American and Exotic Options in a Market with Frictions

Gero Junike[a], Argimiro Arratia[b], Alejandra Cabaña[c], Wim Schoutens[d]

[a]Universitat Autònoma de Barcelona, Department of Mathematics, Building C Science Faculty, 08193, Bellaterra (Barcelona), Spain. This research is partially supported by 13th UAB-PIF scholarship and by grant TIN2008-06582-C03-02 from MINECO (Ministerio de Economía, Industria y Competitividad), Spain; [b]Universitat Politécnica de Catalunya, Department of Computer Science, C. Jordi Girona 1-3, 08034, Barcelona, Spain. Supported by grant TIN2017-89244-R from MINECO and the recognition 2017SGR-856 (MACDA) from AGAUR (Generalitat de Catalunya), Spain; [c]Universitat Autònoma de Barcelona, Department of Mathematics, Building C Science Faculty, 08193, Bellaterra (Barcelona), Spain. Supported by grant TIN2008-06582-C03-02 from MINECO, Spain; [d]KU Leuven, Department of Mathematics, Celestijnenlaan 200 B, 3001 Leuven, Belgium.

ARTICLE HISTORY

Abstract

In a market with frictions, bid and ask prices are described by sublinear pricing functionals, which can be defined recursively using coherent risk measures. We prove the convergence of bid and ask prices for various European and American possible path-dependent options, in particular plain vanilla, Asian, lookback and barrier options in a binomial model with transaction costs. We perform several numerical experiments to confirm the theoretical findings. We apply the results to real market data of American options and compute an implied liquidity to describe the bid-ask spread. This method describes liquidity over time very well, compared to the classical approach of describing bid and ask prices by quoting bid and ask implied volatilities.

KEYWORDS

Market with frictions, bid-ask spread, American options, barrier options, binomial model

1. Introduction

In this article, we obtain closed-form solutions of bid and ask prices of European plain vanilla and barrier options in markets with frictions. Markets with frictions are markets with transaction costs. In contrast to complete financial markets without any imperfections where prices are obtained by a *linear* pricing rule, prices in markets with frictions can be described by *sublinear* pricing functionals, see Jouini (2000). Such pricing functionals may also describe prices in markets with additional or different kind of frictions than (proportional) transaction costs, like short sales costs or constrains, borrowing costs, taxes and other market imperfections, see Jouini and Kallal (2001), Koehl and Pham (2000), Bion-Nadal (2009) and references therein.

Jouini and Kallal (1995, 2001) and Jouini (2000) introduced an axiomatic approach to describe financial markets with frictions. They considered a finite time-horizon $T > 0$ and a multiperiod economy, where investors can trade a riskless and a risky asset. Let N be the number of trading periods in $[0, T]$. Jouini (2000) modelled the bid and ask price processes of the risky asset by adapted processes $0 < S_i^b \leq S_i^a$, $i = 0, .., N$. They postulated the existence of a pricing functional p, which satisfies the following axioms: (i) p is monotone, (ii) p is sub-additive, (iii) p is positively homogeneous, (iv) p does not introduce arbitrage, (v) p is lower-semi-continuous and (vi) for a future random cash flow C, $p(C)$ is less than or equal to the price of the smallest self-financing trading strategy dominating C. The ask price of a contingent claim C is defined by $p(C)$, the bid-price is defined by $-p(-C)$, hence buying the contingent claim C is the same as selling $-C$.

Jouini and Kallal (1995) showed that the market is arbitrage-free, if and only if there exist a measure $\mathbb{Q}$, equivalent to the physical measure $\mathbb{P}$, and a process $Z^{\mathbb{Q}}$, which is a martingale under $\mathbb{Q}$, such that $S^b \leq Z^{\mathbb{Q}} \leq S^a$. This leads to an easy construction

of arbitrage-free financial markets with frictions: we take a complete frictionless market, the binomial model, where the risky underlying is described by some martingale $(S_i)_{i=0,..,N}$ under the risk-neutral measure. Introducing a sequence of dynamic coherent risk measures $(\rho_i)_{i=0,..,N}$ and defining a pricing functional by $p_i(.) := \rho_i(-.)$, we introduce frictions into the market by defining the ask price process of the underlying by $(p_i(S_N))$ and the bid price process by $(-p_i(-S_N))$. The sequence (p_i) based on a sequence of coherent risk measures fulfils axioms (i)-(vi).

Contribution

In a binomial-type model with frictions, bid and ask prices are recursively defined. For efficient numerical applications, we prove convergence of bid and ask prices of European and American plain vanilla and exotic options. We develop closed-form solutions for European plain vanilla and some barrier options and obtain in the limit an extended Black-Scholes formula with a new parameter $\gamma \geq 0$, which adjusts the dividend yield. The greater γ, the greater the bid-ask spread.

The limit bid or ask price of a possible path-dependent option is given by the Black-Scholes price of the option but on a stock with an adjusted dividend yield. Hence existing numerical methods, developed to price options in a Black-Scholes setting in classical finance, can also be used to compute bid and ask prices of such options. No new software need to be written to apply our formulas in financial institutions.

Practical Relevance

We think the main application area of our two-price formulas is the possibility of computing *implicitly* a parameter γ, such that given bid and ask market prices of a European or an American plain vanilla option are exactly matched by our extended Black-Scholes formula.

This idea is comparable to the concept of implied volatility. In principle volatilities could be constant across strikes, maturities, and underlying assets, hence the preference by practitioners for quoting implied volatilities instead of (mid-)prices. Similarly the parameter γ could be constant across all three dimensions, even though there are many

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non-linearities between the bid-ask spread and strikes, maturities, and underlying assets. It should therefore be beneficial to quote an implicitly computed γ instead of the absolute bid-ask spread of a plain vanilla option. Indeed Corcuera et al. (2012) used a setting similar to our model, but in static time, and showed empirically that the liquidity dry up during the period 2007-2009 is described very well by the parameter γ. Our discrete time model makes it possible to analyse also path-dependent and American options.

Up to now, traders quote the difference between implied bid and ask volatilities to describe the current market liquidity of plain vanilla options. Both this heuristic method and our proposal of computing implicitly the parameter γ have the advantage of using only present market data and of being extremely fast in terms of computational time, in both cases one has to invert the Black-Scholes formula. However, we show empirically that our model describes (il)liquidity of American plain vanilla options very well over time compared to the heuristic method of quoting implied bid and ask volatilities. In a static setting, a similar study has been done by Guillaume et al. (2018) for European plain vanilla options.

Limitations

Convergence is only proven for monotone payoffs, for example European or American plain vanilla, lookback, Asian and some barrier options (up-and-out put, down-and-out call, down-and-in put and up-and-in call).

The underlying is essentially modelled by a binomial model in discrete time and by geometric Brownian motion in continuous time. Hence the volatility is assumed to be constant over time and log-returns are assumed to be (approximately) normal distributed.

Future research needs to be done to treat contingent claims which are not monotone with respect to the underlying, e.g. a barrier up-and-out call option and to generalize the market model replacing for example the constant volatility by a mean-reverting stochastic process.

Literature Review

In general, liquidity is effected by many factors like the ability of trading large quantities, by the speed, the cost and the price impact of the trade. Several measures have been developed in literature to capture some or all of these factors. Amihud (2002) defines the liquidity of a stock by the average of the ratio of absolute daily returns to volume, where the average is taken over a month. Acharya and Pedersen (2005) developed a liquidity adjusted capital asset pricing model and measured liquidity using a normalized version of Amihud's liquidity measure. Liu (2006) analyse the relation between liquidity risk and asset pricing using a liquidity measure based on historic data. Goyenko, Holden and Trzcinka (2009) compared several well known liquidity measures using stock data from 1993 to 2005.

In contrast to the above studies, which define (il)liquidity mainly using a historic time series of the stock, our model is well suited to be applied to an option surface and needs only present market data to compute the market implied liquidity parameter γ.

Recently, Madan and Cherny (2010) developed the conic finance theory. Our market model with frictions is connected to conic finance by the common approach of using recursively defined sublinear functionals to describe bid and ask prices. Indeed our discrete market model is closely related to discrete time conic finance models, where bid and ask prices are defined recursively using nonlinear expectations, see Leippold and Schärer (2017), Madan (2010), Madan, Pistorius and Schoutens (2013, 2017) and Madan and Schoutens (2012). Time-consistent nonlinear expectations are connected with solutions to backward stochastic difference equations, see Cohen and Elliott (2010). See Bielecki et al. (2013) and Bielecki, Cialenco and Chen (2015) for a framework incorporating transaction costs in discrete time conic finance models.

Our work is related to Madan, Pistorius and Stadje (2017) who showed in a general context that, under some technical conditions, an iterated spectral risk measure, which is a risk measure in a multiperiod setting based on distortion functions, converges to some g-expectation. A g-expectation is a non-linear expectation proposed by Peng (2004).

Relative to these papers our contribution is to proof convergence of bid and ask prices in a binomial-type model with frictions when the number of trading periods approaches infinity and to obtain closed-form solutions for bid and ask prices for plain vanilla and

barrier options in the limit.

Contents

The remainder of the article is organized as follows. In Section 2, we introduce a discrete time-model for a market with frictions. In Section 3, we present the classical binomial model. In Section 4 we prove convergence of bid and ask prices for European and American possibly path-dependent options. In Section 5, we apply the results to real market data. Section 6 concludes.

2. The Formal Setup

We make the following economic assumptions: we assume all investors have a finite time-horizon and trading can take place only finitely many times. There is a very liquid bank account and a risky-asset whose bid and ask prices can be described by binomial trees. There exists a pricing functional and bid and ask prices of a contingent claim can be computed via the pricing functional. At the end of the time-horizon, the bid-ask spread of all products is assumed to be zero.

Formally, we assume the following framework: Let $T > 0$ be some time-horizon and $N \in \mathbb{N}$ be the number of trading periods, each trading period has length $\frac{T}{N}$. We introduce a frictionless market and extend it to a market with frictions using a pricing functional. Let the risky-asset

$$(S_i)_{i=0,1,..,N}$$

be described by a nonnegative adapted stochastic process on a given filtered probability space

$$\left(\Omega, (\mathcal{F}_i)_{i=0,...,N}, \mathcal{F}, \mathbb{P}\right)$$

satisfying the usual conditions. By

$$B_i = (1 + r)^i, \quad i = 0, ..., N,$$

we denote a risk-free bank account. We assume the market is arbitrage-free and denote by $\mathbb{Q}$ a risk-neutral measure, such that the discounted price process of the risky-asset is a $\mathbb{Q}-$martingale. The process (S_i) describes the risky asset of the underlying frictionless market. In this section we assume that the interest rates are equal to zero, i.e. we work with discounted cash flows, and that the stock is not paying any dividends. Those assumptions are only made to keep the notation simple and will be relaxed in Section 3 and 4. Let

$$L^\infty := L^\infty (\Omega, \mathbb{Q}, \mathcal{F})$$

be the set of $\mathcal{F}-$measurable bounded random variables with respect to the probability measure $\mathbb{Q}$ and $(\rho_i)_{i=0,...,N}$ be a set of dynamic, time-consistent coherent risk measures being continuous from above

$$\rho_i : L^\infty \to L_i^\infty := L^\infty (\Omega, \mathbb{Q}, \mathcal{F}_i).$$

We call the operator

$$p_i(.) := \rho_i(-.), \quad i = 0, ..., N$$

a *price functional*. Then (p_i) fulfils the following properties, see Föllmer and Schied (2011, Definition 11.1, Theorem 11.2. and Lemma 11.11). Let $X, Y \in L^\infty$. It holds for $i = 0, ..., N$,

R1: *Cash invariance:* $p_i(X + X_i) = p_i(X) + X_i$ for any $X_i \in L_i^\infty$.

R2: *Monotonicity:* $X \leq Y \Rightarrow p_i(X) \leq p_i(Y)$.

R3: *Sub-additivity:* $p_i(X + Y) \leq p_i(X) + p_i(Y)$.

R4: *Positive homogeneity:* $p_i(\lambda X) = \lambda p_i(X)$, where $\lambda \in L_i^\infty$ and $0 \leq \lambda$.

R5: *Continuity from below*: It holds $X_n \nearrow X \Rightarrow p_i(X_n) \nearrow p_i(X)$ for any sequence $(X_n) \subset L^\infty$.

R6: *Time-consistency:* $p_i(X) = p_i(p_{i+1}(X)), \;\; i = 0, ..., N-1$.

We additionally assume

R7: *No-arbitrage:* $E_\mathbb{Q}[X \,|\, \mathcal{F}_i] \leq p_i(X)$.

Jouini (2000) modelled the risk-free bank account as perfectly liquid. Property R1 states the same: investors may insert or withdraw any amount of cash to or from the risk-free bank account without transaction costs. Properties R2-R5 have been proposed in similar form and are discussed by Jouini and Kallal (1995, 2001) and Jouini (2000). Time-consistency has been introduced by Peng (2004) for nonlinear expectations. It means that prices behave consistently over time: prices can be computed either directly or using an intermediate instant of time, see Bion-Nadal (2009). Property R7 guarantees that the bid-ask spread is always greater or equal to zero and that the market is arbitrage-free, see Proposition 2.1. Our model of bid and ask prices can be seen as a discrete version of the continuous time model via dynamic convex risk-measures developed by Bion-Nadal (2009).

Bid and ask prices of a contingent claim $C \in L^\infty$ at trading period i are defined by

$$\text{bid}_i(C) := -p_i(-C) \;\; \text{and} \;\; \text{ask}_i(C) := p_i(C), \;\; i = 0, ..., N,$$

i.e. as in Jouini and Kallal (1995), Staum (2004) and Bion-Nadal (2009), we consider that selling C is the same as buying $-C$. By property R1, we assume that at the end of the time-horizon the bid-ask spread of C is zero. We therefore do not have to distinguish between contingent claims with asset delivery and cash settlement. Bid and ask prices of the risky asset are then defined by the processes

$$S_i^b := -p_i(-S_N) \;\; \text{and} \;\; S_i^a := p_i(S_N), \;\; i = 0, ..., N.$$

American contingent claims can be described by adapted stochastic processes, bid and ask prices of such claims are defined in Section 2.3.

We call the tuple $\left((B_i), (S_i^b), (S_i^a), (p_i)\right)$ a *security price model*. We show that our security price model does not admit arbitrage. Furthermore, it is not possible to construct a self-financing portfolio, which super-replicates C but can be bought for less than $p_0(C)$. The proof can be found in the appendix.

Proposition 2.1. *The security price model $\left((B_i), (S_i^b), (S_i^a), (p_i)\right)$ admits no arbitrage and the ask price $p_0(C)$ of a contingent claim $C \in L^\infty$ is less or equal to the price of the smallest self-financing trading strategy dominating C.*

2.1. *Concave Distortion Functions*

In this Section, we introduce a parametric model for the pricing functional (p_i). We will allow the pricing functional to depend on a parameter $\gamma \geq 0$ with the following interpretation: the greater γ, the greater the bid-ask spread; for $\gamma = 0$, the spread is equal to zero. To obtain such parametrization, we let the coherent risk measures, defining the pricing functional, be based on a family of concave distortion functions, which is defined as follows:

Definition 2.2. A family of concave distortion functions (FCDF) $(\Psi^\gamma)_{\gamma \geq 0}$ is a set of functions $\Psi^\gamma : [0,1] \to [0,1]$ that are monotonically increasing, continuous and concave for all $\gamma \geq 0$ and for which $\Psi^\gamma(0) = 0$ and $\Psi^\gamma(1) = 1$. Moreover the family is monotonically increasing and continuous at γ, i.e. it holds that $\Psi^{\gamma_1}(u) \leq \Psi^{\gamma_2}(u)$ for $\gamma_1 \leq \gamma_2$ and the map $\gamma \mapsto \Psi^\gamma(u)$ is continuous for all $u \in [0,1]$.

Additional, we make the following two assumptions:

A1 For $\gamma = 0$, the FCDF is equal to the identity, i.e.

$$\Psi^0(u) = u, \ \ u \in [0,1].$$

A2 The total differential of the FCDF exists at the point $(u, \gamma) = \left(\frac{1}{2}, 0\right)$, i.e.

$$\Psi^{\xi_\gamma}\left(\frac{1}{2} + \xi_p\right) = \frac{1}{2} + \xi_u + \frac{1}{2}\xi_\gamma + o\left(|\xi_u| + |\xi_\gamma|\right), \ \ \xi_u \in (0,1), \ \ \xi_\gamma > 0. \tag{1}$$

Equation (1) holds, if the function $(u, \gamma) \mapsto \Psi^\gamma(u)$ is partial differentiable and all partial derivatives at the point $\left(\frac{1}{2}, 0\right)$ are continuous with

$$\frac{\partial}{\partial u} \Psi^\gamma(u)\bigg|_{(u,\gamma)=(\frac{1}{2},0)} = 1 \quad \text{and} \quad \frac{\partial}{\partial \gamma} \Psi^\gamma(u)\bigg|_{(u,\gamma)=(\frac{1}{2},0)} = \frac{1}{2}. \tag{2}$$

Assumption A2 is used to prove convergence of bid and ask prices. By Assumption A1 the FCDF (Ψ^γ) fulfils $\Psi^0(\frac{1}{2}) = \frac{1}{2}$. All FCDF satisfying Assumption A2 are also (approximately) equal in a small neighbourhood around $(u, \gamma) = (\frac{1}{2}, 0)$. Therefore we will see that the particular choice of the FCDF to model the pricing functional in the discrete time model does not matter when the number of trading periods tend to infinity.

Example 2.3. The family of distortion function corresponding to the expected shortfall can be defined by

$$\Psi^\gamma_{\text{ExpShortfall}}(u) = \min(u(1 + \gamma), 1), \quad u \in [0, 1], \quad \gamma \geq 0.$$

The FCDF corresponding to the expected shortfall satisfies Equation (2) because for u close enough to $\frac{1}{2}$ and γ close enough to zero, it holds

$$\Psi^\gamma_{\text{ExpShortfall}}(u) = u + u\gamma.$$

Example 2.4. Let Φ be the cumulative standard normal distribution function and φ the normal density. The WANG-transform

$$\Psi^\gamma_{\text{WANG}}(u) = \Phi\left(\Phi^{-1}(u) + \frac{1}{2\varphi(0)}\gamma\right), \quad u \in [0, 1], \quad \gamma \geq 0,$$

was introduced by Wang (2000) and fulfils Equation (2). The WANG-transform is widely used in actuarial science and was originally defined without the scaling factor $\frac{1}{2\varphi(0)}$.

Many other FCDF known in literature fulfil Assumptions A1 and A2, in particular the FCDF MINVAR, MAXVAR, MINMAXVAR and MAXMINVAR introduced by Cherny

and Madan (2008).

2.2. *Coherent Risk measures induced by Concave Distortion Functions*

Given a FCDF (Ψ^γ), the map

$$\rho^\gamma(X) := \int\limits_0^\infty \left(\Psi^\gamma\left(\mathbb{Q}\left[X < y\right]\right) - 1\right) dy + \int\limits_{-\infty}^0 \Psi^\gamma\left(\mathbb{Q}\left[X < y\right]\right) dy, \quad \gamma \geq 0 \qquad (3)$$

defines a static coherent risk measure in the sense of Artzner et al. (1999), see Föllmer and Schied (2011, Theorem 4.70). The greater γ, the more conservative the risk measure. As in Madan, Pistorius and Schoutens (2013, 2017) and Leippold and Schärer (2017), we generalize this risk measure to the dynamic case. For $i = 0, ..., N$ let

$$\tilde{\rho}_i^\gamma(X) := \int\limits_0^\infty \left(\Psi^\gamma\left(\mathbb{Q}_i\left[X < y\right]\right) - 1\right) dy + \int\limits_{-\infty}^0 \Psi^\gamma\left(\mathbb{Q}_i\left[X < y\right]\right) dy, \quad \gamma \geq 0,$$

where

$$\mathbb{Q}_i[A] := E_\mathbb{Q}\left[1_A \,\middle|\, \mathcal{F}_i\right], \quad i = 0, ..., N, \quad A \in \mathcal{F},$$

is a conditional probability. Define

$$\rho_N^\gamma := \tilde{\rho}_N^\gamma$$

and recursively

$$\rho_i^\gamma := \tilde{\rho}_i^\gamma(-\rho_{i+1}^\gamma), \quad i = 0, ..., N-1, \quad \gamma \geq 0.$$

The pricing functional used in this article is then defined by

$$p_i^\gamma(.) := \rho_i^\gamma(-.), \quad i = 0, ..., N, \gamma \geq 0. \qquad (4)$$

The recursive definition makes the pricing functional time-consistent. By assumption A1 it holds

$$E_{\mathbb{Q}}[X \mid \mathcal{F}_i] = p_i^0(X).$$

The parameter $\gamma \geq 0$ describes the liquidity of the market: the greater γ, the greater the bid-ask spread. For $\gamma = 0$, bid and ask prices coincide and are identical to the risk neutral price operator.

Due to the time-consistency, for a fixed $\gamma_N \geq 0$, bid and ask prices of a future random cash flow $C^E \in L^\infty$ can be obtained by recursions:

$$
\begin{aligned}
\operatorname{bid}_N\left(C^E\right) &= \operatorname{ask}_N\left(C^E\right) = C^E, \\
\operatorname{bid}_i\left(C^E\right) &= -p_i^{\gamma_N}\left(-\operatorname{bid}_{i+1}\left(C^E\right)\right), \quad i = 0, .., N-1, \\
\text{and} \quad \operatorname{ask}_i\left(C^E\right) &= p_i^{\gamma_N}\left(\operatorname{ask}_{i+1}\left(C^E\right)\right), \quad i = 0, .., N-1.
\end{aligned}
\tag{5}
$$

We explicitly allow the parameter γ_N, which describes the bid-ask spread in the Nth model, to depend on N, in order to obtain convergence results for $N \to \infty$.

Figure 1. Binomial tree to compute the ask price. The up-move probability depends on the sorting of the successive nodes.

Remark 1. The bid and ask prices are recursively defined and can be computed going backwards through a tree. Figure 1 shows a binomial tree with $N = 2$ time-steps. The ask prices at the final nodes are equal to the value of the option at expiration. The ask price at the first node (today) can be computed going iteratively through the tree

using the recursions (5). For example the ask price a_{11} can be computed using the two successive nodes a_{21} and a_{22}.

$$a_{11} = \begin{cases} \Psi^\gamma(1-p)a_{22} + (1 - \Psi^\gamma(1-p))a_{21} & , a_{21} \le a_{22} \\ (1 - \Psi^\gamma(p))a_{22} + \Psi^\gamma(p)a_{21} & , a_{21} > a_{22}, \end{cases}$$

where p denotes the up-move probability in a classical binomial model and (Ψ^γ) is a FCDF. The formula is deduced from the definition of pricing functional, see Equation (4). In contrast to the iterative computation of the risk-neutral price in the classical binomial model, the bid and ask prices depend on the *sorting* of the successive nodes. Therefore in this article we only prove convergence for monotone payoffs, which are precisely defined in Section 2.3. Bid and ask prices of general payoffs can be computed in the discrete time model going backwards through the tree and checking at each node the sorting of the two successive nodes.

2.3. *Payoffs*

The main goal of this article is to prove convergence of bid and ask prices of different European and American contingent claims, when the number of trading periods approaches infinity. We focus on contingent claims which are monotonically increasing or decreasing with respect to the underlying, this has a technical reason, see Remark 1. In the following we precisely define increasing and decreasing European and American contingent claims and provide a selection of examples.

Definition 2.5. A *European contingent claim* C^E is a bounded random variable on $(\Omega, \mathcal{F})$, such that there is a measurable function h, with

$$C^E = h(S_0, ..., S_N).$$

The claim is called *increasing* if

$$h(x_0, ..., x_N) \ge h(y_0, ..., y_N), \quad x_i \ge y_i, \quad i = 0, ..., N$$

and *decreasing* if

$$h(x_0, ..., x_N) \leq h(y_0, ..., y_N), \quad x_i \geq y_i, \quad i = 0, ..., N.$$

An *American contingent claim* C^A is a bounded adapted process

$$C^A = \left(C_i^A \right)_{i=0,..,N},$$

such that for each i there is a measurable function h_i, with

$$C_i^A = h_i(S_0, ..., S_i).$$

The claim is called *increasing* if

$$h_i(x_0, ..., x_i) \geq h_i(y_0, ..., y_i), \quad x_k \geq y_k, \quad i = 0, ..., N, \quad , k = 0, ..., i$$

and *decreasing* if

$$h_i(x_0, ..., x_i) \leq h_i(y_0, ..., y_i), \quad x_k \geq y_k, \quad i = 0, ..., N, \quad , k = 0, ..., i.$$

A European claim C^E can be interpreted as a random payoff at maturity T. For each i, the random variable C_i^A is interpreted as the payoff of the American contingent claim if the claim is exercised after i trading periods. We assume the American option is cash-settled, and the reference price is the process (S_i). If the holder of an American option exercises the option early after i trading periods, she will receive the amount $h_i(S_0, ..., S_i)$, which is independent of the current bid-ask spread or the processes (S_i^b) and (S_i^a). This may in particular hold for cash-settled index options and it holds approximately for options with physically delivery if the transaction costs of trading the stock are small. Similar to European contingent claims, see Equation (5), bid and ask prices of an American contingent claim C^A can be defined recursively, incorporating the

possibility of an early exercise:

$$
\begin{aligned}
\mathrm{bid}_N(C^A) \ &= \mathrm{ask}_N(C^A) = C_N^A, \\
\mathrm{bid}_i(C^A) \ &= C_i^A \vee -p_i^{\gamma_N}(-\mathrm{bid}_{i+1}(C^A)), \ \ i = 0,..,N-1, \\
\text{and} \ \ \mathrm{ask}_i(C^A) \ &= C_i^A \vee p_i^{\gamma_N}(\mathrm{ask}_{i+1}(C^A)), \ \ i = 0,..,N-1.
\end{aligned}
\tag{6}
$$

Let Ξ be the set all American contingent claims. To simplify notation, the operators assigning bid and ask prices to American contingent claims

$$
\mathrm{bid}_i : \Xi \to L_i^\infty \ \ \text{and} \ \ \mathrm{ask}_i : \Xi \to L_i^\infty,
$$

have the same names as the operators describing prices of European contingent claims, which can be seen as functionals from L^∞ to L_i^∞.

We provide some examples of European and American contingent claims. Let $K \geq 0$ be a strike price and $B \geq 0$ be a barrier. By $\mathcal{N}$ we denote the time, the option is exercised. If $\mathcal{N} \in \{0, ..., N\}$ can be chosen by the holder of the option, we speak of an American contingent claim, exercised at time $\mathcal{N}$. If only $\mathcal{N} = N$ is allowed, i.e. the option can only be exercised at maturity, we speak of a European contingent claim.

Example 2.6. The following derivatives are increasing contingent claims.

- Call option: $C_{\mathrm{Call}} = (S_{\mathcal{N}} - K)^+$
- Lookback call option: $C_{\mathrm{LbCall}} = \left(\max\limits_{i \in \{0,...,\mathcal{N}\}} S_i - K \right)^+$
- Asian call option: $C_{\mathrm{AsianCall}} = \left(\frac{1}{\mathcal{N}} \sum_{i=0}^{\mathcal{N}} S_i - K \right)^+$
- Barrier up-and-in call option:

$$
C_{\mathrm{UICall}} =
\begin{cases}
(S_{\mathcal{N}} - K)^+ & , \ \max\limits_{i \in \{0,...,\mathcal{N}\}} S_i \geq B \\
0, & \text{otherwise}
\end{cases}
$$

- Barrier down-and-out call option:

$$C_{\text{DOCall}} = \begin{cases} (S_{\mathcal{N}} - K)^+ & , \min\limits_{i \in \{0,\dots,\mathcal{N}\}} S_i > B \\ 0, & \text{otherwise} \end{cases}$$

Example 2.7. Decreasing payoffs are for example:

- Put option: $C_{\text{Call}} = (K - S_{\mathcal{N}})^+$
- Lookback put option: $C_{\text{LbPut}} = \left(K - \max\limits_{i \in \{0,\dots,\mathcal{N}\}} S_i \right)^+$
- Asian put option: $C_{\text{AsianPut}} = \left(K - \frac{1}{\mathcal{N}} \sum_{i=0}^{\mathcal{N}} S_i \right)^+$
- Barrier up-and-out put option:

$$C_{\text{UOPut}} = \begin{cases} (K - S_{\mathcal{N}})^+ & , \max\limits_{i \in \{0,\dots,\mathcal{N}\}} S_i < B \\ 0, & \text{otherwise.} \end{cases}$$

Example 2.8. The following two derivatives are neither increasing nor decreasing payoffs.

- Barrier up-and-out call option

$$C_{\text{UOCall}} = \begin{cases} (S_{\mathcal{N}} - K)^+ & , \max\limits_{i \in \{0,\dots,\mathcal{N}\}} S_i < B \\ 0, & \text{otherwise} \end{cases}$$

- Barrier down-and-in call option

$$C_{\text{DICall}} = \begin{cases} (S_{\mathcal{N}} - K)^+ & , \min\limits_{i \in \{0,\dots,\mathcal{N}\}} S_i \leq B \\ 0, & \text{otherwise.} \end{cases}$$

3. Classical Binomial Model

In this Section, we recall the classical binomial model. Let $T > 0$ be some time-horizon and assume there are $N \in \mathbb{N}$ trading periods between $[0, T]$, each trading period is of

length $\frac{T}{N}$. There is a riskless bond

$$B_i^{(N)} = (1 + r_N)^i, \quad i = 0, 1, ..., N,$$

paying interest $r_N = \frac{rT}{N} > -1$ in each trading period and just one risky asset, paying dividends $q_N = \frac{qT}{N}$ in each period, and whose price process takes the form

$$S_i^{(N)} = S_0 \prod_{k=1}^{i} (1 + R_k^{(N)}), \quad i = 1, 2, ..., N,$$

where $S_0 > 0$, and the returns

$$R_i^{(N)} = \frac{S_i^{(N)} - S_{i-1}^{(N)}}{S_{i-1}^{(N)}}, \quad i = 1, 2, ..., N$$

are random variables with values in $\{a_N, b_N\} \subset \mathbb{R}$, such that

$$d_N := 1 + a_N = e^{-\sigma\sqrt{\frac{T}{N}}} \quad \text{and} \quad u_N := 1 + b_N = e^{\sigma\sqrt{\frac{T}{N}}}. \tag{7}$$

The market is arbitrage-free and complete if $-1 < a_N < r_N - q_N < b_N$, which holds for N large enough. In this case, the returns $R_1^{(N)}, ..., R_N^{(N)}$ are independent, their distributions are characterized by

$$P_N^*(R_i^{(N)} = b_N) := p_N^* := p_N^q + \varphi(N), \quad i = 1, 2, ..., N, \tag{8}$$

where P_N^* is the unique risk neutral measure, $\varphi(N) \in o\left(\frac{1}{\sqrt{N}}\right)$ and

$$p_N^q := \frac{1}{2} + \frac{r - q}{2\sigma}\sqrt{\frac{T}{N}}. \tag{9}$$

We say the classical binomial model is *characterized by the tuple* $(S_0, T, r, q, \sigma, N)$.

Let $C_E^{(N)}$ be a possibly path-dependent European contingent claim. The discounted

claim

$$H_E^{(N)} = \frac{C_E^{(N)}}{B_N^{(N)}}$$

can be written as

$$H_E^{(N)} = h\left(S_0^{(N)}, ..., S_N^{(N)}\right) \tag{10}$$

for a suitable function h. The value process in the Nth model,

$$V_i^{(N)} = E_{P_N^*}\left[H_E^{(N)} | \mathcal{F}_i\right], \quad i = 1, 2, ..., N,$$

of a replicating strategy for $H_E^{(N)}$ at time $t = \frac{iT}{N}$ is of the form

$$V_i^{(N)}(\omega) = v_i^{(N)}(S_0, S_1(\omega), .., S_i(\omega)),$$

where the function $v_i^{(N)}$ is given by recursion

$$
\begin{aligned}
v_N^{(N)}(x_0, ..., x_N) &= h(x_0, ..., x_N) \\
v_i^{(N)}(x_0, ..., x_i) &= (1 - p_N^*)\, v_{i+1}^{(N)}(x_0, ..., x_i, x_i d_N) \\
&\quad + p_N^* v_{i+1}^{(N)}(x_0, ..., x_i, x_i u_N), \quad i = 0, 1, ..., N-1,
\end{aligned}
$$

see e.g. Föllmer and Schied (2011, Proposition 5.41).

On the other hand, dealing with an American contingent claim

$$C_A^{(N)} = \left(C_{A,i}^{(N)}\right)_{i=0,..,N},$$

and the corresponding discounted claim

$$H_{A,i}^{(N)} = \frac{C_{A,i}^{(N)}}{B_i^{(N)}}, \quad i = 0, .., N,$$

18

for each $i = 0, 1, .., N$ there is a suitable function h_i such that

$$H_{A,i}^{(N)} = h_i(S_0, ..., S_i).$$

By no-arbitrage-arguments, the value process $(V_i)_{i=0,...,N}$ of a replicating strategy for $H_A^{(N)}$ can be found by recursion, compare with Föllmer and Schied (2011, Chapter 6):

$$V_N := H_{A,N}^{(N)}, \;\; V_i := H_{A,i}^{(N)} \vee E_{P_N^*}\left[V_{i+1} | \mathcal{F}_i\right], \;\; i = 0, .., N-1.$$

Hence, there are functions $v_i^{(N)}$ such that

$$V_i = v_i^{(N)}(S_0, ..., S_i), \;\; i = 0, ..., N,$$

namely

$$
\begin{aligned}
v_N^{(N)}(x_0, ..., x_N) &= h_N(x_0, ..., x_N) \\
v_i^{(N)}(x_0, ..., x_i) &= h_i(x_0, ..., x_i) \vee \left\{ (1 - p_N^*) \, v_{i+1}^{(N)}(x_0, ..., x_i, x_i d_N) \right. \\
&\quad \left. + p_N^* v_{i+1}^{(N)}(x_0, ..., x_i, x_i u_N) \right\}, \;\; i = 0, 1, ..., N-1.
\end{aligned}
$$

It is well known that in a classical binomial-tree model, which is characterized by the tuple $(S_0, T, r, q, \sigma, N)$, the risk-neutral price

$$\pi^{(N)}(C^{(N)}, S_0, T, r, q, \sigma) := v_0^{(N)}(S_0) \tag{11}$$

of a European or an American contingent claim $C^{(N)} = C_E^{(N)}$ or $C^{(N)} = C_A^{(N)}$, converge for many products as $N \to \infty$. If the limit exists, we define

$$\pi(C, S_0, T, r, q, \sigma) := \lim_{N \to \infty} \pi^{(N)}(C^{(N)}, S_0, T, r, q, \sigma).$$

Convergence of plain vanilla European options to the Black-Scholes price are discussed in Cox, Ross and Rubinstein (1979). For plain vanilla American options we refer

to Amin and Khanna (1994), for European and American Asian options and lookback options and some other path-dependent options, see Jiang and Dai (2004). For a proof of convergence for European barrier option, see Carbone (2004) and Lin and Palmer (2013) and references therein. Those convergence results can directly be applied to prove convergence of bid and ask prices as Theorem 4.1 shows.

4. Convergence of Bid and Ask Prices

In this Section we prove our main result and show that bid and ask prices of European or American contingent claims converge, if the risk-neutral price of the claim converges in the classical binomial model. The theorem has an important practical implication: it states that bid and ask prices of monotone payoffs, in particular plain vanilla European and American options, can be computed using the classical Black-Scholes model with an adjusted drift. Bid and ask prices of such options can therefore be computed very fast.

Theorem 4.1. *Let* $(\Psi^\gamma)_{\gamma \geq 0}$ *be a FCDF fulfilling Assumption A1 and A2. Let a classical binomial-tree model be given, which is characterized by the tuple*

$$(S_0, T, r, q, \sigma, N).$$

Let $C^{(N)}$ *be an increasing (decreasing) European or American contingent claim. Let* $\gamma \geq 0$ *and*

$$\gamma_N := \gamma \sqrt{\frac{T}{N}}. \tag{12}$$

Define bid and ask prices of a European claim by recursions (5) and of an American claim by recursions (6). If risk-neutral price, defined via Equation (11),

$$\pi^{(N)}(C^{(N)}, S_0, T, r, \tilde{q}, \sigma)$$

converges in the classical binomial model for all dividends

$$\tilde{q} \in [q - \sigma\gamma, q + \sigma\gamma]$$

to some non-negative number $\pi(C, S_0, T, r, \tilde{q}, \sigma)$, then the ask (bid) of the contingent claim converges to

$$\lim_{N \to \infty} \pi^{(N)}(C^{(N)}, S_0, T, r, q - \sigma\gamma, \sigma)$$

and the bid (ask) price converges to

$$\lim_{N \to \infty} \pi^{(N)}(C^{(N)}, S_0, T, r, q + \sigma\gamma, \sigma).$$

Proof. The proof can be found in the appendix. $\qquad\square$

A look at the proof of Theorem 4.1 shows that one could define γ_N in Equation (12) arbitrarily, as long as it converges to zero as fast as $\frac{1}{\sqrt{N}}$. Under our particular choice, γ can be interpreted as a drift-adjustment via the dividend yield in the continuous time limit scaled by the volatility. The drift adjustment is

$$\hat{q}T = (q \pm \sigma\gamma)\, T.$$

Remark 2. In the binomial model, the underlying is modelled by a bounded stochastic process. Therefore there is no restriction of the definition of bid and ask prices via recursions (5) and (6) requiring the contingent claims to be bounded. In particular call options are bounded in discrete time. Bid and ask prices of a contingent claim form a two-dimensional sequence with the natural numbers $1, 2, 3, ..$ as index set. The index N corresponds to the Nth binomial model. In Theorem 4.1, we prove convergence of such a sequence. Hence Theorem 4.1 says that bid and ask prices of a possibly unbounded contingent claim like a European call option in continuous time can be approximated arbitrary closely by the bid and ask prices of a bounded contingent claim in discrete

time.

4.1. *European Plain Vanilla Options*

In the classical Black-Scholes world, there exist closed-form solutions for the risk-neutral price of European plain vanilla and barrier options. By Theorem 4.1, we obtain closed-form solutions for bid and ask prices of European plain vanilla and barrier options, which are stated in the next corollaries, by taking the corresponding closed-form solutions for the risk-neutral price and adjusting the dividend yield.

The Black-Scholes prices of plain vanilla European call and put options with strike K and maturity T are given in closed-form and denoted by

$$\mathrm{BS}_{\mathrm{Call}}(S_0, T, K, r, q, \sigma) = S_0 e^{-qT} \Phi(d_1) - e^{-rT} K \Phi(d_2)$$

and

$$\mathrm{BS}_{\mathrm{Put}}(S_0, T, K, r, q, \sigma) = e^{-rT} K \Phi(-d_2) - S_0 e^{-qT} \Phi(-d_1),$$

where $d_1 = \frac{\log \frac{S_0}{K} + \left(r - q + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$ and $d_2 = d_1 - \sigma\sqrt{T}$ and Φ denotes the distribution function of the standard normal distribution, see Black and Scholes (1973).

Corollary 4.2. *Under the notation of Theorem 4.1, let $C^{(N)}$ be a European plain vanilla option with strike $K > 0$ and maturity T. Bid and ask prices of a put option converge to*

$$\mathrm{bid}_{\mathrm{Put}}^{\gamma} = \mathrm{BS}_{\mathrm{Put}}(S_0, T, K, r, q - \sigma\gamma, \sigma)$$

and

$$\mathrm{ask}_{\mathrm{Put}}^{\gamma} = \mathrm{BS}_{\mathrm{Put}}(S_0, T, K, r, q + \sigma\gamma, \sigma).$$

Bid and ask prices of a call option converge to

$$\text{bid}^\gamma_{\text{Call}} = \text{BS}_{\text{Call}}(S_0, T, K, r, q + \sigma\gamma, \sigma)$$

and

$$\text{ask}^\gamma_{\text{Call}} = \text{BS}_{\text{Call}}(S_0, T, K, r, q - \sigma\gamma, \sigma).$$
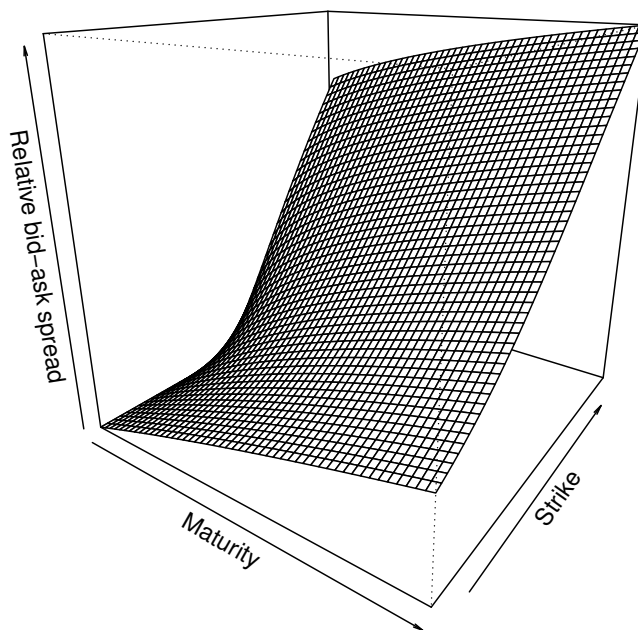
Figure 2. Relative bid-ask spread surface for European plain vanilla call options. We use the following parameters: the underlying is equal to 100, the strikes vary between 50 and 150, annual interest rates are set to 0.01, the dividend yield is assumed to be 0.03, the time left to maturity lays in the interval $[0, 2]$, the annual volatility is 0.2 and the annual γ is set to 0.05.

Figure 2 shows the relative bid-ask spread $\frac{\text{ask}-\text{bid}}{(\text{ask}+\text{bid})\frac{1}{2}}$ surface of European call options over strikes and maturities. Long term options and options being deep out-of-the money are less liquid, the relative bid-ask spread is greater.

Remark 3. Similarly to the existence of an implied volatility smile, there exist an implied liquidity smile. Computing γ implicitly from given bid and ask prices of options, Corcuera et al. (2012) show that there is a non-linear dependence of γ, with respect to the term structure and the moneyness of the option surface. In particular, we cannot

expect to predict the bid-ask spread of one option from given bid and ask prices of another option, if the corresponding strikes and maturities are too distant from each other.

4.2. *Path-dependent and American Options*

In a classical Black-Scholes framework, there exist closed-form solution for many barrier options, see Rubinstein and Reiner (1991) and Cheng (2003). For example the arbitrage-free price of an up-and-in barrier call option with maturity T, strike K and barrier $B > K$ is

$$
\begin{aligned}
\mathrm{BS}_{\mathrm{UICall}}(S_0, T, K, B, r, q, \sigma) &= S_0 e^{-qT} \Phi(x_1) - K e^{-rT} \Phi(x_1 - \sigma\sqrt{T}) \\
&\quad - S_0 e^{-qT} \left(\frac{B}{S_0}\right)^{2m} (\Phi(-y) - \Phi(-y_1)) \\
&\quad + K e^{-rT} \left(\frac{B}{S_0}\right)^{2m-2} (\Phi(-y + \sigma\sqrt{T}) \\
&\quad - \Phi(-y_1 + \sigma\sqrt{T})),
\end{aligned}
$$

where

$$
m = \frac{r - q + \frac{1}{2}\sigma^2}{\sigma^2}, \qquad y = \frac{\log\left(\frac{B^2}{S_0 K}\right)}{\sigma\sqrt{T}} + m\sigma\sqrt{T},
$$

$$
x_1 = \frac{\log\left(\frac{S_0}{B}\right)}{\sigma\sqrt{T}} + m\sigma\sqrt{T}, \qquad y_1 = \frac{\log\left(\frac{B}{S_0}\right)}{\sigma\sqrt{T}} + m\sigma\sqrt{T}.
$$

Corollary 4.3. *Under the notation of Theorem 4.1, let $C^{(N)}$ be an up-and-in barrier call option with maturity T, strike $K > 0$, and barrier $B > K$. The bid price converges to*

$$
\mathrm{bid}_{\mathrm{UICall}}^{\gamma} = BS_{\mathrm{UICall}}(S_0, T, K, B, r, q + \sigma\gamma, \sigma)
$$

and the ask converges to

$$
\mathrm{ask}_{\mathrm{UICall}}^{\gamma} = BS_{\mathrm{UICall}}(S_0, T, K, B, r, q - \sigma\gamma, \sigma).
$$

In the following, we treat bid prices of American put options but the findings can be transferred directly to ask prices of American put options and American call options as well. Let the risk-neutral Black-Scholes prices of a plain vanilla American put option with strike K and maturity T be denoted by

$$\mathrm{BS}_{\mathrm{Put}}^{\mathrm{A}}(S_0, T, K, r, q, \sigma).$$

There are no closed-form solutions for American plain vanilla options in a classical Black-Scholes framework, but there exist efficient numerical methods to approximate $\mathrm{BS}_{\mathrm{Put}}^{\mathrm{A}}$, see for example Barone-Adesi and Whaley (1987) and Bjerksund and Stensland (1993). We denote the numerical approximation by

$$\widetilde{\mathrm{BS}}_{\mathrm{Put}}^{\mathrm{A}}(S_0, T, K, r, q, \sigma)$$

and the error by

$$\varepsilon_{\mathrm{Put}}^{q} := \left| \mathrm{BS}_{\mathrm{Put}}^{\mathrm{A}}(S_0, T, K, r, q, \sigma) - \widetilde{\mathrm{BS}}_{\mathrm{Put}}^{\mathrm{A}}(S_0, T, K, r, q, \sigma) \right|.$$

The next corollary follows immediately:

Corollary 4.4. *Under the notation of Theorem 4.1, let $C^{(N)}$ be an American plain vanilla put option with strike $K > 0$ and maturity T. The bid price converge to*

$$\mathrm{bid}_{\mathrm{Put}}^{\gamma} = \mathrm{BS}_{\mathrm{Put}}^{\mathrm{A}}(S_0, T, K, r, q - \sigma\gamma, \sigma).$$

The error approximating the bid price using $\widetilde{BS}_{Put}^{A}$ as an estimate for BS_{Put}^{A} is less or equal to $\varepsilon_{Put}^{q-\sigma\gamma}$.

The corollary states the following: the bid price of an American put option on a stock with dividend yield q is equal to the risk-neutral price of an American put option but on a stock with dividend yield $q - \sigma\gamma$. The bid price directly inherits the numerical error from the approximation of the the risk-neutral price of the American option by

some numeric algorithm. A similar corollary could easily be stated for other options, like Asian options, which do not have closed-form solutions in the classical Black-Scholes model and can only be approximated for example with Monte Carlo methods. Bid and ask prices can then also be computed using Monte Carlo methods and the absolute error does not increase compared to classical risk-neutral pricing.

4.3. *Numeric Simulations*

In this Section, we try to investigate how fast the recursively defined bid and ask prices converge. We make two approximations: we approximate the concave distortion function by a linear function, see Equation (1) and we approximate the Black-Scholes model by a binomial tree. The error of the first approximation approaches zero faster than $\frac{1}{\sqrt{N}}$, see Equation (13).

The convergence rate of the classical binomial model is well studied in literature for many products: Heston and Zhou (2000) show that the risk-neutral price of a plain vanilla European call option converges at least as fast as $\frac{1}{\sqrt{N}}$. Lamberton (1998) proves that the risk-neutral price of an American put option converges from below and from above at least as fast as $N^{-\frac{2}{3}}$ and $N^{-\frac{3}{4}}$ respectively. Leisen and Reimer (1996) and Leisen (1998) analysed three different approaches to build a binomial tree, in particular the definitions for the returns of one trading period differ. They show that European plain vanilla options converge at least as fast as $\frac{1}{N}$ but American put options may only converge from below as fast as $\frac{1}{\sqrt{N}}$ depending on the exact tree definition. Lin and Palmer (2013) treat barrier options.

In our setting the up-move probability to obtain bid and ask prices has only asymptotically the martingale property, which makes it difficult to directly apply convergence results for classical binomial trees to our framework.

We therefore rely on simulations and compute bid and ask prices of a European call option, an American put option and a European up-and-in call option using recursions (5) and (6) for time-steps ranging from $N \in \{5, .., 2000\}$. We compare the tree-prices to their continuous counterpart, which can be obtained via the Corollaries 4.2, 4.3 and 4.4. Slightly abusing notation, we denote by e_N the absolute difference (error) between the bid or ask price of a contingent claim $C^{(N)}$ in the Nth binomial model to the limiting

bid or ask price. We say the sequence of errors *converges with order* $\rho > 0$, if there is a constant $\kappa > 0$ such that

$$\forall N \in \mathbb{N} \, : \, e_N \leq \frac{\kappa}{N^\rho}.$$

The order of convergence can be indicated straightforwardly by a simulation, see Leisen and Reimer (1996). As

$$\log\left(\frac{\kappa}{N^\rho}\right) = \log(\kappa) - \rho \log(N),$$

the negative slope of a straight line obtained from a log-log plot of the errors e_N against the refinement N can be used as an indicator for ρ. Figure 3 indicates an order of convergence between 1 and $\frac{1}{2}$ of the recursions (5) and (6) for different European and American options.

Convergence Rate for Several Options
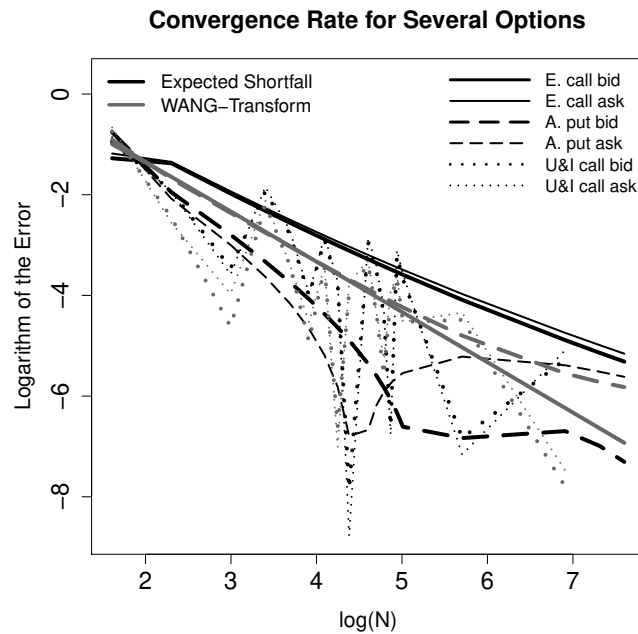
Figure 3. Log-log plot of the binomial tree approximation error for a European call option, an American put option and an up-and-in barrier option with barrier $B = 110$. All option have the strike $K = 100$ and the maturity is set to one year. The stock starts in $S_0 = 100$, annual interest rates are set to 0.01, the dividend yield is assumed to be 0.03, the annual volatility is 0.2 and the annual γ is set to 0.05. N goes in non-equidistant steps from 5 to 2000. The up-and-in barrier option is only simulated up to $N = 1000$.

5. Implied Liquidity (IL)

Change in Liquidity, ATM American Options

Figure 4. This Figure describes the change of the relative bid-ask spread, the IL and the difference of implied bid and ask volatilities from February, 2nd to February, 5th of various American at-the-money put and call options by a multiplicative factor. Maturities are measured in months.

In Guillaume et al. (2018), the concept of *implied liquidity* (IL) is defined, see also Corcuera et al. (2012), Dhaene et al. (2012) and Albrecher, Guillaume, and Schoutens (2013). It is similar to the idea of implied volatility and returns two implicitly computed parameters γ_b and γ_a such that modelled bid and ask prices match real market prices. The benefits of quoting the IL instead of bid-ask spreads are comparable to the benefits of quoting implied volatilities instead of mid-prices: in principle the IL can be constant across strikes, maturities and underlyings and hence makes it possible to compare bid-ask spreads across all three dimensions. Guillaume et al. (2018) computed the IL for European options and compared the defining formula of the IL to the industrial standard of describing bid and ask price by quoting implied bid and ask volatilities. Their main finding is: a change in liquidity is described much more adequately by the IL than by the change of the difference of implied bid and ask volatilities. We repeat this empirical study for American options and obtain similar results.

For February, 2nd and February, 5th, 2018, we obtained end of day bid and ask prices

of 80 plain vanilla, at-the-money American put and call options on the S&P500, or rather the SPDR S&P 500 ETF Trust, an exchange traded fund replicating the S&P500, with maturities ranging from about 3 to 8 month. The option prices were obtained from the Chicago Board Options Exchange and can be found in Table 1.

The CBOE Volatility Index (VIX), tracking short-term market volatility, jumped from 17.31 points on February, 2nd to 37.32 points on February, 5th, thus by 116%, which is the highest daily relative change recorded so far. The S&P500 lost about 4% between the two dates. It is well-known that liquidity dries up, when uncertainty in financial market rises. Therefore the chosen dates are well suited to analyse how different measures for the bid-ask spread behave, when liquidity changes.

For each American option, on both dates, we first compute an implied volatility σ_{Mid} matching exactly the mid-price. Then we use the mid-price implied volatility and compute the IL, $(\gamma_b, \gamma_a) \in \mathbb{R}^2_+$, such that the modelled bid and ask prices match exactly the quoted market bid and ask prices. In particular for an American call option C, we solve numerically

$$\mathrm{bid}_{\text{quoted market price}}(C) = \pi^A_{\mathrm{Call}}\left(S_0, T, K, r, q + \sigma_{\mathrm{Mid}}\gamma_b, \sigma_{\mathrm{Mid}}\right),$$

for γ_b. $\pi^A_{\mathrm{Call}}(S_0, T, K, r, q, \sigma)$ is the risk-neutral price of an American call option in a Black-Scholes setting with strike K and maturity T on a stock with initial value S_0, volatility σ and paying a continuously dividend yield q. The risk-free interest rate is denoted by r. The parameter γ_a for the call option and the IL $(\tilde{\gamma}_b, \tilde{\gamma}_a)$ of put options can be found analogously, see Theorem 4.1. For most options γ_b and γ_a are almost identical.

The average relative bid-ask spread of the American option set is 1.6% on February, 2nd and 6.6% on February, 5th. The relative bid-ask spread changed by the factor 4.1. The average IL $\frac{\gamma_a + \gamma_b}{2}$, changed from an average value of 0.011 for all options on February, 2nd to 0.043 on February, 5th, which corresponds to a change by the factor 3.9. The average difference of implied bid and ask volatilities on the other hand, rose by the factor 6.1, hence about 49% more than the relative bid-ask spread. In Figure 4, we show the multiplicative factor describing the change of the relative bid-ask spread, the
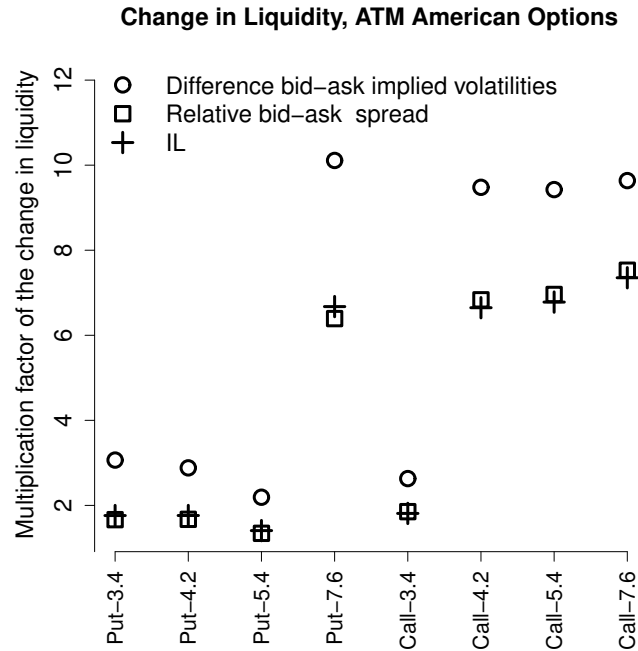
IL and the difference of implied bid and ask volatilities from February, 2nd to February 5th separately for put and call options and different maturities.

The overall picture is the following: a change in liquidity of American options, due to a rise in uncertainty in the market and measured by the change of the relative bid-ask spread, is described by the IL very well. On the other hand, an overestimation of the change of liquidity by 50% and more are no exceptions, when describing the bid-ask spread by the classical way of quoting implied bid and ask volatilities. Our findings for American options are in line with a similar empirical study for European options done by Guillaume et al. (2018).

6. Conclusion

We model a financial market with frictions in discrete time by applying a pricing functional, which is defined recursively via coherent risk measures, to a classical binomial model. Economically, the discrete time model is justified in Jouini and Kallal (1995, 2001) and Jouini (2000). We are able to prove convergence of bid and ask prices for monotone products like European and American plain vanilla, Asian, lookback and (some) barrier options. In the limit we obtain an extended Black-Scholes formula containing a new parameter $\gamma \geq 0$, which adjusts the dividend yield. The greater γ, the greater the bid-ask spread. In particular for European plain vanilla options and barrier options exist closed-form solutions. The convergence results allow us to compute bid and ask prices of an option as fast as the risk-neutral price in a classical Black-Scholes model.

The new model may find a similar application in practise as the classical Black-Scholes model. Trader usually prefer to quote implied volatilities instead of prices, because there are many nonlinearities in prices making comparisons across strikes, maturities, and underlying assets difficult to comprehend and understand. In principle volatilities could be constant across all three dimensions, hence the preference for quoting implied volatilities.

With the same argument it might be more convenient to quote an implicitly computed γ instead of the bid-ask spread. To demonstrate this idea, we computed implicitly two

parameters γ_b and γ_a for a set of American options such that modelled bid and ask prices match real market prices. In principle the tuple (γ_b, γ_a) could be constant across strikes, maturities and underlyings and hence makes it possible to compare bid-ask spreads across all three dimensions.

We compare this new market implied liquidity measure to the common approach of describing the bid-ask spread by the difference between implied bid and ask Black-Scholes volatilities and show empirically that our model describes liquidity of American options over time significantly better. A similar study has been done by Guillaume et al. (2018) for European options using a static model.

A future research could consist of generalizing the binomial model and allowing in particular the parameter γ and the volatility to be possibly correlated stochastic processes.

Biographical Note

ARGIMIRO ARRATIA has obtained his PhD in Mathematics from the University of Wisconsin and holds a professorship in the Computer Science Department of the UPC (Universitat Politècnica de Catalunya). His research interests include computational finance and financial time series analysis. Professor ALEJANDRA CABAÑA is currently working at the Mathematics Department at the UAB (Universitat Autònoma de Barcelona), her research activity is centred in the field of mathematical statistics. GERO JUNIKE has obtained his PhD in Mathematics from the UAB and has several years professional experience at the risk department of a large international bank. WIM SCHOUTENS is a Professor in Financial Engineering at the KU Leuven. He has extensive practical experience of model implementation and is well known for his consulting work to the banking industry.

Acknowledgements

Appendix

Proof of Proposition 2.1.

We trivially have

$$0 \leq S_i^b \leq E_{\mathbb{Q}}\left[S_N | \mathcal{F}_i\right] = S_i \leq S_i^a, \ \ i = 0, ..., N,$$

hence by Jouini and Kallal (1995, Theorem 3.2), the security price model admits no multiperiod free lunch and is hence arbitrage-free. Let

$$\mathcal{A}_0 := \left\{X \in L^{\infty}, \rho_0(X) \leq 0\right\}.$$

For a probability measure Q equivalent to $\mathbb{Q}$, define

$$\alpha_0^{\min}(Q) := \sup_{X \in \mathcal{A}_0} E_Q[X].$$

It holds $\alpha_0^{\min}(\mathbb{Q}) \leq 0$, hence by Föllmer and Penner (2006, Corollary 4.12.), there exist a set of probability measures $(\mathcal{Q}^e)$, such that each element of $\mathcal{Q}^e$ is equivalent to $\mathbb{Q}$ and

$$p_i(X) = \sup_{Q \in \mathcal{Q}^e} E_Q[X | \mathcal{F}_i], \ \ i = 0, .., N.$$

Let $\mathcal{P}$ be a set of probability measures containing all probability measures P which are equivalent to $\mathbb{Q}$ for which exist a $P-$martingale $\left(Z_i^P\right)$ with

$$S_i^b \leq Z_i^P \leq S_i^a, \ \ i = 0, .., N.$$

It follows $\mathcal{Q}^e \subseteq \mathcal{P}$: for each $Q^e \in \mathcal{Q}^e$ there is a Q^e-martingale $\left(Z_i^{Q^e}\right)$, namely

$$Z_i^{Q^e} := E_{Q^e}\left[S_N | \mathcal{F}_i\right],$$

such that

$$S_i^b \leq Z_i^{Q^e} \leq S_i^a, \quad i = 0, .., N.$$

Let $C \in L^\infty$. By Jouini (2000, Theorem 1.), the value $p^*(C) := \sup_{P \in \mathcal{P}} E_P[C]$ is less or equal to the price of the smallest self-financing trading strategy dominating C. As $p_0(C) \leq p^*(C)$, we conclude. $\square$

Proof of Theorem 4.1

We first assume $C^{(N)}$ models a European contingent claim and can be described by a function h as in Equation (10). Let u_N, d_N and p_N^q be defined as in Section 3. Then it holds for the processes describing the ask price $(A_i)_{i=0,1,..,N}$ and the bid price $(B_i)_{i=0,1,..,N}$ of $C^{(N)}$,

$$A_i(\omega) = a_i((S_0, S_1(\omega), .., S_i(\omega))),$$

and

$$B_i(\omega) = b_i((S_0, S_1(\omega), .., S_i(\omega))),$$

where the functions a_i and b_i are recursively defined:

$$b_N(x_0, ..., x_N) = a_N(x_0, ..., x_N) = h(x_0, ..., x_N)$$

and for $i = 0, 1, .., N-1$, if the European contingent claim is increasing

$$\begin{aligned}
a_i(x_0, ..., x_i) &= (1 - \Psi^{\gamma_N}(p_N^q + \varphi(N))) \, a_{i+1}(x_0, ..., x_i, x_i d_N) \\
&\quad + \Psi^{\gamma_N}(p_N^q + \varphi(N)) \, a_{i+1}(x_0, ..., x_i, x_i u_N)
\end{aligned}$$

and

$$b_i(x_0, ..., x_i) = \Psi^{\gamma_N} \left(1 - (p_N^q + \varphi(N))\right) b_{i+1} \left(x_0, ..., x_i, x_i d_N\right)$$
$$+ \left(1 - \Psi^{\gamma_N} \left(1 - (p_N^q + \varphi(N))\right)\right) b_{i+1} \left(x_0, ..., x_i, x_i u_N\right).$$

If h defines a decreasing European contingent claim, it follows for $i = 0, .., N-1$,

$$a_i(x_0, ..., x_i) = \Psi^{\gamma_N} \left(1 - (p_N^q + \varphi(N))\right) a_{i+1} \left(x_0, ..., x_i, x_i d_N\right)$$
$$+ \left(1 - \Psi^{\gamma_N} \left(1 - (p_N^q + \varphi(N))\right)\right) a_{i+1} \left(x_0, ..., x_i, x_i u_N\right)$$

and

$$b_i(x_0, ..., x_i) = \left(1 - \Psi^{\gamma_N} \left(p_N^q + \varphi(N)\right)\right) b_{i+1} \left(x_0, ..., x_i, x_i d_N\right)$$
$$+ \Psi^{\gamma_N} \left(p_N^q + \varphi(N)\right) b_{i+1} \left(x_0, ..., x_i, x_i u_N\right).$$

The ask (bid) price of an increasing European payoff and the bid (ask) price of a decreasing European payoff at level $\gamma_N \geq 0$ are exactly defined as the risk-neutral price in the classical binomial model, when replacing the up-move probability in Equation (8), i.e,

$$p_N^q + \varphi(N),$$

by

$$\Psi^{\gamma_N} \left(p_N^q + \varphi(N)\right),$$

respectively by

$$\left(1 - \Psi^{\gamma_N} \left(1 - (p_N^q + \varphi(N))\right)\right).$$

This observation can directly be carried forward to American contingent claims and is explained by the structure of the binomial model and the recursive definition of bid and ask prices.

By Assumption A2, there is a sequence $\tilde{\varphi}(N) \in o\left(\frac{1}{\sqrt{N}}\right)$ such that

$$
\begin{aligned}
\Psi^{\gamma_N}\left(p_N^q + \varphi(N)\right) &= \frac{1}{2} + \frac{r-q}{2\sigma}\sqrt{\frac{T}{N}} + \frac{\gamma}{2}\sqrt{\frac{T}{N}} + \tilde{\varphi}(N) \\
&= \frac{1}{2} + \frac{r-(q-\sigma\gamma)}{2\sigma}\sqrt{\frac{T}{N}} + \tilde{\varphi}(N) \\
&= p_N^{q-\sigma\gamma} + \tilde{\varphi}(N).
\end{aligned}
$$

Similarly it holds for a suitable $\hat{\varphi}(N) \in o\left(\frac{1}{\sqrt{N}}\right)$,

$$
1 - \Psi^{\gamma_N}\left(1 - \left(p_N^q + \varphi(N)\right)\right) = p_N^{q+\sigma\gamma} + \hat{\varphi}(N).
$$

Hence the up-move probability of the distorted binomial model describing bid and ask prices can be expressed as in the classical binomial model with an adjusted dividend yield:

$$
\frac{1}{2} + \frac{r-(q\pm\sigma\gamma)}{2\sigma}\sqrt{\frac{T}{N}} + o\left(\frac{1}{\sqrt{N}}\right). \tag{13}
$$

As the up and down moves u_N and d_N remain unchanged compared to the classical binomial model, we conclude. $\qquad\square$

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Table 1. End of day bid and ask prices of the American option data-set. The underlying is the SPDR S&P 500 ETF Trust.

Today	Maturity	Strike	Bid-call	Ask-call	Bid-put	Ask-put
02/02/2018	18/05/2018	273	10.1	10.25	6.99	7.14
02/02/2018	15/06/2018	273	11.23	11.39	8.15	8.27
02/02/2018	20/07/2018	273	12.32	12.51	9.29	9.48
02/02/2018	21/09/2018	273	14.85	15.08	11.32	11.51
02/02/2018	18/05/2018	274	9.42	9.57	7.32	7.46
02/02/2018	15/06/2018	274	10.55	10.7	8.48	8.61
02/02/2018	20/07/2018	274	11.66	11.84	9.64	9.83
02/02/2018	21/09/2018	274	14.21	14.41	11.66	11.86
02/02/2018	18/05/2018	275	8.77	8.9	7.67	7.82
02/02/2018	15/06/2018	275	9.9	10.03	8.84	8.97
02/02/2018	20/07/2018	275	11.01	11.2	9.99	10.18
02/02/2018	21/09/2018	275	13.56	13.76	12.02	12.2
02/02/2018	18/05/2018	276	8.13	8.26	8.03	8.2
02/02/2018	15/06/2018	276	9.25	9.37	9.2	9.33
02/02/2018	20/07/2018	276	10.38	10.55	10.35	10.55
02/02/2018	21/09/2018	276	12.93	13.11	12.38	12.57
02/02/2018	18/05/2018	277	7.51	7.64	8.42	8.58
02/02/2018	15/06/2018	277	8.62	8.74	9.59	9.72
02/02/2018	20/07/2018	277	9.75	9.92	10.74	10.94
02/02/2018	21/09/2018	277	12.29	12.48	12.75	12.95
05/02/2018	18/05/2018	262	13.07	13.44	12.69	13.15
05/02/2018	15/06/2018	262	13.83	15.2	13.76	14.11
05/02/2018	20/07/2018	262	14.79	16.48	14.8	15.2
05/02/2018	21/09/2018	262	17.07	18.98	16.62	18.4
05/02/2018	18/05/2018	263	12.42	12.77	13.04	13.49
05/02/2018	15/06/2018	263	13.16	14.52	14.1	14.46
05/02/2018	20/07/2018	263	14.15	15.81	15.14	15.55
05/02/2018	21/09/2018	263	16.44	18.32	16.98	18.79
05/02/2018	18/05/2018	264	11.76	12.11	13.4	13.84
05/02/2018	15/06/2018	264	12.54	13.84	14.46	14.81
05/02/2018	20/07/2018	264	13.52	15.15	15.51	15.91
05/02/2018	21/09/2018	264	15.81	17.65	17.35	19.2
05/02/2018	18/05/2018	265	11.12	11.47	13.76	14.2
05/02/2018	15/06/2018	265	11.92	13.19	14.83	15.18
05/02/2018	20/07/2018	265	12.91	14.5	15.88	16.29
05/02/2018	21/09/2018	265	15.2	17.01	17.73	19.63
05/02/2018	18/05/2018	266	10.52	10.83	14.13	14.59
05/02/2018	15/06/2018	266	11.3	12.15	15.2	15.55
05/02/2018	20/07/2018	266	12.32	13.85	16.26	16.67
05/02/2018	21/09/2018	266	14.6	16.37	18.11	20.04

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