SOME REMARKS ON THE FAST SPATIAL GROWTH/DECAY IN EXTERIOR REGIONS

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Abstract: In this paper we investigate the spatial behavior of the solutions to several partial differential equations/systems for exterior or cone-like regions. Under certain conditions for the equations we prove that the growth/decay estimates are faster than any exponential depending linearly on the distance to the origin. This kind of spatial behavior has not been noticed previously for parabolic problems and exterior or cone-like regions. The results obtained in this work apply in particular for the linear case.

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1. Introduction

Determining the spatial damping of the solutions to a partial differential equation (or a system of PDEs) has become a topic of much interest in a lot of contexts. Probably, the main reason of that interest lies in the attempt to clarify and better understand the Saint-Venant principle in elasticity [31, 32] and in the heat conduction [1] as well as in other thermomechanical situations. The first approaches and contributions to this problem considered the study of equations/systems in a semi-infinite cylinder. The challenge is to obtain the rate of growth/decay of the solutions when the axial variable tends to infinity. It is accepted that for elliptic equations this rate is given by an exponential depending linearly on the distance from the cross-section to the finite end of the cylinder [6]. However, in the linear hyperbolic case, the solutions satisfy a certain *domain of influence* property, i. e. after a finite period of time the solutions cease to exist for a finite distance with respect to the finite end of the cylinder [7, 4]. For linear parabolic problems the rate of decay is faster than for the elliptic problems, but slower than for the hyperbolic ones [12, 13].

In this work we intend to make a step forward in these studies considering exterior or cone-like regions. That is, one would like to see how the solutions growth/decay when the problem is defined on a region which is determined by the exterior of a bounded domain and the distance to the origin increases. It is worth recalling that the first contributions in this line were proposed by Horgan and Payne [10, 11] and Knops, Rionero and Payne [16] who restricted their attention to the elasticity (see also [24, 26]). Recently Knops and Quintanilla [14] obtained estimates for the static thermoelasticity in a class of problems where the cross-section becomes unbounded,

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that is, they considered elliptic systems. Bofill and Quintanilla [2] extended the results for the linear hyperbolic problem to exterior or cone-like regions and they proved that the behavior of the solutions is similar to the one obtained for cylinders. In other words: the decay is extremely fast. As far as we know few attention has been paid to this problem in the parabolic case [15, 25].

In this contribution we study the spatial behavior of the solutions to a class of parabolic problems in exterior (or cone-like) regions. Our main purpose is to give growth/decay estimates for these solutions. To this end we mainly use the weighted energy method. The use of this method is not a novelty: it has been used by many authors for continuum thermomechanics problems [8, 9, 3]. However, we consider the possibility that the usual ω parameter introduced in this approach tends toward infinity.

As a consequence, we obtain that the rate of growth/decay of the solutions is faster than any exponential depending on the distance to the origin in a linear way. This is a fact that is worth underlying.

2. Preliminaries

In this section we describe the basic elements that we need to propose our problems.

Here, and from now on, we use the usual summation convention, that is summation over repeated indices is assumed. We also use the standard notation in which a colon followed by a subscript means the partial derivation with respect to the corresponding coordinate.

We consider an *exterior region* $\Omega \subset \mathbb{R}^3$. We recall here that such a region can be seen as the complementary of a ball of radius $R_0 > 0$ centred at the origin. We intend to study the spatial evolution of the solutions as the distance to the origin $r = (x_i x_i)^{1/2}$ increases.

First, we study the problem determined by the equation

$$c(\mathbf{x})u_t = (\rho(\mathbf{x}, t, u, u_{,i})u_{,i})_{,i}, \tag{2.1}$$

with the initial data

$$u(\mathbf{x}, 0) = 0, \ \mathbf{x} \in \Omega, \tag{2.2}$$

and the boundary data

$$u(\mathbf{x}, t) = f(\mathbf{x}, t), \quad \text{when } x_i x_i = R_0^2. \tag{2.3}$$

In this note we develop the analysis for exterior regions, but it is worth noting that the analysis and results can be adapted without difficulties to cone-like regions or even cylinder regions whenever we impose homogeneous Dirichlet boundary conditions on the lateral face of the region.

Although it is usual the use of the Poincaré inequality on the cross-sections in the studies of the spatial behavior, we note that our approach does not make use of this inequality.

The problem proposed here and the ones we will study later are *ill-posed* in the sense of Hadamard. It is clear that uniqueness of solutions cannot be guaranteed. In general it is possible to obtain solutions to our problem with different behavior at the infinity to the same initial-boundary problem. Nevertheless, we here are looking for Phragmen-Lindelof alternatives for the solutions whenever the existence is assumed.

In some cases, adding appropriate constrains to the solutions a well posed problem can be set. It is what happens, for instance, if we impose that the solutions tend to zero at the infinity.

Therefore, the existence and uniqueness of solutions can be easily obtained (at least) in the linear cases (see for example [5, 29]).

We consider the problem in the three-dimensional case, but in fact our analysis is independent of the dimension and it can be conveniently adapted to one or two dimensions.

We will distinguish several situations. In the next section we will work with the previously stated problem in a general way. In Section 4 we will restrict our attention to a subclass of equations and, using a quite different argument, an alternative measure is proposed. In Section 5 we consider another class of equations related with viscoelastic problems (also type III heat conduction). A parabolic phase-lag equation is also considered in a remark. Finally, in Section 6 we deal with a system of equations that arises from the thermoelasticity. It is worth noting that even in this case, where the system is a combination of parabolic and hyperbolic equations, our arguments can still be used. A remark concerning phase-lag thermoelasticity is also included. We finish the contribution by giving some conclusions.

3. First case

Consider the problem determined by equation (2.1), with initial condition (2.2) and boundary conditions (2.3). We will suppose that the thermal capacity and the thermal conductivity satisfy the following conditions:

(a) $0 < c_0 \leq c(\mathbf{x})$.

(b) $0 < \rho \leq M < \infty$,

for some positive real numbers c_0 and M.

Lin and Payne [20] imposed also condition (b). They also studied the two dimensional problem at [19]. They restricted their attention to cylinders or strips. We will see that our approach allows to improve the rate of growth/decay.

To illustrate the fact that these two conditions are by no means exceptional, we give a couple of examples.

Example 3.1. In the first example we consider the linear problem, meanwhile the second corresponds to a nonlinear situation.

- If $c(\mathbf{x})$ and ρ are both positive constants, therefore we are facing the well known Fourier heat equation
$$cu_t = \rho \triangle u.$$
Of course, conditions (a) and (b) are satisfied in this case.
- If $c(\mathbf{x})$ is a positive constant and
$$\rho = \frac{\nu}{\sqrt{1 + \nu^* |\frac{\nabla u}{u}|^2}}$$
where ν and ν^* are two positive constants, therefore we obtain the so-called *relativistic heat equation*. As in the previous example, it is clear that conditions (a) and (b) are satisfied.

We start the process of giving an estimate for the spatial behavior of the solutions to the problem (2.1), (2.2) and (2.3) by considering for each positive constant ω, the function

$$F_{\omega}(r,t)=\int_0^t\int_{D(r)}\exp(-2\omega s)\rho u u_{,i}\frac{x_i}{r}dads. \tag{3.1}$$

Here, and from now on, $D(r)=\{\mathbf{x}, x_i x_i=r^2\}$ is the exterior surface to the sphere of radius r centred at the origin.

If we apply the Divergence theorem and take into account the differential equation and the initial data we obtain

$$F_{\omega}(r+h,t)-F_{\omega}(r,t)=\frac{\exp(-2\omega t)}{2}\int_{B(r+h,r)}c(\mathbf{x})u^2dv \tag{3.2}$$

$$+\int_0^t\int_{B(r+h,r)}\exp(-2\omega s)(\rho|\bigtriangledown u|^2+\omega c(\mathbf{x})u^2)dvds,$$

where $B(r+h,r)=\{\mathbf{x}, r<(x_i x_i)^{1/2}<r+h\}$.

Therefore, we obtain that

$$\frac{\partial F_{\omega}}{\partial r}=\frac{\exp(-2\omega t)}{2}\int_{D(r)}c(\mathbf{x})u^2da+\int_0^t\int_{D(r)}\exp(-2\omega s)(\rho|\bigtriangledown u|^2+\omega c(\mathbf{x})u^2)dads. \tag{3.3}$$

We now want to estimate the absolute value of F_{ω} in terms of the spatial derivative of the function.

To this end, we use the Holder inequality, the arithmetic-geometric mean inequality and the assumptions on the constitutive functions to obtain the following inequalities:

$$|F_{\omega}(r,t)|\leq\left(\int_0^t\int_{D(r)}\exp(-2\omega s)\rho^2|\bigtriangledown u|^2dads\right)^{1/2}\left(\int_0^t\int_{D(r)}\exp(-2\omega s)u^2dads\right)^{1/2} \tag{3.4}$$

$$\leq\left(\frac{M}{\omega c_0}\right)^{1/2}\left(\int_0^t\int_{D(r)}\exp(-2\omega s)\rho|\bigtriangledown u|^2dads\right)^{1/2}\left(\int_0^t\int_{D(r)}\exp(-2\omega s)\omega c u^2dads\right)^{1/2}$$

$$\leq\left(\frac{M}{4\omega c_0}\right)^{1/2}\left(\int_0^t\int_{D(r)}\exp(-2\omega s)(\rho|\bigtriangledown u|^2+\omega c u^2)dads\right)$$

$$\leq\left(\frac{M}{4\omega c_0}\right)^{1/2}\frac{\partial F_{\omega}}{\partial r}.$$

This kind of inequality is well known in the study of spatial estimates. In particular it implies that

$$F_{\omega}(r,t)\leq\left(\frac{M}{4\omega c_0}\right)^{1/2}\frac{\partial F_{\omega}}{\partial r}, \tag{3.5}$$

and

$$-F_{\omega}(r,t)\leq\left(\frac{M}{4\omega c_0}\right)^{1/2}\frac{\partial F_{\omega}}{\partial r}. \tag{3.6}$$

The consequences of these two inequalities are also known (see [6]). If there exists a value r_0 such that $F_\omega(r_0,t)>0$, then the function $F_\omega(r,t)$ is increasing from r_0 in the spatial variable. Hence, it can be proved that the function

$$F^*_\omega(r,t)=\frac{\exp(-2\omega t)}{2}\int_{B(r,R_0)} cu^2dv+\int_0^t\int_{B(r,R_0)}\exp(-2\omega s)(\rho|\bigtriangledown u|^2+\omega cu^2)dvds \tag{3.7}$$

blows up as

$$\exp\left[\left(\frac{4\omega c_0}{M}\right)^{1/2} r\right]. \tag{3.8}$$

On the contrary, if $F_\omega(r,t)\le 0$ for every $r\ge R_0$, we obtain that

$$-F_\omega(r,t)\le -F_\omega(R_0,t)\exp\left[\left(-\frac{4\omega c_0}{M}\right)^{1/2}(r-R_0)\right]. \tag{3.9}$$

These two results represent a classical alternative of Phragmen-Lindelof type.

It is worth recalling that $F_\omega(r,t)$ tends to zero when r tends to infinity, then if we introduce the function

$$E_\omega(r,t)=\frac{\exp(-2\omega t)}{2}\int_{B(r)} cu^2dv+\int_0^t\int_{B(r)}\exp(-2\omega s)(\rho|\bigtriangledown u|^2+\omega cu^2)dvds, \tag{3.10}$$

where $B(r)=\{\mathbf{x},x_ix_i>r^2\}$, the estimate (3.9) can be written as

$$E_\omega(r,t)\le E_\omega(R_0,t)\exp\left[\left(-\frac{4\omega c_0}{M}\right)^{1/2}(r-R_0)\right]. \tag{3.11}$$

We summarize the above analysis in the following statement.

Theorem 3.2. *Let $u(\mathbf{x},t)$ be a solution to the problem determined by equation (2.1), the initial conditions (2.2) and the boundary condition (2.3). Let us also assume that conditions (a) and (b) hold. Therefore $u(\mathbf{x},t)$ either blows up as (3.8) for the expression (3.7) or the decay estimate (3.11) holds.*

As a consequence, whenever the estimate (3.11) holds the function

$$\mathcal{E}(r,t)=\frac{1}{2}\int_{B(r)} cu^2dv+\int_0^t\int_{B(r)}\rho|\bigtriangledown u|^2dvds, \tag{3.12}$$

satisfies the estimate

$$\mathcal{E}(r,t)\le E_\omega(R_0,t)\exp(2\omega t)\exp\left[\left(-\frac{4\omega c_0}{M}\right)^{1/2}(r-R_0)\right]. \tag{3.13}$$

Notice that this estimate works for a fixed time. We can see that the term $\exp(2\omega t)$ only affects the amplitude term. The spatial decay is determined by the exponential depending on the variable r. A similar comment applies for the decay estimates obtained later in this paper.

Remark 3.3. Notice also that when $R_0\to 0$ and that we are considering bounded solutions, therefore it must be $u(\mathbf{x},t)\equiv 0$.

Remark 3.4. We note that the rate of growth/decay obtained previously depends on the square root of ω. But as ω is an arbitrary positive constant we have that the rate will be bigger than every exponential depending linearly on the distance to the origin. This is a striking fact because what it is usual in many problems defined in exterior regions is that the rate of growth/decay should be of polynomial type. The result about growth is new in the linear case for cylinders. The rate of decay is not new in the case of cylinders (see [12, 13]), but as far as the author knows there were no results concerning the fact that the rate of growth is faster than every exponential depending linearly on the distance to the origin.

We note that when the solution blows-up we see that

$$F_1^*(r,t) \geq \omega^{-1} F_\omega^*(r,t), \tag{3.14}$$

assuming that $\omega \geq 1$, which implies that

$$\lim_{r\to\infty} \exp(-\lambda r) F_1^*(r,t) > 0, \tag{3.15}$$

for every positive λ. At the same time inequality (3.13) implies that when the solution converges to zero the asymptotic behavior satisfies

$$\lim_{r\to\infty} \exp(\lambda r) \mathcal{E}(r,t) = 0, \tag{3.16}$$

for every positive λ. These two limits show that the asymptotic behavior of the solutions cannot be controlled by any linear expression of the exponential of the distance with respect to the origin, but the rate of growth/decay should be faster.

Remark 3.5. The analysis proposed in this section can be adapted to study the equation

$$c(\mathbf{x})u_t = (\rho_{ij}(\mathbf{x}, t, u, u_{,i})u_{,i})_{,j}, \tag{3.17}$$

whenever $c(\mathbf{x})$ satisfies the condition (a) and the function ρ_{ij} satisfy the inequality

$$\rho_{ij}\rho_{kj}u_{,i}u_{,k} \leq M\rho_{ij}u_{,i}u_{,j},$$

where M is a positive constant.

In this case we may define

$$F_\omega(r,t) = \int_0^t \int_{D(r)} \exp(-2\omega s)\rho_{ij}uu_{,i}\frac{x_j}{r}dads. \tag{3.18}$$

We see that

$$\frac{\partial F_\omega}{\partial r} = \frac{\exp(-2\omega t)}{2}\int_{D(r)} c(\mathbf{x})u^2 da + \int_0^t \int_{D(r)} \exp(-2\omega s)(\rho_{ij}u_{,i}u_{,j} + \omega c(\mathbf{x})u^2)dads. \tag{3.19}$$

If we note that

$$|F_\omega(r,t)| \leq \left(\int_0^t \int_{D(r)} \exp(-2\omega s)\rho_{ij}\rho_{kj}u_{,i}u_{,k}dads\right)^{1/2} \left(\int_0^t \int_{D(r)} \exp(-2\omega s)u^2 dads\right)^{1/2}, \tag{3.20}$$

it is clear that we can adapt the previous analysis to this case.

An easy example of equation satisfying the conditions proposed in Remark 3.5 is the linear heat equation

$$c(\mathbf{x})u_t = (k_{ij}(\mathbf{x})u_{,i})_{,j},$$

where k_{ij} is a symmetric, upper bounded and positive definite tensor.

4. Second case

In this section we consider a particular subclass of equation (2.1). We still assume condition (a), but now we suppose that ρ only depends on the variables $\mathbf{x}$ and $|\bigtriangledown u|^2$. That is, our equation can be written as

$$c(\mathbf{x})u_t = (\rho(\mathbf{x}, |\bigtriangledown u|^2)u_{,i})_{,i}. \tag{4.1}$$

Instead of assumption (b) we need to impose:

(b*) $0 < \rho$ and $\rho^2 |\bigtriangledown u|^2 \leq MW(|\bigtriangledown u|^2)$, where M is a positive constant and

$$W(|\bigtriangledown u|^2) = \int_0^{|\bigtriangledown u|^2} \rho(\varsigma)d\varsigma.$$

As before, there are interesting examples in the literature of functions satisfying condition (b*).

Example 4.1. Let us give a couple of them.

- If ρ is defined as
$$\rho(|\bigtriangledown u|^2) = \frac{1}{\sqrt{1 + |\bigtriangledown u|^2}},$$
it is not difficult to see that
$$W(|\bigtriangledown u|^2) \geq \rho^2 |\bigtriangledown u|^2,$$
and condition (b*) is satisfied with $M = 1$.
- Let us consider
$$\rho = \left(1 + \frac{b}{n}|\bigtriangledown u|^2\right)^{n-1},$$
where $b > 0$ and $0 < n \leq 1$. The interested reader can see that the condition (b*) is satisfied (see [27]).

In this case the analysis starts by considering the function

$$G_\omega(r, t) = \int_0^t \int_{D(r)} \exp(-2\omega s)\rho u_s u_{,i} \frac{x_i}{r} dads. \tag{4.2}$$

It is worth noting that

$$\frac{d}{dt} W(\mathbf{x}, |\bigtriangledown u|^2) = 2\rho(\mathbf{x}, |\bigtriangledown u|^2)u_{,i}u_{,it},$$

Therefore, we have that

$$G_\omega(r + h, t) - G_\omega(r, t) = \frac{\exp(-2\omega t)}{2} \int_{B(r+h,r)} W(|\bigtriangledown u|^2)dv + \int_0^t \int_{B(r+h,r)} \exp(-2\omega s)(\omega W(|\bigtriangledown u|^2) + c(\mathbf{x})u_s^2)dvds. \tag{4.3}$$

Therefore

$$\frac{\partial G_\omega}{\partial r} = \frac{\exp(-2\omega t)}{2} \int_{D(r)} W(|\bigtriangledown u|^2)da + \int_0^t \int_{D(r)} \exp(-2\omega s)(\omega W(|\bigtriangledown u|^2) + c(\mathbf{x})u_s^2)dads. \tag{4.4}$$

As in the previous section we can see that

$$|G_\omega(r,t)| \leq \left(\int_0^t \int_{D(r)} \exp(-2\omega s)\rho^2 |\bigtriangledown u|^2 dads\right)^{1/2} \left(\int_0^t \int_{D(r)} \exp(-2\omega s) u_s^2 dads\right)^{1/2} \tag{4.5}$$

$$\leq \left(\frac{M}{\omega c_0}\right)^{1/2} \left(\int_0^t \int_{D(r)} \exp(-2\omega s)\omega W(|\bigtriangledown u|^2) dads\right)^{1/2} \left(\int_0^t \int_{D(r)} \exp(-2\omega s) c(\mathbf{x}) u_s^2 dads\right)^{1/2}$$

$$\leq \left(\frac{M}{4\omega c_0}\right)^{1/2} \frac{\partial G_\omega}{\partial r}.$$

We can follow, *mutatis mutandis*, the same argument given in Section 3 to obtain a Phragmen-Lindelof alternative.

In fact, in this case we obtain that the solutions satisfy the asymptotic behavior

$$\lim_{r\to\infty} \exp(-\lambda r) \left(\int_{B(r,R_0)} W(|\bigtriangledown u|^2) dv + \int_0^t \int_{B(r,R_0)} (W(|\bigtriangledown u|^2) + c(\mathbf{x}) u_s^2) dvds\right) > 0, \tag{4.6}$$

for every positive λ, or the decay estimate

$$\mathcal{E}^*(r,t) \leq \mathcal{E}^*(R_0,t)\exp(2\omega t)\exp\left[\left(-\frac{4\omega c_0}{M}\right)^{1/2} (r - R_0)\right], \tag{4.7}$$

for every positive ω, where

$$\mathcal{E}^*(r,t) = \frac{1}{2}\int_{B(r)} W(|\bigtriangledown u|^2) dv + \int_0^t \int_{B(r)} (\omega W(|\bigtriangledown u|^2) + c(\mathbf{x}) u_s^2) dvds. \tag{4.8}$$

Hence we have proved the following result.

Theorem 4.2. *Let $u(\mathbf{x},t)$ be a solution to the problem determined by equation (4.1), the initial conditions (2.2) and the boundary condition (2.3). Let us also assume that conditions (a) and (b*) hold. Therefore $u(\mathbf{x},t)$ either satisfies the asymptotic condition (4.6) or the decay estimate (4.7) holds.*

It is worth noting that the decay estimate also implies that

$$\lim_{r\to\infty} \exp(\lambda r) \left(\frac{1}{2}\int_{B(r)} W(|\bigtriangledown u|^2) dv + \int_0^t \int_{B(r)} (W(|\bigtriangledown u|^2) + c(\mathbf{x}) u_s^2) dvds\right) = 0, \tag{4.9}$$

for every positive λ.

Remark 4.3. The equation

$$c(\mathbf{x}) u_t = (k_{ij}(\mathbf{x}) u_{,i})_{,j},$$

where k_{ij} is a symmetric, upper bounded and positive definite tensor also admits a similar treatment.

If we define

$$G_\omega(r,t) = \int_0^t \int_{D(r)} \exp(-2\omega s) k_{ij} u_s u_{,i} \frac{x_j}{r} dads, \tag{4.10}$$

we have that

$$\frac{\partial G_\omega}{\partial r} = \frac{\exp(2\omega t)}{2} \int_{D(r)} k_{ij} u_{,i} u_{,j} da + \int_0^t \int_{D(r)} \exp(-2\omega s)(\omega k_{ij} u_{,i} u_{,j} + c(\mathbf{x}) u_s^2) dads. \tag{4.11}$$

Therefore we can adapt the previous analysis to this situation.

5. Viscoelastic type

In this section we extend the previous results to equations coming from viscoelastic problems. To be precise, we consider equations of the type

$$c(\mathbf{x})u_{tt} = (\rho(\mathbf{x}, |\bigtriangledown u|^2)u_{,i})_{,i} + \gamma u_{,jjt}, \tag{5.1}$$

where $c(\mathbf{x})$ satisfies condition (a), $\rho(\mathbf{x}, |\bigtriangledown u|^2)$ satisfies condition (b*) and γ is a positive constant. We note that this kind of equations are also present in the study of the type III heat conduction. The linear version of the type III heat conduction satisfies our assumptions. We also note that in the two-dimensional case, if ρ is the second function proposed in the example 4.1, then equation (5.1) corresponds to the anti-plane shear dynamics deformations for a subclass of power-law materials.

To study equation (5.1) we impose the initial condition (2.1) and also

$$u_t(\mathbf{x}, 0) = 0, \;\; \mathbf{x} \in \Omega. \tag{5.2}$$

The analysis starts with the function

$$M_\omega(r,t) = \int_0^t \int_{D(r)} \exp(-2\omega s)(\rho u_{,i} + \gamma u_{,it}) u_s \frac{x_i}{r} da ds. \tag{5.3}$$

We see that

$$M_\omega(r+h,t) - M_\omega(r,t) = \frac{\exp(-2\omega t)}{2} \int_{B(r+h,r)} (c(\mathbf{x})u_t^2 + W(|\bigtriangledown u|^2)) dv \tag{5.4}$$

$$+ \int_0^t \int_{B(r+h,r)} \exp(-2\omega s)(\omega W(|\bigtriangledown u|^2) + \omega c(\mathbf{x}) u_s^2 + \gamma |\bigtriangledown u|^2) dv ds,$$

and

$$\frac{\partial M_\omega}{\partial r} = \frac{\exp(-2\omega t)}{2} \int_{D(r)} (c(\mathbf{x})u_t^2 + W(|\bigtriangledown u|^2)) da \tag{5.5}$$

$$+ \int_0^t \int_{D(r)} \exp(-2\omega s)(\omega W(|\bigtriangledown u|^2) + \omega c(\mathbf{x}) u_s^2 + \gamma |\bigtriangledown u_s|^2) da ds.$$

We have

$$|M_\omega(r,t)| \leq \left(\int_0^t \int_{D(r)} \exp(-2\omega s)\rho^2 |\bigtriangledown u|^2 da ds \right)^{1/2} \left(\int_0^t \int_{D(r)} \exp(-2\omega s) u_s^2 da ds \right)^{1/2} \tag{5.6}$$

$$+ \left(\int_0^t \int_{D(r)} \exp(-2\omega s)\gamma^2 |\bigtriangledown u_s|^2 da ds \right)^{1/2} \left(\int_0^t \int_{D(r)} \exp(-2\omega s) u_s^2 da ds \right)^{1/2}$$

$$\leq \left(\frac{2M}{\omega^2 c_0} \right)^{1/2} \left(\int_0^t \int_{D(r)} \exp(-2\omega s)\omega W(|\bigtriangledown u|^2) da ds \right)^{1/2} \left(\int_0^t \int_{D(r)} \frac{\omega}{2} \exp(-2\omega s) c(\mathbf{x}) u_s^2 da ds \right)^{1/2}$$

$$+ \left(\frac{2\gamma}{\omega c_0} \right)^{1/2} \left(\int_0^t \int_{D(r)} \exp(-2\omega s)\gamma |\bigtriangledown u_s|^2 da ds \right)^{1/2} \left(\int_0^t \int_{D(r)} \frac{\omega}{2} \exp(-2\omega s) c(\mathbf{x}) u_s^2 da ds \right)^{1/2}$$

$$\leq f(\omega)\frac{\partial M_\omega}{\partial r},$$

where

$$f(\omega)=\max\left(\left(\frac{M}{2\omega^2 c_0}\right)^{1/2},\left(\frac{\gamma}{2\omega c_0}\right)^{1/2}\right). \tag{5.7}$$

As ω can be selected as large as we want, we see that $f(\omega)^{-1}$ becomes unbounded when ω increases. Therefore we obtain another alternative of the type

(5.8)

$$\lim_{r\to\infty}\exp(-\lambda r)\Big(\int_{B(r,R_0)}(c(\mathbf{x})u_t^2+W(|\triangledown u|^2))dv+\int_0^t\int_{B(r,R_0)}(W(|\triangledown u|^2)+c(\mathbf{x})u_s^2+\gamma|\triangledown u_s|^2)dvds\Big)>0,$$

for every positive λ, or the decay estimate

$$\mathcal{E}^{**}(r,t)\leq\mathcal{E}^{**}(R_0,t)\exp(2\omega t)\exp\left(-f(\omega)^{-1}(r-R_0)\right), \tag{5.9}$$

for every positive ω, where

(5.10)

$$\mathcal{E}^{**}(r,t)=\frac{1}{2}\int_{B(r)}(c(\mathbf{x})u_t^2+W(|\triangledown u|^2))dv+\int_0^t\int_{B(r)}(\omega W(|\triangledown u|^2)+\omega c(\mathbf{x})u_s^2+\gamma|\triangledown u_s|^2)dvds.$$

In this case, we have obtained the following theorem.

Theorem 5.1. *Let $u(\mathbf{x},t)$ be a solution to the problem determined by equation (5.1), the initial conditions (2.2) and (5.2) and the boundary condition (2.3). Let us also assume that conditions (a) and (b*) hold. Therefore $u(\mathbf{x},t)$ either satisfies the assymptotic condition (5.8) or the decay estimate (5.9) holds.*

Remark 5.2. If $R_0\to 0$, the only bounded solution is $u(\mathbf{x},t)\equiv 0$.

Remark 5.3. Again the rate of growth/decay obtained in this case is arbitrarily large. We see that the rate will be faster than any exponential depending linearly on the distance to the origin. To be precise we can see that

$$\lim_{r\to\infty}\exp(\lambda r)\Big(\frac{1}{2}\int_{B(r)}(c(\mathbf{x})u_t^2+W(|\triangledown u|^2))dv+\int_0^t\int_{B(r)}(W(|\triangledown u|^2)+c(\mathbf{x})u_s^2+\gamma|\triangledown u_s|^2)dvds\Big)=0,$$

for every positive λ.

Remark 5.4. It is clear that the above analysis can be adapted without difficulties to the equation

$$au_t+bu_{tt}=c\triangle u+\triangle u_t$$

for any positive constants a,b and c. This equation models heat conduction at low temperatures [22].

Remark 5.5. The spatial behavior of the equation

$$u_{tt}+\tau_q u_{ttt}=k^*\triangle u+\tau_\nu^*\triangle u_t+k\tau_u\triangle u_{tt}$$

has been studied for a cylinder whenever k,k^*,τ_q,τ_u and τ_ν^* are positive constants and certain conditions among them are satisfied. We can apply our arguments to this equation by assuming null initial conditions for u,u_t and u_{tt} for every point of the region [28].

In this case the analysis starts by considering the function

$$M_\omega(r,t)=\int_0^t\int_{D(r)}\exp(-2\omega s)\hat{u}_{,i}\tilde{u}\frac{x_i}{r}dads \tag{5.11}$$

where

$$\tilde{u}=u_s+\tau_q u_{ss} \text{ and } \hat{u}=k^*u+\tau_\nu^* u_s+k\tau_u u_{ss}.$$

We have

(5.12)

$$\frac{\partial M_\omega}{\partial r}=\frac{\exp(-2\omega t)}{2}\int_{D(r)}\left((\tilde{u})^2+k^*|\bigtriangledown(u+\tau_q u_t)|^2+(\tau_q(\tau_{\nu^*}-k^*\tau_q)+k\tau_T)|\bigtriangledown u_t|^2\right)da$$

$$+\int_0^t\int_{D(r)}\exp(-2\omega s)(\omega\Xi_1+\Xi_2)dads$$

where

$$\Xi_1=(\tilde{u})^2+k^*|\bigtriangledown(u+\tau_q u_s)|^2+(\tau_q(\tau_{\nu^*}-k^*\tau_q)+k\tau_T)|\bigtriangledown u_s|^2,$$

and

$$\Xi_2=(\tau_\nu^*-k^*\tau_q)|\bigtriangledown u_s|^2+k\tau_T\tau_q|\bigtriangledown u_{ss}|^2.$$

It is clear that the analysis can be adapted here, if $\tau_q(\tau_{\nu^*}-k^*\tau_q)+k\tau_T$ is stricitly positive.

Remark 5.6. The equation

$$c(\mathbf{x})u_{tt}=(k_{ij}^*(\mathbf{x})u_{,i})_{,j}+(k_{ij}(\mathbf{x})u_{,i})_{,jt},$$

where k_{ij} and k_{ij}^* are two symmetric, upper bounded and positive definite tensors admits an analogous analysis.

In fact, by following the ideas proposed in [30] it is possible to extend this argument to equations of the form[1]

$$c_0(\mathbf{x})u+c_1(\mathbf{x})u^{(1)}+...+c_{n+1}(\mathbf{x})u^{(n+1)}=(k_{ij}^0(\mathbf{x})u_{,i})_{,j}+(k_{ij}^1(\mathbf{x})u_{,i}^{(1)})_{,j}+...+(k_{ij}^n(\mathbf{x})u_{,i}^{(n)})_{,j}, \tag{5.13}$$

where $c_n(\mathbf{x})\geq c_0>0$, k_{ij}^n is positive definite and k_{ij}^l are symmetric and upper bounded for every l. In this case we need to assume that the initial conditions for $u,u^{(1)},...u^{(n)}$ are homogeneous.

If we define

$$M_\omega(r,t)=\int_0^t\int_{D(r)}\exp(-2\omega s)\left(k_{ij}^0u_{,i}+k_{ij}^1u_{,i}^{(1)}+...+k_{ij}^nu_{,i}^{(n)}\right)u^{(n)}\frac{x_j}{r}dads, \tag{5.14}$$

we see that

$$\frac{\partial M_\omega}{\partial r}(r,t)=\frac{1}{2}\int_{D(r)}\exp(-2\omega t)c_{n+1}|u^{(n)}|^2da \tag{5.15}$$

$$+\int_0^t\int_{D(r)}\exp(-2\omega s)\left(Q+\omega c_{n+1}|u^{(n)}|^2+k_{ij}^nu_{,i}^{(n)}u_{,j}^{(n)}\right)dads$$

where

$$Q=(k_{ij}^0u_{,i}+k_{ij}^1u_{,i}^{(1)}+...+k_{ij}^{n-1}u_{,i}^{(n-1)})u_{,i}^{(n)}+(c_0(\mathbf{x})u+c_1(\mathbf{x})u^{(1)}+...+c_n(\mathbf{x})u^{(n)})u^{(n)}.$$

[1]Here and from now on $u^{(i)}=\partial^iu/\partial t^i$.

By taking ω large enough and making use of the weigthed Poincaré inequality (see [17]) we can obtain that (see also [30])

$$\frac{\partial M_\omega}{\partial r}(r,t) \geq \frac{1}{2}\int_{D(r)} \exp(-2\omega t) c_{n+1}|u^{(n)}|^2 da \tag{5.16}$$

$$+\frac{1}{2}\int_0^t \int_{D(r)} \exp(-2\omega s)\left(\omega c_{n+1}|u^{(n)}|^2 + k_{ij}^{(n)} u_{,i}^{(n)} u_{,j}^{(n)}\right) dads.$$

An inequality of the type (5.6) can be obtained, where $f(\omega)$ is again a function which tends to zero when ω tends to infinity. Therefore we can see that the rate of growth/decay is faster than any exponential depending linearly on the distance to the origin.

6. Thermoelasticity

We consider now a classical system of equations that appear in the thermoelasticity context. We want to determine the spatial behavior of the solutions to the system of equations given by

$$c(\mathbf{x}) u_{i,tt} = (\frac{\partial \Xi}{\partial e_{ij}})_{,j}, \quad -\frac{d}{dt}\frac{\partial \Xi}{\partial \theta} = (k_{ij}(\mathbf{x})\theta_{,i})_{,j}. \tag{6.1}$$

Here u_i is the displacement vector, θ is the relative temperature, $e_{ij} = (u_{i,j} + u_{j,i})/2$ is the strain tensor, k_{ij} is the thermal conductivity and Ξ plays the role of the free energy and it depends on the material point $\mathbf{x}$, the thermal variable and the strain tensor.

To this system of equations we adjoin the initial conditions:

$$u_i(\mathbf{x},0) = u_{i,t}(\mathbf{x},0) = \theta(\mathbf{x},0) = 0. \tag{6.2}$$

We also assume boundary conditions

$$u_i(\mathbf{x},t) = f_i(\mathbf{x},t), \quad \theta(\mathbf{x},t) = \vartheta(\mathbf{x},t), \quad \text{when } x_i x_i = R_0^2. \tag{6.3}$$

Let us consider the function

$$\Psi(e_{ij},\theta) = \Xi(e_{ij},\theta) - \theta\frac{\partial \Xi}{\partial \theta}, \tag{6.4}$$

which corresponds to the internal energy in the case of linear thermoelasticity.

Through this section we assume again condition (a), that is $0 < c_0 \leq c(\mathbf{x})$. We also assume the symmetry, upper bound and positivity of the thermal conductivity. This last condition says that there exists a positive constant K such that

$$k_{ij}\xi_i\xi_j \geq K\xi_i\xi_i, \tag{6.5}$$

for every vector (ξ_i). Therefore, we can guarantee the existence of $C^* > 0$ such that

$$k_{ij}\xi_i k_{lj}\xi_l \leq C^* k_{ij}\xi_i\xi_j.$$

We also assume that there exists a positive constant C such that

$$\frac{\partial \Xi}{\partial e_{ij}}\frac{\partial \Xi}{\partial e_{ij}} + \theta^2 \leq C\Psi. \tag{6.6}$$

This condition is the natural extension to the thermoelastic context of the one proposed in [7] for the isothermal case. In the linear case it guarantees that the internal energy is positive definite with respect to the strain tensor and the temperature.

In this situation the analysis starts by considering the function

$$L_\omega(r,t)=\int_0^t\int_{D(r)}\exp(-2\omega s)\left(\frac{\partial\Xi}{\partial e_{ij}}u_{i,s}+k_{ij}\theta_{,i}\theta\right)n_j dads. \tag{6.7}$$

We see that

$$L_\omega(r+h,t)-L_\omega(r,t)=\frac{\exp(2\omega t)}{2}\int_{B(r+h,r)}(c(\mathbf{x})u_{i,t}u_{i,t}+\Psi(e_{ij},\theta))dv \tag{6.8}$$
$$+\int_0^t\int_{B(r+h,r)}\exp(-2\omega s)(\omega c(\mathbf{x})u_{i,s}u_{i,s}+\omega\Psi(e_{ij},\theta)+k_{ij}\theta_{,i}\theta_{,j})dvds,$$

and

$$\frac{\partial L_\omega}{\partial r}=\frac{\exp(2\omega t)}{2}\int_{D(r)}(c(\mathbf{x})u_{i,t}u_{i,t}+\Psi(e_{ij},\theta))da \tag{6.9}$$
$$+\int_0^t\int_{D(r)}\exp(-2\omega s)(\omega c(\mathbf{x})u_{i,s}u_{i,s}+\omega\Psi(e_{ij},\theta)+k_{ij}\theta_{,i}\theta_{,j})dads.$$

We have that
(6.10)

$$|L_\omega(r,t)|\leq\left(\int_0^t\int_{D(r)}\exp(-2\omega s)\frac{\partial\Xi}{\partial e_{ij}}\frac{\partial\Xi}{\partial e_{ij}}dads\right)^{1/2}\left(\int_0^t\int_{D(r)}\exp(-2\omega s)u_{i,t}u_{i,t}dads\right)^{1/2}$$
$$+\left(\int_0^t\int_{D(r)}\exp(-2\omega s)k_{ij}\theta_{,i}k_{lj}\theta_{,l}dads\right)^{1/2}\left(\int_0^t\int_{D(r)}\exp(-2\omega s)\theta^2 dads\right)^{1/2}.$$
$$\leq\left(\frac{2C}{c_0\omega^2}\right)^{1/2}\left(\int_0^t\int_{D(r)}\exp(-2\omega s)\frac{\omega}{2}\Psi dads\right)^{1/2}\left(\int_0^t\int_{D(r)}\exp(-2\omega s)\omega c u_{i,t}u_{i,t}dads\right)^{1/2}$$
$$+\left(\frac{2CC^*}{\omega}\right)^{1/2}\left(\int_0^t\int_{D(r)}\exp(-2\omega s)k_{ij}\theta_{,i}\theta_{,j}dads\right)^{1/2}\left(\int_0^t\int_{D(r)}\exp(-2\omega s)\frac{\omega}{2}\Psi dads\right)^{1/2}.$$

If we denote by

$$g(\omega)=\max\left(\left(\frac{C}{2c_0\omega^2}\right)^{1/2},\left(\frac{CC^*}{2\omega}\right)^{1/2}\right),$$

we obtain that

$$|L_\omega|\leq g(\omega)\frac{\partial L_\omega}{\partial r}, \tag{6.11}$$

ω can be selected as large as we want. Therefore, we have that $g(\omega)^{-1}$ becomes unbounded when ω increases. We then obtain the alternative

$$\lim_{r\to\infty}\exp(-\lambda r)\Big(\int_{B(r,R_0)}(c(\mathbf{x})u_{i,t}u_{i,t}+\Psi(e_{ij},\theta))dv \tag{6.12}$$
$$+\int_0^t\int_{B(r,R_0)}(c(\mathbf{x})u_{i,s}u_{i,s}+\Psi(e_{ij},\theta)+k_{ij}\theta_{,i}\theta_{,j})dvds\Big)>0.$$

for every $\lambda>0$, or the decay estimate

$$\mathcal{E}^{***}(r,t) \leq \mathcal{E}^{***}(R_0,t)\exp(2\omega t)\exp\left(-g(\omega)^{-1}(r-R_0)\right), \tag{6.13}$$

is satisfied for every positive ω , where

$$\mathcal{E}^{***}(r,t) = \frac{1}{2}\int_{B(r)}(c(\mathbf{x})u_{i,t}u_{i,t}+\Psi(e_{ij},\theta))dv+\int_0^t\int_{B(r)}(\omega c(\mathbf{x})u_{i,s}u_{i,s}+\omega\Psi(e_{ij},\theta)+k_{ij}\theta_{,i}\theta_{,j})dvds. \tag{6.14}$$

Theorem 6.1. *Let $(u_i(\mathbf{x},t),\theta(\mathbf{x},t))$ be a solution to the problem determined by the system (6.1), the initial conditions (6.2) and the boundary condition (6.3). Therefore $(u_i(\mathbf{x},t),\theta(\mathbf{x},t))$ either satisfies the assymptotic condition (6.12) or the decay estimate (6.13) holds.*

Again as ω can be selected as large as we want we see that

$$\lim_{r\to\infty}\exp(\lambda r)\Big(\frac{1}{2}\int_{B(r)}(c(\mathbf{x})u_{i,t}u_{i,t}+\Psi(e_{ij},\theta))dv+\int_0^t\int_{B(r)}(c(\mathbf{x})u_{i,s}u_{i,s}+\Psi(e_{ij},\theta)+k_{ij}\theta_{,i}\theta_{,j})dvds\Big)=0,$$

for every positive λ.

Remark 6.2. Again, when $R_0 \to 0$ the only bounded solution defined in the whole space is the null solution. On the other hand, the rate of growth/decay of the solutions is also faster than any exponential of a linear expression of the distance to the origin.

Remark 6.3. The system determined by the equations

$$c(\mathbf{x})u_{i,tt} = (C_{ijkl}(\mathbf{x})u_{k,l}-\beta_{ij}(\mathbf{x})\theta)_{,j} \tag{6.15}$$

$$c_0(\mathbf{x})\frac{d}{dt}(a_0\theta+a_1\theta^{(1)}+...+a_n\theta^{(n)})+\beta_{ij}(\mathbf{x})(a_0v_{i,j}+a_1v_{i,j}^{(1)}+...+a_nv_{i,j}^{(n)}) \tag{6.16}$$
$$= (k_{ij}^0(\mathbf{x})\theta_{,i})_{,j}+(k_{ij}^1(\mathbf{x})\theta_{,i}^{(1)})_{,j}+...+(k_{ij}^n(\mathbf{x})\theta_{,i}^{(n)})_{,j},$$

can also be treated in a similar way if $C_{ijkl}=C_{klij}$ is a positive definite tensor, a_n is stricly positive, k_{ij}^n is a symmetric positive definite tensor and k_{ij}^l are symmetric for every l. We assume that (6.2) holds and that $\theta^{(1)},...,\theta^{(n)}$ also vanish at $t=0$[2].

For that case the analysis starts by definining

$$L_\omega(r,t) = \int_0^t\int_{D(r)}\exp(-2\omega s)(L_a^j+L_b^j)\frac{x_j}{r}dads, \tag{6.17}$$

where

$$L_a^j = (C_{ijkl}\tilde{u}_{k,l}-\beta_{ij}\tilde{\theta})\tilde{u}_i,\;\; L_b^j = (k_{ij}^0(\mathbf{x})\theta_{,i}+k_{ij}^1(\mathbf{x})\theta_{,i}^{(1)}+...+k_{ij}^n(\mathbf{x})\theta_{,i}^{(n)})\tilde{\theta},$$

and

$$\tilde{g} = (a_0g+a_1g^{(1)}+...+a_ng^{(n)}).$$

It is not difficult to see that

$$\frac{\partial L_\omega}{\partial r} = \frac{\exp(-2\omega t)}{2}\int_{D(r)}\left(c_0(\mathbf{x})|\tilde{\theta}|^2+c(\mathbf{x})\dot{\tilde{u}}_i\dot{\tilde{u}}_i+C_{ijkl}\tilde{u}_{i,j}\tilde{u}_{k,l}\right)da \tag{6.18}$$
$$+\omega\int_0^t\int_{D(r)}\exp(-2\omega s)\left(c_0(\mathbf{x})|\tilde{\theta}|^2+c(\mathbf{x})\dot{\tilde{u}}_i\dot{\tilde{u}}_i+C_{ijkl}\tilde{u}_{i,j}\tilde{u}_{k,l}\right)dads$$

[2]An existence result for the solutions of this problem can be obtained by adapting the semigroup arguments proposed in [21].

$$+\int_0^t \int_{D(r)} \exp(-2\omega s) \left(Q^* + k_{ij}^n a_n \theta_{,i}^{(n)} \theta_{,j}^{(n)} \right) dads,$$

where

$$Q^* = k_{ij}^0 (a_0 \theta_{,i} \theta_{,j} + a_1 \theta_{,i} \theta_{,j}^{(1)} + ... + a_n \theta_{,i} \theta_{,j}^{(n)}) +$$
$$+ k_{ij}^1 (a_0 \theta_{,i}^{(1)} \theta_{,j} + a_1 \theta_{,i}^{(1)} \theta_{,j}^{(1)} + ... + a_n \theta_{,i}^{(1)} \theta_{,j}^{(n)})$$
$$+ ... +$$
$$+ k_{ij}^n (a_0 \theta_{,i}^{(n)} \theta_{,j} + a_1 \theta_{,i}^{(n)} \theta_{,j}^{(1)} + ... + a_n \theta_{,i}^{(n)} \theta_{,j}^{(n-1)}).$$

Using again the weigthed Poincaré inequality (see [18]) we obtain

$$\frac{\partial L_\omega}{\partial r} \geq \frac{\exp(-2\omega t)}{2} \int_{D(r)} \left(c_0(\mathbf{x}) |\tilde{\theta}|^2 + c(\mathbf{x}) \dot{\tilde{u}}_i \dot{\tilde{u}}_i + C_{ijkl} \tilde{u}_{i,j} \tilde{u}_{k,l} \right) da \tag{6.19}$$
$$+ \omega \int_0^t \int_{D(r)} \exp(-2\omega s) \left(c_0(\mathbf{x}) |\tilde{\theta}|^2 + c(\mathbf{x}) \dot{\tilde{u}}_i \dot{\tilde{u}}_i + C_{ijkl} \tilde{u}_{i,j} \tilde{u}_{k,l} \right) dads$$
$$+ \frac{1}{2} \int_0^t \int_{D(r)} \exp(-2\omega s) k_{ij}^n a_n \theta_{,i}^{(n)} \theta_{,j}^{(n)} dads,$$

for ω large enough. From this point we can obtain again that the growth/decay behavior is faster than any exponential depending linearly on the distance to the origin.

In fact, we can prove the following instability result

$$\lim_{r \to \infty} \exp(-\lambda r) \Big(\frac{1}{2} \int_{B(r,R_0)} \left(c_0(\mathbf{x}) |\tilde{\theta}|^2 + c(\mathbf{x}) \dot{\tilde{u}}_i \dot{\tilde{u}}_i + C_{ijkl} \tilde{u}_{i,j} \tilde{u}_{k,l} \right) da$$
$$+ \int_0^t \int_{B(r,R_0)} \left(c_0(\mathbf{x}) |\tilde{\theta}|^2 + c(\mathbf{x}) \dot{\tilde{u}}_i \dot{\tilde{u}}_i + C_{ijkl} \tilde{u}_{i,j} \tilde{u}_{k,l} \right) dads + \frac{3}{2} \int_0^t \int_{B(r,R_0)} k_{ij}^n a_n \theta_{,i}^{(n)} \theta_{,j}^{(n)} dads \Big) > 0.$$

However to prove the decay result, we need to explain a little bit the analysis. It is clear that

$$\lim_{r \to \infty} \exp(\lambda(\omega) r) \Big(\frac{\exp(-2\omega t)}{2} \int_{B(r)} \left(c_0(\mathbf{x}) |\tilde{\theta}|^2 + c(\mathbf{x}) \dot{\tilde{u}}_i \dot{\tilde{u}}_i + C_{ijkl} \tilde{u}_{i,j} \tilde{u}_{k,l} \right) da$$
$$+ \int_0^t \int_{B(r)} \exp(-2\omega s) \left(c_0(\mathbf{x}) |\tilde{\theta}|^2 + c(\mathbf{x}) \dot{\tilde{u}}_i \dot{\tilde{u}}_i + C_{ijkl} \tilde{u}_{i,j} \tilde{u}_{k,l} \right) dads$$
$$+ \frac{1}{2} \int_0^t \int_{B(r)} \exp(-2\omega s) k_{ij}^n a_n \theta_{,i}^{(n)} \theta_{,j}^{(n)} dads \Big) = 0,$$

where $\lambda(\omega)$ becomes unbounded when ω increases. In particular we can obtain that

$$\lim_{r \to \infty} \exp(\lambda r) \int_0^t \int_{B(r)} k_{ij}^n a_n \theta_{,i}^{(n)} \theta_{,j}^{(n)} dads = 0,$$

for every positive λ and then we see that this measure for the temperature tends very fast to zero. We want to see a similar effect for the mechanical part. To this end it is important to note that

$$\int_0^t \int_{B(r)} \exp(-2\omega s) c(\mathbf{x}) \dot{\tilde{u}}_i \dot{\tilde{u}}_i dvds \geq a_n^2 (1 - \epsilon) \int_0^t \int_{B(r)} \exp(-2\omega s) c(\mathbf{x}) \dot{u}_i^{(n)} \dot{u}_i^{(n)} dvds,$$

after the use of the weighted Poincaré inequality whenever ω is large enough and ϵ is a positive parameter as small as we want. Even more, when ω increases ϵ becomes smaller. We can take for every ω large enough $\epsilon = 1/2$, then we will obtain that

$$\lim_{r\to\infty} \exp(\lambda r) \int_0^t \int_{B(r)} c(\mathbf{x}) \dot{u}_i^{(n)} \dot{u}_i^{(n)} dvds = 0,$$

for every positive λ. In a similar way we can prove that

$$\lim_{r\to\infty} \exp(\lambda r) \int_0^t \int_{B(r)} C_{ijkl} u_{i,j}^{(n)} u_{k,l}^{(n)} dvds = 0,$$

and hence we obtain the alternative of Phragmen-Lindelof type which is again faster than any exponential depending linearly on the distance to the origin.

7. Conclusions

In this paper we have presented several situations where the spatial growth/decay of the solutions is faster than any exponential depending linearly on the distance to the origin. In particular we have shown this effect for several parabolic equations including the Fourier heat conduction, anti-plane shear dynamic deformations for power-law materials and the type III heat conduction, phase-lag heat conduction as well as for the classical system of linear thermoelasticity and the phase-lag thermoelasticity. We emphasize that this behavior differs from the behavior obtained for elliptic equations in exterior regions. We have presented our results and analysis for exterior regions, but the extension to cone-like regions or even cylinder regions is direct whenever we assume homogeneous Dirichlet or Neumann boundary conditions on the lateral side of the cone or cylinder.

It is worth noting that the approach that we propose here has been done other times in the study of linear thermoelastic solids. However it is suitable to say that in our approach the parameter ω is considered as a variable which can be unbounded and then the asymptotic behavior of the solutions becomes faster than any exponential depending linearly on the spatial variable. We want to finish these conclusions by underlying that fact that our results apply (in particular) for linear problems.

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