed across time.

Setting and Sample

Population Definition

The larger population of interest for this study was first grade classrooms in Idaho. Within this larger population, the study consisted of first grade classrooms from five school districts. Two of the school districts serve between 15,650 to 26,240 students, and three of the districts serve between 600 to 1725 students. There were over 2600 students with Limited English Proficiency (LEP) comprising approximately 8% of the total districts. In these districts, the student demographics were 79.3% white, 10.3% Hispanic/Latino, 5.9% Asian, 3.3% Black, 0.9% Native American, and 0.8% Pacific

Islander. For this study, the sampling frame will be first grade classrooms in Idaho chosen on the basis of similarly matched demographics related to students who receive free and reduced lunch assistance.

Participants

School district approval to implement a supplemental mathematics resource, collect and analyze students' scores from the PMA-S was attained through district level administrators. Approval then allowed district mathematics curriculum directors to recruit first grade classroom teachers for the EIS and Traditional groups.

Tables 1 and 2 display the EIS and Traditional groups' demographics.

Table 1: EIS group demographics

Table 1 *EIS Group*						
DMT Modules						
School	%FRL	White	Hispanic	Other	Teacher	Experience
Parkton	63%	72%	19%	9%	Mrs. Aura	10 years
					Ms. Commons	5 years
Lagunitas	75%	43%	27%	30%	Mr. Hops	4 years
					Mrs. Velitan	12 years
					Mrs. Shippy	8 years
Firestone	88%	59%	24%	17%	Mrs. Odell	4 years
					Mr. Antilla	6 years
					Ms. Hope	8 years
Sienna	15%	83%	14%	3%	Mrs. Joplin	12 years
					Ms. Wiley	15 years

Table 2: Traditional Group Demographics

Table 2 *Traditional Group*						
District Curricula						
School	%FRL	White	Hispanic	Other	Teacher	Experience
Murray	49%	86%	12%	2%	Mrs. Sterling	16 years
					Mrs. Eppe	6 years
					Mrs. Jameson	8 years
Hillside	83%	58%	41%	1%	Mrs. Bentley	12 years
					Mrs. Jeppeson	8 years
Wallace	87%	61%	18%	21%	Mrs. Gallon	12 years
					Mrs. Smeade	6 years
Eastman	13%	87%	9%	4%	Mrs. Jonni	16 years
					Mrs. Sumpter	13 years
					Ms. Yarbow	9 years
					Ms. Deming	6 years
					Mr. Boyd	3 years

Instrumentation

Independent Variable

The independent variable was separated into two groups—instruction focused on the integration of enactive, iconic, symbolic representation for the EIS group, and the implementation of district adopted curriculum aligned with Common Core Standards for the Traditional group. Teachers in the EIS condition taught a set of eight mathematics unit modules designed to encourage teachers to engage learners in the progression of enactive, iconic, and symbolic representation of their mathematical thinking to build relational thinking and spatial reasoning in first grade. Teachers in the comparison group taught mathematics lessons aligned with the scope and sequence from their curriculum.

DMT modules. The independent variable was comprised of eight modules for teachers to use in their mathematics instruction. The Developing Mathematical Thinking (DMT) framework (Brendefur, 2008) includes five critical dimensions to improve students' mathematical understanding: (a) taking student's ideas seriously, (b) pressing students conceptually, (c) encouraging multiple strategies and models, (d) addressing misconceptions, and (e) focusing on the structure of mathematics. The framework was developed with the understanding that students' mathematical understandings develop over time, beginning with the use of informal strategies and models for problem solving. Teachers are encouraged to use student thinking and models as a starting point, then press them to adopt a more formal and abstract set of strategies through a process called progressive formalization (Treffers, 1987; Gravemeijer & van Galen, 2003). The modules are designed with Bruner's (1966) enactive, iconic, and symbolic (E/I/S) modes of representation to develop conceptual understanding of the equal sign through a series of lessons, which incorporate spatial reasoning and relational thinking tasks. Students construct meaning as they problem solve using enactive and iconic models to conceptualize the given task, represent their thinking, and draw conclusions about their solutions. The DMT modules (DMTI Inc, 2016) incorporate the E/I/S progression to help students see the relationships between concepts and equip them with the mathematical understandings to choose which strategies work best for a given task (Brendefur, 2008).

Within each of the lessons, students are encouraged to discuss their mathematical thinking with peers, as well as reason through other students' solution strategies. The DMT modules were designed to develop the following mathematical concepts for first grade—number and place value, measurement, geometry, and data analysis. The subsequent sections will describe the way in which relational thinking and spatial reasoning tasks incorporate enactive and iconic models to increase conceptual understanding.

Number and place value. The number and place value modules develop students' ability to make connections between strategies and models utilizing the E/I/S progression to formalize their thinking. Module one, three, five, and seven sequentially build number, counting and place value concepts through the use of word problems, physically building numbers with place value cubes, tracing of a unit of 1 and a unit of 10 with unifix cubes, decomposing units of place value, tree diagrams and bar models, correcting student errors, and the introduction of number lines. Table 3 highlights the lessons for each module. Samples from the lessons are included in the appendix.

Table 3: Number and Place Value descriptions for modules one, three, and five (DMTI, 2016).

Table 3 *Number and Place Value*	
Module Lessons	
Module	Focus
1. Building Number, Counting, and Place Value	The focus for Module one is on counting forward and backwards using unifix cubes to represent each quantity to build place value understanding of numbers 0-20. Contextual story problems are used to introduce addition and subtraction for join and separate problem types. Each of the lessons incorporates enactive, iconic, and symbolic models.
3. Number: Counting and Place Value	The focus for Module three is on adding and subtracting using part-whole and compare problem types through the use of the bar model. Each lesson incorporates enactive, iconic, and symbolic models to build understanding of the commutative and associate properties.
5. Number: Join and Separate, and Place Value	The focus for Module five is on adding and subtracting using join and separate problem types through the use of a bar model and number line. Each lesson incorporates enactive, iconic, and symbolic models to build understanding of the commutative and associate properties, in addition to regrouping when adding numbers within 0-100.

Measurement and Geometry. The measurement and geometry modules develop students' ability to make connections between strategies and models utilizing the E/I/S progression to formalize their thinking. Module two, four, six, and eight sequentially build an understanding of informal linear measurement through iteration. The first lessons in module one lay the foundation for the use of a more formal measurement tool, the ruler. The following lessons build on comparing lengths of objects and analyzing student misconceptions using units of ten and one. Table 4 highlights the lessons for each module. Samples from the lessons can be found in the appendix.

Table 4: Measurement and Geometry descriptions for modules two and four (DMTI, 2016).

Table 4 Measurement and Geometry	
Module Lessons	
Module	Focus
2. Informal Linear Measurement Through Iteration	The focus for Module two is on making informal linear measurements using paper strips, paper clips, and cubes. The number line is introduced as a linear tool for measurement, where the bar model is used to develop understanding of comparisons between measurements. Each lesson includes addition and subtraction contextual problems to make comparisons between measurements.
4. Composing Shapes and Space	The focus for Module four is on composing and decomposing shapes and 3-D objects. Lessons examine the way shapes can be configured to compose new figures using pattern blocks, square tiles, and 3-D shapes. Other lessons include the opportunity for shape classification based upon particular attributes with enactive and iconic models.

District Adopted Curricula. The comparison groups used a variety of k-5 curriculum recently published with the Common Core State Standards and the inclusion of mathematical practices to help students develop conceptual understandings of the mathematics. Two of the main curricula used by the comparison group was Bridges in Mathematics (Frykholm, 2016), and Math in Focus, Singapore Math (Fong et al., 2015). The primary grades, specifically kindergarten and first grade utilize multiple visual models to deepen students' mathematical understandings, such as the number rack, ten-frame, dominos, unifix cubes, bundles and sticks, and the number line. The curricula support teachers' instruction to improve mathematical dialog, critical thinking and problem solving. Students were given multiple opportunities to use concrete manipulatives to represent solutions for given tasks. Each of the models were used to visually represent how to compose and decompose numbers, see the relationships between numbers, addition and subtraction, and discover place value through grouping strategies of tens and ones.

Authors from both curricula note the importance of developing algebraic thinking through flexibly solving a variety of equations. To promote algebraic thinking and the notion that the equal sign was more than just a signal to perform an operation, students play fact family games to determine the relationships between quantities. The teacher displays the dot card and tells them the total number of dots, but then only reveals part of the dot card. Students are encouraged to look at the dot patterns and make assumptions of the possible combinations based upon the particular view. Figure 4 shows an example of the game and the suggested dialog of the teacher.

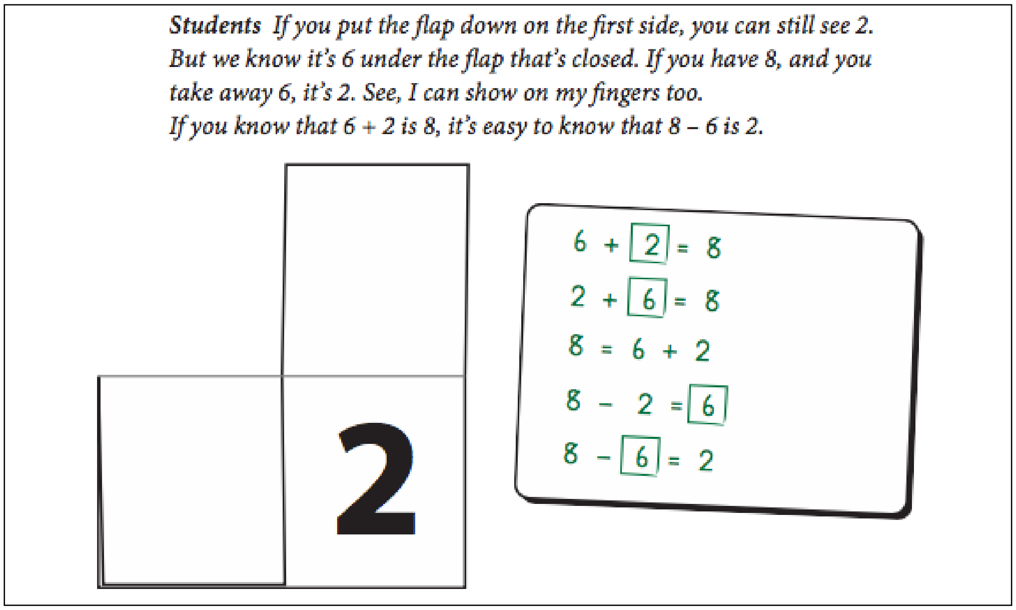

Figure 10. **Double-Flap Dot Card game to encourage algebraic thinking.**

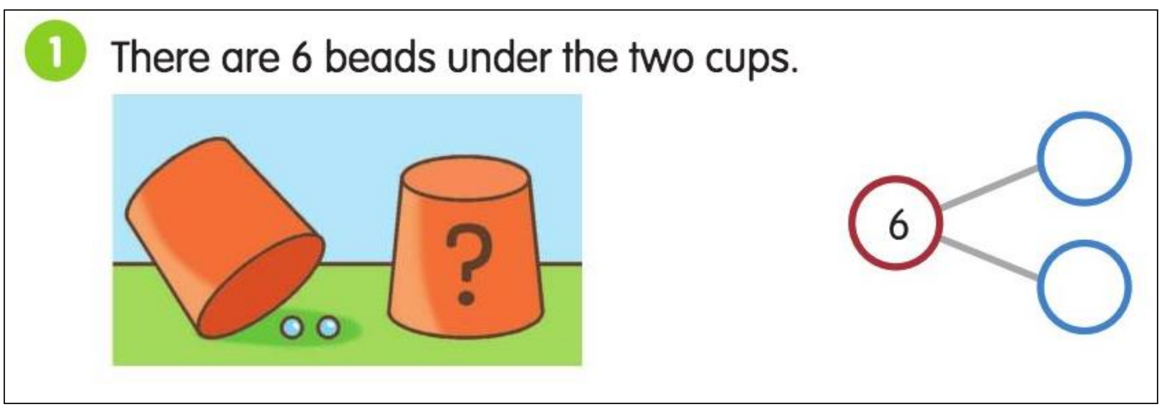

Figure 11. **Number bonds to encourage algebraic thinking.**

Many of the lessons focus on the use of concrete representations paired with the symbolic representations, however the iconic representations are pre-drawn, and in some instances the actual manipulative was photographed for students to refer back to. For example, students are asked to represent their thinking using a number rack (refer to figure four for example) when presented with equations. The textbook provides the teacher with a variety of possible student responses to the equation 2 + 4 = ___ using the number rack, and the corresponding equation. Students are instructed to use the number rack as their tool for solving the problems given by the teacher. Although this tool was helpful in creating visual representation, the iconic representation was neglected in the lesson.

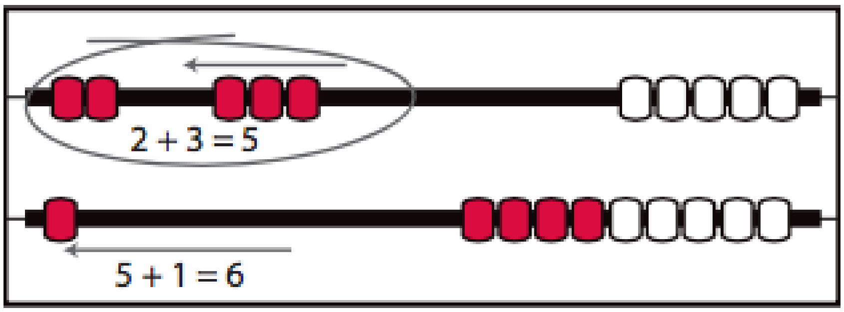

Figure 12. Number rack for solving various problem types.

As was commonly noted in many textbooks by Seo and Ginsburg (2003) and McNeil (2014), the equal sign was first introduced in the context of addition with the traditional form of a + b = c. For example, in one lesson the teacher poses a problem where students are expected to represent the following context symbolically “Sage has 2 green Popsicles in her left hand and 4 purple Popsicles in her right hand. How many Popsicles does she have in all?” When students observe number sentences such as this, they will most likely interpret the equal sign as a function to perform an action, and reinforce their operational view. See figure 3 for an example of the expected student work scripted in the teacher’s guide.

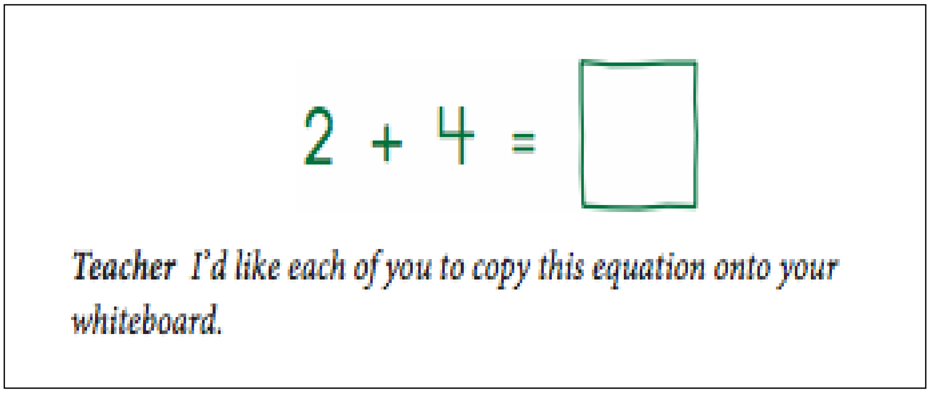

Figure 13. Student work in Bridges for Mathematics teacher's guide.

In other instances, the curriculum assumes that a pre-constructed model of unifix cubes will transfer as an iconic representation for solving a subtraction problem.

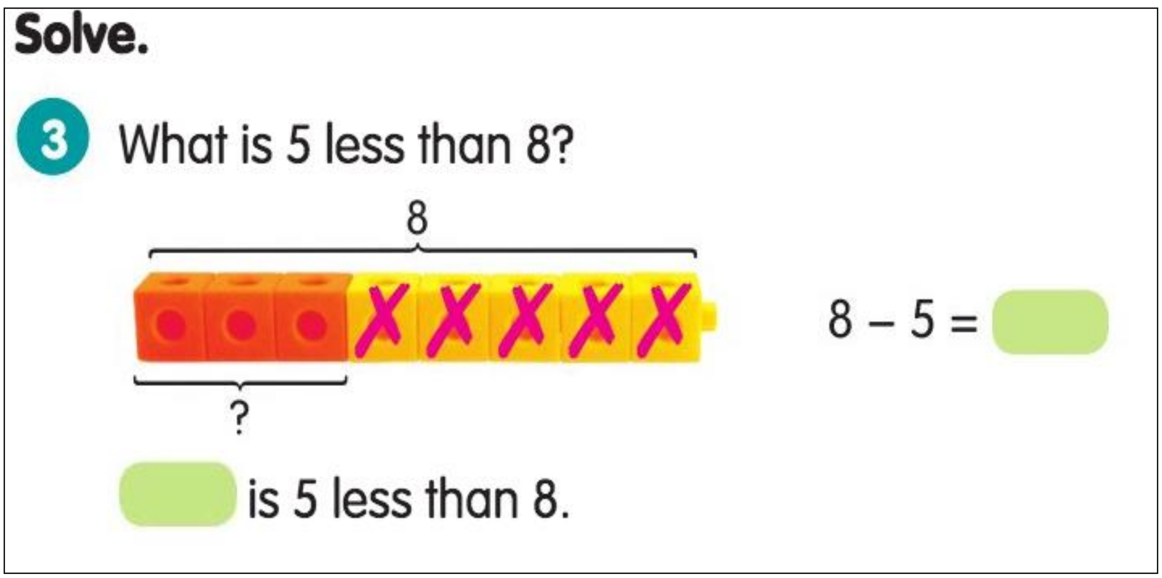

Figure 14. Iconic representation to develop knowledge of various problem types.

Both curricula heavily emphasize the use of visual representations as a tool for solving problems. Overall, each lesson strategically provides opportunities for students to engage in the enactive and symbolic modes of representation. However, with a few exceptions where a number line, hundred chart or unifix cubes are presented, the curriculum does not emphasize the use of students representing their own mathematical thinking through iconic modeling. Although geometry topics are included in the Bridges curriculum, the emphasis remains on identifying, describing and comparing the attributes of two-dimensional shapes. As Cheng and Mix (2014) and Fyfe et. al. (2014) revealed through their research, the need to connect relational thinking tasks and spatial reasoning is critical for the development of students' conceptual knowledge of the equal sign. Curriculum has shown to be a major contributor to students' misconceptions and lack of algebraic knowledge (Seo & Ginsburg, 2003; McNeil, 2014). The DMT curriculum was developed through a framework designed to promote relational thinking and spatial reasoning by including the opportunities for students to communicate their mathematical ideas by drawing their own iconic models.

Dependent Variable

The dependent variable of this study was the difference in scores from the Primary Mathematics Assessment (PMA) developed for grades kindergarten through second grade (Brendefur, Strother, & Thiede, 2012) to assess students' knowledge of mathematics with a short, comprehensive and predictive screener. The Primary Mathematics Assessment Screener (PMA-S) was administered at the beginning of the study in September as a pretest, and again in May as a posttest.

Primary Mathematics Assessment. The Primary Mathematics Assessment screener (Brendefur, 2012) is a formative assessment designed to test students' knowledge of mathematics with a short, comprehensive and predictive screener. The Primary Mathematics Assessment screener (PMA-S) builds a profile of students' strengths and weaknesses for 6 dimensions: number sense and sequencing, number facts, contextual problems, relational thinking, measurement, and spatial reasoning. As stated in the hypothesis, the study will focus specifically on relational thinking and spatial reasoning, although the other four dimensions will be highlighted in the analysis. Scores on the PMA-S are grouped into quintiles. Figure 15 is an example of a class report from the PMA-S administered in the fall to first grade students.

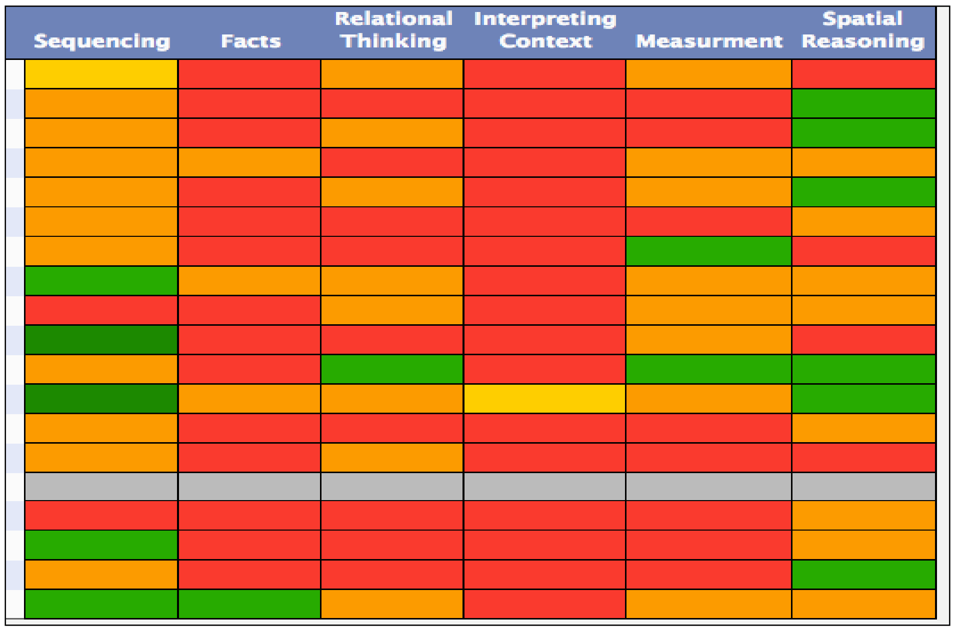

Figure 15. PMA-S class report for teacher.

The Primary Mathematics Assessment Screener (PMA-S) will be administered at the beginning of the study as a pretest, and then again as a posttest at the end of the study. Spacing of the pretest and posttest will allow adequate time for the EIS and Traditional groups to teach the four modules or units, and then assess to determine student growth.

The PMA-S has been thoroughly examined to ensure reliability, validity, and security with a series of pilot studies, review and psychometric analysis (Siebert & Brendefur, 2018). The subscale questions were written to include Webb's (2002) first three depths of knowledge categories: recall, skill/concept, and strategic thinking. The readability of the test questions was rigorously field tested by trained test administrators with teaching experience in kindergarten, first and second grade. Internal consistency was evaluated for the psychometric properties of the PMA giving a Cronbach's alpha between .80 or above for the kindergarten and first grade assessment. Each item within the six dimensions was tested using Rasch analysis to verify that within each of the dimensions, the items were testing similar ideas.

In this study, students were given the PMA-S as the pretest to all of their students in the first month of school, and again in early spring as the post-test. Spatial reasoning and relational thinking have shown to be the most predictive of later success in students' mathematical performance. Therefore, it is necessary to further explain the dimensions of spatial reasoning and relational thinking.

Relational Thinking. The screener includes a series of questions for greater than or less than, quantitative sameness, identifying missing parts in a bar model, open number sentence, and true/false number sentences. Each of the questions are explained in the subsequent sections.

Greater than, less than. This section aims to determine students' understanding of number magnitude and place value. The questions are written in symbolic notation with a prompt directing students to determine which number is bigger or smaller. For example, one prompt asks students to view three numbers "66, 59, 61" on the screen and then choose which of the numbers is more.

66	59	61

Figure 16. Sample greater than, less than question of the PMA-S.

Quantitative sameness. This section of the assessment has two different images for students to view to determine quantitative sameness. The first image has a two dimensional picture of eggs separated into two different baskets where students are prompted to count the eggs in each basket, and next students tell how many more need to be added so that each basket has the same quantity. The other image consists of dot patterns separated by a line where students are prompted to count the dots, and then determine if each side has the same amount.

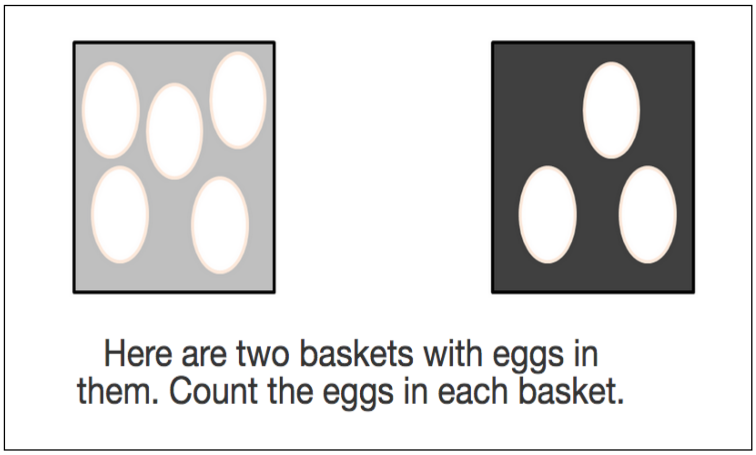

Figure 17. Sample quantitative sameness question from the PMA-S.

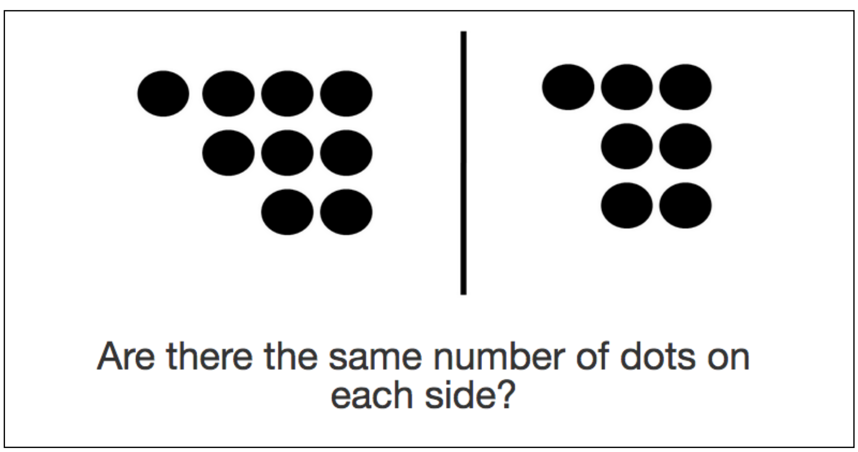

Figure 18. Sample quantitative sameness question from the PMA-S.

Bar model and missing part. The goal of this section was to see how well students understand the concept of part-part whole represented by the bar model. Each question has a number represented in the top bar model with two parts spatially drawn to show the relative size of each part. Students are prompted to determine the number in each of the spaces marked with a question mark.

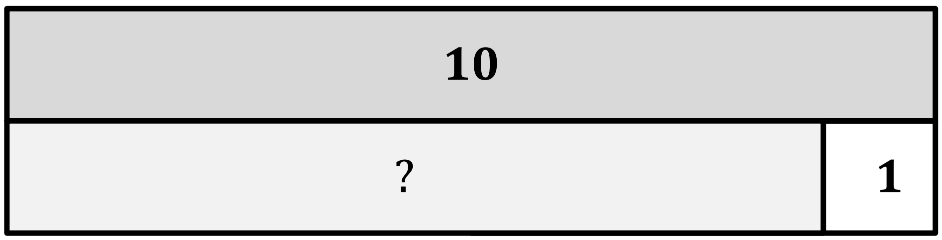

Figure 19. Sample bar model and missing part question from the PMA-S.

Open number sentences. Similar to the quantitative sameness section, however, the open number sentences are written in symbolic form. Students are asked to think of a number for the box so that each side of the equation computes to the same amount.

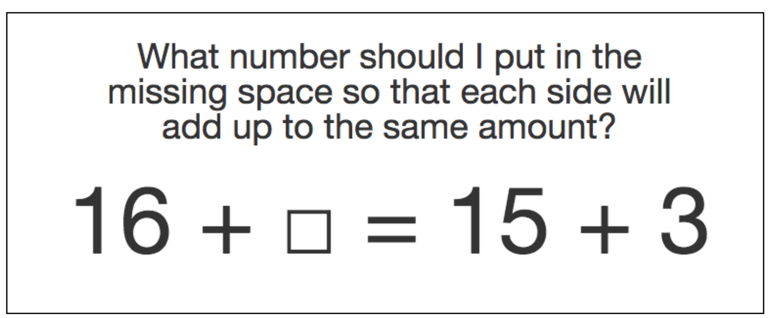

Figure 20. Sample open number sentences question from the PMA-S.

True/False number sentences. This section is also presented in symbolic form where students look at the given number sentence and determine if the numbers compute to be equal or not. Students are asked to use the terms true or false when giving an answer for each number sentence.

$$\boxed{41 + 8 = 42 + 9}$$

Figure 21. Sample true/false number sentences question from the PMA-S.

Spatial Reasoning. The diagnostic includes a series of questions for shape composition. There are three subsections which include—shape composition without the need to rotate, composing a figure requiring overlapping of pieces during translations, and composing a figure by filling in a missing space. Each of the questions is explained in the subsequent sections.

Composing a shape without the need to rotate. The diagnostic for spatial reasoning begins by asking students to look at three shapes and determine whether those three shapes can be used to compose a new shape. Figure 7 shows an example of one question from the PMA-D spatial reasoning assessment. Figure 8 shows an example of one question asking students if the given pieces can be used to compose the given shape.

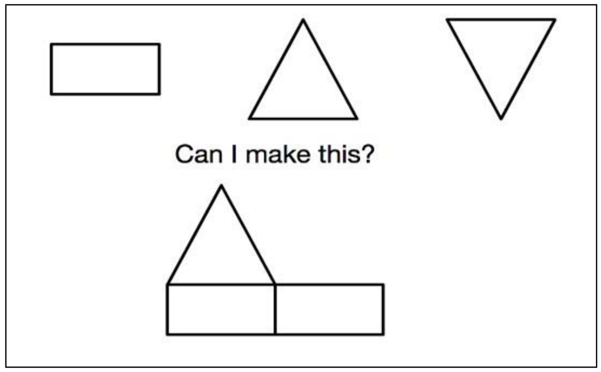

Figure 22. **Sample shape composition question for the PMA-S.**

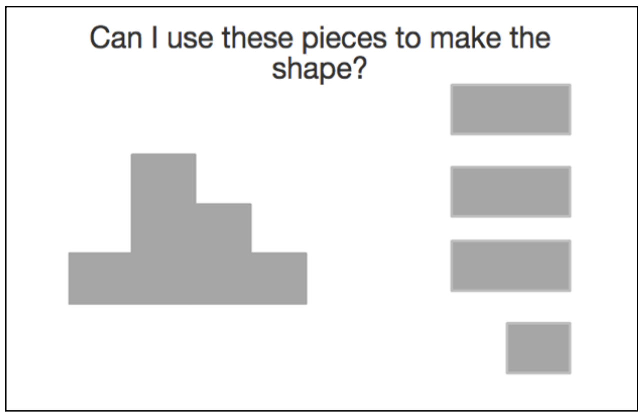

Figure 23. **Sample shape composition question for the PMA-S.**

Composing a figure requiring overlap during translation. The second section asks students to determine which of the given shapes can be used to compose a given figure.

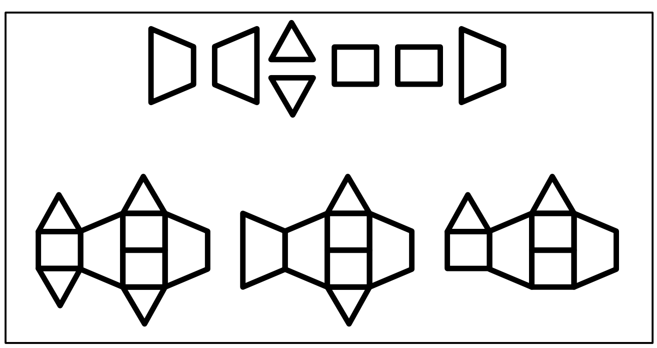

Figure 24. **Sample transformation question for the PMA-S.**

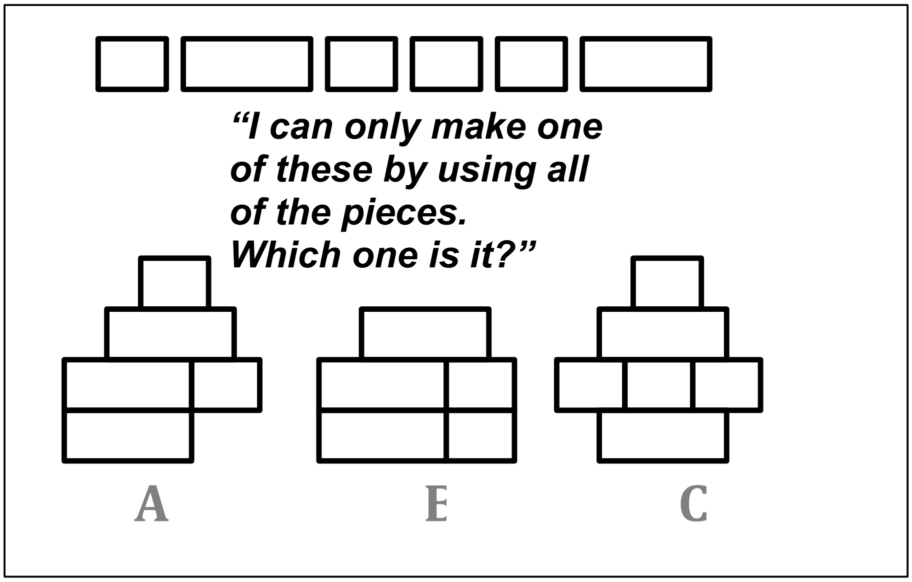

Figure 25. **Sample transformation question for the PMA-S.**

Composing a figure within a missing space. The third section asks students to decide whether the given shapes can fill the empty space.

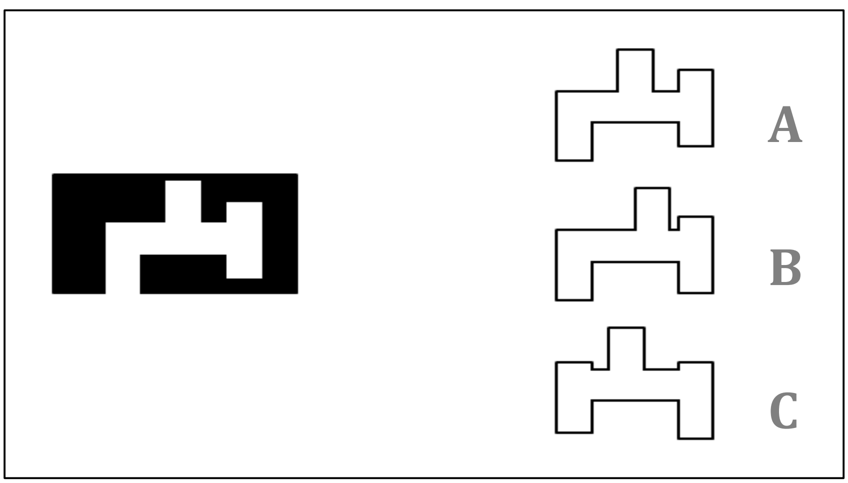

Figure 26. **Sample transformation question for the PMA-S.**

Timeline and Data Collection Procedures

<u>Data Collection</u>

Following district approval, expedited IRB approval was granted under #108-SB16-128. This study took place over a nine-month period – from September through

May. Data for both groups was collected in September with the pre-test and May with the post-test. Students in the EIS group were instructed with the same unit modules developed by the DMT researcher (Brendefur, 2016). Students in the comparison group were instructed with district adopted curriculum aligned with the Common Core Standards. In both groups, lessons were taught every day, with each lesson lasting approximately forty-five minutes in length to students in the general education mathematics class.

The PMA-S was administered to students in both in September before any mathematics lessons were taught, and mid-May at the conclusion of the mathematics lessons. The PMA-S was administered on laptops by trained proctors to students in both groups. Discussion about the fidelity of the testing procedures follows.

Internal Validity

Multiple considerations were taken to maintain fidelity and avoid possible threats to the validity of the study. Scores from the PMA-S were collected in the fall and spring. Trained proctors administered the PMA-S. Teachers were trained on how to implement the unit modules. Lastly, observations, note taking, and email correspondence were conducted throughout the study.

Next, careful consideration was taken to ensure all students were present for each of the lessons. Teachers with students who received special education or other services were mindful of the timing of their mathematics lessons to not interfere with those outside services. All students in the EIS group were instructed using all eight modules, lasting approximately three to four weeks each, for the duration of the study.

Additionally, the teachers in the EIS group were part of a small training session, which included ways to develop relational thinking and spatial reasoning through the use of the modules. Teachers were provided with ways to supply multiple opportunities for students to build the necessary conceptual understanding of relational thinking and spatial reasoning through multiple representations including enactive, iconic and symbolic models. They were also given ways to present many different problem-solving situations for students to draw upon the modes of representation, and flexibly work through situations that involved spatial reasoning and relational thinking. The varied practice in each module provided students situations for all modes of representation to be utilized in problem solving situations. In addition to the modules, the teachers and the researcher met to discuss any questions regarding the lesson, preparation of necessary materials, and pacing for each lesson.

Lastly, to ensure consistency during each test, independently trained individuals proctored the PMA-S. For testing, each student was taken to a quiet area within the school to eliminate as many distractions as possible. Total testing time for each participant took approximately eight minutes. Tests were read aloud from a laptop computer to each student, with the proctor marking the given answer on the test to avoid accidental miscues.

Data Analysis

All data analyses were conducted using SPSS Version 24 (IBM, 2016) and/or Microsoft Excel for Mac Version 15 (2016). Scores from the PMA-S (DMT, Brendefur, 2015) database for pre-and posttest were provided as a spreadsheet. All student data files were transmitted via email to researcher’s university email account that is password protected. Data were sorted, analyzed, and stored on the researcher's password protected computer.

Student names and scores were compared between the pre-and posttest to ensure accurate reporting of students' having taken both pre-and posttest for total score and within the six subset dimensions. Any incomplete data sets in the pre-and posttest were removed. The data were then merged into one single spreadsheet file.

A two-way design was used to explore the main effects on the different treatments, EIS instruction and Traditional instruction, and their interactions under different conditions, pretest and posttest. The research question, "What is the effect of integrating iconic representation through student drawings in conjunction with the enactive, iconic and symbolic teaching methodology into mathematics instruction on first grade students' relational thinking and spatial reasoning performance?" was analyzed using a 2 x 2 analysis of variance (ANOVA) to explore whether scores on the pre-and posttest was dependent upon the type of instruction.

Repeated measures analysis of variance (ANOVA) allows a look at change over time using the PMA-S given two times over nine months of instruction with different conditions (EIS and Traditional instruction). Main effects and interactions were analyzed on the independent variables (EIS and Traditional instruction and time) from the dependent variable PMA-S scores. There are six assumptions that must be met concerning the statistical population of the data set in order to use the repeated measures ANOVA. The first three are related to the study design, while the other three relate to how the data fits the model. The dependent variable must be measured at a continuous level. The scores on the PMA-S range from 0-3, thus meeting the first assumption. One of the independent variables must be categorical, while the other independent variable must reflect two or more time slots. The first grade students in the study were taught with one of two types of instruction, EIS or traditional. Both groups were given a pre-test prior to instruction, and again as a post-test at the end of instruction. The third assumption requires that any outliers be removed from the data because they distort the differences, and cause problems when generalizing the results of the sample to the population. Data found to be +/-3 standard deviations from the mean score are considered to be outliers, however, the data remained within the confines of +/-3 standard deviations. The fourth assumption is that of confounding data, where an outside variable influences the EIS and Traditional groups and the scores on the PMA-S, causing interpretation of the data to make false conclusions. There were no other mathematics curriculum or external resources used in either the EIS or the Traditional groups. The fifth assumption is that of normality. This assumes the averages for Relational Thinking and Spatial Reasoning on the PMA-S are normally distributed with the pre-and posttest. According to Kesselman (1998) analysis such as this are generally robust to the violation of normality, and can be overlooked with a larger sample size such as this study. The final assumption is the homogeneity of variance. This assumes the squared standard deviations of both the EIS and Traditional groups are clustered near the mean, and that their variances are equal. For Relational Thinking homogeneity of variance was met ($p > .05$, posttest $p = .889$) except for at the beginning of the study ($p < .001$), as assessed by the Levene's Test of homogeneity of variance, which can also be violated with large, equal sized groups (Maxwell & Delaney, 2003).

Table 5: **Tests of Normality for Relational Thinking**

Tests of Normality						
	Levene's homogeneity of Variance			Shapiro-Wilk		
	Statistic	df	Sig.	Statistic	df	Sig.
Pretest	29.5	449	.000	.821	451	.000
Posttest	.02	449	.889	.862	451	.000

Summary

In summary, the design of the study was to understand whether first grade students' relational thinking and spatial reasoning changed over time when given either EIS or Traditional mathematics instruction. Both groups were tested using the Primary Mathematics Assessment Screener (PMA-S; Brendefur, 2012) in September, prior to the mathematics instruction, and again mid-May after the mathematics instruction. An analysis was used to understand the interaction between the two independent variables, pre-and posttest (time), and EIS or traditional instruction (group) on the dependent variable, scores from the PMA-S. Teachers in the EIS group implemented eight DMT modules to progressively press students' informal strategies and models to a more formal way of conceptual understanding using Bruner's EIS progression of representation. Teachers in the Traditional group implemented district adopted curricula aligned with Common Core Standards. The next chapter will explain the results from PMA-S given the implementation of both independent variables.

CHAPTER 4: RESULTS

Overview of the Study

The study was conducted to investigate whether there was a significant difference in first grade students' performance in relational thinking and spatial reasoning when they learn to construct and compare numbers using iconic modeling. The study compared relational thinking and spatial reasoning for first grade students whose teacher received a curriculum designed to increase use of enactive, iconic, symbolic representation in teaching (EIS group) and those whose teacher received district adopted curriculum and taught in a more traditional manner (Traditional group). Both groups were tested using the Primary Mathematics Assessment Screener (PMA-S; Brendefur, 2012) in September, prior to the mathematics instruction, and again mid-May after the mathematics instruction; therefore, student performance was also compared across time. Thus, this study used a 2 (EIS group versus Comparison group) x 2 (pretest versus posttest) design. The dependent variable was the students' knowledge of relational thinking and spatial reasoning measured with the PMA-S. The goal of this study was to determine whether student achievement on the PMA-S differed between the EIS and Traditional groups, and whether achievement differed across time.

The study attempted to answer the following question and null hypothesis:

1: What is the effect of integrating iconic representation through student drawings in conjunction with the enactive, iconic and symbolic teaching methodology into mathematics instruction on first grade students’ relational thinking and spatial reasoning performance?

*H*1: There is a positive effect on integrating enactive, iconic and symbolic teaching methodology into mathematics instruction on first grade students’ relational thinking and spatial reasoning performance.

$H_0$1: There is not a positive effect on integrating iconic teaching methodology into mathematics instruction on first grade students’ relational thinking and spatial reasoning performance.

Data Analyses

The analysis compared the mean differences between groups split on two independent variables, EIS/Traditional instruction and Time. The purpose was to understand if there was an interaction between the two independent variables, EIS/Traditional and Time, on the dependent variable, PMA-S. The analysis was used to determine whether there were differences between the groups over time. The primary purpose of carrying out this analysis was to understand if there was a two-way interaction between the EIS/Traditional groups and the pre-and posttest. The goal of the study was to understand if first grade students’ relational thinking and spatial reasoning knowledge changed over time when given EIS or traditional mathematics instruction and how their knowledge changed over time. Understanding this requires the analysis of the two-way interaction effect. The analysis allows the researcher to distinguish between the effects of different types of instruction over time.

PMA-S Scores

Data from the Relational Thinking and Spatial Reasoning PMA-S scores were analyzed using a repeated measures analysis of variance (ANOVA) to determine whether there was a significant difference in growth between the EIS and the Traditional groups over nine months of instruction. The EIS group had 208 scores, and the Traditional had 243 scores. Students in the EIS and Traditional groups stayed in their same classroom for the duration of the study, thus meeting the assumption of independence to be met.

A two way repeated measures ANOVA was conducted to determine whether there was a significant difference in growth between the EIS group and the Traditional group for relational thinking and spatial reasoning. The PMA-S screened four other subset dimensions, Facts, Context, Sequence, and Measurement which were not included in the design of the study. The results of these other four dimensions provide a deeper understanding of students' mathematical understandings, therefore, follow up analyses are presented after Relational Thinking and Spatial Reasoning.

Relational Thinking

For the Relational Thinking subtest, there was a main effect for TIME with a statistically significant difference for both groups (EIS and Traditional)—scores increase from pretest to posttest, $F(1, 449) = 105.2$, $MSe = .9$, $p < .001$. There is also a main effect for Group with a statistically significant difference between EIS and Traditional, $F(1, 449) = 5.6$, $MSe = 1.2$, $p = .019$.

There was a statistically significant interaction between both groups and time on relational thinking, $F(1, 449) = 13.2$, $MSe = .9$, $p < .001$, $\eta^2 = .03$. This indicates the difference of change in students' knowledge of relational thinking in the EIS and

Traditional group was dependent upon the type of mathematical instruction. Based on the profile plots of estimated marginal means of Relational Thinking in Figure 28, EIS (group 1) and Traditional (group 2), EIS and Traditional groups' trajectory indicate different patterns of mean scores over time. The p-value for the two-way interaction effect is < .001, indicating mean Relational Thinking changed differently over time depending on whether students were in EIS or Traditional.

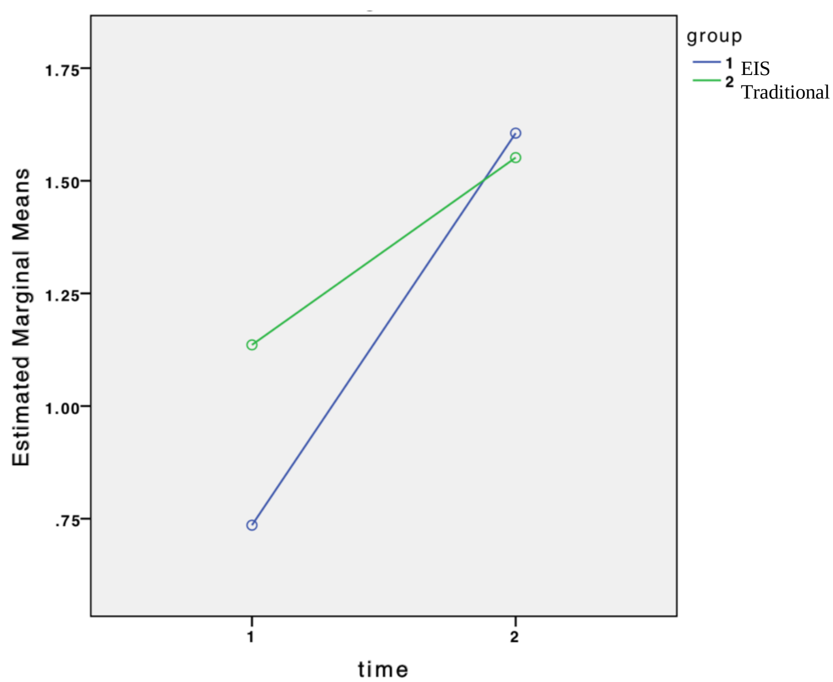

Figure 27. **Estimated marginal means of Relational Thinking**

To better understand the interaction, tests of simple effects were conducted. These results showed for the EIS group, scores on the Relational Thinking scale increased significantly from pretest to posttest, $t(242) = 10.2$, $p < .001$. For the Traditional group, scores on the Relational Thinking scale also increased significantly from pretest to posttest, $t(242) = 4.6$, $p < .001$. Thus, for both groups, scores increased from pretest to posttest. The EIS and Traditional groups were also compared separately on the pretest and then the posttest. These results showed that for the pretest, the groups differed significantly, t(449) = 4.5, $p < .001$. For the posttest, the groups were not significantly different, t(449) = .53, p = .6. For the pretest scores were greater for the Traditional group than for the EIS group.

Taken all together the results of these analyses show that scores on the Relational Thinking subtest scores did not differ across groups. However, significant interaction suggests that the change from pretest to posttest was not the same for the two groups. As seen in Table 6, the change was greater for the EIS group than for the Traditional group. Here, the EIS group began the study with significantly lower scores on the Relational Thinking subtests, coupled with a greater posttest score confirming EIS has a positive effect.

Table 6: Relational Thinking Descriptive statistics.

Relational Thinking				
	Pretest		Posttest	
Group	Mean	SD	Mean	SD
EIS	.74	.77	1.61	1.2
Traditional	1.14	1.1	1.55	1.1

Spatial Reasoning

For the Spatial Reasoning subtest, there was a main effect for TIME with a statistically significant difference for both groups (EIS and Traditional)—scores increase from pretest to posttest, F(1, 449) = 85.2, MSe = .6, $p < .001$. There is also a main effect for Group with a statistically significant difference between EIS and Traditional, $F(1, 449) = 3.9$, $MSe = .9$, $p = .05$.

There was a marginal significant interaction between both groups and time on relational thinking, $F(1, 449) = 3.3$, $MSe = .6$, $p < .071$, $\eta^2 = .01$. This indicates the difference of change in students' knowledge of spatial reasoning in the EIS and Traditional group was dependent upon the type of mathematical instruction. Based on the profile plots of estimated marginal means of Spatial Reasoning, EIS and Traditional groups' trajectory indicate slightly different patterns of mean scores over time.

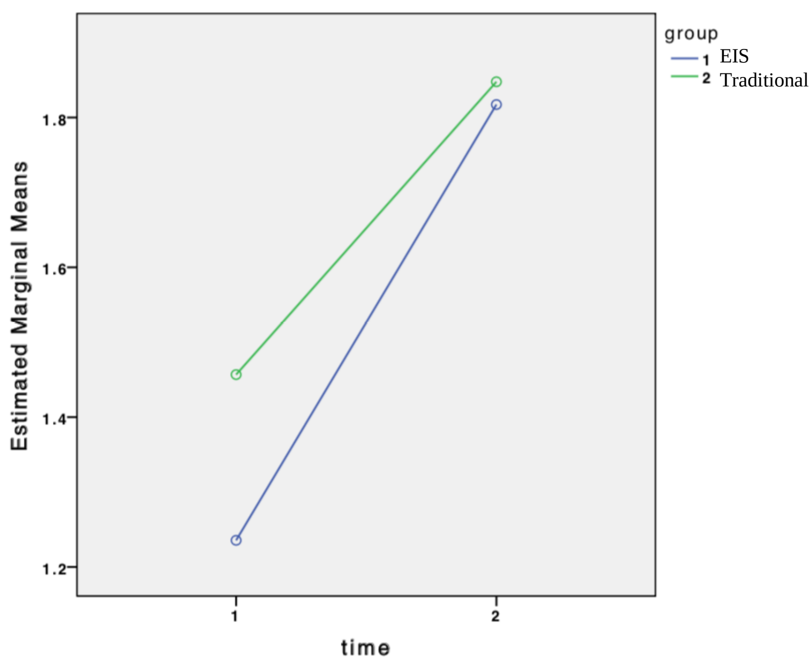

Figure 28. Estimated marginal means of Spatial Reasoning

To better understand the interaction, tests of simple effects were conducted. These results showed for the EIS group, scores on the Spatial Reasoning scale increased significantly from pretest to posttest, $t(207) = 7.4$, $p < .001$. For the Traditional group, scores on the Spatial Reasoning scale also increased significantly from pretest to posttest, *t*(242) = 5.5, p < .001. Thus, for both groups, scores increased from pretest to posttest. The EIS and Traditional groups were also compared separately on the pretest and then the posttest. These results showed that for the pretest, the groups differed significantly, *t*(449) = 2.8, p < .01. For the posttest, the groups were not significantly different, *t*(449) = .36, p = .72. For the pretest, scores were greater for the Traditional than for the EIS group, and on the posttest, scores were the same across both groups.

Taken all together the results of these analyses show that scores on the Spatial Reasoning subtest were equal on the posttest across both groups. However, the marginally significant interaction suggests that the change from pretest to posttest was not the same for the two groups. As seen in Table 7, the change was greater for the EIS group than for the Traditional group. The EIS group began the study with significantly lower scores on the Spatial Reasoning subtests. The EIS group shows statistically higher gains than the Traditional, thus confirming EIS has an effect.

Table 7: Spatial Reasoning Descriptive statistics.

Spatial Reasoning				
	Pretest		Posttest	
Group	Mean	SD	Mean	SD
EIS	1.24	.803	1.82	.871
Traditional	1.46	.905	1.85	.912

Summary

In summarizing the data, the instructional method (EIS vs. Traditional) did have a significant effect on first grade students' relational thinking and spatial reasoning. The study demonstrated statistical significance between the treatment group who implemented the EIS instruction and comparison group who used traditional mathematics instruction. The next chapter will provide details of the interpretation of findings, practical implications for educators, and recommendations for further study.

CHAPTER 5: DISCUSSION, CONCLUSIONS, AND RECOMMENDATIONS

Summary

Purpose

The purpose of this study was to investigate whether there was a significant difference in first grade students' performance in relational thinking and spatial reasoning when they learn to construct and compare numbers using iconic modeling. The study compared relational thinking and spatial reasoning for first grade students whose teacher received professional development to increase use of enactive, iconic, symbolic representation in teaching (EIS group) and those whose teacher received no professional development and taught in a more traditional manner (Traditional group). Both groups were tested using the Primary Mathematics Assessment Screener (PMA-S; Brendefur, 2012) in September, prior to the mathematics instruction, and again mid-May after the mathematics instruction; therefore, student performance was also compared across time. Thus, this study used a 2 (EIS group versus Comparison group) x 2 (pretest versus posttest) design.

To address the primary purpose of this research study, the following research question and hypothesis was investigated:

1. What is the effect of integrating iconic representation through student drawings in conjunction with the enactive, iconic and symbolic teaching methodology into mathematics instruction on first grade students' relational thinking and spatial reasoning performance?

*H*1: There is a positive effect on integrating enactive, iconic and symbolic teaching methodology into mathematics instruction on first grade students' relational thinking and spatial reasoning performance.

$H_0$1: There is not a positive effect on integrating iconic teaching methodology into mathematics instruction on first grade students' relational thinking and spatial reasoning performance.

The dependent variable was the students' knowledge of relational thinking and spatial reasoning measured with the PMA-S. The goal of this study was to determine whether student achievement on the PMA-S differed between the EIS and Traditional groups, and whether achievement differed across time.

Connection Back to Literature Review

The review of literature aimed to understand more deeply the relationship among relational thinking, spatial reasoning, mathematical models, and their effect on students' conceptual understandings in early elementary school mathematics. Often times, instructional practices are centered around lessons which strengthen an operational view of the equal sign. As students continue to formulate ideas about the equal sign over the course of their elementary years, the ability to reverse the entrenched ideas becomes much more challenging (Chesney & McNeil, 2014). Simple arithmetic problems in elementary school promote operational thinking, often times making it difficult for students to generalize beyond the given problem. Altering the elementary school curriculum with a relational view of the equal sign can build students schema and improve their mathematical performance (McNeil & Alibali, 2005). Therefore, k-12 reform has included an integration of meaningful lessons designed to enhance algebraic thinking into the primary years of school across all mathematical domains, pressing students to use critical thinking (Kaput, 2000). Much of the mathematical instruction is limited to the use of connecting student understanding with the equal sign in a very abstract, symbolic way.

Mathematical modeling gives students a visual representation to explain their mathematical thinking (Erbas et al., 2014). The use of mathematical modeling should connect through a progression of concrete, visual representation to an iconic model and then to the formal, abstract symbols of an equation (Fyfe et al., 2015).

A review of literature exposed how curriculum should support students' conceptual understandings through the integration of relational thinking, spatial reasoning and mathematical models by incorporating Bruner's EIS framework. There is currently a lack of research into the effects of integrating the iconic representation into mathematics on students' relational thinking and spatial reasoning.

Design of Study

This study used a two-way repeated measures analysis to find the effect of two different independent variables (EIS representation versus Traditional instruction) measured by the dependent variable (PMA-S) given once in September and again in May. Analysis of variance (ANOVA) is a statistical model used to explain the effects of different components and make predictions about the behavior of the statistics. The observed variance within the dependent variables is partitioned among the various groups to determine if the means of several groups are significantly different or equally distributed from the mean. The two-way ANOVA analyzes two independent variables under two different conditions. A repeated measures looks at change over time with different conditions. Therefore, the two-way repeated measures ANOVA is used to determine the main effects for both independent variables and their interactions across two different conditions.

Interpretation of Findings

The primary focus of the study was to look into the effects that integrating the EIS representation into first grade mathematics lessons had on students' conceptual understandings of relational thinking and spatial reasoning. Data from the PMA-S pretests and posttest were analyzed across the six dimensions of mathematical competency—Relational Thinking, Spatial Reasoning, Facts, Sequence, Context, and Measurement.

The research question "What is the effect of integrating iconic teaching methodology into mathematics instruction on first grade students' relational thinking and spatial reasoning performance?" was analyzed with a 2x2 repeated measures ANOVA to determine whether there was a significant difference in pre-and posttest scores for the two groups (EIS representation and Traditional instruction). The result of this test was separated into its six subsets.

6 Dimensions

For the six subtests—Relational Thinking, Spatial Reasoning, Facts, Context, Sequence, and Measurement, the scores increase from pretest to posttest ($p < .001$), which suggests there was an interaction between group and time (Facts, $p < .001$; Context, $p = .07$; Sequence, $p < .001$; Measurement, $p = .075$; Relational Thinking, $p < .001$; Spatial Reasoning, $p = .071$). The simple effects of the EIS (pre-and posttest) and Traditional (pre-and posttest) showed significant effects for both EIS time and

Traditional time (Facts, $p < .001$; Context, $p = .03$; Sequence, $p = .12$; Measurement, $p = .03$; Relational Thinking, $p < .001$; Spatial Reasoning, $p = .72$). When comparing mean scores from pretest to posttest, there is very little difference in the posttest means for all dimensions. The Traditional group starts out higher than the EIS group in pretest scores, but the posttest EIS had a larger effect than the Traditional with significantly higher gains shown in Table 13.

Table 8: Dimensions Gain Scores from pretest to posttest.

6 Dimensions Posttest Mean Scores		
Group	EIS	Traditional
Relational Thinking	1.27	.89
Spatial Reasoning	.58	.39
Facts	1.02	.64
Context	.66	.51
Sequence	1.26	0.8
Measurement	.51	.33

The EIS group performed statistically higher in Relational Thinking than the Traditional group, doubling mean scores from pretest (.74) to posttest (1.27). Previous work has shown students who are instructed to solve equations strictly in symbolic form struggle with algebraic thinking (Falkner et al., 1999; Seo & Ginsburg, 2003). Integrating EIS representation into first grade mathematics lessons with a balanced set of equations has shown to be effective at developing students relational thinking.

As Cheng and Mix (2014) revealed through their research, the need to integrate spatial reasoning tasks is critical for the development of students' conceptual knowledge.

Similar claims can be made based off results from this study. The EIS group started lower on the pretest, yet had higher gains in Spatial Reasoning (.58) and Measurement (.51) than the Traditional group. This study concluded the integration of spatial reasoning had positive effects on first grade students' spatial reasoning skills, the development of conceptual understanding, and mathematical competency.

The Findings for Facts and Context support the notion that the integration of EIS representation into mathematics lessons offers students sufficient conceptual knowledge to develop number operations and mathematical competency (Stephens et al., 2015). Gain scores in Facts and Context are found to be consistent with earlier works from Carbonneau, et al. (2013), who suggests mathematics instruction should refrain from isolated skill and procedural practice in lieu of the development of conceptual understanding. Instruction designed to include a progression of enactive, iconic, and symbolic form of an equation support conceptual understanding (Bruner, 1966; Fyfe et. al., 2014; Gravemeijer, 2003). Students in the EIS group were instructed to enactively build and iconically represent their math facts simultaneously. In doing so, they increased conceptual understandings of the mathematics occurring quickly solve math facts. In addition to recalling facts, fluency with first grade math facts helped students solve context questions with ease.

With scores from the pretest being heavily skewed, one possible rival explanation could be results occurred only by happenstance. Given scores of the EIS group were considerably low from the start of the study, any sort of mathematics instruction would have had a positive effect. However, K-12 reform has included an integration of meaningful lessons designed to enhance algebraic thinking across all mathematical domains, and altering the curriculum to include relational thinking and spatial reasoning tasks has shown to improve mathematical performance (Kaput, 2000; McNeil & Alibali, 2005). This study has shown first grade mathematics lessons designed to integrate EIS representation have a positive effect on students' relational thinking, spatial reasoning, and overall mathematical competency.

Implications for Social Change

It has been shown that students as young as kindergarten and first grade have informal knowledge of number relations, however, the mathematics presented in traditional textbooks do not explicitly draw out these relations, allow time for the relations to organically emerge, or instruct students to determine how the ideas can be generalized (Blanton & Kaput, 2005). Consequently, there is a need for mathematics instruction to incorporate more than just the traditional format of equations into daily lessons, and include ways to represent relational equivalence (Ellis, 2011; Molina et al., 2005). The DMT modules offer practical instruction, well aligned with the current body of research in support of integration of relational thinking, spatial reasoning and the progression of EIS representation.

Results from the study support an immediate need for other school districts to advocate the need for learners to have access to a curriculum, which supports relational thinking, spatial reasoning, and the progression of EIS representation for mathematics instruction. The dissemination of the findings is vitally important to share with mathematics educators, administration, and school district officials. A collaborative effort by members of the mathematics community must be made aware of the effect of implementing accessible mathematics instruction to all learners. This can be achieved by the ongoing professional development and instruction provided by our local universities and state department.

Recommendations for Further Study

More research of sustained instruction integrating relational thinking, spatial reasoning, and EIS representation is needed (Blanton et.al, 2015). This study served as a preliminary springboard for more research to be conducted across elementary grade levels to determine the longitudinal effects from a whole school implementation. The current body of literature would benefit from a longitudinal analysis to determine the long-term effects on students' conceptual understandings of equivalence with instructional practices integrating relational thinking, spatial reasoning, and visual models for algebraic reasoning and computational. Implementation of such instruction assumes it will in turn effect students' overall success in the later years of schooling, which require deep understanding of algebraic concepts (Britt & Irwin, 2008). Further research is necessary to determine the effects of sustained instructional practices included in this study over the course for the duration of students' elementary school experience. Based on the conclusions of this study, further research on the integration of EIS teaching should include:

1. How does the implementation of EIS representation effect at-risk students' spatial reasoning and relational thinking skills, and overall mathematical competency?

2. What is the effect of integrating iconic representation through student drawings in conjunction with the enactive, iconic and symbolic teaching methodology into mathematics instruction on fifth grade students' mathematical performance?

Conclusion

The purpose of the study was to investigate the effectiveness of a curriculum focused on the integration of Bruner's (1966) EIS progression into first grade mathematics lessons, and the change in students' conceptual understandings of relational thinking and spatial reasoning. Findings from this research strongly support the integration of such lessons, not only to improve relational thinking and spatial reasoning, but also develop an overall level of mathematical competency including fact fluency, sequence of numbers, measurement, and contextual problems. Students have a better opportunity to develop conceptual understanding of relational thinking and spatial reasoning in their early years of school when instructed with the progression of enactive, iconic, and symbolic representation.

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