Multiplicity and Poincaré series for mixed multiplier ideals

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Let X be a complex surface with at most a rational singularity at a point $O \in X$ and $\mathfrak{m} = \mathfrak{m}_{X,O}$ be the maximal ideal of the local ring $\mathcal{O}_{X,O}$ at O. Given a tuple of $\mathfrak{m}$-primary ideals $\boldsymbol{\mathfrak{a}} := (\mathfrak{a}_1, ..., \mathfrak{a}_r) \subseteq (\mathcal{O}_{X,O})^r$ we will consider a common *log-resolution*, that is a birational morphism $\pi : X' \to X$ such that X' is smooth, $\mathfrak{a}_i \cdot \mathcal{O}_{X'} = \mathcal{O}_{X'}(-F_i)$ for some effective Cartier divisors F_i, $i = 1, \ldots, r$ and $\sum_{i=1}^r F_i + E$ is a divisor with simple normal crossings where $E = Exc(\pi)$ is the exceptional locus. Actually, the divisors F_i are supported on the exceptional locus since the ideals are $\mathfrak{m}$-primary.

We define the *mixed multiplier ideal* at a point $\mathbf{c} := (c_1, .., c_r) \in \mathbb{R}^r_{\geqslant 0}$ as [1]

$$\mathcal{J}(\boldsymbol{\mathfrak{a}}^{\mathbf{c}}) := \mathcal{J}(\mathfrak{a}_1^{c_1} \cdots \mathfrak{a}_r^{c_r}) = \pi_* \mathcal{O}_{X'}(\lceil K_\pi - c_1 F_1 - \cdots - c_r F_r \rceil)$$

where $\lceil \cdot \rceil$ denotes the *round-up* and $K_\pi = \sum_{i=1}^s k_j E_j$ is the *relative canonical divisor*, a $\mathbb{Q}$-divisor on X' supported on the exceptional locus E which is characterized by the property $(K_\pi + E_i) \cdot E_i = -2$ for every exceptional component E_i, $i = 1, \ldots, s$.

Associated to any point $\boldsymbol{c} \in \mathbb{R}^r_{\geqslant 0}$, we consider the *region* of $\boldsymbol{c}$ as:

$$\mathcal{R}_{\boldsymbol{\mathfrak{a}}}(\boldsymbol{c}) = \left\{ \boldsymbol{c}' \in \mathbb{R}^r_{\geqslant 0} \;\middle|\; \mathcal{J}\left(\boldsymbol{\mathfrak{a}}^{\boldsymbol{c}'}\right) \supseteq \mathcal{J}(\boldsymbol{\mathfrak{a}}^{\boldsymbol{c}}) \right\} .$$

The boundary of the region $\mathcal{R}_{\boldsymbol{\mathfrak{a}}}(\boldsymbol{c})$ is what we call the *jumping wall* associated to $\boldsymbol{c}$. From now on we will denote by $\mathbf{JW}_{\boldsymbol{\mathfrak{a}}}$ the set of jumping walls of $\boldsymbol{\mathfrak{a}}$.

1 Multiplicities of jumping points

We define the multiplicity attached to a point $\boldsymbol{c} \in \mathbb{R}^r_{\geqslant 0}$ as the codimension of $\mathcal{J}(\boldsymbol{\mathfrak{a}}^{\boldsymbol{c}})$ in $\mathcal{J}\left(\boldsymbol{\mathfrak{a}}^{(1-\varepsilon)\boldsymbol{c}}\right)$ for $\varepsilon > 0$ small enough, i.e.

$$m(\boldsymbol{c}) := \dim_{\mathbb{C}} \frac{\mathcal{J}\left(\boldsymbol{\mathfrak{a}}^{(1-\varepsilon)\boldsymbol{c}}\right)}{\mathcal{J}(\boldsymbol{\mathfrak{a}}^{\boldsymbol{c}})} .$$

One can compute these multiplicities using the following result:

[1] By an abuse of notation, we will also denote $\mathcal{J}(\boldsymbol{\mathfrak{a}}^{\mathbf{c}})$ its stalk at O so we will omit the word "sheaf" if no confusion arises.

Theorem 1 *Let $\mathfrak{a} \subseteq (\mathcal{O}_{X,O})^r$ be a tuple of $\mathfrak{m}$-primary ideals, $\boldsymbol{c} \in \mathbb{R}^r_{>0}$ a point and $H_{\boldsymbol{c}}$ the reduced divisor defined as $H_{\boldsymbol{c}} = \lceil K_\pi - (1-\varepsilon)c_1F_1 - \cdots - (1-\varepsilon)c_rF_r \rceil - \lceil K_\pi - c_1F_1 - \cdots - c_rF_r \rceil$ for a sufficiently small $\varepsilon > 0$. Then,*

$$m(\boldsymbol{c}) = (\lceil K_\pi - c_1F_1 - \cdots - c_rF_r \rceil + H_{\boldsymbol{c}}) \cdot H_{\boldsymbol{c}} + \# \{\textit{connected components of } H_{\boldsymbol{c}}\}.$$

2 Poincaré series of mixed multiplier ideals

Given a $\mathfrak{m}$-primary ideal $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$, Galindo and Montserrat in 2010 introduced its *Poincaré series* as

$$P_{\mathfrak{a}}(t) = \sum_{c \in \mathbb{R}_{>0}} m(c)\, t^c.$$

For a tuple of $\mathfrak{m}$-primary ideals $\mathfrak{a} = (\mathfrak{a}_1, \ldots, \mathfrak{a}_r) \subseteq (\mathcal{O}_{X,O})^r$ we are going to give a generalization of this series by considering a sequence of mixed multiplier ideals indexed by points in a ray $L : \boldsymbol{c}_0 + \mu\mathbf{u}$ in the positive orthant $\mathbb{R}^r_{>0}$ with a vector $\mathbf{u} = (u_1, \ldots, u_r) \in \mathbb{Z}^r_{\geqslant 0}$, $\mathbf{u} \neq \mathbf{0}$ and $\boldsymbol{c}_0 \in \mathbb{Q}^r_{\geqslant 0}$. Here we are considering, for simplicity, a point $\boldsymbol{c}_0$ belonging to a coordinate hyperplane but not necessarily being the origin and $\mu \in \mathbb{R}_{>0}$. Namely, we consider the sequence of mixed multiplier ideals

$$\mathcal{J}(\mathfrak{a}^{\boldsymbol{c}_0}) \supsetneq \mathcal{J}(\mathfrak{a}^{\boldsymbol{c}_1}) \supsetneq \mathcal{J}(\mathfrak{a}^{\boldsymbol{c}_2}) \supsetneq \cdots \supsetneq \mathcal{J}(\mathfrak{a}^{\boldsymbol{c}_i}) \supsetneq \cdots$$

where $\{\boldsymbol{c}_i\}_{i>0} = L \cap \mathbf{JW}_{\mathfrak{a}}$ or equivalently $\{\boldsymbol{c}_i\}_{i>0}$ is the set of jumping points of this sequence. Then we define the *Poincaré series of $\mathfrak{a}$ alongside the ray L* as

$$P_{\mathfrak{a}}(\underline{t}; L) = \sum_{\boldsymbol{c} \in L} m(\boldsymbol{c})\, \underline{t}^{\boldsymbol{c}}$$

where $\underline{t}^{\boldsymbol{c}} := t_1^{c_1} \cdots t_r^{c_r}$.

Theorem 2 *Let $\mathfrak{a} = (\mathfrak{a}_1, \ldots, \mathfrak{a}_r) \subseteq (\mathcal{O}_{X,O})^r$ be a tuple of $\mathfrak{m}$-primary ideals and let $L : \boldsymbol{c}_0 + \mu\mathbf{u}$ be a ray in the positive orthant $\mathbb{R}^r_{\geqslant 0}$ with $\mathbf{u} \in \mathbb{Z}_{\geqslant 0}$, $\mathbf{u} \neq \mathbf{0}$. The Poincaré series of $\mathfrak{a}$ alongside L can be expressed as*

$$P_{\mathfrak{a}}(\underline{t}; L) = \underline{t}^{\boldsymbol{c}_0} \sum_{\mu \in [0,1)} \left(\frac{m(\boldsymbol{c}_0 + \mu\mathbf{u})}{1 - \underline{t}^{\mathbf{u}}} + \rho_{\boldsymbol{c}_0 + \mu\mathbf{u}, \mathbf{u}} \frac{\underline{t}^{\mathbf{u}}}{(1 - \underline{t}^{\mathbf{u}})^2} \right) \underline{t}^{\mu\mathbf{u}}.$$

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References

[1] M. Alberich-Carramiñana, J. Àlvarez Montaner, F. Dachs-Cadefau and V. González-Alonso, *Multiplicities of jumping points for mixed multiplier ideals*, To appear.