COTANGENT MODELS FOR INTEGRABLE SYSTEMS ON b-SYMPLECTIC MANIFOLDS

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Abstract. In this paper we give cotangent models for integrable systems in symplectic and b-symplectic manifolds. The proof of the existence of such (semilocal) models boils down to the corresponding action-angle coordinate theorems in these settings. The theorem of Liouville-Mineur-Arnold establishes the existence of action-angle coordinates in a neighbourhood of a Liouville torus [A74] of a symplectic manifold. An action-angle theorem for a class of Poisson manifolds called b-symplectic manifolds was proven in [KMS16]. This viewpoint of cotangent models provides a new machinery to produce examples of integrable systems on b-symplectic manifolds and revisit known examples. At the end of the paper we introduce non-degenerate singularities as lifted cotangent models on b-symplectic manifolds.

1. Introduction

The action-angle theorem of Liouville-Mineur-Arnold ([A74], [D80]) can be reformulated as a symplectic equivalence in a neighbourhood of a Liouville torus to an integrable system determined by the cotangent lift of the actions by translations on the Liouville torus. Having such a cotangent lift model for integrable systems is useful to produce examples as lifts of abelian actions on the base. The Hamiltonian nature of the lifted action is automatic (see [GS90]) and the fact that the action on the base is given by an abelian group automatically yields an integrable system on the total space.

The first result in this paper is a reformulation of the existence of action-angle coordinates for integrable systems in a symplectic manifold as semilocal equivalence of the system with the model given by the cotangent lift of the action by rotations on a Liouville torus to the cotangent bundle of this torus. To our knowledge this point of view is new.

This paper is also devoted to establishing the analogous results in the context of a class of Poisson manifolds called b-Poisson manifolds. b-Symplectic/b-Poisson manifolds have been the object of study of recent works in Poisson Geometry (cf. [GMP11], [GMP12], [GMPS13], [GMW15], [GMPS14],

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[GL14], [GLPR14], [KMS16] and [DKM15]). For these manifolds, such a reformulation is possible replacing the cotangent bundle by the b-cotangent bundle and considering the dual Liouville form obtaining what we call the *twisted b-cotangent lift*. The equivalence to these models is given by the existence of action-angle coordinates [KMS16]. This new point of view turns out to be very fruitful because it provides a handful of examples of b-integrable systems which was missing in the literature. One of the families of examples is produced by considering abelian symmetries of affine manifolds and applying the twisted b-cotangent lift recipe to generate b-integrable systems. This class of examples brings back reminiscences of the obstructions to the global existence of action-angle coordinates in the symplectic context studied by Duistermaat [D80]. We also consider the canonical b-cotangent lift which can be used to furnish examples of Hamiltonian actions on b-symplectic manifolds.

Singularities of integrable systems are present in mechanical systems and they correspond to equilibria of Hamiltonian systems. From a topological point of view, an integrable system on a compact manifold must have singularities. In [E84], [E90], [Mi14], [Mi03], [MZ04] in total analogy with Liouville-Mineur-Arnold theorem, a symplectic Morse-Bott theory is constructed in a neighbourhood of a point of a compact invariant manifold.

In the last section of the paper we present non-degenerate singular integrable systems in the b-symplectic case as twisted b-cotangent lifts of actions by abelian groups which have fixed points on the base or are non-compact. This provides several examples with different kinds of singularities (elliptic, hyperbolic, focus-focus). This last section is an invitation to the study of singularities of integrable systems on b-symplectic manifolds. We plan to study normal form theorems for these singularities in b-symplectic manifolds as equivalence to the twisted b-cotangent models in the future, thus readdressing the normal form theory already initiated in [GMP12].

2. Preliminaries

2.1. Integrable systems and action-angle coordinates on symplectic manifolds.

Let (M^{2n}, ω) be a symplectic manifold. An **integrable system** is given by n functions $f_1, \dots, f_n$ in involution with respect to the Poisson bracket associated to the symplectic form ω and which are functionally independent on a dense set. Recall that the Poisson bracket associated to ω is defined via

$$\{f, g\} := \omega(X_f, X_g), \qquad f, g \in C^\infty(M),$$

where for a function f the vector field X_f is the **Hamiltonian vector field** of f defined by $\iota_{X_f}\omega = -df$.

The expression *integrable* refers to integrability of the system of differential equations associated to a function H which can be chosen as one of the commuting functions: Integrability of the system in the sense described above (also called Liouville integrability) is related to actual integration of the system by quadratures [L1855].

The local structure of a symplectic manifold is described by the Darboux theorem. In the context of integrable systems the Darboux-Carathéodory theorem states that we can find a special Darboux chart in which half the coordinate functions are the integrals of the system (locally around a point where the integrals are independent).

The theorem of Liouville-Mineur-Arnold goes one step further and establishes a *semi-local* result in a neighbourhood of a compact level set ("Liouville torus") of the integrable system:

Theorem 1. (Liouville-Mineur-Arnold Theorem)
Let (M^{2n}, ω) be a symplectic manifold. Let $F = (f_1, \dots, f_n)$ be an n-tuple of functions on M which are functionally independent (i.e. $df_1 \wedge \cdots \wedge df_n \neq 0$) on a dense set and which are pairwise in involution. Assume that m is a regular point[1] of F and that the level set of F through m, which we denote by $\mathcal{F}_m$, is compact and connected.

Then $\mathcal{F}_m$ is a torus and on a neighborhood U of $\mathcal{F}_m$ there exist $\mathbb{R}$-valued smooth functions $(p_1, \dots, p_n)$ and $\mathbb{R}/\mathbb{Z}$-valued smooth functions $(\theta_1, \dots, \theta_n)$ such that:

1. *The functions $(\theta_1, \dots, \theta_n, p_1, \dots, p_n)$ define a diffeomorphism $U \simeq \mathbb{T}^n \times B^n$.*
2. *The symplectic structure can be written in terms of these coordinates as*
$$\omega = \sum_{i=1}^{n} d\theta_i \wedge dp_i.$$
3. *The leaves of the surjective submersion $F = (f_1, \dots, f_s)$ are given by the projection onto the second component $\mathbb{T}^n \times B^n$, in particular, the functions $f_1, \dots, f_s$ depend only on $p_1, \dots, p_n$.*

The coordinates p_i are called action coordinates; the coordinates θ_i are called angle coordinates.

Remark 2. In physics, usually one of the integrals f_i of Theorem 1 is the energy H, e.g. $f_1 = H$, and motion is given by the flow of the Hamiltonian vector field of H. Statement (3) in Theorem 1 implies that H is constant along the level sets of the functions f_i. Moreover, since $df_i(X_H) = \{f_i, H\} = 0$, the vector field X_H is tangent to the level sets. More precisely, in the action-angle coordinate chart, the flow of X_H is linear on the invariant tori.

Many important examples of dynamical systems in physics are integrable. A first class of examples is given by any 2-dimensional Hamiltonian system with $dH \neq 0$ on a dense set, e.g. the mathematical pendulum. Other examples are,

[1] i.e. the differentials df_i are independent at m.

Example 3. *The two-body problem: A system consisting of two bodies which interact through a potential V that depends only on their distance, i.e. the system with configuration space $\mathbb{R}^3 \times \mathbb{R}^3$ and Hamiltonian $H(q_1, q_2, p_1, p_2) = \frac{p_1}{2m_1} + \frac{p_2}{2m_2} + V(|q_1 - q_2|)$ is integrable with first integrals the energy H, the total momentum $p_1 + p_2$ and the total angular momentum $q_1 \times p_1 + q_2 \times p_2$. The case where V is the gravitational potential is the well-known Kepler problem; in physical examples the two bodies can e.g. be a satellite and a planet, a planet and a star etc.*

Example 4. *N bodies in a central force field: The above example can be reduced to the problem of a single body in a central force field via appropriate changes of coordinates. More generally, we can consider the problem of $N \geq 1$ bodies under the influence of a central force, e.g. the motion of planets in the gravitational field of the sun. If we neglect any interaction between the bodies, this is an integrable system as well. In contrast, the classical N body problem, where mutual interaction between all bodies is allowed, is not integrable for $N \geq 3$, as was shown in 1887 by H. Bruns*[2].

Example 5. *A rigid body fixed at its centre of gravity in a constant gravitational field: The system of a rigid body fixed at a point has configuration space $SO(3)$. Therefore, in addition to energy, one more constant of motion is needed to obtain integrability. If the fixed point is the centre of gravity, then such an integral is given by $\|M\|$, the norm of the total angular momentum. Instead of assuming that the fixed point is the centre of gravity, other (non-trivial) conditions can be given which guarantee integrability. A list of these cases is given in* [A99].

2.2. **Background on b-symplectic geometry.** In [GMP11] and [GMP12] the concept of b-symplectic manifolds is introduced and studied. These manifolds are symplectic away from a hypersurface Z; along Z the symplectic form has a certain controlled singularity. We first want to describe this singularity from the Poisson viewpoint:

b-Poisson manifolds. A symplectic structure ω, which is a section of $\bigwedge^2 T^*M$ induces a "dual" bivector field Π, i.e. a section of $\bigwedge^2 TM$:

$$\Pi(df, dg) := \omega(X_f, X_g) = \{f, g\}, \qquad f, g \in C^\infty(M).$$

It can be shown that the bivector field Π associated to a symplectic form satisfies the Jacobi identity, which means that it is a *Poisson* bivector field. Now consider the case where we start with a symplectic form on $M \setminus Z$ whose dual Poisson structure vanishes along Z in the following controlled way:

Definition 6. *Let (M^{2n}, Π) be an oriented Poisson manifold. If the map*

$$p \in M \mapsto (\Pi(p))^n \in \bigwedge^{2n}(TM)$$

[2]Bruns indeed showed it is not algebraically integrable.

is transverse to the zero section, then Π *is called a* b***-Poisson structure*** *on* M*. The hypersurface* $Z = \{p \in M | (\Pi(p))^n = 0\}$ *is the* **critical hypersurface** *of* Π*. The pair* (M, Π) *is called a* b***-Poisson manifold****.*

We want to develop a concept that allows us to extend the symplectic structure from $M \backslash Z$ to the whole manifold M. This singular form will be called a "b-symplectic" form on M

Definition 7. *A* b***-manifold*** *is a pair* (M^N, Z) *of an oriented manifold* M *and an oriented hypersurface* $Z \subset M$*. A* b***-vector field*** *on a* b*-manifold* (M, Z) *is a vector field which is tangent to* Z *at every point* $p \in Z$*.*

If x is a local defining function for Z on some open set $U \subset M$ and $(x, y_1, \ldots, y_{N-1})$ is a chart on U, then the set of b-vector fields on U is a free $C^\infty(M)$-module with basis

$$(x\frac{\partial}{\partial x}, \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_N}).$$

We call the vector bundle associated to this locally free $C^\infty(M)$-module the b**-tangent bundle** and denote it ${}^bTM$.

We define the b**-cotangent bundle** ${}^bT^*M$ of M to be the vector bundle dual to ${}^bTM$.

Given a defining function f for Z, let $\mu \in \Omega^1(M \setminus Z)$ be the one-form $\frac{df}{f}$. If v is a b-vector field then the pairing $\langle v, \mu \rangle \in C^\infty(M \setminus Z)$ extends smoothly over Z and hence μ itself extends smoothly over Z as a section of ${}^bT^*M$. For ease of notation, we will write $\mu = \frac{df}{f}$, even though the expression on the right hand side is not well-defined at points in Z.

For each $k > 0$, let ${}^b\Omega^k(M)$ denote the space of b**-de Rham** k**-forms**, i.e., sections of the vector bundle $\Lambda^k({}^bT^*M)$. The usual space of de Rham k-forms sits inside this space in a natural way; for f a defining function of Z every b-de Rham k-form can be written as

$$\omega = \alpha \wedge \frac{df}{f} + \beta, \text{ with } \alpha \in \Omega^{k-1}(M) \text{ and } \beta \in \Omega^k(M). \tag{1}$$

The decomposition (1) enables us to extend the exterior d operator to ${}^b\Omega(M)$ by setting

$$d\omega = d\alpha \wedge \frac{df}{f} + d\beta.$$

The right hand side is well defined and agrees with the usual exterior d operator on $M \backslash Z$ and also extends smoothly over M as a section of $\Lambda^{k+1}({}^bT^*M)$. Note that $d^2 = 0$, which allows us to define the complex of b-forms, the b-de Rham complex.

In order for this complex to admit a Poincaré lemma, it is convenient to enlarge the set of smooth functions and consider the set of b**-functions** ${}^bC^\infty(M)$, which consists of functions with values in $\mathbb{R} \cup \{\infty\}$ of the form

$$c \log|f| + g,$$

where $c \in \mathbb{R}$, f is a defining function for Z, and g is a smooth function. For ease of notation, from now on we identify $\mathbb{R}$ with the completion $\mathbb{R} \cup \{\infty\}$.

We define the differential operator d on this space in the obvious way:

$$d(c \log|f| + g) := \frac{c\, df}{f} + dg \in {}^b\Omega^1(M),$$

where dg is the standard de Rham derivative.

Moreover, we define the Lie derivative of b-forms via the **Cartan formula**:

$$\mathcal{L}_X\omega = \iota_X(d\omega) + d(\iota_X\omega) \in^b \Omega^k(M), \tag{2}$$

where $\omega \in^b \Omega^k(M)$ and X is a b-vector field.

2.2.1. *b-symplectic manifolds.* Instead of working with b-Poisson structures we can dualize them and work with b-forms. In that sense, a b-symplectic form is just a symplectic form modeled over a different Lie algebroid (the b-cotangent bundle instead of the cotangent bundle):

Definition 8. *Let (M^{2n}, Z) be a b-manifold and $\omega \in {}^b\Omega^2(M)$ a closed b-form. We say that ω is **b-symplectic** if ω_p is of maximal rank as an element of $\Lambda^2({}^bT_p^*M)$ for all $p \in M$.*

Definition 9 (**b-Hamiltonian vector field**)**.** *Let (M, ω) be a b-symplectic manifold. Given a b-function $H \in^b C^\infty(M)$ we denote by X_H the (smooth) vector field satisfying*

$$\iota_{X_H}\omega = -dH.$$

Obviously, the flow of a b-Hamiltonian vector field preserves the b-symplectic form and hence the Poisson structure, so b-Hamiltonian vector fields are in particular Poisson vector fields.

The classical Darboux theorem for symplectic manifolds has its analogue in the b-symplectic case ([GMP12]):

Theorem 10 (**b-Darboux theorem**)**.** *Let (M, Z, ω) be a b-symplectic manifold. Then, on a neighborhood of a point $p \in Z$, there exist coordinates $(x_1, y_1, \ldots, x_{n-1}, y_{n-1}, z, t)$ centered at p such that*

$$\omega = \sum_{i=1}^{n-1} dx_i \wedge dy_i + \frac{1}{z}\, dz \wedge dt.$$

Remark 11. As is clear from the proof of the b-Darboux theorem in [GMP12], we can specify a particular local defining function z of the critical hypersurface around m and complete it to a coordinate system $(x_1, y_1, \ldots x_{n-1}, y_{n-1}, z, t)$ such that the above holds.

2.2.2. *The topology and geometry of Z.* For the rest of this paper we will also assume that Z is **compact**. For a given volume form Ω on a Poisson manifold M the associated **modular vector field** u^Ω_{mod} is defined as the following derivation:

$$C^\infty(M) \to \mathbb{R} : f \mapsto \frac{\mathcal{L}_{X_f}\Omega}{\Omega}.$$

It can be shown (see for instance [We97]) that this is indeed a derivation and, moreover, a Poisson vector field. Furthermore, for different choices of volume form Ω, the resulting vector fields only differ by a Hamiltonian vector field.

The topology of the exceptional hypersurface Z of a b-symplectic structure has been studied in [GMP11] and [GMP12]. In [GMP11] it was shown that if Z is compact and connected, then it is the mapping torus of any of its symplectic leaves $\mathcal{L}$ by the flow of any choice of modular vector field u:

$$Z = (\mathcal{L} \times [0,k])/_{(x,0)\sim(\phi(x),k)},$$

where k is a certain positive real number and ϕ is the time-k flow of u. In particular, all the symplectic leaves inside Z are symplectomorphic.

In the transverse direction to the symplectic leaves, all the modular vector fields flow with the same speed. This allows the following definition:

Definition 12 (Modular period)**.** *Taking any modular vector field u^Ω_{mod}, the* **modular period** *of Z is the number k such that Z is the mapping torus*

$$Z = (\mathcal{L} \times [0,k])/_{(x,0)\sim(\phi(x),k)},$$

and the time-t flow of u^Ω_{mod} is translation by t in the $[0,k]$ factor above.

2.3. **Hamiltonian $\mathbb{T}^n$-actions on b-symplectic manifolds.** Hamiltonian $\mathbb{T}^n$-actions will play a key role in the definition of the cotangent model for b-symplectic manifolds. These actions were studied in [GMPS14]. We recall the definitions and results:

Definition 13. *An action of $\mathbb{T}^r$ on a b-symplectic manifold (M^{2n},ω) is* ***Hamiltonian*** *if for all $X, Y \in \mathfrak{t}$:*

- *the b-one-form $\iota_{X^\#}\omega$ is exact;*
- *$\omega(X^\#, Y^\#) = 0$.*

Here, $\mathfrak{t}$ denotes the Lie algebra of $\mathbb{T}^r$ and $X^\#$ is the fundamental vector field of X. The primitive of the exact b-one-form $\iota_{X^\#}\omega$ is defined via the moment map $\mu : M \to \mathfrak{t}^*$:

$$\iota_{X^\#}\omega|_p = d\langle \mu(p), X\rangle.$$

In other words, $X^\#$ is the b-Hamiltonian vector field of $-\langle \mu(p), X\rangle$.

2.3.1. *Modular weights.* When a b-function $f \in C^\infty(M)$ is expressed as $c \log|y| + g$ locally near some point of a component Z' of Z, the number $c_{Z'}(f) := c \in \mathbb{R}$ is uniquely determined by f, even though the functions y and g are not.

Definition 14 (Modular weight)**.** *Given a Hamiltonian $\mathbb{T}^r$-action on a b-symplectic manifold, the* ***modular weight*** *of a connected component Z' of Z is the map*

$$v_{Z'} : \mathfrak{t} \to \mathbb{R}$$

given by $v_{Z'}(X) = c_{Z'}(H_X)$. This map is linear and therefore we can regard it as an element of the dual of the Lie algebra $v_{Z'} \in \mathfrak{t}^*$. We denote the kernel of $v_{Z'}$ by $\mathfrak{t}_{Z'} \subset \mathfrak{t}$.

2.4. **b-integrable systems.** In [KMS16] we introduced a definition of integrable systems for b-symplectic manifolds, where we allow the integrals to be b-functions. Such a "b-integrable system" on a $2n$-dimensional manifold consists of n integrals, just as in the symplectic case. More precisely we have the following

Definition 15 (**b-integrable system**)**.** *A b-integrable system on a $2n$-dimensional b-symplectic manifold (M^{2n}, ω) is a set of n pairwise Poisson commuting b-functions $F = (f_1, \dots, f_{n-1}, f_n)$ (i.e, $\{f_i, f_j\} = 0$), satisfying, $df_1 \wedge \cdots \wedge df_n$ is nonzero as a section of $\wedge^n({}^bT^*(M))$ on a dense subset of M and on a dense subset of Z. We say that a point in M is* **regular** *if the vector fields $X_{f_1}, \dots, X_{f_n}$ are linearly independent (as* smooth *vector fields) at it.*

Notice that if a point on Z is regular, then at least one of the f_i must be non-smooth there.

On the set of regular points, the distribution given by $X_{f_1}, \dots, X_{f_n}$ defines a foliation $\mathcal{F}$. We denote the integral manifold through a regular point $m \in M$ by $\mathcal{F}_m$. If $\mathcal{F}_m$ is compact, then it is an n-dimensional torus (also referred to as "(standard) **Liouville torus**"). Because the X_{f_i} are b-vector fields and are therefore tangent to Z, any Liouville torus that intersects Z actually lies inside Z. Two (b-)integrable systems F and F' are called **equivalent** if there is a map $\mu : \mathbb{R}^n \supset F(M) \to \mathbb{R}^n$ taking one system to the other: $F' = \mu \circ F$. We will not distinguish between equivalent integrable systems.

Remark 16. Near a regular point of Z, a b-integrable system on a b-symplectic manifold is equivalent to one of the type $F = (f_1, \dots, f_{n-1}, f_n)$, where $f_1, \dots, f_{n-1}$ are C^∞ functions and f_n is a b-function. In fact, we may always assume that $f_n = c \log|t|$, where $c \in \mathbb{R}$ and t is a global defining function for Z.

In analogy to the Liouville-Mineur-Arnold theorem, we have the following *action-angle coordinates* theorem for b-integrable systems [KMS16]:

Theorem 17 (Action-angle coordinates for b-integrable systems)**.** *Let*

$$(M, \omega, F = (f_1, \dots, f_{n-1}, f_n = \log|t|))$$

be a b-integrable system, and let $m \in Z$ be a regular point for which the integral manifold containing m is compact, i.e. a Liouville torus $\mathcal{F}_m$. Then there exists an open neighborhood U of the torus $\mathcal{F}_m$ and coordinates

$$(\theta_1, \dots, \theta_n, p_1, \dots, p_n) : U \to \mathbb{T}^n \times B^n$$

such that

$$\omega|_U = \sum_{i=1}^{n-1} d\theta_i \wedge dp_i + \frac{c}{p_n} d\theta_n \wedge dp_n, \tag{3}$$

where the coordinates $p_1, \dots, p_n$ depend only on F and the number c is the modular period of the component of Z containing m.

3. Cotangent models for integrable systems

3.1. **General facts about cotangent lifts.** Let G be a Lie group and let M be any smooth manifold. Given a group action $\rho : G \times M \longrightarrow M$, we define its cotangent lift as the action on T^*M given by $\hat{\rho}_g := \rho^*_{g^{-1}}$ where $g \in G$. We then have a commuting diagram

$$\begin{array}{ccc} T^*M & \xrightarrow{\hat{\rho}_g} & T^*M \\ \Big\downarrow \pi & & \Big\downarrow \pi \\ M & \xrightarrow{\rho_g} & M \end{array} \tag{4}$$

where π is the canonical projection from T^*M to M.

The cotangent bundle T^*M is a symplectic manifold endowed with the symplectic form $\omega = -d\lambda$, where λ is the Liouville one-form. The latter can be defined intrinsically:

$$\langle \lambda_p, v \rangle := \langle p, (\pi_p)_*(v) \rangle \tag{5}$$

with $v \in T(T^*M), p \in T^*M$.

A straightforward argument [GS90] shows that the cotangent lift $\hat{\rho}$ is Hamiltonian with moment map $\mu : T^*M \to \mathfrak{g}^*$ given by

$$\langle \mu(p), X \rangle := \langle \lambda_p, X^{\#}|_p \rangle = \langle p, X^{\#}|_{\pi(p)} \rangle,$$

where $p \in T^*M$, X is an element of the Lie algebra $\mathfrak{g}$ and we use the same symbol $X^{\#}$ to denote the fundamental vector field of X generated by the action on T^*M or M.

An easy computation shows that the Liouville one-form is invariant under the action, i.e. $\hat{\rho}_g^* \lambda = \lambda$. It is known that invariance of λ implies equivariance of the moment map μ, meaning that

$$\mu \circ \hat{\rho}_g = Ad^*_{g^{-1}} \circ \mu.$$

A consequence is that the moment map is Poisson (cf. Proposition 7.1 in [CW99]).

To distinguish the construction of this section from other types of cotangent lifts that we will define later on, we will also refer to it as the **symplectic cotangent lift**.

3.2. **Symplectic cotangent lift of translations on the torus.** In the special case where the manifold M is a torus $\mathbb{T}^n$ and the group is $\mathbb{T}^n$ acting by translations, we obtain the following explicit structure: Let $\theta_1, \dots, \theta_n$ be the standard (S^1-valued) coordinates on $\mathbb{T}^n$ and let

$$\underbrace{\theta_1, \dots, \theta_n}_{=:\theta}, \underbrace{a_1, \dots, a_n}_{:=a} \tag{6}$$

be the corresponding chart on $T^*\mathbb{T}^n$, i.e. we associate to the coordinates (6) the cotangent vector $\sum_i a_i d\theta_i \in T^*_\theta \mathbb{T}^n$. The Liouville one-form, which we defined intrinsically above, is given in these coordinates by

$$\lambda = \sum_{i=1}^n a_i d\theta_i$$

and its negative differential is the standard symplectic form on $T^*\mathbb{T}^n$:

$$\omega_{can} = \sum_{i=1}^n d\theta_i \wedge da_i.$$

We denote by τ_β the translation by $\beta \in \mathbb{T}^n$ on $\mathbb{T}^n$; its lift to $T^*\mathbb{T}^n$ is given by

$$\hat{\tau}_\beta : (\theta, a) \mapsto (\theta + \beta, a).$$

The moment map $\mu_{can} : T^*\mathbb{T}^n \to \mathfrak{t}^*$ of the lifted action with respect to the canonical symplectic form is

$$\mu_{can}(\theta, a) = \sum_i a_i d\theta_i,$$

where the θ_i on the right hand side are understood as elements of $\mathfrak{t}^*$ in the obvious way. Even simpler, if we identify $\mathfrak{t}^*$ with $\mathbb{R}^n$ by choosing the standard basis $\frac{\partial}{\partial\theta_1}, \dots, \frac{\partial}{\partial\theta_n}$ of $\mathfrak{t}$ then the moment map is just the projection onto the second component of $T^*\mathbb{T}^n \cong \mathbb{T}^n \times \mathbb{R}^n$. We will adopt this viewpoint from now on. Note that the components of μ naturally define an integrable system on $T^*\mathbb{T}^n$.

3.3. ***b*-Cotangent lifts of $\mathbb{T}^n$.** As before, let $T^*\mathbb{T}^n$ be endowed with the standard coordinates (θ, a), $\theta \in \mathbb{T}^n$, $a \in \mathbb{R}^n$ and consider again the action on $T^*\mathbb{T}^n$ induced by lifting translations of the torus $\mathbb{T}^n$.

We now want to view this action as a b-Hamiltonian action with respect to a suitable b-symplectic form. In analogy to the classical Liouville one-form we define the following non-smooth one-form away from the hypersurface $Z = \{a_1 = 0\}$:

$$c \log |a_1| d\theta_1 + \sum_{i=2}^{n} a_i d\theta_i.$$

The negative differential of this form extends to a b-symplectic form on $T^*\mathbb{T}^n$, which we call the **twisted b-symplectic form** on $T^*\mathbb{T}^n$ (we will explain the terminology below):

$$\omega_{tw,c} := \frac{c}{a_1} d\theta_1 \wedge da_1 + \sum_{i=2}^{n} d\theta_i \wedge da_i. \tag{7}$$

The moment map of the lifted action with respect to this b-symplectic form is then given by

$$\mu_{tw,c} := (c \log |a_1|, a_2, \ldots, a_n),$$

where we identify $\mathfrak{t}^*$ with $\mathbb{R}^n$ as before.

We call this lift together with the b-symplectic form (7) the **twisted b-cotangent lift** with modular period c. Note that, in analogy to the symplectic case, the components of the moment map define a b-integrable system on $(T^*\mathbb{T}^n, \omega_{tw,c})$.

Remark 18. We use the term "twisted b-symplectic form" to distinguish our construction from the canonical b-symplectic form on ${}^bT^*M$, where M is any smooth manifold. The latter is obtained naturally if we use the intrinsic definition of the Liouville one-form (5) in the b-setting (see e.g. [NT96]). More precisely, for M a b-manifold, we define a b-form λ on ${}^bT^*M$ via

$$\langle \lambda_p, v \rangle := \langle p, (\pi_p)_*(v) \rangle, \tag{8}$$

where $v \in^b T({}^bT^*M)$ and $p \in^b T^*M$. The negative differential

$$\omega = -d\lambda$$

is the canonical b-symplectic form on ${}^bT^*M$. Here, we view ${}^bT^*M$ as a b-manifold with hypersurface $\pi^{-1}(Z)$ where

$$\pi : {}^bT^*M \to M$$

is the canonical projection. Choosing a local set of coordinates $x_1, \ldots, x_n$ on M, where x_1 is a defining function for Z we have a corresponding chart

$$(x_1, \ldots, x_n, p_1, \ldots, p_n)$$

on T^*M, given by identifying the $2n$-tuple above with the b-cotangent vector

$$p_1 \frac{dx_1}{x_1} + \sum_{i=2}^{n} p_i dx_i \in^b T_x^*M.$$

In these coordinates

$$\lambda = p_1 \frac{dx_1}{x_1} + \sum_{i=2}^{n} p_i dx_i \in^b T^*({}^bT^*M).$$

Note that the singularity here is given by the coordinate x_1 on the base manifold whereas in our "twisted" construction it is given by a fiber coordinate, which is what we require for the description of b-integrable systems.

3.4. **Models.** With the notation introduced above we define the following models of integrable systems, which we will use below to give a semilocal description of integrable and b-integrable systems. We write an integrable system as a triple (M, ω, F) where M is a manifold, ω a (b-)symplectic form and F the set of integrals.

(a) $(T^*\mathbb{T}^n)_{can} := (T^*\mathbb{T}^n, \omega_{can}, \mu_{can})$
(b) $(T^*\mathbb{T}^n)_{tw,c} := (T^*\mathbb{T}^n, \omega_{tw,c}, \mu_{tw,c})$

We say that two (b-)integrable systems (M_1, ω_1, F_1) and (M_2, ω_2, F_2) are **equivalent** if there exists a Poisson diffeomorphism ψ and a map $\varphi : \mathbb{R}^s \to \mathbb{R}^s$ such that we have a commuting diagram:

$$\begin{array}{ccc} (M_1, \omega_1) & \xrightarrow{\psi} & (M_2, \omega_2) \\ & \underset{F_1}{\searrow} & \downarrow{\scriptstyle \varphi \circ F_2} \\ & & \mathbb{R}^s \end{array}$$

4. Description of (b-)integrable systems in terms of cotangent models

4.1. **Symplectic case.** We restate the Liouville-Mineur-Arnold theorem (Theorem 1) in terms of the symplectic cotangent model:

Theorem 19. *Let $F = (f_1, \dots, f_n)$ be an integrable system on the symplectic manifold (M, ω). Then semilocally around a regular Liouville torus the system is equivalent to the cotangent model $(T^*\mathbb{T}^n)_{can}$ restricted to a neighborhood of the zero section $(T^*\mathbb{T}^n)_0$ of $T^*\mathbb{T}^n$.*

Proof. Let $\mathcal{T}$ be a regular Liouville torus of the system. The action-angle coordinate theorem (Theorem 1) implies that there exists a neighborhood U of $\mathcal{T}$ and a symplectomorphism

$$\psi : U \to (\mathbb{T}^n \times B^n, \omega_{can})$$

such that the "action coordinates", i.e. the projections onto B^n, depend only on the integrals $f_1, \dots, f_n$, hence their composition with ψ yields an equivalent integrable system on U. We know that the projections onto B^n correspond to the moment map μ_{can} of the cotangent lifted action on $T^*\mathbb{T}^n \cong \mathbb{T}^n \times \mathbb{R}^n$ (restricted to $\mathbb{T}^n \times B^n$ and understood with respect to the canonical basis on $\mathfrak{t}^*$), hence we can write

$$\begin{array}{ccc} U & \xrightarrow{\psi} & (T^*\mathbb{T}^n, \omega_{can}) \\ & \overset{F}{\searrow} & \downarrow{\scriptstyle \varphi\circ\mu} \\ & & \mathbb{R}^n \end{array}$$

where φ is the map that establishes the dependence of the action coordinates on $f_1, \ldots, f_n$. □

4.2. ***b*-symplectic case.** The model of twisted b-cotangent lift allows us to express the action-angle coordinate theorem for b-integrable systems in the following way:

Theorem 20. *Let $F = (f_1, \ldots, f_n)$ be a b-integrable system on the b-symplectic manifold (M, ω). Then semilocally around a regular Liouville torus $\mathcal{T}$, which lies inside the exceptional hypersurface Z of M, the system is equivalent to the cotangent model $(T^*\mathbb{T}^n)_{tw,c}$ restricted to a neighborhood of $(T^*\mathbb{T}^n)_0$. Here c is the modular period of the connected component of Z containing $\mathcal{T}$.*

Proof. The proof is the same as above using the action-angle coordinate theorem for b-integrable systems (Theorem 17): Around the Liouville torus $\mathcal{T}$ we have a Poisson diffeomorphism

$$\psi : U \to \mathbb{T}^n \times B^n$$

taking the b-symplectic form on U to

$$\sum_{i=1}^{n-1} d\theta_i \wedge dp_i + \frac{c}{p_n} d\theta_n \wedge dp_n,$$

where $(\theta_1, \ldots, \theta_n, p_1, \ldots, p_n)$ are the standard coordinates on $\mathbb{T}^n \times B^n$, and such that $p_1, \ldots, p_n$ only depend on the integrals. Hence in the language of Section 3.3 we have a commuting diagram

$$\begin{array}{ccc} U & \xrightarrow{\psi} & (T^*\mathbb{T}^n, \omega_{tw,c}) \\ & \overset{F}{\searrow} & \downarrow{\scriptstyle \varphi\circ\mu_{tw,c}} \\ & & \mathbb{R}^n \end{array}$$

□

4.3. ***b*-Cotangent lifts in the general setting.** Above we focused on the case where the manifold M is a torus and the action is by rotations of the torus on itself, since this is the model that describes (b-)integrable systems semilocally around a Liouville torus.

To obtain a wider class of examples, we now consider any manifold M and the action of any Lie group G on M:

$$\rho : G \times M \to M : (g, m) \mapsto \rho_g(m). \tag{9}$$

As described in Section 3.1 we can lift the action to an action $\hat{\rho}$ on T^*M, which is Hamiltonian with respect to the standard symplectic structure on T^*M. We want to investigate modifications of this construction, which lead to Hamiltonian actions on b-symplectic manifolds.

Canonical b-cotangent lift. Connecting with Remark 18, assume that M is an n-dimensional b-manifold with distinguished hypersurface Z. Instead of T^*M consider the b-cotangent bundle ${}^bT^*M$ endowed with the canonical b-symplectic structure as described in the remark. Moreover, assume that the action of G on M preserves the hypersurface Z, i.e. ρ_g is a b-map for all $g \in G$. Then the lift of ρ to an action on ${}^bT^*M$ is well-defined:

$$\hat{\rho} : G \times {}^bT^*M \to {}^bT^*M : (g, p) \mapsto \rho^*_{g^{-1}}(p).$$

We call this action on ${}^bT^*M$, endowed with the canonical b-symplectic structure, the **canonical b-cotangent lift**.

Proposition 21. *The canonical b-cotangent lift is Hamiltonian with equivariant moment map given by*

$$\mu : {}^bT^*M \to \mathfrak{g}^*, \quad \langle \mu(p), X \rangle := \langle \lambda_p, X^\#|_p \rangle = \langle p, X^\#|_{\pi(p)} \rangle, \tag{10}$$

where $p \in {}^bT^*M$, $X \in \mathfrak{g}$, $X^\#$ is the fundamental vector field of X under the action on ${}^bT^*M$ and the function $\langle \lambda, X^\# \rangle$ is smooth because $X^\#$ is a b-vector field.

Proof. The proof of Equation (10) for the moment map is exactly the same as in the symplectic case: Using the implicit definition of λ, Equation (8), we show that λ is invariant under the action:

$$\langle (\hat{\rho}_g^* \lambda)_p, v \rangle = \langle \lambda_{\hat{\rho}_g(p)}, (\hat{\rho}_g)_* v \rangle = \langle \hat{\rho}_g(p), (\pi_{\hat{\rho}_g(p)})_* ((\hat{\rho}_g)_* v) \rangle =$$

$$= \langle \rho^*_{g^{-1}}(p), (\rho_{g^{-1}})_* ((\pi_p)_*(v)) \rangle = \langle p, (\pi_p)_*(v) \rangle.$$

In going from the first to the second line we have used the definition of $\hat{\rho}$ and applied the chain rule to $\pi_{\hat{\rho}_g(p)} \circ \hat{\rho}_g = \rho_{g^{-1}} \circ \pi_p$.

Hence we have $\mathcal{L}_{X^\#} \lambda = 0$ and applying the Cartan formula for b-symplectic forms, Equation (2), we obtain

$$\iota_{X^\#} \omega_p = -\iota_{X^\#} d\lambda_p = d(\iota_{X^\#} \lambda_p),$$

which proves the expression for the moment map stated above.

Equivariance of μ is a consequence of the invariance of λ:

$$\langle (Ad^*_{g^{-1}} \circ \mu)(p), X \rangle = \langle \mu(p), Ad_{g^{-1}} X \rangle = \langle \lambda_p, \underbrace{(Ad_{g^{-1}} X)^\#}_{=(\hat{\rho}_g)_* X^\#} |_p \rangle =$$

$$= \langle \hat{\rho}_g^* \lambda_p, X^\#|_{\hat{\rho}_{g^{-1}}(p)} \rangle = \langle \lambda_{\hat{\rho}_{g^{-1}}(p)}, X^\#|_{\hat{\rho}_{g^{-1}}(p)} \rangle = \langle \mu(\hat{\rho}_{g^{-1}}(p)), X \rangle$$

for all $g \in G$, $X \in \mathfrak{g}$, $p \in T^*M$, where in the first equality of the second line we have used that λ is invariant. □

Remark 22. The condition that the action preserves Z means that all fundamental vector fields are tangent to Z and therefore at a point in Z the maximum number of independent fundamental vector fields is $n-1$. This means that the moment map of such an action never defines a b-integrable system on ${}^bT^*M$ since this would require n independent functions.

Twisted b-cotangent lift. We have already defined the twisted b-cotangent lift on the cotangent space of a torus $T^*\mathbb{T}^n$ in Section 3.3. In particular, on T^*S^1 with standard coordinates (θ, a) we have the logarithmic Liouville one-form $\lambda_{tw,c} = \log|a|d\theta$ for $a \neq 0$.

Now consider any $(n-1)$-dimensional manifold N and let λ_N be the standard Liouville one-form on T^*N. We endow the product $T^*(S^1 \times N) \cong T^*S^1 \times T^*N$ with the product structure $\lambda := (\lambda_{tw,c}, \lambda_N)$ (defined for $a \neq 0$). Its negative differential $\omega = -d\lambda$ is a b-symplectic structure with critical hypersurface given by $a = 0$.

Let K be a Lie group acting on N and consider the component-wise action of $G := S^1 \times K$ on $M := S^1 \times N$ where S^1 acts on itself by rotations. We lift this action to T^*M as described in the beginning of this section. This construction, where T^*M is endowed with the b-symplectic form ω, is called the **twisted b-contangent lift**.

If $(x_1, \dots, x_{n-1})$ is a chart on N and $(x_1, \dots, x_{n-1}, y_1, \dots, y_{n-1})$ the corresponding chart on T^*N we have the following local expression for λ

$$\lambda = \log|a|d\theta + \sum_{i=1}^{n-1} y_i dx_i.$$

Just as in the symplectic case and in the case of the canonical b-cotangent lift, this action is Hamiltonian with moment map given by contracting the fundamental vector fields with λ:

Proposition 23. *The twisted b-cotangent lift on $M = S^1 \times N$ is Hamiltonian with equivariant moment map μ given by*

$$\langle \mu(p), X \rangle := \langle \lambda_p, X^\#|_p \rangle, \tag{11}$$

where $X^\#$ is the fundamental vector field of X under the action on T^*M.

Proof. As in the proof of Proposition 21, we show that the action preserves the logarithmic Liouville one-form $\lambda = (\lambda_{tw,c}, \lambda_N)$. Since the action splits this amounts to showing invariance of $\lambda_{tw,c}$ under S^1; the invariance of λ_N under K is the classical symplectic result. The former is easy to see:

$$(\hat{\tau})^*_\varphi \lambda_{tw,c} = \log|a \circ \hat{\tau}_\varphi| d(\underbrace{\theta \circ \hat{\tau}_\varphi}_{=\theta+\varphi}) = \log|a|d\theta,$$

where τ is the action of S^1 on itself by rotations and $\varphi \in S^1$.

This shows that $\mathcal{L}_{X^\#}\lambda = 0$ and as before we conclude the proof by using Cartan's formula.

□

Remark 24. A special case of a manifold $S^1 \times N$ is a cylinder $\mathbb{T}^k \times \mathbb{R}^{n-k}$. We will use the construction in this case in Section 4.4.2

Remark 25. For computing the moment map it is convenient to observe that the expression $\langle \lambda, X^* \rangle$ remains unchanged when we replace the fundamental vector field X^* of the action on T^*M by any vector field on T^*M that projects to the same vector field on M (namely the fundamental vector field of the action on M). This follows immediately from the definition of λ.

4.4. **Examples of integrable systems on b-symplectic manifolds.** As an application of the models above we can construct examples of ($b-$)integrable systems:

Theorem 26. *Let M be a smooth manifold of dimension n and let G be a n-dimensional abelian Lie group acting on M effectively. Pick a basis $X_1, \ldots, X_n$ of the Lie algebra of G. Consider the moment map $\mu : T^*M \to \mathfrak{g}^*$ of one of the following Hamiltonian actions:*

(1) the (symplectic) cotangent lift on T^*M
(2) the twisted b-cotangent lift on T^*M, where we assume that $M = S^1 \times N$ and $G = S^1 \times K$ for N an $n-1$ dimensional manifold and K a Lie algebra and that the action splits with S^1 acting on itself by rotations.

Then the components of the moment map with respect to the basis X_i define an (1) integrable resp. (2) b-integrable system on T^*M.

Proof. Denote the components of the moment map by $f_i := \langle \mu, X_i \rangle$. Effectiveness of the action implies that the f_i are linearly independent everywhere. Moreover, since μ is a Poisson map and the elements X_i commute, we obtain $\{f_i, f_j\} = 0$.

□

4.4.1. ***The geodesic flow.*** A special case of a $\mathbb{T}^n$-action is obtained in the case of a Riemannian manifold M which is assumed to have the property that all its geodesics are closed, so-called **P-manifolds**. Then the geodesics admit a common period (see e.g. [Be12], Lemma 7.11); hence their flow induces an S^1-action on M and we can use the twisted b-cotangent lift to obtain a b-Hamiltonian S^1-action on T^*M. The moment map then corresponds to a *non-commutative* b-integrable system on T^*M, which is a generalization of the systems studied here and will be explored in a future work. In dimension two, examples of P-manifolds are Zoll and Tannery surfaces (see Chapter 4 in [Be12]). Given an S^1-action on such a surface, via the cotangent lift we immediately obtain examples of (b-)integrable systems on its cotangent bundle.

4.4.2. ***Affine manifolds.*** A smooth manifold M is called **flat** if it admits a flat (i.e. zero curvature) connection. It is called **affine** if moreover the connection is torsion-free.

It is well-known that a simply connected flat manifold is parallelizable, i.e. it admits a basis of vector fields that are everywhere independent. Such a basis is called parallel. The relation between flatness (in the sense that the curvature is zero) and parallelizability was studied in [T65]. We are not assuming that the affine manifold is compact.

Bieberbach [Bi1911] proved in 1911 that any complete affine Riemannian manifold is a finite quotient of $\mathbb{R}^k \times \mathbb{T}^{n-k}$.

Theorem 27. *Let M be a cylinder $\mathbb{R}^k \times \mathbb{T}^{n-k}$. Then for any choice of parallel basis $X_1, \ldots X_n$, we obtain a (b-)integrable system on T^*M.*

Proof. Let $X_1, \ldots, X_n$ be a global basis of parallel vector fields. Since the torsion of the connection is zero and the vector fields X_i are parallel, the expression $\nabla_{X_i} X_j - \nabla_{X_j} X_i - [X_i, X_j] = T^\nabla(X_i, X_j) = 0$ yields $[X_i, X_j] = 0$. In other words, the flows of the vector fields commute. Let us denote by $\Phi_{X_j}^{s_j}$ the s_j-time flow of the vector field X_j. Since the manifold is complete, the joint flow of the vector fields X_i then defines an $\mathbb{R}^n$-action[3],

$$\begin{aligned} \Phi : \mathbb{R}^n \times M &\to M \\ \big((s_1, \ldots, s_n), (x)\big) &\mapsto \Phi_{X_1}^{s_1} \circ \cdots \circ \Phi_{X_n}^{s_n}((x)). \end{aligned}$$

By the construction defined in Section 4.3 we obtain a (b-)Hamiltonian action on T^*M and the components of the moment map of this action define a (b-)integrable system (Theorem 26). □

Remark 28. We proved the above result only for cylinders $\mathbb{R}^k \times \mathbb{T}^{n-k}$. It will be interesting to explore whether a similar construction is possible for finite quotients of $\mathbb{R}^k \times \mathbb{T}^{n-k}$, which by Bieberbach's result would correspond to all complete affine Riemannian manifolds.

Remark 29. Even if this yields examples of b-integrable systems on non-compact manifolds, we may consider Marsden-Weinstein reduction to obtain compact examples. Reduction in the b-setting is already plotted in [GMPS13] for abelian groups. The general scheme follows similar guidelines.

Remark 30. In [BJ04] the authors use the cotangent lift to $T^*(G/H)$ to construct examples of non-commutative integrable systems, this approach can also be exported to the context of b-symplectic manifolds to obtain examples of non-commutative integrable systems.

[3]Depending on the topology of the fiber, this action may descend to a $\mathbb{T}^n$-action or more generally to a $\mathbb{R}^k \times \mathbb{T}^{n-k}$-action.

5. b-INTEGRABLE SYSTEMS WITH SINGULARITIES

In this framework we will introduce non-degenerate integrable systems as twisted b-cotangent lifts of some action. This section is intended as an invitation to the study of singularities of integrable systems.

5.1. **The harmonic oscillator.** Let us consider the 2-dimensional harmonic oscillator, i.e. the coupling of two simple harmonic oscillators. The configuration space is $\mathbb{R}^2$ with standard coordinates $x = (x_1, x_2)$. The phase space is $T^*(\mathbb{R}^2)$ endowed with symplectic form $\omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$. H is the sum of potential and kinetic energy,

$$H = \frac{1}{2}(y_1^2 + y_2^2) + \frac{1}{2}(x_1^2 + x_2^2)$$

The level set $H = h$ is a sphere S^3. We have rotational symmetry on this sphere. Thus another first integral is the angular momentum $L = x_1y_2 - x_2y_1$ which corresponds to the action by rotations lifted to the cotangent bundle.

Its Hamiltonian vector field is $X_L = (-x_2, x_1, -y_2, y_1)$. This yields $X_L(H) = \{L, H\} = 0$ thus proving integrability of the system.

To construct a b-integrable system, we consider $S^1 \times S^1$ acting on $M := S^1 \times \mathbb{R}^2$, where the first S^1 component acts on itself by rotations and the second one acts on $\mathbb{R}^2$ by rotations. We lift the action to T^*M, which we endow with the twisted b-symplectic form, see Section 4.3. The moment map with respect to the standard basis of the Lie algebra of $S^1 \times S^1$ is then given by

$$T^*M \cong S^1 \times \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2 : (\theta, x_1, x_2, a, y_1, y_2) \mapsto (\log |a|, x_1y_2 - x_2y_1).$$

Note that the second component is the angular momentum L. We can complete these two functions to a b-integrable system by adding the energy H, i.e. the system is given by $(\log |a|, L, H)$.

5.2. **Hyperbolic singularities.** Consider the group $G := S^1 \times \mathbb{R}^+$ acting on $M := S^1 \times \mathbb{R}$ in the following way:

$$(\varphi, g) \cdot (\theta, x) := (\theta + \varphi, gx),$$

i.e. on the S^1 component we have rotations and on the $\mathbb{R}$ component we have multiplications. Then the Lie algebra basis $(\frac{\partial}{\partial \theta}, \frac{\partial}{\partial g})$ induces the following fundamental vector fields on M:

$$X_1 := \frac{\partial}{\partial \theta}, \qquad X_2 := x\frac{\partial}{\partial x}.$$

As defined in Section 4.3 we consider the twisted b-cotangent lift on T^*M, i.e. the b-symplectic structure $\omega = -d\lambda$ where

$$\lambda := \log |p| d\theta + y dx$$

and (θ, p, x, y) are the standard coordinates on T^*M. As we showed in Proposition 23, the lifted action on T^*M is b-Hamiltonian with moment map given by $\mu := (f_1, f_2)$:

$$f_1 = \langle \lambda, X_1^{\#} \rangle = \log |p|,$$
$$f_2 = \langle \lambda, X_2^{\#} \rangle = xy.$$

This type of singular b-integrable system is known as *hyperbolic singularity.*

Definition 31 (b-integrable system with hyperbolic singularity)**.** *Let (f_1, f_2) be a b-integrable system on a b-symplectic manifold (M, ω). Consider a point $p \in M$ where the system is singular, i.e. the Hamiltonian vector fields X_{f_i} are not independent there. We say that the singularity is of hyperbolic type if there is a chart (t, z, x, y) centred at p such that the critical hypersurface of ω is locally around p given by $t = 0$ and the integrals are*

$$f_1 = c \log |t|, \quad f_2 = xy.$$

5.3. **Focus-focus singularities.** Consider the group $G := S^1 \times \mathbb{R}^+ \times S^1$ acting on $M := S^1 \times \mathbb{R}^2$ in the following way:

$$(\varphi, a, \alpha) \cdot (\theta, x_1, x_2) := (\theta + \varphi, aR_\alpha(x_1, x_2)),$$

where R_α is the matrix corresponding to rotation by α in the plane. In other words, on $\mathbb{R}^2$ we have $\mathbb{R}^+$ acting by radial contractions/expansions and S^1 acting by rotations.

Using the coordinates above, the Lie algebra basis $(\frac{\partial}{\partial\theta}, \frac{\partial}{\partial a}, \frac{\partial}{\partial\alpha})$ induces the following fundamental vector fields on M:

$$X_1 := \frac{\partial}{\partial\theta}, \qquad X_2 := x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2}, \qquad X_3 := x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1}.$$

As above, we consider the twisted b-cotangent lift. The logarithmic Liouville one-form is

$$\lambda := \log |p| d\theta + y_1 dx_1 + y_2 dx_2$$

and the moment map is, according to Proposition 23, given by $\mu := (f_1, f_2, f_3)$ with

$$f_1 = \langle \lambda, X_1^{\#} \rangle = \log |p|,$$
$$f_2 = \langle \lambda, X_2^{\#} \rangle = x_1 y_1 + x_2 y_2,$$
$$f_3 = \langle \lambda, X_3^{\#} \rangle = x_1 y_2 - y_1 x_2.$$

In the theory of singular integrable systems on symplectic manifolds, the last two components define the well-known focus-focus singularity if we extend the manifold M to include points with $(x_1, x_2) = 0$.

In a future work we will study focus-focus singularities in the b-symplectic context. Here we only give the definition

Definition 32 (b-integrable system with focus-focus singularity)**.** *Let (f_1, f_2, f_3) be a b-integrable system on a b-symplectic manifold (M, ω). Consider a point $p \in M$ where the system is singular, i.e. the Hamiltonian vector fields X_{f_i} are not independent there. We say that the singularity is of focus-focus* *type if there is a chart* $(t, z, x_1, y_1, x_2, y_2)$ *centred at* p *such that the critical hypersurface of* ω *is locally around* p *given by* $t = 0$ *and the integrals are*

$$f_1 = c \log |t|, \quad f_2 = x_1 y_1 + x_2 y_2, \quad f_3 = x_1 y_2 - y_1 x_2.$$

Remark 33. By simplifying the examples above and eliminating the "b-part" we may also introduce non-degenerate singularities of integrable systems on symplectic manifolds in the sense of [E90], [E84], [Z96], [Mi14], [Mi03] and view them as cotangent lifts.

Remark 34. We may define non-degenerate singularities of integrable systems on b-symplectic manifolds as Cartan subalgebras of $\mathfrak{sp}(2n-1, \mathbb{R}) \oplus \mathbb{R}$.

Remark 35. We may obtain general $(0, k_h, k_f)$-Williamson type[4] singularities of integrable systems and view them as b-cotangent lifts by coupling the examples in Subsections 5.2 and 5.3.

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