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# NONLINEAR D'ALEMBERT FORMULA FOR DISCRETE PSEUDOSPHERICAL SURFACES

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ABSTRACT. On the basis of loop group decompositions (Birkhoff decompositions), we give a discrete version of the nonlinear d'Alembert formula, a method of separation of variables of difference equations, for discrete constant negative Gauss curvature (pseudospherical) surfaces in Euclidean three space. We also compute two examples by this formula in detail.

## INTRODUCTION

The study of constant negative Gauss curvature surfaces (pseudospherical surfaces, PS surfaces in this paper) in Euclidean three space  $\mathbb{E}^3$  was important in classical differential geometry, see for example [1]. On the one hand the method of integrable systems in theory of differential equations, which started from 1960s, gave a new aspect for the study of PS surfaces: The structure equation for PS surfaces is a famous integrable system *sine-Gordon* equation, that is,  $u_{xy} - \sin u = 0$  under the *asymptotic Chebyshev coordinates*  $(x, y) \in \mathbb{R}^2$  for PS surfaces.

Moreover, PS surfaces can be characterized as the existence of a moving frame with an additional parameter  $\lambda \in \mathbb{R}^\times$ , the so-called *spectral parameter*, and the compatibility condition of the moving frame becomes the sine-Gordon equation, that is, PS surfaces can be characterized by the existence of a family of flat connections on a trivial bundle  $\mathbb{R}^2 \times \mathrm{SU}_2$ . Such moving frame is called the *extended frame*, and it is known that derivative of the extended frame with respect to the spectral parameter reproduces the original PS surface. It is fundamental to see that the extended frame takes values in the loop group of  $\mathrm{SU}_2$  and thus the loop group method can be applied for PS surfaces.

In [16, 21, 6], it was shown that loop group decompositions (Birkhoff decompositions, see Theorem 1.2) of the extended frame  $F$  of a PS surface induced a pair of 1-forms  $(\xi_+, \xi_-)$ , that is,  $F = F_+F_- = G_-G_+$  with  $\xi_+ = F_+^{-1}dF_+$  and  $\xi_- = G_-^{-1}dG_-$ . Then it was proved that  $\xi_+$  and  $\xi_-$  depended only on  $x$  and  $y$ , respectively. Conversely it was shown that solving the pair of ordinary differential equations  $dF_+ = F_+\xi_+$  and  $dG_+ = G_+\xi_-$  and using the loop group decomposition, the extended frame could be recovered. This construction is called the *nonlinear d'Alembert formula* for PS surfaces.

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A discrete analogue of smooth PS surfaces was defined in [2]: First of all discrete surfaces are maps from quadrilateral lattice  $(n, m) \in \mathbb{Z}^2$  into  $\mathbb{E}^3$ . Then discrete PS surfaces are defined in terms of some geometric properties which are analogous to the case of smooth PS surfaces. It is essential to see that the discrete PS surfaces have a discrete extended frame which is an exact analogue of the smooth extended frame of a PS surface. Therefore it is natural to expect that the loop group method can be applied. In fact in [18], the discrete PS surfaces were obtained by a loop group action, and in [13], several examples were discussed with taking into account a loop group structure.

In this paper, we show that there exists a discrete version of nonlinear d'Alembert formula for discrete PS surfaces: Decomposing the discrete extended frame  $F$  by loop group decompositions (Birkhoff decompositions), that is  $F = F_+F_-$  and  $F = G_-G_+$ , the discrete extended frame  $F$  induces a pair of discrete potentials  $(\xi_+, \xi_-)$  with  $\xi_+ = F_+^{-1}(n, m)F_+(n+1, m)$  and  $\xi_- = G_-^{-1}(n, m)G_-(n, m+1)$ . It will be shown that  $\xi_+$  and  $\xi_-$  depend only on  $n \in \mathbb{Z}$  and  $m \in \mathbb{Z}$ , respectively, see Theorem 2.1. Conversely using the loop group decomposition and a pair of ordinary difference equations  $F_+(n+1) = F_+(n)\xi_+(n)$  and  $G_-(m+1) = G_-(m)\xi_-(m)$ , one has the discrete extended frame  $F$  of the original discrete PS surface, see Theorem 2.2. Finally derivative of the discrete extended Frame reproduces the original discrete PS surface. We call this construction a *nonlinear d'Alembert formula* for discrete PS surfaces. Moreover using the nonlinear d'Alembert formula, we consider two examples of discrete PS surfaces in detail. It is interesting that difference equations obtained by these examples (see equations (3.2) and (3.10)) are the same difference equations obtained by previous works [19, 13].

This paper is organized as follows: In Section 1, basic results about smooth/discrete PS surfaces will be recalled. In particular we will recall the nonlinear d'Alembert formula for smooth PS surfaces in Theorem 1.3. In Section 2, we will give a nonlinear d'Alembert formula for discrete PS surfaces using loop group decompositions. Theorems 2.1 and 2.2 are the main results of this paper. In Section 3, two examples of discrete PS surfaces are discussed in detail. In Appendix A, we will discuss a relation between a smooth/discrete flow of constant torsion space curves and a smooth/discrete PS surface in  $\mathbb{E}^3$ .

## 1. PRELIMINARIES

In this section, we recall basic notation and results about smooth and discrete PS surfaces in  $\mathbb{E}^3$  [17, 2] and the nonlinear d'Alembert formula for smooth PS surfaces [21, 9, 16]. Throughout this paper, we use the Pauli matrices  $\sigma_j$  ( $j = 1, 2, 3$ ) as follows:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \text{and} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

**1.1. Pseudospherical surfaces.** We first identify  $\mathbb{E}^3$  with the Lie algebra of the special unitary group  $SU_2$ , which will be denoted by  $\mathfrak{su}_2$ :

$$(x, y, z)^t \in \mathbb{E}^3 \longleftrightarrow \frac{i}{2}x\sigma_1 - \frac{i}{2}y\sigma_2 + \frac{i}{2}z\sigma_3 \in \mathfrak{su}_2.$$

Let  $f$  be a PS surface with Gaussian curvature  $K = -1$  in  $\mathbb{E}^3$ . It is known that there exist the *asymptotic Chebyshev coordinates*  $(x, y) \in \mathbb{R}^2$  for  $f$ , that is, they are asymptotic coordinates normalized by  $|f_x| = |f_y| = 1$ . Here the subscripts  $x$  and  $y$  denote the  $x$ - and

$y$ -derivatives  $\partial_x$  and  $\partial_y$ , respectively. Let  $\{e_1, e_2, e_3\}$  be the Darboux frame rotating on the tangent plane clockwise angle  $u$ , where  $u$  is the angle between two asymptotic lines. Then there exists a  $\tilde{F}$  taking values in  $SU_2$  such that

$$(1.1) \quad e_1 = -\frac{i}{2}\tilde{F}\sigma_1\tilde{F}^{-1}, \quad e_2 = -\frac{i}{2}\tilde{F}\sigma_2\tilde{F}^{-1} \quad \text{and} \quad e_3 = -\frac{i}{2}\tilde{F}\sigma_3\tilde{F}^{-1},$$

see for example [9]. Without loss of generality, at some base point  $(x_*, y_*) \in \mathbb{R}^2$ , we have  $\tilde{F}(x_*, y_*) = \text{Id}$ . Then there exists a family of frames  $F$  parametrized by  $\lambda \in \mathbb{R}_+ := \{r \in \mathbb{R} \mid r > 0\}$  satisfying the following system of partial differential equations:

$$(1.2) \quad F_x = FU \quad \text{and} \quad F_y = FV,$$

where

$$(1.3) \quad U = \frac{i}{2} \begin{pmatrix} -u_x & \lambda \\ \lambda & u_x \end{pmatrix}, \quad V = -\frac{i}{2} \begin{pmatrix} 0 & \lambda^{-1}e^{iu} \\ \lambda^{-1}e^{-iu} & 0 \end{pmatrix}.$$

The parameter  $\lambda \in \mathbb{R}_+$  will be called the *spectral parameter*. We choose  $F$  such that

$$F|_{\lambda=1} = \tilde{F} \quad \text{and} \quad F|_{(x_*, y_*)} = \text{Id}.$$

The compatibility condition of the system in (1.2), that is  $U_y - V_x + [V, U] = 0$ , becomes a version of the sine-Gordon equation:

$$(1.4) \quad u_{xy} - \sin u = 0.$$

It turns out that the sine-Gordon equation is the Gauss-Codazzi equations for PS surfaces. Thus from the fundamental theorem of surface theory there exists a family of PS surfaces parametrized by the spectral parameter  $\lambda \in \mathbb{R}_+$ , see [9] for a clear explanation of it.

Then it is easy to see that  $F$  defined in (1.2) together with the condition  $F|_{(x_*, y_*)} = \text{Id}$  is an element in the *twisted  $SU_2$ -loop group*:

$$(1.5) \quad \Lambda SU_2 := \left\{ g : \mathbb{R}^\times \cup S^1 \rightarrow SL_2\mathbb{C} \mid g \text{ is smooth, } g(\lambda) = \overline{g(\bar{\lambda})}^{t-1} \text{ and } \sigma g(\lambda) = g(-\lambda) \right\},$$

where  $\sigma X = \text{Ad}(\sigma_3)X = \sigma_3 X \sigma_3^{-1}$ , ( $X \in SL_2\mathbb{C}$ ) is an involution on  $SL_2\mathbb{C}$ . In order to make the above group a Banach Lie group, we restrict the occurring matrix coefficients to the Wiener algebra  $\mathcal{A} = \{f(\lambda) = \sum_{n \in \mathbb{Z}} f_n \lambda^n : S^1 \rightarrow \mathbb{C} \mid \sum_{n \in \mathbb{Z}} |f_n| < \infty\}$ . Then the Wiener algebra is a Banach algebra relative to the norm  $\|f\| = \sum |f_n|$  and the loop group  $\Lambda SU_2$  is a Banach Lie group, [11].

Then the family of frames  $F$  will be called the *extended frame* for  $f$ . We note that in fact the extended frame  $F$  of a smooth PS surface is an element in the subgroup of  $\Lambda SU_2$ , consisting of loops which extend to  $\mathbb{C}^\times := \mathbb{C} \setminus \{0\}$  as analytic functions of  $\lambda$ .

From the extended frame  $F$ , a family of PS surfaces  $f^\lambda$ , ( $\lambda \in \mathbb{R}_+$ ) is given by the so-called *Sym formula*, [20]:

$$(1.6) \quad f^\lambda = \lambda \left. \frac{\partial F}{\partial \lambda} F^{-1} \right|_{\lambda \in \mathbb{R}_+}.$$

*Remark 1.1.*

- (1) The immersion  $f^\lambda|_{\lambda=1}$  is the original PS surface  $f$  up to rigid motion. The one-parameter family  $\{f^\lambda\}_{\lambda \in \mathbb{R}_+}$  will be called the *associated family* of  $f$ .

- (2) It is known that a surface of negative Gauss curvature is a PS surface if and only if the unit normal of the surface (Gauss map) is Lorentz harmonic. Note that the Lorentz structure is induced by the second fundamental form of a surface.

**1.2. Nonlinear d'Alembert formula.** In this subsection, we recall the nonlinear d'Alembert formula for smooth PS surfaces in [16, 21, 9]. Let  $\mathbb{D}^+$  and  $\mathbb{D}^-$  be the interior of the unit disk in the complex plane and the union of the exterior of the unit disk in the complex plane and infinity, respectively. We first define two subgroups of  $\Lambda\text{SU}_2$ :

$$(1.7) \quad \Lambda^+\text{SU}_2 = \{g \in \Lambda\text{SU}_2 \mid g \text{ can be analytically extend to } \mathbb{D}^+\},$$

$$(1.8) \quad \Lambda^-\text{SU}_2 = \{g \in \Lambda\text{SU}_2 \mid g \text{ can be analytically extend to } \mathbb{D}^-\}.$$

Then  $\Lambda_*^+\text{SU}_2$  and  $\Lambda_*^-\text{SU}_2$  denote subgroups of  $\Lambda^+\text{SU}_2$  and  $\Lambda^-\text{SU}_2$  normalized at  $\lambda = 0$  and  $\lambda = \infty$ , respectively:

$$\Lambda_*^+\text{SU}_2 = \{g \in \Lambda^+\text{SU}_2 \mid g(\lambda = 0) = \text{Id}\} \quad \text{and} \quad \Lambda_*^-\text{SU}_2 = \{g \in \Lambda^-\text{SU}_2 \mid g(\lambda = \infty) = \text{Id}\}.$$

The following decomposition theorem is fundamental.

**Theorem 1.2** (Birkhoff decomposition, [11, 4]). *The multiplication maps*

$$\Lambda_*^+\text{SU}_2 \times \Lambda^-\text{SU}_2 \rightarrow \Lambda\text{SU}_2 \quad \text{and} \quad \Lambda_*^-\text{SU}_2 \times \Lambda^+\text{SU}_2 \rightarrow \Lambda\text{SU}_2$$

*are diffeomorphisms onto  $\Lambda\text{SU}_2$ , respectively.*

We note that the Birkhoff decomposition is in general only diffeomorphism onto the open dense subset. However, in [4], it was proved that if the group is compact semisimple then the diffeomorphism extends to the everything.

From now on, for simplicity, we assume that the base point is  $(x_*, y_*) = (0, 0)$  and the extended frame  $F$  at the base point is identity:

$$F(0, 0, \lambda) = \text{Id}.$$

The nonlinear d'Alembert formula for PS surfaces is summarized as follows.

**Theorem 1.3** ([21, 9]). *Let  $F$  be the extended frame for a PS surface  $f$  in  $\mathbb{E}^3$ . Moreover, let  $F = F_+F_-$  and  $F = G_-G_+$  be the Birkhoff decompositions given in Theorem 1.2, respectively. Then  $F_+$  and  $G_-$  do not depend on  $y$  and  $x$ , respectively, and the Maurer-Cartan forms of  $F_+$  and  $G_-$  are given as follows:*

$$(1.9) \quad \begin{cases} \xi_+ = F_+^{-1}dF_+ = \frac{i}{2}\lambda \begin{pmatrix} 0 & e^{-i\alpha(x)} \\ e^{i\alpha(x)} & 0 \end{pmatrix} dx, \\ \xi_- = G_-^{-1}dG_- = -\frac{i}{2}\lambda^{-1} \begin{pmatrix} 0 & e^{i\beta(y)} \\ e^{-i\beta(y)} & 0 \end{pmatrix} dy, \end{cases}$$

where, using the angle function  $u(x, y)$ ,  $\alpha$  and  $\beta$  are given by

$$\alpha(x) = u(x, 0) - u(0, 0) \quad \text{and} \quad \beta(y) = u(0, y).$$

Conversely, let  $\xi_{\pm}$  be a pair of 1-forms defined in (1.9) with functions  $\alpha(x)$  and  $\beta(y)$  satisfying  $\alpha(0) = 0$ . Moreover, let  $F_+$  and  $G_-$  be solutions of the pair of following ordinary differential equations:

$$\begin{cases} dF_+ = F_+\xi_+, \\ dG_- = G_-\xi_-, \end{cases}$$

with  $F_+(x=0, \lambda) = G_-(y=0, \lambda) = \text{Id}$ . Moreover let  $D = \text{diag}(e^{-\frac{i}{2}\alpha}, e^{\frac{i}{2}\alpha})$  and decompose  $(F_+D)^{-1}G_-$  by the Birkhoff decomposition in Theorem 1.2:

$$(F_+D)^{-1}G_- = V_-V_+^{-1},$$

where  $V_- \in \Lambda_*^- \text{SU}_2$  and  $V_+ \in \Lambda^+ \text{SU}_2$ . Then  $F = G_-V_+ = F_+DV_-$  is the extended frame of some PS surface in  $\mathbb{E}^3$ .

The pair of 1-forms  $(\xi_+, \xi_-)$  in (1.9) will be called the *pair of normalized potentials*.

*Remark 1.4.* In [9], it was shown that the extended frames of PS surfaces can be also constructed from the following pair of 1-forms:

$$(1.10) \quad \eta^x = \sum_{j=-\infty}^1 \eta_j^x \lambda^j dx \quad \text{and} \quad \eta^y = \sum_{j=-1}^{\infty} \eta_j^y \lambda^j dy,$$

where  $\eta_j^x$  and  $\eta_j^y$  take values in  $\mathfrak{su}_2$ , and each entry of  $\eta_j^x$  (resp.  $\eta_j^y$ ) is smooth on  $x$  (resp.  $y$ ), and  $\det \eta_1^x \neq 0, \det \eta_{-1}^y \neq 0$ . Moreover  $\eta_j^x$  and  $\eta_j^y$  are diagonal (resp. off-diagonal) if  $j$  is even (resp. odd). This pair of 1-forms  $(\eta^x, \eta^y)$  is a generalization of the normalized potentials  $(\xi_+, \xi_-)$  in (1.9) and will be called the *pair of generalized potentials*, see also [5].

**1.3. Discrete pseudospherical surfaces.** It is known that a discrete analogue of PS surfaces and its loop group formulation was defined in [2]. In this subsection, instead of the smooth coordinates  $(x, y) \in \mathbb{R}^2$ , we use the quadrilateral lattice  $(n, m) \in \mathbb{Z}^2$ , that is, all functions depend on the lattice  $(n, m) \in \mathbb{Z}^2$ . The subscripts 1 and 2 (resp.  $\bar{1}$  and  $\bar{2}$ ) denote the forward (resp. backward) lattice points with respect to  $n$  and  $m$ : For a function  $f(n, m)$  of the lattice  $(n, m) \in \mathbb{Z}^2$ , we define  $f_1, f_2, f_{\bar{1}}$  and  $f_{\bar{2}}$  by

$$f_1 = f(n+1, m), \quad f_{\bar{1}} = f(n-1, m), \quad f_2 = f(n, m+1) \quad \text{and} \quad f_{\bar{2}} = f(n, m-1).$$

The *discrete extended frame*  $F$  of a discrete PS surface can be defined by the following partial difference system, see [3, Section 3.2] and [2]<sup>1</sup>:

$$(1.11) \quad F_1 = FU \quad \text{and} \quad F_2 = FV,$$

where

$$(1.12) \quad U = \frac{1}{\Delta_+} \begin{pmatrix} e^{-\frac{i}{2}(u_1-u)} & \frac{i}{2}p\lambda \\ \frac{i}{2}p\lambda & e^{\frac{i}{2}(u_1-u)} \end{pmatrix}, \quad V = \frac{1}{\Delta_-} \begin{pmatrix} 1 & -\frac{i}{2}qe^{\frac{i}{2}(u_2+u)}\lambda^{-1} \\ -\frac{i}{2}qe^{-\frac{i}{2}(u_2+u)}\lambda^{-1} & 1 \end{pmatrix},$$

with  $\Delta_+ = \sqrt{1 + (p/2)^2 \lambda^2}$  and  $\Delta_- = \sqrt{1 + (q/2)^2 \lambda^{-2}}$ . Here  $u$  is a real function depending on both  $n$  and  $m$ , and  $p \neq 0$  and  $q \neq 0$  are real functions depending only on  $n$  and  $m$ , respectively:

$$u = u(n, m), \quad p = p(n), \quad \text{and} \quad q = q(m).$$

<sup>1</sup>Since the identification of  $\mathbb{E}^3$  and  $\mathfrak{su}_2$  is different in [3], the matrices  $\mathcal{U}$  and  $\mathcal{V}$  of (3.22) and (3.23) in [3] and our  $U, V$  are related by  $U = \text{Ad}(\text{offdiag}(1, 1))^t \mathcal{U}$  and  $V = \text{Ad}(\text{offdiag}(1, -1))^t \mathcal{V}$  with  $h_{n,m} = u/2, p_n = p/2, q_m = q/2$ . On the one hand if we normalize  $p = q = \delta$ , then  $U$  and  $V$  are the same as  $\Omega_{n,m}$  and  $\Theta_{n,m}$  in [18, Theorem 3.1].

The compatibility condition of the system in (1.11), that is  $VU_2 = UV_1$ , gives the so-called *discrete sine-Gordon equation*:

$$(1.13) \quad \sin\left(\frac{u_{12} - u_1 - u_2 + u}{4}\right) = \frac{pq}{4} \sin\left(\frac{u_{12} + u_1 + u_2 + u}{4}\right).$$

The equation (1.13) was first found by Hirota in [12] and also called the *Hirota equation*. Then a discrete PS surface  $f$  can be given by the so-called Sym formula, [2]:

$$(1.14) \quad f^\lambda = \lambda \left. \frac{\partial F}{\partial \lambda} F^{-1} \right|_{\lambda \in \mathbb{R}_+}.$$

The family of frames  $F$  defined by (1.11) with  $F|_{(n_*, m_*)} = \text{Id}$  will be called the *discrete extended frame* for a discrete PS surface.

*Remark 1.5.* The geometric characterization of a discrete PS surface  $f$  was also given in [2, 3]. A discrete surface  $f$  is a discrete PS surface if and only if the following two conditions hold:

- (1) For each point  $f$ , there is a plane  $P$  such that

$$f, f_1, f_{\bar{1}}, f_2, f_{\bar{2}} \in P.$$

- (2) The length of the opposite edge of an elementary quadrilateral is equal:

$$|f_1 - f| = |f_{12} - f_2| = a(n) \neq 0, \quad |f_2 - f| = |f_{12} - f_1| = b(m) \neq 0.$$

It is easy to see that the map  $f^\lambda$  defined in (1.14) has these properties and  $f^\lambda$  gives a family of discrete PS surfaces, see [3, Theorem 3].

We now introduce a discrete analogue of the smooth angle function between all edges at point  $(n, m) \in \mathbb{Z}^2$  as follows, see [2]:

$$\begin{aligned} \phi^{(1)} &\equiv -\frac{u_2}{2} - \frac{u_1}{2}, & \phi^{(2)} &\equiv \frac{u_{\bar{1}}}{2} + \frac{u_2}{2}, \\ \phi^{(3)} &\equiv -\frac{u_{\bar{2}}}{2} - \frac{u_{\bar{1}}}{2}, & \phi^{(4)} &\equiv \frac{u_1}{2} + \frac{u_{\bar{2}}}{2}, \end{aligned}$$

where  $\equiv$  means that the equalities hold modulo  $2\pi$ . Then from the symmetry of the quadrilateral and the fact that the sum of angles around a vertex is equal to  $2\pi$  by (1) in Remark 1.5, we have

$$\phi^{(1)} = \phi_{12}^{(3)}, \quad \phi_1^{(2)} = \phi_2^{(4)} \quad \text{and} \quad \phi^{(1)} + \phi^{(2)} + \phi^{(3)} + \phi^{(4)} \equiv 0.$$

Moreover, setting  $Q = \exp(i\phi^{(1)})$  and  $k = pq/4$ , the discrete sine-Gordon equation (1.13) can be rephrased as

$$(1.15) \quad Q_{12}Q = \frac{Q_2 - k_2}{1 - k_2 Q_2} \frac{Q_1 - k_1}{1 - k_1 Q_1},$$

see for example [3, Section 3.4]. We note that the other discrete angle functions  $\exp(i\phi^{(j)})$  ( $j \in \{2, 3, 4\}$ ) also satisfy the same equation (1.15).

## 2. NONLINEAR D'ALEMBERT FORMULA FOR DISCRETE PS SURFACES

In this section, we derive a discrete version of nonlinear d'Alembert formula for discrete PS surfaces.

**2.1. Nonlinear d'Alembert formula.** From now on we assume that the base point is  $(n_*, m_*) = (0, 0)$  and the discrete extended frame  $F$  at the base point is identity:

$$F(0, 0, \lambda) = \text{Id}.$$

Moreover, we also assume that the functions  $p$  and  $q$  in (1.12) satisfy the inequalities

$$(2.1) \quad 0 < \left| \frac{p}{2} \right| < 1 \quad \text{and} \quad 0 < \left| \frac{q}{2} \right| < 1.$$

These inequalities imply that the zeros of  $\Delta_+$  and  $\Delta_-$  are in the exterior and the interior of the unit disk, respectively. Moreover it is easy to see that zeros of  $\Delta_{\pm}$  are not on  $\mathbb{R} \cup \{\infty\}$ . Then the discrete extended frame  $F$  takes values in the twisted loop group  $\Lambda \text{SU}_2$ .

The following theorem gives a pair of matrices, which depend only on one lattice variables and it is a discrete version of Theorem 1.3.

**Theorem 2.1.** *Let  $f$  be a discrete PS surface and  $F$  the corresponding discrete extended frame. Decompose  $F$  according to the Birkhoff decomposition in Theorem 1.2:*

$$F = F_+ F_- = G_- G_+,$$

where  $F_+ \in \Lambda_*^+ \text{SU}_2$ ,  $F_- \in \Lambda^- \text{SU}_2$ ,  $G_- \in \Lambda_*^- \text{SU}_2$  and  $G_+ \in \Lambda^+ \text{SU}_2$ . Then  $F_+$  and  $G_-$  do not depend on  $m \in \mathbb{Z}$  and  $n \in \mathbb{Z}$ , respectively, and the discrete Maurer-Cartan forms of  $F_+$  and  $G_-$  are given as follows:

$$(2.2) \quad \begin{cases} \xi_+ = F_+^{-1}(F_+)_1 = \frac{1}{\Delta_+} \begin{pmatrix} 1 & \frac{i}{2} p e^{-i\alpha} \lambda \\ \frac{i}{2} p e^{i\alpha} \lambda & 1 \end{pmatrix}, \\ \xi_- = G_-^{-1}(G_-)_2 = \frac{1}{\Delta_-} \begin{pmatrix} 1 & -\frac{i}{2} q e^{i\beta} \lambda^{-1} \\ -\frac{i}{2} q e^{-i\beta} \lambda^{-1} & 1 \end{pmatrix}, \end{cases}$$

where  $\Delta_+ = \sqrt{1 + (p/2)^2 \lambda^2}$  and  $\Delta_- = \sqrt{1 + (q/2)^2 \lambda^{-2}}$ , the functions  $p$  and  $q$  are given in (1.12), and  $\alpha$  and  $\beta$  are functions of  $n \in \mathbb{Z}$  and  $m \in \mathbb{Z}$ , respectively. Moreover using the function  $u(n, m)$  in (1.12),  $\alpha(n)$  and  $\beta(m)$  are given by

$$(2.3) \quad \begin{cases} \alpha(n) = \frac{1}{2} u(n+1, 0) + \frac{1}{2} u(n, 0) - u(0, 0), \\ \beta(m) = \frac{1}{2} u(0, m+1) + \frac{1}{2} u(0, m). \end{cases}$$

*Proof.* Let  $F = F_+ F_- = G_- G_+$  be the Birkhoff decomposition of the discrete extended frame. Since  $F_+$  takes values in  $\Lambda_*^+ \text{SU}_2$ ,  $F_+^{-1}(F_+)_2$  also takes values in  $\Lambda_*^+ \text{SU}_2$ . On the one hand, we have

$$F_+^{-1}(F_+)_2 = F_- F^{-1} F_2 (F_-)_2^{-1} = F_- V (F_-)_2^{-1}.$$

Since  $V$  takes values in  $\Lambda_*^- \text{SU}_2$ ,  $F_- V (F_-)_2^{-1}$  can be expanded with respect to  $\lambda$  as

$$F_- V (F_-)_2^{-1} = V_{(0)} + \lambda^{-1} V_{(-1)} + \lambda^{-2} V_{(-2)} + \lambda^{-3} V_{(-3)} + \cdots.$$

Thus

$$(2.4) \quad F_+^{-1}(F_+)_2 = \text{Id}.$$



Therefore  $F_+$  depends only on  $n$ . Similarly  $F_+^{-1}(F_+)_1$  takes values in  $\Lambda_*^+ \text{SU}_2$  and  $F_+^{-1}(F_+)_1 = F_- F^{-1} F_1 (F_-)_1^{-1} = F_- U (F_-)_1^{-1}$ . Multiplying  $\Delta_+$  on both sides, we have

$$(2.5) \quad \Delta_+ F_+^{-1}(F_+)_1 = U_{(0)} + \lambda U_{(1)} = F_- (\Delta_+ U) (F_-)_1^{-1},$$

where we use the expansion  $F_- (\Delta_+ U) (F_-)_1^{-1} = \lambda U_{(1)} + U_{(0)} + \lambda^{-1} U_{(-1)} + \dots$ . Since  $F_+$  takes values in  $\Lambda_*^+ \text{SU}_2$ ,  $U_{(0)} = \text{Id}$  and  $U_{(1)}$  has the off diagonal form with two entries are minus complex conjugate each other and  $\det(U_{(0)} + \lambda U_{(1)}) = \Delta_+^2$ . Then

$$(2.6) \quad \xi_+ = F_+^{-1}(F_+)_1 = \frac{1}{\Delta_+} \begin{pmatrix} 1 & \ell(n)\lambda \\ -\ell(n)\lambda & 1 \end{pmatrix}, \quad \text{with } |\ell(n)|^2 = p(n)^2.$$

Thus we have the form  $\xi_+$  in (2.2). We now look the expansion of  $F_- (\Delta_+ U) (F_-)_1^{-1}$  more closely. Since  $F_- \in \Lambda^- \text{SU}_2$ , we can set  $F_-(\lambda = \infty) = \text{diag}(e^{\frac{i}{2}f}, e^{-\frac{i}{2}f})$ . Then from (2.4), we see that

$$\frac{f_2}{2} = \frac{f}{2} \pmod{2\pi}.$$

Then consider (2.5) multiplying  $\Delta_+^{-1}$  on both sides and set  $m = 0$ :

$$(2.7) \quad \xi_+ = F_+^{-1}(F_+)_1 = F_+^{-1}(F_+)_1|_{m=0} = F_- U (F_-)_1^{-1}|_{m=0}.$$

Let  $\tilde{F}_+$  be the solution of  $\tilde{F}_+^{-1}(\tilde{F}_+)_1 = U|_{m=0}$  with  $\tilde{F}_+(n=0) = \text{Id}$ . It is easy to see that  $\tilde{F}_+$  does not depend on  $m \in \mathbb{Z}$  and takes values in  $\Lambda^+ \text{SU}_2$ . Moreover,  $\tilde{F}_+ F_-^{-1}|_{m=0}$  satisfies that  $(\tilde{F}_+ F_-^{-1})^{-1}(\tilde{F}_+ F_-^{-1})_1|_{m=0} = F_- U (F_-)_1^{-1}|_{m=0}$ . Since  $\tilde{F}_+ F_-^{-1}|_{(n,m)=(0,0)} = F_+|_{(n,m)=(0,0)} = \text{Id}$  and by (2.7), we have

$$\tilde{F}_+ F_-^{-1}|_{m=0} = F_+|_{m=0}.$$

Thus  $F_-|_{m=0} (= F_+^{-1} \tilde{F}_+|_{m=0})$  takes values in  $\Lambda^+ \text{SU}_2 \cap \Lambda^- \text{SU}_2$  and therefore  $F_-|_{m=0}$  does not depend on  $\lambda$ . A straightforward computation shows that

$$F_- U (F_-)_1^{-1}|_{m=0} = \frac{1}{\Delta_+} \begin{pmatrix} e^{-\frac{i}{2}(u_1-u) - \frac{i}{2}(f_1-f)} & \frac{i}{2} p e^{\frac{i}{2}(f_1+f)} \lambda \\ \frac{i}{2} p e^{-\frac{i}{2}(f_1+f)} \lambda & e^{\frac{i}{2}(u_1-u) + \frac{i}{2}(f_1-f)} \end{pmatrix} \Big|_{m=0}.$$

Thus comparing this with  $\xi_+$  in (2.6) and setting  $\alpha = -\frac{1}{2}(f_1 + f)|_{m=0}$ , we have  $-(u_1 - u)|_{m=0} = (f_1 - f)|_{m=0}$  and  $\ell = \frac{i}{2} p e^{-i\alpha}$ . The equation  $-(u_1 - u)|_{m=0} = (f_1 - f)|_{m=0}$  can be easily solved to  $f$  with the initial condition  $f(0, 0) = 0$ :

$$f(n, 0) = u(0, 0) - u(n, 0).$$

Note that  $f(0, 0) = 0$  is satisfied since  $F_-(\lambda = \infty) = \text{Id}$ . Therefore  $\alpha$  can be explicitly computed as

$$\alpha(n) = \frac{1}{2} u(n+1, 0) + \frac{1}{2} u(n, 0) - u(0, 0).$$

For the case of  $G_-$ , the verbatim argument shows that  $G_-$  does not depend on  $n$  and  $\xi_- = G_-^{-1}(G_-)_2$  has the following form:

$$\xi_- = G_-^{-1}(G_-)_2 = \frac{1}{\Delta_-} \begin{pmatrix} 1 & r(m)\lambda^{-1} \\ -r(m)\lambda^{-1} & 1 \end{pmatrix} \quad \text{with } |r(m)|^2 = q(m)^2.$$

Thus we have the form  $\xi_-$  in (2.2). Consider

$$(2.8) \quad \xi_- = G_-^{-1}(G_-)_2|_{n=0} = G_+ V (G_+)_2^{-1}|_{n=0}.$$

Let  $\tilde{G}_-$  be the solution of  $\tilde{G}_-^{-1}(\tilde{G}_-)_2 = V|_{n=0}$  with  $\tilde{G}_-(m=0) = \text{Id}$ . It is easy to see that  $\tilde{G}_-$  does not depend on  $n \in \mathbb{Z}$  and takes values in  $\Lambda^- \text{SU}_2$ . Moreover,  $\tilde{G}_- G_+^{-1}|_{n=0}$  satisfies  $G_+ \tilde{G}_-^{-1}(\tilde{G}_-)_2(G_+)_2^{-1}|_{n=0} = G_+ V(G_+)_2^{-1}|_{n=0}$ . Since  $\tilde{G}_+ G_+^{-1}|_{(n,m)=(0,0)} = G_-|_{(n,m)=(0,0)} = \text{Id}$  and by (2.8), we have

$$\tilde{G}_- G_+^{-1}|_{n=0} = G_-|_{n=0}.$$

Therefore  $G_+|_{n=0} (= G_-^{-1} \tilde{G}_-|_{n=0})$  takes values in  $\Lambda^+ \text{SU}_2 \cap \Lambda^- \text{SU}_2$  and thus  $G_+|_{n=0}$  does not depend on  $\lambda$ . Moreover, since  $G_-^{-1}(G_-)_2$  and  $V$  are in  $\Lambda_*^- \text{SU}_2$  and from (2.8), we have  $G_+|_{n=0} = \text{Id}$ . A straightforward computation shows that

$$G_+ V(G_+)_2^{-1}|_{n=0} = \frac{1}{\Delta_-} \left( \begin{array}{cc} 1 & -\frac{i}{2} q e^{\frac{i}{2}(u_2+u)} \lambda^{-1} \\ -\frac{i}{2} q e^{-\frac{i}{2}(u_2+u)} \lambda^{-1} & 1 \end{array} \right) \Big|_{n=0}.$$

Thus setting  $\beta = \frac{1}{2}(u_2 + u)|_{n=0}$ , we have  $r = -\frac{i}{2} q e^{i\beta}$ . Therefore we have

$$\beta(m) = \frac{1}{2} u(0, m+1) + \frac{1}{2} u(0, m).$$

This completes the proof.  $\square$

**Definition 1.** The pair of matrices  $(\xi_-, \xi_+)$  given in (2.2) will be called the *pair of discrete normalized potentials*.

Conversely, the pair of discrete normalized potentials give some discrete PS surface through the Birkhoff decomposition.

**Theorem 2.2.** Let  $\xi_{\pm}$  be a pair of discrete normalized potentials defined in (2.2) with arbitrary functions  $\alpha = \alpha(n), \beta = \beta(m)$  with  $\alpha(0) = 0$  and  $p = p(n), q = q(m)$  satisfying the conditions (2.1). Moreover let  $F_+ = F_+(n, \lambda)$  and  $G_- = G_-(m, \lambda)$  be the solutions of the ordinary difference equations

$$(F_+)_1 = F_+ \xi_+ \quad \text{and} \quad (G_-)_2 = G_- \xi_-,$$

with  $F_+(n=0, \lambda) = G_-(m=0, \lambda) = \text{Id}$  and set a matrix  $D = \text{diag}(e^{\frac{i}{2}k}, e^{-\frac{i}{2}k}) \in U_1$ , where  $k(0) = 0$  and  $k(n) = 2 \sum_{j=0}^{n-1} (-1)^{j+n} \alpha(j)$  for  $n \geq 1$ . Decompose  $(F_+ D)^{-1} G_-$  by the Birkhoff decomposition in Theorem 1.2:

$$(2.9) \quad (F_+ D)^{-1} G_- = V_- V_+^{-1},$$

where  $V_- \in \Lambda_*^- \text{SU}_2, V_+ \in \Lambda^+ \text{SU}_2$ . Then  $F = G_- V_+ = F_+ D V_-$  is the discrete extended frame of some discrete PS surface in  $\mathbb{E}^3$ . Moreover the solution  $u = u(n, m)$  of the discrete sine-Gordon for the discrete PS surface satisfies the relations in (2.3).

*Proof.* Set  $F = G_- V_+ = F_+ D V_-$  as in the statement of Theorem 2.2. Let us compute the discrete Maurer-Cartan form  $F^{-1} F_1$  and  $F^{-1} F_2$ , respectively. A direct computation shows that

$$F^{-1} F_1 = V_-^{-1} D^{-1} \xi_+ D_1 (V_-)_1 = V_+^{-1} G_-^{-1} (G_-)_1 (V_+)_1 = V_+^{-1} (V_+)_1.$$

Here we use the property that  $G_-$  does not depend on  $n \in \mathbb{Z}$ . Thus  $F^{-1} F_1$  takes values in  $\Lambda^+ \text{SU}_2$  and using the expansion  $V_-^{-1} D^{-1} (\Delta_+ \xi_+) D_1 (V_-)_1 = * \lambda + * \lambda^0 + * \lambda^{-1} + \dots$ , we have

$$\Delta_+ F^{-1} F_1 = F_{(0)} + \lambda F_{(1)}.$$

Here  $F_{(0)}$  and  $F_{(1)}$  are diagonal and off-diagonal matrices, respectively. Moreover, from the form of  $\xi_+$  and  $V_+(\lambda = 0) = \text{diag}(e^{\frac{i}{2}h}, e^{-\frac{i}{2}h})$  and  $\alpha + \frac{1}{2}(k_1 + k) = 0$ , we have

$$(2.10) \quad F^{-1}F_1 = \frac{1}{\Delta_+} \begin{pmatrix} e^{\frac{i}{2}(h_1-h)} & \frac{i}{2}p\lambda \\ \frac{i}{2}p\lambda & e^{-\frac{i}{2}(h_1-h)} \end{pmatrix}.$$

Similarly we compute  $F^{-1}F_2$  as

$$F^{-1}F_2 = V_+^{-1}\xi_-(V_+)_2 = V_-^{-1}(F_+D)^{-1}(F_+D)_2(V_-)_2 = V_-^{-1}(V_-)_2.$$

Here we use the property that  $F_+D$  does not depend on  $m \in \mathbb{Z}$ . Thus, noting that  $F^{-1}F_2 \in \Lambda_*^+ \text{SU}_2$ , we have

$$\Delta_- F^{-1}F_2 = \text{Id} + \lambda^{-1}F_{(-1)}.$$

Since  $F \in \Lambda \text{SU}_2$ ,  $F_{(-1)}$  is an off-diagonal matrix. Moreover, since  $F$  is unitary on  $\lambda \in \mathbb{R}_+$  and the form of  $\xi_-$ , we have

$$(2.11) \quad F^{-1}F_2 = \frac{1}{\Delta_-} \begin{pmatrix} 1 & -\frac{i}{2}qe^{i\beta - \frac{i}{2}(h+h_2)}\lambda^{-1} \\ -\frac{i}{2}qe^{-i\beta + \frac{i}{2}(h+h_2)}\lambda^{-1} & 1 \end{pmatrix}.$$

From the compatibility condition of  $F$ , that is  $F_{12} = F_{21}$ , we have

$$(2.12) \quad -X_2 + Y = X + Y_1 \quad \text{and} \quad e^{\frac{i}{2}X_2} + \frac{pq}{4}e^{\frac{i}{2}Y} = e^{\frac{i}{2}X} + \frac{pq}{4}e^{-\frac{i}{2}Y_1},$$

where  $X = h_1 - h$  and  $Y = 2\beta - (h_2 + h)$ . The first equation in (2.12) can be solved by a function  $u = u(n, m)$  as

$$(2.13) \quad h_1 - h = -u_1 + u \quad \text{and} \quad 2\beta - (h_2 + h) = u_2 + u.$$

Then the second equation in (2.12) is the discrete sine-Gordon equation and  $F$  is the extended frame for some discrete PS surface. Let us look at the Birkhoff decomposition in (2.9) with  $n = 0$ :

$$(F_+D)^{-1}G_-|_{n=0} = V_-V_+^{-1}|_{n=0}.$$

Then since  $F_+D|_{n=0} = \text{Id}$  and  $G_-$  takes values in  $\Lambda_*^- \text{SU}_2$ , we have

$$V_-|_{n=0} = G_-|_{n=0} \quad \text{and} \quad V_+|_{n=0} = \text{Id}.$$

Thus  $h|_{n=0} = 0$  and the second equation in (2.13) is equivalent with the second equation in (2.3). Similarly consider the Birkhoff decomposition in (2.9) with  $m = 0$ :

$$(F_+D)^{-1}G_-|_{m=0} = V_-V_+^{-1}|_{m=0}.$$

Then since  $G_-|_{m=0} = \text{Id}$  and  $F_+D$  takes values in  $\Lambda^+ \text{SU}_2$ , we have

$$V_-|_{m=0} = \text{Id} \quad \text{and} \quad V_+|_{m=0} = F_+D|_{m=0}.$$

Thus  $h|_{m=0} = k$  and from the first equation in (2.13) is equivalent with  $k_1 - k = (-u_1 + u)|_{m=0}$ . Since  $k(n) = 2 \sum_{j=0}^{n-1} (-1)^{j+n} \alpha(j)$ , we can solve the equation for  $\alpha$ , that is,  $\alpha(n) = \frac{1}{2}u(n+1, 0) + \frac{1}{2}u(n, 0) - u(0, 0)$ , which is the first equation in (2.3).  $\square$

**2.2. Pairs of discrete generalized potentials.** In Section 2.1, we discussed the construction of discrete PS surfaces from pairs of discrete normalized potentials. In this subsection, we generalize the pairs of discrete normalized potentials to pairs of discrete generalized potentials and show that those pairs produce also discrete PS surfaces. Let  $\xi_{\pm}$  be a pair of discrete normalized potentials and set

$$(2.14) \quad \eta_n = P_-^l \xi_+ P_-^r, \quad \eta_m = P_+^l \xi_- P_+^r.$$

Here we assume that  $P_{\pm}^{\star}$  ( $\star = l$  or  $r$ ) take values in  $\Lambda^{\pm} \text{SU}_2$  and do not depend on  $m$  and  $n$ , respectively, that is,  $P_-^{\star} = P_-^{\star}(n, \lambda)$  and  $P_+^{\star} = P_+^{\star}(m, \lambda)$ . Thus the  $\eta_n$  and  $\eta_m$  do not depend on  $m$  and  $n$ , respectively:

$$\eta_n = \eta_n(n, \lambda), \quad \eta_m = \eta_m(m, \lambda).$$

Then the pair of discrete generalized potentials  $(\eta_n, \eta_m)$  produces a discrete PS surface as follows.

**Theorem 2.3.** *Let  $(\eta_n, \eta_m)$  be the pair of discrete generalized potentials defined in (2.14). Moreover let  $F_n$  and  $G_m$  be the solutions of the ordinary difference equations*

$$(F_n)_1 = F_n \eta_n \quad \text{and} \quad (G_m)_2 = G_m \eta_m,$$

with some initial condition  $F_n(n=0)$  and  $G_m(m=0)$  taking values in  $\Lambda \text{SU}_2$  and let  $D = \text{diag}(e^{\frac{i}{2}k}, e^{-\frac{i}{2}k}) \in \text{U}_1$ , where  $k = k(n)$  is

$$k(0) = 0, \quad k(n) = \sum_{j=0}^{n-1} (-1)^{j+n} (2\alpha(j) - \theta^l(j) + \theta^r(j)) \quad \text{for } n \geq 1.$$

Here  $\theta^l$  and  $\theta^r$  are defined by  $P_-^l(n, \lambda = \infty) = \text{diag}(e^{\frac{i}{2}\theta^l(n)}, e^{-\frac{i}{2}\theta^l(n)})$  and  $P_-^r(n, \lambda = \infty) = \text{diag}(e^{\frac{i}{2}\theta^r(n)}, e^{-\frac{i}{2}\theta^r(n)})$ , respectively. Then decompose  $(F_n D)^{-1} G_m$  by the Birkhoff decomposition in Theorem 1.2:

$$(F_n D)^{-1} G_m = V_- V_+^{-1},$$

where  $V_- \in \Lambda_*^- \text{SU}_2, V_+ \in \Lambda^+ \text{SU}_2$ . Then  $F = G_m V_+ = F_n D V_-$  is the discrete extended frame of some discrete PS surface in  $\mathbb{E}^3$ .

*Proof.* Set  $F = G_m V_+ = F_n D V_-$  as in the statement of Theorem 2.3. Let us compute the discrete Maurer-Cartan form  $F^{-1} F_1$  and  $F^{-1} F_2$ , respectively. A direct computation shows that

$$F^{-1} F_1 = V_-^{-1} D^{-1} \eta_n D_1 (V_-)_1 = V_+^{-1} G_m^{-1} (G_m)_1 (V_+)_1.$$

Since  $G_m$  does not depend on  $n$ , the right hand side of the equation becomes  $V_+^{-1} (V_+)_1$  which takes values in  $\Lambda^+ \text{SU}_2$ . Therefore,  $\Delta_+ F^{-1} F_1$  has the form

$$\Delta_+ F^{-1} F_1 = F_{(0)} + \lambda F_{(1)}.$$

From the twisted property of  $F_n$  and  $G_m$ ,  $V_{\pm}$  and  $F$  also have the same property. Thus  $F_{(0)}$  and  $F_{(1)}$  are diagonal and off-diagonal matrices, respectively.

Moreover, from the form of  $D^{-1} \eta_n D_1$  and using  $\alpha - \frac{1}{2}(\theta^l - \theta^r) + \frac{1}{2}(k_1 + k) = 0$ , we have

$$F^{-1} F_1 = \frac{1}{\Delta_+} \begin{pmatrix} e^{\frac{i}{2}(h_1-h)} & \frac{i}{2} p \lambda \\ \frac{i}{2} p \lambda & e^{-\frac{i}{2}(h_1-h)} \end{pmatrix},$$

where we set  $V_+(\lambda = 0) = \text{diag}(e^{\frac{i}{2}h}, e^{-\frac{i}{2}h})$ . Similarly we compute  $F^{-1}F_2$  as

$$F^{-1}F_2 = V_+^{-1}\eta_m(V_+)_2 = V_-^{-1}(F_n D)^{-1}(F_n D)_2(V_-)_2 = V_-^{-1}(V_-)_2.$$

Thus, noting that  $V_-^{-1}(V_-)_2 \in \Lambda_*^+ \text{SU}_2$ , we have

$$\Delta_- F^{-1}F_2 = \text{Id} + \lambda^{-1}F_{(-1)}.$$

From the twisted property of  $F_n$  and  $G_m$ ,  $V_\pm$  and  $F$  also have the same property. Thus  $F_{(-1)}$  is an off-diagonal matrix. Moreover, from the form of  $\eta_m$ , we have

$$F^{-1}F_2 = \frac{1}{\Delta_-} \begin{pmatrix} 1 & -\frac{i}{2}q e^{i\beta - \frac{i}{2}(h+h_2 - \omega^l + \omega^r)} \lambda^{-1} \\ -\frac{i}{2}q e^{-i\beta + \frac{i}{2}(h+h_2 - \omega^l + \omega^r)} \lambda^{-1} & 1 \end{pmatrix},$$

where we set  $P_+^l(\lambda = 0) = \text{diag}(e^{\frac{i}{2}\omega^l}, e^{-\frac{i}{2}\omega^l})$  and  $P_+^r(\lambda = 0) = \text{diag}(e^{\frac{i}{2}\omega^r}, e^{-\frac{i}{2}\omega^r})$ . Since  $F^{-1}F_1$  and  $F^{-1}F_2$  have the forms in (2.10) and (2.11), respectively. Thus we can use the same argument in the proof of Theorem 2.2 and there exists a function  $u$  such that

$$h_1 - h = -u + u, \quad 2\beta - (h + h_2 + \omega^l + \omega^r) = u_2 + u,$$

and  $u$  satisfies the discrete sine-Gordon equation. Then  $F$  is the discrete extended frame of some discrete PS surface in  $\mathbb{E}^3$ , which completes the proof.  $\square$

**Definition 2.** The pair  $(\eta_n, \eta_m)$  given in (2.14) will be called the *pair of discrete generalized potentials*.

*Remark 2.4.* The pair of normalized potentials  $(\xi_+, \xi_-)$  and the corresponding pair of discrete generalized potentials  $(\eta_n, \eta_m)$  in (2.14) give in general different discrete PS surfaces.

### 3. EXAMPLES

In this section, we discuss two examples using the nonlinear d'Alembert formula.

**3.1. Discrete pseudospherical surfaces of revolution.** We recall that the construction of smooth PS surfaces of revolution via the nonlinear d'Alembert formula. Let  $(\eta^x, \eta^y)$  be a pair of generalized potentials as follows:

$$(3.1) \quad \eta^x = A dx, \quad \eta^y = -A dy, \quad \text{with } A = \begin{pmatrix} 0 & i(\lambda + \lambda^{-1}) \\ i(\lambda + \lambda^{-1}) & 0 \end{pmatrix}.$$

It is easy to see that solutions of the pair of ordinary differential equations  $dF^x = F^x \eta^x$  and  $dG^y = G^y \eta^y$  with  $F^x(x = 0, \lambda) = G^y(y = 0, \lambda) = \text{Id}$  can be computed as

$$F^x(x) = \exp(xA) \quad \text{and} \quad G^y(y) = \exp(-yA).$$

Consider the Birkhoff decomposition in Theorem 1.2 as  $(F^x)^{-1}G^y = V_- V_+^{-1}$  and set

$$F = F^x V_- = G^y V_+,$$

where  $V_- \in \Lambda_*^- \text{SU}_2$  and  $V_+ \in \Lambda_*^+ \text{SU}_2$ . Then  $F$  is the extended frame for a PS surface of revolution: Let  $\gamma$  be a translation of  $(x, y)$  to  $(x + p, y - p)$  with some  $p \in \mathbb{R}$ . Then

$$\begin{aligned} \gamma^* F^x &:= F^x(x + p, y - p, \lambda) = \exp(pA) F^x, \\ \gamma^* G^y &:= G^y(x + p, y - p, \lambda) = \exp(pA) G^y. \end{aligned}$$

Thus  $(\gamma^*F^x)^{-1}\gamma^*G^y = (F^x)^{-1}G^y$ , and  $V_-^{-1}\gamma^*V_- = V_+^{-1}\gamma^*V_+$ . The left-hand and the right hand side take values in  $\Lambda_*^-\text{SU}_2$  and  $\Lambda^+\text{SU}_2$ , respectively. Thus we have

$$\gamma^*V_- = V_- \quad \text{and} \quad \gamma^*V_+ = V_+.$$

Therefore, we conclude that

$$\gamma^*F = \exp(pA)F.$$

Inserting  $\gamma^*F$  into the Sym formula (1.6), it is easy to see that the resulting surface is a PS surface of revolution. The sine-Gordon equation with coordinates  $(\tilde{x}, \tilde{y}) = (x + y, x - y)$  becomes

$$u_{\tilde{x}\tilde{x}} - \sin u = 0.$$

Note that  $u$  depends only on  $\tilde{x}$  and the above equation is an ordinary differential equation. It is well known that the above ordinary differential equation can be explicitly solved by an elliptic function, see for example [15].

A discrete analogue of the above argument holds. Let  $(\eta_n, \eta_m)$  be a pair of matrices as follows:

$$\eta_n = \eta_m^{-1} = A_+LA_-, \quad \text{with} \quad A_\pm = \frac{1}{\Delta_\pm} \begin{pmatrix} 1 & \pm \frac{i}{2}q\lambda^{\pm 1} \\ \pm \frac{i}{2}q\lambda^{\pm 1} & 1 \end{pmatrix} \quad \text{and} \quad L = \text{diag}(e^{ic}, e^{-ic}),$$

where  $\Delta_\pm = \sqrt{1 + (q/2)^2\lambda^{\pm 2}}$ ,  $q$  ( $0 < |q/2| < 1$ ) and  $c = \pi\ell^{-1}$  ( $\ell \in \mathbb{Z}_+$ ) are some constants. Here we can consider  $\xi_+ = A_+$ ,  $P_-^r = LA_-$  and  $P_-^l = \text{Id}$  for  $\eta_n$ , and  $\xi_- = A_-^{-1}$ ,  $P_+^r = L^{-1}A_+^{-1}$  and  $P_+^l = \text{Id}$  for  $\eta_m$ , respectively. Thus  $(\eta_n, \eta_m)$  is a pair of discrete generalized potentials in (2.14). It is easy to see that solutions of the pair of ordinary difference equations  $(F_n)_1 = F_n\eta_n$  and  $(G_m)_2 = G_m\eta_m$  with  $F_n(n=0) = G_m(m=0) = \text{Id}$  as

$$F_n(n) = (A_+LA_-)^n \quad \text{and} \quad G_m(m) = (A_+LA_-)^{-m}.$$

Here we denote  $X^{-m} = (X^{-1})^m$  and  $X^0 = \text{Id}$  for a matrix  $X$ . Since  $P_-^r|_{\lambda=\infty} = LA_-|_{\lambda=\infty} = \text{diag}(e^{ic}, e^{-ic})$ , we can define the matrix  $D$  as in Theorem 2.3. Consider the Birkhoff decomposition in Theorem 1.2:

$$(F_nD)^{-1}G_m = V_-V_+^{-1},$$

where  $V_- \in \Lambda_*^-\text{SU}_2$ ,  $V_+ \in \Lambda^+\text{SU}_2$ , and set  $F = F_nDV_- = G_mV_+$ . Let  $\gamma$  be a translation of  $(n, m) \in \mathbb{Z}^2$  to  $(n + p, m - p) \in \mathbb{Z}^2$  with some  $p \in \mathbb{Z}$ . Then  $\gamma^*F_n = (A_+LA_-)^pF_n$ ,  $\gamma^*G_m = (A_+LA_-)^pG_m$  and  $\gamma^*$ . Thus  $(\gamma^*F_nD)^{-1}\gamma^*G_m = (F_nD)^{-1}G_m$ , and we have

$$\gamma^*V_- = V_- \quad \text{and} \quad \gamma^*V_+ = V_+.$$

Therefore, we conclude that

$$\gamma^*F = (A_+LA_-)^pF\gamma^*D.$$

Note that  $\gamma^*D \in U_1$ . We now set  $M = (A_+LA_-)^p$  and insert  $\gamma^*F$  into the Sym formula (1.6):

$$\gamma^*f = MfM^{-1} + \lambda(\partial_\lambda M)M^{-1}.$$

Then a direct computation shows that  $MfM^{-1}|_{\lambda=1}$  represents a rotation around an axis  $L = (1, 0, -3/4)$  and an angle  $2p\ell$ . Moreover an another direct computation shows that  $\lambda(\partial_\lambda M)M^{-1}|_{\lambda=1}$  perpendicular to the axis  $L$ . Thus  $\gamma^*f|_{\lambda=1}$  represents a rotation around the axis  $L$  through a point  $p_0$ . In fact  $p_0$  can be explicitly computed as  $p_0 = -1/2 \cot \ell$  ( $\ell \in \mathbb{Z}_+$ ).

Therefore the resulting surface is a discrete PS surface of revolution with period  $\ell \in \mathbb{Z}_+$ . The discrete sine-Gordon equation in (1.15) becomes

$$(3.2) \quad Q_1 Q_{\bar{1}} = \left( \frac{Q - k}{1 - kQ} \right)^2,$$

where  $k = q^2/4$ .

*Remark 3.1.* It is known that Equation (3.2) is a special case of the dP<sub>III</sub> equation discussed in [19].

**3.2. Discrete Amsler surface and discrete Painlevé III equation.** We recall a construction of smooth Amsler surfaces via the nonlinear d'Alembert formula. Let  $\xi_{\pm}$  be a pair of normalized potentials as follows:

$$\xi_+ = \lambda A dx, \quad \xi_- = -\lambda^{-1} A dy \quad \text{with} \quad A = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Let  $F_+$  and  $G_-$  be a pair of solutions of ordinary differential equations  $dF_+ = F_+ \xi_+$  and  $dG_- = G_- \xi_-$  with initial conditions  $F_+(x = 0) = \text{diag}(e^{is}, e^{-is})$  and  $G_-(y = 0) = \text{diag}(e^{i\ell}, e^{-i\ell})$  with  $s \neq \ell$ . It is easy to compute the solutions as

$$F_+(x, \lambda) = \text{diag}(e^{is}, e^{-is}) \exp(x\lambda A), \quad G_-(y, \lambda) = \text{diag}(e^{i\ell}, e^{-i\ell}) \exp(-y\lambda^{-1} A).$$

Consider the Birkhoff decomposition in Theorem 1.2:

$$F_+^{-1} G_- = V_- V_+^{-1},$$

where  $V_- \in \Lambda_*^- \text{SU}_2$  and  $V_+ \in \Lambda^+ \text{SU}_2$ , and set  $F = F_+ V_- = G_- V_+$ . Let  $\gamma$  be a transformation of  $(x, y, \lambda)$  to  $(xp, yp^{-1}, \lambda p^{-1})$  with  $p \in \mathbb{R}^\times$ . Then it is easy to see that

$$\gamma^* F_+ = F_+ \quad \text{and} \quad \gamma^* G_- = G_-.$$

Thus  $\gamma^*(V_- V_+^{-1}) = F_+^{-1} G_- = V_- V_+^{-1}$ , that is,

$$V_-(xp, yp^{-1}, \lambda p^{-1}) V_+(xp, yp^{-1}, \lambda p^{-1})^{-1} = V_-(x, y, \lambda) V_+(x, y, \lambda)^{-1}.$$

Then  $V_-(xp, yp^{-1}, \lambda p^{-1})$  and  $V_+(xp, yp^{-1}, \lambda p^{-1})$  clearly take values in  $\Lambda_*^- \text{SU}_2$  and  $\Lambda^+ \text{SU}_2$ , respectively, and thus  $\gamma^* V_- = V_-$  and  $\gamma^* V_+ = V_+$ . Finally, we have

$$\gamma^* F = F.$$

This implies that  $F$  depends only on  $r = xy$  and  $\lambda$ , and the resulting surface is a Amsler surface. The sine-Gordon equation becomes the so-called Painlevé III equation in trigonometric form with fixed parameters [10, p. 443]:

$$\tilde{u}''(r) + \frac{1}{r} \tilde{u}'(r) + \sin \tilde{u}(r) = 0,$$

where  $\tilde{u}(r) = u(x, y)$  and  $r = 2\sqrt{-xy}$ . We note that if the initial condition  $F_+(x = 0) = G_-(y = 0)$ , that is  $s = \ell$ , then the resulting surface is a line.

Moreover, the Amsler surface has an additional reflection symmetry as follows: We first define a transformation  $\delta$  as

$$\delta^* L(x, y, \lambda) = \overline{L(y, x, \bar{\lambda}^{-1})}.$$

It is easy to see that  $\delta^*F_+ = C_0G_-$  and  $\delta^*G_- = C_0F_+$  with  $C_0 = \text{diag}(e^{-i(\ell+s)}, e^{i(\ell+s)})$ . Then a direct computation shows that  $(\delta^*(F_+^{-1}G_-))^{-1} = (\delta^*G_-)^{-1}\delta^*F_+ = F_+^{-1}G_-$ . On the one hand, from the decomposition  $F_+^{-1}G_- = V_-V_+^{-1}$ , we have  $V_-V_+^{-1} = \delta^*V_+(\delta^*V_-)^{-1}$ :

$$\delta^*V_- = V_+d \text{ and } \delta^*V_+ = V_-d,$$

where  $d^{-1} = V_+(\lambda = 0) \in U_1$ . Finally using  $\delta^*F_+ = C_0G_-$ , we have

$$\delta^*F = \delta^*(F_+V_-) = C_0G_-V_+d = C_0Fd.$$

Inserting this symmetry into the Sym formula, we have

$$\delta^*f = \overline{f(y, x, \bar{\lambda}^{-1})} = -C_0f(x, y, \lambda)C_0^{-1}.$$

Thus, at  $\lambda = 1$ , this implies

$$f(y, x) = -C_0^{-1}\overline{f(x, y)}C_0.$$

This represents a reflection through a plane. Thus we have

$$u(y, x) = u(x, y) \pmod{2\pi}.$$

We now give a discrete analogue of the above argument as follows. Let  $\xi_{\pm}$  be a pair of discrete normalized potentials

$$(3.3) \quad \xi_+ = A_+, \quad \xi_- = A_- \quad \text{with} \quad A_{\pm} = \frac{1}{\Delta_{\pm}} \begin{pmatrix} 1 & \pm \frac{i}{2}q\lambda^{\pm 1} \\ \pm \frac{i}{2}q\lambda^{\pm 1} & 1 \end{pmatrix},$$

where  $\Delta_{\pm} = \sqrt{1 + (q/2)^2\lambda^{\pm 2}}$  and  $q$  ( $0 < |q/2| < 1$ ) is some real constant. Note that  $\alpha(n) = \beta(m) = 0$  for  $\xi_{\pm}$  in (2.2), thus the matrix  $D$  in Theorem 2.2 is identity matrix. It is easy to see that solutions of the pair of ordinary difference equations  $(F_+)_1 = F_+\xi_+$  and  $(G_-)_2 = G_-\xi_-$  with initial conditions  $F_+(n=0) = \text{diag}(e^{is}, e^{-is})$  and  $G_-(m=0) = \text{diag}(e^{i\ell}, e^{-i\ell})$ , ( $\ell \neq s$ ) as

$$(3.4) \quad F_+(n) = \text{diag}(e^{is}, e^{-is})A_+^n \quad \text{and} \quad G_-(m) = \text{diag}(e^{i\ell}, e^{-i\ell})A_-^m.$$

Here we use notation  $A_{\pm}^0 = \text{Id}$ . Then a straightforward computation shows

$$\begin{aligned} F_+^{-1}G_- &= A_+^{-n} \text{diag}(e^{i(\ell-s)}, e^{-i(\ell-s)})A_-^m \\ &= \frac{1}{\Delta_+^n \Delta_-^m} \begin{pmatrix} 1 & -\frac{i}{2}q\lambda \\ -\frac{i}{2}q\lambda & 1 \end{pmatrix}^n \begin{pmatrix} e^{i(\ell-s)} & 0 \\ 0 & e^{-i(\ell-s)} \end{pmatrix} \begin{pmatrix} 1 & -\frac{i}{2}q\lambda^{-1} \\ -\frac{i}{2}q\lambda^{-1} & 1 \end{pmatrix}^m. \end{aligned}$$

Consider the Birkhoff decomposition in Theorem 1.2 as  $F_+^{-1}G_- = V_-V_+^{-1}$  and set

$$F = F_+V_- = G_-V_+,$$

where  $V_- \in \Lambda_*^- \text{SU}_2$  and  $V_+ \in \Lambda^+ \text{SU}_2$ . Then the discrete extended frame  $F$  has the special property as follows.

**Theorem 3.2.** *The  $\lambda$ -derivative of the discrete extended frame  $F$  has the following form:*

$$(3.5) \quad \lambda F^{-1} \partial_{\lambda} F = \frac{1}{\Delta_+^2 \Delta_-^2} \begin{pmatrix} a & b\lambda^{-1} + c\lambda \\ -\bar{b}\lambda^{-1} - \bar{c}\lambda & -a \end{pmatrix},$$

where  $a \in i\mathbb{R}$  and  $b, c \in \mathbb{C}$  are functions of  $(n, m) \in \mathbb{Z}^2$ .



*Proof.* Since  $F = F_+V_- = G_-V_+$ , we have

$$(3.6) \quad \lambda F^{-1}\partial_\lambda F = V_-^{-1}F_+^{-1}(\lambda\partial_\lambda F_+)V_- + V_-^{-1}\lambda\partial_\lambda V_- = V_+^{-1}G_-^{-1}(\lambda\partial_\lambda G_-)V_+ + V_+^{-1}\lambda\partial_\lambda V_+.$$

Using the explicit forms of  $F_+$  and  $G_-$  in (3.4), we compute

$$(3.7) \quad \lambda F_+^{-1}\partial_\lambda F_+ = \frac{inq\lambda}{2\Delta_+^2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \lambda G_-^{-1}\partial_\lambda G_- = \frac{imq\lambda^{-1}}{2\Delta_-^2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Therefore, multiplying  $(\Delta_+\Delta_-)^2$  on the both sides of (3.6), we have the equality

$$(3.8) \quad \begin{aligned} & \frac{inq\lambda}{2}\Delta_-^2V_-^{-1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} V_- + (\Delta_+\Delta_-)^2V_-^{-1}\lambda\partial_\lambda V_- \\ &= \frac{imq\lambda^{-1}}{2}\Delta_+^2V_+^{-1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} V_+ + (\Delta_+\Delta_-)^2V_+^{-1}\lambda\partial_\lambda V_+. \end{aligned}$$

Moreover, since  $V_- \in \Lambda_*^-\text{SU}_2$  and  $V_+ \in \Lambda^+\text{SU}_2$ , we have

$$\begin{aligned} \lambda V_-^{-1}\partial_\lambda V_- &= \lambda(\text{Id} + \lambda^{-1}\check{V}_{(-1)} + \lambda^{-2}\check{V}_{(-2)} + \cdots)(-\lambda^{-2}\check{V}_{(-1)} - 2\lambda^{-3}\check{V}_{(-2)} + \cdots), \\ &= -\lambda^{-1}\check{V}_{(-1)} + \lambda^{-2} * + \cdots, \end{aligned}$$

and

$$\begin{aligned} \lambda V_+^{-1}\partial_\lambda V_+ &= \lambda(\hat{V}_{(0)} + \lambda\hat{V}_{(1)} + \lambda^2\hat{V}_{(2)} + \cdots)(V_{(1)} + 2\lambda V_{(2)} + \cdots), \\ &= \lambda\hat{V}_{(0)}V_{(1)} + \lambda^2 * + \cdots. \end{aligned}$$

Here we set  $V_- = \text{Id} + \lambda^{-1}\check{V}_{(-1)} + \lambda^{-2}\check{V}_{(-2)} + \cdots$  and  $V_+ = V_{(0)} + \lambda V_{(1)} + \lambda^2 V_{(2)} + \cdots$ . Noting that  $(\Delta_+\Delta_-)^2 = q^2(\lambda^2 + \lambda^{-2})/4 + q^4/16 + 1$ , we have

$$(\Delta_+\Delta_-)^2F^{-1}\lambda\partial_\lambda F = \lambda^{-1}F_{(-1)} + F_{(0)} + \lambda F_{(1)}.$$

Since  $F$  takes values in  $\Lambda\text{SU}_2$ ,  $\lambda F^{-1}\partial_\lambda F$  takes values in  $\Lambda\mathfrak{su}_2$ , and  $F_{(-1)}$  and  $F_{(1)}$  are off-diagonal matrices and  $F_{(0)}$  is a diagonal matrix. This completes the proof.  $\square$

The *discrete Amsler surface* was defined in [13] using the discrete sine-Gordon equation constrained by the equation [13, Equation (4.26)]:

$$(3.9) \quad (m-n)(Q - Q_{\bar{1}\bar{2}}) = (n+m) \left( \frac{k - Q_{\bar{1}}}{1 - kQ_{\bar{1}}} - \frac{k - Q_{\bar{2}}}{1 - kQ_{\bar{2}}} \right),$$

where  $k = q^2/4$ . This is equivalent with that the  $\lambda$ -derivative of  $F$  has the form in (3.5), [13, Theorem 4.3]. Thus, we have the following corollary.

**Corollary 3.3.** *The surface constructed by the pair of discrete normalized potentials in (3.3) is actually the discrete Amsler surface. The discrete sine-Gordon equation for the discrete Amsler surface is given as follows:*

$$(3.10) \quad (m+n)QQ_{\bar{1}\bar{2}} + (m-n)(Q_{\bar{1}\bar{2}} - Q) \left( \frac{Q_{\bar{1}} - k}{1 - kQ_{\bar{1}}} \right) = (m+n) \left( \frac{Q_{\bar{1}} - k}{1 - kQ_{\bar{1}}} \right)^2,$$

where  $k = q^2/4$ .

*Proof.* In the discrete sine-Gordon equation given in (1.15), we eliminate  $Q_{\bar{2}}$  by (3.9). Then the equation (3.10) follows.  $\square$

We finally show that the resulting discrete surface has an additional reflection symmetry and contains two straightlines as follows. Therefore it is natural to call the resulting surface  $f$  a discrete Amsler surface.

**Theorem 3.4.** *The resulting discrete PS surface  $f$  given by the pair of normalized potential  $(\xi_+, \xi_-)$  in (3.3) has a reflection symmetry:*

$$f(m, n) = -C_0^{-1} \overline{f(n, m)} C_0, \quad u(n, m) = u(m, n) \pmod{2\pi},$$

where  $C_0 = \text{diag}(e^{-i(\ell+s)}, e^{i(\ell+s)})$  and  $u$  is the solution to the discrete sine-Gordon equation. Moreover  $f$  contains two straightlines.

*Proof.* The argument for proof is similar to the case of the smooth Amsler surface: Define a transformation  $\delta$  on a loop

$$\delta^* L(n, m, \lambda) = \overline{L(m, n, \bar{\lambda}^{-1})}.$$

It is easy to see that  $\delta^* F_+ = C_0 G_-$  and  $\delta^* G_- = C_0 F_+$  with  $C_0 = \text{diag}(e^{-i(\ell+s)}, e^{i(\ell+s)})$ . Then a direct computation shows that  $(\delta^*(F_+^{-1} G_-))^{-1} = (\delta^* G_-)^{-1} \delta^* F_+ = F_+^{-1} G_-$ . On the one hand, from the decomposition  $F_+^{-1} G_- = V_- V_+^{-1}$ , we have

$$V_- V_+^{-1} = \delta^* V_+ (\delta^* V_-)^{-1}, \quad \text{that is, } \delta^* V_- = V_+ d, \quad \delta^* V_+ = V_- d,$$

where  $d^{-1} = V_+(\lambda=0) \in U_1$ . Finally using  $\delta^* F_+ = C_0 G_-$ , we have

$$\delta^* F = \delta^*(F_+ V_-) = C_0 G_- V_+ d = C_0 F d.$$

Inserting this symmetry into the Sym formula, we have

$$\delta^* f = \overline{f(y, x, \bar{\lambda}^{-1})} = -C_0 f(x, y, \lambda) C_0^{-1}.$$

Thus, at  $\lambda = 1$ , this implies

$$f(m, n) = -C_0^{-1} \overline{f(n, m)} C_0.$$

This represents a reflection through a plane and the angle function  $u$  has the symmetry  $u(n, m) = u(m, n) \pmod{2\pi}$ .

Let us consider the case  $n = 0$ . Then

$$F_+^{-1} G_-|_{n=0} = \text{diag}(e^{i(\ell-s)}, e^{-i(\ell-s)}) A_-^m,$$

and the Birkhoff decomposition can be computed explicitly as

$$V_-|_{n=0} = \text{diag}(e^{i(\ell-s)}, e^{-i(\ell-s)}) A_-^m \text{diag}(e^{-i(\ell-s)}, e^{i(\ell-s)}), \quad V_+|_{n=0} = \text{diag}(e^{-i(\ell-s)}, e^{i(\ell-s)}).$$

Thus  $F|_{n=0} = F_+ V_-|_{n=0} = G_- V_+|_{n=0} = \text{diag}(e^{i\ell}, e^{-i\ell}) A_-^m \text{diag}(e^{-i(\ell-s)}, e^{i(\ell-s)})$  and

$$V|_{n=0} = F^{-1} F_2|_{n=0} = \text{Ad}(\text{diag}(e^{i(\ell-s)}, e^{-i(\ell-s)})) A_-.$$

Therefore  $u(0, m) = 4(\ell - s) - u(0, 0)$  if  $m$  is odd, and  $u(0, m) = u(0, 0)$  if  $m$  is even. Using the  $\lambda$ -derivative of  $G_-$  in (3.7), we compute that

$$f(0, m) = \lambda(\partial_\lambda F) F^{-1}|_{n=0, \lambda=1} = \frac{2imq}{4+q^2} \begin{pmatrix} 0 & e^{2i\ell} \\ e^{-2i\ell} & 0 \end{pmatrix}.$$

Thus  $f(0, m)$  represents a line in the  $(x, y)$ -plane through origin.

Similarly let us consider the case  $m = 0$ . Then

$$F_+^{-1}G_-|_{m=0} = A_+^{-n} \text{diag}(e^{i(\ell-s)}, e^{-i(\ell-s)}),$$

and the Birkhoff decomposition can be computed as

$$V_-|_{m=0} = \text{Id}, \quad V_+|_{m=0} = \text{diag}(e^{i(s-\ell)}, e^{-i(s-\ell)})A_+^n.$$

Thus  $F|_{m=0} = F_+V_-|_{m=0} = G_-V_+|_{m=0} = \text{diag}(e^{is}, e^{-is})A_+^n$  and

$$U|_{m=0} = F^{-1}F_1|_{m=0} = A_+.$$

Therefore  $u(n, 0) = u(0, 0)$ . Using the  $\lambda$ -derivative of  $F_+$  in (3.7), we compute that

$$f(n, 0) = \lambda(\partial_\lambda F)F^{-1}|_{m=0, \lambda=1} = \frac{2inq}{4+q^2} \begin{pmatrix} 0 & e^{2is} \\ e^{-2is} & 0 \end{pmatrix}.$$

Thus  $f(n, 0)$  also represents a line in the  $(x, y)$ -plane through origin. This completes the proof.  $\square$

## APPENDIX A. DISCRETE FLOW OF DISCRETE CURVES AND DISCRETE PSEUDOSPHERICAL SURFACES

It is known that the sine-Gordon equation  $u_{st} - \sin u = 0$  (or equivalently PS surfaces) has another geometric interpretation; the flow of a constant torsion curve  $\gamma$  in  $\mathbb{E}^3$ , see [7, 8]. In [14], a discrete analogue of the flow of constant torsion curves was introduced. In this appendix we show that the discrete flow of a discrete constant torsion curve can be identified with a discrete PS surface.

**A.1. Pseudospherical surfaces and the flows of constant torsion space curves.** Let  $\gamma$  be a constant torsion curve  $\gamma$  in  $\mathbb{E}^3$  and denote by  $\kappa$  and  $\tau$  the curvature and the torsion of  $\gamma$ , respectively. We use the parameter  $s$  for the curve  $\gamma$ . Moreover, set

$$\kappa = u_s \quad \text{and} \quad \lambda = -i\tau.$$

Then the Frenet frame for the curve  $\gamma$  taking values in  $\text{SU}_2$  is described in the first equation in (A.1). We now consider a special flow given by the second equation in (A.1) with the flow parameter  $t$ , see [8]:

$$(A.1) \quad \tilde{F}_s = \tilde{F}\tilde{U} \quad \text{and} \quad \tilde{F}_t = \tilde{F}\tilde{V},$$

where

$$(A.2) \quad \tilde{U} = \frac{1}{2} \begin{pmatrix} \lambda & -u_s \\ u_s & -\lambda \end{pmatrix}, \quad \tilde{V} = \frac{1}{2\lambda} \begin{pmatrix} \cos u & -\sin u \\ -\sin u & -\cos u \end{pmatrix}.$$

Then the compatibility condition of the above system is the sine-Gordon equation:

$$u_{st} - \sin u = 0.$$

Moreover, the constant torsion space curve  $\gamma$  with curvature  $\kappa = u_s$  and torsion  $\tau$  can be represented by the following formula:

$$(A.3) \quad \gamma(s, t) = i \frac{\partial \tilde{F}}{\partial \lambda} \tilde{F}^{-1} \Big|_{\lambda=-i\tau}.$$

The  $\lambda = -i\tau$  can be considered as a parameter of the system (A.1) and  $\tilde{F}$  is an element of the loop group, which is not in  $\Lambda\text{SU}_2$ , since the twisted condition is not satisfied. We call  $\tilde{F}$  the *extended frame* of the curve  $\gamma$ .

The extended frame  $\tilde{F}$  and the formula for the curve  $\gamma$  in (A.3) is similar to the case of PS surfaces in Section 1.1. In fact they are gauge equivalent as follows, see also [7, Section 2.1].

**Proposition A.1.** *The extended frame  $F$  of a PS surface and the extended frame  $\tilde{F}$  of the flow of a constant torsion curve are conjugate by some constant matrix  $C$ :*

$$(A.4) \quad F(x, y, \lambda) = C^{-1}\tilde{F}(x, y, i\lambda)C, \quad \text{with } C = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{\frac{i\pi}{4}} & e^{\frac{i\pi}{4}} \\ -e^{-\frac{i\pi}{4}} & e^{-\frac{i\pi}{4}} \end{pmatrix}.$$

Moreover, the immersion  $f$  in (1.6) and the flow of constant torsion curve  $\gamma$  in (A.3) are related by rotation of the matrix  $C$ :

$$f^\lambda|_{\lambda=-\tau} = (-i\tau)C^{-1}\gamma C.$$

*Proof.* It is straightforward to compute  $F_x = FU$  and  $F_y = FV$  in terms of  $\tilde{F}$  as follows:

$$\tilde{F}_x = \tilde{F}(CUC^{-1}) \quad \text{and} \quad \tilde{F}_y = \tilde{F}(CVC^{-1}).$$

Then it is easy to see that  $CUC^{-1} = \tilde{U}$  and  $CVC^{-1} = \tilde{V}$  with  $\lambda$  replaced by  $i\lambda$ .  $\square$

*Remark A.2.*  $\tilde{F}$  is an element of some loop group, but it does not satisfy the usual twisted condition, that is,  $\text{Ad}_{\sigma_3}g(\lambda) = g(-\lambda)$ . However, it satisfies another twisted condition  $\text{Ad}_{(\sigma_3 C^{-1})}g(\lambda) = g(-\lambda)$ , where  $C$  is given in (A.4).

**A.2. Discrete flow of a discrete constant torsion space curve.** From the result in [14, Theorem 6.2], we discuss a discrete flow of a discrete constant torsion space curve. Let  $\gamma$  be a discrete constant torsion curve parametrized by  $n \in \mathbb{Z}$  and denote the torsion by  $\tau$ . Then we evolve the curve  $\gamma$  as follows:

$$\gamma_2 = \gamma + \delta(\cos wT + \sin wN), \quad \delta = \frac{b}{1 + (b/2)^2\lambda^2},$$

$$\tan \frac{w_1 + k_1}{2} = \frac{b+a}{b-a} \tan \frac{w}{2},$$

where  $T$  and  $N$  are the tangent and the principal normal vectors to  $\gamma$ , respectively, and  $a$  and  $c$  depend only on  $n$  and  $m$ , respectively<sup>2</sup>:

$$a = a(n), \quad b = b(m).$$

It is known that  $k = k(n, m)$  and  $w = w(n, m)$  can be rephrased by  $u = u(n, m)$  [14, (8.3) and (6.22)] as

$$k = \frac{u_1 - u_{\bar{1}}}{2}, \quad \text{and} \quad w = -\frac{u_2 + u_1}{2}.$$

<sup>2</sup>Note that in [14], the identification of  $\mathbb{E}^3$  and  $\mathfrak{su}_2$  is different from our paper and they use different notation; the relations are given by  $a(n) = a_n$  and  $c(m) = 1/b_m$ .

The discrete Frenet frame  $\tilde{\Phi} = (T, N, B)$  taking values in  $\text{SO}_3$  with the binormal vector  $B$  of the curve  $\gamma$  satisfies [14, (6.14) and (6.16)]

$$\tilde{\Phi}_1 = \tilde{\Phi}\tilde{L}, \quad \tilde{\Phi}_2 = \tilde{\Phi}\tilde{M},$$

with

$$\tilde{L} = \tilde{R}^{(1)}(-\nu_1)\tilde{R}^{(3)}(\kappa_1), \quad \tilde{M} = \tilde{R}^{(3)}(w)\tilde{R}^{(1)}(\mu)\tilde{R}^{(3)}(-w_1 - \kappa_1),$$

where  $\nu_1 = 2 \arctan \frac{a\lambda}{2}$ ,  $\mu = 2 \arctan \frac{2}{b\lambda}$  and  $\tilde{R}^{(j)}(x)$  denotes the rotation matrix around the vector  $e_j$  with angle  $x$ . Then there is a 2 to 1 correspondence from  $\text{SU}_2$  to  $\text{SO}_3$  such that the corresponding matrices  $R^{(j)} \in \text{SU}_2$  have the following forms:

$$\text{SO}_3 \ni \tilde{R}^{(1)}(x) \longleftrightarrow R^{(1)}(x) = \begin{pmatrix} \cos \frac{x}{2} & -i \sin \frac{x}{2} \\ -i \sin \frac{x}{2} & \cos \frac{x}{2} \end{pmatrix} \in \text{SU}_2,$$

$$\text{SO}_3 \ni \tilde{R}^{(3)}(x) \longleftrightarrow R^{(3)}(x) = \text{diag} \left( e^{-i\frac{x}{2}}, e^{i\frac{x}{2}} \right) \in \text{SU}_2.$$

Thus we translate the Frenet frame  $\tilde{\Phi}$  in  $\text{SO}_3$  into the corresponding frame  $\Phi$  in  $\text{SU}_2$  see for example [14, Section 2] for the explicit correspondence. From this correspondence, it is clear that the discrete Frenet frame  $\Phi$  satisfies

$$(A.5) \quad \Phi_1 = \Phi L, \quad \Phi_2 = \Phi M,$$

where

$$L = R^{(1)}(-\nu_1)R^{(3)}\left(\frac{u_{11} - u}{2}\right), \quad M = R^{(3)}\left(-\frac{u_2 + u_1}{2}\right)R^{(1)}(\mu)R^{(3)}\left(\frac{u_{12} + u}{2}\right),$$

where  $u_{11} = u(n+2, m)$ . Then we have the following proposition. It is already mentioned in [14, Remark 8.3], but we give a proof for completeness.

**Proposition A.3.** *The discrete Frenet frame  $\Phi$  in (A.5) and the discrete extended frame  $F$  in (1.11) are related by*

$$F = \Phi G, \quad G = \text{diag} \left( e^{i\frac{u_1 - u}{4}}, e^{-i\frac{u_1 - u}{4}} \right).$$

Moreover both  $F$  and  $\Phi$  define a same discrete PS surface.

*Proof.* We first rephrase the matrices  $U$  and  $V$  in (1.12) as

$$U = \begin{pmatrix} e^{-\frac{i}{2}(u_1 - u)} \cos \frac{p^\lambda}{2} & i \sin \frac{p^\lambda}{2} \\ i \sin \frac{p^\lambda}{2} & e^{\frac{i}{2}(u_1 - u)} \cos \frac{p^\lambda}{2} \end{pmatrix}, \quad V = \begin{pmatrix} \cos \frac{q^\lambda}{2} & -ie^{-\frac{i}{2}(u_2 + u)} \sin \frac{q^\lambda}{2} \\ -ie^{\frac{i}{2}(u_2 + u)} \sin \frac{q^\lambda}{2} & \cos \frac{q^\lambda}{2} \end{pmatrix},$$

where  $p^\lambda = 2 \arccos \left( 1/\sqrt{1 + (p/2)^2 \lambda^2} \right)$  and  $q^\lambda = 2 \arccos \left( 1/\sqrt{1 + (q/2)^2 \lambda^{-2}} \right)$ . Let us compute the moving frame equations for  $\Phi G$ :

$$(\Phi G)^{-1}(\Phi G)_1 = G^{-1}LG_1, \quad (\Phi G)^{-1}(\Phi G)_2 = G^{-1}MG_2.$$

Since  $G$  can be rephrased by  $G = R^{(3)}\left(-\frac{u_1 - u}{2}\right)$  and using the property of the rotation matrix, we compute

$$G^{-1}LG_1 = R^{(3)}\left(\frac{u_1 - u}{2}\right)R^{(1)}(-\nu_1)R^{(3)}\left(\frac{u_1 - u}{2}\right), \quad G^{-1}MG_2 = R^{(3)}\left(-\frac{u_2 + u}{2}\right)R^{(1)}(\mu)R^{(3)}\left(\frac{u_2 + u}{2}\right).$$

Then a straightforward computation shows that  $G^{-1}LG_1 = U$  and  $G^{-1}MG_2 = V$  under the identification  $a = p$  and  $b = 4/q$ . Here we use the identity  $\arctan(x) = \arccos(1/\sqrt{1 + x^2})$  for  $x > 0$ . This completes the proof.  $\square$

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