

# Explicit Integration of some Integrable Systems of Classical Mechanics

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# Chapter 1

## Introduction

The thesis is devoted to the analytical and geometrical study of some integrable finite-dimensional dynamical systems of classical mechanics, namely, the classical generalization of the Euler top: the Zhukovski–Volterra system describing the free motion of a gyrostat, the Steklov–Lyapunov integrable case of the Kirchhoff equation and generalization of Steklov–Lyapunov system Rubanovskii system.

Studying integrable systems of classical mechanics had been a principal task in the area of dynamical systems in XIX-th century. Moreover, during the last decades a considerable progress in this direction was made due to the relation to some nonlinear partial differential equations and discovery of new methods of integration, in particular, the Lax representation approach.

However, some little-known integrable problems, which may have important applications, remained practically without consideration, among them the gyroscopic generalization of the Euler and the Steklov–Lyapunov systems. It appears that the methods that allow to integrate the original systems are not directly applicable to these generalizations, and some non-trivial modifications or changes of variables are required.

The main objective of the thesis is twofold. First, to perform explicit integration of these systems by means of their reduction, separation of variables, and inverting the quadratures. Second, to give a description of bifurcation diagram of the Zhukovski–Volterra system, the Steklov–Lyapunov integrable case of the Kirchhoff equation and its generalization - Rubanovskii system.

## 1.1 Objectives of the Thesis

### 1.1.1 Explicit solution of the Zhukovsky–Volterra gyrostat

The first objective is the explicit integration and a qualitative study of behavior of the classical generalization of the Euler top: the system describing the free motion of a gyrostat: a rigid body carrying inside a symmetric rotator whose axis is fixed in the body. As was shown by N.E.Zhukovsky [64] and, independently, by V. Volterra [58], the system can be reduced to the following equations describing the evolution of the total angular momentum vector  $M \in \mathbb{R}^3$ :

$$\dot{M} = M \times (aM - g), \quad M = (M_1, M_2, M_3)^T, \quad (1.1)$$

where  $g \in \mathbb{R}^3$  is a constant vector characterizing the axial angular momentum of the rotator. In the case  $g = 0$ , these equation reduce to the classical Euler top problem.

The motion of the gyrostat in space is then described by solutions of the Poisson equations

$$\dot{\gamma} = \gamma \times \omega(t), \quad \text{where} \quad \omega = aM(t) - g,$$

$\omega$  being the angular velocity vector.

Like the Euler top, the system (1.1) has two quadratic integrals, which, however, are not all homogeneous in  $M_i$ ,

$$f_1(M) = M_1^2 + M_2^2 + M_3^2 = k, \quad (1.2)$$

$$f_2(M) = a_1 M_1^2 + a_2 M_2^2 + a_3 M_3^2 - 2M_1 g_1 - 2M_2 g_2 - 2M_3 g_3 = l \quad k, l = \text{const.} \quad (1.3)$$

Then, according to a theorem of the algebraic geometry (see e.g., [23]), for generic values of the constants  $k, l$ , the complex invariant variety of the Zhukovsky–Volterra (ZV) system (1.2) is an elliptic curve  $\mathcal{E}$ . However, in contrast to the classical Euler top, explicit integration of the ZV system and, especially, the explicit description of the motion of the gyrostat in space given by solutions of the Poisson equations in practice appears to be a much more complicated problem.

In [58] Volterra presented expressions for the components of the momenta  $M$  and of the rotation matrix of the gyrostat in terms of sigma-functions and exponents, however these expressions include several undetermined parameters and only provide the structure of the solution, but not explicit formulas.

We bridge these gaps, namely,

- 1) To find expressions for the components of momenta  $M_i$  as elliptic functions on the curve  $\mathcal{E}$  by using a new rational parametrization for  $M_i$  in terms of some canonical coordinates on  $\mathcal{E}$ , whose dependence on time is known;
- 2) To derive all explicit solutions of the Poisson equations and obtain trigonometric functions of the Euler angles as functions of time  $t$ .

It is expected that the obtained explicit formulas provide an effective description of the motion of the gyrostat in space and be useful in practice.

Zhukovski-Volterra system described above has the property of being bi-Hamiltonian. By using this property and applying the new scheme for topological analysis of bi-Hamiltonian system [12] we construct bifurcation diagram of momentum mapping, given by integrals  $f_1, f_2$ :

$$\Phi(f_1, f_2) : \mathbb{R}^3(M_1, M_2, M_3) \rightarrow \mathbb{R}^2(f_1, f_2).$$

We describe the set of critical points, verify the non-degeneracy condition and determine the stability of equilibrium points. By using the parametric description of critical points, we obtained the bifurcation diagram of momentum mapping  $\Phi$  and analyze the topological type of common level of integrals  $f_1, f_2$ .

Notice, that the standard scheme for describing set of critical point and analysis of their stability consists of the Jacobi matrix and Hessian of the restriction  $f_1$  onto symplectic leaf, whereas this new technics has allowed us to answer all this questions without difficult computations.

### 1.1.2 Separation of variables, explicit integration and bifurcation analysis of the Steklov–Lyapunov systems

The classical Steklov integrable case of the Kirchhoff equations

$$\dot{M} = M \times \frac{\partial H}{\partial M} + p \times \frac{\partial H}{\partial p}, \quad \dot{p} = p \times \frac{\partial H}{\partial p}, \quad (1.4)$$

where  $M, p \in \mathbb{R}^3$  are the vectors of the impulsive momentum and the impulsive force, and  $H = H(M, p)$  is the Hamiltonian, which is quadratic in  $M, p$ . given by

$$H = \frac{1}{2} \sum_{\alpha=1}^3 \left( b_{\alpha} M_{\alpha}^2 + 2\nu b_{\beta} b_{\gamma} M_{\alpha} p_{\alpha} + \nu^2 b_{\alpha} (b_{\beta} - b_{\gamma})^2 p_{\alpha}^2 \right), \quad (1.5)$$

$b_i$  and  $\nu$  being arbitrary constants, was first integrated in terms of theta-functions of 2 arguments by F. Kötter [38] in 1900. However, the method of integration was not indicated in that paper and, moreover, the solutions presented contain several undetermined parameters, which make impossible to apply them in practice.

The second group of objectives of the thesis are

- 1) To revise the separation of variables and explicit integration of the classical Steklov–Lyapunov systems. Namely, we give a geometric interpretation of the separating variables;
- 2) then, applying the Weierstrass root functions, obtain an explicit theta-function solution to the problem.
- 3) construct and analyze the bifurcation diagram for the Steklov-Lyapunov system by using the bi-Hamiltonian properties of the system [12] and then, indicate on the plane  $(h_1, h_2)$  the domains of real motion, describe the type of the special motion for each segment of the bifurcation curves and do stability analysis for critical periodic solutions.

### 1.1.3 Bifurcation analysis of Rubanovskii sistem

Apart from the classical Steklov integrable case of the Kirchhoff equations, in [50] V. Rubanovsky found its gyroscopic generalization describing the motion of a gyrost in an ideal fluid and also the rigid body in presence of non-zero circulation. In contrast to the Steklov Hamiltonian (1.5), the Hamiltonian of the gyroscopic generalization contains linear terms in  $M, p$  and has the form

$$H_1 = \frac{1}{2} \sum_{\alpha=1}^3 \left( b_\alpha (M_\alpha - 2g_\alpha)^2 + 2\nu b_\beta b_\gamma M_\alpha p_\alpha + \nu^2 b_\alpha (b_\beta - b_\gamma)^2 p_\alpha^2 + 4\nu (b_\beta + b_\gamma) g_\alpha p_\alpha \right), \quad (1.6)$$

$$\nu, g_1, g_2, g_3 = \text{const.}$$

Like in the case of the Zhykovsky–Volterra gyrost, here the vector  $g$  characterizes the axial angular momentum of the rotator inside the body.

The Kirchhoff equations with the Hamiltonian (1.6) possess a second integral quadratic in  $M, p$ .

It can be observed that under the change of variables  $M \rightarrow z$

$$2z_\alpha = M_\alpha - (b_\beta + b_\gamma) p_\alpha, \quad \alpha = 1, 2, 3, \quad (\alpha, \beta, \gamma) = (1, 2, 3)$$

these equations take the form

$$\dot{z} = z \times (Bz - g) - Bp \times (Bz - g), \quad \dot{p} = p \times (Bz - g).$$

and, as was shown in [27], the latter equations admit the following Lax pair with skew-symmetric matrices and an elliptic parameter  $s$

$$\dot{L}(s) = [L(s), A(s)], \quad L(s), A(s) \in so(3), \quad s \in \mathbb{C}, \quad (1.7)$$

$$L(s)_{\alpha\beta} = \varepsilon_{\alpha\beta\gamma} \left( \sqrt{s - b_\gamma} (z_\gamma + sp_\gamma) + g_\gamma / \sqrt{s - b_\gamma} \right),$$

$$A(s)_{\alpha\beta} = \varepsilon_{\alpha\beta\gamma} \frac{1}{s} \sqrt{(s - b_\alpha)(s - b_\beta)} (b_\gamma z_\gamma - g_\gamma). \quad (1.8)$$

Writing out the characteristic equation for  $L(s)$  we arrive at the following quadratic integrals

$$J_1 = \langle p, p \rangle, \quad J_2 = 2\langle z, p \rangle - \langle Bp, p \rangle, \quad H_1 = \frac{1}{2} \langle z, Bz \rangle - \langle z, g \rangle,$$

$$H_2 = \frac{1}{2} \langle z, z \rangle - \langle Bz, p \rangle + \langle p, g \rangle, \quad B = \text{diag}(b_1, b_2, b_3).$$

Lax-pair to the Rubanovskii case [27] allows to describe a bi-Hamiltonian structure corresponding to this system. Using the fact that Rubanovskii system is bi-Hamiltonian and applying new techniques [12] we solve the following problems:

- 1) description of the singularities of the momentum mapping defined by four integrals

$$\Phi : \mathbb{R}^6(z, p) \rightarrow \mathbb{R}^4(J_1, J_2, H_1, H_2);$$



- 2) stability analysis for closed trajectories;
- 3) non-degeneracy and stability analysis for equilibria;
- 4) some property of bifurcation diagram of Rubanovskii system.

## 1.2 Background and the main tools

Here we quote some fundamental and recent results, as well as some methods in the area of integrable systems, which we are going to use to achieve the objectives of the thesis.

### 1.2.1 Theorems of integration

The most fundamental notion of integrability of a dynamical system is *integrability by quadratures*, that is, finding its solutions using finitely many "algebraic" operations (including inverting functions) and calculation of integrals of known function.

If, moreover, the system on  $n$ -dimensional manifold  $M^n$  is Hamiltonian and its vector field can be written as  $I dH$ , then the main tool is the following theorem

**Theorem 1.2.1** (Liouville) *Suppose that the smooth functions  $F_1, \dots, F_n : M^n \rightarrow \mathbb{R}$  are pairwise in involution and  $\dim M = 2n$ . If*

- 1). *their differentials are linearly independent at each point of  $M_f$ ,*
- 2). *the Hamiltonian vector fields generated by  $F_i$  ( $1 \leq i \leq n$ ) are complete on  $M_f$ ,*

*then*

- a) *each connected component of  $M_f$  is diffeomorphic to a cylinder  $\mathbb{T}^k \times \mathbb{R}^{n-k}$ , or, in the particular case  $k = n$ , to a torus  $\mathbb{T}^n$ ;*
- b) *on  $\mathbb{T}^k \times \mathbb{R}^{n-k}$  there exist coordinates  $\{\varphi_1, \dots, \varphi_k \mid \text{mod } 2\pi\}$  and  $\{y_1, \dots, y_{n-k}\}$  such that Hamilton equations on  $M_f$  takes the form*

$$\dot{\varphi}_m = \omega_{mi}, \quad \dot{y}_s = c_{si}, \quad \omega_{mi}, c_{si} = \text{const.}$$

The Hamiltonian system with the Hamilton function  $F_i$  for each  $i = 1, \dots, n$  is said to be completely integrable.

**A system of differential equations with invariant measure.** We consider a system of differential equations

$$\dot{x} = f(x), x \in M \tag{1.9}$$

and let  $\{g^t\}$  be its phase flow. Suppose that equation (1.9) has an integral invariant with some smooth density  $M(x)$ , which means that

$$\int_{g^t} M(x) dx = \int_D M(x) dx \tag{1.10}$$

for any measurable domain  $D \subset M$  and for all  $t$ . Recall the well-known assertion of Liouville on the existence of an integral invariant.

**Proposition 1.2.2** *A smooth function  $\mu : M \rightarrow \mathbb{R}$  is the density of the invariant  $\int \mu(x) dx$  if and only if*

$$\operatorname{div}(\mu f) = 0.$$

*If  $M(x) > 0$  for all  $x$ , then formula (1.10) defines a measure in  $M$  that is invariant under the action of  $\{g^t\}$ .*

The existence of an invariant measure simplifies integration of the differential equation.

**Theorem 1.2.3** (Euler-Jacobi) *Suppose that the system of equation (1.9) with an invariant measure (1.10) has  $n - 2$  first integrals  $F_1, \dots, F_{n-2}$ . Suppose that the differentials of functions  $F_1, \dots, F_{n-2}$  are linearly independent on an invariant set*

$$M_c = \{x \in M : F_s = c_s, 1 \leq s \leq n - 2\}.$$

*Then*

1. *The solution of equation (1.9) lying on  $M_c$  can be found by quadratures.*
- 2). *If  $L_c$  is a compact connected component of the level set  $M_c$  and  $f(x) \neq 0$  on  $L_c$ ,*

*then*

*$L_c$  is a smooth manifold diffeomorphic to the two-dimensional torus;*

*On  $L_c$  there exist angle coordinates  $x, y \bmod 2\pi$  such that in these variables equation (1.9) on  $L_c$  takes the form*

$$\dot{x} = \frac{\lambda}{\Phi(x, y)}, \quad \dot{y} = \frac{\mu}{\Phi(x, y)},$$

*where  $\lambda, \mu = \text{const}$  and  $\Phi$  is a smooth function  $2\pi$ -periodic in  $x, y$ .*

It should be mentioned that, historically, this theorem had been formulated before the Liouville theorem and it was successfully applied to prove the integrability of several classical systems, like the Jacobi geodesic problem or the motion of a heavy rigid body.

Various generalizations of this theorem that use the existence of integral invariants of different kind, as well as symmetry fields were constructed in [33].

### 1.2.2 Examples of Completely Integrable Systems.

**Integrable cases of the motion of a heavy rigid body about a fixed point.**

The motion is described by the Euler-Poisson equations

$$A\dot{\omega} = A\omega \times \omega + e \times r, \quad \dot{e} = e \times \omega,$$

where  $\omega, e \in \mathbb{R}^3$  are the angular velocity and the a unit vector fixed in space. These equations contain six parameters: three eigenvalues  $A_1, A_2, A_3$  of the inertia operator

$A$  and three coordinates of the center of mass  $r = (r_1, r_2, r_3)^T$  with respect to the principal axes. These equations are Hamiltonian on the four-dimensional invariant symplectic manifolds

$$M_c = \{(\omega, e) \in \mathbb{R}^6 : \langle A\omega, e \rangle = c, \quad \langle e, e \rangle = 1\}.$$

There always exists one integral of these equations on  $M_c$ , the energy integral. Thus, by the Liouville theorem, for the complete integrability it is sufficient to have one more independent integral.

We list the known integrable cases, when such integrals exist:

1) *The Euler case*:  $r_1 = r_2 = r_3 = 0$  (the center of mass coincides with the fixed point). The new integral  $\langle A\omega, A\omega \rangle$  is the square of the magnitude of the angular momentum.

2) *The Lagrange case*:  $A_1 = A_2$  and  $r_1 = r_2 = 0$  (the body has an axial symmetry, which contains the mass center). The new integral  $\omega_3$  is the projection of the angular velocity onto the axis of symmetry.

3) *The Kovalevskaya case*:  $A_1 = A_2 = 2A_3$  and  $r_3 = 0$ . Choose coordinate axes in the plane perpendicular to the axis of dynamical symmetry such that  $r_2 = 0$ .

4) *The Goryachev–Chaplygin case*:  $A_1 = A_2 = 4A_3$  and  $r_3 = 0, c = \langle A\omega, e \rangle = 0$ . In contrast to the cases 1)-3), here we have an integrable Hamiltonian system only on the integral level  $M_0$ .

**Integrable cases of the Kirchhoff equations.** These equations describe the motion of a rigid body in an ideal fluid and have the form

$$\dot{M} = M \times \frac{\partial H}{\partial M} + p \times \frac{\partial H}{\partial p}, \quad \dot{p} = p \times \frac{\partial H}{\partial p}, \quad (1.11)$$

where  $M, p \in \mathbb{R}^3$  are the vectors of the impulsive pair and the impulsive force respectively and  $H = H(M, p)$  is the Hamiltonian, which is quadratic in  $M, p$ .

The system possesses first integrals  $\langle M, p \rangle, \langle p, p \rangle, H(M, p)$  and the following integrable cases are known:

1) *The Clebsch case*:  $H = \frac{1}{2}\langle M, AM \rangle + \frac{\mu}{2}\langle p, A^{-1}Mp \rangle$ ,  $A$  being an arbitrary diagonal matrix.

2) *The Steklov–Lyapunov case*:

$$H = \frac{1}{2} \sum_{\alpha=1}^3 \left( b_\alpha M_\alpha^2 + 2\nu b_\beta b_\gamma M_\alpha p_\alpha + \nu^2 b_\alpha (b_\beta - b_\gamma)^2 p_\alpha^2 \right), \quad (1.12)$$

$b_i$  and  $\nu$  being arbitrary constants.

In both cases equations (1.11) have an extra integral, also quadratic in  $M, p$ .

As was recently shown by V. Sokolov, apart from these classical cases, there exist two other quadratics Hamiltonian, for which the equations have *quartic* additional integrals (see, e.g., [16]).

**The Jacobi geodesic problem.** Consider geodesic motion on an ellipsoid  $Q$  in  $\mathbb{R}^3$  given by equation

$$\frac{x_1^2}{a_1} + \frac{x_2^2}{a_2} + \frac{x_3^2}{a_3} = 1, \quad a_1 > a_2 > a_3 > 0. \quad (1.13)$$

The equations of motion in the redundant coordinates  $x$  are

$$\ddot{\vec{x}} = \lambda \vec{n}, \quad (1.14)$$

where  $\vec{n}$  is a normal vector for  $Q$  at the point  $\vec{x}$ , that is,  $\vec{n} = a^{-1}\vec{x}$ , and  $\lambda$  is Lagrange multiplier, which can be found by differentiating the condition  $\langle \dot{\vec{x}}, \vec{n} \rangle = 0$ :

$$\langle \ddot{\vec{x}}, a^{-1}\vec{x} \rangle + \langle \dot{\vec{x}}, a^{-1}\dot{\vec{x}} \rangle = 0.$$

Then, using (1.14) we find

$$\lambda = -\frac{\langle \dot{\vec{x}}, a^{-1}\dot{\vec{x}} \rangle}{\langle \vec{x}, a^{-2}\vec{x} \rangle}.$$

Equations (1.14) defines a Hamiltonian flow on the 4-dimensional tangent bundle  $TQ$  and has two independent integrals

$$L = \frac{1}{2}\langle \dot{\vec{x}}, \dot{\vec{x}} \rangle, \quad I = \langle \dot{\vec{x}}, a^{-1}\dot{\vec{x}} \rangle \langle \vec{x}, a^{-2}\vec{x} \rangle.$$

In addition, the system (1.14) preserves an invariant measure on  $TQ$ , hence it is integrable by the Euler–Jacobi theorem.

### 1.2.3 Some Methods of integration of Hamiltonian systems.

**Separation of variables via the Stäckel theorem.** The simplest and most effective method of integration is the separation of variables. In the case of a Hamiltonian system on  $M^{2n}$  with the symplectic coordinates  $p_1, \dots, p_n, q_1, \dots, q_n$  there is a special case, when the separation of variables can be performed straightforward.

**Theorem 1.2.4** (Stäckel, [51]) . *Let  $\Phi$  be a determinant of the matrix  $(\varphi_{ij}(q_j))$ ,  $(1 \leq i, j \leq n)$  and  $\Phi_{ij}$  be the cofactor of the entry  $\varphi_{ij}$ . Suppose that the Hamiltonian function has the form*

$$H(p, q) = \sum_{s=1}^n \frac{\Phi_{1s}(q) f_s(p_s, q_s)}{\Phi(q)}, \quad (1.15)$$

then the following  $n$  functions

$$F_k = \sum_{s=1}^n \frac{\Phi_{ks} f(p_s, q_s)}{\Phi(q)}$$

form a complete involutive set of integrals of the system on  $M^{2n}$  and the Hamilton equations are integrable by the Liouville theorem.

Note that there is no general rule for finding separable variables for an integrable system. However, if a given integrable Hamiltonian system admits a Lax representation with a rational spectral parameter, then, according to [1, 2], one can find such variables in a systematic, although quite tedious, way.

**Example: The Jacobi geodesic problem.** One of the classical examples of a successful application of the method is related to elliptic coordinates in  $\mathbb{R}^n$  (or, in general, in  $\mathbb{R}^n$ ), which allows to reduce to quadratures the Jacobi geodesic problem described in the previous section. A similar approach, using spheroconical coordinates allows reducing to quadratures the Neumann problem describing the motion of a mass point on  $S^2$  in a quadratic potential field ([45]).

The Cartesian coordinates  $x$  of a point on the ellipsoid  $Q$  given by (1.13) are related to the *Jacobi elliptic coordinates*  $\lambda_1, \lambda_2$  as follows

$$x_i^2 = \frac{a_i (a_i - \lambda_1) (a_i - \lambda_2)}{(a_i - a_j) (a_i - a_k)} \quad (1.16)$$

Then the Lagrangian function  $L = \frac{1}{2} (\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2)$  describing the free motion on the ellipsoid, takes the form

$$L = \frac{1}{8} \left( \frac{(\lambda_1 - \lambda_2) \lambda_1}{(\lambda_1 - a_1) (\lambda_1 - a_2) (\lambda_1 - a_3)} \dot{\lambda}_1^2 + \frac{(\lambda_2 - \lambda_1) \lambda_2}{(\lambda_2 - a_1) (\lambda_2 - a_2) (\lambda_2 - a_3)} \dot{\lambda}_2^2 \right). \quad (1.17)$$

Now we pass to the conjugated momenta  $p_i = \partial L / \partial \dot{\lambda}_i$ ,  $i = 1, 2$ , that is,

$$\begin{aligned} p_1 &= \frac{1}{4} \frac{(\lambda_1 - \lambda_2) \lambda_1}{(\lambda_1 - a_1) (\lambda_1 - a_2) (\lambda_1 - a_3)} \dot{\lambda}_1, \\ p_2 &= \frac{1}{4} \frac{(\lambda_2 - \lambda_1) \lambda_2}{(\lambda_2 - a_1) (\lambda_2 - a_2) (\lambda_2 - a_3)} \dot{\lambda}_2 \end{aligned} \quad (1.18)$$

The energy of the free motion on the ellipsoid takes the form

$$H = 2 \left( \frac{(\lambda_1 - a_1) (\lambda_1 - a_2) (\lambda_1 - a_3)}{(\lambda_1 - \lambda_2) \lambda_1} p_1^2 + \frac{(\lambda_2 - a_1) (\lambda_2 - a_2) (\lambda_2 - a_3)}{(\lambda_2 - \lambda_1) \lambda_2} p_2^2 \right). \quad (1.19)$$

It is seen that this Hamiltonian has the Stäckel form (1.15) with the matrix

$$\varphi = \begin{pmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{pmatrix}, \quad \det \varphi = \lambda_2 - \lambda_1$$

and the cofactor of an element  $\varphi_{11}$  is  $\Phi_{11} = \lambda_2$ , the cofactor of an element  $\varphi_{12}$  is  $\Phi_{12} = -\lambda_1$ .

According to the Stäckel theorem, the system with the Hamiltonian (1.19) has a first integral  $I_1(p, q)$ , which commutes with  $H$  and, when written in terms of  $\lambda, \dot{\lambda}$ , takes the form

$$I_1 = \frac{(\lambda_1 - \lambda_2) \lambda_1 \lambda_2}{(\lambda_1 - a_1) (\lambda_1 - a_2) (\lambda_1 - a_3)} \dot{\lambda}_1^2 + \frac{(\lambda_2 - \lambda_1) \lambda_2 \lambda_1}{(\lambda_2 - a_1) (\lambda_2 - a_2) (\lambda_2 - a_3)} \dot{\lambda}_2^2. \quad (1.20)$$

Now we set  $H(p, \lambda) = L(\lambda, \dot{\lambda}) = h$ ,  $I_1(\lambda, \dot{\lambda}) = l$ , where  $h, l$  are constants of motion, and combining the integrals (1.20) and (1.19), we obtain the following quadratures

$$\frac{\lambda_1 d\lambda_1}{\sqrt{R_5(\lambda_1)}} + \frac{\lambda_2 d\lambda_2}{\sqrt{R_5(\lambda_2)}} = 0, \quad \frac{\lambda_1^2 d\lambda_1}{\sqrt{R_5(\lambda_1)}} + \frac{\lambda_2^2 d\lambda_2}{\sqrt{R_5(\lambda_2)}} = dt,$$

where  $R_5(\lambda) = -\lambda(\lambda - a_1)(\lambda - a_2)(\lambda - a_3)(\lambda - c)$ ,  $c = \frac{l}{h}$ .

Following Weierstrass [61], under the reparameterization  $dt = -\lambda_1 \lambda_2 d\tau$ , the equations take the form

$$\frac{d\lambda_1}{2\sqrt{R(\lambda_1)}} + \frac{d\lambda_2}{2\sqrt{R(\lambda_2)}} = d\tau, \quad \frac{\lambda_1 d\lambda_1}{2\sqrt{R(\lambda_1)}} + \frac{\lambda_2 d\lambda_2}{2\sqrt{R(\lambda_2)}} = 0,$$

which, upon integration, lead to the following quadratures

$$\begin{aligned} \int_{\lambda_0}^{\lambda_1} \frac{d\lambda}{2\sqrt{R(\lambda)}} + \int_{\lambda_0}^{\lambda_2} \frac{d\lambda}{2\sqrt{R(\lambda)}} &= d\tau, \\ \int_{\lambda_0}^{\lambda_1} \frac{\lambda d\lambda}{2\sqrt{R(\lambda)}} + \int_{\lambda_0}^{\lambda_2} \frac{\lambda d\lambda}{2\sqrt{R(\lambda)}} &= 0. \end{aligned} \tag{1.21}$$

The latter are a particular case of the *Abel-Jacobi equations*, which arise as quadratures in many classical integrable problems of dynamics. In order to obtain their explicit solutions, one has to invert these quadratures. The corresponding method will be briefly considered in subsection 1.4.

**Method of Lax-pairs.** Assume that a system of differential equations

$$\frac{d}{dt}x_i = F_i(x_1, \dots, x_n), \quad i = 1, \dots, n. \tag{1.22}$$

can be obtained from a matrix commutator equation

$$\frac{d}{dt}L(x, \lambda) = [L(x, \lambda), A(x, \lambda)], \tag{1.23}$$

where the matrices  $L, A$  depend in a rational way on an extra parameter  $\lambda \in \mathbb{C}$ .

It follows from (1.23) that  $L(x(t))$  undergoes the similarity transformation

$$L(t) = \mathcal{T}(t)L(0)\mathcal{T}^{-1}(t), \quad A(t) = \dot{\mathcal{T}}(t)\mathcal{T}^{-1}(t).$$

Hence, the eigenvalues of  $L(x(t), \lambda)$  do not depend on  $t$ , that is, the coefficients of the polynomials  $\text{tr}(L^k(x, \lambda))$ ,  $k \in \mathbb{N}$  are first integrals of the system (1.22).

The equation (1.23) is called a *Lax pair*. In modern approach to integrable systems it plays one of the main roles. If it produces sufficiently large number of independent integrals, then the corresponding system (1.22) is completely integrable.

**Example.** The best known example is the Lax pair for the Euler–Frahm equations (generalized Euler top) describing the free motion of an  $m$ -dimensional rigid body

$$\dot{M} + [\Omega, M] = 0, \quad (1.24)$$

where  $M, \Omega \in so(m)$  are the matrices of the angular momentum and the angular velocity of the body respectively. This system possesses "trivial" integrals  $I_k = \text{tr } M^k, k = 2, 4, \dots$ , the invariants of the Lie algebra  $so(m)$ .

A first Lax pair for the multi-dimensional body problem had been discovered by Manakov [42] in the form

$$\begin{aligned} \frac{d}{dt}(M + \lambda U) &= [M + \lambda U, \Omega + \lambda V], & \lambda \in \mathbb{C} & \\ U &= \text{diag}(a_1, \dots, a_m), & V &= \text{diag}(b_1, \dots, b_m), & U, V &= \text{const.} \end{aligned} \quad (1.25)$$

Under the condition  $[M, V] = [\Omega, U]$ , the coefficients at  $\lambda^0$  in (1.25) give the system (1.24) with

$$\Omega_{ij} = \frac{b_i + b_j}{a_i + a_j} M_{ij}.$$

The coefficients of the polynomials  $\text{tr } (M + \lambda U)^k, k = 2, \dots, m$  provide a complete set of first integrals, which is sufficient to prove the integrability by the Liouville theorem ([43]).

Another type of Lax pair for the Euler–Frahm equations that involves an elliptic parameter was indicated in [27].

If a dynamical system can be written in a Lax form (1.23), in which both matrices  $L, A$  depend on parameter  $\lambda$ , then using algebraic geometry methods developed in [19, 20, 1] and some other paper, one can write directly the generic solution to the system in terms of the theta-functions associated to the spectral algebraic curve  $S \in \mathbb{C}^2 = (\lambda, \mu)$  given by the characteristic equation  $|L(x, \lambda) - \mu \mathbf{I}| = 0$ .

However, in many integrable problems, including quite simple ones, this approach leads to too complicated theta-function solutions, which require a simplification. Hence, the alternative classical approach of separation of variables and reduction to quadratures remains to be important.



### 1.2.4 Theta-function solution of the Jacobi inversion problem

**Abelian differentials and Jacobian varieties.** Recall the definition of *Jacobian variety* of a regular Riemann surface  $\Gamma$  of genus  $g$ . Namely, consider a differential 1-forms (differential)  $\omega = \varphi(\tau) d\tau$  on  $\Gamma$ , where  $\tau$  is a local parameter at a point  $P \in \Gamma$ . The differential  $\omega$  is called *holomorphic* (or an Abelian differential of the first kind), if  $\varphi(\tau)$  is a holomorphic function for any point  $P$ . It is known that on a genus  $g$  surface  $\Gamma$ , there exist exactly  $g$  independent holomorphic differentials  $\omega_1, \dots, \omega_g$ , each of them having  $2g - 2$  zeros. Let us choose a canonical basis of cycles  $a_1, \dots, a_g, b_1, \dots, b_g$  on  $\Gamma$  such that

$$a_i \circ a_j = b_i \circ b_j = 0, \quad a_i \circ b_j = \delta_{ij}, \quad i, j = 1, \dots, g,$$

where  $\gamma_1 \circ \gamma_2$  denotes the intersection index of the cycles  $\gamma_1, \gamma_2$  (see an example in Figure (1.2.1)).

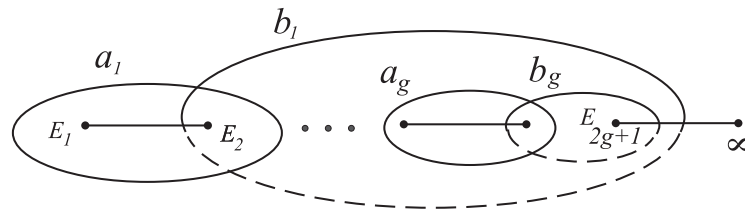


Figure 1.2.1:

Let us calculate the  $g \times g$  period matrix

$$A_{ij} = \oint_{a_j} \omega_i.$$

Since  $\omega_1, \dots, \omega_g$  are independent,  $A$  is non-degenerate. Hence we can uniquely find a basis of the *normalized* holomorphic differentials  $\bar{\omega}_1, \dots, \bar{\omega}_g$  such that

$$\oint_{a_j} \bar{\omega}_i = 2\pi j \delta_{ij}, \quad \text{that is,} \quad \bar{\omega}_i = 2\pi j \sum_{k=1}^g C_{ki} \omega_k, \quad C = A^{-1}. \quad (1.26)$$

Then the matrix of  $b$ -periods  $B_{ij} = \oint_{b_j} \omega_i$  is symmetric and has a negative definite real part. Consider the period lattice  $\Lambda^0 = \{2\pi j \mathbb{Z}^g + B \mathbb{Z}^g\}$  of rank  $2g$  in  $\mathbb{C}^g = (z_1, \dots, z_g)$ . The complex torus  $\text{Jac}(\Gamma) = \mathbb{C}^g / \Lambda^0$  is called the *Jacobi variety* (*Jacobian*) of the curve  $\Gamma$ . It is a compact principally polarized Abelian variety. Notice that for  $g = 2$ , any principally polarized Abelian variety that does not contain elliptic curves is the Jacobian of a hyperelliptic Riemann surface.

For points  $P, P_0$  on a nonsingular Riemann surface  $\Gamma$ , we define the vector function

$$\mathcal{A}(P) = \left( \int_{P_0}^P \omega_1, \dots, \int_{P_0}^P \omega_g \right)^T$$

in which the integration paths on  $\Gamma$  are all the same. The function describes a correctly defined holomorphic mapping  $\Gamma \rightarrow \text{Jac}(\Gamma)$  with the basepoint  $P_0$ .

Now consider  $g$ -th symmetric product  $S^g\Gamma$  of the curve  $\Gamma$ . A point on  $S^g\Gamma$  is represented by a disordered set  $D = \{P_1, \dots, P_g\}$  of  $g$  points on  $\Gamma$  called a *divisor* of points. The mapping  $\Gamma \rightarrow \text{Jac}(\Gamma)$  can be extended to the mapping

$$S^g\Gamma \rightarrow \text{Jac}(\Gamma) : D \rightarrow \mathcal{A}(D) = \mathcal{A}(P_1) + \dots + \mathcal{A}(P_g),$$

which is called *the Abel-Jacobi mapping*. The latter gives rise to the system of  $g$  *Abel-Jacobi equations*

$$\int_{P_0}^{P_1} \bar{\omega} + \dots + \int_{P_0}^{P_g} \bar{\omega} = z, \quad (1.27)$$

$$\bar{\omega} = (\bar{\omega}_1, \dots, \bar{\omega}_g)^T, \quad z = (z_1, \dots, z_g)^T \in \mathbb{C}^g.$$

Under the mapping, functions on  $S^g\Gamma$ , i.e., symmetric functions of the coordinates of the points  $P_1, \dots, P_g$  are  $2g$ -fold periodic functions of the complex variables  $z_1, \dots, z_g$  with the period lattice  $\Lambda^0$  (Abelian functions). Thus we arrive at the celebrated *Jacobi inversion problem*: to express meromorphic functions on  $S^g\Gamma$  in terms of  $z$ , or, in a geometric formulation, given a point on the Jacobian  $\mathbb{T}^g$  with coordinates  $z$ , to recover the corresponding divisor  $D = \{P_1, \dots, P_g\}$ . We call a divisor  $D$  *nonspecial*, if in the neighborhood of  $\mathcal{A}(D)$  the Abel-Jacobi mapping is uniquely invertible. Otherwise,  $D$  is called a *special* divisor.

Now we concentrate on hyperelliptic Riemann surfaces of genus  $g$  represented in standard forms

$$\Gamma = \{w^2 = R_{2g+1}(\lambda) = (\lambda - E_1) \cdots (\lambda - E_{2g+1})\},$$

$$\Gamma' = \{w^2 = R_{2g+2}(\lambda) = (\lambda - E_1) \cdots (\lambda - E_{2g+2})\},$$

which we call the odd order form and the even order form respectively. The curves are represented as 2-fold ramified coverings of  $\bar{\mathbb{C}} = \{\lambda\} \cup \infty$ . Let us choose canonical cycles  $a_1, \dots, a_g, b_1, \dots, b_g$  as shown in Figure (1.2.1), where segments of cycles on the lower sheet of  $\bar{\mathbb{C}}$  are depicted by dashed lines.

In this case a natural basis of holomorphic (non-normalized) differentials is

$$\omega_k = \frac{\lambda^{k-1} d\lambda}{w}, \quad k = 1, \dots, g. \quad (1.28)$$

Let  $\tau = 1/\sqrt{\lambda}$  be a local parameter in a neighborhood of  $\infty \in \Gamma : \tau(\infty) = 0$ . Then  $d\lambda = -2d\tau/\tau^3$ , and in the same neighborhood the differentials (1.28) take the form

$$\omega_k = \left( -2\tau^{g-k} + o(\tau^{g-k}) \right) d\tau, \quad k = 1, \dots, g.$$

Analogously, let  $\tilde{\tau} = 1/\lambda$  be a local parameter in a neighborhood of  $\infty_-$  on  $\Gamma' : \tilde{\tau}(\infty_-) = 0$ . Then  $d\lambda = -d\tilde{\tau}/\tilde{\tau}^2$ , and in this neighborhood

$$\omega_k = \left( -\tilde{\tau}^{g-k} + o(\tilde{\tau}^{g-k}) \right) d\tilde{\tau}, \quad k = 1, \dots, g.$$

It is seen that  $\omega_1, \dots, \omega_{g-1}$  have zeros at  $\infty$  in the odd order case and at  $\infty_-, \infty_+$  in the even order case. In view of the condition (1.26), for the chosen canonical basis of cycles, the basis of normalized holomorphic differentials is given by

$$\bar{\omega}_k = \sum_{j=1}^g C_{kj} \frac{\lambda^{j-1} d\lambda}{w}, \quad C = A^{-1}, \quad A_{kj} = 2 \int_{E_{2k-1}}^{E_{2k}} \frac{\lambda^{j-1} d\lambda}{w}, \quad (1.29)$$

**Solution to the Jacobi inversion problem.** The problem of inversion of the Abel–Jacobi map (1.27) is solved by means of theta-function  $\theta(z|B)$  of the Riemann surface  $\Gamma$  with the period matrix  $B$ , defined by the series

$$\theta(z|B) = \sum_{M \in \mathbb{Z}^g} \exp((BM, M) + (M, z)), \quad (1.30)$$

$$(M, z) = \sum_{i=1}^g M_i z_i, \quad (BM, M) = \sum_{i,j=1}^g B_{ij} M_i M_j.$$

It is convergent everywhere in  $\mathbb{C}^g$  provided the real part of  $B$  is negative definite. The function  $\theta(z|B)$  enjoys the following quasiperiodic property

$$\theta(z + 2\pi j K + BM|B) = \exp\{-(BM, M)/2 - (M, z)\} \theta(z|B), \quad (1.31)$$

$$K, M \in \mathbb{Z}^g.$$

The vectors  $2\pi j e_1, \dots, 2\pi j e_g, B e_1, \dots, B e_g$  and their linear combinations with integer coefficients are called *quasiperiods* of  $\theta(z|B)$ . Notice that, up to multiplication by a constant,  $\theta(z|B)$  is a unique entire function of  $z_1, \dots, z_g$  satisfying the condition (1.31). On the torus  $\mathbb{C}^g/\Lambda_0$ ,  $\Lambda_0 = \{2\pi j \mathbb{Z}^g + B \mathbb{Z}^g\}$ , equation  $\theta(z|B) = 0$  defines a codimension one subvariety  $\Theta$  (for  $g > 2$  with singularities) called *theta-divisor*. Notice that if an Abelian variety  $(\mathbb{C}^g/\Lambda_0, \mathcal{D})$  is principally polarized, then the divisor  $\mathcal{D}$  is a union of translates of  $\Theta$ .

Let  $\alpha = (\alpha_1, \dots, \alpha_g)^T$ ,  $\beta = (\beta_1, \dots, \beta_g)^T$  be arbitrary real vectors. Define *theta-functions with characteristics*, which are obtained from  $\theta(z)$  by shifting the argument  $z$  and multiplying by an exponent:

$$\theta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (z) \equiv \theta \begin{bmatrix} \alpha_1 & \dots & \alpha_g \\ \beta_1 & \dots & \beta_g \end{bmatrix} (z) = \exp\{(B\alpha, \alpha)/2 + (z + 2\pi j \beta, \alpha)\} \theta(z + 2\pi j \beta + B\alpha)$$

(here and below we omit  $B$  in the theta-functional notation). As a consequence, for a pair of characteristics we obtain the following useful relations

$$\theta \begin{bmatrix} \alpha + \alpha' \\ \beta + \beta' \end{bmatrix} (z) = \exp\{(B\alpha', \alpha')/2 + (z + 2\pi j \beta + 2\pi j \beta', \alpha')\} \theta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (z + 2\pi j \beta' + B\alpha'). \quad (1.32)$$

Clearly,  $\theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (z) = \theta(z)$ . The quasiperiodic law for  $\theta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (z)$  has the form

$$\theta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (z + 2\pi j K + BM) = \exp(2\pi j \epsilon) \exp\{-(BM, M)/2 - (M, z)\} \theta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (z), \quad (1.33)$$

$$\epsilon = (\alpha, K) - (\beta, M),$$

which differs from the quasiperiodic law (1.31) only by multiplication by a root of unit.

Notice that characteristics  $\alpha_k, \beta_k$  are defined modulo 1.

An important particular case is represented by theta-functions with half-integer characteristics  $\alpha, \beta \in \frac{1}{2}\mathbb{Z}^g/\mathbb{Z}^g$ . For example, the function  $\theta \begin{bmatrix} 0 & 1/2 \\ 0 & 0 \end{bmatrix} (z)$  on  $\mathbb{C}^2$  has quasi-periods  $2\pi j e_1, 4\pi j e_2, B e_1, B e_2$  and changes sign after a shift by  $2\pi j e_2$ . The  $4^g$  points  $\frac{1}{2}(2\pi j \mathbb{Z}^g + B \mathbb{Z}^g)/\mathbb{Z}^g$  on  $\mathbb{T}^g/\Lambda_0$  are called *half-periods* (or *second order points*). A half-integer characteristic  $\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$  and the corresponding half-period  $2\pi j \beta + B \alpha$  are said to be even if the integer  $4(\alpha, \beta)$  is even, and odd if  $4(\alpha, \beta)$  is odd. As follows from (1.33), the function  $\theta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (z)$  is even (resp. odd) if  $\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$  is even (odd). In particular,  $\theta(z)$  is even. The clue to the solution is given by the following theorem:

**Theorem 1.2.5** *The theta-divisor  $\{\theta(z|B) = 0\}$  admits parametrization*

$$\Theta = \{z = \mathcal{A}(P_1) + \cdots + \mathcal{A}(P_{g-1}) + \mathcal{K}\}, \quad (1.34)$$

where  $P_1, \dots, P_{g-1}$  are arbitrary points on  $\Gamma$ , and  $\mathcal{K} = (\mathcal{K}_1, \dots, \mathcal{K}_g)^T$  is the vector of the Riemann constants

$$\mathcal{K}_j = \frac{1}{2}(2\pi j + B_{jj}) - \frac{1}{2\pi j} \sum_{l \neq j}^g \left( \oint_{a_l} \bar{\omega}_l(P) \int_{P_0}^P \bar{\omega}_j \right), \quad j = 1, \dots, g, \quad (1.35)$$

$P_0$  being the basepoint of the Abel mapping  $\mathcal{A}(P)$ .

Observe that  $\mathcal{K}$  depends only on  $P_0$  and on the choice of canonical cycles on  $\Gamma$ . It turns out that under an appropriate choice of  $P_0$ , the vector  $\mathcal{K}$  becomes a half-period on  $\text{Jac}(\Gamma)$  (see [26] and expressions (1.36) below).

It passes through the six odd half-periods of the 16 half-periods on the Jacobian. In the simplest case  $g = 1$ ,  $\text{Jac}(\Gamma)$  coincides with the elliptic curve  $\Gamma$ , and  $\Theta$  is one of the four second order points on the Jacobian.

Now introduce the function  $F(P) = \theta(\mathcal{A}(P) - e)$ , where  $P \in \Gamma$  and  $e$  is an arbitrary vector in  $\mathbb{C}^g$ . The function is single-valued and analytic in a simply connected domain  $\tilde{\Gamma}$ , the dissection of the surface  $\Gamma$  along its canonical cycles having a single common point.

**Lemma 1.2.6** 1). *The function  $F(P)$  equals zero identically on  $\Gamma$  if and only if  $e = \mathcal{A}(Q_1) + \cdots + \mathcal{A}(Q_g) + \mathcal{K}$ , where the divisor  $\{Q_1, \dots, Q_g\}$  is special.*

2). *If  $F(P)$  does not vanish identically, then it has precisely  $g$  zeros on  $\Gamma$  (possibly, with multiplicity).*

As a corollary of Theorem 1.2.5 and Lemma 1.2.6 we obtain

**Theorem 1.2.7** (Riemann) *Let  $D = \{P_1, \dots, P_g\}$  be a nonspecial divisor on  $\Gamma$ . Then the function  $F(P) = \theta(\mathcal{A}(P) - z - \mathcal{K})$  has precisely  $g$  zeros  $P_1, \dots, P_g$  on  $\Gamma$  giving the solution to the problem of inversion of the mapping (1.27).*

Indeed, since  $D$  is a nonspecial divisor,  $F(P)$  does not vanish identically on  $\Gamma$ . If  $P$  coincides with one of the points  $P_1, \dots, P_g$ , the argument of the theta-function admits the parameterization (1.34) for the divisor  $\Theta$ , i.e.,  $F(P) = 0$ . In addition, by item 2) of Lemma 1.33,  $F(P)$  cannot have zeros except  $P_1, \dots, P_g$ .

In the hyperelliptic case the vector of the Riemann constants  $\mathcal{K}$  can easily be found in an explicit form. For the basepoint  $P_0 = E_{2g+2}$  or  $\infty$  and the above choice of canonical cycles, the formula (1.35) gives (see e.g., [26, 44])

$$\mathcal{K} = 2\pi i \Delta'' + B \Delta' \pmod{\Lambda}, \quad \Delta'', \Delta' \in \frac{1}{2} \mathbb{Z}^g / \mathbb{Z}^g, \quad (1.36)$$

$$\Delta' = (1/2, \dots, 1/2)^T, \quad \Delta'' = (g/2, (g-1)/2, \dots, 1, 1/2)^T \pmod{1}.$$

### 1.2.5 Hyperelliptic root functions and solutions

In order to give explicit theta-functional solutions of many integrable systems one can apply remarkable relations between roots of certain functions on symmetric products of hyperelliptic curves and quotients of theta-functions with half-integer characteristics, which are historically referred to as *root functions* (Wurzelfunktionen), see e.g., [17, 35, 36]. The root functions give the shortest way to obtain theta-functional solution in a great variety of integrable systems admitting separation of variables. It is these functions which have been used by Kowalewski, Weber, Kötter, and other mathematicians in their works devoted to integration of equations of classical dynamics.

Consider first an odd-order hyperelliptic surface

$$\Gamma = \{w^2 = R(\lambda)\}, \quad R(\lambda) = (\lambda - E_1) \cdots (\lambda - E_{2g+1})$$

of genus  $g$ , a divisor of points  $P_1 = (\lambda_1, w_1), \dots, P_g = (\lambda_g, w_g)$  on it, and the Abel–Jacobi mapping  $z = \mathcal{A}(P_1, \dots, P_g)$  with the basepoint  $\infty$ . Let  $E_i$  briefly denote the branch point  $(E_i, 0)$  on  $\Gamma$ . Introduce the polynomial  $U(\lambda, s) = (s - \lambda_1) \cdots (s - \lambda_g)$ . The square root of it can be regarded as a single-valued function on the symmetric product of  $g$  coverings of  $\Gamma$ . Then under the above mapping, the following relations hold

$$\sqrt{U(\lambda, E_i)} \equiv \sqrt{(E_i - \lambda_1) \cdots (E_i - \lambda_g)} = c_i \frac{\theta[\delta + \eta_i](z)}{\theta[\delta](z)}, \quad (1.37)$$

$$\sum_{k=1}^g \frac{\sqrt{R(\lambda_k)} \sqrt{U(\lambda, E_i)} \sqrt{U(\lambda, E_j)}}{\prod_{l \neq k} (\lambda_k - \lambda_l) (E_i - \lambda_k)(E_j - \lambda_k)} = c_{ij} \frac{\theta[\delta + \eta_{ij}](z)}{\theta[\delta](z)}, \quad (1.38)$$

$$i, j = 1, \dots, 2g - 1, \quad i \neq j,$$

where  $c_i, c_{ij}$  are certain constants depending on the periods of  $\Gamma$  only and  $\Delta, \eta_i$  are half-integer theta-characteristics such that

$$\delta = \begin{pmatrix} \delta' \\ \delta'' \end{pmatrix}, \quad \eta_i = \begin{pmatrix} \eta'_i \\ \eta''_i \end{pmatrix}, \quad 2\pi \eta''_i + B \eta'_i = \mathcal{A}(E_i) \equiv \int_{\infty}^{E_i} \bar{\omega} \pmod{\Lambda}, \quad (1.39)$$

$$\eta_{ij} = \eta_i + \eta_j \pmod{\mathbb{Z}^{2g}}.$$

For the chosen canonical basis of cycles  $a_1, \dots, a_g, b_1, \dots, b_g$  on  $\Gamma$ , the characteristic  $\Delta$  is given by (1.36). Next, in view of relations

$$\int_{\infty}^{E_{2g+1}} \bar{\omega} = \frac{1}{2} \left( \oint_{a_1} \bar{\omega} + \cdots + \oint_{a_g} \bar{\omega} \right), \quad \int_{\infty}^{E_{2k}} \bar{\omega} = \int_{\infty}^{E_{2k-1}} \bar{\omega} + \frac{1}{2} \oint_{a_k} \bar{\omega},$$

from (1.36) we have

$$\begin{pmatrix} \eta'_{2k-1} \\ \eta''_{2k-1} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad \begin{pmatrix} \eta'_{2g+1} \\ \eta''_{2g+1} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 0 & \cdots & 0 \\ 1 & \cdots & 1 \end{pmatrix},$$

$$\begin{pmatrix} \eta'_{2k} \\ \eta''_{2k} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 1 & \cdots & 1 & 1 & 0 & \cdots & 0 \end{pmatrix}, \quad k = 1, \dots, g \quad (1.40)$$

where the unit in the upper rows for  $\eta_1, \dots, \eta_{2g}$  stands at  $k$ -th position.

Apparently, relations (1.38) were first obtained in the explicit form by Königsberger ([34]). Earlier, the root functions (1.37) had been considered by K. Weierstrass in [59] as generalizations of the Jacobi elliptic functions  $sn(z)$ ,  $cn(z)$ , and  $dn(z)$ , and later in [44]. More nontrivial root functions that are represented as quotients of theta-functions including a sum of half-integer theta-characteristics of the type  $\eta_{i_1} + \dots + \eta_{i_k}$  are indicated in [3] and the modern book [7].

According to the quasiperiodic law (1.33), the quotients

$$\frac{\theta[\Delta + \eta_i](z)}{\theta[\Delta](z)}, \quad \frac{\theta[\Delta + \eta_j](z)}{\theta[\Delta](z)}$$

are not single-valued functions on the Jacobian  $\text{Jac}(\Gamma) = \mathbb{C}^g/\Lambda_0$ ,  $\Lambda_0 = \{2\pi j\mathbb{Z}^g + B\mathbb{Z}^g\}$  but on its certain unramified covering  $\mathbb{C}^g/\Lambda$ , where the lattice  $\Lambda$  is obtained by multiplying some of the periods of  $\Lambda_0$  by 2 (under a shift of  $z$  by a period of  $\Lambda_0$  some of the theta-quotients change sign).

To indicate analogs of the root functions for the case of even-order hyperelliptic curve  $\Gamma' = \{w^2 = R(\lambda)\}$ ,  $R(\lambda) = (\lambda - E_1) \cdots (\lambda - E_{2g+2})$ , it is natural to consider the Abel–Jacobi mapping (1.27) with basepoint  $P_0 = E_{2g+2}$ . First, notice that the rational function  $f(P) = \lambda(P) - E_i$  on  $\Gamma'$  has a double zero at the branch point  $(E_i, 0)$  and two single poles at  $\infty_-, \infty_+$ .

$$U(\lambda, E_i) \equiv (\lambda_1 - E_i) \cdots (\lambda_g - E_i) = \tilde{\kappa}_i \frac{\theta^2[\Delta + \eta_i](z)}{\theta[\Delta](z - q/2)\theta[\Delta](z + q/2)}, \quad (1.41)$$

$$q = \int_{\infty_-}^{\infty_+} \bar{\omega} = 2 \int_{E_{2g+2}}^{\infty_+} \bar{\omega}, \quad \tilde{\kappa}_i = \text{const}, \quad i = 1, \dots, 2g + 2.$$

The half-integer characteristics  $\Delta, \eta_i$  are defined in (1.36), (1.40). Under the Abel–Jacobi mapping (1.27) including the normalized holomorphic differentials of  $\Gamma'$ , the following analogs of the root function (1.37), (1.38) hold:

$$\frac{\sqrt{U(\lambda, E_i)}}{\sqrt{U(\lambda, E_j)}} = \kappa'_{ij} \frac{\theta[\Delta + \eta_i](z)}{\theta[\Delta + \eta_j](z)}, \quad (1.42)$$

$$\sum_{k=1}^g \frac{\sqrt{R(\lambda_k)}}{\prod_{l \neq k} (\lambda_k - \lambda_l)} \frac{\sqrt{U(\lambda, E_i)} \sqrt{U(\lambda, E_j)}}{(E_i - \lambda_k)(E_j - \lambda_k)} = c'_{ij} \frac{\theta[\delta + \eta_{ij}](z) \theta[\delta](z)}{\mathcal{N}(z)}, \quad (1.43)$$

$$\mathcal{N} = \theta[\Delta](z - q/2) \theta[\Delta](z + q/2),$$

$$\sum_{k=1}^g \frac{\sqrt{R(\lambda_k)}}{\prod_{l \neq k} (\lambda_k - \lambda_l)} \frac{\sqrt{U(\lambda, E_i)} \sqrt{U(\lambda, E_j)}}{(E_i - \lambda_k)(E_j - \lambda_k)(E_s - \lambda_k)} = \kappa'_{ijs} \frac{\theta[\Delta + \eta_{ij} + \eta_s](z)}{\theta[\Delta + \eta_s](z)}, \quad (1.44)$$

$$\kappa'_{ij}, \kappa'_{ijs}, c'_{ij} = \text{const}, \quad \eta_{ij} = \eta_i + \eta_j \pmod{\mathbb{Z}^{2g}} \quad i, j, s = 1, \dots, 2g + 2$$

These relations can be deduced directly from (1.37), (1.38) and (1.41) by making a fractionally-linear transformation of  $\lambda$  that sends the Weierstrass point  $E_{2g+2}$  on  $\Gamma'$  to infinity.

### 1.2.6 Compatible Poisson brackets and bi-Hamiltonian systems

Many completely integrable Hamiltonian systems arising in mechanics, mathematical physics and geometry have the remarkable property of being bi-Hamiltonian, i.e., they are Hamiltonian systems with respect to two different Poisson structures at the same time (e.g., see [9, 16, 18, 27, 13, 30, 22, 41, 40]). Very often, these structures are compatible, and the system in question is Hamiltonian with respect to any of their linear combinations (with constant coefficients).

Since the pioneering work by Franco Magri [41] it has been well known that integrability of many systems is closely related to their bi-Hamiltonian nature [8, 9, 10, 18, 40, 53]. The bi-Hamiltonian structure have been observed in many classical systems and, at the same time, by using the bi-Hamiltonian technics, many new interesting and non-trivial examples of integrable systems have been discovered [16, 22, 54, 46, 47]. Moreover, this approach, based on a very simple, natural and elegant notion of compatible Poisson structures, proved to be very powerful in the theory of integrable systems not only for finding new examples, but also for explicit integration and description of analytical properties of solutions.

It turns out (see [12]) that the bi-Hamiltonian approach can also be extremely effective in the study of bifurcations and singularities of integrable systems, especially in the case of many degrees of freedom when using other methods often leads to serious computational problems. However, as we shall see below, even for two degrees of freedom systems these ideas are very useful too.

Speaking of singularities of integrable Hamiltonian systems, we mean those integral trajectories which lie outside the set of Liouville tori or, in other words, which belong to the singular set that corresponds to those points where the first integrals of a given system become functionally dependent. The analysis of the system on this set and in its neighborhood is undoubtedly very important because the singular set usually contains the most interesting trajectories, in particular, equilibrium points, and its topological structure is closely related to the bifurcations of Liouville tori, monodromy phenomena and other global effects.

The main idea introduced and developed in [12] can be explained as follows: the structure of singularities of a bi-Hamiltonian system is determined by that of the corresponding compatible Poisson brackets. Since in many examples the underlying bi-Hamiltonian structure has a natural algebraic interpretation, the technology developed in in [12] allows one to reformulate rather non-trivial analytic and topological questions related to the dynamics of a given system into pure algebraic language, which often leads to quite simple and natural answers.

Below, we recall briefly some basic notions and results related to the bi-Hamiltonian approach to integrability and sketch some idea from [12].

**Definition 1** A skew-symmetric tensor field  $\mathcal{A} = (\mathcal{A}^{ij})$  of type  $(2, 0)$  on a smooth manifold  $M$  is called a *Poisson structure* if the operation on  $C^\infty(M)$  defined by

$$\{f, g\} = \mathcal{A}^{ij} \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j}$$



satisfies the Jacobi identity:

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0 \quad \text{for all } f, g, h \in C^\infty(M).$$

In this situation, the space of smooth functions has the natural structure of an infinite-dimensional Lie algebra, and the operation  $\{ , \}$  is called the Poisson bracket.

**Definition 2** Let  $f$  be a smooth function. A vector field of the form

$$(X_H)^j = A^{ij} \frac{\partial H}{\partial x^i}$$

is said to be *Hamiltonian*, and the function  $H$  is called a *Hamiltonian*. The invariant definition states:  $X_H$  is a vector field such that any smooth function  $H$  satisfies the identity  $X_H(f) = \{H, f\}$ .

The *rank of the Poisson structure (bracket)  $\mathcal{A}$  at a point  $x$*  is the rank of the skew-symmetric matrix  $\mathcal{A}^{ij}(x)$ . Speaking of the *rank of  $\mathcal{A}$  on the manifold  $M$*  as a whole, we mean its rank at a generic point, i.e.,

$$\text{rank } \mathcal{A} = \max_{x \in M} \text{rank } \mathcal{A}^{ij}(x).$$

Below we confine ourselves with real-analytic Poisson structures so that generic points always form an open everywhere dense subset in  $M$ .

If  $\mathcal{A}(x)$  is non-degenerate at each point  $x \in M$ , then we can consider the inverse tensor to  $\mathcal{A}^{ij}$  as a differential 2-form  $\omega = \omega_{ij} dx_i \wedge dx_j$ ,  $\omega_{ij} \mathcal{A}^{jk} = \delta_i^k$ . This form, as is easy to see, is a symplectic structure, i.e., it is nondegenerate and closed. However, below we are going to deal with degenerate Poisson structures only, i.e.  $\text{rank } \mathcal{A} < \dim M$ .

**Definition 3** A function  $f : M \rightarrow \mathbb{R}$  is a *Casimir function* of a Poisson structure  $\mathcal{A}$  if  $\{f, g\} \equiv 0$  for any smooth function  $g$ . We shall denote the space of such functions by  $\mathcal{Z}(\mathcal{A})$ .

A Casimir function  $f$  can be characterized by the following condition:  $df(x) \in \text{Ker } \mathcal{A}(x)$  at each point  $x \in M$ .

If the Poisson structure  $\mathcal{A}$  is degenerate, then locally in a neighborhood of a generic point, Casimir functions always exist and the number of functionally independent Casimir functions is exactly the *corank of the Poisson structure*  $\text{corank } \mathcal{A} = \dim M - \text{rank } \mathcal{A}$ , i.e., the differentials of Casimir functions generate the kernel of  $\mathcal{A}(x)$  at generic points  $x$ .

**Example 1.** The simplest example of a Poisson structure is a constant one:  $\mathcal{A}^{ij}(x) = A^{ij}$  where  $A^{ij} \in \mathbb{R}$  are certain constants. If  $\text{rank } A^{ij} < \dim M = n$  (here  $M$  is an open domain in  $\mathbb{R}^n$ ), then the Casimir functions are linear  $f(x) = \sum c_i x^i$ , where  $c_i \in \mathbb{R}$  are defined from the system of linear equations  $\sum A^{ij} c_j = 0$ ,  $j = 1, \dots, n$ .

**Example 2.** One of the most important examples of Poisson brackets are linear Poisson brackets or PoissonLie brackets. Linearity means that the coefficients of the tensor field  $A_{ij}(x)$  are linear functions of the coordinates  $x^1, \dots, x_n$  (here it is convenient to interchange the superscripts and the subscripts). It is easy to see that there is a natural one-to-one correspondence between such brackets and Lie algebras. Indeed, let  $\mathfrak{g}$  be a finite-dimensional Lie algebra, and  $\mathfrak{g}^*$  be its dual space. On  $\mathfrak{g}^*$  we define the Poisson bracket by the formula

$$\{f, g\}(x) = x([df(x), dg(x)]), \quad x \in \mathfrak{g}^*, \quad df(x), dg(x) \in (\mathfrak{g}^*)^* = \mathfrak{g}. \quad (1.45)$$

Equivalently, in coordinates, this bracket can be written as

$$\{f, g\}(x) = \sum c_{ij}^k x_k \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j}$$

where the  $c_{ij}^k$  are the components of the structural tensor of the algebra  $\mathfrak{g}$  in the basis corresponding to the coordinates  $x_1, \dots, x_n$ .

Conversely, if we have a linear Poisson bracket, i.e., if  $A^{jk}(x) = c_{ij}^k x_k$ , then  $c_{ij}^k$  is the structural tensor of some Lie algebra. The Casimir functions of the PoissonLie bracket are exactly the invariants of the coadjoint representation of the corresponding Lie group  $G$  on  $\mathfrak{g}^*$ .

Two examples of Lie-Poisson brackets are particularly important for applications in classical mechanics.

For the Lie algebra  $\mathfrak{so}(3)$ , the corresponding Poisson-Lie bracket in coordinates  $M_1, M_2, M_3$  is defined as:

$$\{M_i, M_j\} = \varepsilon_{ijk} M_k,$$

or, in matrix form:

$$\mathcal{A} = \mathcal{A}(M_1, M_2, M_3) = \begin{pmatrix} 0 & M_3 & -M_2 \\ -M_3 & 0 & M_1 \\ M_2 & -M_1 & 0 \end{pmatrix} \quad (1.46)$$

This bracket is degenerate of corank 1, and the Casimir function is

$$f(M) = M_1^2 + M_2^2 + M_3^2$$

For the Lie algebra  $\mathfrak{e}(3) = \mathfrak{so}(3) \oplus \mathbb{R}^3$ , the corresponding Poisson-Lie bracket in coordinates  $M_1, M_2, M_3, \gamma_1, \gamma_2, \gamma_3$  is defined as:

$$\{M_i, M_j\} = \varepsilon_{ijk} M_k, \quad \{M_i, \gamma_j\} = \varepsilon_{ijk} \gamma_k, \quad \{\gamma_i, \gamma_j\} = 0.$$

or, in matrix form:

$$\mathcal{A} = \mathcal{A}(M, \gamma) = \begin{pmatrix} 0 & M_3 & -M_2 & 0 & \gamma_3 & -\gamma_2 \\ -M_3 & 0 & M_1 & -\gamma_3 & 0 & \gamma_1 \\ M_2 & -M_1 & 0 & \gamma_2 & -\gamma_1 & 0 \\ 0 & \gamma_3 & -\gamma_2 & 0 & 0 & 0 \\ -\gamma_3 & 0 & \gamma_1 & 0 & 0 & 0 \\ \gamma_2 & -\gamma_1 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (1.47)$$

This bracket is degenerate of corank 2, and its independent Casimir functions are

$$J_1 = \langle \gamma, \gamma \rangle = \gamma_1^2 + \gamma_2^2 + \gamma_3^2,$$

$$J_2 = \langle M, \gamma \rangle = \gamma_1 M_1 + \gamma_2 M_2 + \gamma_3 M_3.$$

**Definition 4** Two Poisson structures  $\mathcal{A}$  and  $\mathcal{B}$  are said to be *compatible* if their sum  $\mathcal{A} + \mathcal{B}$  (or, equivalently, an arbitrary linear combination of  $\mathcal{A}$  and  $\mathcal{B}$  with constant coefficients) is again a Poisson structure.

The non-trivial and essential part of the compatibility condition is that the sum of two Poisson brackets also satisfies the Jacobi identity. This From the analytical viewpoint, this condition is equivalent to the fact that the so-called Schouten bracket  $\{\{\mathcal{A}, \mathcal{B}\}\}$  of the Poisson structures  $\mathcal{A}$  and  $\mathcal{B}$  vanishes, which amounts to a rather non-trivial system of PDEs. A local description of compatible Poisson brackets can be found in (see [30, 48]).

**Example 3.** Any two constant Poisson brackets are compatible.

**Example 4.** Let  $\mathfrak{g}$  be a finite-dimensional (real) Lie algebra and  $\mathfrak{g}^*$  its dual space endowed with the standard Lie–Poisson bracket (1.45)

Along with this standard Lie–Poisson bracket, on the dual space  $\mathfrak{g}^*$  we can define a constant bracket  $\{ , \}_a$  for any  $a \in \mathfrak{g}^*$  by

$$\{f, g\}_a(x) = a([df(x), dg(x)]) = c_{jk}^i a_i \frac{\partial f}{\partial x_j} \frac{\partial g}{\partial x_k}. \quad (1.48)$$

It can be easily verified that (1.45) and (1.48) are compatible.

**Example 5.** Let  $\mathfrak{g} = \mathfrak{so}(n)$  be considered as the space of skew-symmetric  $(n \times n)$ -matrices. As usual, we identify  $\mathfrak{so}(n)$  and  $\mathfrak{so}(n)^*$  by means of the Killing form. Along with the standard commutator  $[X, Y] = XY - YX$  we introduce on  $\mathfrak{so}(n)$  another operation:

$$[X, Y]_B = XBY - YBX, \quad (1.49)$$

where  $A$  is a symmetric matrix.

It is easy to see that  $[X, Y]_B$  satisfies the Jacobi identity and is compatible with the standard commutator in the sense that any of linear combinations  $\lambda[ , ] + \lambda'[ , ]_B = [ , ]_{\lambda E + \lambda' B}$  still defines a Lie algebra structure on  $\mathfrak{so}(n)$  (considered as the space of skew-symmetric matrices). Such an algebraic structure (i.e. linear family of Lie algebras) is called a Lie pencil.

Interpreting this observation into the dual language, we may say that on  $\mathfrak{so}(n) = \mathfrak{so}(n)^*$  there is a pencil of compatible Poisson brackets  $\{ , \}_{B + \lambda E}$  related to the commutators  $[ , ]_{B + \lambda E}$ .

**Example 6.** In some (very exceptional!) cases Examples 4 and 5 can be combined as follows. Let

- $\{ , \}$  be the standard  $\mathfrak{so}(3)$  bracket (1.46),

- $\{ \cdot, \cdot \}_B$  be the bracket related to the commutator (1.49) where  $B$  is a diagonal  $3 \times 3$  matrix and
- $\{ \cdot, \cdot \}_g$  be the constant bracket defined by the matrix

$$\begin{pmatrix} 0 & g_3 & -g_2 \\ -g_3 & 0 & g_1 \\ g_2 & g_1 & 0 \end{pmatrix}$$

Then these three brackets are all compatible in the sense that any linear combination

$$\lambda \{ \cdot, \cdot \} + \mu \{ \cdot, \cdot \}_B + \nu \{ \cdot, \cdot \}_g$$

is still a Poisson bracket. This family of Poisson bracket is closely related to the Zhukovskii-Volterra systems discussed below. Notice that for  $n > 3$  this construction fails: the brackets described above are not compatible in general.

Many integrable dynamical systems in mathematical physics and mechanics possess the property of being *bi-Hamiltonian*, i.e., Hamiltonian with respect to two compatible Poisson brackets  $\mathcal{A}$  and  $\mathcal{B}$  (or with respect to any non-trivial combination  $\lambda'\mathcal{A} + \lambda\mathcal{B}$ ). This property can be understood as an additional symmetry of a given system which leads to the existence of a big algebra of commuting first integrals. These integrals can be constructed by using the so-called Magri–Lenard scheme (see [41, 40]). Here we recall one of its versions.

Consider a family (pencil) of compatible Poisson brackets  $\mathcal{J} = \{\lambda'\mathcal{A} + \lambda\mathcal{B} \mid \lambda', \lambda \in \mathbb{R}\}$  on a manifold  $M$ .

**Convention.** In this theory, one considers linear combinations  $\lambda'\mathcal{A} + \lambda\mathcal{B}$  up to proportionality, so that we may assume that  $\lambda' = 1$ , but  $\lambda$  may have value  $\infty$ . Thus we shall use notation  $\mathcal{A}_\lambda = \mathcal{A} + \lambda\mathcal{B}$  (assuming that  $\lambda \in \overline{\mathbb{R}}$  or  $\lambda \in \overline{\mathbb{C}}$ ) and shall sometimes refer to  $\mathcal{B}$  as  $\mathcal{A}_\infty$ .

Assume that all  $\mathcal{A}_\lambda \in \mathcal{J}$  are degenerate, i.e.  $\text{rank } \mathcal{A}_\lambda < \dim M$ . By definition, we set the *rank of the pencil*  $\mathcal{J}$  to be

$$\text{rank } \mathcal{J} = \max_{\lambda \in \overline{\mathbb{R}}} \text{rank } \mathcal{A}_\lambda.$$

If  $\text{rank } \mathcal{A}_\lambda$  is maximal in the family  $\mathcal{J}$ , i.e.,  $\text{rank } \mathcal{A}_\lambda = \text{rank } \mathcal{J}$ , we shall say that  $\mathcal{A}_\lambda$  is *generic*.

The next statement gives a recipe for constructing a large family of commuting functions on  $M$ .

**Theorem 1.2.8** *Let  $\mathcal{F}_\mathcal{J}$  be the algebra generated (with respect to usual multiplication of functions) by Casimir functions of all generic Poisson structures  $\mathcal{A}_\mu \in \mathcal{J}$ .*

- 1)  $\mathcal{F}_\mathcal{J}$  is commutative with respect to every Poisson structure  $\mathcal{A}_\lambda \in \mathcal{J}$ .
- 2) If  $\dot{x} = v(x)$  is a dynamical system which is Hamiltonian with respect to every generic Poisson structure  $\mathcal{A}_\mu \in \mathcal{J}$ , then each function from  $\mathcal{F}_\mathcal{J}$  is its first integral.

For simplicity, we shall assume that the pencil  $\mathcal{F}_{\mathcal{J}}$  satisfies the following natural conditions (which are fulfilled for almost all known examples)

- $M$  and  $\mathcal{J}$  are real-analytic.
- The Casimir functions of every Poisson structure  $\mathcal{A}_{\lambda} \in \mathcal{J}$  are globally defined and they distinguish all symplectic leaves of maximal dimension. More precisely, we assume the following: if  $x \in M$  is a point of maximal rank for  $\mathcal{A}_{\lambda}$ , then the kernel of  $\mathcal{A}_{\lambda}(x)$  is generated by the differentials of Casimir functions  $f \in \mathcal{Z}(\mathcal{A}_{\lambda})$ :

$$\text{Ker } \mathcal{A}_{\lambda}(x) = \text{span} \{df(x) \mid f \in \mathcal{Z}(\mathcal{A}_{\lambda})\}.$$

- The family  $\mathcal{F}_{\mathcal{J}}$  admits a basis, i.e., a collection of functionally independent functions  $f_1, \dots, f_N \in \mathcal{F}_{\mathcal{J}}$  such that every (basic) Casimir function  $f_{\lambda} \in \mathcal{Z}(\mathcal{A}_{\lambda})$  can be expressed as a smooth function  $f_{\lambda} = F(f_1, \dots, f_N)$  (this property must hold for almost all  $\lambda \in \mathbb{R}$ ).

Notice that Theorem 1.2.8 says nothing about the number  $N$  of functionally independent integrals in the family  $\mathcal{F}_{\mathcal{J}}$ . Recall that the completeness condition for  $\mathcal{F}_{\mathcal{J}}$  that guarantees Liouville integrability is  $N = \frac{1}{2}(\dim M + \text{corank } \mathcal{J})$ . A necessary and sufficient condition for completeness is given by the following

**Theorem 1.2.9 ([8])** *The family  $\mathcal{F}_{\mathcal{J}}$  is complete if and only if for a generic point  $x \in M$  the following maximal rank condition holds:*

$$\text{rank } \mathcal{A}_{\lambda}(x) = \text{rank } \mathcal{J} \quad \text{for all } \lambda \in \overline{\mathbb{C}}. \quad (1.50)$$

There is a natural and efficient principle that allows us to verify this completeness condition. To formulate it, we first notice that if the family  $\mathcal{F}_{\mathcal{J}}$  is complete, then all the structures  $\mathcal{A}_{\lambda}$  must be of the same rank on  $M$ , but, for each  $\lambda$ , the rank of  $\mathcal{A}_{\lambda}(x)$  may drop on a certain singular set  $S_{\lambda} = \{x \in M \mid \text{rank } \mathcal{A}_{\lambda}(x) < \text{rank } \mathcal{J}\}$ . From the viewpoint of completeness these points are “bad”. Condition (1.50) simply says that for completeness there must exist “good” points which belong to none of  $S_{\lambda}$ ’s. For such points to exist, it is sufficient to require that singular sets  $S_{\lambda}$  have codimension at least two. Then the union of sets  $S_{\lambda}$  over all  $\lambda$ ’s will have codimension at least one and its complement will consist of “good” points, as needed.

In this “*codimension two principle*” there is a subtle point: we consider a real manifold  $M$ , but the parameter  $\lambda$  is complex so that from the real point of view, the “space of parameters” is not one-, but two-dimensional. However, in concrete examples we have to deal with, the difference between “real” and “complex”, in fact, disappears. The point is that we usually work with algebraic objects (manifolds, Poisson structures, Casimir functions) which can be naturally complexified: we can introduce a new complex manifold  $M^{\mathbb{C}}$  endowed with the complex Poisson pencil  $\mathcal{J}^{\mathbb{C}}$  and construct the corresponding family of complex functions  $\mathcal{F}_{\mathcal{J}}^{\mathbb{C}}$ . In all natural situations, the complex functions that generate  $\mathcal{F}_{\mathcal{J}}^{\mathbb{C}}$  are obtained from the real functions  $f(x_1, \dots, x_n)$  generating  $\mathcal{F}_{\mathcal{J}}$  just by replacing real variables  $x_i$  with complex ones  $z_i \in \mathbb{C}$ . If such a complexification is well-defined, then we have the following

**Theorem 1.2.10 (Codimension Two Principle)** *Let all the brackets  $\mathcal{A}_\lambda$ ,  $\lambda \in \mathbb{C}$ , have the same rank and  $\text{codim } S_\lambda \geq 2$  for almost all  $\lambda \in \mathbb{C}$ . Then  $\mathcal{F}_\mathcal{J}^\mathbb{C}$  is complete. The completeness of  $\mathcal{F}_\mathcal{J}^\mathbb{C}$  is equivalent to the completeness of  $\mathcal{F}_\mathcal{J}$ .*

**Critical points of the momentum mapping** As was already noticed, the structure of the set of critical points of the momentum mapping plays an important role in the study of topological properties of integrable Hamiltonian systems.

Suppose that we have  $n$  commuting independent (almost everywhere) integrals  $F_1, \dots, F_n : M \rightarrow \mathbb{R}$  of a Hamiltonian system given on a symplectic manifold  $(M^{2n}, \omega)$ . Then we can naturally define the momentum mapping

$$\Phi : M \rightarrow \mathbb{R}^n, \quad \Phi(x) = (F_1(x), \dots, F_n(x)).$$

**Definition 5** *We will say that a point  $x \in M$  is a critical point of the momentum mapping if  $\text{rank } d\Phi(x) < n$ .*

In real problems we have to deal with, the situation may often be slightly different. First of all, the phase space of a system is often not symplectic, but Poisson. In this case, it is natural to add Casimir functions to a given family  $\mathcal{F}$  of first integrals and consider them all together. Also, for some families  $\mathcal{F}$  there is no canonical method for choosing a basis. To avoid this ambiguity, it is convenient to work with the Poisson algebra generated by the given commuting integrals and Casimir functions. Since we do not add any essentially new integrals, we will use the same notation  $\mathcal{F}$  for this “wider” algebra of first integrals.

**Definition 6** *Let  $(M, \mathcal{A})$  be a Poisson manifold and let  $\mathcal{F} \subset C^\infty(M)$  be a complete commutative Poisson algebra of functions on  $M$ . We will say that a point  $x \in M$  is a critical point for  $\mathcal{F}$  if the subspace  $d\mathcal{F}(x) \subset T_x^*M$  generated by the differentials  $df(x)$  of all functions  $f \in \mathcal{F}$  is not maximal isotropic with respect to  $\mathcal{A}$ .*

It is clear that the standard Definition 5 of a critical point of the momentum mapping is a particular case of Definition 6. The reason for such a modification is that now we don’t need to fix any universal basis in the algebra of integrals, but may choose appropriate basis integrals depending on a point  $x \in M$  under consideration which can be quite convenient. The modification of the other definitions and constructions discussed below to the case of Poisson manifolds is straightforward, and we will follow the standard “symplectic” setting.

Consider the *set of critical points* of the momentum mapping:

$$\mathcal{K} = \{x \in M \mid \text{rank } d\Phi(x) < n\}.$$

Its image  $\Sigma = \Phi(\mathcal{K}) \subset \mathbb{R}^n$  is called the *bifurcation diagram* of  $\Phi$ .

If  $a \notin \Sigma$ , then its preimage  $\Phi^{-1}(a)$  is a disjoint union of Liouville tori. These tori transform smoothly in  $M$  under any continuous change of  $a$  outside  $\Sigma$ , however, if  $a$  passes through  $\Sigma$ , then Liouville tori undergo a bifurcation.

It is clear that the topological properties of the momentum mapping  $\Phi$ , its singular set  $\mathcal{K}$ , and bifurcation diagram  $\Sigma$  keep very important information about qualitative behavior of a given dynamical system both in local and in global. Roughly speaking, they help us to understand and to describe the structure of the fibration  $\mathcal{L}$ , which, in turn, can be viewed as a portrait of the system and contains almost all qualitative information we usually want to know about the system (number and types of equilibrium points, stability of solutions, bifurcations of tori, Hamiltonian monodromy and so on).

Let us recall some basic notions and terminology related to this subject (see [11] for details).

We say that  $x \in M$  is a *critical point of corank  $k$*  (or, equivalently, *of rank  $(n - k)$* ) if  $\text{rank } d\Phi(x) = n - k$ . This condition is equivalent to the fact that the orbit  $\iota(x)$  of the  $\mathbb{R}^n$ -action generated by the integrals passing through  $x$  has dimension  $n - k$ . A singular fiber  $L$  of the Lagrangian fibration  $\mathcal{L}$  may contain several orbits of different dimension (the standard situation is that this fiber is a stratified manifold whose strata are those orbits). If  $n - k = \min_{x \in L} \dim \iota(x)$ , we shall say that  $L$  is a *singularity of corank  $k$* .

First of all, as usual in the theory of singularities, one distinguishes the class of generic (or non-degenerate) singularities.

We recall this definition first for critical points  $x \in M$  of corank  $n$ . In other words, we assume that the Hamiltonian vector fields of the integrals  $F_1, \dots, F_n$  all vanish at  $x$ . From the dynamical viewpoint, such points can usually be characterized as isolated equilibria of the system.

**Definition 7** *Let  $\text{rank } d\Phi(x) = 0$ . Then the critical point  $x$  is called non-degenerate if the Hessians  $d^2F_1(x), \dots, d^2F_n(x)$  are linearly independent and there exists a linear combination  $\lambda_1 d^2F_1(x) + \dots + \lambda_n d^2F_n(x)$  such that the roots of its “characteristic polynomial”*

$$\chi(t) = \det \left( \sum_{i=1}^n \lambda_i d^2F_i(x) - t \cdot \omega \right) \quad (1.51)$$

*are all distinct.*

*In a more abstract terminology, the non-degeneracy condition (for a critical point of corank  $n$ ) means that the linearizations of the Hamiltonian vector fields  $\text{sgrad } F_1, \dots, \text{sgrad } F_n$  at the point  $x$  generate a Cartan subalgebra in the symplectic Lie algebra  $\text{sp}(T_x M)$ .*

It is not hard to generalize this definition to the case of arbitrary rank of  $d\Phi(x)$  (see, for example, [11]). But in this work, we shall discuss two degrees of freedom systems only, so in addition to Definition 7, we’ll need the definition of non-degeneracy for critical points of corank 1 only.

**Definition 8** *Let  $x \in \mathcal{K}$  be a critical point of corank 1, i.e.,  $\text{rank } d\Phi(x) = n - 1$ . This point is called non-degenerate if there exists a function  $f \in \mathcal{F}$  such that  $df(x) = 0$  and the linearization of the Hamiltonian vector field  $\text{sgrad } f$  at  $x$  has at least one non-zero eigenvalue.*

Equivalently, this condition means that the restriction of  $f$  onto the common level of arbitrary  $n - 1$  independent integrals  $f_1, \dots, f_{n-1} \in \mathcal{F}$  passing through  $x$  is a Bott function.

In the case of two degrees of freedom, non-degenerate singularities of corank one represent closed trajectories. They can be of two kinds: hyperbolic and elliptic depending on the type of eigenvalues which can be respectively real and pure imaginary. In the latter case, the trajectory is stable, hyperbolic trajectories are unstable.

Non-degenerate critical points of the momentum mapping possess a number of remarkable properties. One of them is the existence of a very simple and natural local normal form, see [21].

**Theorem 1.2.11 (Eliasson Theorem, [21])** *Let  $x$  be a non-degenerate critical point of rank  $l$ . Then in a neighborhood of  $x$ , there exist symplectic coordinates  $p_1, \dots, p_n, q_1, \dots, q_n$  and a diffeomorphic transformation of the integrals*

$$\tilde{F}_1 = \tilde{F}_1(F_1, \dots, F_n), \quad \dots, \quad \tilde{F}_n = \tilde{F}_n(F_1, \dots, F_n)$$

such that

$$\tilde{F}_1 = p_1, \quad \dots, \quad \tilde{F}_l = p_l,$$

and  $\tilde{F}_i$  for  $i = l + 1, \dots, n$  has one of the following forms:

- 1)  $\tilde{F}_i = p_i^2 + q_i^2$  (elliptic case),
- 2)  $\tilde{F}_i = p_i q_i$  (hyperbolic case),
- 3)  $\begin{aligned} \tilde{F}_i &= p_i q_{i+1} - p_{i+1} q_i \\ \tilde{F}_{i+1} &= p_i q_i + p_{i+1} q_{i+1} \end{aligned}$  (focus-focus case).

In the case of two degrees of freedom, in addition to nondegenerate closed trajectories, there are non-degenerate equilibrium points of four types depending on the roots of the characteristic polynomial (1.51):

- 1) two pairs of real roots  $\lambda, -\lambda, \mu, -\mu$  (saddle-saddle type),
- 2) pair of real and pair of imaginary roots  $\lambda, -\lambda, i\mu, -i\mu$  (saddle-center type),
- 3) two pairs of imaginary roots  $\lambda, -\lambda, \mu, -\mu$  (center-center type),
- 4) four complex roots  $\lambda + i\mu, \lambda - i\mu, -\lambda + i\mu, -\lambda - i\mu$  (focus-focus).

Thus, the Eliasson theorem shows that the local structure of non-degenerate singularities (up to a symplectomorphism) can be uniquely characterised by its *type*, i.e., its (co)rank and the number of elliptic, hyperbolic and focus-focus components.

Before starting any global topological analysis for a specific integrable system we have, as a rule, to deal with the following tasks:

- 1) describe the set of critical points;
- 2) verify the non-degeneracy condition for observed singularities;
- 3) find the type of non-degenerate singularities.



The straightforward approach is just to take the Jacobi matrix

$$d\Phi(x) = \begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \dots & \frac{\partial F_1}{\partial x_{2n}} \\ \frac{\partial F_2}{\partial x_1} & \dots & \frac{\partial F_2}{\partial x_{2n}} \\ \dots & \dots & \dots \\ \frac{\partial F_n}{\partial x_1} & \dots & \frac{\partial F_n}{\partial x_{2n}} \end{pmatrix}, \quad (1.52)$$

compute all its  $(n \times n)$ -minors, and then find those points where all of them vanish. Then for each critical point we need to analyze the Hessians of the integrals in order to verify non-degeneracy and determine the type of the corresponding singularity.

It turns out that the property of being bi-Hamiltonian affects the structure of singularities of a system and helps to simplify its topological analysis by reformulating the above questions in terms of compatible Poisson brackets.

**Singularities of bi-Hamiltonian systems** We now consider a bi-Hamiltonian dynamical system and take the algebra  $\mathcal{F}_{\mathcal{J}}$  of its integrals generated by the Casimir functions of the pencil of Poisson brackets  $\mathcal{J} = \{\mathcal{A} + \lambda\mathcal{B} \mid \lambda \in \mathbb{R}\}$ . Suppose that this algebra is complete and therefore according to the general construction, all the brackets in the pencil are of the same rank.

Under the natural assumptions formulated above, the *set of critical points for the family  $\mathcal{F}_{\mathcal{J}}$*

$$\mathcal{K}_{\mathcal{J}} = \{x \in M \mid \dim d\mathcal{F}_{\mathcal{J}}(x) < \frac{1}{2}(\dim M + \text{corank } \mathcal{J})\}$$

can be described as follows.

As before, for each  $\lambda \in \mathbb{C}$ , we consider the *set of singular points of the Poisson structure  $\mathcal{A}_{\lambda}$  in  $M$*

$$S_{\lambda} = \{x \in M \mid \text{rank}(\mathcal{A}(x) + \lambda\mathcal{B}(x)) < \text{rank } \mathcal{J}\}.$$

In addition, we formally set  $S_{\infty} = \{x \in M \mid \text{rank } \mathcal{B}(x) < \text{rank } \mathcal{J}\}$ . Also consider the *set of singular points of the pencil  $\mathcal{J}$* :

$$S_{\mathcal{J}} = \bigcup_{\lambda \in \overline{\mathbb{C}}} S_{\lambda}.$$

**Theorem 1.2.12 ([12])** *A point  $x \in M$  is a critical point for  $\mathcal{F}_{\mathcal{J}}$  if and only if there exists  $\lambda \in \overline{\mathbb{C}}$  such that  $x \in S_{\lambda}$ . In other words,  $\mathcal{K}_{\mathcal{J}} = S_{\mathcal{J}}$ .*

Thus, in the case of bi-Hamiltonian systems the set of critical points of the momentum mapping has a natural description in terms of the singular sets of the Poisson structures  $\mathcal{A}_{\lambda}$ ,  $\lambda \in \overline{\mathbb{C}}$ . Here we see the following general principle: the singularities of the Lagrangian fibration associated with a bi-Hamiltonian system are defined by the singularities of the pencil  $\mathcal{J}$ .

Now let  $x$  belong to a regular symplectic leaf of the Poisson structure  $A = A_0$ . We say that  $x$  is a common equilibrium point for  $\mathcal{F}_{\mathcal{J}}$  if  $\text{sgrad } f(x) = A df(x) = 0$  for any  $f \in \mathcal{F}_{\mathcal{J}}$ .

**Theorem 1.2.13** ([12]) *A point  $x \in M$  is a common equilibrium point for  $\mathcal{F}_{\mathcal{J}}$  if and only if the kernels of all (regular) brackets at this point coincide*

Equivalently, for  $x$  to be a common equilibrium it is sufficient to require that the kernels of *just two* brackets coincide:  $\text{Ker } \mathcal{A}_{\mu}(x) = \text{Ker } \mathcal{A}_{\lambda}(x)$ ,  $\mu \neq \lambda$ .

Now assume that  $x \in M$  is a critical point of corank 1. This means that the dimension of the subspace  $d\mathcal{F}_{\mathcal{J}}(x)$  is  $k - 1$ , where  $k$  is the dimension of the maximal isotropic subspace in  $T_x^*M$ . In this case, there exists a unique  $\lambda \in \mathbb{R}$  such that the rank of  $\mathcal{A}(x) + \lambda\mathcal{B}(x)$  is not maximal.

According to the Weinstein theorem [63], the Poisson structure  $\mathcal{A}_{\lambda}$  in a small neighborhood of  $x$  splits into direct product of the transversal Poisson structure and the non-degenerate Poisson structure defined on the symplectic leaf through  $x$ . The non-degeneracy condition is formulated in terms of this transversal structure. For simplicity we shall assume that the Poisson structure  $\mathcal{A}_{\lambda}$  is semisimple in the sense that  $M$  has a natural identification with a real semisimple Lie algebra  $\mathfrak{g} = \mathfrak{g}^*$  endowed with the standard Lie–Poisson bracket.

Then  $x$  is a singular element in the semisimple Lie algebra  $\mathfrak{g}$  and the dimension of its centralizer (which, as we know, coincides with the kernel of  $\mathcal{A}_{\lambda}(x)$ ) is  $\text{ind } \mathfrak{g} + 2$ . For simplicity, we shall assume that  $x \in \mathfrak{g}$  is a semisimple element (this is obviously a generic case). Then the centralizer of  $x$  in  $\mathfrak{g}$  is a Lie subalgebra of the form  $\mathfrak{u} \oplus \mathbb{R}^{l-1}$ , where  $\mathfrak{u}$  is a three-dimensional real semisimple Lie algebra and  $l = \text{ind } \mathfrak{g}$ .

Consider another bracket  $\mathcal{A}_{\mu}$  from  $\mathcal{J}$ ,  $\mu \neq \lambda$ , and take the restriction of  $\mathcal{A}_{\mu}(x)$  to  $\text{Ker } \mathcal{A}_{\lambda}(x) = \mathfrak{u} \oplus \mathbb{R}^{l-1}$ . Then  $\text{Ker } (\mathcal{A}_{\mu}(x)|_{\text{Ker } \mathcal{A}_{\lambda}(x)})$  has codimension 2 in  $\text{Ker } \mathcal{A}_{\lambda}(x) = \mathfrak{u} \oplus \mathbb{R}^{l-1}$ . It can be easily checked that the center  $\mathbb{R}^{l-1}$  belongs to  $\text{Ker } (\mathcal{A}_{\mu}(x)|_{\text{Ker } \mathcal{A}_{\lambda}(x)})$ . This means that the restriction of  $\mathcal{A}_{\mu}$  to  $\mathfrak{u}$  has rank 2 and  $\text{Ker } (\mathcal{A}_{\mu}(x)|_{\mathfrak{u}})$  is generated by some vector  $\xi \in \mathfrak{u}$ . It turns out that non-degeneracy condition can be naturally formulated in terms of this vector  $\xi$ . Namely, the following holds.

**Theorem 1.2.14** ([12]) *Let  $x$  be a corank 1 critical point of  $\mathcal{F}_{\mathcal{J}}$ . Suppose that*

- 1) *there exists unique  $\lambda \in \overline{\mathbb{R}}$  such that  $\text{rank } \mathcal{A}_{\lambda}(x) < \text{rank } \mathcal{J}$ ,*
- 2) *the bracket  $\mathcal{A}_{\lambda}$  is semisimple, i.e.,  $(M, \mathcal{A}_{\lambda})$  has a natural identification with the dual space  $\mathfrak{g}^*$  of a real semisimple Lie algebra  $\mathfrak{g}$  endowed with the standard Lie–Poisson bracket,*
- 3)  *$x$  is a semisimple singular element in  $\mathfrak{g}^* = \mathfrak{g}$ , and  $\text{Ker } \mathcal{A}_{\lambda}(x) = \mathfrak{u} \oplus \mathbb{R}^{l-1}$ , where  $\mathfrak{u}$  is a three-dimensional semisimple subalgebra,  $l = \text{ind } \mathfrak{g}$ ,*
- 4)  *$\text{Ker } (\mathcal{A}_{\mu}(x)|_{\mathfrak{u}})$  is generated by  $\xi \in \mathfrak{u}$ ,  $\xi \not\subseteq 0$ ,  $\mu \not\subseteq \lambda$ .*

*If  $\xi$  is semisimple element in  $\mathfrak{u}$ , then  $x$  is non-degenerate. Moreover, if  $\langle \xi, \xi \rangle > 0$ , then the singularity is hyperbolic, and if  $\langle \xi, \xi \rangle < 0$ , then the singularity is elliptic, where  $\langle \cdot, \cdot \rangle$  is the Killing form on  $\mathfrak{u}$ .*

The verification of non-degeneracy condition for higher corank singular points can be done in a similar way. In the case of two degrees of freedom, the points of corank 2 (i.e., equilibria) of the family  $\mathcal{F}_{\mathcal{J}}$  are characterized by the following condition: there are two values  $\lambda_1$  and  $\lambda_2$  for which the rank of  $\mathcal{A}_{\lambda} = \mathcal{A} + \lambda\mathcal{B}$

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drops (sometimes  $\lambda_1 = \lambda_2$  may represent a double characteristic number of the pencil). The sufficient non-degeneracy condition is that  $\lambda_1 \neq \lambda_2$  and for each  $\lambda = \lambda_i$  separately the assumptions 2), 3) and 4) of Theorem 1.2.14 are fulfilled.

## Chapter 2

# Explicit solution of the Zhukovski–Volterra gyrostat

The Chapter 2 is devoted to explicit integration of the classical generalization of the Euler top: the Zhukovski–Volterra system describing the free motion of a gyrostat. We revise the solution for the components of the angular momentum first obtained by Volterra in [58] and present an alternative solution based on an algebraic parametrization of the invariant curves. This also enables us to give an effective description of the motion of the body in space. The results of this problem was published in [6].

### 2.1 Introduction

One of the simplest known integrable systems of classical mechanics is the Euler top, which describes the motion of a free rigid body about a fixed point. Let  $\omega$  be the vector of the angular velocity of the body,  $J$  be its tensor of inertia and  $\bar{M} = (\bar{M}_1, \bar{M}_2, \bar{M}_3)^T = J\omega \in \mathbb{R}^3$  be the vector of the angular momentum. Then the evolution of  $\bar{M}$  is given by the well known Euler equations

$$\dot{\bar{M}} = \bar{M} \times a\bar{M}, \quad a = J^{-1} = \text{diag}(a_1, a_2, a_3), \quad (2.1)$$

which have two independent integrals

$$\langle \bar{M}, a\bar{M} \rangle = l, \quad \langle \bar{M}, \bar{M} \rangle = k^2, \quad l, k = \text{const}. \quad (2.2)$$

It is also well-known (see e.g., [62]) that the solution of this system is expressed in terms of elliptic functions associated to the elliptic curve  $E_0$  given by the equation

$$\mu^2 = -(\lambda - a_1)(\lambda - a_2)(\lambda - a_3)(\lambda - c), \quad c = l/k^2 \quad (2.3)$$

In the real motion, when  $a_1 < a_2 < a_3$ , one has  $c \in [a_1, a_3]$ .

Next, let  $\gamma$  be a unit vector fixed in space. The motion of the top in space is described by 3 independent solutions of the Poisson equations

$$\dot{\gamma} = \gamma \times \omega \equiv \gamma \times a\bar{M}(t).$$

The latter were completely solved by Jacobi [37], who gave explicit expressions for the Euler angles and the components of the rotation matrix of the body in terms of sigma-functions and exponents (see also [62]).

On the other hand, N. Zhukovski [64] and, independently, V. Volterra [58] investigated the problem of the motion of the rigid body with a cavity filled with an ideal incompressible liquid. They showed that in the frame attached to the body the evolution of its angular velocity  $\omega$  is described by the equations

$$\frac{d}{dt}(J\omega) = (J\omega + d) \times \omega,$$

where  $d$  is a constant vector characterizing the cyclical motion of the liquid in the cavity. From the form of the equation it is seen that the generalized angular momentum  $M = J\omega + d$  remains fixed in space. By setting

$$\omega = aM - g, \quad g = (g_1, g_2, g_3)^T = J^{-1}d,$$

one can rewrite this system in the form (1.1)

Like the Euler top, the Zhykovski–Volterra (ZV) system (1.1) has two quadratic integrals, which, however, are not all homogeneous in  $M_i$  (see(1.2)).

Then, for generic values of the constants  $k, l$ , the complex invariant variety of the system is again an elliptic curve (see formula (2.5) below).

However, in contrast to the classical Euler top, an explicit integration of the ZV system and, especially, the explicit description of the motion of the gyrostat in space appear to be a much more complicated problem.

In [58] Volterra presented expressions for the components of the momenta  $M$  and of the rotation matrix of the gyrostat in terms of sigma-functions and exponents, however, on our opinion, these expressions include several undetermined parameters and only provide the structure of the solution, but not explicit formulas.

An alternative method of integration of the ZV equations (1.1) only, which is based on a trigonometric parametrization of the intersection of two quadrics was proposed in [60] (see also [16]).

To obtain the evolution of the ZV top in space, one could also apply the powerful method of Baker–Akhiezer functions based on a representation of the system (1.1) in a Lax form. Following [27], there exists the following Lax pair with a parameter ranging over the elliptic curve  $\{w^2 = (s - a_1)(s - a_2)(s - a_3)\}$ , namely

$$\dot{L}(s) = [L(s), A(s)], \quad L, A \in so(3), \quad (2.4)$$

$$L_{\alpha\beta}(s) = \varepsilon_{\alpha\beta\gamma} \left( \sqrt{(s - a_\gamma)} M_\gamma + g_\gamma / \sqrt{(s - a_\gamma)} \right),$$

$$A_{\alpha\beta}(s) = \varepsilon_{\alpha\beta\gamma} \sqrt{(s - a_\alpha)(s - a_\beta)} M_\gamma, \quad (\alpha, \beta, \gamma) = (1, 2, 3),$$

where  $\varepsilon_{\alpha\beta\gamma}$  is the Levi-Civita symbol.

As calculations show, the spectral curve of the matrix  $L(s)$  has genus 5. On the other hand, it is known that the complex invariant manifold is an algebraic curve of genus 1. Hence, a direct application of the method seems to be ineffective, since it will lead to a complicated process of reduction of theta-functions to elliptic functions.

**Contains of the Chapter.** In Section 2.2, following Volterra [58], we briefly derive the explicit solution of the Euler equations (1.1) in terms of elliptic functions.

In Section 2.3 we provide a new alternative solution of these equations by using an algebraic parametrization of the momentum  $M$  in terms of coordinates of the invariant elliptic curve (Proposition 2.3.1).

In Section 2.4, by using this parametrization, we obtain explicit expressions for the Euler angles describing the motion of the gyrostat in space.

## 2.2 Volterra's solution of the ZV system

To give the explicit solution of the ZV system we note that its complex invariant variety is the intersection of the two quadrics in  $\mathbb{C}^3 = (M_1, M_2, M_3)$  defined by the integrals (1.2). Thus, according to a theorem of the algebraic geometry (see e.g., [23]), in the generic case this intersection is an open subset of a spatial elliptic curve, which is birationally equivalent to the plane curve

$$E = \{w^2 = P_4(z)\}, \quad (2.5)$$

where  $P_4(z)$  is a polynomial of degree 4 given by the discriminant equation

$$P_4 = \begin{vmatrix} z - a_1 & 0 & 0 & g_1 \\ 0 & z - a_2 & 0 & g_2 \\ 0 & 0 & z - a_3 & g_3 \\ g_1 & g_2 & g_3 & l - kz \end{vmatrix} = -k(z - \lambda_1)(z - \lambda_2)(z - \lambda_3)(z - \lambda_4). \quad (2.6)$$

The curve  $E$  can be viewed as 2-fold cover of the Riemann sphere  $\mathbb{P} = \mathbb{C} \cup \infty$  ramified at the roots  $\lambda_j$  of  $P_4(z)$ .

In the sequel we assume that the roots  $\lambda_\alpha$  are all distinct (otherwise  $E$  becomes singular and the corresponding solutions are asymptotic). In the real motion, when the quadrics defined by (1.2) have a nonempty real intersection, all  $\lambda_\alpha$  are either real or two of them are real and the other two are complex conjugated.

For future purposes it is also convenient to represent the curve  $E$  in the canonical Weierstrass form

$$W^2 = 4(Z - e_1)(Z - e_2)(Z - e_3), \quad e_1 + e_2 + e_3 = 0,$$

which is parameterized by the elliptic  $\wp$ -function of Weierstrass and its derivative:

$$Z = \wp(u|\omega_1, \omega_3), \quad W = \frac{d}{du}\wp(u|\omega_1, \omega_3).$$

Here  $\omega_1, \omega_3$  are half-periods of  $\wp$  (note that  $\omega_2 = -\omega_1 - \omega_3$ ) and  $e_i = \wp(\omega_i)$ .

Let  $\sigma(u|2\omega_1, 2\omega_3), \sigma_1(u), \sigma_2(u), \sigma_3(u)$ ,  $u \in \mathbb{C}$  be the sigma functions of Weierstrass such that

$$\sqrt{\wp(u|\omega_1, \omega_3) - e_\alpha} = \frac{\sigma_\alpha(u)}{\sigma(u)}, \quad \alpha = 1, 2, 3.$$

Then, following V. Volterra [58], we have

**Theorem 2.2.1** *The explicit complex solution of the ZV equations (1.1) with the constants of motion  $k, l$  has the form*

$$M_i(t) = \frac{\sum_{\alpha=1}^3 A_{i\alpha} \mu_\alpha \sigma_\alpha(u) + A_{i4} \mu_4 \sigma(u)}{\sum_{\alpha=1}^3 A_{4\alpha} \mu_\alpha \sigma_\alpha(u) + A_{44} \mu_4 \sigma(u)}, \quad i = 1, 2, 3, \quad u = \delta t + u_0, \quad (2.7)$$

where  $u_0$  is a constant phase of the solution and

$$A_{i\alpha} = \frac{g_i}{(a_i - \lambda_\alpha) \sqrt{\Delta_\alpha}}, \quad i = 1, 2, 3, \quad A_{4\alpha} = \frac{\sqrt{-k}}{\sqrt{\Delta_\alpha}}, \quad \alpha = 1, 2, 3, 4, \quad (2.8)$$

$$\Delta_\alpha = \sum_{i=1}^3 \frac{g_i^2}{(a_i - \lambda_\alpha)^2} - k, \quad \mu_\alpha = \frac{\delta}{h_1 \sqrt{(\lambda_j - \lambda_i)(\lambda_k - \lambda_i)}}. \quad (2.9)$$

The parameters of the curve  $E$  in the canonical form are given by

$$e_i = \frac{\rho}{\delta^2} (\lambda_i \lambda_4 + \lambda_j \lambda_k), \quad \rho = [108(\lambda_4 - \lambda_1)^2 (\lambda_4 - \lambda_2)^2 (\lambda_4 - \lambda_3)^2]^{-1/3}. \quad (2.10)$$

**Remark 1.** Note that due to the invariance property  $\sigma(\lambda u | \lambda \omega_1, \lambda \omega_3) = \lambda \sigma(u | \omega_1, \omega_3)$  the rescaling parameter  $\delta$  can be chosen arbitrary nonzero.

**Remark 2.** As follows from (2.6), in the limit  $g \rightarrow 0$  one has  $\{\lambda_1, \dots, \lambda_4\} \rightarrow \{a_1, a_2, a_3, c\}$ . Assuming, without loss of generality, that  $\lambda_i \rightarrow a_i$ ,  $\lambda_4 \rightarrow c$ , one can show that

$$\lim_{g \rightarrow 0} \frac{g_i}{(a_i - \lambda_i)} = \infty, \quad \lim_{g \rightarrow 0} \frac{g_j}{(a_j - \lambda_i)} = 0 \quad (j \neq i)$$

and, therefore,

$$\Delta_1, \Delta_2, \Delta_3 \rightarrow \infty, \quad \Delta_4 \rightarrow -k, \quad A_{i\alpha} = \delta_{i\alpha}, \quad A_{44} = 1.$$

Then in this limit, the solution (2.7) transforms to

$$M_\alpha(t) = \mu_\alpha \frac{\sigma_\alpha(u)}{\sigma(u)}, \quad \alpha = 1, 2, 3,$$

which coincides with the solution for the Euler top equations (2.1).

Using the quasi-periodic properties of  $\sigma_1(u), \dots, \sigma(u)$  one can show that the set of solutions (2.7) are elliptic functions of  $u$  with the minimal periods  $4\omega_1, 4\omega_3$  and that they have the same poles  $q_1, \dots, q_4$  in the corresponding parallelogram of periods. As a result, the solutions are not single-valued functions on the curve  $E_0$ , but on its 4-fold unramified covering  $E$  obtained by doubling the two periods of  $E_0$ . Therefore, the curves  $E_0$  and  $E$  are *isomorphic* and can be described by the same equation (2.2).

According to the Abel theorem, the momenta  $M_i$ , as functions of  $u$ , can also be written in the form

$$M_i(u) = \varkappa_i \frac{\bar{\sigma}(u - p_1^{(i)}) \bar{\sigma}(u - p_2^{(i)}) \bar{\sigma}(u - p_3^{(i)}) \bar{\sigma}(u - p_4^{(i)})}{\bar{\sigma}(u - q_1) \bar{\sigma}(u - q_2) \bar{\sigma}(u - q_3) \bar{\sigma}(u - q_4)}, \quad i = 1, 2, 3, \quad (2.11)$$

where  $\bar{\sigma}(u) = \sigma(u|4\omega_1, 4\omega_3)$  is the sigma-function with the doubled quasi-periods, the numbers  $p_1^{(i)}, \dots, p_4^{(i)}$  are the zeros of  $M_i$  such that, modulo the period lattice  $\{4\omega_1\mathbb{Z} + 4\omega_3\mathbb{Z}\}$ ,

$$p_1^{(i)} + \dots + p_4^{(i)} = q_1 + \dots + q_4,$$

and  $\varkappa_i$  are certain constants that depend on the periods  $4\omega_\alpha$  only.

The relation between the numbers  $p_\alpha^{(i)}, q_\alpha$  and the corresponding points  $P_\alpha^{(i)}, Q_\alpha$  on the "big" curve  $E$  is described by the elliptic integrals,

$$p_1^{(i)} = \mathcal{A}(P_1^{(i)}), \quad q_i = \mathcal{A}(Q_i), \quad \mathcal{A}(P) = \int_\infty^P \frac{dZ}{W}.$$

**Remark 3.** The solution in the form (2.11) was used by Volterra to give a description of the motion of the gyrostat in space. Unfortunately, the solution (2.7) does not provide the information about position of poles  $q_i$  and zeros  $p_i^{(j)}$  of  $M_i(t)$  in the above parallelogram of periods. More precisely, the zeros and poles appear as solutions of transcendental equations, obtained by equating to zero the denominator and the numerators of (2.7).

For this reason, in the next section we present another, algebraic solution of the ZV system.

*Proof of Theorem 2.2.1* As noticed by Volterra [58], equations (1.1) admit representation in the following symmetric form

$$\frac{dM_i}{dt} = \frac{\partial(f_1, f_2)}{\partial(M_j, M_k)} \equiv \begin{vmatrix} \frac{\partial f_1}{\partial M_j} & \frac{\partial f_1}{\partial M_k} \\ \frac{\partial f_2}{\partial M_j} & \frac{\partial f_2}{\partial M_k} \end{vmatrix}, \quad (2.12)$$

$f_1, f_2$  being the integrals (1.2). Here and below  $(i, j, k)$  is a cyclic permutation of the indices  $(1, 2, 3)$ .

Now let us set  $f_1(M) = k, f_2(M) = l, k, l = \text{const}$  and introduce projective coordinates  $z_1, \dots, z_4$  by formulas

$$M_i = \frac{z_i}{z_4}, \quad i = 1, 2, 3, \quad (z_1, z_2, z_3, z_4) \in C^4 \setminus \{0\}. \quad (2.13)$$

Substituting (2.13) into equations (2.12) we find

$$\frac{z_4 dz_i - z_i dz_4}{d\tau} = \frac{\partial(\varphi_1, \varphi_2)}{\partial(z_j, z_k)}, \quad (2.14)$$

where  $\varphi_l$  are homogenous quadratic forms in  $z_i$ ,

$$\varphi_l(z_1, z_2, z_3, z_4) = z_4^2 \left( f_l \left( \frac{z_1}{z_4}, \frac{z_2}{z_4}, \frac{z_3}{z_4} \right) - h_l \right), \quad l = 1, 2,$$

that is,

$$\begin{aligned} \varphi_1 &= z_1^2 + z_2^2 + z_3^2 - kz_4^2, \\ \varphi_2 &= a_1 z_1^2 + a_2 z_2^2 + a_3 z_3^2 - 2g_1 z_1 z_4 - 2g_2 z_2 z_4 - 2g_3 z_3 z_4 - lz_4^2. \end{aligned} \quad (2.15)$$



Lineal combinations of the relations (2.14) also imply

$$\frac{z_i dz_j - z_j dz_i}{d\tau} = \frac{\partial(\varphi_1, \varphi_2)}{\partial(z_j, z_4)}. \quad (2.16)$$

As was also noticed by Volterra, the relations (2.14), (2.16) are invariant with respect to any linear non-degenerate transformation of  $z_i$ . On the other hand, according to a well-known theorem of lineal algebra, there exists a unique transformation

$$z_r = \sum A_{rs} \xi_s, \quad r = 1, 2, 3, 4, \quad (2.17)$$

which reduces  $\varphi_1, \varphi_2$  to a diagonal form simultaneously:

$$\varphi_1 = \xi_1^2 + \dots + \xi_4^2, \quad \varphi_2 = \lambda_1 \xi_1^2 + \dots + \lambda_4 \xi_4^2, \quad (2.18)$$

$\lambda_r$  being the roots of (2.5). One easily derives the values of  $A_{rs}$  in the form (2.8).

Due to the invariance of the relations (2.14), (2.16), in the new coordinates one obtains

$$\xi_4 \dot{\xi}_i - \xi_i \dot{\xi}_4 = k(\lambda_r - \lambda_j) \xi_j \xi_r, \quad \xi_i \dot{\xi}_j - \xi_j \dot{\xi}_i = k(\lambda_r - \lambda_4) \xi_4 \xi_r. \quad (2.19)$$

It was observed in [58] that the system (2.14), (2.16) has the same structure as the system of differential equations for the four Weierstrass sigma-functions  $\sigma_1, \sigma_2, \sigma_3, \sigma_4$  of the complex argument  $u$

$$\sigma'_i \sigma_j - \sigma'_j \sigma_i = (e_j - e_i) \sigma \sigma_r, \quad \sigma'_i \sigma - \sigma' \sigma_i = -\sigma_j \sigma_r, \quad (2.20)$$

where

$$\sigma' = \frac{d\sigma(u)}{du}, \quad (i, j, k) = (1, 2, 3).$$

Then it is natural to look for the solution of the system (2.19) in the form

$$\xi_\alpha = \mu_\alpha \sigma_\alpha(u), \quad \alpha = 1, 2, 3, \quad \xi_4 = \mu_4 \sigma(u), \quad u = \delta t + u_0, \quad \delta = \text{const}, \quad (2.21)$$

where, without loss of generality, one can set  $\mu_4 = 1$ . Substituting (2.21) into (2.18) and using (2.20) we obtain the following system for the constants  $\mu_\alpha, e_\alpha, \delta$ :

$$\mu_i \delta = (\lambda_j - \lambda_k) \mu_j \mu_k, \quad \mu_i \mu_j \delta (e_j - e_i) = \mu_k, \quad (i, j, k) = (1, 2, 3). \quad (2.22)$$

Taking into account the condition  $e_1 + e_2 + e_3 = 0$ , we obtain the expressions (2.9) and (2.10). This proves the theorem.

Theorem 2.2.1 actually establishes the following relation between generic solutions of the ZV system and the Euler top, which will be used in the next section.

**Proposition 2.2.2** *Let  $M(t)$  be a solution of the ZV system with constants of motion  $k, l$  and the corresponding roots  $\lambda_1, \dots, \lambda_4$  defined in (2.6), and let  $\bar{M}(t)$  be the solution of the Euler equations (2.1) with the parameters  $a_\alpha = \lambda_\alpha$  and  $k = 1, l = \lambda_4$ . Then these solutions are related by the projective transformations*

$$M_i = g_i \frac{\sum_{\alpha=1}^3 \frac{\bar{M}_\alpha}{(a_i - \lambda_\alpha) \sqrt{\Delta_\alpha}} + \frac{1}{(a_i - \lambda_4) \sqrt{-\Delta_4}}}{\sum_{\alpha=1}^3 \frac{\bar{M}_\alpha}{\sqrt{\Delta_\alpha}} + 1/\sqrt{-\Delta_4}}, \quad (2.23)$$

where  $\Delta_i$  are defined in (2.9).

Indeed, the transformation (2.23) is equivalent to the composition of the substitutions  $\bar{M}_i = \xi_i/(\sqrt{-1}\xi_4)$ , (2.17), and (2.13). Under this composition the quadrics (1.2) transform to the homogeneous form

$$\bar{M}_1^2 + \bar{M}_2^2 + \bar{M}_3^2 = 1, \quad \lambda_1 \bar{M}_1^2 + \lambda_2 \bar{M}_2^2 + \lambda_3 \bar{M}_3^2 = \lambda_4, \quad (2.24)$$

which coincide with the Euler top integrals (2.2) if we set  $a_\alpha = \lambda_\alpha$  and  $k = 1, l = \lambda_4^1$ .

Next, since the equations (2.12) are invariant under the transformation (2.23), the variables  $\bar{M}_i$  satisfy these equations with  $f_1, f_2$  replaced by the left hand sides of (2.24), that is, the Euler equations (2.1).

**A numerical example.** Consider the motion of the ZV top with

$$a_1 = 3, \quad a_2 = 4, \quad a_3 = 5, \quad g = (0.5, 0.5, 0.5)^T$$

and the initial conditions  $M = (4, 4, 2)^T$ . Then the constants of motion will be  $l = 122, k = 36$  and up to  $10^{-8}$ ,

$$\{\lambda_1, \dots, \lambda_4\} = \{3.018214867, 3.386547967, 3.98841998, 4.995706073\}.$$

This implies that invariant curve of the ZV top with given initial conditions has two connected components. We then choose  $\lambda_1, \lambda_2, \lambda_3$  close to the values of  $a_i$ , that is,

$$\lambda_1 = 3.018214867, \quad \lambda_2 = 3.98841998, \quad \lambda_3 = 4.995706073, \quad \lambda_4 = 3.386547967.$$

Then

$$\begin{aligned} \Delta_1 &= 717.83121275, & \Delta_2 &= 1828.8243593, \\ \Delta_3 &= 13523.41092, & \Delta_4 &= -33.5664995. \end{aligned}$$

Substituting these values into (2.23), we obtain the following relation between the trajectories of both systems.

$$\begin{aligned} M_1 &= 0.5 \frac{-11.871 \bar{M}_1 - 0.13731 \bar{M}_2 - 2.4999 \times 10^{-2} \bar{M}_3 - 2.5873}{0.21606 \bar{M}_1 + 0.13572 \bar{M}_2 + 4.9891 \times 10^{-2} \bar{M}_3 + 1.0} \\ M_2 &= 0.5 \frac{0.22006 \bar{M}_1 + 11.700 \bar{M}_2 - 5.0107 \times 10^{-2} \bar{M}_3 + 1.6300}{0.21606 \bar{M}_1 + 0.13572 \bar{M}_2 + 4.9891 \times 10^{-2} \bar{M}_3 + 1.0} \\ M_3 &= 0.5 \frac{0.10902 \bar{M}_1 + 0.13416 \bar{M}_2 + 11.603 \bar{M}_3 + 0.61977}{0.21606 \bar{M}_1 + 0.13572 \bar{M}_2 + 4.9891 \times 10^{-2} \bar{M}_3 + 1.0} \end{aligned}$$

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<sup>1</sup>Note that in contrast to the "physical" Euler top, the parameters  $\lambda_i$  are not necessarily real positive.

### 2.3 Alternative parametrization of the ZV solution

Apart from the solution in terms of sigma-functions (2.7), one can give a rational parametrization of the momenta  $M_i$  in terms of the coordinates  $z, w$  on the elliptic curve  $E$  given by equation (2.5), that is,

$$w^2 = P_4(z) \equiv -k(z - \lambda_1)(z - \lambda_2)(z - \lambda_3)(z - \lambda_4).$$

For this purpose we first note that the coordinate  $w$  on has 4 simple zeros in the roots of  $\lambda_1, \dots, \lambda_4$  and 2 double poles in the two infinite points of the curve (denoted as  $\infty_-, \infty_+$ ), while the coordinate  $z$  has 2 simple zeros at the points  $O_{\pm} = (0, \pm\sqrt{P_4(0)})$  and two simple poles at  $\infty_-, \infty_+$ . The next proposition describes the structure of this parametrization.

**Proposition 2.3.1** 1) *The components of momenta  $M_i$  have the following natural parametrization in terms of the coordinates  $z, w$  on  $E$ :*

$$M_i = \frac{\alpha_i w + U_i(z)}{w + U_0(z)}, \quad U_i = u_{i2}z^2 + u_{i1}z + u_{i0}, \quad i = 0, 1, 2, 3, \quad (2.25)$$

where  $\alpha_i, u_{ij}$  are certain constants depending only on the values of the integrals (1.2), which are determined below in (2.45);

2) *The evolution of  $z$  is described by the quadrature*

$$\frac{dz}{w} \equiv \frac{dz}{\sqrt{-k(z - \lambda_1)(z - \lambda_2)(z - \lambda_3)(z - \lambda_4)}} = dt. \quad (2.26)$$

*Proof.* It is sufficient to show that the right hand sides of (2.25) has precisely 4 simple poles and zeros on the curve  $E$ , as required by the structure of the Volterra solution (2.7) or (2.11). Indeed, the zeros of the common denominator and numerators in (2.7) are uniquely defined by the equations

$$\begin{aligned} \sqrt{-k(z - \lambda_1)(z - \lambda_2)(z - \lambda_3)(z - \lambda_4)} + u_{02}z^2 + u_{01}z + u_{00} &= 0, \\ \sqrt{-k(z - \lambda_1)(z - \lambda_2)(z - \lambda_3)(z - \lambda_4)} + u_{i2}z^2 + u_{i1}z + u_{i0} &= 0, \end{aligned} \quad (2.27)$$

which, in general, lead to algebraic equations of degree 4 in  $z$ , whose roots are  $z$ -coordinates of the points  $Q_1, \dots, Q_4, P_1^{(j)}, \dots, P_4^{(j)} \in E$ . Finally, in view of the analytic behavior of  $z, w$  described above, at the infinite points  $\infty_-, \infty_+ \in \mathcal{E}$  both the denominator and the numerators in (2.7) have poles of second order, which implies that  $M_i$  are finite at  $\infty_{\pm}$ . ■

Thus, given the values of  $\alpha_i, u_{ij}$ , the parametrization (2.25) allows to determine the poles and zeros of the solution (2.11) in an algebraic way. That is, solving the equations (2.27) and obtaining the points  $Q_1, \dots, Q_4$  and  $P_1^{(j)}, \dots, P_4^{(j)}$ , one gets the required poles and zeros in (2.11) by calculating the integrals

$$q_{\alpha} = \mathcal{A}(Q_{\alpha}), \quad p_{\alpha}^{(j)} = \mathcal{A}(P_{\alpha}^{(j)}), \quad \text{where} \quad \mathcal{A}(P) = \int_c^P \frac{dz}{w(z)}.$$

**Determination of the constants  $\alpha_i, u_{ij}$ .** To calculate these constants, we first obtain a similar rational parametrization for the solutions  $\bar{M}_i$  of the standard Euler top equations (2.1).

Recall that the latter solutions admit the following irrational parametrization in terms of the coordinates on the elliptic curve (2.3) (see, e.g., [29])

$$\bar{M}_i = k \sqrt{\frac{(a_j - c)(a_k - c)}{(a_i - a_j)(a_i - a_k)}} \sqrt{\frac{\lambda - a_i}{\lambda - c}}, \quad (2.28)$$

$$(i, j, k) = (1, 2, 3), \quad \lambda \in \mathbb{C},$$

and the evolution of  $\lambda$  is given by the equation

$$\dot{\lambda} = 2\sqrt{-(\lambda - a_1)(\lambda - a_2)(\lambda - a_3)(\lambda - c)}.$$

Note that, according to this parametrization, the squares  $\bar{M}_i^2$  are rational functions on  $E_0$  with a pole at the origin  $O = \mathcal{A}(c)$  and zeros at the half periods  $\mathcal{A}(a_i)$ . However, due to the irrationality in (2.28), the momenta  $\bar{M}_i$  themselves are not meromorphic on  $E_0$ , but on its 4-fold unramified covering  $E$ , obtained by doubling of the both periods of  $E_0$ . Then the "big" curve  $E$  is also elliptic and is described by the same equation as  $E_0$ . To distinguish between the curves  $E_0$  and  $E$ , we use different letters, namely

$$E = \{w^2 = -k(z - a_1)(z - a_2)(z - a_3)(z - c)\}. \quad (2.29)$$

The curve can be represented in the canonical Weierstrass form

$$E_0 = \{W^2 = 4(Z - e_1)(Z - e_2)(Z - e_3) = 4Z^3 - g_2Z - g_3\}, \quad (2.30)$$

$$e_1 + e_2 + e_3 = 0$$

by the birational transformation

$$Z = \frac{\gamma z + \alpha}{z - c}, \quad W = \frac{w}{(z - c)^2} \quad (2.31)$$

such that  $Z(z = c) = \infty$  and  $Z(z = a_i) = e_i$ . Substituting (2.31) into (2.30) and comparing with (2.3), we obtain the following system of equations for  $e_1, e_2, e_3, \alpha, \gamma$

$$\begin{aligned} \alpha + e_1 c &= -a_1(\gamma - e_1) \\ \alpha + e_2 c &= -a_2(\gamma - e_2) \\ \alpha + e_3 c &= -a_3(\gamma - e_3) \\ e_1 + e_2 + e_3 &= 0 \\ -4(\gamma - e_1)(\gamma - e_2)(\gamma - e_3) &= 1, \end{aligned}$$

which yields

$$e_i = \rho(\Sigma_2 + c\Sigma_1 - 3(a_j a_k + c a_i)), \quad (i, j, k) = (1, 2, 3), \quad (2.32)$$

$$\gamma = \rho(\Sigma_2 - 2c\Sigma_1 + 3c^2), \quad \alpha = \rho(2c\Sigma_2 - c^2\Sigma_1 - 3\Sigma_3), \quad (2.33)$$

where

$$\Sigma_1 = a_1 + a_2 + a_3, \quad \Sigma_2 = a_1a_2 + a_3a_1 + a_2a_3, \quad \Sigma_3 = a_1a_2a_3, \quad (2.34)$$

$$\rho = 3 [2(c - a_1)^2(c - a_2)^2(c - a_3)^2]^{-1/3}. \quad (2.35)$$

Since the "big" curve  $E$  is obtained by doubling of the two periods of  $E_0$ , it can be written in the same form (2.30) but with different coordinates  $\tilde{Z}, \tilde{W}$ :

$$E = \left\{ \tilde{W}^2 = 4(\tilde{Z} - e_1)(\tilde{Z} - e_2)(\tilde{Z} - e_3) \right\} \quad (2.36)$$

These coordinates admit parameterizations

$$\tilde{Z} = \wp(u \mid \Omega_1, \Omega_3), \quad \tilde{W} = \wp'(u \mid \Omega_1, \Omega_3), \quad u = \int_{\infty}^{(\tilde{Z}, \tilde{W})} \frac{d\tilde{Z}}{\tilde{W}}, \quad (2.37)$$

where  $2\Omega_1 = 4\omega_1, 2\Omega_3 = 4\omega_3$  are the periods of  $E^2$ . Then  $\wp(\Omega_i \mid \Omega_1, \Omega_3) = e_i$ .

**Proposition 2.3.2** *The momenta  $\bar{M}_i$  of the Euler top admit the following rational parameterizations:*

1). in terms of the coordinates  $\tilde{Z}, \tilde{W}$  of the canonical curve (2.36):

$$\bar{M}_i = \beta_i \frac{\tilde{Z}^2 - 2e_i\tilde{Z} + e_i e_j + e_i e_k - e_j e_k}{\tilde{W}}, \quad (2.38)$$

$$\beta_i = \frac{k \sqrt[6]{2(c - a_1)(c - a_2)(c - a_3)}}{\sqrt{(a_i - a_j)(a_i - a_k)}}, \quad (i, j, k) = (1, 2, 3), \quad (2.39)$$

where  $e_i$  are given by the expressions (2.32),

2) and in terms of the coordinates  $z, w$  on the degree 4 curve (2.29):

$$\begin{aligned} \bar{M}_i = \frac{k}{2w \sqrt{-(a_i - a_j)(a_i - a_k)}} & \left[ (a_j + a_k - a_i - c)z^2 + 2z(ca_i - a_j a_k) \right. \\ & \left. + c(a_j a_k - a_i a_j - a_i a_k) + a_1 a_2 a_3 \right]. \end{aligned} \quad (2.40)$$

*Proof of Proposition 2.3.2* 1). As follows from the irrational parametrization (2.28), on each of the 4 copies of the small curve  $E_0$  the function  $\bar{M}_i$  has a simple pole at the origin of  $E_0$  and a simple zero at the half-period  $\omega_i$  and does not have zeros or poles elsewhere. It follows that on the big curve  $E = \mathbb{C}/\{2\Omega_1\mathbb{Z} + 2\Omega_2\mathbb{Z}\}$  the function  $\bar{M}_i$  has

- 1). 4 simple poles at  $u = 0, \Omega_1, \Omega_3, \Omega_1 + \Omega_3$ ;
- 2). 4 simple zeros obtained from these poles by shifting them by the quarter-period  $\Omega_i/2 = \omega_i$ . In particular,  $\bar{M}_1$  has zeros at

$$\frac{\Omega_1}{2}, \frac{3\Omega_1}{2}, \frac{\Omega_1}{2} + \Omega_3, \frac{3\Omega_1}{2} + \Omega_3, \quad (2.41)$$

as depicted on Figure (2.3) (a).

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<sup>2</sup>As usual, it is assumed that  $\Omega_2 = -\Omega_1 - \Omega_3$ .

Note that, in view of the pairness  $\wp(-u) = \wp(u)$ , we have

$$\wp\left(\frac{\Omega_1}{2}\right) = \wp\left(\frac{3\Omega_1}{2}\right), \quad \wp\left(\frac{\Omega_1}{2} + \Omega_3\right) = \wp\left(\frac{3\Omega_1}{2} + \Omega_3\right),$$

which means that the pair of points  $\{u = \Omega_1/2, u = 3\Omega_1/2\}$  have the same  $\tilde{Z}$ -coordinate, as well as the pair of points  $\{u = \Omega_1/2 + \Omega_3, u = 3\Omega_1/2 + \Omega_3\}$ .

Now we want to find a rational expression for  $M_i$  in terms of  $\tilde{Z}, \tilde{W}$  that has the above zeros and poles. It is seen easily that such expressions must have the structure

$$\bar{M}_1 = \beta_1 \frac{(\tilde{Z} - \wp(\Omega_1/2))(\tilde{Z} - \wp(\Omega_1/2 + \Omega_3))}{\tilde{W}}, \quad \beta_1 = \text{const} \quad (2.42)$$

and similar expressions for  $\bar{M}_2, \bar{M}_3$ . Indeed, when  $u$  coincides with one of the above four quarter-periods, the numerator of (2.42) has a simple zero and, as a second order polynomial in  $\tilde{Z}$ , it does not vanish elsewhere. Next, as seen from (2.36), the denominator  $\tilde{W}$  has simple zeros when  $\tilde{Z} = e_1, e_2, e_3$ , that is, when  $u = \Omega_1, \Omega_3, \Omega_1 + \Omega_3$  and does not vanish elsewhere. Finally, take into account that in a neighborhood of  $u = 0$

$$\tilde{Z} = \frac{1}{u^2} + O(1), \quad \tilde{W} = -\frac{2}{u^3} + O(u). \quad (2.43)$$

Therefore, at  $u = 0$  all the quotients (2.42) have a simple pole as well. As a result, they have the required zeros and poles on the curve  $E$ .

Next, using the formulas of multiplication/division of the argument of  $\wp$  by 2 (see, e.g., [39]), we find

$$\{\wp(\Omega_1/2), \wp(\Omega_1/2 + \Omega_3)\} = \left\{e_1 \pm \sqrt{(e_1 - e_2)(e_1 - e_3)}\right\} \quad (2.44)$$

and similar expressions for other quarter-periods on  $E$ . Substituting these expressions into (2.42), we obtain the fractions in (2.38).

Next, the constants  $\beta_i$  are chosen such that the expressions (2.38), (2.39) satisfy the momentum and energy integrals (2.2) for any  $\tilde{Z} \in \mathbb{C}$ . Namely, in view of the expressions (2.32) and the equation (2.30) of the curve  $E$ ,

$$\begin{aligned} \sum_i \bar{M}^2 &= k^2 (2(c - a_1)(c - a_2)(c - a_3))^{1/3} \sum_{i=1}^3 \frac{\tilde{Z}^4 - 4e_i \tilde{Z}^3 + O(\tilde{Z}^2)}{(a_i - a_j)(a_i - a_k) \tilde{W}^2} \\ &= k^2 (2(c - a_1)(c - a_2)(c - a_3))^{1/3} 3\rho, \\ \sum_i a_i \bar{M}^2 &= k^2 (2(c - a_1)(c - a_2)(c - a_3))^{1/3} \sum_{i=1}^3 \frac{a_i \tilde{Z}^4 - 4a_i e_i \tilde{Z}^3 + O(\tilde{Z}^2)}{(a_i - a_j)(a_i - a_k) \tilde{W}^2} \\ &= k^2 (2(c - a_1)(c - a_2)(c - a_3))^{1/3} 3c\rho, \end{aligned}$$

which, due to (2.35), equal  $k^2$  and  $l$  respectively.

2). Making in the parametrization (2.38) the substitution (2.31) and taking into account (2.32), (2.33), after simplification we get the parameterizations (2.40) in terms of the coordinates  $z, w$  of the curve (2.3). The proposition is proved.  $\blacksquare$

**Real part of the parametrization for the Euler top.** Due to the structure of formulas (2.40), when  $a_i, c \in \mathbb{R}$ , the momenta  $\bar{M}_i$  cannot be all real when the coordinate  $z$  is real.

It appears that the two real commented components of the trajectory  $\bar{M}(t) \subset \mathbb{R}^3$  are parametrized by two loops  $\mathcal{R}_+, \mathcal{R}_-$  on  $E$ , which, under the projection  $(z, w) \rightarrow z$ , are mapped to an oval  $\mathcal{R}$  on the complex plane  $z$ . Under the Abel map  $\mathcal{A} : E \mapsto \mathbb{C} = \{u\}$ , the loops  $\mathcal{R}_\pm$  pass to two real lines passing through some of the quarter-periods  $\Omega_i/2 = \omega_i$ , whereas the branch points  $z = a_i, z = c$  pass to the half-periods  $0, \Omega_1, \Omega_2, \omega_3$ , that is, the poles of  $\bar{M}_i$ <sup>3</sup>.

In particular, if  $a_1 < c < a_2 < a_3$ , the real trajectory  $\bar{M}(t)$  has 2 components, on each of them the momentum  $\bar{M}_1$  does not vanish, whereas  $M_2, M_3$  has two simple zeros. These components are images of the ovals on the upper and lower  $z$ -sheets of  $E$  that embrace the branch points  $z = a_1, z = c$ , as shown in Figure 2.3.1 (b). The white dots on the ovals stand for pairs of real zeros of  $\bar{M}_2(z)$  and complex conjugated zeros of  $\bar{M}_3(z)$ . On  $\mathbb{C} = \{u\}$  these ovals correspond to the two (dashed) lines parallel to the real axis, as shown in Figure 2.3.1 (a). These two lines pass through zeros of  $\bar{M}_2, \bar{M}_3$ , but do not contain neither the zeros (2.41) of  $\bar{M}_3$ , nor the common poles.

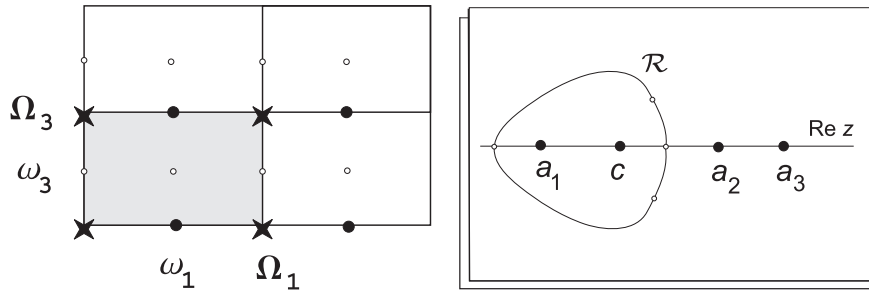


Figure 2.3.1: a) The big curve  $E$  as a 4-fold unramified covering of  $E_0$  (grey area) on the complex plane  $u$ . The zeros and poles of  $\bar{M}_1$  are represented by black dots and crosses respectively. b) The big curve  $E$  as a 2-fold ramified covering of  $\mathbb{C} = \{z\}$  and the oval  $\mathcal{R}$ , which corresponds to 2 real trajectories  $\bar{M}(t)$  for the case  $a_1 < c < a_2 < a_3$ .

Now, in order to calculate the coefficients  $\alpha_i, u_{i2}, u_{i1}, u_{i0}$  in the rational parametrization (2.25), we consider the momenta  $\bar{M}_i$  above as the image of the projective transformation  $M \mapsto \bar{M}$  described by Proposition 2.2.2 and set  $a_i = \lambda_i, c = \lambda_4$ . Then, substituting the parametrization (2.40) for  $\bar{M}$  with  $k = 1$  into

<sup>3</sup>This contrasts to the irrational parametrization (2.28), when real trajectories  $\bar{M}(t)$  correspond to segments of the *real axis* on the complex plane  $\lambda$ .

formulas (2.23), we obtain the parameterizations (2.25) for the momenta  $M_i$  with

$$\left. \begin{aligned} u_{i2} &= g_i \sqrt{-\Delta_4} \sum_{\alpha=1}^3 \frac{\lambda_\beta + \lambda_\gamma - \lambda_\alpha - \lambda_4}{2\sqrt{\Delta_\alpha}(a_i - \lambda_\alpha)\sqrt{-(\lambda_\alpha - \lambda_\beta)(\lambda_\alpha - \lambda_\gamma)}}, \\ u_{i1} &= 2g_i \sqrt{-\Delta_4} \sum_{\alpha=1}^3 \frac{\lambda_\alpha \lambda_4 - \lambda_\beta \lambda_\gamma}{2\sqrt{\Delta_\alpha}(a_i - \lambda_\alpha)\sqrt{-(\lambda_\alpha - \lambda_\beta)(\lambda_\alpha - \lambda_\gamma)}}, \\ u_{i0} &= g_i \sqrt{-\Delta_4} \sum_{\alpha=1}^3 \frac{(\lambda_\beta \lambda_\gamma - \lambda_\alpha \lambda_\beta - \lambda_\alpha \lambda_\gamma)\lambda_4 + \lambda_\alpha \lambda_\beta \lambda_\gamma}{2\sqrt{\Delta_\alpha}(a_i - \lambda_\alpha)\sqrt{-(\lambda_\alpha - \lambda_\beta)(\lambda_\alpha - \lambda_\gamma)}}, \\ \alpha_i &= \frac{g_i}{a_i - \lambda_4}, \\ u_{02} &= \sqrt{-\Delta_4} \sum_{\alpha=1}^3 \frac{\lambda_\beta + \lambda_\gamma - \lambda_\alpha - \lambda_4}{2\sqrt{\Delta_\alpha}\sqrt{-(\lambda_\alpha - \lambda_\beta)(\lambda_\alpha - \lambda_\gamma)}}, \\ u_{01} &= 2\sqrt{-\Delta_4} \sum_{\alpha=1}^3 \frac{\lambda_\alpha \lambda_4 - \lambda_\beta \lambda_\gamma}{2\sqrt{\Delta_\alpha}\sqrt{-(\lambda_\alpha - \lambda_\beta)(\lambda_\alpha - \lambda_\gamma)}}, \\ u_{00} &= \sqrt{-\Delta_4} \sum_{\alpha=1}^3 \frac{(\lambda_\beta \lambda_\gamma - \lambda_\alpha \lambda_\beta - \lambda_\alpha \lambda_\gamma)\lambda_4 + \lambda_\alpha \lambda_\beta \lambda_\gamma}{2\sqrt{\Delta_\alpha}\sqrt{-(\lambda_\alpha - \lambda_\beta)(\lambda_\alpha - \lambda_\gamma)}} \end{aligned} \right\} \quad (2.45)$$

where in summations  $(\alpha, \beta, \gamma) = (1, 2, 3)$  and, as above,

$$\Delta_\alpha = \sum_{i=1}^3 \frac{g_i^2}{(a_i - \lambda_\alpha)^2} - k.$$

**A numerical example.** For the ZV gyrost at with given initial conditions parameterization for the momenta  $M_i$  has the form

$$M_1 = \frac{-1.2937\omega + (0.10912 - 41.54i)z - (2.2188 \times 10^{-2} - 5.5206i)z^2 - (0.10269 - 76.596i)}{\omega - (0.21571 - 1.3411i)z + (4.3862 \times 10^{-2} - 0.17686i)z^2 + (0.203 - 2.8432i)}$$

$$M_2 = \frac{0.81500\omega - (9.298 - 0.85747i)z + (1.8906 - 0.11466i)z^2 + (8.7501 - 1.5778i)}{\omega - (0.21571 - 1.3411i)z + (4.3862 \times 10^{-2} - 0.17686i)z^2 + (0.203 - 2.8432i)}$$

$$M_3 = \frac{0.30989\omega - (0.10662 + 19.678i)z + (2.1680 \times 10^{-2} + 2.7764i)z^2 + (0.10034 + 35.538i)}{\omega - (0.21571 - 1.3411i)z + (4.3862 \times 10^{-2} - 0.17686i)z^2 + (0.203 - 2.8432i)}$$

## 2.4 Determination of the motion of the gyrost at in space

The main objective of this section is to determine the motion of the gyrost at in the space by making use the parameterizations (2.25) and the relation (2.26).



Let us choose a *fixed* in space orthonormal frame  $O \mathbf{e}_1 \mathbf{e}_2, \mathbf{e}_3$  such that the third axis is directed along the constant momentum vector  $M$  of the gyrostat, and  $\theta, \psi, \phi$  be the Euler angles of nutation, precession, and rotation with respect to this frame. Then, according to the definition of the angles,

$$M_1 = -|M| \sin \theta \sin \phi, \quad M_2 = |M| \sin \theta \cos \phi, \quad M_3 = |M| \cos \theta. \quad (2.46)$$

These expressions allow to determine trigonometric functions of  $\theta$  and  $\psi$  in terms of  $M_i$  and, in view of the solution (2.7), as functions of time  $t$ .

Next, as follows from the Euler kinematical equations (see e.g. [62])

$$\dot{\psi} = \frac{-\omega_1 \cos \phi + \omega_2 \sin \phi}{\sin \theta}.$$

Substituting here the expressions for  $\cos \psi, \sin \psi$  from (2.46) and using  $|M| = k = \text{const}$ , we find

$$\dot{\psi} = -\frac{\omega_1 M_1 + \omega_2 M_2}{k \sin^2 \theta} \equiv -k \frac{\omega_1 M_1 + \omega_2 M_2}{M_1^2 + M_2^2}.$$

Next, expressing  $\omega_i = a_i M_i - g_i$ ,  $i = 1, 2$  and fixing the value  $l$  of the energy integral in (1.2), we obtain

$$\begin{aligned} \dot{\psi} &= -k \frac{a_1 M_1 + a_2 M_2 - g_1 M_1 - g_2 M_2}{k^2 - M_3^3} = -k \frac{l + g_1 M_1 + g_2 M_2 + 2g_3 M_3 - a_3 M_3^2}{k^2 - M_3^3} \\ &= -k \frac{l - a_3 k^2 + g_1 M_1 + g_2 M_2 + 2g_3 M_3}{k^2 - M_3^2} - k a_3. \end{aligned} \quad (2.47)$$

This form suggests introducing new angle  $\tilde{\psi} = \psi + k a_3 t$ . In view of the relation (2.26) between  $dt$  and  $dz$ , we then get

$$d\tilde{\psi} = -\frac{l - a_3 k^2 + g_1 M_1 + g_2 M_2 + 2g_3 M_3}{(k - M_3)(k + M_3)} \frac{dz}{\sqrt{-(z - \lambda_1)(z - \lambda_2)(z - \lambda_3)(z - \lambda_4)}}. \quad (2.48)$$

Now, in view of the parametrization (2.25) for  $\bar{M}_i$  in terms of  $z, w$ , we see that (2.48) is a meromorphic differential of the third kind on the big elliptic curve  $E$  with

- 1). 4 pairs of simple poles  $\mathcal{D}_1^\pm, \dots, \mathcal{D}_4^\pm$  given by equations  $M_3 = \pm k$ , that is, their  $z$ -coordinates are the solutions of

$$(\alpha_3 \mp k) \sqrt{-k(z - \lambda_1)(z - \lambda_2)(z - \lambda_3)(z - \lambda_4)} = U_3(z) \mp k U_0(z). \quad (2.49)$$

Using the expressions (2.45) for the coefficients of  $U_i(z), U_0(z)$ , one can also show that the residues of  $d\tilde{\psi}$  at  $\mathcal{D}_i^\pm$  are  $\pm\sqrt{-1}$ .

- 2). 4 simple zeros as the zeros of the numerator  $l - a_3 k^2 + g_1 M_1 + g_2 M_2 + 2g_3 M_3$ , and 4 simple zeros  $Q_1, \dots, Q_4$ , which are the poles of  $M_3$ .

There are no other poles or zeros of  $d\tilde{\psi}$ . Note that the conjugated poles  $\mathcal{D}_i^+$  and  $\mathcal{D}_i^-$  have different  $z$ -coordinates, that is, they are not symmetric with respect to the involution  $w \rightarrow -w$  on  $E$ .

It follows that the latter differential can be represented as a sum of 4 elementary differentials of the third kind:

$$d\tilde{\psi} = \sqrt{-1}(\Omega_1 + \cdots + \Omega_4), \quad (2.50)$$

where  $\Omega_i$  has only simple poles at  $\mathcal{D}_i^\pm$  with the residues  $\pm 1$ .

Now we use the following classical result (see e.g. [39]): Let  $\Omega_{X,Y}$  be the meromorphic differential of the third kind on an elliptic curve  $E$  having a pair of simple poles at  $X, Y \in E$  (only) with the residues  $\pm 1$  respectively. Then, for  $u = \mathcal{A}(P) = \int_{P_0}^P \frac{dz}{w}$  with a fixed basepoint  $P_0 \in E$ , one has

$$\int_{P_0}^P \Omega_{X,Y} = \log \frac{\sigma(u - \mathcal{A}(Y))}{\sigma(u - \mathcal{A}(X))} + \nu u + \log \frac{\sigma(\mathcal{A}(X))}{\sigma(\mathcal{A}(Y))}, \quad (2.51)$$

where the constant factor  $\nu$  is determined from the condition that both sides of (2.51) have the same periods on  $E$ .

Let now  $\mathcal{A}$  denote the Abel map with the basepoint  $P_0 = (\lambda_4, 0)$  on the degree 4 curve (2.5) and set  $d_i^\pm = \mathcal{A}(\mathcal{D}_i^\pm)$ . Let also  $\sigma(u) = \sigma(u \mid \Omega_1, \Omega_3)$ . Integrating the differential (2.50) and using formula (2.51), we can express the new angle  $\tilde{\psi}$  as the following function of  $u$ :

$$\tilde{\psi} = \sqrt{-1} \log \frac{\sigma(u - d_1^+) \cdots \sigma(u - d_4^+)}{\sigma(u - d_1^-) \cdots \sigma(u - d_4^-)} + Vu + C, \quad C = \sqrt{-1} \log \frac{\sigma(d_1^-) \cdots \sigma(d_4^-)}{\sigma(d_1^+) \cdots \sigma(d_4^+)}. \quad (2.52)$$

In order to find the factor  $V$ , we use the quasi-periodicity property  $\sigma(u + 2\Omega_i) = -e^{2\eta_i(u + \Omega_i)}\sigma(u)$ ,  $\eta_i$  being the quasi-period of the Weierstrass function  $\zeta(u)$  and, changing in (2.52)  $u \rightarrow u + 2\Omega_1$ , get the condition

$$\mathcal{I} \equiv 2 \int_{\lambda_4}^{\lambda_1} d\tilde{\psi} = 2\eta_1 \sum_{j=1}^4 (d_j^+ - d_j^-) + 2\Omega_1 V.$$

Here the value of the integral  $\mathcal{I}$  can be calculated numerically by considering  $\bar{M}_i$  in (2.48) as functions of  $z, w$ , that is, by using the parametrization (2.25). Once  $\mathcal{I}$  is known, we also calculate

$$V = \frac{1}{\Omega_1} (\mathcal{I}/2 - \eta_1 \sum_{j=1}^4 (d_j^+ - d_j^-)).$$

As a result, for the original precession angle  $\psi$  we obtain

$$e^{\sqrt{-1}\psi} = \frac{\sigma(u - d_1^-) \cdots \sigma(u - d_4^-)}{\sigma(u - d_1^+) \cdots \sigma(u - d_4^+)} e^{(V - ka_3)u + C}. \quad (2.53)$$

This formula allows to express

$$\cos \psi = \left( e^{\sqrt{-1}\psi} + e^{-\sqrt{-1}\psi} \right) / 2, \sin \psi = \left( e^{\sqrt{-1}\psi} - e^{-\sqrt{-1}\psi} \right) / (2\sqrt{-1})$$

as meromorphic functions of the complex variable  $u$  and, therefore, the components of the unit vectors  $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$  in the body as functions of time  $t$ .

**Remark 4.** The above procedure follows the same lines as that first used by K. Jacobi [37] in calculation of the rotation matrix for the Euler top. As I understood, in [58] Volterra used similar approach, however, as was mentioned in Introduction, a method of explicit calculation of the poles  $d_i^\pm$  and of the constant angular velocity  $V$  in (2.53) was not indicated there.

In given chapter we presented a new rational parametrization of the classical Euler top in terms of coordinates  $z, w$  of the appropriate elliptic curve (Proposition 2.3.2). It was observed that the real trajectories  $\bar{M}(t)$  do not correspond to real  $z$ , but to oval on the complex plane  $z$ . In this sense, this parametrization is essentially complex.

## Chapter 3

# Bifurcation analysis of the Zhukovskii-Volterra system via bi-Hamiltonian approach.

### 3.1 Introduction

The main goal of Chapter 3 consists of the bifurcation analysis for the ZV system (1.1,1.2), including analysis of stability of solutions. To be more precise, we state the problem as follows:

- 1) find the equilibrium points of the system;
- 2) check the non-degeneracy condition for the equilibrium points in the sense of the singularity theory of Hamiltonian systems;
- 3) determine the types of equilibrium points and verify whether they are stable or not;
- 4) do bifurcation analysis of the ZV system, i.e., construct and analyze the bifurcation diagram of the momentum mapping given by the Hamiltonian  $H$  and the integral  $F$ ;
- 5) describe topological type of common levels of integrals  $\{H = h_0, F = f_0\}$  for different values of parameters  $h_0, f_0$ ; find out how many connected components this level contains and how it transforms if the point  $(h_0, f_0)$  passes through the bifurcation diagram.

The result of this chapter was published in ([5]).

In the case of two degrees of freedom, similar problems have been discussed in many papers (for references see, for example, [11]). Here we deal, in fact, with just one degree of freedom, but the problem still remains quite interesting and substantial. Although all calculations can be done directly, we propose to proceed in a different way: we use the fact that the ZV system is bi-Hamiltonian and apply new techniques for analysis of singularities of bi-Hamiltonian systems, which have

been recently developed by A. Bolsinov and A. Oshemkov in [12]. The ZV system can serve as a very good illustrating example for this new method.

Thus the goal of this work is double: to study the system and to demonstrate some new techniques. It is a remarkable fact that using the bi-Hamiltonian property makes it possible to answer all the above questions practically without any computations (however, all the results of this work presented below have been also checked by independent direct computation).

These techniques have been developed in [12] to study the case of multidimensional integrable systems where other methods are less effective because of computational difficulties. In the case which we discuss below, the dimension of the system is just 3, and all the results can, of course, be obtained by more direct methods. However, Zhukovskii-Volterra system is a very good example to illustrate new techniques from [12] and to show how some very natural and classical constructions in the theory dynamical systems are reflected in the algebraic mirror.

### 3.2 Bi-Hamiltonian structure for the Zhukovski-Volterra system

Lax representation with spectral parameter and bi-Hamiltonian structure for Zhukovski-Volterra system is the property, which helps to hold bifurcation analysis practically without any calculation.

We assume  $a_1 < a_2 < a_3$ .

The family of compatible Poisson brackets is:

$$\{\cdot, \cdot\}_s = s\{\cdot, \cdot\} + (\{\cdot, \cdot\}_B - \{\cdot, \cdot\}_g);$$

The bracket  $\{\cdot, \cdot\}_s$  is given by the skew-symmetric matrix:

$$\mathcal{A}_s = \begin{pmatrix} 0 & sM_3 + a_3M_3 - g_3 & -sM_2 - a_2M_2 + g_2 \\ -sM_3 - a_3M_3 + g_3 & 0 & sM_1 + a_1M_1 - g_1 \\ sM_2 - a_2M_2 + g_2 & -sM_1 - a_1M_1 + g_1 & 0 \end{pmatrix} \quad (3.1)$$

First of all we describe the algebraic type of these brackets and the corresponding Casimir functions, assuming that  $g_i \neq 0$ .

**Proposition 3.2.1** *The bracket  $\{\cdot, \cdot\}_s$  is semisimple except for three values of the parameter  $s$ , namely  $s = -a_i$ ,  $i = 1, 2, 3$ . Moreover, for  $s \in (-\infty, -a_3) \cup (-a_1, +\infty)$  the corresponding Lie algebra is isomorphic to  $so(3)$ , whereas for  $s \in (-a_3, -a_2) \cup (-a_2, -a_1)$  it is isomorphic to  $sl(2)$ . The Casimir function for  $\{\cdot, \cdot\}_s$  is*

$$f_s = \frac{1}{2} \sum_{i=1}^3 (s + a_i) M_i^2 \pm \sum_{i=1}^3 g_i M_i$$

If  $s = -a_i$ , then  $\{\cdot, \cdot\}_s$  is diffeomorphic to a constant Poisson bracket. To see this for  $s = -a_3$ , for example, it is sufficient to consider the following (everywhere non-degenerate!) change of variables:

$$\tilde{M}_1 = M_1, \quad \tilde{M}_2 = M_2, \quad \tilde{M}_3 = f_{-a_3}(M_1, M_2, M_3)$$

In this coordinate system, we obviously have

$$\mathcal{A}_s = \begin{pmatrix} 0 & -g_3 & 0 \\ g_3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

In particular, the singular set  $S_{-a_i}$  for each of these three exceptional brackets is empty.

### 3.3 Set of critical points and equilibria

Consider the integrals

$$\begin{aligned} H(\mathbf{M}) &= \frac{1}{2}(a_1 M_1^2 + a_2 M_2^2 + a_3 M_3^2) - (g_1 M_1 + g_2 M_2 + g_3 M_3) \\ F(\mathbf{M}) &= M_1^2 + M_2^2 + M_3^2 \end{aligned} \quad (3.2)$$

of our system and define the momentum mapping

$$\Phi = (H, F) : \mathbb{R}^3(M_1, M_2, M_3) \rightarrow \mathbb{R}^2(H, F) \quad (3.3)$$

Consider the set of critical points of  $\Phi$ :

$$K = \{\mathbf{M} = (M_1, M_2, M_3) \in \mathbb{R}^3 \mid dH(\mathbf{M}) \text{ and } dF(\mathbf{M}) \text{ are linearly dependent}\}$$

According to the general scheme (Theorem 1), this singular set can be characterized as the union of the singular sets for all the brackets from this family. The singular set for a particular Poisson bracket  $\{ , \}$  consists, by definition, of those points where the rank of  $\mathcal{A}_s$  is less than 2, in other words, the matrix  $\mathcal{A}_s$  just vanishes:  $\mathcal{A}_s = 0$ . Notice that this may happen only for real values of the parameter  $s$ .

Thus, we immediately obtain the parametrization for for the singular set:

**Theorem 3.3.1** *The critical set  $K$  consists of the points  $\mathbf{M}_s$  with coordinates*

$$M_i(s) = \frac{g_i}{s + a_i}, \quad i = 1, 2, 3, \quad s \in \overline{\mathbb{R}} \setminus \{-a_1, -a_2, -a_3\} \quad (3.4)$$

It is easy to see that for  $s = \pm\infty$  we obtain, in fact, the same point  $\mathbf{M} = (0, 0, 0)$  and moreover the two families of critical points corresponding to the intervals  $(-\infty, -a_3)$  and  $(-a_1, +\infty)$  are smoothly attached to each other at this point. Thus, we have 3 families of critical points (connected components of  $K$ ) for  $s \in (-\infty, -a_3) \cup (-a_1, +\infty)$ ,  $s \in (-a_3, -a_2)$ , and  $s \in (-a_2, -a_1)$ .

It is worth noticing that the pencil  $\mathcal{A}_\lambda$ ,  $\lambda \in \overline{\mathbb{R}}$  consists of three exceptional brackets and three families of semisimple brackets. The three connected components of  $K$  correspond exactly to these semisimples families.

Notice that the standard scheme for describing  $K$  consists of the analysis of the Jacobi matrix of  $\Phi$ , whereas Theorem (3.4) has allowed us to describe it without any computation.

Since the symplectic leaves are two-dimensional, all point of  $K$  are common equilibria of the Zhukovski-Volterra system. ‘‘Common’’ means that at  $x \in K$ , the Hamiltonian vector fields generated by  $H$  and  $F$  both vanish.

### 3.4 Non-degeneracy and topological types of equilibria

The Zhukovskii-Volterra system “lives” on a 3-dimensional Poisson manifold and, being restricted to symplectic leaf, can be treated as a Hamiltonian system with 1 degree of freedom. The non-degeneracy condition for such systems becomes very simple and can be formulated as follows (in this case “non-degeneracy” in the sense of integrable systems coincides with that in the sense of the Morse theory).

**Definition 9** Let  $x$  be an equilibrium point for a Hamiltonian system with a Hamiltonian  $H$  given on a 3-dimensional Poisson manifold  $(M, \mathcal{A})$  and  $\text{rank } \mathcal{A}(x) = 2$ . This point is said to be *non-degenerate* if the Hessian of the restriction of  $H$  onto the symplectic leaf through  $x$  is non-degenerate, i.e.  $\det d^2(H|_{O(x)}) \neq 0$ . The equilibrium point is *elliptic* if  $\det d(H|_{O(x)}) > 0$  and *hyperbolic* if  $\det d^2(H|_{O(x)}) < 0$ .

Notice that a non-degenerate equilibrium point is stable if and only if it is elliptic.

We are going to verify the non-degeneracy condition and stability by using Proposition ???. In our particular case its statement becomes much simpler. For each critical point  $M_s$  we consider two objects: the kernel of  $\mathcal{A}_s$  and the kernel of any other bracket  $\mathcal{A}_t$ ,  $t \neq s$ . We can choose, for example,  $\mathcal{A}_\infty$ , and assume, that the kernel at point  $M_s$  is generated by vector  $\xi_s$ . Clearly,  $\text{Ker } \mathcal{A}_s = \mathbb{R}^3$  and  $\text{Ker } \mathcal{A}_\infty$  is one-dimensional. It is important that  $\mathbb{R}^3$  carries the natural structure of the Lie algebra  $\mathfrak{g}_s$  defined by  $\mathcal{A}_s$ . The following statement is just an adaptation of Proposition ??? to our particular situation.

**Proposition 3.4.1**  $M_s$  is non-degenerate if and only if  $\xi_s \in \mathfrak{g}_s$  is semisimple. Moreover,  $M_s$  is stable if and only if  $\xi_s$  is of elliptic type.

It is easy to see that the kernel of  $\mathcal{A}_\infty$  at the point  $M_s$  is generated by  $M_s$  itself, the commutator in  $\mathfrak{g}_s$  is given by (3.1) where the constant terms  $g_i$  are omitted. In other words,  $\mathfrak{g}_s$  is the space of skew-symmetric matrices with the commutator given by

$$[X, Y]_s = X(sE + B)Y - Y(sE + B)X.$$

Thus the question is to determine the type of the matrix

$$M_s = \begin{pmatrix} 0 & \frac{g_3}{s+a_3} & -\frac{g_2}{s+a_2} \\ -\frac{g_3}{s+a_3} & 0 & \frac{g_1}{s+a_1} \\ \frac{g_2}{s+a_2} & -\frac{g_1}{s+a_1} & 0 \end{pmatrix}$$

in this Lie algebra.

We use the following simple criterion.

**Proposition 3.4.2** Let  $\mathfrak{g}$  be a semisimple real Lie algebra of dimension 3 (i.e., either  $so(3)$ , or  $sl(2)$ ), and  $\xi \in \mathfrak{g}$  be one of its elements. Let  $\langle \cdot, \cdot \rangle$  be the Killing form on  $\mathfrak{g}$ . Then

1.  $\xi$  is elliptic if  $\langle \xi, \xi \rangle < 0$ ;

2.  $\xi$  is hyperbolic if  $\langle \xi, \xi \rangle > 0$ ;
3.  $\xi$  is nilpotent if  $\langle \xi, \xi \rangle = 0$ .

It is easy to see that (up to a constant positive factor) the Killing form form  $\mathfrak{g}_s$  is defined by

$$\langle X, Y \rangle_s = \text{Tr } X(sE + B)Y(sE + B)$$

Hence, we immediately obtain the following description for the equilibrium points of the Zhukovskii–Volterra system.

**Theorem 3.4.3** *The equilibrium point  $M_s$  given by (3.4) is non-degenerate if and only if*

$$\langle M_s, M_s \rangle_s = -(s + a_1)(s + a_2)(s + a_3) \sum_{i=1}^3 \frac{g_i^2}{(s + a_i)^3} \neq 0$$

*Moreover, this point is stable if and only if the above expression is negative.*

Notice that for the first family of critical points, i.e., for  $s \in (-\infty, -a_3) \cup (-a_1, +\infty)$  the Lie algebra is isomorphic to  $so(3)$  and all of its elements are elliptic. The same conclusion follows, of course, from Theorem 3.4.3 because  $\varphi(s) = \langle M_s, M_s \rangle_s$  is obviously negative for all  $s \in (-\infty, -a_3) \cup (-a_1, +\infty)$ . On the other two intervals the situation is opposite: the function  $\varphi(s) = \langle M_s, M_s \rangle_s$  changes sign and therefore the point  $M_s$  changes its stability type. The stability intervals are those where  $\varphi(s) < 0$ .

### 3.5 Bifurcation diagram.

To get a parametric description of the bifurcation diagram  $\Sigma = \Phi(K) \subset \mathbb{R}^2(H, F)$  we simply substitute the parametric equations (3.4) of  $K$  to (3.2).

**Theorem 3.5.1** *The bifurcation diagram of the momentum mapping (3.3) is given parametrically by*

$$H(s) = \frac{1}{2} \sum_{i=1}^3 a_i \left( \frac{g_i}{s + a_i} \right)^2 - \sum_{i=1}^3 \frac{g_i^2}{s + a_i},$$

$$F(s) = \sum_{i=1}^3 \left( \frac{g_i}{s + a_i} \right)^2$$

The result is shown in Figure 3.5.1. The image of the momentum mapping is inside of the curve  $\mathcal{G}_1$ . The boundary of the momentum mapping (i.e., the curve  $\mathcal{G}_1$ ) is the image of equilibria points corresponding to  $s \in (-\infty, -a_3) \cup (-a_1, +\infty)$ . According to Theorem 3.4.3 all of them are stable. The two other curves  $\mathcal{G}_2$  and  $\mathcal{G}_3$  correspond to  $s \in (-\infty, -a_3) \cup (-a_1, +\infty)$ . The lost of stability happens exactly at those points where these curves have cusp singularity.



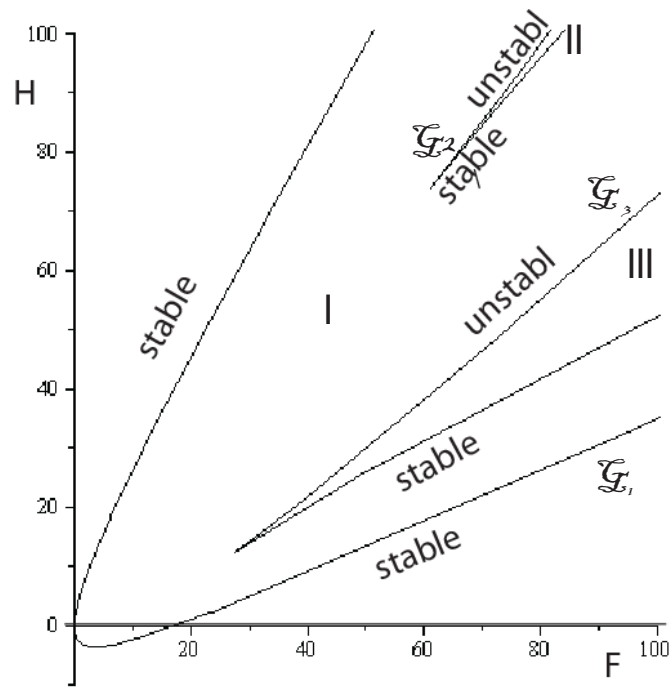


Figure 3.5.1: Bifurcation diagram of Zhukovskii-Volterra system

The bifurcation diagram divides the image of the momentum mapping into three zones denoted in Figure 3.5.1 by Roman digits I, II and III. Based on the above presented analysis we can now determine how many connected components each common level of integrals  $H$  and  $F$  contains. If  $(h_0; f_0) \in I$ , then the level  $H(M) = h_0; F(M) = f_0$  is connected and represents a closed trajectory of the system. If  $(h_0; f_0) \in II$ , or III, then the level  $H(M) = h_0; F(M) = f_0$  is disconnected and contains two periodic trajectories.

Also it is not hard to see if  $(h_0; f_0)$  belongs to the unstable part of the curve  $\gamma_i, i = 2, 3$ , then the level  $H(M) = h_0; F(M) = f_0$  represents a eight-figure curve, i.e., consists of a hyperbolic equilibrium point and two asymptotic trajectories.

## Chapter 4

# Separation of variables and explicit theta-function solution of the classical Steklov–Lyapunov systems: A geometric and algebraic geometric background.

In the Chapter 3 we revise the separation of variables and explicit integration of the classical Steklov–Lyapunov systems, which was first made by F. Kötter in 1900. Namely, we give a geometric interpretation of the separating variables and, then, applying the Weierstrass root functions, obtain an explicit theta-function solution to the problem. All results of this chapter were presented in ([25]).

The motion of a rigid body in the ideal incompressible fluid is described by the classical Kirchhoff equations (1.4). Note that this system always possesses two trivial integrals (Casimir functions of the coalgebra  $e^*(3)$ )  $\langle M, p \rangle, \langle p, p \rangle$  and the Hamiltonian itself is also a first integral.

Steklov [52] noticed that the classical Kirchhoff equations are integrable under certain conditions i.e., when the Hamiltonian has the form (1.5). Under the Steklov condition, the equations possess fourth additional integral

$$H_2 = \frac{1}{2} \sum_{\alpha=1}^3 \left( K_\alpha^2 - 2\nu b_\alpha K_\alpha p_\alpha + \nu^2 (b_\beta - b_\gamma)^2 p_\alpha^2 \right). \quad (4.1)$$

Later Lyapunov [55] discovered an integrable case of the Kirchhoff equations whose Hamiltonian was a linear combination of the additional integral (1.10) and the two trivial integrals. Thus, the Steklov and Lyapunov integrable systems actually define different trajectories on the same invariant manifolds, two-dimensional tori. This fact was first noticed in [32].

In the sequel, without loss of generality, we assume  $\nu = 1$  (this can always be made by an appropriate rescaling  $p \rightarrow p/\nu$ ).

The Kirchhoff equations with the Hamiltonians (1.5), (4.1) were first solved explicitly by Kötter [38], who used the change of variables  $(K, p) \rightarrow (z, p)$ :

$$2z_\alpha = K_\alpha - (b_\beta + b_\gamma)p_\alpha, \quad \alpha = 1, 2, 3, \quad (\alpha, \beta, \gamma) = (1, 2, 3), \quad (4.2)$$

which transforms the Steklov–Lyapunov systems to the form

$$\dot{z} = z \times Bz - Bp \times Bz, \quad \dot{p} = p \times Bz, \quad B = \text{diag}(b_1, b_2, b_3) \quad (4.3)$$

and, respectively,

$$\dot{z} = p \times Bz, \quad \dot{p} = p \times (z - Bp). \quad (4.4)$$

Kötter implicitly showed that the above systems admit the following Lax representation with  $3 \times 3$  skew-symmetric matrices and a spectral parameter

$$\begin{aligned} \dot{L}(s) &= [L(s), A(s)], \quad L(s), A(s) \in so(3), \quad s \in \mathbb{C}, \\ L(s)_{\alpha\beta} &= \varepsilon_{\alpha\beta\gamma} \left( \sqrt{s - b_\gamma} (z_\gamma + sp_\gamma) \right), \end{aligned} \quad (4.5)$$

where  $\varepsilon_{\alpha\beta\gamma}$  is the Levi-Civita tensor. Equations (4.3) and (4.4) are generated by the operators

$$\begin{aligned} A(s)_{\alpha\beta} &= \frac{\varepsilon_{\alpha\beta\gamma}}{s} \sqrt{(s - b_\alpha)(s - b_\beta)} b_\gamma z_\gamma \quad \text{resp.} \\ A(s)_{\alpha\beta} &= \varepsilon_{\alpha\beta\gamma} \sqrt{(s - b_\alpha)(s - b_\beta)} p_\gamma. \end{aligned} \quad (4.6)$$

The radicals in (4.5)–(4.6) are single-valued functions on the elliptic curve  $\widehat{\mathcal{E}}$ , the 4-sheeted unramified covering of the plane curve  $\mathcal{E} = \{w^2 = (s - b_1)(s - b_2)(s - b_3)\}$ . For this reason, the Lax representation has an elliptic spectral parameter.

Writing out the characteristic equation for  $L(s)$ , we arrive at the following family of quadratic integrals

$$\mathcal{F}(s) = \sum_{\gamma=1}^3 (s - b_\gamma)(z_\gamma + sp_\gamma)^2 \equiv J_1 s^3 + J_2 s^2 + 2sH_2 - 2H_1, \quad (4.7)$$

where

$$H_1 = \frac{1}{2} \langle z, Bz \rangle, \quad H_2 = \frac{1}{2} \langle z, z \rangle - \langle Bz, p \rangle, \quad J_2 = 2 \langle z, p \rangle - \langle Bp, p \rangle, \quad J_1 = \langle p, p \rangle. \quad (4.8)$$

It is seen that under the Kötter substitution (4.2) the functions  $J_1, J_2$  transform into invariants of the coalgebra  $e^*(3)$ , whereas the integrals  $H_1(z, p), H_2(z, p)$  (up to a linear combination of the invariants) become the Hamiltonians (1.5), (4.1).

## 4.1 Separation of variables by F. Kötter.

The explicit solution of the Steklov–Lyapunov systems in the generic case was given by Kötter in the brief communication [38], where he presented the following scheme.

Let us fix the constants of motion, then the invariant polynomial (4.7) can be written as

$$\mathcal{F}(s) = c_0(s - c_1)(s - c_2)(s - c_3), \quad c_0, c_1, \dots, c_3 = \text{const.} \quad (4.9)$$

Assume, without loss of generality, that  $b_1 < b_2 < b_3$ . Then for real  $z, p$ :

**Proposition 4.1.1** *The roots  $c_1, c_2, c_3$  are either all real or 2 of them are complex conjugated. If  $c_\alpha$  is real, then it belongs to the segment  $[b_1, b_3]$ .*

*Proof.* Indeed, setting in  $F(s)$   $s = c_\alpha$ , we obtain

$$\sum_{\gamma=1}^3 (c_\alpha - b_\gamma)(z_\gamma + c_\alpha p_\gamma)^2 = 0. \quad (4.10)$$

If  $z_i, p_i$  are real, then  $(z_\gamma + c_\alpha p_\gamma)^2$  are all non-negative. Moreover, since  $z_\gamma + c_\alpha p_\gamma$  are not integrals of the motion, at certain time their squares are all positive. Hence, the above sum can be zero iff  $b_1 \leq c_\alpha \leq b_3$ . This holds for any real  $c_\alpha$ .  $\square$

**Proposition 4.1.2** *In the real case, the separating variables  $\lambda_1, \lambda_2$  are also real and*

$$\lambda_1 \in [b_1, b_2], \quad \lambda_2 \in [b_2, b_3].$$

Next, when no one of  $c_\alpha$  coincides with  $b_1, b_2, b_3$ , the level variety of the four first integrals of the problem (given by the coefficients at  $s^3, s^2, s, s^0$ ) is a union of two-dimensional tori in  $\mathbb{R}^6 = (z, p)$ . We restrict ourselves to this generic situation, excluding the other cases, which correspond to special motions.

Let  $\lambda_1, \lambda_2$  be the roots of the equation

$$f(\lambda) = \sum_{i=1}^3 \frac{(z_j p_k - z_k p_j)^2}{\lambda - b_i} = 0, \quad (i, j, k) = (1, 2, 3), \quad (4.11)$$

where, when all  $c_\alpha$  are real,

$$\lambda_1 \in [b_1, c_1], \quad \lambda_2 \in [c_3, b_3]. \quad (4.12)$$

Then for fixed  $c_0, c_1, c_2, c_3$  the variables  $z, p$  can be expressed in terms of  $\lambda_1, \lambda_2$  in such a way that for any  $s \in \mathbb{C}$  the following relation holds (see formula (7) in [38])

$$z_i + s p_i = \sqrt{c_0} \frac{x_i \sum_{\alpha=1}^3 (s - c_\alpha) \frac{\sqrt{-(\lambda_1 - c_\alpha)(\lambda_2 - c_\alpha)}}{(c_\alpha - c_\beta)(c_\alpha - c_\gamma)} \left( \frac{\sqrt{\Phi(\lambda_1)\psi(\lambda_2)}}{(\lambda_1 - b_i)(\lambda_2 - c_\alpha)} - \frac{\sqrt{\Phi(\lambda_2)\psi(\lambda_1)}}{(\lambda_2 - b_i)(\lambda_1 - c_\alpha)} \right)}{(\lambda_1 - \lambda_2) \sum_{\alpha=1}^3 \frac{\sqrt{-(\lambda_1 - c_\alpha)(\lambda_2 - c_\alpha)}}{(c_\alpha - c_\beta)(c_\alpha - c_\gamma)}}, \quad (4.13)$$

where

$$\Phi(\lambda) = (\lambda - b_1)(\lambda - b_2)(\lambda - b_3), \quad \psi(\lambda) = (\lambda - c_1)(\lambda - c_2)(\lambda - c_3), \quad (4.14)$$

$$x_i = \frac{\sqrt{(\lambda_1 - b_i)(\lambda_2 - b_i)}}{\sqrt{(b_i - b_j)(b_i - b_k)}}, \quad (4.15)$$

$$(i, j, k) = (1, 2, 3), \quad (\alpha, \beta, \gamma) = (1, 2, 3).$$

Setting in the above expression  $s \rightarrow \infty$  and  $s = 0$ , one obtains the corresponding formulas for  $p_i, z_i$ .

Note that for real  $z_i, p_i$ , in the case (1) (all  $c_\alpha$  are real), in view of the condition (4.12) all the expressions under the radicals in (4.13) are non-negative. In the rest of the cases the roots can be complex. For any  $\alpha = 1, 2, 3$ , the branches of  $\sqrt{-(\lambda_1 - c_\alpha)(\lambda_2 - c_\alpha)}$  in the numerator and the denominator of (4.13) must be the same.

Next, the evolution of  $\lambda_1, \lambda_2$  is described by the quadratures

$$\begin{aligned} \frac{d\lambda_1}{\sqrt{R(\lambda_1)}} + \frac{d\lambda_2}{\sqrt{R(\lambda_2)}} &= \delta_1 dt, \\ \frac{\lambda_1 d\lambda_1}{\sqrt{R(\lambda_1)}} + \frac{\lambda_2 d\lambda_2}{\sqrt{R(\lambda_2)}} &= \delta_2 dt, \\ R(\lambda) &= -\Phi(\lambda)\psi(\lambda) \end{aligned} \quad (4.16)$$

with certain constants  $\delta_1, \delta_2$  depending on the choice of the Hamiltonian only. Note that the paper [38] does not describe explicitly this dependence, which can be found in [14], [57].

The above quadratures rewritten in the integral form

$$\int_{\lambda_0}^{\lambda_1} \frac{d\lambda}{2\sqrt{R(\lambda)}} + \int_{\lambda_0}^{\lambda_2} \frac{d\lambda}{2\sqrt{R(\lambda)}} = u_1, \quad (4.17)$$

$$\begin{aligned} \int_{\lambda_0}^{\lambda_1} \frac{\lambda d\lambda}{2\sqrt{R(\lambda)}} + \int_{\lambda_0}^{\lambda_2} \frac{\lambda d\lambda}{2\sqrt{R(\lambda)}} &= u_2, \\ u_1 &= \delta_1 t + u_{10}, \quad u_2 = \delta_2 t + u_{20}, \end{aligned} \quad (4.18)$$

which represent the Abel–Jacobi map associated to the genus 2 hyperelliptic curve  $\mu^2 = -\Phi(\lambda)\psi(\lambda)$ . Inverting the map (4.17) and substituting symmetric functions of  $\lambda_1, \lambda_2, \mu_1, \mu_2$  into (4.13), one finally finds  $z, p$  as functions of time.

Everyone who had read paper [38] might be surprised by how Kötter managed to invent the intricate substitution  $(z, p) \rightarrow (\lambda_1, \lambda_2, c_0, c_1, c_2, c_3)$  and to represent the result in the symmetric form (4.13). Unfortunately, the author of the paper gave no explanations of his computations. Nevertheless, it is clear that behind the striking formulas there must be a certain geometric idea, which we try to reconstruct in the next section.

## 4.2 A geometric background of Kötter's solution.

Let  $(x_1 : x_2 : x_3)$  be homogeneous coordinates in  $\mathbb{P}^2$  defined up to multiplication by the same non-zero factor. Consider a line  $l$  in  $\mathbb{P}^2 = (x_1 : x_2 : x_3)$  defined by equation

$$y_1x_1 + y_2x_2 + y_3x_3 = 0.$$

Following Plücker (see e.g., [31]), the coefficients  $y_1, y_2, y_3$  can be regarded as homogeneous coordinates of a point in the dual projective space  $(\mathbb{P}^2)^*$ . Now let  $l_1, l_2$  be two intersecting lines in  $\mathbb{P}^2$  with the Plücker coordinates  $(y_1^{(1)}, y_2^{(1)}, y_3^{(1)})$ ,  $(y_1^{(2)}, y_2^{(2)}, y_3^{(2)})$ .

Then, for any constants  $\lambda, \mu \in \mathbb{C}$  not vanishing simultaneously, the linear combination  $\lambda y_\alpha^{(1)} + \mu y_\alpha^{(2)}$  are also Plücker coordinates of a line  $l_{\lambda, \mu} \in \mathbb{P}^2$ . Hence, we arrive at an important geometric object, a *pencil of lines* in  $\mathbb{P}^2$ , i.e., a one-parameter family  $l_{\lambda, \mu}$ . It is remarkable that all the lines of a pencil intersect at the same point  $\mathcal{P} \in \mathbb{P}^2$ . The point  $\mathcal{P}$  is called *the focus* of the pencil.

**Theorem 4.2.1** ([31]) *Let  $l_{\lambda, \mu}$  be a pencil of lines in  $\mathbb{P}^2$  defined by Plücker coordinates  $\lambda y_\alpha^{(1)} + \mu y_\alpha^{(2)}$ ,  $(\lambda : \mu) \in \mathbb{P}$ . Then the homogeneous coordinates of the focus are*

$$\mathbf{P} = \left( y_2^{(1)} y_3^{(2)} - y_3^{(1)} y_2^{(2)} : y_1^{(1)} y_3^{(2)} - y_3^{(1)} y_1^{(2)} : y_1^{(1)} y_2^{(2)} - y_2^{(1)} y_1^{(2)} \right).$$

Next, consider the family of confocal quadrics in  $\mathbb{P}^2$

$$Q(s) = \left\{ \frac{x_1^2}{s - b_1} + \frac{x_2^2}{s - b_2} + \frac{x_3^2}{s - b_3} = 0 \right\} \quad (4.19)$$

and a fixed point  $P = (X_1 : X_2 : X_3)$ . Then one defines the spheroconical coordinates  $\lambda_1, \lambda_2$  of this point (with respect to  $Q(s)$ ) as the roots of the equation

$$\frac{X_1^2}{\lambda - b_1} + \frac{X_2^2}{\lambda - b_2} + \frac{X_3^2}{\lambda - b_3} = 0.$$

Now, going back to the Steklov–Lyapunov systems, we make the following observation.

**Proposition 4.2.2** *The separating variables  $\lambda_1, \lambda_2$  defined by formula (4.11) are spheroconical coordinates of the focus  $\mathbf{P}$  of the pencil of lines in  $\mathbb{P}^2$  with the Plücker coordinates  $z + sp = (z_1 + sp_1 : z_2 + sp_2 : z_3 + sp_3)$ ,  $s \in \mathbb{P}$  with respect to the family of quadrics (4.19).*

*Proof.* According to Theorem 4.2.1, the homogeneous coordinates of the focus  $\mathbf{P}$  are

$$(z_2p_3 - z_3p_2 : z_3p_1 - z_1p_3 : z_1p_2 - z_2p_1), \quad (4.20)$$

hence, the spheroconical coordinates of  $\mathbf{P}$  with respect to the family (4.19) are precisely the roots of the equation (4.11), i.e.,  $\lambda_1, \lambda_2$ .  $\square$

Note also the following property: for  $\alpha = 1, 2, 3$ , the line  $l_\alpha$  with the Plücker coordinates  $z + c_\alpha p$  is tangent to the quadric  $Q_\alpha = Q(c_\alpha)$ . Indeed, setting in the right hand side of (4.7)  $s = c_\alpha$ , we obtain

$$\sum_{i=1}^3 (c_\alpha - b_i)(z_i + c_\alpha p_i)^2 = 0,$$

which represents the condition of tangency of the line  $l_\alpha$  and the quadric  $Q_\alpha$ .

As a result, the following configuration holds: *the three lines  $l_1, l_2, l_3$  in  $\mathbb{P}^2$  intersect at the same point  $\mathbf{P}$  and are tangent to the quadrics  $Q_1, Q_2, Q_3$  respectively.* An example of such a configuration is shown in Fig.4.2.1.

It follows that a solution  $z(t), p(t)$  defines a trajectory of the focus  $\mathbf{P}$  on  $\mathbb{P}^2$  or on  $S^2 = \{x_1^2 + x_2^2 + x_3^2 = 1\}$ , and it is natural to suppose that the Steklov–Lyapunov systems define dynamical systems on the sphere. Indeed, some of these systems were studied in [57] and were shown to be related to a generalization of the Neumann system with a quartic potential.

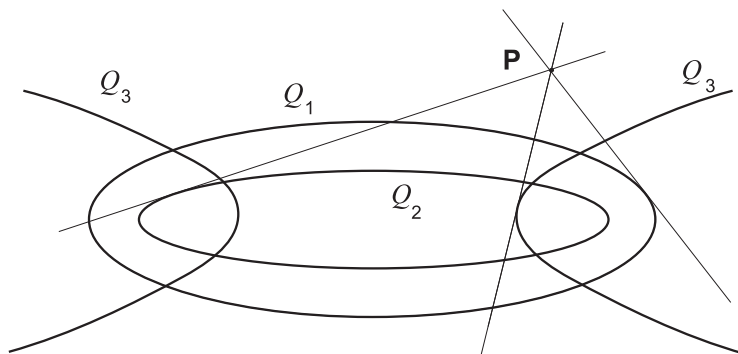


Figure 4.2.1: A configuration of tangent lines in  $\mathbb{R}^2 = \left(X_1 = \frac{x_1}{x_3}, X_2 = \frac{x_2}{x_3}\right)$  for the case  $b_1 < c_1 < b_2 < c_2 < c_3 < b_3$ , when the quadrics  $Q_\alpha$  are two ellipses and a hyperbola.

In the sequel our main goal will be to recover the variables  $z$  and  $p$  as functions of the spheroconical coordinates of the focus  $\mathbf{P}$ , that is, to reconstruct the Kötter formula (4.13). Obviously, the solution is not unique: to each pair  $(\lambda_1, \lambda_2), \lambda_k \neq b_1, b_2, b_3$  there correspond 4 points on  $\mathbb{P}^2$ , and for each point  $\mathbf{P}$  that does not lie on any of the quadrics  $Q(c_\alpha)$ ,  $2^3 = 8$  different configurations of tangent lines  $l_1, l_2, l_3$  are possible (Fig.4.2.1 shows just one of them). Thus, under the above generality conditions, a pair  $(\lambda_1, \lambda_2)$  gives 32 different tangent configurations.

**Reconstruction of  $z, p$  in terms of the separating variables.** Let  $(\mathbb{P}^2)^* = (G_1 : G_2 : G_3)$  be the dual space to  $\mathbb{P}^2 = (x_1 : x_2 : x_3)$ , ( $G_i$  being the Plücker coordinates of lines in  $\mathbb{P}^2$ ). It is convenient to regard  $G_i$  also as Cartesian coordinates in the space  $(\mathbb{C}^3)^* = (G_1, G_2, G_3)$ . The pencil  $\sigma(\mathbf{P})$  of lines in  $\mathbb{P}^2$  with the focus

(4.20) is represented by a line in  $(\mathbb{P}^2)^*$  or by plane

$$\pi = \{(z_2 p_3 - z_3 p_2)G_1 + (z_3 p_1 - z_1 p_3)G_2 + (z_1 p_2 - z_2 p_1)G_3 = 0\} \subset (\mathbb{C}^3)^*.$$

Consider the line  $\bar{\sigma}(\mathbf{P}) = \{z + sp | s \in \mathbb{R}\} \subset (\mathbb{C}^3)^*$ . Obviously,  $\{z + sp\} \subset \pi$ . Now let us use the condition for the three lines  $\ell_1, \ell_2, \ell_3$  defined by the points  $z + c_1 p, z + c_2 p, z + c_3 p$  in  $(\mathbb{P}^2)^*$  to be tangent to the quadrics  $Q(c_1), Q(c_2), Q(c_3)$  respectively. Let  $\mathbf{V}_\alpha = (V_{\alpha 1}, V_{\alpha 2}, V_{\alpha 3}) \subset \pi$ ,  $\alpha = 1, 2, 3$  be some vectors in  $(\mathbb{C}^3)^*$  representing these points, so that  $\ell_\alpha = \{V_{\alpha 1} x_1 + V_{\alpha 2} x_2 + V_{\alpha 3} x_3 = 0\}$ . Then we have

$$z + c_1 p - \mu_1 \mathbf{V}_1 = 0, \quad z + c_2 p - \mu_2 \mathbf{V}_2 = 0, \quad z + c_3 p - \mu_3 \mathbf{V}_3 = 0 \quad (4.21)$$

for some indefinite factors  $\mu_\alpha$ . This system is equivalent to a homogeneous system of 9 scalar equations for 9 variables  $z_\alpha, p_\alpha, \mu_\alpha$ ,  $\alpha = 1, 2, 3$ . Thus the variables can be found up to multiplication by a common factor. Eliminating  $z, p$  from (4.21), we obtain the following homogeneous system for  $\mu_1, \mu_2, \mu_3$

$$(c_2 - c_3)V_{\alpha 1}\mu_1 + (c_3 - c_1)V_{\alpha 2}\mu_2 + (c_1 - c_2)V_{\alpha 3}\mu_3 = 0, \quad \alpha = 1, 2, 3,$$

which has a nontrivial solution, since  $\det \|V_{\alpha i}\| = 0$  (the vectors  $\mathbf{V}_\alpha$  lie in the same hyperplane  $\pi$ ). It follows, for example, that

$$\mu_1 = \mu \Sigma_1 / (c_2 - c_3), \quad \mu_2 = \mu \Sigma_2 / (c_3 - c_1), \quad \mu_3 = \mu \Sigma_3 / (c_1 - c_2), \quad (4.22)$$

$$\Sigma_1 = V_{22}V_{33} - V_{32}V_{23}, \quad \Sigma_2 = V_{32}V_{13} - V_{33}V_{12}, \quad \Sigma_3 = V_{12}V_{23} - V_{13}V_{22}, \quad (4.23)$$

$\mu \neq 0$  being an arbitrary factor. Substituting these expressions into (4.21) and using the obvious identity

$$\Sigma_1 \mathbf{V}_1 + \Sigma_2 \mathbf{V}_2 + \Sigma_3 \mathbf{V}_3 = 0,$$

after transformations we find

$$p = \frac{\mu}{(c_1 - c_2)(c_2 - c_3)(c_3 - c_1)} (c_1 \Sigma_1 \mathbf{V}_1 + c_2 \Sigma_2 \mathbf{V}_2 + c_3 \Sigma_3 \mathbf{V}_3), \quad (4.24)$$

$$z = \frac{\mu}{(c_1 - c_2)(c_2 - c_3)(c_3 - c_1)} (c_2 c_3 \Sigma_1 \mathbf{V}_1 + c_1 c_3 \Sigma_2 \mathbf{V}_2 + c_1 c_2 \Sigma_3 \mathbf{V}_3). \quad (4.25)$$

As a result,

$$z + sp = \frac{\mu}{(c_1 - c_2)(c_2 - c_3)(c_3 - c_1)} \sum_{\alpha=1}^3 (c_\alpha s + c_\beta c_\gamma) \Sigma_\alpha \mathbf{V}_\alpha. \quad (4.26)$$

Now we calculate the components of  $\mathbf{V}_\alpha$ . Up to an arbitrary nonzero factor, they can be found from the system of equations

$$V_{\alpha 1} x_1 + V_{\alpha 2} x_2 + V_{\alpha 3} x_3 = 0, \quad \sum_{i=1}^3 (c_\alpha - b_i) V_{\alpha i}^2 = 0, \quad \alpha = 1, 2, 3, \quad (4.27)$$

which represent the conditions that the line  $\ell_\alpha$  passes through the focus  $\mathbf{P} = (x_1 : x_2 : x_3)$  and touches the quadric  $Q(c_\alpha)$ .



In the sequel we apply the normalization  $x_1^2 + x_2^2 + x_3^2 = 1$ , which gives rise to expressions (4.15).

For  $\mathbf{P} \notin Q(c_\alpha)$ , this system possesses two different solutions, and for  $\mathbf{P} \in Q(c_\alpha)$  a single one (the line touches  $Q(c_\alpha)$  at the point  $\mathbf{P}$ ). In the latter case we can just put

$$V_{\alpha i} = x_i / (c_\alpha - b_i). \quad (4.28)$$

Next, it is obvious that under reflection  $(x_1 : x_2 : x_3) \rightarrow (-x_1 : x_2 : x_3)$ , a solution  $(V_{\alpha 1} : V_{\alpha 2} : V_{\alpha 3})$  transforms to  $(-V_{\alpha 1} : V_{\alpha 2} : V_{\alpha 3})$  (similarly, for the two other reflections). Let us seek solutions of equations (4.27) in the form of symmetric functions of the complex coordinates  $\lambda_1, \lambda_2$  such that

- 1) for  $\lambda_1 = c_\alpha$  or  $\lambda_2 = c_\alpha$  (i.e., when  $\mathbf{P} \in Q(c_\alpha)$ ) there is a unique solution proportional to (4.28);
- 2) if  $\lambda_1$  or  $\lambda_2$  circles around the point  $\lambda = c_\alpha$  on the complex plane  $\lambda$ , the two solutions transform into each other;
- 3) for  $\lambda_1 = b_i$  or  $\lambda_2 = b_i$  (i.e., when  $x_i = 0$ ),  $V_{\alpha i}$  does not vanishes.

Using the Jacobi identities

$$\sum_{i=1}^n \frac{a_i^k}{\prod (a_i - a_j)} = \begin{cases} 0, & k < n - 1 \\ 1, & k = n - 1 \\ \sum_{i=1}^n a_i, & k = n, \end{cases} \quad (4.29)$$

one can check that the following expressions satisfy equations (4.27) and the above three conditions

$$V_{\alpha i} = x_i \left( \frac{\sqrt{\Phi(\lambda_1)(\lambda_2 - c_\alpha)}}{\lambda_1 - b_i} + \frac{\sqrt{\Phi(\lambda_2)(\lambda_1 - c_\alpha)}}{\lambda_2 - b_i} \right), \quad x_i = \frac{\sqrt{(\lambda_1 - b_i)(\lambda_2 - b_i)}}{\sqrt{(b_i - b_j)(b_i - b_k)}}. \quad (4.30)$$

Then, using again the identities (4.29), we have

$$\langle \mathbf{V}_\alpha, \mathbf{V}_\beta \rangle \equiv (\lambda_2 - \lambda_1) \left( \sqrt{(\lambda_2 - c_\alpha)(\lambda_2 - c_\beta)} - \sqrt{(\lambda_1 - c_\alpha)(\lambda_1 - c_\beta)} \right). \quad (4.31)$$

and, in particular,  $\langle \mathbf{V}_\alpha, \mathbf{V}_\alpha \rangle = (\lambda_1 - \lambda_2)^2$  for  $\alpha = 1, 2, 3$ .

Next, substituting (4.30) into (4.23) and applying the symbolic multiplication rule  $\sqrt{ab}\sqrt{ac} = a\sqrt{bc}$ , we find the factors  $\Sigma_\alpha$  in form

$$\Sigma_\alpha = (\lambda_1 - \lambda_2)x_1 \left( \sqrt{-(\lambda_1 - c_\gamma)(\lambda_2 - c_\beta)} - \sqrt{-(\lambda_1 - c_\beta)(\lambda_2 - c_\gamma)} \right), \quad (4.32)$$

$$(\alpha, \beta, \gamma) = (1, 2, 3).$$

Further, putting (4.30), (4.32) into (4.26), we obtain

$$\begin{aligned}
z_i + sp_i &= \frac{\mu(\lambda_1 - \lambda_2)x_1}{(c_1 - c_2)(c_2 - c_3)(c_3 - c_1)} x_i \cdot \sum_{\alpha=1}^3 (c_\alpha s + c_\beta c_\gamma) \\
&\cdot \left[ \frac{\sqrt{\Phi(\lambda_1)\psi(\lambda_2)}}{\lambda_1 - b_i} \left( \sqrt{\frac{\lambda_1 - c_\gamma}{\lambda_2 - c_\gamma}} - \sqrt{\frac{\lambda_1 - c_\beta}{\lambda_2 - c_\beta}} \right) + \frac{\sqrt{\Phi(\lambda_2)\psi(\lambda_1)}}{\lambda_2 - b_i} \left( \sqrt{\frac{\lambda_2 - c_\gamma}{\lambda_1 - c_\gamma}} - \sqrt{\frac{\lambda_1 - c_\beta}{\lambda_2 - c_\beta}} \right) \right] \\
&\equiv \mu(\lambda_1 - \lambda_2)x_1 x_i \sum_{\alpha=1}^3 (s - c_\alpha) \frac{\sqrt{-(\lambda_1 - c_\alpha)(\lambda_2 - c_\alpha)}}{(c_\alpha - c_\beta)(c_\alpha - c_\gamma)} \left( \frac{\sqrt{\Phi(\lambda_1)\psi(\lambda_2)}}{(\lambda_1 - b_i)(\lambda_2 - c_\alpha)} - \frac{\sqrt{\Phi(\lambda_2)\psi(\lambda_1)}}{(\lambda_2 - b_i)(\lambda_1 - c_\alpha)} \right),
\end{aligned} \tag{4.33}$$

which, up to multiplication by a common factor, coincides with the numerator in Kötter's formula (4.13).

To determine the factor  $\mu$  in (4.26) and in (4.33), we apply the condition  $\langle p, p \rangle = c_0$  which follows from (4.9). Then, from (4.24) we get

$$\frac{c_0}{\mu^2} = \frac{|c_1 \Sigma_1 \mathbf{V}_1 + c_2 \Sigma_2 \mathbf{V}_2 + c_3 \Sigma_3 \mathbf{V}_3|^2}{(c_1 - c_2)^2 (c_2 - c_3)^2 (c_3 - c_1)^2}. \tag{4.34}$$

Using the expressions (4.31), we obtain

$$\begin{aligned}
\left| \sum_{\alpha=1}^3 c_\alpha \Sigma_\alpha \mathbf{V}_\alpha \right|^2 &\equiv \sum_{\alpha=1}^3 [c_\alpha^2 \Sigma_\alpha^2 \langle \mathbf{V}_\alpha, \mathbf{V}_\alpha \rangle + 2c_\beta c_\gamma \Sigma_\beta \Sigma_\gamma \langle \mathbf{V}_\beta, \mathbf{V}_\gamma \rangle] \\
&= (\lambda_1 - \lambda_2)^3 x_1^2 \sum_{\alpha=1}^3 \left[ c_\alpha^2 (\lambda_1 - \lambda_2) \left( \sqrt{-(\lambda_1 - c_\gamma)(\lambda_2 - c_\beta)} - \sqrt{-(\lambda_1 - c_\beta)(\lambda_2 - c_\gamma)} \right)^2 \right. \\
&\quad + 2c_\beta c_\gamma \left( \sqrt{-(\lambda_1 - c_\gamma)(\lambda_2 - c_\beta)} - \sqrt{-(\lambda_1 - c_\beta)(\lambda_2 - c_\gamma)} \right) \\
&\quad \cdot \left( \sqrt{-(\lambda_1 - c_\alpha)(\lambda_2 - c_\gamma)} - \sqrt{-(\lambda_1 - c_\gamma)(\lambda_2 - c_\alpha)} \right) \\
&\quad \left. \cdot \left( \sqrt{-(\lambda_2 - c_\beta)(\lambda_2 - c_\gamma)} - \sqrt{-(\lambda_1 - c_\beta)(\lambda_1 - c_\gamma)} \right) \right].
\end{aligned}$$

Simplifying the above expression and again using symbolic multiplication of square roots, one can verify that it is a full square:

$$\left| \sum_{\alpha=1}^3 c_\alpha \Sigma_\alpha \mathbf{V}_\alpha \right|^2 = x_1^2 (\lambda_1 - \lambda_2)^4 \left( \sum_{\alpha=1}^3 (c_\beta - c_\gamma) \sqrt{-(\lambda_1 - c_\alpha)(\lambda_2 - c_\alpha)} \right)^2.$$

Hence, from (4.34) we find

$$\frac{\sqrt{c_0}}{\mu} = x_1 (\lambda_1 - \lambda_2)^2 \sum_{\alpha=1}^3 \frac{\sqrt{-(\lambda_1 - c_\alpha)(\lambda_2 - c_\alpha)}}{(c_\alpha - c_\beta)(c_\alpha - c_\gamma)}.$$

Combining this with (4.33), we finally arrive at (4.13).

Thus, we derived the remarkable Kötter formula by making use of the geometric interpretation of the variables  $\lambda_1, \lambda_2$ . We also note that the expressions (4.13) are symmetric in  $\lambda_1, \lambda_2$ .

**Remark 1.** As noticed above, a disordered generic pair  $(\lambda_1, \lambda_2)$  gives 32 different configurations of tangent lines to the quadrics  $Q(c_1), Q(c_2), Q(c_3)$ . Since the common factor  $\mu$  in (4.26) is defined up to sign flip, we conclude that, according to the formula (4.13), *to each generic pair  $(\lambda_1, \lambda_2)$  there correspond 64 different points  $(z, p)$  on the invariant manifold (a union of 2-dimensional tori) defined by the constants  $c_0, c_1, c_2, c_3$ .* This ambiguity corresponds to different signs of the square roots in the Kötter formula.

In the next section we shall use the expressions (4.13) and the quadratures (4.17) to find explicit theta-functional solutions for the Steklov–Lyapunov systems.

### 4.3 Explicit theta-function solution of the Steklov-Lyapunov systems

**The root functions.** To obtain theta-functions solution of Steklov-Lyapunov systems we apply the *root functions* described in Introduction (1.2.5).

For our purposes it is sufficient to quote only several root functions for the particular case  $g = 2$  and the even-order hyperelliptic curve

$$\Gamma = \{\mu^2 = R(\lambda)\}, \quad R(\lambda) = (\lambda - E_1) \cdots (\lambda - E_6).$$

Let us introduce the polynomial  $U(\lambda, s) = (s - \lambda_1)(s - \lambda_2)$ .

**Proposition 4.3.1** *Under the Abel–Jacobi mapping (1.27) with  $g = 2$  and the base-point  $P_0 = E_6$  the following relations hold*

$$U(\lambda, E_i) \equiv (\lambda_1 - E_i)(\lambda_2 - E_i) = \kappa_i \frac{\theta^2[\Delta + \eta_i](z)}{\theta[\Delta](z - q/2) \theta[\Delta](z + q/2)}, \quad (4.35)$$

$$q = \int_{\infty_-}^{\infty_+} \bar{\omega} = 2 \int_{E_6}^{\infty_+} \bar{\omega}, \quad \kappa_i = \text{const}, \quad i = 1, \dots, 6,$$

$$\begin{aligned} \frac{1}{\lambda_1 - \lambda_2} & \left( \frac{\sqrt{R(\lambda_1)}}{(E_i - \lambda_1)(E_j - \lambda_1)(E_s - \lambda_1)} - \frac{\sqrt{R(\lambda_2)}}{(E_i - \lambda_2)(E_j - \lambda_2)(E_s - \lambda_2)} \right) \\ & = \kappa_{ijs} \frac{\theta[\Delta + \eta_i + \eta_j + \eta_s](z) \theta[\Delta](z - q/2) \theta[\Delta](z + q/2)}{\theta[\Delta + \eta_i](z) \theta[\Delta + \eta_j](z) \theta[\Delta + \eta_s](z)}, \end{aligned} \quad (4.36)$$

$$\begin{aligned} \frac{\sqrt{U(\lambda, E_i)} \sqrt{U(\lambda, E_j)}}{\lambda_1 - \lambda_2} & \left( \frac{\sqrt{R(\lambda_1)}}{(E_i - \lambda_1)(E_j - \lambda_1)(E_s - \lambda_1)} - \frac{\sqrt{R(\lambda_2)}}{(E_i - \lambda_2)(E_j - \lambda_2)(E_s - \lambda_2)} \right) \\ & = \kappa'_{ijs} \frac{\theta[\Delta + \eta_i + \eta_j + \eta_s](z)}{\theta[\Delta + \eta_s](z)}, \end{aligned} \quad (4.37)$$

$$\kappa_{ijs}, \kappa'_{ijs} = \text{const}, \quad i, j, s = 1, \dots, 6, \quad i \neq j \neq s \neq i,$$

where

$$\begin{aligned} \Delta &= \begin{pmatrix} 1/2 & 1/2 \\ 0 & 1/2 \end{pmatrix}, & \Delta + \eta_1 &= \begin{pmatrix} 0 & 1/2 \\ 0 & 1/2 \end{pmatrix}, & \Delta + \eta_2 &= \begin{pmatrix} 0 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}, \\ \Delta + \eta_3 &= \begin{pmatrix} 1/2 & 0 \\ 1/2 & 1/2 \end{pmatrix}, & \Delta + \eta_4 &= \begin{pmatrix} 1/2 & 0 \\ 1/2 & 0 \end{pmatrix}, & \Delta + \eta_5 &= \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 0 \end{pmatrix}. \end{aligned}$$

and  $\infty_+, \infty_-$  are the infinite points of the compactified curve  $\Gamma$ . The constant factors  $\kappa_i, \kappa_{ijs}, \kappa'_{ijs}$  depend on the modulo of  $\Gamma$  only.

The constants  $\kappa_i, \kappa_{ijs}, \kappa'_{ijs}$  can be calculated explicitly in terms of the coordinates  $E_1, \dots, E_6$  and theta-constants by equating  $\lambda_1, \lambda_2$  to certain  $E_i$  and the argument  $z$  to the corresponding half-period in  $\text{Jac}(\Gamma)$  (see, e.g., [15]).

**Explicit solution.** Now we are able to write explicit solution for the Steklov–Lyapunov systems by comparing the root functions (4.35), (4.37) with the Kötter expression (4.13).

Namely, let  $\Gamma = \{\mu^2 = \Phi(\lambda)\varphi(\lambda)\}$  where the polynomials  $\phi$  and  $\varphi$  are defined in (4.14) and identify (without ordering) the sets

$$\{E_1, E_2, E_3, E_4, E_5, E_6\} = \{b_1, b_2, b_3, c_1, c_2, c_3\}.$$

By  $\eta_{b_i}, \eta_{c_\alpha}$  we denote the half-integer characteristics corresponding to the branch points  $(b_i, 0), (c_\alpha, 0)$  respectively, according to formula (1.40).

**Theorem 4.3.2** For fixed constants of motion  $c_1, c_2, c_3$  the variables  $z, p$  can be expressed in terms of theta-functions of the curve  $\Gamma$  in a such a way that for any  $s \in \mathbb{C}$

$$z_i + sp_i = \frac{\sum_{\alpha=1}^3 k_{i\alpha} (s - c_\alpha) \theta[\Delta + \eta_{c_\beta} + \eta_{c_\gamma} + \eta_{b_i}](z)}{\sum_{\alpha=1}^3 k_{0\alpha} \theta[\Delta + \eta_{c_\alpha}](z)}, \quad (\alpha, \beta, \gamma) = (1, 2, 3), \quad (4.38)$$

where  $k_{i\alpha}, k_{0\alpha}$  are certain constants depending on the modulo of  $\Gamma$  only, and the components of the argument  $z$  are linear functions of  $t$ :

$$z_j = C_{j1}\delta_1 + C_{j2}\delta_2 + z_{j0}, \quad z_{j0} = \text{const}, \quad C = A^{-1} \quad (4.39)$$

$A$  being is the matrix of  $\mathbf{a}$ -periods of the differentials  $d\lambda/\mu, \lambda d\lambda/\mu$  on  $\Gamma$ .

Thus, we have recovered the theta-function solution of the systems obtained by Kötter in [38].

**Remark 2.** In view of the definition of theta-function with characteristics, under the argument shift  $z \rightarrow z - \mathcal{K}$  the special characteristic  $\Delta$  is killed and the solutions (4.38) are simplified to

$$z_i + sp_i = \frac{\sum_{\alpha=1}^3 \bar{k}_{i\alpha} (s - c_\alpha) \theta[\eta_{c_\beta} + \eta_{c_\gamma} + \eta_{b_i}](z)}{\sum_{\alpha=1}^3 \bar{k}_{0\alpha} \theta[\eta_{c_\alpha}](z)}, \quad (\alpha, \beta, \gamma) = (1, 2, 3), \quad (4.40)$$

where the constants  $\bar{k}_{i\alpha}, \bar{k}_{0\alpha}$  coincide with  $k_{i\alpha}, k_{0\alpha}$  in (4.38) up to multiplication by a quartic root of unity. In each concrete case of position of  $b_i, c_\alpha$ , one can also simplify the sums of characteristics in the numerator of (4.40) by using the relations (??).

*Proof of Theorem 4.3.2.* The summands in the numerator of the Kötter solution (4.13), when divided by  $\lambda_1 - \lambda_2$ , can be written as

$$\begin{aligned} & \frac{s - c_\alpha}{(c_\alpha - c_\beta)(c_\alpha - c_\gamma)} \frac{\sqrt{-(\lambda_1 - c_\alpha)(\lambda_2 - c_\alpha)}}{\lambda_1 - \lambda_2} \cdot \left( \frac{\sqrt{\Phi(\lambda_1)\psi(\lambda_2)}}{(\lambda_1 - b_i)(\lambda_2 - c_\alpha)} - \frac{\sqrt{\Phi(\lambda_2)\psi(\lambda_1)}}{(\lambda_2 - b_i)(\lambda_1 - c_\alpha)} \right) \\ &= \frac{s - c_\alpha}{(c_\alpha - c_\beta)(c_\alpha - c_\gamma)} \frac{\sqrt{-(\lambda_1 - c_\beta)(\lambda_2 - c_\beta)}\sqrt{-(\lambda_1 - c_\gamma)(\lambda_2 - c_\gamma)}}{\lambda_1 - \lambda_2} \\ & \quad \times \left( \frac{\mu_1}{(\lambda_1 - b_i)(\lambda_1 - c_\beta)(\lambda_1 - c_\gamma)} - \frac{\mu_2}{(\lambda_2 - b_i)(\lambda_2 - c_\beta)(\lambda_2 - c_\gamma)} \right), \\ & \mu_1 = \sqrt{\Phi(\lambda_1)\psi(\lambda_1)}, \quad \mu_2 = \sqrt{\Phi(\lambda_2)\psi(\lambda_2)}. \end{aligned}$$

The right hand sides have the form of the root function (4.37). Hence, up to a constant factor, they are equal to

$$(s - c_\alpha) \frac{\theta[\Delta + \eta_{c_\beta} + \eta_{c_\gamma} + \eta_{b_i}](z)}{\theta[\Delta + \eta_{b_i}](z)}.$$

Next, in view of (4.35), we obtain

$$\begin{aligned} x_i &= \varkappa_i \frac{\theta[\Delta + \eta_{b_i}](z)}{\sqrt{\theta[\Delta](z - q/2)\theta[\Delta](z + q/2)}}, \\ \sqrt{-(\lambda_1 - c_\alpha)(\lambda_2 - c_\alpha)} &= \varkappa_\alpha \frac{\theta[\Delta + \eta_{c_\alpha}](z)}{\sqrt{\theta[\Delta](z - q/2)\theta[\Delta](z + q/2)}}, \quad (4.41) \\ \varkappa_i, \varkappa_\alpha &= \text{const.} \end{aligned}$$

Combining the above expressions, we rewrite the right hand side of (4.13) in the form

$$\sqrt{c_0} \frac{\frac{\theta[\Delta + \eta_{b_i}](z)}{\theta[\Delta](z - q/2)\theta[\Delta](z + q/2)} \sum_{\alpha=1}^3 \frac{k_{i\alpha}(s - c_\alpha)\theta[\Delta + \eta_{c_\beta} + \eta_{c_\gamma} + \eta_{b_i}](z)}{\theta[\Delta + \eta_{b_i}](z)}}{\sum_{\alpha=1}^3 \frac{k_{0\alpha}\theta[\Delta + \eta_{c_\alpha}](z)}{\theta[\Delta](z - q/2)\theta[\Delta](z + q/2)}},$$

which, after simplifications, gives (4.38).

Formulas (4.39) follow from the relation  $(\bar{\omega}_1, \bar{\omega}_2)^T = C(d\lambda/\mu, \lambda d\lambda/\mu)$ , where, as above,  $\bar{\omega}_j$  are the normalized holomorphic differentials on  $\Gamma$ , and the functions (4.18).  $\square$

In given Chapter 3 we gave a justification of the separation of variables and the theta-function solution of the Steklov–Lyapunov systems obtained by F. Kötter [38].



follows:

$$z = -b_k p + \lambda e_k, \quad \lambda \in \mathbb{R}$$

where  $e_k$  denotes the  $k$ th basis vector.

The corresponding part of the bifurcation diagram is the image of this set. To describe it we substitute the above relation between  $z$  and  $p$  to the formulas for first integrals:

$$\begin{aligned} \langle p, p \rangle &= J_1, \\ \langle 2z, p \rangle - \langle Bp, p \rangle &= J_2, \\ \langle z, z \rangle - 2\langle z, Bp \rangle &= H_2, \\ \langle z, Bz \rangle &= H_1. \end{aligned}$$

This substitution gives:

$$\begin{aligned} J_1 &= \langle p, p \rangle, \\ J_2 &= \langle 2z, p \rangle - \langle Bp, p \rangle = -2b_k \langle p, p \rangle + 2\lambda p_k - \langle Bp, p \rangle, \\ h_2 &= \langle z, z \rangle - 2\langle z, Bp \rangle = b_k^2 \langle p, p \rangle - 4\lambda b_k p_k + \lambda^2 + 2b_k \langle Bp, p \rangle, \\ h_1 &= \langle z, Bz \rangle = b_k^2 \langle p, Bp \rangle - 2\lambda b_k^2 p_k + b_k \lambda^2. \end{aligned}$$

Simplifying, we obtain:

$$\begin{aligned} h_2 &= \langle z, z \rangle - 2\langle z, Bp \rangle = -3b_k^2 J_1 - 2b_k J_2 + \lambda^2, \\ h_1 &= \langle z, Bz \rangle = -2b_k^3 J_1 - b_k^2 J_2 + b_k \lambda^2, \end{aligned}$$

which gives, by excluding  $\lambda$ , one linear relation between the integrals:

$$h_1 = b_k^3 + b_k^2 J_2 + b_k h_2.$$

For fixed value  $J_2$ , this relation defines a line  $\ell_i$  of the bifurcation diagram  $\mathcal{D}$ . Assume  $J_1 = 1$ .

If  $s \neq b_i$  it appears from the form of the matrix (5.1) that the rang of the matrix also drops if  $z_i + sp_i = 0$ . Thus, this implies description of critical points:

**Theorem 5.1.1** *For fixed value  $J_2$ , the bifurcation diagram is a subset of the curve  $\mathcal{D}$  given parametrically  $s \in \mathbb{R}$  by the formulas:*

$$\begin{aligned} h_2(s) &= -3s^2 - 2sJ_2, \\ h_1(s) &= -2s^3 - s^2J_2 \end{aligned} \tag{5.2}$$

and three lines  $\ell_i$

$$\begin{aligned} h_2 &= -3b_k^2 - 2b_k J_2 + \lambda^2, \\ h_1 &= -2b_k^3 - b_k^2 J_2 + b_k \lambda^2, \end{aligned}$$

where  $\lambda \in \mathbb{R}$ .

For our further purposes we need the implicit form of the above curve  $\mathcal{D}$ , that under transformation take the form

$$\mathcal{D} = \{-27h_2^2 - 18J_2h_2h_1 + J_2^2h_1^2 + 4J_2^3h_2 - 4h_1^3 = 0\}$$

The following Proposition presents geometrical properties of the above curve  $\mathcal{D}$  and three lines  $\ell_i, i = 1, 2, 3$ .

**Proposition 5.1.2** 1) *The curve  $\mathcal{D}$  has a unique return point (a cusp)  $C$  at  $(h_2 = J_2^2/3, h_1 = -J_2^3/27)$ , and is tangent to all the lines. It is also tangent to the horizontal axis  $h_1$  at the origin.*

2) *For  $J_2 = -(b_1 + b_2 + b_3)$  the three lines  $\ell_i$  have a unique intersection point with the coordinates  $h_2 = b_1b_2 + b_2b_3 + b_1b_3, h_1 = b_1b_2b_3$ .*

*For the other values  $J_2$  the lines  $\ell_i$  form a triangle  $\Delta = A_1A_2A_3$  with vertices  $A_k = \ell_i \cap \ell_j, (i, j, k) = (1, 2, 3)$ . The triangle  $\Delta$  lies completely inside the domain  $\mathcal{R}$  of real roots. As a consequence, the cusp  $C$  cannot be inside  $\Delta$ .*

3) *The cusp  $C$  lies on the line  $\ell_i$  iff  $J_2 = -3b_i$ . In this case the curve  $\mathcal{D}$  has a double tangency with  $\ell_i$  at  $C$  and there are no other points of intersection of  $\mathcal{D}$  and  $\ell_i$ .*

Examples of the bifurcation lines and curve are given in Figure 5.1.2.

*Proof of Proposition 5.1.2.*

1) The coordinates of the cusp  $C$  are calculated in the standard way, by determining singular points of  $\mathcal{D}$ . Next, substituting the equation of  $\ell_i$  into that of  $\mathcal{D}$ , we obtain an equation for  $h_1$ , which always has a double root  $h_1 = -2J_2b_i - 3b_i^2$ . This implies that at the corresponding point  $(h_1, h_2)$  the curve  $\mathcal{D}$  is tangent to  $\ell_i$ .

2) For generic different  $b_i$ , the condition for  $\ell_1, \ell_2, \ell_3$  to intersect at the same point gives a system of three equations  $b_i^3 + J_2b_i^2 + b_ih_1 - h_2 = 0$  on  $h_1, h_2, J_2$ , solving this equations we find the coordinates of intersecting point and condition on  $J_2$ , for which the intersection is possible (see an example in Figure 5.1.2)

Next, at the vertices  $A_k$  of the triangle  $\Delta$  we have  $c_\alpha = b_i, c_\beta = b_j$ , that is, 2 roots are real. Hence, at  $A_k$  all  $c_\alpha$  must be real, and all the vertices must belong to the domain  $\mathcal{R}$ . Finally, in view of 2), the curve  $\mathcal{D}$  cannot cross two times any of the edges of  $\Delta$  since otherwise it will not be tangent to  $\ell_i$ . As a result, the triangle itself belongs to  $\mathcal{R}$ .

3) Substituting the coordinates of the cusp  $C$  into the equation for  $\ell_i$  we obtain the above condition on  $J_2$ . Then it is shown directly that under this condition  $\mathcal{D}$  and  $\ell_i$  have a unique triple intersection at  $C$ .  $\square$

The bifurcation diagram divides the image of the momentum mapping into different zones denoted in Figures by Roman digits I, II and III, IV. If the point  $(h_1, h_2)$  passes through these zones the number of Liouville tori changes. Now we describe which part of  $\mathbb{R}^2 = (h_1, h_2)$  correspond to real  $z_i, p_i$ .



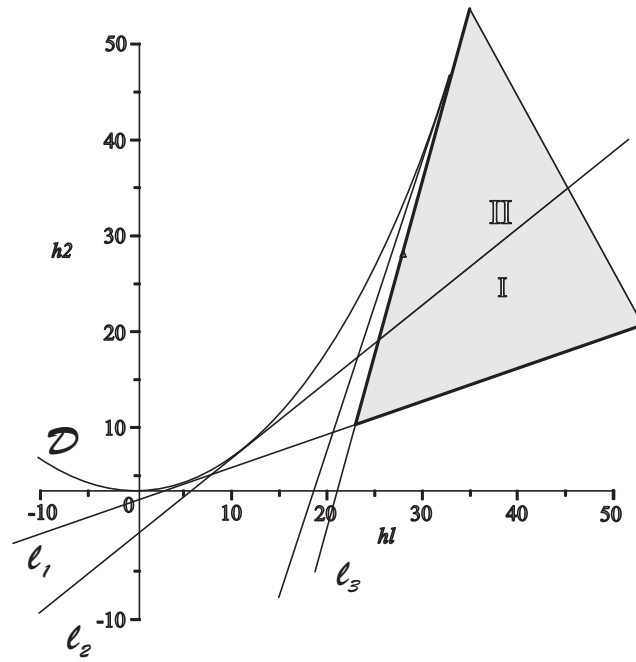


Figure 5.1.1:  $J_2 = -8, b_1 = 0.5, b_2 = 1, b_3 = 3$ .

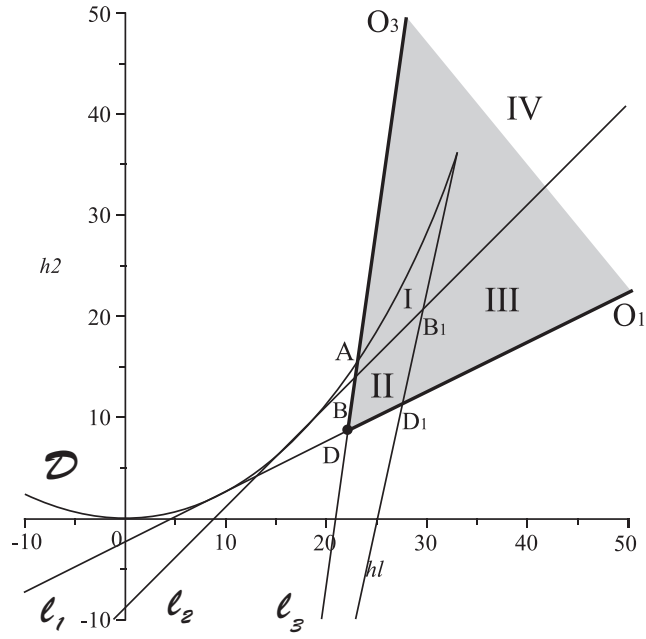


Figure 5.1.2:  $J_2 = -10, b_1 = 0.5, b_2 = 1, b_3 = 7$ .

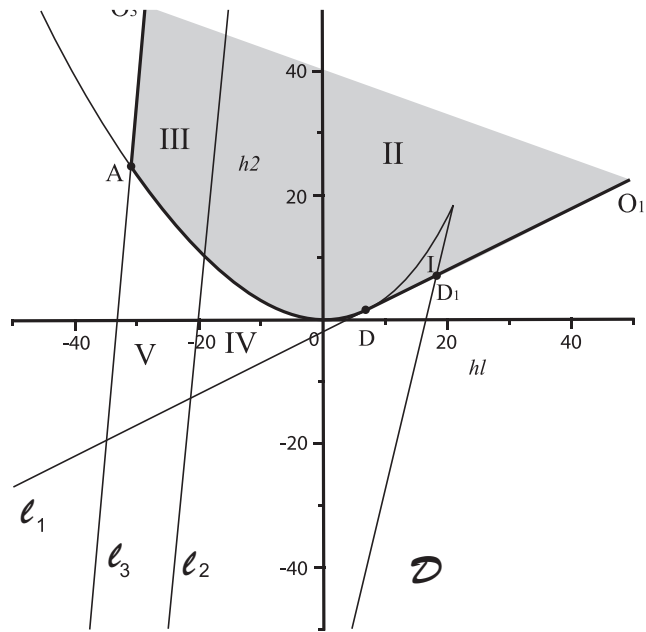


Figure 5.1.3:  $J_2 = -8, b_1 = 0.5, b_2 = 10, b_3 = 11$ .

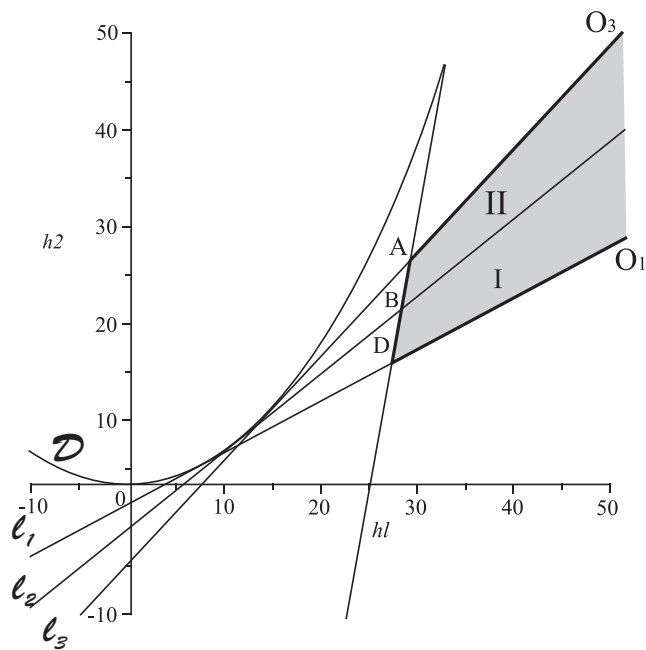


Figure 5.1.4:  $J_2 = -10, b_1 = 0.4, b_2 = 0.6, b_3 = 0.8$ .

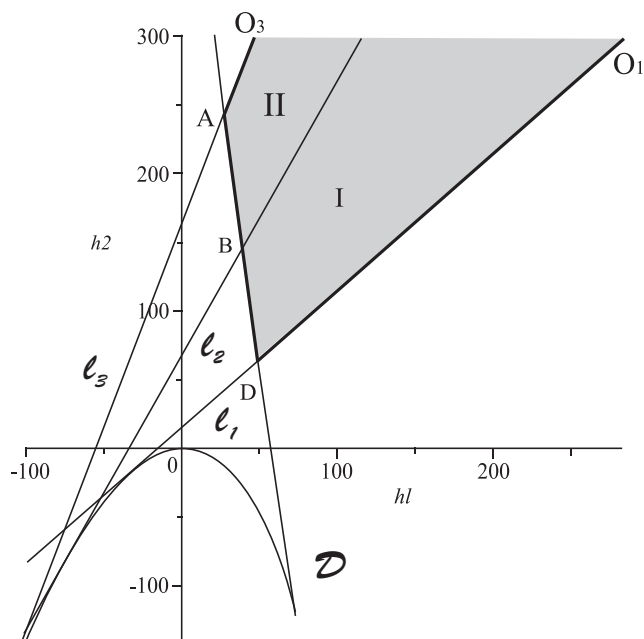


Figure 5.1.5:  $J_2 = 15, b_1 = 1, b_2 = 2, b_3 = 3$ .

## 5.2 Description of zones of real motions

Let  $\mathcal{S}$  be a subset of the "positive" sector restricted by the lines  $\ell_1, \ell_3$ . The domain of real motion depends on the relative position of  $\mathcal{D}$  and the sector  $\mathcal{S}$ .

Now let  $S_{i,2}$  be the part of  $\mathcal{S}$  restricted by  $\ell_i$  and  $\ell_2$ . Let  $\ell_1^+, \ell_3^+$  be the corresponding rays with the vertex  $O = \ell_1 \cap \ell_3$ . Summarizing the above observations, we arrive at

**Proposition 5.2.1** *There are 5 different domains of real motion depending on the value of  $J_2$ .*

*The segment  $[-3b_3, -3b_1]$  necessarily consists of 3 parts corresponding to different domains of real motion:*

- 1) If  $J_2 \in (-3b_3, \frac{b_1^2 + b_1 b_3 - 2b_3^2}{b_3 - b_1})$ , then  $\mathcal{D}$  is tangent to  $\ell_3$  inside  $\mathcal{S}$ . (See Figure 5.1.1)  
In this case the domain of real motion is  $\mathcal{S} - \mathcal{Q}$ , where  $\mathcal{Q}$  is a subset of the sectors, restricted by rays  $\ell_1^+$  and  $\ell_3^+$  and the segment of the curve  $\mathcal{D}$  between the intersection point of the  $\mathcal{D}$  and  $\ell_3$  and tangency point of the  $\mathcal{D}$  to  $\ell_1$ ;
- 2) If  $J_2 \in (\frac{b_1^2 + b_1 b_3 - 2b_3^2}{b_3 - b_1}, \frac{b_3^2 + b_1 b_3 - 2b_1^2}{b_1 - b_3})$ , then  $\mathcal{D}$  is tangent neither to  $\ell_3$  nor  $\ell_1$  inside  $\mathcal{S}$  (See Figure 5.1.2) The domain of real motions coincides with the sector  $\mathcal{S}$ ;
- 3) If  $J_2 \in (\frac{b_3^2 + b_1 b_3 - 2b_1^2}{b_1 - b_3}, -3b_1)$ , then  $\mathcal{D}$  is tangent to  $\ell_1$  inside  $\mathcal{S}$ . (See Figure 5.1.3) The domain of real motion is  $\mathcal{S} - \mathcal{Q}$ ;
- 4) If  $J_2 < -3b_3$ , cusp  $C \notin \mathcal{S}$  and it is above the line  $\ell_3$  (see Figure 5.1.4.);
- 5) If  $J_2 > -3b_1$ , cusp  $C \notin \mathcal{S}$  and it is below the line  $\ell_1$  (see Figure 5.1.5.) In both cases 4) and 5) the curve  $\mathcal{D}$  crosses all the 3 rays  $\ell_i^+$  of the sector  $\mathcal{S}$ , and the domain of real motions is the truncated sector  $\mathcal{S} - \mathcal{S} \cap \mathcal{R}$ .

*Proof.*

1), 3). The coordinate  $h_2$  of the point  $O = \ell_1 \cap \ell_3$  (the vertex of the sector  $\mathcal{S}$ ) is

$$h_2^* = -J_2(b_1 + b_3) - (b_1^2 + b_1 b_3 + b_3^2)$$

The coordinate  $h_1$  of the tangency point  $T_3 = \mathcal{D} \cap \ell_3$  is

$$h_2 = -2J_2 b_3 - 3b_3^2$$

Let us set  $J_2 = -3b_3$ , that is, the cusp  $C$  lies on  $\ell_3$ . Substituting this into the above formulas, we get  $h_2 = 3b_3^2$  and  $h_2^* = 2b_3^2 + 2b_1 b_3 - b_1^2$ . Then

$$h_2 - h_2^* = b_3^2 - 2b_1 b_3 + b_1^2 = (b_3 - b_1)^2 > 0,$$

hence the tangency point  $T_3$  is to the *right* of the vertex  $O$  for any  $b_1 < b_2 < b_3$ .

Now change  $J_2$  by setting  $J_2 = -3b_3 + e$ , that is, adding a small number  $e$ . Then the cusp  $C$  is inside of the sector  $\mathcal{S}$  and, by continuity, the tangency point  $T_3$  on  $\ell_3$  is again to the *right* of the vertex  $O$ , that is, it belongs to  $\mathcal{S}$ .

The same argumentation works in the case when  $\mathcal{D}$  is tangent to  $\ell_1$ .

There are three real roots  $c_1, c_2, c_3$  inside of the sectors  $\mathcal{Q}$ , but one of them does not belong to  $[b_1, b_3]$ .

Hence, all the points of the sector  $\mathcal{Q}$  do not correspond to real  $z_i, p_i$  and for this case the domain of real motion is  $\mathcal{S} - \mathcal{Q}$ .

Note that in the case (2) it is possible that  $\mathcal{D}$  can be or can not be tangent to  $\ell_2$  inside the sector  $\mathcal{S}$ . This does not changes the domain of real motions, but the tangency point  $\mathcal{D} \cap \ell_2$  inside  $\mathcal{S}$  gives a very special motion (periodic or equilibria).

2). As follows from the assumption  $b_1 < b_2 < b_3$  and item 4) of Proposition 5.1.2, under the condition  $-3b_2 \leq J_2 \leq -3b_1$  the cusp  $C$  must lie between the lines  $\ell_1$  and  $\ell_2$  outside of the triangle  $\Delta$ . Hence, it belongs to the subsector  $\mathcal{S}_{1,2}$ . Next, due to item 2) of Proposition 5.1.2, at the cusp  $C$  the polynomial  $F(s)$  has the triple root  $c_1 = c_2 = c_3 = -J_2/3$ , which implies  $b_1 \leq c_1 = c_2 = c_3 \leq b_2$ . As  $(h_1, h_2)$  moves away from  $C$ , but does not cross the rays  $\ell_1^+, \ell_3^+$ , either the roots  $c_1, c_2, c_3$ , when all real, stay inside  $[b_1, b_3]$  or  $c_1, c_2$  are complex and  $c_3 \in [b_1, b_3]$ . The same argumentation holds if  $C \in \mathcal{S}_{2,3}$ . Hence, all the points of the sector  $\mathcal{S}$  correspond to real  $z_i, p_i$ . For the points  $(h_1, h_2)$  inside the area of intersection the domain of real motion with  $\mathcal{R}$  all the 3 roots  $c_\alpha$  are real and belong to  $[b_1, b_3]$ , for the points  $(h_1, h_2)$  outside that area there is only one real root  $c_\alpha$  and this root belongs to  $[b_1, b_3]$ .

4),5). Due to items 2), 4) of Proposition 5.1.2, when  $J_2 > -3b_1$  the cusp  $C$  is below  $\ell_1$ , i.e., outside of the sector  $\mathcal{S}$ . Then at  $C$  we have  $c_1 = c_2 = c_3 < b_1$ . When the point  $(h_1, h_2)$  leaves  $C$ , but stays inside  $\mathcal{R}$  and outside of  $\mathcal{S}$ , all  $c_\alpha$  are real and less than  $b_1$ .

When  $(h_1, h_2)$  crosses  $\ell_1^+$  and enters the area  $\mathcal{S} \cap \mathcal{R}$ , one of the  $c_\alpha$  enters  $[b_1, b_3]$ , but the other real roots remain less than  $b_1$ . Hence, due to Proposition 4.1.1, the points of this area cannot correspond to real  $z_i, p_i$ . Finally, when  $(h_1, h_2)$  crosses  $\mathcal{D}$  and enters the truncated sector  $\mathcal{S} - \mathcal{S} \cap \mathcal{R}$ , there is one real root  $c_\alpha \in [b_1, b_3]$  and 2 complex roots, and  $z_i, p_i$  are real.

Similar argumentation holds when  $J_2 < -3b_3$  and the cusp  $C$  is above the line  $\ell_3$ .

Finally, since the discriminant curve  $\mathcal{D}$  cannot cross the triangle  $\Delta$ , it must cross all the rays of the sector  $\mathcal{S}$ .

For the points  $(h_1, h_2)$  inside  $\mathcal{S} - \mathcal{S} \cap \mathcal{R}$  only one root  $c_\alpha$  is real and it belongs to  $[b_1, b_3]$ .  $\square$

Now we present a geometrical description of the different types of the bifurcation diagram.

As was noticed before,  $\mathcal{D}$  has one tangency point and one intersection point with each of the lines  $l_i, i = 1, 2, 3$ . To find the values of the parameter  $s$  where the tangency and intersection take place, we substitute (5.2) into the equation of  $l_i$ . The solutions of the cubic equation so obtained are easy to find. We have one double root  $s = b_i$  (that corresponds to the tangency point) and one simple root  $s = -\frac{J_2 + b_i J_1}{2}$  (that corresponds to the intersection point).

The parameter  $s$  corresponding to the cusp point can be found from (5.2) by differentiation, i.e., from the equations  $\frac{\partial H_1}{\partial s} = 0$ ,  $\frac{\partial H_2}{\partial s} = 0$ . We obtain  $s = -\frac{J_2}{3J_1}$ .

The form of the bifurcation diagram is determined by the mutual position of these 7 mentioned points, namely, 3 intersection points, 3 tangency points and one cusp point. This position changes as we vary  $J_2$ . It is convenient to use the scheme shown in Figure 5.2.1, which illustrates the mutual position of these points on the part of  $\mathcal{D}$  corresponding to real motion. Each of these seven points is now presented as a line on the  $(s, J_2)$ -plane:

- $s = b_i$  for the tangency points,  $i = 1, 2, 3$ ;
- $s = -(J_2 + b_i)/2$  for the intersection points,  $i = 1, 2, 3$ ;
- $s = -J_2/3$  for the cusp point.

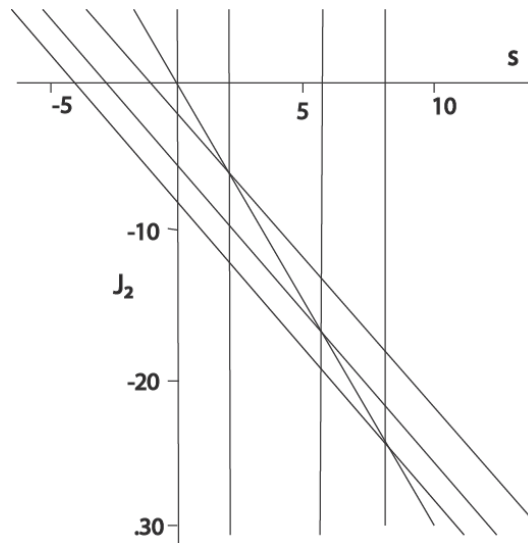


Figure 5.2.1:

As was shown in Proposition 5.2.1, this real motion zone is always located between the lines  $\ell_1$  and  $\ell_3$ . In other words,  $s \in [-(J_2 + b_3)/2, -(J_2 + b_1)/2]$ . In Figure 5.2.1 this zone is shown as a horizontal segment between two straight lines  $s = -(J_2 + b_3)/2$  and  $s = -(J_2 + b_1)/2$ . As  $J_2$  decreasing, this segments moves down always remaining between these lines. The bifurcation diagram changes its form when this horizontal segment passes through one of nine points of intersection of the seven lines shown in Figure 5.2.1. Thus, for fixed geometric parameters  $b_1, b_2, b_3$  we always have 10 different types of bifurcation diagram. The corresponding 10 horizontal segments are shown in Figure 5.2.1 too.

Using Figure 5.2.1 one can easily describe the position of  $\mathcal{D}$  relative to  $\ell_1$ ,  $\ell_2$  and  $\ell_3$ . For example, the upper horizontal segment in Figure 5.2.1 “says” that  $\mathcal{D}$  first intersects transversally  $\ell_1$ , then  $\ell_2$  and  $\ell_3$  and exits from the real zone. Next, on  $\mathcal{D}$

there appear the cusp and, finally, three tangency points with  $\ell_1$ ,  $\ell_2$  and  $\ell_3$ . This is exactly the situation which we see in Figure 5.1.5

The next segment (second from above in Figure 5.2.1 “says” that  $\mathcal{D}$  transversally intersects  $\ell_3$ , then  $\ell_2$ . After this comes the tangency point with  $\ell_1$ , then the cusp and, finally, the intersection point with  $\ell_1$ . The tangency points with  $\ell_2$  and  $\ell_3$  are located out of the real zone. This is exactly the situation shown in Fig. 5.1.3

Each “real motion” part of the bifurcation curve corresponds to a certain singular periodic motion of the system (the same point can correspond, of course, to several (two or more) periodic solutions. Our next goal is to analyze their stability depending on the location on the bifurcation diagram.

### 5.3 Stability analysis for critical periodic solutions

We start with the stability analysis for critical solutions corresponding to the discriminant curve  $\mathcal{D}$ .

**Proposition 5.3.1** *Each (non-exceptional) point of the “real motion” part of the discriminant curve  $\mathcal{D}$  corresponds to exactly two critical periodic solutions of the Steklov-Lyapunov system.*

*Proof.* From the definition of  $\mathcal{D}$ , it easily follows that for the point  $P \in \mathcal{D}$  with parameter  $s$  (see (5.2)) we have

$$F(s) = \sum_{\gamma=1}^3 (s - b_\gamma)(z_\gamma + sp_\gamma)^2 \equiv J_1 s^3 + J_2 s^2 + H_2 s - H_1 = 0$$

The preimage of this point is a singular integral surface that contains one or more critical periodic solutions on which the first integrals  $J_1, J_2, H_1$  and  $H_2$  become functionally dependent. From the above form of  $\mathcal{F}(f)$  we can conclude that their critical points are characterized by a very simple condition

$$z_\gamma + sp_\gamma = 0, \quad \gamma = 1, 2, 3.$$

Here we assume that  $s$  does not take exceptional values that correspond to tangency, intersection or cusp points. If we substitute these three equations into

$$J_1 = \langle p, p \rangle, \quad J_2 = 2\langle z, p \rangle - \langle Bp, p \rangle$$

we obtain two equations for  $p$ :

$$\langle p, p \rangle = J_1, \quad \langle Bp, p \rangle = -J_2 - 2sJ_1. \quad (5.3)$$

These equations define two closed curves being the projection of the corresponding periodic solutions onto the sphere  $\langle p, p \rangle = J_1$ . The solutions can be obtained from the projection by letting  $z = -sp$ .  $\square$

Notice that these critical solutions can be found without integration by solving the system of two algebraic equations (5.3). We now want to analyze their stability.

As well known, generic critical periodic solutions can be of two different types: elliptic and hyperbolic. Elliptic ones are stable and can be characterized by the fact that the corresponding integral surface  $J_i = j_i$ ,  $H_i = h_i$ , ( $i = 1, 2$ ) is one dimensional and coincides with this solution itself. The integral surfaces that contain hyperbolic periodic solutions are two-dimensional and always contain separatrices attached to these solutions. This observation makes it possible to analyze the stability by analyzing the dimension of singular integral surfaces. The formal justification of this method can be found in [4], where is proved the following criterion for stability of any closed trajectory of an integrable Hamiltonian system :

**Theorem 5.3.2** *A closed trajectory of an integrable Hamiltonian system is stable if and only if the connected component of the integral manifold containing this trajectory coincides with the trajectory itself.*

In order to apply this method, it is convenient to use the following change of variables:  $\tilde{z} = z + sp$  ( $p$  remains unchanged). After this change, the critical solution will be characterized by the condition  $\tilde{z} = 0$ . We also use following elementary algebraic

**Proposition 5.3.3** *Consider the restriction of the quadratic form  $ax^2 + by^2 + cz^2$  to the plane  $Ax + By + Cz = 0$ . This restriction*

- *is positive or negative definite if  $abc \left( \frac{A^2}{a} + \frac{B^2}{b} + \frac{C^2}{c} \right) > 0$ ,*
- *is indefinite if  $abc \left( \frac{A^2}{a} + \frac{B^2}{b} + \frac{C^2}{c} \right) < 0$ ,*
- *is degenerate if  $abc \left( \frac{A^2}{a} + \frac{B^2}{b} + \frac{C^2}{c} \right) = 0$ .*

So, we have the following result

**Theorem 5.3.4** *Let a critical periodic solution of the Steklov-Lyapunov system passes through a point  $(z, p)$  with  $z + sp = 0$  (these solutions correspond to the discriminant curve  $\mathcal{D}$ ). This solution is stable depending on the sign of the following expression*

$$(b_1 - s)(b_2 - s)(b_3 - s) \sum_{i=1}^3 (b_i - s)p_i^2 = (b_1 - s)(b_2 - s)(b_3 - s)(\langle Bp, p \rangle - s\langle p, p \rangle) \quad (5.4)$$

*Namely, if this expression is positive then we have stability, if this expression is negative, then this solution is unstable.*

*Proof.*

It is easy to see that the corresponding integral surface

$$\begin{aligned} \langle p, p \rangle &= J_1, \\ \langle 2z, p \rangle - \langle Bp, p \rangle &= J_2, \\ \langle z, z \rangle - 2\langle z, Bp \rangle &= H_2, \\ \langle z, Bz \rangle &= H_1, \end{aligned}$$



in these new variables takes the form

$$\begin{aligned}\langle p, p \rangle &= J_1, \\ \langle 2\tilde{z}, p \rangle - \langle Bp, p \rangle &= J_2 + 2sJ_1 \\ \langle \tilde{z}, \tilde{z} \rangle - 2s\langle \tilde{z}, (B-s)p \rangle &= 0 \\ \langle \tilde{z}, (B-s)\tilde{z} \rangle &= 0\end{aligned}$$

If we consider  $p$  as parameter, then two last equations can be easily analyzed to verify whether or not they admit local solutions different from  $\tilde{z} = 0$ . If not, then the integral surface coincides with the critical trajectory and therefore the latter is stable. Otherwise, it is unstable.

At the point  $\tilde{z} = 0$ , the equation  $\langle \tilde{z}, B\tilde{z} \rangle - 2s\langle \tilde{z}, (B-s)p \rangle = 0$  represents a regular two-dimensional surface with the tangent space  $T$  defined by  $\langle \tilde{z}, (B-s)p \rangle = 0$ . If we restrict the quadratic function  $\langle \tilde{z}, (B-s)\tilde{z} \rangle$  on  $T$  we can have three possibilities: this restriction can be either positive definite, or indefinite, or negative definite. Positive and negative definiteness correspond to the stable situation, the indefinite case means instability. Applying the Proposition 5.3.3, in our case  $a = b_1 - s$ ,  $b = b_2 - s$ ,  $c = b_3 - s$  and  $A = (b_1 - s)p_1$ ,  $B = (b_2 - s)p_2$ ,  $A = (b_3 - s)p_3$ . Thus we have result of theorem.  $\square$

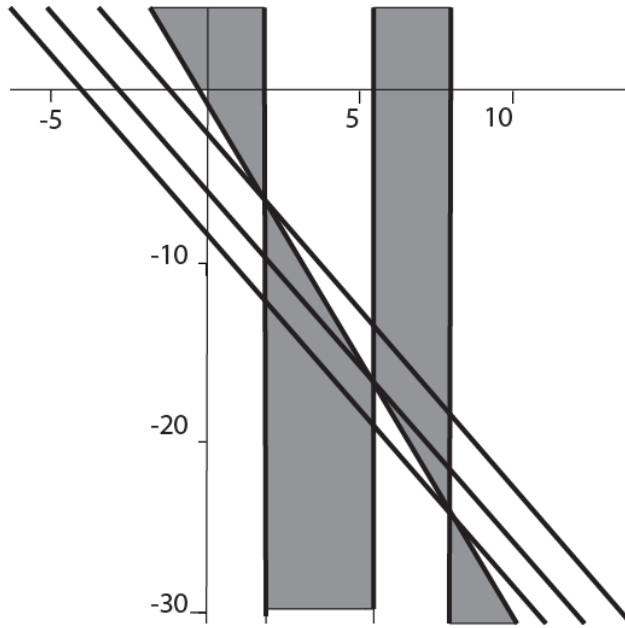


Figure 5.3.1:

This criterion has a natural interpretation in terms of the bifurcation diagram.

**Theorem 5.3.5** *A point  $P$  lying on the "real motion" part of the discriminant curve  $\mathcal{D}$  corresponds to a stable periodic solution if and only if*

$$(b_1 - s)(b_2 - s)(b_3 - s)(-J_2 - 3sJ_1) > 0.$$

*Proof.* Notice that for our critical solution we have  $z = -sp$ , so

$$J_2 = 2\langle z, p \rangle - \langle Bp, p \rangle = -2s\langle p, p \rangle - \langle Bp, p \rangle, \quad \text{i.e. } \langle Bp, p \rangle = -J_2 - 2J_1.$$

This implies that the expression  $\langle Bp, p \rangle - s\langle p, p \rangle = -J_2 - 3sJ_1$  changes sign for  $s = -\frac{3J_2}{J_1}$ , i.e. exactly at the cusp point.

In other words, the whole expression (5.4) changes sign if and only if the point  $P \in \mathcal{D}$  passes either through the tangency points with parameters  $s = b_i$ , or through the cusp point with parameter  $s = -\frac{3J_2}{J_1}$ . However, the stability does not change if the point  $P \in \mathcal{D}$  passes through the transversal intersection points with parameters  $s = -(J_2 + b_i J_1)/2$ . The stability zone on the  $(s, J_2)$ -plane is shown in Figure 5.3.1 as a shadowed region.  $\square$

As an example, consider the case presented in Figure 5.1.3. Theorem 5.3.5 says that the stable periodic solutions correspond to the part of  $\mathcal{D}$  located between the points  $A$  and  $D$ , then between  $D$  and the cusp point we have unstable periodic solutions, and finally between the cusp point and  $D_1$  these solutions become stable again.

The stability analysis for the periodic solutions corresponding the three straight lines  $\ell_1, \ell_2$  and  $\ell_3$  is simpler, namely we have the following

**Theorem 5.3.6** *The periodic solutions corresponding to the lines  $\ell_1$  and  $\ell_3$  are stable, whereas those corresponding to  $\ell_2$  are unstable.*

*Proof.* This analysis is based on the same ideas. We need to analyze the relation

$$\mathcal{F}(s) = \sum_{\gamma=1}^3 (s - b_\gamma)(z_\gamma + sp_\gamma)^2 \equiv J_1 s^3 + J_2 s^2 + H_2 s - H_1 = 0$$

for  $s = b_1, b_2, b_3$ .

Let  $s = b_1$ . Then we have

$$\mathcal{F}(b_1) = (b_1 - b_2)(z_2 + b_1 p_2)^2 + (b_1 - b_3)(z_3 + b_1 p_3)^2$$

The corresponding critical points are defined by the following relations:

$$z_2 + b_1 p_2 = 0, \quad z_3 + b_1 p_3 = 0$$

This is an invariant four-dimensional subspace in  $\mathbb{R}^6(z, p)$ .

To describe the critical solutions and the corresponding integral surface we, as before, use the following change of variables  $\tilde{z} = z + b_1 p$  ( $p$  remains unchanged). Then the integral surface is defined by:

$$\begin{aligned} \langle p, p \rangle &= J_1, \\ \langle 2\tilde{z}, p \rangle - \langle Bp, p \rangle &= J_2 + 2sJ_1 \\ \langle \tilde{z}, \tilde{z} \rangle - 2s\langle \tilde{z}, (B - s)p \rangle &= H_2 + 2b_1 J_2 + 3b_1^2 J_1 \\ \langle \tilde{z}, (B - s)\tilde{z} \rangle &= 0 \end{aligned} \tag{5.5}$$

Since critical periodic solutions are characterized by  $\tilde{z}_2 = \tilde{z}_3 = 0$ , then we can find them by solving the following system of three equations for  $p$  and  $\tilde{z}_1$ :

$$\begin{aligned} \langle p, p \rangle &= J_1, \\ 2\tilde{z}_1 p_1 - \langle Bp, p \rangle &= J_2 + 2sJ_1 \\ \tilde{z}_1^2 &= H_2 + 2b_1 J_2 + 3b_1^2 J_1 \end{aligned} \tag{5.6}$$

The third equation simply means that  $\tilde{z}_1$  is constant along each critical solution. The remaining two equations define a curve in the 3-dimensional  $p$ -space which is the intersection of two quadrics. This curve may have one or two connected components. Taking into account the fact that  $\tilde{z}_1$  is defined up to sign, we see that a point  $P \in l_1$  may correspond to two or four critical periodic solutions. Notice that the system (5.6) allows one to describe the corresponding critical solutions of the Steklov-Lyapunov system without integration by means of solving a system of algebraic (quadratic) equations.

To analyze the stability of these solutions we consider the system (5.5). The last equation

$$(b_2 - b_1)\tilde{z}_2^2 + (b_3 - b_1)\tilde{z}_3^2 = 0$$

immediately implies that  $\tilde{z}_2 = \tilde{z}_3 = 0$ , i.e. the integral surface belongs to the critical set and therefore coincides locally with the critical solution, in particular, this surface is not two-, but one-dimensional. This condition guarantees the stability.

In the case of  $\ell_3$  the situation is similar because  $(b_2 - b_3)$  and  $(b_1 - b_3)$  are of the same sign. In the case of  $\ell_2$  the situation is opposite because  $(b_1 - b_2)$  and  $(b_3 - b_2)$  are of different signs, so the equation  $(b_1 - b_2)\tilde{z}_1^2 + (b_3 - b_2)\tilde{z}_3^2 = 0$  has non-trivial solutions (two intersecting lines) and therefore, the corresponding integral surface has dimension two which means instability. This completes the proof.

## Chapter 6

# Bifurcation analysis of the Rubanovskii sistem via bi-Hamiltonian approach

### 6.1 Bi-Hamiltonian structure and integrals for the Rubanovskii case

The generalization of the “Kötter type” Lax-pair to the Rubanovskii system (1.8) discovered by Y. N. Fedorov [27] makes it possible in a natural way to describe a bi-Hamiltonian structure corresponding to this system. It is, in fact, an obvious generalization of the bi-Hamiltonian structure for the Steklov-Lyapunov case (see [13]). It can be also derived from the paper by A. Tsyganov [56] where the classification of all bi-Hamiltonian structures related the Lie algebras  $so(4)$  and  $e(3)$  was obtained. The bi-Hamiltonian structure we are interested in has a natural algebraic interpretation, but for our purposes it would be more convenient to give its definition just in coordinates.

**Theorem 6.1.1** *The Rubanovskii system is Hamiltonian with respect to any linear combination of the following Poisson brackets given in the space  $\mathbb{R}^6(z, p)$ :*

$$\Pi_0 = \begin{pmatrix} 0 & z_3 - b_3 p_3 & -z_2 + b_2 p_2 & 0 & p_3 & -p_2 \\ -z_3 + b_3 p_3 & 0 & z_1 - b_1 p_1 & -p_3 & 0 & p_1 \\ z_2 - b_2 p_2 & -z_1 + b_1 p_1 & 0 & p_2 & -p_1 & 0 \\ 0 & p_3 & -p_2 & 0 & 0 & 0 \\ -p_3 & 0 & p_1 & 0 & 0 & 0 \\ p_2 & -p_1 & 0 & 0 & 0 & 0 \end{pmatrix}$$
$$\Pi_1 = \begin{pmatrix} 0 & b_3 z_3 - g_3 & -b_2 z_2 + g_2 & 0 & 0 & 0 \\ -b_3 z_3 + g_3 & 0 & b_1 z_1 - g_1 & 0 & 0 & 0 \\ b_2 z_2 - g_2 & -b_1 z_1 + g_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & p_3 & -p_2 \\ 0 & 0 & 0 & -p_3 & 0 & p_1 \\ 0 & 0 & 0 & p_2 & -p_1 & 0 \end{pmatrix}$$



Coming back to the initial variables, we obtain:

$$F_\lambda = \sum_{i=1}^3 (\lambda - b_i) \left( z_i + \lambda p_i + \frac{g_i}{\lambda - b_i} \right)^2 \quad (6.2)$$

We simplify this expression and expand it (excluding the constant term) in powers of  $\lambda$

$$\begin{aligned} F_\lambda &= \lambda^3 \sum_{i=1}^3 p_i^2 + \lambda^2 \sum_{i=1}^3 (2z_i p_i - b_i p_i^2) + \lambda \sum_{i=1}^3 (z_i^2 - 2b_i z_i p_i + 2g_i p_i) + \\ &+ \sum_{i=1}^3 (-b_i z_i^2 + 2g_i z_i) + \sum_{i=1}^3 \frac{g_i^2}{\lambda - b_i} = \\ &= J_1 \lambda^3 + J_2 \lambda^2 + 2\lambda H_2 - 2H_1 + \sum_{i=1}^3 \frac{g_i^2}{\lambda - b_i}, \end{aligned} \quad (6.3)$$

where

$$J_1 = \langle p, p \rangle, \quad J_2 = 2\langle z, p \rangle - \langle Bp, p \rangle, \quad H_1 = \frac{1}{2} \langle z, Bz \rangle - \langle z, g \rangle,$$

$$H_2 = \frac{1}{2} \langle z, z \rangle - \langle Bz, p \rangle + \langle p, g \rangle, \quad B = \text{diag}(b_1, b_2, b_3).$$

The integrals we have to deal with are:  $J_1$ ,  $J_2$ ,  $H_1$  and  $H_2$ . Of course, the same integrals arise from the Lax representation with the spectral parameter  $s$ . The first two of them,  $J_1$  and  $J_2$ , are Casimir functions of the bracket  $\Pi_0$ . After the Kötter transformation they become standard invariants of the Lie algebra  $e(3)$ . These functions are considered below as parameters, and  $H_1$ ,  $H_2$  as nontrivial integrals.

Our further goal is to describe the singularities of the momentum mapping defined by these four integrals

$$\Phi : \mathbb{R}^6(z, p) \rightarrow \mathbb{R}^4(J_1, J_2, H_1, H_2). \quad (6.4)$$

or of its restriction to one of the symplectic leaves of  $\Pi_0$ , which is defined by fixing values  $J_1$  and  $J_2$ :

$$\Phi_{j_1, j_2} : M_{j_1, j_2} = \{(z, p) \in \mathbb{R}^6, \quad J_1(z, p) = j_1, J_2(z, p) = j_2\} \rightarrow \mathbb{R}^2(H_1, H_2).$$

It is convenient to treat it exactly this way, i.e. to allow  $J_1$  be not only 1 (what is motivated by physical meaning) but to take any positive values. The point is that the dynamics on the level  $J_1 = c \neq 1$  is connected with the dynamics on the level  $J_1 = 1$  by means of some transformation, which involves not only transformation of dynamical variables but also some change of geometrical parameters  $g$  and  $b$ . Therefore the case  $J_1 = c \neq 1$  has, in fact, some physical meaning as well.

## 6.2 Description of critical points of the momentum mapping

To describe the critical points  $(z, p) \in \mathbb{R}^6$  of the momentum mapping (6.4) we will use the technics developed in [12].

According to the results of this paper, a point  $(z, p) \in \mathbb{R}^6$  is singular if and only if the rank of the bracket  $\Pi_1 - \lambda\Pi_0$  drops for some  $\lambda \in \mathbb{C}$ .

Such points are easy to describe. Preceding this description we state an obvious remark. The rank of any bracket from our family drops, if  $p = 0$ . But from the physical viewpoint this case is of no interest for us, and below we don't consider it at all. In other words, we suppose from now on that  $p \neq 0$ . Next, we notice that for  $\lambda = b_i$  the rank of the bracket  $\Pi_1 - \lambda\Pi_0$  is everywhere equal to 4 and cannot drop (recall that we suppose  $g_i \neq 0$ ,  $p \neq 0$ , otherwise the rank can drop!). Thus, we have to consider only the case  $\lambda \neq b_i$ , for which, as we have seen above, the bracket can be brought to the block-diagonal form. It is clear, that its rank drops if and only if the upper block vanishes, i. e.  $\tilde{z}_i = 0$ .

This immediately implies the complete description of all critical points of the momentum mapping (6.4).

**Theorem 6.2.1** *A point  $(z, p) \in \mathbb{R}^6$  is a singular point of the momentum mapping (6.4) if and only if there exists  $\lambda \in \mathbb{C} \setminus \{b_1, b_2, b_3\}$  such that*

$$z_i + \lambda p_i + \frac{g_i}{\lambda - b_i} = 0, \quad i = 1, 2, 3. \quad (6.5)$$

If  $\lambda$  is a real number, then equation (6.5) determines some three-dimensional subspace. Varying  $\lambda$ , we can say that the set of critical points is represented as one parameter family of three-dimensional subspaces (or better to say: four separate families, because  $\lambda$  changes not on the whole real line but on four disjoint intervals). Note that  $\lambda$  in this theorem is not necessarily a real number. It may well happen that the written equation is fulfilled for a pair of complex conjugate values  $\lambda, \bar{\lambda}$ . These points do not get into the subspaces mentioned above. From the dynamical viewpoint, they can be characterized by the property that the rank of the momentum mapping at these points drops by two. This implies in particular that they are equilibria of the system.

Since we deal with two degrees of freedom, the rank of the momentum mapping can drop either by one or by two. If the rank drops by two, then we have an equilibrium of the system (more precisely, a common equilibrium for the both integrals  $H_1$  and  $H_2$ ). The next step is to distinguish equilibria of the system (i. e., points of rank zero) in the set of singular points of the momentum mapping.

**Theorem 6.2.2** *A point  $(z, p) \in \mathbb{R}^6$  is a common equilibrium for the integrals  $H_1$  and  $H_2$  if and only if the rank of the matrix*

$$\mathcal{C} = \begin{pmatrix} p_1 & z_1 - b_1 p_1 & g_1 - b_1 z_1 \\ p_2 & z_2 - b_2 p_2 & g_2 - b_2 z_2 \\ p_3 & z_3 - b_3 p_3 & g_3 - b_3 z_3 \end{pmatrix} \quad (6.6)$$

is equal to 1. One can reformulate this constraint by saying that the vectors  $p$ ,  $z - Bp$  and  $g - Bz$  are proportional.

**Proof.** Let  $(z, p)$ ,  $p \neq 0$  be a common equilibrium for all the integrals simultaneously (we consider the Hamiltonian dynamics generated by the bracket  $\Pi_0$ ). According to [12], these points are characterized by the fact that the kernels of all (regular) brackets from the pencil coincide at these points. In particular, the kernels of the initial Poisson brackets  $\Pi_0$  and  $\Pi_1$  must be the same. For  $\Pi_1$ , the kernel is generated by the vectors  $(g - Bz, 0)$  and  $(0, p)$ . The latter vector belongs to the kernel of the bracket  $\Pi_0$  automatically, and the assumption that the first vector also belongs to the kernel of  $\Pi_0$  amounts to the fact that the vector  $g - Bz$  is proportional both to  $p$  and  $z - Bp$  which is exactly the statement of Theorem 6.2.2.

Since the first column of matrix (6.6) is always supposed to be non-zero, it suffices to consider the system of vector equations  $z - Bp = \alpha p$ ,  $g - Bz = \beta p$  in order to find all solutions. This system can be easily solved:

$$p = (B^2 + \alpha B + \beta)^{-1}g, \quad z = (B^2 + \alpha B + \beta)^{-1}(B + \alpha)g.$$

This solution can be interpreted as a two parameter family of points which has two real numbers  $\alpha$  and  $\beta$  as parameters.

It is useful to notice that not every point  $p$  is admissible. Those points  $p$  for which the equilibria  $(z, p)$  really exist, comply with a natural restriction obtained by excluding  $\alpha$  and  $\beta$  from the equations written above. Namely:

$$\det \begin{pmatrix} p_1 & b_1 p_1 & b_1^2 p_1 - g_1 \\ p_2 & b_2 p_2 & b_2^2 p_2 - g_2 \\ p_3 & b_3 p_3 & b_3^2 p_3 - g_3 \end{pmatrix} = 0$$

Now we give another description of those points where the rank of the momentum mapping drops by one. We first notice that the equation (6.5) can be rewritten in vector form:

$$\lambda^2 p + \lambda(z - Bp) + g - Bz = 0,$$

whereof it follows that the vectors  $p$ ,  $z - Bp$  and  $g - Bz$  are linearly dependent. In particular, the vanishing of the determinant of matrix (6.6) is a necessary condition for  $(z, p)$  to be a critical point. This condition is not sufficient because we need the coefficients of the nontrivial linear combination between the vectors  $p$ ,  $z - Bp$  and  $g - Bz$  to have a special form, namely  $\lambda^2, \lambda, 1$ .

In order to rewrite this additional condition as an algebraic equation, we note that the triple of  $2 \times 2$  subdeterminants of matrix (6.6):

$$\left( \begin{vmatrix} z_1 - b_1 p_1 & -b_1 z_1 + g_1 \\ z_2 - b_2 p_2 & -b_2 z_2 + g_2 \end{vmatrix}, \begin{vmatrix} p_1 & b_1 z_1 + g_1 \\ p_2 & b_2 z_2 + g_2 \end{vmatrix}, \begin{vmatrix} p_1 & z_1 - b_1 p_1 \\ p_2 & z_2 - b_2 p_2 \end{vmatrix} \right)$$

obliges to be proportional to the triple  $(\lambda^2, \lambda, 1)$ . This evidently implies the condition

$$\begin{vmatrix} p_1 & b_1 z_1 + g_1 \\ p_2 & b_2 z_2 + g_2 \end{vmatrix}^2 = \begin{vmatrix} z_1 - b_1 p_1 & -b_1 z_1 + g_1 \\ z_2 - b_2 p_2 & -b_2 z_2 + g_2 \end{vmatrix} \cdot \begin{vmatrix} p_1 & z_1 - b_1 p_1 \\ p_2 & z_2 - b_2 p_2 \end{vmatrix}.$$



Analogously, for all the other indices:

$$\begin{vmatrix} p_i & b_i z_i + g_i \\ p_j & b_j z_j + g_j \end{vmatrix}^2 = \begin{vmatrix} z_i - b_i p_i & -b_i z_i + g_i \\ z_j - b_j p_j & -b_j z_j + g_j \end{vmatrix} \cdot \begin{vmatrix} p_i & z_i - b_i p_i \\ p_j & z_j - b_j p_j \end{vmatrix} \quad (6.7)$$

The above conditions are not algebraically independent. However, the following statement holds true:

**Proposition 6.2.3** *The rank of the momentum mapping (6.4) drops exactly by one at a point  $(z, p)$  if and only if the rank of matrix (6.6) is equal to 2 and at the same time the conditions (6.7) are fulfilled.*

Once again, we sum up the obtained results.

**Theorem 6.2.4** *A point  $(z, p)$  is a common equilibrium of the Hamiltonians  $H_1$  and  $H_2$  if and only if the vectors  $z - Bp$  and  $g - Bz$  are proportional to the vector  $p$ . In other words, the rank of the matrix*

$$\begin{pmatrix} p_1 & z_1 - b_1 p_1 & g_1 - b_1 z_1 \\ p_2 & z_2 - b_2 p_2 & g_2 - b_2 z_2 \\ p_3 & z_3 - b_3 p_3 & g_3 - b_3 z_3 \end{pmatrix}$$

is equal to 1.

*The rank of the momentum mapping at a point  $(z, p)$  drops by one if the rank of this matrix is equal to 2 and there exists  $\lambda \in \mathbb{R}$  such that*

$$\begin{pmatrix} p_1 & z_1 - b_1 p_1 & g_1 - b_1 z_1 \\ p_2 & z_2 - b_2 p_2 & g_2 - b_2 z_2 \\ p_3 & z_3 - b_3 p_3 & g_3 - b_3 z_3 \end{pmatrix} \begin{pmatrix} \lambda^2 \\ \lambda \\ 1 \end{pmatrix} = 0.$$

*We can say this in a little bit different way as follows: the nontrivial solution  $(x_1, x_2, x_3)$  of the system of homogeneous equations*

$$\begin{pmatrix} p_1 & z_1 - b_1 p_1 & g_1 - b_1 z_1 \\ p_2 & z_2 - b_2 p_2 & g_2 - b_2 z_2 \\ p_3 & z_3 - b_3 p_3 & g_3 - b_3 z_3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

*satisfies the condition  $x_1 x_3 = x_2^2$ .*

This theorem supplies a simple and effective method to verify the criticality condition for a given point. We should also make one more remark. Any singular point of rank one is characterized by one real parameter  $\lambda$ . A critical point of rank zero is characterized by two parameters  $\lambda_{i,j}$  which represent the roots of the quadratic equation  $\lambda^2 + \alpha\lambda + \beta = 0$ , where  $\alpha$  and  $\beta$  are the proportionality coefficients between the vectors  $p$ ,  $z - Bp$  and  $p$ ,  $g - Bz$ , respectively, i. e.  $z - Bp = \alpha p$ ,  $g - Bz = \beta p$ .

The parameter  $\lambda$  (or the pair of parameters  $(\lambda_1, \lambda_2)$ ) is convenient for the characterization of dynamics, amongst others for the stability conditions.

### 6.3 Construction of the bifurcation diagram

Now let us discuss the question about the image of the critical set of the momentum mapping, that is, the bifurcation diagram.

Note that we have already described all critical trajectories of the Rubanovskii system together with related stability conditions. At the same time (it may seem to be surprising at the first glance) we did not whatever consider the system of dynamical equations itself and did not discuss its first integrals. It's been sufficient for us to analyze the pencil of brackets  $\Pi_1 - \lambda\Pi_0$ .

Now, since we are going to describe the bifurcation diagram, we have to consider the integrals (otherwise the discussion about the bifurcation diagram becomes almost meaningless).

The integrals have been described above by means of formulae (6.2) and (6.3).

The singular points of the momentum mapping are also known, and we now intend to describe its bifurcation diagram in terms of integrals  $H_1$  and  $H_2$  thinking of  $J_1$  and  $J_2$  as parameters.

We will define the branches of the bifurcation diagram in parametric form taking  $\lambda$  as a parameter.

First, we call our attention to formula (6.2). Since every singular point is characterized by the condition  $z_i + \lambda p_i + \frac{g_i}{\lambda - b_i} = 0$  for some  $\lambda \in \mathbb{R}$ , we can see that  $F_\lambda$  vanishes at this point. Hence we obtain the relation

$$\lambda^3 J_1 + \lambda^2 J_2 + 2\lambda H_2 - 2H_1 + \sum_{i=1}^3 \frac{g_i^2}{\lambda - b_i} = 0.$$

However, we have yet another relation. The point is that not only the function  $F_\lambda$  itself vanishes at the given point but its differential as well. This readily implies that we have one more condition (that can be verified directly) at every singular point:

$$\frac{\partial F_\lambda}{\partial \lambda} = 3\lambda^2 J_1 + 2\lambda J_2 + 2H_2 - \sum_{i=1}^3 \frac{g_i^2}{(\lambda - b_i)^2} = 0$$

In terms of differential geometry, it means that the bifurcation diagram represents the enveloping surface for the family of hyperplanes  $F_\lambda = 0$  in the space  $\mathbb{R}^4(J_1, J_2, H_1, H_2)$ . The desired description can be immediately obtained by solving the system of equations (from which we want to express  $H_1$  and  $H_2$  as functions of  $J_1, J_2$  and  $\lambda$ ):

$$\begin{cases} \lambda^3 J_1 + \lambda^2 J_2 + 2\lambda H_2 - 2H_1 + \sum_{i=1}^3 \frac{g_i^2}{\lambda - b_i} = 0 \\ 3\lambda^2 J_1 + 2\lambda J_2 + 2H_2 - \sum_{i=1}^3 \frac{g_i^2}{(\lambda - b_i)^2} = 0 \end{cases}$$

This system is linear in  $H_1$  and  $H_2$  and easy to solve. However, to formulate the answer, we have to keep in view the fact that the conditions we obtain are necessary but by no means sufficient. This yields that the bifurcation diagram lies on the curves defined below but does not coincide with them. Some parts of these curves should be removed afterwards.

**Theorem 6.3.1** For fixed values of  $J_1$  and  $J_2$ , the bifurcation diagram is contained in the union of four curves (branches) given parametrically by the same formulas:

$$H_2(\lambda) = -\frac{3}{2}\lambda^2 J_1 - \lambda J_2 + \frac{1}{2} \sum_{i=1}^3 \frac{g_i^2}{(\lambda - b_i)^2}$$

$$H_1(\lambda) = -2\lambda^3 J_1 - \lambda^2 J_2 + \sum_{i=1}^3 \frac{(2\lambda - b_i)g_i^2}{(\lambda - b_i)^2}$$

The parameter  $\lambda$  runs over the following intervals  $(-\infty, b_1)$ ,  $(b_1, b_2)$ ,  $(b_2, b_3)$ ,  $(b_3, +\infty)$ .

Example of such a curve is given in Figure 6.6.

There is an important observation which is helpful to complete the reconstruction (i. e., for removing the redundant parts) of the bifurcation diagram:

If  $\lambda > b_3$ , then all the expressions  $(\lambda - b_i)$  are also positive, therefore the function  $F_\lambda$  is always positive as well.

This means that the image of the momentum mapping is situated on the same side of the straight line

$$2\lambda H_2 - 2H_1 + \left( \lambda^3 J_1 + \lambda^2 J_2 + \sum_{i=1}^3 \frac{g_i^2}{\lambda - b_i} \right) = 0.$$

Since this straight line is tangent to the bifurcation curve, then the image of the momentum mapping will always lie on the same side of this curve. The same will be true for  $\lambda < b_1$  (in this case the function  $F_\lambda$  will be strictly negative).

For various values of parameters of problem the bifurcation diagrams may differ substantially from each other.

## 6.4 Stability analysis for closed trajectories

First of all, note that those closed trajectories of the Rubanovskii system that are not critical cannot be stable. Therefore it suffices to make the stability analysis only for the trajectories laying in the critical set described in Theorem (6.2.1).

As we have already shown above, the relation

$$\lambda^2 p + \lambda(z - Bp) + g - Bz = 0$$

is fulfilled (identically) on any critical trajectory, where  $\lambda$  is a uniquely defined real number which belongs to one of the intervals into which the real line is divided by the points  $b_1, b_2, b_3$ .

If a critical trajectory is given (it is sufficient to define its initial conditions), then we can easily compute  $\lambda$  and therefore from the very beginning we can think of  $\lambda$  as a given parameter and use it in our formulations.

According to the general construction, the non-degeneracy and stability of this trajectory should be proved as follows. First we should consider the kernel of the

bracket  $\Pi_1 - \lambda\Pi_0$  at the point  $(z, p)$ . In our case, this kernel appears to be four-dimensional and can be naturally identified with a Lie algebra, which is isomorphic to either  $so(3) \oplus \mathbb{R}$  or  $sl(2) \oplus \mathbb{R}$ . Then we have to distinguish the semisimple part and restrict to it any other bracket from the pencil. If the kernel of this restriction turns out to be one-dimensional then we should consider its generating vector as an element of the corresponding semisimple Lie algebra. The trajectory is non-degenerate if and only if this element is semisimple, i. e. if the Killing form on this element is non-zero. The trajectory is stable if this form is negative, and unstable if it is positive.

This general algebraic scheme can be essentially simplified in the case under consideration. It would be reasonable to clarify this scheme by means of a simple example. Without going into algebraic particulars we can illustrate the situation as follows.

Assume that we have two compatible Poisson brackets in the three-dimensional space  $\mathbb{R}^3(x_1, x_2, x_3)$ , one of these brackets vanishes at some point, whereas the other is of rank two. Let us consider their Casimir functions  $f$  and  $g$  respectively. We need to characterize the singularity of the mapping  $\Phi = (f, g) : \mathbb{R}^3(x_1, x_2, x_3) \rightarrow \mathbb{R}^2(f, g)$  at this point. This can be easily done in terms of these brackets only. We shall comment on the most important special case. Let the first bracket be linear and defined by a matrix of the form (any semisimple bracket can be brought to such a form):

$$A = \begin{pmatrix} 0 & c_3x_3 & -c_2x_2 \\ -c_3x_3 & 0 & c_1x_1 \\ c_2x_2 & -c_1x_1 & 0 \end{pmatrix},$$

and the second one be of the form:

$$B = \begin{pmatrix} 0 & a_3 + \dots & -a_2 + \dots \\ -a_3 + \dots & 0 & a_1 + \dots \\ a_2 + \dots & -a_1 + \dots & 0 \end{pmatrix},$$

where “dots” denote the higher-order terms and  $a_1, a_2, a_3 = \text{const}$ . The following statement can be easily proved.

**Proposition 6.4.1** *The singularity of the mapping  $\Phi = (f, g) : \mathbb{R}^3(x_1, x_2, x_3) \rightarrow \mathbb{R}^2(f, g)$  at the origin is non-degenerate if and only if*

$$c_1c_2c_3 \sum_{i=1}^3 \frac{a_i^2}{c_i} \neq 0.$$

*Moreover, if  $c_1c_2c_3 \sum_{i=1}^3 \frac{a_i^2}{c_i} > 0$ , then the singularity is of elliptic type, and if  $c_1c_2c_3 \sum_{i=1}^3 \frac{a_i^2}{c_i} < 0$ , then it is of hyperbolic type.*

Recall that the elliptic singularities are diffeomorphic to those of the form  $f = x^2 + y^2$ ,  $g = z$ , and the hyperbolic ones are diffeomorphic to the singularities of the form  $f = x^2 - y^2$ ,  $g = z$ . Elliptic singularities correspond to stable critical

trajectories of an integrable Hamiltonian system, and hyperbolic singularities to unstable ones.

In the case in question, the situation will coincide with that we've just described if we make a coordinate change bringing the bracket  $\Pi_1 - \lambda\Pi_0$  to the canonical form (6.1). Our singular point is thereby defined by the condition  $\tilde{z} = 0$ . Thus the upper left  $3 \times 3$ -block of this matrix corresponds exactly to a three-dimensional subalgebra (which is always semisimple but can be isomorphic either to  $so(3)$  or to  $sl(2)$  depending on  $\lambda$ )

After the change  $z \rightarrow \tilde{z}$ , the brackets  $\Pi_0$  and  $\Pi_1$  take the following form:

$$\begin{aligned} \Pi_0 &= \begin{pmatrix} z - Bp & p \\ p & 0 \end{pmatrix} \mapsto \begin{pmatrix} \tilde{z} + (\lambda - B)p - (\lambda - B)^{-1}g & p \\ p & 0 \end{pmatrix} \\ \Pi_1 &= \begin{pmatrix} Bz - g & 0 \\ 0 & p \end{pmatrix} \mapsto \begin{pmatrix} B\tilde{z} + \lambda((\lambda - B)p - (\lambda - B)^{-1}g) & \lambda p \\ \lambda p & p \end{pmatrix} \end{aligned}$$

Here, in order to shorten our notation, we use the standard identification of vectors in a three-dimensional space with  $3 \times 3$  skew-symmetric matrices.

According to the general construction, we have to restrict all forms contained in the considered pencil to the three-dimensional subspace corresponding to the upper left block and examine the singularity of the resulting three-dimensional pencil. The point  $p$  plays the role of a parameter.

Using Proposition 6.4.1, where the upper left blocks of the Poisson brackets  $\Pi_1 - \lambda\Pi_0$  and  $\Pi_0$  are taken as  $A$  and  $B$  (one can also take  $\Pi_1$  instead of  $\Pi_0$ , the result remains the same):

$$A = \begin{pmatrix} 0 & (b_3 - \lambda)\tilde{z}_3 & -(b_2 - \lambda)\tilde{z}_2 \\ -(b_3 - \lambda)\tilde{z}_3 & 0 & (b_1 - \lambda)\tilde{z}_1 \\ (b_2 - \lambda)\tilde{z}_2 & -(b_1 - \lambda)\tilde{z}_1 & 0 \end{pmatrix}$$

and

$$B = \begin{pmatrix} 0 & (\lambda - b_3)p_3 - \frac{g_3}{\lambda - b_3} + \dots & -(\lambda - b_2)p_2 + \frac{g_2}{\lambda - b_2} + \dots \\ -(\lambda - b_3)p_3 + \frac{g_3}{\lambda - b_3} + \dots & 0 & (\lambda - b_1)p_1 - \frac{g_1}{\lambda - b_1} + \dots \\ (\lambda - b_2)p_2 - \frac{g_2}{\lambda - b_2} + \dots & -(\lambda - b_1)p_1 + \frac{g_1}{\lambda - b_1} + \dots & 0 \end{pmatrix}$$

we obtain the following result ( $\tilde{z}_i$  are considered as variables for this block, and  $p_i$  are used as parameters).

**Theorem 6.4.2** *Let a critical closed trajectory of the Rubanovskii system pass through a point  $(z, p)$  with a parameter  $\lambda$ . This trajectory is non-degenerate if and only if the condition*

$$(\lambda - b_1)(\lambda - b_2)(\lambda - b_3) \sum_{i=1}^3 \left( (\lambda - b_i)p_i - \frac{g_i}{\lambda - b_i} \right)^2 \frac{1}{\lambda - b_i} \neq 0 \quad (6.8)$$

*is fulfilled. The trajectory is stable if this expression is greater than zero, and unstable if it is less than zero.*

**Remark.** It is noteworthy to mention the case when the expressions in brackets  $(\lambda - b_i)p_i - \frac{g_i}{\lambda - b_i}$  vanish simultaneously. It is not difficult to prove that in this case the point  $(z, p)$  turns out to be an equilibrium with  $\lambda$  being its multiple value of parameter. Thus, on the critical trajectories the expressions  $(\lambda - b_i)p_i - \frac{g_i}{\lambda - b_i}$  cannot vanish simultaneously.

We also note that the condition (6.8) can be rewritten in terms of the first integrals which could be useful to interpret the corresponding singularity by means of the bifurcation diagram.

## 6.5 Non-degeneracy and stability analysis for equilibria

The non-degeneracy condition for the equilibria of the Rubanovskii system can be verified in the same way as we did for closed trajectories. One should just remember the fact that an equilibrium is characterized not by one but two values of parameter  $\lambda$ . The condition (6.9) should be verified for each of them.

**Theorem 6.5.1** *Let  $(z, p)$  be a common equilibrium for the Hamiltonian  $H_1$  and  $H_2$  which corresponds to parameters  $\lambda_1$  and  $\lambda_2$ . If the conditions*

$$c_\alpha = (\lambda_\alpha - b_1)(\lambda_\alpha - b_2)(\lambda_\alpha - b_3) \sum_{i=1}^3 \left( (\lambda_\alpha - b_i)p_i - \frac{g_i}{\lambda_\alpha - b_i} \right)^2 \frac{1}{\lambda_\alpha - b_i} \neq 0, \quad \alpha = 1, 2. \quad (6.9)$$

*are fulfilled, then the equilibrium is a non-degenerate singular point of the momentum mapping.*

*Its type is determined by the signs of  $c_1$  and  $c_2$ . For real  $\lambda_1, \lambda_2$ , we have three possibilities. Namely,*

1. *if  $c_1 > 0, c_2 > 0$ , then the singular point is of center–center type and stable;*
2. *if  $c_1 > 0, c_2 < 0$  (or  $c_1 < 0, c_2 > 0$ ), then the singular point is of center–saddle type and unstable;*
3. *if  $c_1 < 0, c_2 < 0$ , then the singular point is of saddle–saddle type and unstable.*

*Finally, if  $\lambda_1$  and  $\lambda_2$  are a pair of complex conjugate numbers then the singular point is of focus–focus type.*

It is important to emphasize that all listed possibilities can be realized for some appropriately chosen parameters. Hereby is rather surprising that focus-focus singular points appear in the Rubanovskii case in contrast to the Zhukovskii-Volterra and Steklov- Lyapunov system where no focus singularities exist.

## 6.6 An open problem

As our computer experiments show, the number of different types of bifurcation diagrams for Rubanovskii case is quite large. This happens because the bifurcation

diagram essentially depends on three additional parameters  $g_1, g_2, g_3$  (in contrast to the Steklov-Lyapunov case where the only essential parameter is the integral  $J_2$ ). It is still an open problem to classify all possible types of the bifurcation diagram for the Rubanovskii case and to describe their dependence on parameters  $g_1, g_2, g_3$ .

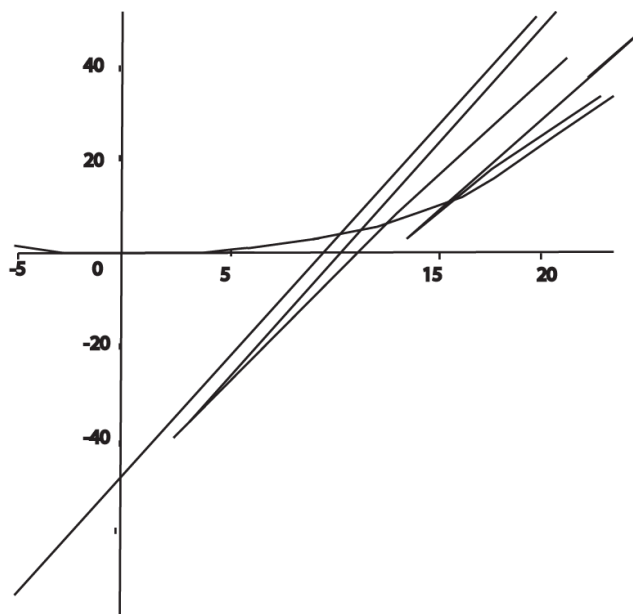


Figure 6.6.1: The bifurcation curve in the plan  $\mathbb{R}^2 = (H_1, H_2)$  for the case  $g_1 = 0.1$ ,  $g_2 = 0.5$ ,  $g_3 = 0.2$ ,  $b_1 = 5$ ,  $b_2 = 4$ ,  $b_3 = 3$ ,  $J_1 = 1$ ,  $J_2 = -7$ .

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