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Homoclinic phenomena in conservative systems

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To my family

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Contents

Introduction

This thesis is devoted to the study of dynamical phenomena related to the existence of homoclinic orbits of conservative systems. We consider homoclinic (bi-asymptotic) orbits either to saddle periodic orbits or to whiskered tori. Such type orbits, called homoclinic by Poincaré, are of great interest in the theory of dynamical systems since their presence implies complicated dynamics.

We deal with a range of problems in two quite different topics related to the homoclinic phenomena in conservative systems:

- Bifurcations of homoclinic tangencies in area-preserving maps (APMs)
- Exponentially small splitting of separatrices for whiskered tori with several frequencies in Hamiltonian systems

The first topic is related to the study of the behavior of orbits near a given homoclinic trajectory, while the second topic consists in the detection of homoclinic orbits arising from a perturbation of a Hamiltonian system with a homoclinic connection (separatrix). Both topics are well known among specialists in dynamical systems, and any result obtained is very relevant for theoretical aspects as well as for applications, but many questions in these topics still remain unsolved. The problems of this thesis, on the one hand, are new and in line with modern research of chaotic dynamics in conservative systems. On the other hand, their statements go back to classic problems by H. Poincaré, J. Hadamard and other researchers of 19th century.

Before explaining the historical remarks and the main contributions to each topic, let us give some basic definitions and properties. It is well known from the theory of dynamical systems that an orbit or a trajectory is the ordered set of states determined by the evolution rule of the system considered at time t. When the time t is continuous, the evolution rule is the flow defined by an ordinary differential equation and an orbit is a curve. In the case of discrete time, the evolution rule is a map (diffeomorphism) and an orbit is a sequence of iterations of the map. Let a system have a hyperbolic invariant object (for instance, saddle equilibrium, saddle fixed point, hyperbolic periodic orbit, whiskered torus, normally hyperbolic invariant manifold, etc). Then, as well-known, there are two invariant manifolds associated to this invariant object: the stable one formed by all incoming orbits and the unstable one composed of outgoing orbits. Sometimes the invariant manifolds coincide (see Figure 1a), then

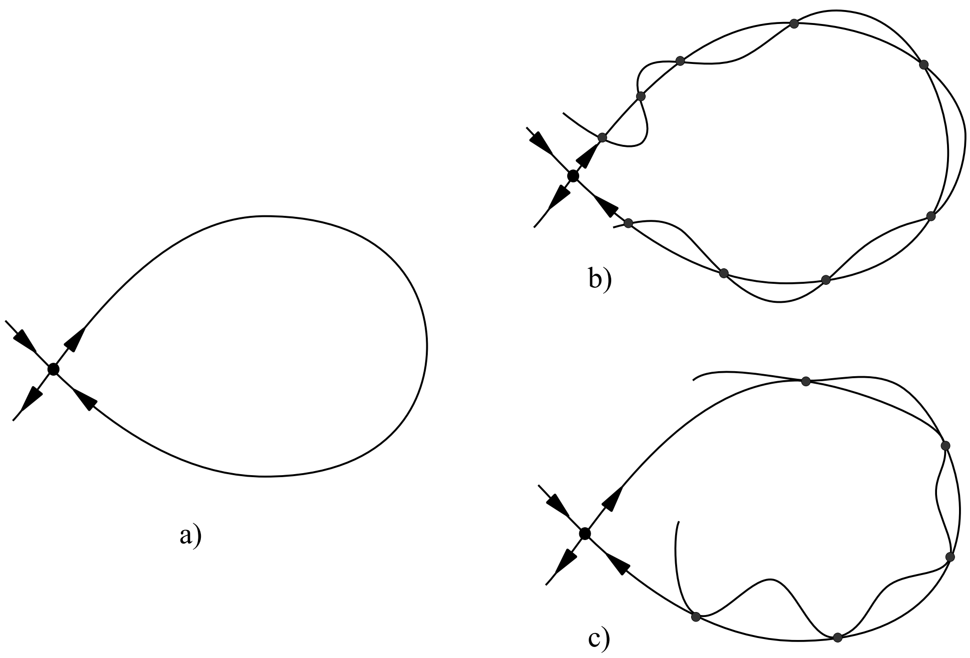

Figure 1: Examples of homoclinic orbits to a saddle point: a) homoclinic connection, b) transverse homoclinic orbit, c) nontransversal homoclinic orbit (homoclinic tangency)

we say that we have a *homoclinic connection*, also called *separatrix* or, in the case of two-dimensional flow, *separatrix loop* and *homoclinic loop*. This configuration is usual for integrable systems. In other cases, the stable and unstable invariant manifolds do not coincide and can intersect along a homoclinic orbit. When the stable and unstable invariant manifolds intersect transversally (at nonzero angle), the homoclinic orbit is called *transverse* (Figure 1b). Otherwise, we deal with a *homoclinic tangency* (a *non-transversal* homoclinic trajectory), see Figure 1c. Note that if the stable and unstable invariant manifolds of different hyperbolic objects intersect, the corresponding points and orbits are called *heteroclinic*.

Background and state of the art

The phenomenon of the transverse intersection of stable and unstable invariant manifolds was first discovered by the French mathematician H. Poincaré in his celebrated work [Poi90] while he was studying the problem on stability of the solar system. He considered the restricted 3-body problem Sun-Earth-Moon with the Moon as a small mass. Since without Moon the problem reduced to the Kepler problem, Poincaré described the problem Sun-Earth-Moon by means of a Hamiltonian system with 2 degrees of freedom as a small perturbation of the Kepler problem with the mass of the Moon as the perturbation parameter. In the memoir he introduced new different tools and ideas that laid down to the foundations of the area and are still popular nowadays: Poincaré (first return) maps, integral invariants, the Poincaré recurrence theorem, etc. He established that indeed the stable and unstable invariant manifolds intersected (splitting of separatrices) in the perturbed system and called the orbits passing through the intersection points as *doubly-asymptotic* (later in [Poi99] he gave them the name *homoclinic solutions*). Poincaré said:

> *Que l'on cherche à se représenter la figure formée par ces deux courbes et leurs intersections en nombre infini dont chacune correspond à une solution doublement asymptotique, ces intersections forment une sorte de treillis, de tissu, de réseau à mailles infiniment serrées ; chacune des deux courbes ne doit jamais se recouper elle-même, mais elle doit se replier sur elle-même d'une manière trés complexe pour venir recouper une infinité de fois toutes les mailles du réseau. On sera frappé de la complexité de cette figure, que je ne cherche même pas à tracer.*

(in a free translation into English: *If one attempts to represent the figure formed by these two curves and their infinitely many intersections, each of which corresponds to a doubly-asymptotic solution, these intersections form a kind of lattice or tissue or web with infinitely tight loops. Each of these curves must never intersect itself, but it must fold upon itself in a very complicated manner in order to intersect all the loops of the web infinitely many times. One is struck by the complexity of this figure, which I will not even attempt to draw*). This can be considered as the first mathematical description of chaotic motions. In [Poi99] he proved that the presence of at least one transverse homoclinic point implied the existence of infinitely many homoclinic points. He conjectured on the complexity of the dynamics near a transverse homoclinic trajectory and noted also that in some cases the splitting should be exponentially small with respect to the perturbation parameter. Also we would like to mention that to find the intersection points of the invariant manifolds Poincaré developed a method, known today as the Poincaré-Melnikov-Arnold method.

It is a curious fact that Poincaré considered the problem of description of homoclinic structures as not very interesting, since, as he correctly assumed, the corresponding orbits are all unstable.[1] However, the problems connected with the splitting of separatrices, and, in a wide sense, with studying systems close to integrable ones, were entitled by him as "the main problem of dynamics".

After Poincaré, the investigations of homoclinic structures were continued by G.D. Birkhoff. In his memoir of 1935 [Bir35], he proved that, in the case of two-dimensional

[1]Moreover, he knew well the paper [Had98] by another French mathematician J. Hadamard in which similar problem on dynamics of systems on surfaces of negative curvature was studied and the instability of geodesics was proved. Note that in this Hadamard's work the methods of symbolic dynamics were first applied.

area-preserving and analytic diffeomorphisms, the set $\mathcal{N}$ of the orbits entirely lying in a neighborhood of a transverse homoclinic trajectory contains infinitely many periodic orbits. Moreover, he proposed an important idea about the possibility of the complete description of the set $\mathcal{N}$ by means of the symbolic dynamics. This problem (on the description of the set $\mathcal{N}$) is called the *Poincaré-Birkhoff problem*.

Later the next most significant result in the field was achieved by S. Smale [Sma65]. He had introduced his famous *horseshoe map*, nowadays widely known as *Smale horseshoe*, that was the first example of dynamical system which is structurally stable (or *rough* in the terminology of Andronov and Pontryagin [AP37]) and had infinitely many periodic orbits. As Smale wrote in [Sma63] it was the answer to the *Andronov question*: "Can rough systems have infinitely many periodic orbits?" In [Sma65] the idea of the horseshoe was used to study the complicated behavior of orbits near a transverse homoclinic orbit of a multidimensional diffeomorphism T. He discovered a nontrivial hyperbolic subset on which T is topologically conjugate to the Bernoulli shift (bi-infinite sequences on two symbols) and showed that the Smale horseshoe is contained in $\mathcal{N}$. Naturally, the Smale horseshoe had the great influence on the theory of dynamical systems and now its presence is considered as a landmark of the chaotic dynamics. However, Smale imposed additional conditions on the linearization near the saddle that are not always fulfilled in some resonant cases (for example, in a quite wide range of systems as Hamiltonian systems and symplectic maps).

Two years later, L.P. Shilnikov [Shi67] used symbolic dynamics to give the complete solution of the Poincaré-Birkhoff problem on the description of the set $\mathcal{N}$ of the orbits entirely lying in a neighborhood of a transverse homoclinic trajectory to a saddle equilibrium. To overcome the obstacles due to the Smale's conditions on linearization, Shilnikov employed a new technique consisting in the resolution of a *boundary problem* near the saddle using *cross-coordinates* (also often called *Shilnikov coordinates*). He proved that the set $\mathcal{N}$ is a locally maximal invariant hyperbolic set and described $\mathcal{N}$ in terms of the Bernoulli shifts. Furthermore, this approach was improved by his students and collaborators to solve other problems on transverse homoclinic orbits [SSTC98, GS07].

One more work [Shi68] of Shilnikov on transverse homoclinic orbits is of great importance for the theory of dynamical chaos. In this paper, Shilnikov considers homoclinic trajectories to hyperbolic invariant tori and proves a result which is quite analogous to the case of a saddle fixed point. Among other works on this topic we note the famous works of V. M. Alekseev [Ale68, Ale69, Ale76] in which the Shilnikov's results were generalized to hyperbolic sets described by topological Markov chains with arbitrary (finite) number of states. Further, Alekseev used these tools and methods to describe various hyperbolic sets (and to discover the new ones) in celestial mechanics [Ale81]. Note also that L.M. Lerman and L.P. Shilnikov gave a solution of the Poincaré-Birkhoff problem both for the infinite dimensional case [LS88] and for the case of non-autonomous flows [LS92].

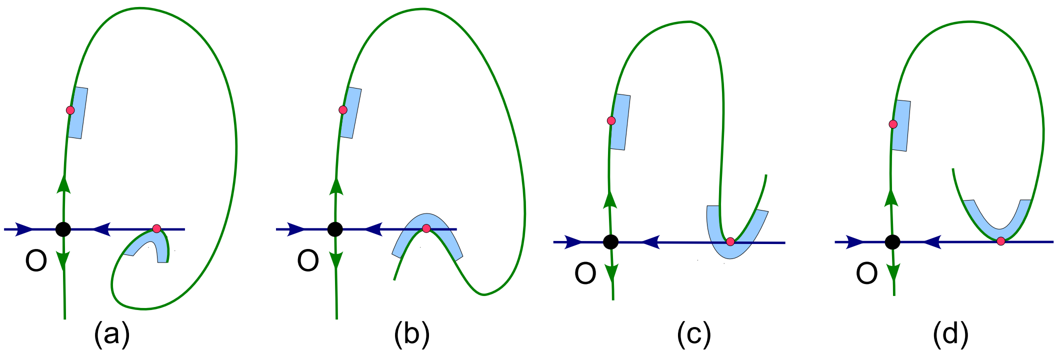

Figure 2: Examples of homoclinic tangencies (a) and (b) of the first type; (c) of the second type; (d) of the third type.

Bifurcations of nontransversal homoclinic orbits

Afterwards, it was natural to study nontransversal intersections of invariant manifolds whose case is more intricate. The systematic study of bifurcations of homoclinic tangencies was started by N.K. Gavrilov and L.P. Shilnikov [GS72, GS73] in the case of a two-dimensional dissipative diffeomorphism f_0 (three-dimensional flows) with a saddle periodic orbit O whose stable and unstable invariant manifolds were quadratically tangent along a homoclinic orbit Γ_0. The saddle orbit O had the multipliers $|\lambda| < 1, |\gamma| > 1$ and the saddle value $\sigma = |\lambda\gamma| < 1$. They considered a parameter-dependent family, also called as *general unfoldings*, f_μ, containing f_0 at $\mu = 0$. They discovered many remarkable dynamical phenomena and below we give a short description of them.

Classification of homoclinic tangencies. The diffeomorphisms f_0 with homoclinic tangencies were subdivided into three types depending of the structure of the set $\mathcal{N}_0$ of the orbits entirely lying in a neighborhood of Γ_0, see Figure 2. The first type: $\mathcal{N}_0$ has a trivial structure, $\mathcal{N}_0 = \{O, \Gamma_0\}$; the second type: $\mathcal{N}_0$ has the complete description in terms of the symbolic dynamics; the third type: $\mathcal{N}_0$ has a nontrivial (chaotic) structure. Such classification was extended later to the multidimensional case [GS86] as well as to the case $\sigma = 1$ [GS87] including the conservative one [GS01].

Existence of nontrivial hyperbolic subsets. Let $\mathcal{N}_\mu$ be the set of orbits of f_μ entirely lying in a small neighborhood of Γ_0. It was shown in [GS72] that $\mathcal{N}_\mu$ contains a subset $\tilde{\mathcal{N}}_\mu$ that is hyperbolic and has a nontrivial orbit structure (except for systems of the first type). Moreover, a description of the subsets $\tilde{\mathcal{N}}_\mu$ in terms of the symbolic dynamics was given in [GS72]

Homoclinic Ω-explosion. It was established in [GS73] that the system f_0 with a homoclinic tangency of the first type could belong to the boundary of Morse-Smale systems and, thus, separate systems with simple and chaotic dynamics. At the transition through this boundary ($\mu = 0$), the complicated dynamics appears immediately, "by explosion" (for that reason such bifurcations were called (homoclinic) Ω-*explosion*)[2]: before the tangency (at $\mu < 0$) the system has a simple dynamics: $\mathcal{N}_\mu = \{O\}$; there is only one (nontransversal) homoclinic orbit $\mathcal{N}_0 = \{O \cup \Gamma_0\}$ at the moment of the tangency ($\mu = 0$); and infinitely many Smale horseshoes appear just after the homoclinic tangency splits (at $\mu > 0$) into two transversal homoclinic orbits: $\mathcal{N}_\mu$ is nontrivial. In more detail, this phenomenon was later studied in papers of S. Newhouse and J. Palis [NP76], J. Palis and F. Takens [PT85], L. Shilnikov and O. Stenkin [SS98] etc.

Theorem on cascade of periodic sinks. This theorem is one of the fundamental results in homoclinic dynamics and plays quite important rôle in the theory of *dissipative* chaos. It states that, in the family f_μ, there exist (nonintersecting) intervals of values of μ accumulating to $\mu = 0$ such that the corresponding diffeomorphism of the family has an asymptotically stable periodic orbit (periodic sink). This result was extended to the multidimensional case by S. Newhouse [New74] and S. Gonchenko [Gon83], and general criteria for the existence of stable periodic orbits near a homoclinic tangency were pointed out by S. Gonchenko, L. Shilnikov and D. Turaev in [GST93a, GST96a].

Theory of moduli of topological and Ω-conjugacy of diffeomorphisms with homoclinic tangencies. The authors explained the importance of homoclinic tangencies of different types for the global dynamics of systems. As we said before, systems with homoclinic tangencies of the first type can belong to the boundary of Morse-Smale systems. In [GS73] it was shown that systems with homoclinic tangencies of the second type can belong to the boundary of hyperbolic systems. Also in [GS73] it was established that diffeomorphisms with homoclinic tangencies of the third type possess Ω-moduli, i.e. continuous invariants of topological conjugacy on the set of non-wandering orbits of f_μ. The main Ω-modulus $\theta = -\ln|\lambda|/\ln|\gamma|$ was introduced in [GS73], where it was shown that varying θ leads to bifurcations of periodic orbits of f_μ. Further investigations of this topic (see e.g. [Gon89, GS90, GST91, GST93a, GST96a, GST99, Kal00, DN05, GST08]) have laid to the creation of a very interesting and rich theory of homoclinic bifurcations which provides a theoretical basis of the dynamical chaos.

Simultaneously, S. Newhouse obtained a series of fundamental results [New70, New74, New79] related to the theory of homoclinic bifurcations in two-dimensional nonconservative diffeomorphisms. He wanted to see what happens in a one-parameter unfolding, when a homoclinic tangency splits, and discovered *wild hyperbolic sets*, i.e. nontrivial, transitive and uniformly hyperbolic sets whose the stable and unstable invariant manifolds have an irremovable nondegenerate tangency (in the sense that al-

[2] The first example of the Ω-explosion was given by Shilnikov in the work [Shi69] where bifurcations of a three-dimensional flow with several homoclinic loops to a saddle-saddle equilibrium were studied. Recall that the saddle-saddle is an equilibrium with eigenvalues $\lambda_1 = 0, \lambda_2 < 0, \lambda_3 > 0$ having also nonzero the first Lyapunov value l_1 (the simplest example is given by system $\dot{x}_1 = l_1 x_1^2, \dot{x}_2 = \lambda_2 x_2, \dot{x}_3 = \lambda_3 x_3$).

though the given homoclinic tangency is removed by a small perturbation of the system, one cannot avoid the appearance of new homoclinic tangencies). It is important to note that the wild hyperbolic sets exist for diffeomorphisms close (in the C^2-topology) to any diffeomorphism with a homoclinic tangency, [New79], and, hence, there exist open regions, the so-called *Newhouse regions*, where diffeomorphisms with homoclinic tangencies are dense. Later, the existence of Newhouse regions near any system with a homoclinic tangency was proved in [GST93b] for the general multidimensional case.

The dynamics in Newhouse regions for various kinds of systems was studied in a series of papers by S. Gonchenko, L. Shilnikov and D. Turaev [GST93c, GST97, GST99, GST07], who established the impossibility of providing a complete study of homoclinic bifurcations within the framework of finite parameter families.

These results were obtained for general systems. However, some genericity conditions exclude from considerations such very important classes of systems as conservative, reversible, Hamiltonian ones etc. The study of such systems with additional structures is of great interest and requires often special tools and methods. Some quite important results on homoclinic bifurcations of such systems were also obtained. We mention a series of papers [GG00, GG04, Gon02, GKM05, GOT12] related to bifurcations of diffeomorphisms with quadratic homoclinic tangencies in the case $\sigma = 1$, where very interesting homoclinic phenomena passing between the cases $\sigma < 1$ and $\sigma > 1$ were studied; in [GMO06] bifurcations of three-dimensional diffeomorphisms with quadratic homoclinic tangencies to a saddle-focus fixed point with Jacobian equal 1 was studied and the birth of Lorenz-like strange attractors was proved (see also [GST09, GO10] where analogous results were obtained).

Rather interesting results were obtained recently for two-dimensional reversible maps with homoclinic and heteroclinic tangencies. Thus, J. Lamb and O. Stenkin [LS04] proved the existence of Newhouse regions (in the class of reversible maps) in which maps possessing simultaneously infinitely many asymptotically stable (attracting), saddle, completely unstable (repelling) and elliptic periodic orbits are dense, extending the results of [GST97]. They considered the case of reversible and *a priori nonconservative maps* (i.e. maps having two symmetric saddle fixed points with the Jacobian different to 1). Symmetry breaking bifurcations leading to the appearance of attracting and repelling periodic orbits in reversible maps having a nontransversal heteroclinic cycle containing two saddle fixed points on the symmetry line were studied in [DGL06]. This paper gave a method of detecting elements of nonconservative dynamics in reversible systems.

Concerning the conservative case, we mention, first, the paper of Newhouse [New77], where the appearance of 1-elliptic periodic points[3] under bifurcation of homoclinic tangency was proved for symplectic multidimensional maps. Area-preserving maps (APMs) with homoclinic tangencies were studied by L. Mora, N. Romero [MR97] who proved the existence of a cascade of generic elliptic points. Also in the papers of

[3]That is, points having exactly one pair of multipliers $e^{\pm i\varphi}$. Note that the birth of 2-elliptic points in four-dimensional symplectic maps with homoclinic tangencies was established in [GST98, GST04].

S.Gonchenko, L. Shilnikov [GS97, GS00], conditions of the coexistence of infinitely many generic elliptic points were found for APMs with nontransversal heteroclinic cycles and in [GS01, GS03] a phenomenon of *global resonance* was discovered when an APM with a homoclinic tangency had infinitely many generic elliptic points of all successive (sufficiently large) periods.

In this thesis (Part I) we continue these investigations for APMs with homoclinic tangencies and give, in particular, in a sense a complete description for bifurcations of single-round periodic orbits (read, fixed points of first return maps defined near a homoclinic tangency) including construction of the corresponding bifurcation diagrams. Moreover, non-orientable APMs with homoclinic tangencies are also considered.

The methods of the study of the orbit behavior near homoclinic and heteroclinic tangencies are based, first of all, on the construction of return maps. In a series of papers [TY86, BS89, GST93a, GG00, GS01, GGT02, GSS02, GST02, GS03, GG04] it was shown that the corresponding rescaled first return maps are of the form of Hénon-like maps (standard Hénon maps, generalized Hénon maps, cubic Hénon maps, three-dimensional Hénon maps, etc).

Exponentially small splitting of separatrices in Hamiltonian systems

In general, if a Hamiltonian system with an object having the coincident stable and unstable invariant manifolds (separatrix) is perturbed, the invariant manifolds intersect at points of homoclinic orbits without coincidence. This phenomenon has got the name of *splitting of separatrices* and the problem of measuring the splitting has become classic since the work by H. Poincaré [Poi90] where this phenomenon was discovered. Many researchers devoted to finding estimates for the splitting in different settings both for flows and maps. The splitting of separatrices can be measured by several quantities such as: the maximal distance between the two invariant manifolds, the angle between the invariant manifolds at a homoclinic point, the area of the lobe between two consecutive homoclinic points, the homoclinic (Lazutkin) invariant as well as the width of the chaotic zone.

The most popular tool to measure the splitting is the Poincaré-Melnikov perturbative method, introduced by Poincaré in [Poi90] and rediscovered 70 years later by Melnikov and Arnold [Mel63, Arn64] (also called shortly as Melnikov method). The distance between the invariant manifolds is given by a function called *splitting function*. This method provides to it a first order approximation with respect to a perturbation parameter given by an integral known as *Melnikov function*, whose simple zeros give rise to transversal intersections between the stable and unstable perturbed manifolds. For n-dimensional whiskered tori, it was established [Eli94, DG00] that the splitting function and the Melnikov function, which are defined on n-dimensional tori, are the gradients of scalar functions: the splitting potential and the Melnikov potential, respectively. This means that the transverse homoclinic orbits correspond to the nondegenerate critical points of the splitting potential.

In the case of exponentially small splitting, the error of the method may overcome the main term and an additional study is required to ensure that the Poincaré-Melnikov approximation dominates the error term.

The first exponentially small upper bound was obtained by Neishtadt [Nei84] in one and a half degrees of freedom Hamiltonian systems. Later, similar estimates were found in [HMS88, Fon93, Fon95] for the rapidly perturbed pendulum. Also Fontich and Simó [FS90] obtained upper bounds for the splitting in the case of area-preserving maps close to identity. In the case of whiskered tori with 2 or more frequencies, several authors gave also exponentially small upper bounds [Sim94, Gal94, BCG97, BCF97, DGJS97]. In [DGS04] accurate upper bounds for the case of Diophantine n-dimensional whiskered tori were obtained by introducing flow-box coordinates.

In general, establishing lower bounds is usually more difficult, but some results have been obtained also by several methods.

First, the case of one-dimensional whiskered tori (periodic orbits) was considered [Laz84, DS92, Gel97, Tre97, DS97, DR98]. Here, V.F.Lazutkin [Laz84] introduced new tools studying splitting of separatrices in the Chirikov standard map. The invariant manifolds are parameterized analytically in a complex strip whose size is defined by the singularities of the unperturbed homoclinic orbit. Lazutkin used *flow box coordinates* around one of the manifolds and obtained in the complex strip the splitting function, an analytic periodic function. Using the analytic properties, one obtains, in the real domain, exponentially small bounds for the splitting. The same technique was used to justify the Poincaré-Melnikov method in a Hamiltonian with one and a half degrees of freedom [DS92, DS97] and an area-preserving map [DR98].

When the dimension of the whiskered torus is greater than 1, it turns out that the arithmetic properties of its frequencies play important rôle and influence on the expression of the quasiperiodic splitting function in which the small divisors are presented. This was first detected by Simó [Sim94] and then rigorously proved by Delshams et al. [DGJS97] in the quasi-periodically forced pendulum.

Later, several authors studied the splitting of separatrices for two-dimensional whiskered tori in 3 degrees of freedom Hamiltonian systems. For instance, Simó and Valls [SV01] studied the Arnold example (introduced by Arnold in [Arn64] to illustrate the transition chain mechanism which is crucial in the study of the Arnold diffusion) and also considered the homoclinic bifurcations that can occur. Lochak, Marco and Sauzin [Sau01, LMS03], Rudnev and Wiggins [RW00] used a different technique, namely the parametrization of the whiskers by two different solutions of Hamilton-Jacobi equation, to study a generalized Arnold model and proved the exponential smallness of the splitting for some intervals of the perturbation parameter ε. Pronin and Treschev [PT00] gave exponentially small bounds for a slow-fast system using another method called continuous averaging.

In [DG03, DG04] Delshams and Gutiérrez studied a generalization of the Arnold's example, a Hamiltonian system with 3 degrees of freedom having a two-dimensional whiskered torus whose frequency ratio is the *golden mean* $(\sqrt{5}-1)/2$ or other few quadratic number. They applied the theory of continued fractions to select primary resonances related to the small divisors that appear in the dominant harmonics of the Melnikov function. It was shown that the dominant harmonics of the splitting function correspond to the dominant harmonics of the Melnikov function, providing the asymptotic estimates (and, hence, lower bounds) for the splitting. With these estimates they proved that in the case of the quadratic golden frequencies, there exist exactly four transverse homoclinic orbits to the whiskered torus for all the sufficiently small values of the perturbation parameter.

The asymptotic estimates were done for two-dimensional whiskered tori with few quadratic frequencies [DG03, DG04], and one of the objectives in this thesis is to generalize these results to other quadratic numbers in the two-dimensional case and also to the three-dimensional case. It is worth mentioning that there is no standard theory of continued fractions for the case of three or more frequency vectors. This is the reason to consider a particular case of *cubic frequency vector*, just to be able to provide some results on exponentially small splitting of separatrices for 3 frequencies for the first time.

Notice that when the Poincaré-Melnikov approach cannot be validated, other techniques can be applied to get exponentially small estimates. For example, the parametrization of the invariant manifolds by solutions of the so-called *inner equation*, introduced by Lazutkin [Laz84], with the subsequent application of the *complex matching* technique [Bal06, OSS03, MSS11, MSS11b], and "beyond all orders" asymptotic methods [Lom00]. Also in [Tre97] an asymptotic formula for the splitting was given in the case of a "pendulum with a suspension point" using continuous averaging.

Structure and main results

This thesis is organized into two parts according to the topic considered, and each part is subdivided into a number of chapters which contain the main results and appendices with some complementary facts. Usually every chapter is devoted to a different problem.

Bifurcations of homoclinic tangencies in area-preserving maps

In the first part we study area-preserving maps (whose Jacobian is ± 1) with a homoclinic tangency to a saddle fixed point (see, for example, Figure 3). In order to know how trajectories behave in a neighborhood of a nontransversal homoclinic orbit, we study their bifurcations, i.e. we consider parameter dependent families of maps close to the initial one (which possesses the homoclinic tangency) and observe how the behavior of nearby trajectories changes qualitatively as the maps approach to or move away from the initial map (varying the parameters). Usually the initial map corre-

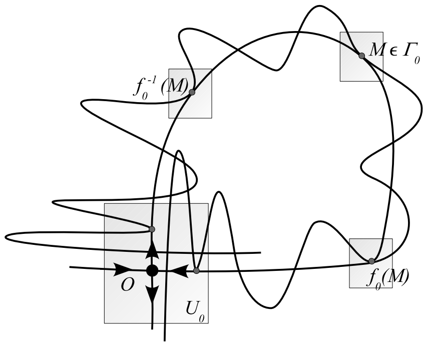

Figure 3: An example of area-preserving map having a quadratic homoclinic tangency along of a homoclinic orbit Γ_0.

sponds to a bifurcation value of the parameters and divides the family into subfamilies with qualitatively different phase portraits. In particular, we want to see what happens with the so-called *single-round periodic orbits*, i.e. periodic orbits which entirely lie in a neighborhood of the nontransversal homoclinic orbit and pass close to it only once. To this end, we construct *first return maps*, for which we use *finitely-smooth normal forms* of the saddle maps, containing only resonant monomials in nonlinearities up to some order $n \geq 3$, and introduce cross-coordinates (see details in Chapter 4). The fixed points of the first return maps correspond to single-round periodic orbits of the maps under consideration. Applying rescaling methods (see the Rescaling Lemmas in every chapter) we derive the first return maps to the Hénon-like maps whose bifurcations are well known. Thus, translating the results obtained for the fixed points of the return maps to the periodic orbits, we prove the main results. We also study the phenomenon of the coexistence of infinitely many single-round periodic orbits of different large periods (called *global resonance*) and prove a two parameter version of the theorem on cascades of elliptic periodic points.

More precisely, we consider the following problems:

Chapter 1. We consider two-dimensional symplectic maps, i.e. area-preserving maps which are also orientation-preserving (the Jacobian is equal to 1). The initial map f_0 has a saddle fixed point O with multipliers λ and λ^{-1} and possesses a quadratic homoclinic tangency Γ_0. Let $\mathcal{H}_s$ be a (codimension one) bifurcation surface composed of symplectic C^r-maps close to f_0 and such that every map of $\mathcal{H}_s$ has a nontransversal homoclinic orbit close to Γ_0. We consider one parameter general unfoldings f_μ of symplectic maps, where μ is the parameter of splitting of the homoclinic tangency, and we require that family f_μ is transverse to $\mathcal{H}_s$ at $\mu = 0$.

Note that the initial map f_0 possesses also a homoclinic invariant τ (introduced in (1.15)) that is responsible for the presence of the chaotic dynamics. The point is that the value $\tau = 0$ can be "bifurcational", even without splitting the initial tangency

[GS01]: if $\tau > 0$, f_0 has infinitely many Smale horseshoes, while if $\tau < 0$, then dynamics of f_0 is trivial: the set $\mathcal{N}_0$ of orbits entirely lying in a small neighborhood of Γ_0 contains only the saddle point O and the homoclinic orbit Γ_0, i.e. $\mathcal{N}_0 = O \bigcup \Gamma_0$.

By Rescaling Lemma 1.4, p. 48, we deduce the first return maps to the normal forms which take the form of a conservative Hénon-like map (with a small cubic term) and establish the one parameter theorem on cascade of elliptic periodic points (see more details in Theorem 1.1, p. 37):

Theorem 0.1. *Let f_0 be a symplectic map with a homoclinic tangency to a saddle point and f_μ be a one parameter general unfolding as described above. Then the following statements take place:*

1. *In any segment $[-\mu_0, \mu_0]$ of values μ, there are infinitely many open intervals δ_k, $k = \bar{k}, \bar{k}+1, \ldots$ ($\bar{k}$ is some integer), such that $\delta_k \to 0$ as $k \to +\infty$ and the map f_μ has a single-round elliptic periodic orbit at $\mu \in \delta_k$;*
2. *At the border points $\mu = \mu_k^+$ and $\mu = \mu_k^-$ of δ_k, f_μ has a single-round parabolic periodic orbit with double multipliers $+1$ and -1, respectively;*
3. *The elliptic orbit is generic (KAM-stable) for $\mu \in \delta_k$, except for exactly two values corresponding to the strong resonances 1:3 and 1:4, i.e. when the multipliers are $e^{\pm i2\pi/3}$ and $e^{\pm i\pi/2}$;*
4. *When $\tau \neq 0$, the intervals δ_i and δ_j do not intersect for sufficiently large integers $i \neq j$.*

Note that analogous results related to items 1, 2 and 3 of Theorem 0.1 were proved in [Bir87], [BS89] and [MR97], but the coexistence of single-round elliptic periodic orbits of different periods (the global resonance) was not considered. Item 4 of Theorem 0.1 shows that, in general ($\tau \neq 0$), such elliptic orbits of different and large periods cannot coexist.

The case $\tau = 0$ is exceptional and requires a further study within the framework of two-parameter unfoldings: $f_{\mu,\tau}$. It turns out that the phenomenon of the global resonance depends strongly on the geometry of the initial homoclinic tangency of f_0. We distinguish two cases: the case I with homoclinic tangencies similar to Figure 2(a),(c) and the case II with homoclinic tangencies as in Figure 2(b),(d). Thus, we prove a new two parameter version of the theorem on cascade of elliptic periodic points (Theorem 1.2, p. 38):

Theorem 0.2. *Let f_0 be a symplectic map with a homoclinic tangency to a saddle point and $f_{\mu,\tau}$ be a two parameter general unfolding as described above. Then the following statements take place:*

1. *In any neighborhood of the origin in the (τ, μ)-plane, there are infinitely many open domains Δ_k, for k starting with some integer $\bar{k}$, such that the map $f_{\mu,\tau}$ has a single-round periodic elliptic orbit in Δ_k;*

2. The domains Δ_k accumulate to the axis $\mu = 0$ as $k \to \infty$;

3. The boundaries of Δ_k are two curves L_k^+ and L_k^- where the map $f_{\mu,\tau}$ has a parabolic single-round periodic orbit with double multipliers either $+1$ and -1, respectively;

4. The elliptic orbit is generic (KAM-stable) for all values of $(\tau, \mu) \in \Delta_k$, except for those which belong to curves $L_k^{2\pi/3}$ and $L_k^{\pi/2}$ when resonances 1:3 and 1:4 occur, respectively;

5. In the case I, the domains Δ_i and Δ_j do not intersect for any sufficiently large and different integers i and j;

6. In the case II, the domains Δ_i and Δ_j are necessarily crossed and they intersect the axis $\mu = 0$; Moreover, all domains Δ_k with sufficiently large k contain the origin $(\tau = 0, \mu = 0)$, provided some condition (1.16) is satisfied.

See Figure 4 for an illustration of Theorem 0.2 where the planar domains Δ_k in the cases I and II are represented.

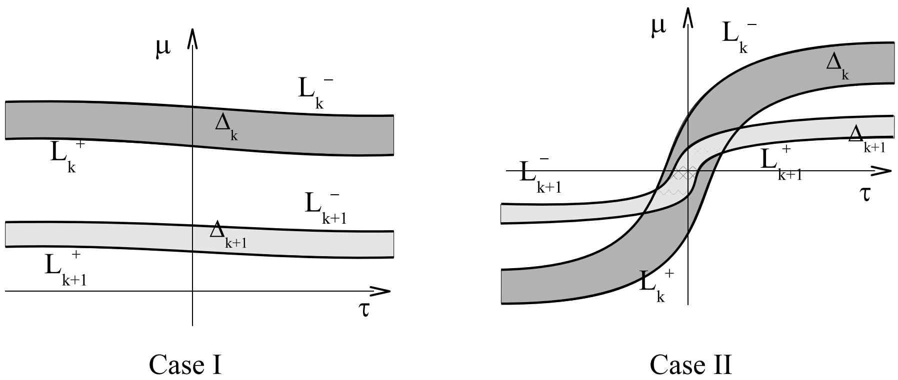

Figure 4: Domains Δ_k of Theorem 0.2 in the cases I and II

In case II, it follows from item 6 of Theorem 0.2 that at $\tau = 0$ all domains Δ_k, for k starting with some integer $\bar{k}$, intersect and, moreover, under certain conditions (see Corollary 1.1) all the domains contain $\mu = 0$ – this means that the map f_0 has infinitely many coexisting generic elliptic periodic points of all successive periods $k = \bar{k}, \bar{k}+1, \ldots$ (the global resonance).

In the next theorem we describe the character of bifurcations when μ varies inside the intervals δ_k of Theorem 0.1 and find the conditions under which the bifurcations through the strong resonances 1:3 and 1:4 are non-degenerate (Theorem 1.3, p. 40).

Theorem 0.3. *The bifurcations of fixed points in the first return map of f_μ follow the same scenario as the one observed in the conservative generalized Hénon map*

$$\bar{x} = y, \quad \bar{y} = M - x - y^2 + \nu_k y^3, \tag{1}$$

where $M \sim \lambda^{-2k}(\mu - \alpha_k)$ and $\alpha_k \sim \lambda^k$, $\nu_k \sim \lambda^k$ are small coefficients. For this map the resonance $1:3$ is non-degenerate for all values of ν_k, while the resonance $1:4$ is non-degenerate if $\nu_k \neq 0$

The reader is referred to equations (1.18) to see the exact formulae for M and ν_k as well as to Figure 1.7 of Chapter 1 to see the corresponding bifurcations of the map (1).

In this chapter we also provide a classification of quadratic homoclinic tangencies in the symplectic case (see Section 1.2).

Chapter 2. We consider f_0 an area-preserving map that does not preserve orientation (the Jacobian is -1). It has a saddle fixed point O with multipliers $0 < |\lambda| < 1 < |\gamma|$, $|\lambda\gamma| = 1$. The stable and unstable invariant manifolds of O have a homoclinic tangency along a homoclinic orbit Γ_0. We divide such APMs f_0 into 2 groups:

- the *globally non-orientable* maps with an orientable saddle (the saddle value is $\lambda\gamma = 1$) on a non-orientable manifold (Möbius strip, Klein bottle, etc), see an example of such a map in Figure5;

- the *locally non-orientable* maps with non-orientable saddles (the saddle value is $\lambda\gamma = -1$).

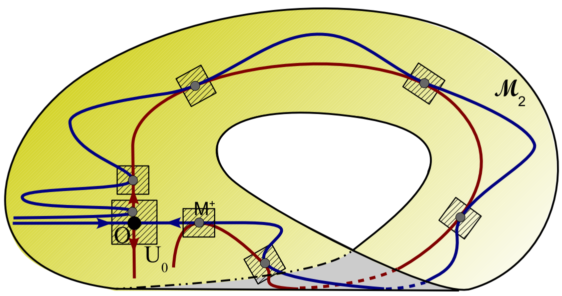

Figure 5: An example of non-orientable area-preserving map (on a Möbius strip) with a quadratic homoclinic tangency along a homoclinic orbit Γ_0.

We also consider one and two parameter families f_μ, $f_{\mu,\alpha}$ and $f_{\mu,\hat{\alpha}}$, where μ is still the parameter of splitting of the homoclinic tangency and α and $\hat{\alpha}$ (introduced in (2.6)) are analogs of τ in the symplectic case, that is homoclinic invariants responsible for the presence of the chaotic dynamics in f_0.

It turns out that in the globally non-orientable case the first return maps do not have elliptic fixed points, but a period two elliptic point appears which corresponds to a double-round periodic orbit; whereas, in the locally non-orientable maps, there exist intervals of the parameter μ where the first return maps have elliptic fixed points and other intervals where the first return maps have period 2 points. Thus, we establish the existence of cascades of elliptic points (see details in Theorem 2.1, p. 62):

Theorem 0.4. *Let f_0 be a non-orientable APM and f_μ be a one parameter family of close to f_0 APMs as described above. For any interval $(-\mu_0, \mu_0)$, there exists such a positive integer $\bar{k}$ such that the following holds:*

1. (a). *In the globally non-orientable case, the maps f_μ have no single-round elliptic periodic orbits, while there exist infinitely many intervals e_k^2, $k = \bar{k}, \bar{k}+1, \dots,$ where f_μ has a double-round elliptic orbit.*
   (b) *In the locally non-orientable case, there exist infinitely many alternating intervals e_{2m} and e_{2m+1}^2 such that the map f_μ has a single-round elliptic periodic orbit at $\mu \in \mathsf{e}_{2m}$ and has a double-round elliptic periodic orbit at $\mu \in \mathsf{e}_{2m+1}^2$.*

2. *The intervals e_k as well as e_k^2 accumulate to $\mu = 0$ as $k \to \infty$ and do not intersect for sufficiently large and different integer k if $\alpha \neq 0$ and $\hat{\alpha} \neq 0$.*

3. *Any interval e_k has border points $\mu = \mu_k^+$ and $\mu = \mu_k^-$ where the map f_μ has a single-round periodic orbit with double multiplier $+1$ and with double multiplier -1, respectively. Any interval e_k^2 has border points $\mu = \mu_k^{2+}$ and $\mu = \mu_k^{2-}$ where the map f_μ has a single-round periodic orbit with multipliers $+1$ and -1 at $\mu = \mu_k^{2+}$ and a double-round periodic orbit with double multiplier -1 at $\mu = \mu_k^{2-}$.*

4. *The elliptic orbit is generic (KAM-stable) in e_k and e_k^2, except for strong resonances 1:3 and 1:4.*

We also consider the question on the coexistence of elliptic periodic points by means of two parameter families $f_{\mu,\alpha}$ and $f_{\mu,\hat{\alpha}}$.

Theorem 0.5. *For two parameter families $f_{\mu,\alpha}$ and $f_{\mu,\hat{\alpha}}$ there exist infinitely many open domains, E_k^2 in the globally non-orientable case and domains E_{2m} and E_{2m+1}^2 in the locally non-orientable case, such that*

1. *The map $f_{\mu,\alpha}$ and $f_{\mu,\hat{\alpha}}$ have a single-round periodic elliptic orbit in E_k, and have a double-round elliptic periodic orbit in E_k^2;*

2. *The domains E_k and E_k^2 accumulate to the axis $\mu = 0$ as $k \to \infty$;*

3. *Any domain E_k has two boundaries, bifurcation curves L_k^+ and L_k^-, corresponding to a single-round nondegenerate periodic orbit with double multipliers $+1$ and -1, respectively;*

4. *Any domain E_k^2 has two boundaries, bifurcation curves L_k^{2+} and L_k^{2-}, corresponding to a single-round nondegenerate periodic orbit with multipliers ± 1 and a double-round nondegenerate periodic orbit with double multipliers -1, respectively;*

5. *In the globally non-orientable case, the domains E_i^2 and E_j^2 with sufficiently large $i \neq j$ are crossed in the (μ, α)-plane and they intersect the axis $\mu = 0$.*

6. *In the locally non-orientable case, in the (μ, α)-plane, the domains E_{2i} and E_{2j} are crossed for sufficiently large $i \neq j$ and intersect all domains E_{2m+1}^2 as well as the axis $\mu = 0$, but the domains E_{2i+1}^2 and E_{2j+1}^2 do not intersect for $i \neq j$. Otherwise, in the $(\mu, \hat{\alpha})$-plane, the domains E_{2i+1}^2 and E_{2j+1}^2 are crossed and they intersect all domains E_{2m} as well as the axis $\mu = 0$, but the domains E_{2i} and E_{2j} do not intersect for $i \neq j$.*

See Figure 6 for an illustration of the theorem.

From items 5 and 6 of Theorem 0.5, one can conclude that the domains of Theorem 0.5 can intersect for different k (and, hence, elliptic periodic orbits of different periods coexist) and all of them contain the origin ($\mu = \alpha = 0$ or $\mu = \hat{\alpha} = 0$. This means that f_0 has infinitely many elliptic orbits of different periods and, thus, the global resonance is observed for non-orientable maps too. The conditions under which the global resonance occurs, are pointed out in more details in the corresponding Theorem 2.3, p. 65.

Chapter 3. We study bifurcations in two-dimensional symplectic maps f_0 with a homoclinic orbit Γ_0 along which the stable and unstable invariant manifolds to a saddle fixed point have a cubic homoclinic tangency. We distinguish two types of cubic homoclinic tangencies: "incoming from above" and "incoming from below", see Figure 7.

We consider a two-parameter family f_{μ_1,μ_2}, where μ_1 and μ_2 are the parameters of splitting of the initial cubic tangency. The bifurcational diagram of the splitting is as in Figure 8 and it turns out that in the parameter plane there is a curve B_0 at which f_{μ_1,μ_2} has a quadratic homoclinic tangency together with another transverse homoclinic orbit. At passing through B_0, one transverse homoclinic orbit (close to Γ_0) breaks up into three transverse homoclinic orbits

The "incoming from above" and "incoming from below" cases give different first return maps derived by the cubic Rescaling lemma (see Lemma 3.8, p. 83) to diverse cubic conservative Hénon maps with quite different bifurcation diagrams (see the corresponding bifurcational diagrams in Figures 3.4– 3.5). In this way, we prove the following theorem:

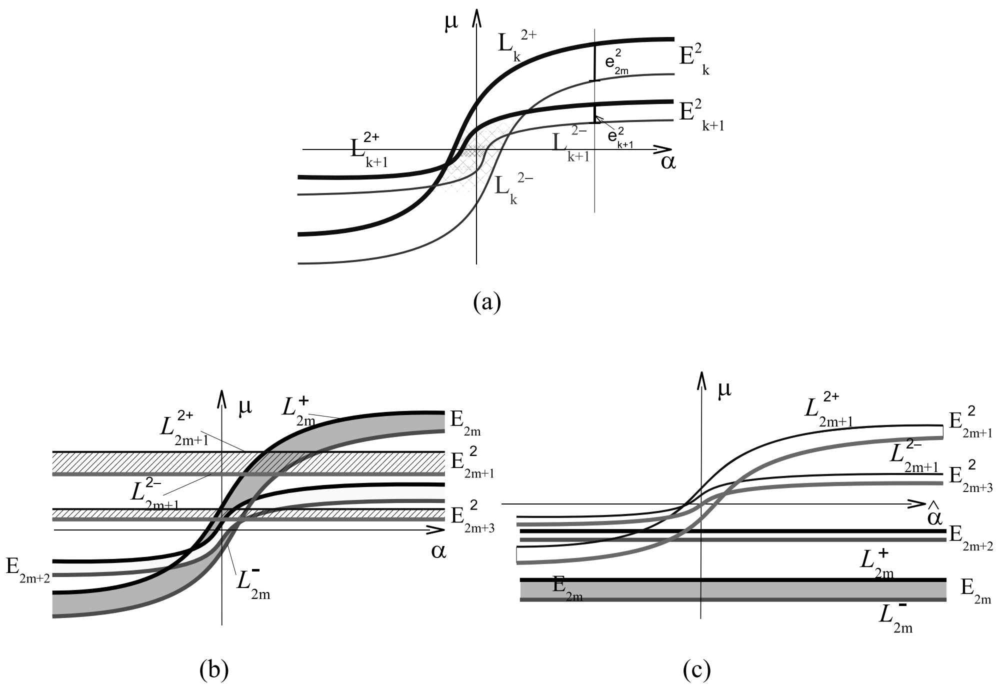

Figure 6: Domains E_k and E_k^2 of Theorem 0.5 (a) for globally non-orientable $f_{\mu,\alpha}$; (b) for locally non-orientable $f_{\mu,\alpha}$; (c) for locally non-orientable $f_{\mu,\hat{\alpha}}$.

Theorem 0.6 (On the structure of the bifurcational diagram in f_{μ_1,μ_2})**.** *Let f_{μ_1,μ_2} be a two parameter family of symplectic maps close to f_0 with a cubic homoclinic tangency. Then*
1) *In any neighborhood of the origin in the (μ_1,μ_2)-plane, there exist infinitely many bifurcation curves L_k^+ and L_k^- as well as $C_{1,2}^{k+}$ and $C_{1,2}^{k-}$ that accumulate to the curve B_0 as $k \to \infty$. The map f_{μ_1,μ_2} has a parabolic single-round periodic orbit with multipliers $\nu_1 = \nu_2 = +1$ (respectively, $\nu_1 = \nu_2 = -1$) at $\mu \in L_k^+$ (respectively, $\mu \in L_k^-$), a double-round periodic orbit with multipliers $\nu_1 = \nu_2 = +1$ (respectively, $\nu_1 = \nu_2 = -1$) at $\mu \in C_{1,2}^{k+}$ (respectively, $\mu \in C_{1,2}^{k-}$).*
2) *For any sufficiently large k, in the (μ_1,μ_2)-plane there is a domain E_k between the curves L_k^+ and L_k^- where the map f_{μ_1,μ_2} has a single-round elliptic periodic orbit at $\mu \in E_k$. This point is generic (KAM-stable) for all such μ except for the ones corresponding to strong resonances $1:3$ and $1:4$. $\nu_{1,2} = e^{\pm i\pi/2}$ or $\nu_{1,2} = e^{\pm i2\pi/3}$.*

See an illustration of Theorem 0.6 in Figure 9). Note that the results of this chapter are a generalization to the conservative case of the results obtained for the dissipative case in [Gon85].

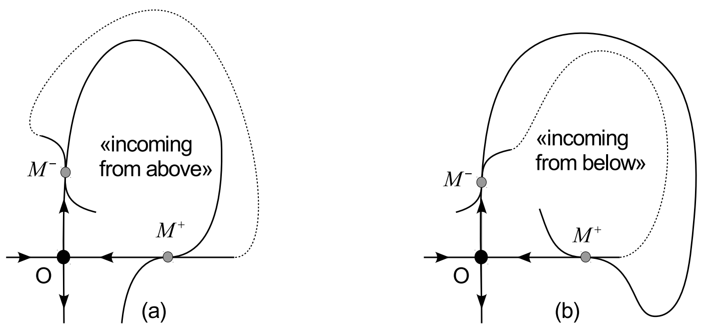

Figure 7: Two types of cubic homoclinic tangencies.

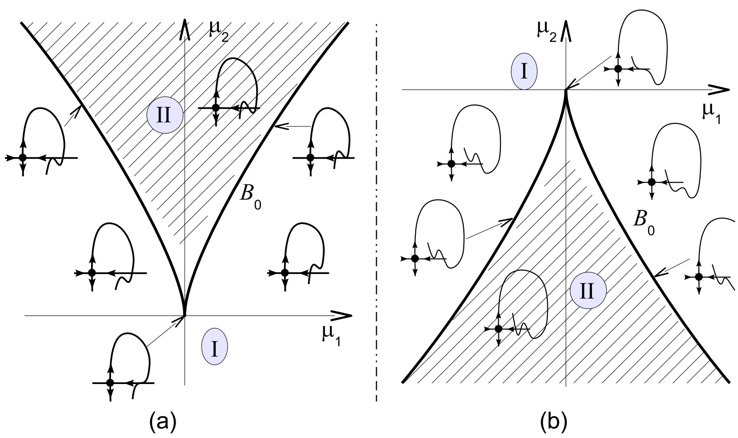

Figure 8: The bifurcation curve B_0 in (a) "incoming from above" and (b) "incoming from below" cases. Map f_μ has only one transverse homoclinic orbit in I and three such orbits in II.

Chapter 4. This chapter is devoted to the proof of the technical results (Lemmas 1.1, 1.2 and 2.2) which allow us to derive the area-preserving maps from Chapters 1 and 2 to the finitely-smooth normal forms. These finitely-smooth normal forms are used to construct the first return maps.

Appendix A. The structure of 1:4 resonance is analyzed for some conservative Hénon-like maps. Namely, we study bifurcations of fixed points with multipliers $e^{\pm i\pi/2}$ for the standard conservative Hénon map and both types of conservative cubic Hénon map.

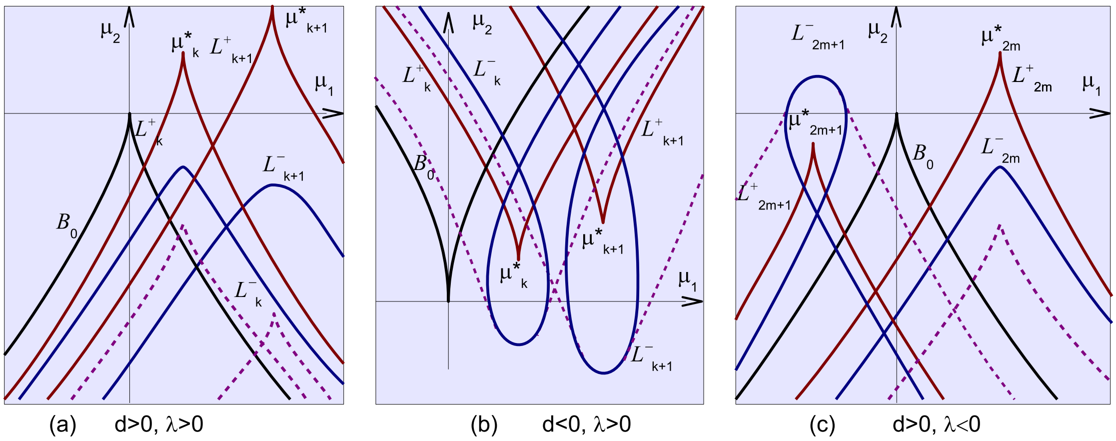

Figure 9: Main elements of bifurcation diagram for the families f_{μ_1,μ_2} in different cases.

Exponentially small splitting of separatrices for whiskered tori with various frequencies in Hamiltonian systems

The second part of the thesis is dedicated to the study of splitting of separatrices arising from a perturbation of a Hamiltonian system possessing a homoclinic connection. We consider a perturbation of an integrable Hamiltonian system having whiskered tori with coincident stable and unstable whiskers. Generally, in the perturbed system, the whiskers do not coincide anymore and our goal is to detect the transverse homoclinic orbits associated to the persistent whiskered tori. The perturbed system turns out to be not integrable due to the presence of these homoclinic trajectories and, consequently, there is chaotic dynamics near them. We give a suitable (Lazutkin) parametrization to the whiskers to determine the distance between them (the splitting function $\mathcal{M}(\theta)$), and the simple zeros of the splitting function give rise to transverse homoclinic orbits. We use the Poincaré-Melnikov approach to measure the splitting, although in the case of exponential smallness we have to ensure that the first order approximation overcome the error term.

Chapter 5. This is a preliminary chapter where we describe the nearly-integrable Hamiltonian system under study and give the statement of the problems. In particular, we consider an example of a *singular* or *weakly hyperbolic* (*a priori stable*) Hamiltonian with $n+1$ degrees of freedom defined by[4]

$$\begin{aligned} &H(x,y,\varphi,I) = H_0(x,y,I) + \mu H_1(x,\varphi), \quad (x,y,\varphi,I) \in \mathbb{T}\times\mathbb{R}\times\mathbb{T}^n\times\mathbb{R}^n \\ &H_0(x,y,I) = \langle \omega_\varepsilon, I\rangle + \tfrac{1}{2}\langle \Lambda I, I\rangle + y^2/2 + \cos x - 1, \quad H_1(x,\varphi) = h(x)f(\varphi), \end{aligned} \tag{2}$$

with two parameters ε and μ linked by a relation of kind $\mu = \varepsilon^p (p > 0)$.

[4]We study the cases $n = 2$ and $n = 3$ in Chapters 6 and 7, respectively

The unperturbed Hamiltonian H_0 possesses whiskered (hyperbolic) invariant tori with coincident stable and unstable invariant manifolds. We will focus our attention on the torus, located at $I = 0$, whose frequency vector is ω_ε that is a vector of *fast frequencies* given by a n-dimensional vector ω:

$$\omega_\varepsilon = \omega/\sqrt{\varepsilon}. \tag{3}$$

We will consider vector ω and a symmetric $n \times n$ matrix Λ such that H_0 satisfies the *Diophantine condition* of constant type

$$|\langle k, \omega \rangle| \geq \frac{\gamma}{|k|^\tau}, \quad \forall k \in \mathbb{Z}^n \backslash \{0\}$$

with some $\gamma > 0$, and the condition of *isoenergetic nondegeneracy*

$$\det \begin{pmatrix} \Lambda & \omega \\ \omega^\top & 0 \end{pmatrix} \neq 0.$$

We denote $\mathcal{W}_0$ the homoclinic whisker associated to this torus and consider the parameterization to it

$$\mathcal{W}_0 : (x_0(s), y_0(s), \theta, 0), \quad s \in \mathbb{R}, \theta \in \mathbb{T}^2,$$

where

$$x_0(s) = 4 \arctan e^s, \quad y_0(s) = \frac{2}{\cosh s}$$

When perturbing ($\mu \neq 0$), the *hyperbolic KAM theorem* implies that, for μ small enough, the whiskered torus persists, although the whiskers do not coincide anymore, in general. The problem consists in the detection of this splitting. As in the famous example by Arnold [Arn64], we choose the perturbation H_1 having the form of a product: $H_1(x, \varphi) = h(x) f(\varphi)$ with

$$h(x) = \cos x - \nu, \quad \text{with } \nu = 0 \text{ or } \nu = 1, \quad f(\varphi) = \sum_{k \in \mathbb{Z}^n} e^{-\rho|k|} \cos(\langle k, \varphi \rangle - \sigma_k), \text{ with } \sigma_k \in \mathbb{T}, \tag{4}$$

where the constant $\rho > 0$ in the Fourier expansion of $f(\varphi)$ gives the complex width of analyticity of $f(\varphi)$. The phases σ_k can be chosen arbitrarily, in principle, although some conditions on these phases have to be fulfilled for the validity of our results. The difference between two values of ν in (4) is the following: in the case $\nu = 0$ the whiskered torus persists with some shift and deformations, whereas in the case $\nu = 1$ it remains fixed under the perturbation, though the whiskers deform.

In Chapter 5 we explain in detail the Poincaré-Melnikov method that gives the first order in μ approximation for the splitting function $\mathcal{M}(\theta)$ whose simple zeros give rise to transverse homoclinic orbits. The point of the problem is that since this approximation is exponentially small in ε, we have to justify the method in our case

$\mu = \varepsilon^p$ and show that the remainder is smaller than the main term providing the corresponding estimates.

Note that due to the form of f, the splitting function $\mathcal{M}$ as well as its Poincaré-Melnikov approximation are readily represented in their Fourier series, and for each value of ε only a finite number of dominant harmonics is relevant to find the simple zeros.

Chapter 6. We study the splitting of whiskers for a two-dimensional (the case $n = 2$) whiskered torus in the Hamiltonian system (2) with 3 degrees of freedom. We consider the whiskered torus with frequencies (3) given by

$$\omega = (1, \Omega),$$

where Ω is a quadratic irrational number, i.e. a real root of a quadratic polynomial with integer coefficients. We deal with numbers whose continued fractions satisfy certain arithmetic properties (see (6.14)) which give us 24 cases for consideration:

$$\Omega_1, \Omega_2, \ldots, \Omega_{13}, \Omega_{1,2}, \ldots, \Omega_{1,12}, \tag{5}$$

where we denote an irrational quadratic number by Ω_a according to its periodic part in the continued fraction, for instance, $\Omega_1 = [1, 1, 1, \ldots] = [\overline{1}] = (\sqrt{5} - 1)/2$ is the famous *golden number*, $\Omega_{1,12} = [1, 12, 1, 12, \ldots] = [\overline{1, 12}] = 4\sqrt{3} - 6$.

We show that the Poincaré-Melnikov method can be applied to detect the splitting as long as we choose the exponent $p > p^*$, where p^* depends on the value of ν in the function $h(x)$ in (4). First, we give an asymptotic estimate for the maximal distance of the splitting by means of the maximum size in modulus of the splitting function $\mathcal{M}(\theta)$ (see details in Theorem 6.1, p. 125). We use the notation $f \sim g$ if we can bound $c_1|g| \le |f| \le c_2|g|$ with positive constant c_1, c_2 not depending on ε, μ.

Theorem 0.7 ((Maximal) splitting distance)**.** *For the Hamiltonian system (2-4) with $n = 2$, assume that ε is small enough and $\mu = \varepsilon^p$, $p > p^*$ with $p^* = 2$ if $\nu = 1$ and $p^* = 3$ if $\nu = 0$. Then, for the 24 quadratic numbers (5), the following estimate holds*

$$\max_{\theta \in \mathbb{T}^2} |\mathcal{M}(\theta)| \sim \frac{\mu}{\sqrt{\varepsilon}} \exp\left\{ -\frac{C_0 h_1(\varepsilon)}{\varepsilon^{1/4}} \right\}$$

where C_0 is a positive constant, given in (6.12), and the function $h_1(\varepsilon)$ is a periodic function in $\ln \varepsilon$ which satisfies $\min h_1(\varepsilon) = 1$ and $\max h_1(\varepsilon) = A_1 > 1$.

We also find 4 simple zeros of $\mathcal{M}$ and, hence, establish the existence of 4 homoclinic orbits to the whiskered tori. To show the simplicity of these zeros we need 2 essential dominant harmonics (see Definition 6.1 of essential dominant harmonics in p. 126). We provide estimates for the dominant harmonics as well as for the remaining harmonics and give also an asymptotic estimate for the minimal eigenvalue (in modulus) of the splitting matrix $\partial_\theta \mathcal{M}$ at each zero. This estimate provides a measure of transversality of the homoclinic orbits. See also Theorem 6.2, p. 130.

Theorem 0.8 (Transversality of the splitting)**.** *Under the hypotheses of Theorem 0.7, one has:*

- *the Melnikov function* $\mathcal{M}(\theta)$ *has exactly 4 zeros* θ_**, all simple, for all* ε *except for some small neighborhood of some geometric sequences of* ε *(given in (6.13) and (6.21)).*
- *The minimal eigenvalue of* $\partial_\theta \mathcal{M}(\theta_*)$ *satisfies*

$$m_* \sim \mu\varepsilon^{1/4} \exp\left\{-\frac{C_0 h_2(\varepsilon)}{\varepsilon^{1/4}}\right\}$$

where $h_2(\varepsilon)$ *is a positive periodic in* $\ln \varepsilon$ *function.*

In Figure 10 there is a schematic illustration of the functions $h_1(\varepsilon)$ and $h_2(\varepsilon)$ presented as exponents in the corresponding estimates of Theorem 0.7 and 0.8. It is worth mentioning that the expression of h_1 and h_2 depends on the specific quadratic number chosen from (5)

Notice that the geometric sequences mentioned in Theorem 0.8 are those where the splitting function has more than 2 essential dominant harmonics because the second essential dominant harmonic coincides with the third one, and this requires a special study. As an illustration, we carry out this study for the *silver* number $\Omega_2 = [2, 2, 2, \ldots] = [\overline{2}] = \sqrt{2} - 1$ and show (imposing some conditions on the phases σ_k of f in (4)) the *continuation* (without bifurcations) of the 4 homoclinic orbits for *all* values of $\varepsilon \to 0$, see also Theorem 6.3, p. 153.

Theorem 0.9 (Transversality of the splitting for Ω_2)**.** *For the Hamiltonian system (2-4) with* $n = 2$ *and* $\Omega = \Omega_2$ *in (3), assume that* $\varepsilon > 0$ *is small enough and* $\mu = \varepsilon^p$*,* $p > p^*$ *with* $p^* = 2$ *if* $\nu = 1$ *and* $p^* = 3$ *if* $\nu = 0$*, then if* $\sigma_k = 0$ *for all* $k \in \mathbb{Z}^2 \backslash \{0\}$*, one has:*

- *the Melnikov function* $\mathcal{M}(\theta)$ *has exactly 4 zeros* θ_**, all simple, for all* ε*;*
- *The minimal eigenvalue of* $\partial_\theta \mathcal{M}(\theta_*)$ *satisfies*

$$m_* \sim \mu\varepsilon^{1/4} \exp\left\{-\frac{C_0 h_2(\varepsilon)}{\varepsilon^{1/4}}\right\}$$

where $h_2(\varepsilon)$ *is a positive periodic function in* $\ln \varepsilon$*.*

Chapter 7. We consider the case $n = 3$ of the Hamiltonian system (2-4) and generalize the results obtained for two-dimensional tori with quadratic frequencies to a

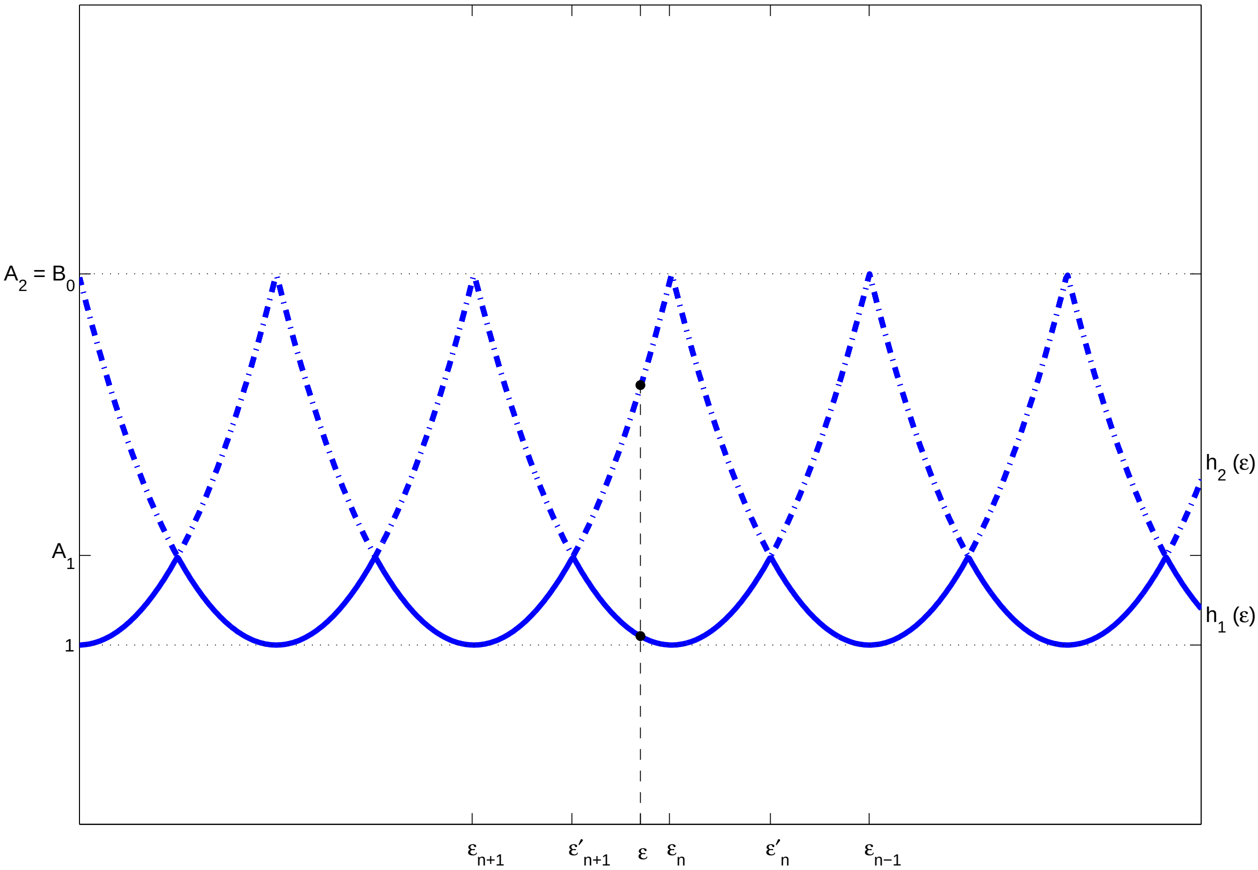

Figure 10: Plots of the functions h_1 and h_2 for Ω_2.

three-dimensional whiskered torus with *cubic* frequencies. To fix ideas, we consider a frequency vector of the form

$$\omega = (1, \Omega, \Omega^2),$$

where Ω is a cubic irrational number, i.e. a real root of a cubic polynomial with integer coefficients. We consider the so-called *complex case*, i.e. the components of the frequency vector lie in a cubic field, generated by a cubic irrational number whose two conjugates are not real, and show an oscillatory behavior of their principal small divisors, that did not take place for the quadratic frequencies.

First, in Section 7.1 we study the arithmetic properties of the cubic frequencies and give a classification of the associated resonances ($k \in \mathbb{Z}^3 \backslash \{0\}$ such that $\gamma_k := |\langle k, \omega\rangle| |k|^2$ is small). The idea is to construct a unimodular matrix T with the cubic frequency vector ω as one of its eigenvectors, and, thus, classify the resonances into *primary* and *secondary* ones, see for more details Section 7.1. Unfortunately, for cubic irrational numbers there is no standard theory of continued fractions like the one that was applied for quadratic numbers (in the quadratic case the periodicity of the continued fractions is used to construct the matrix T). Therefore, only some concrete cubic numbers can be considered for which the matrix T is known, see for example [Cha02]. In particular, we pay special attention to the *cubic golden number*, the real root of $\Omega^3 + \Omega = 1 (\Omega \approx 0.6823)$.

We prove that the Poincaré-Melnikov method can be applied choosing an appropriate $p > p^*$ and provide an asymptotic estimate for the maximal size of the splitting function $\mathcal{M}(\theta)$ (see also Theorem 7.1, p. 171):

Theorem 0.10 ((Maximal) splitting distance)**.** *For the Hamiltonian system (2-4) with $n = 3$, assume that ε is small enough and $\mu = \varepsilon^p$, $p > p^*$ with $p^* = 2$ if $\nu = 1$ and $p^* = 3$ if $\nu = 0$, then the following asymptotic estimate holds*

$$\max_{\theta \in \mathbb{T}^3} |\mathcal{M}(\theta)| \sim \frac{\mu}{\sqrt[3]{\varepsilon}} \exp\left\{-\frac{C_0 h_1(\varepsilon)}{\varepsilon^{1/6}}\right\},$$

where C_0 is the constant given in (7.11) and the function $h_1(\varepsilon)$ satisfies the following bounds:

- *"Constant bound": $0 < C_1^- \le h_1(\varepsilon) \le C_2^+$ with constants C_1^- and C_2^+, defined in (7.22);*
- *"Periodic bound": $0 < h_1^-(\varepsilon) \le h_1(\varepsilon) \le h_1^+(\varepsilon)$, where $h^-(\varepsilon)$, $h^+(\varepsilon)$ are a $3\ln\lambda$-periodic functions in $\ln\varepsilon$; $\min h_1^- = C_1^-, \max h_1^- = C_2^-, \min h_1^+ = C_1^+, \max h_1^+ = C_2^+$, the constants C_2^-, C_1^+ are defined in (7.22).*

In contrast to the quadratic case, the function $h_1(\varepsilon)$ is not periodic and has a more complicated form (see Figure 11 where one can suspect that $h_1(\varepsilon)$ is a quasiperiodic function).

Also we establish the following numerical result about the existence and the transversality of 8 homoclinic orbits to the whiskered torus. We prove that for ε small enough the 8 simple zeros of the splitting function $\mathcal{M}$ are determined by its 3 dominant harmonics if the vectors of indexes S_1, S_2, S_3 corresponding to these terms are independent, and, otherwise, by 4 dominant harmonics if not (see also Theorem 7.2, p. 173):

Theorem 0.11 (Transversality)**.** *Under the hypotheses of Theorem 0.10, one has:*

- *If $\det(S_1, S_2, S_3) \neq 0$, the Melnikov function $\mathcal{M}(\theta)$ has exactly 8 zeros θ_*, all simple, for all ε except for some small neighborhood of a discrete set of ε. The minimal eigenvalue of $\partial_\theta \mathcal{M}(\theta_*)$ satisfies*

$$m_* \sim \mu\varepsilon^{1/2} \exp\left\{-\frac{C_0 h_3(\varepsilon)}{\varepsilon^{1/6}}\right\}.$$

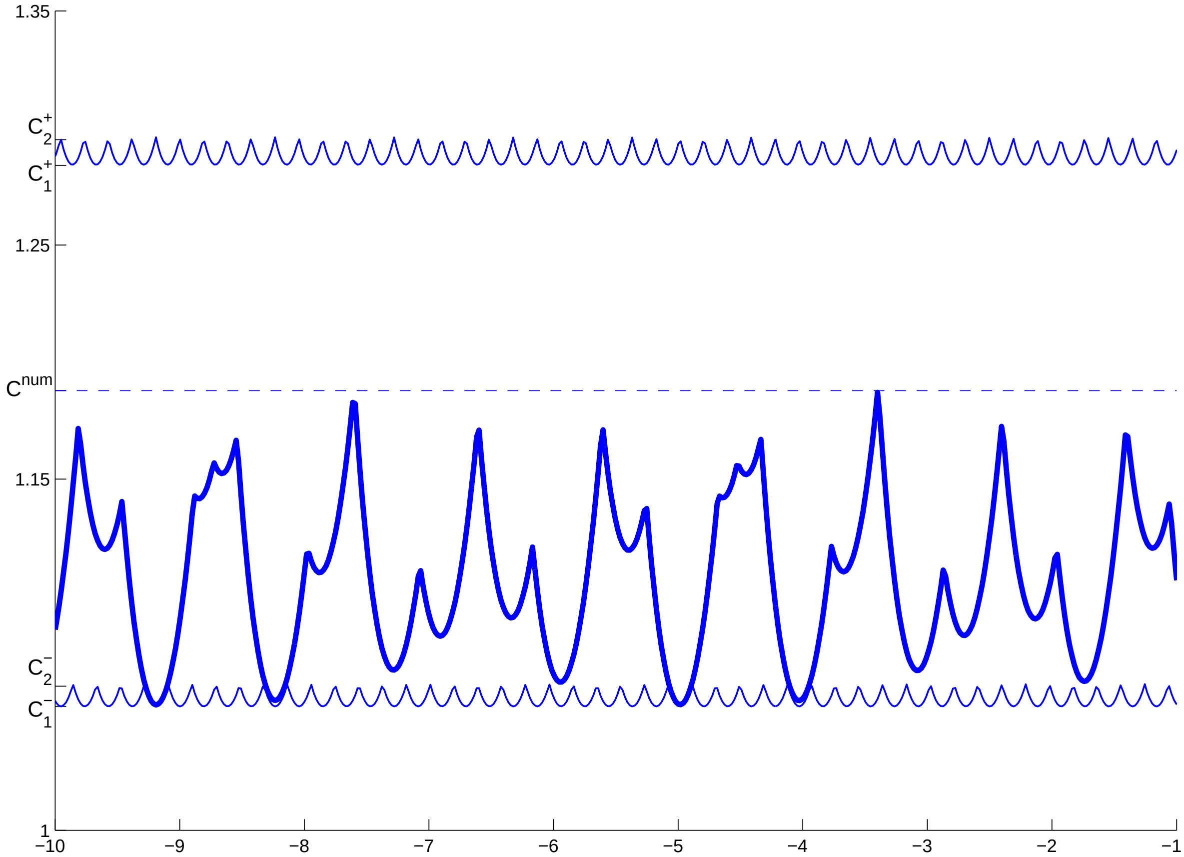

Figure 11: Plots of the functions $h_1(\varepsilon)$ (thick solid) and $h_1^{\pm}(\varepsilon)$ (solid) using a logarithmic scale for ε. Notice that the upper bound C_2^+ is not sharp, but a numerical sharp upper bound C^{num} can be obtained.

- *If* $\det(S_1, S_2, S_3) = 0$, *but* $\det(S_1, S_2, S_4) \neq 0$, *the Melnikov function* $\mathcal{M}(\theta)$ *has exactly 8 zeros* θ_*, *all simple, for all* ε *except for some small neighborhood of a discrete set of* ε. *The minimal eigenvalue of* $\partial_\theta \mathcal{M}(\theta_*)$ *satisfies*

$$m_* \sim \mu \varepsilon^{1/2} \exp\left\{ -\frac{C_0 h_4(\varepsilon)}{\varepsilon^{1/6}} \right\}.$$

The proof of this theorem is more difficult than the one of the quadratic case, since the functions presented in the exponents of the estimates are not periodic in $\ln \varepsilon$ and, actually, we cannot point out exactly (analytically) the discrete set where the theorem fails.

It is worth mentioning that, as we know, these are the first asymptotic results for the problem of splitting of separatrices for whiskered tori with 3 frequencies, and that many open problems still remain.

Appendix B. We prove some auxiliary lemmas related to the Fixed Point theorem. These lemmas enable us to find the critical points of the splitting potential $\mathcal{L}(\theta)$ and the zeros of the splitting function $\mathcal{M}(\theta)$.

Conclusions and future work

In this section we summarize the main achievements of the thesis and suggest some open problems to investigate in the nearest future.

Bifurcations of homoclinic tangencies in area-preserving maps

- We have studied bifurcations of a quadratic homoclinic tangency for two-dimensional symplectic (Chapter 1) and area-preserving non-orientable (Chapter 2) saddle maps and proved the existence of cascades of elliptic periodic orbits near the homoclinic orbit within the framework of one and two parameter general unfoldings.

- We have considered the question of the coexistence of elliptic periodic orbits of different periods for the symplectic and area-preserving non-orientable saddle maps and established the phenomenon of the global resonance.

- We have studied bifurcations of a cubic homoclinic tangency for two-dimensional symplectic maps and discovered the structure of the bifurcational diagram in two parameter general unfoldings (Chapter 3).

- We have constructed finitely-smooth normal forms for two-dimensional symplectic and area-preserving non-orientable saddle maps (Chapter 4).

- We have established the structure of $1:4$ resonance for some conservative Hénon-like maps (Appendix A).

Exponentially small splitting of separatrices for whiskered tori with several frequencies in Hamiltonian systems

- We have studied exponentially small splitting of separatrices for two-dimensional whiskered tori with quadratic frequencies. We have found 23 new quadratic numbers for which the Poincaré-Melnikov method can be applied and established the existence of 4 transverse homoclinic orbits.

- We have studied the continuation of the homoclinic orbits for all $\varepsilon \to 0$ in the case of the silver number $\Omega_2 = \sqrt{2} - 1$.

- We have established the existence of exponentially small splitting of separatrices for three-dimensional whiskered tori with cubic golden frequency vector and detected the transversality of 8 homoclinic orbits.

Future work

In the closest future we plan to continue investigations in these topics.

Regarding the first topic, we are going to translate the obtained results to the case of reversible maps. Namely, we would like

- To study bifurcations of cubic tangencies in reversible maps, putting a special emphasis on symmetry-breaking bifurcations.
- To adapt the results obtained for symplectic maps to the case of reversible maps.
- To understand which mechanisms of asymmetry are caused by a transverse homoclinic trajectory.
- To analyze global bifurcations of area-preserving maps with a transverse homoclinic orbit to a parabolic fixed point.

For the second topic, we plan in the future

- In the two-dimensional case, to study the continuation of the homoclinic orbits for all sufficiently small ε in the case of the quadratic numbers (5) introduced in Chapter 6.
- To find new quadratic numbers for which the technique developed in Chapter 6 can be applied to detect splitting of separatrices.
- In the three-dimensional case, to consider other concrete cubic numbers in the complex case and apply the technique of Chapter 7 to establish splitting of separatrices.
- To consider cubic numbers in the so-called *real* case, (i.e. the components of the frequency vector lie in a cubic field, generated by a cubic irrational number whose two conjugates are real). This study requires a different approach. In this case, the behavior of the associated small divisors seems to be different to the complex case considered in Chapter 7, and will require intensive numerical high-precision simulations in order to establish the properties of such vectors, and then try to obtain rigorous asymptotic estimates for the splitting.
- To consider noble numbers $\Omega = [a_1, a_2, \ldots, a_m, 1, 1, 1, \ldots]$ related to the golden mean Ω_1, to apply the results obtained in this thesis to establish the phenomenon of Arnold diffusion.

Part I

Bifurcations of homoclinic tangencies in area-preserving maps

Chapter 1

Bifurcations of quadratic homoclinic tangencies for two-dimensional symplectic maps

1.1 Statement of the problem and main results

Consider a C^r-smooth ($r \geq 3$) symplectic map f_0 satisfying the following conditions:

A. f_0 has a saddle fixed point O with multipliers λ and λ^{-1}, where $|\lambda| < 1$.

B. f_0 has a homoclinic orbit Γ_0 at whose points the stable and unstable invariant manifolds of the saddle O have a quadratic tangency (see Figure 1).

Let $\mathcal{H}_s$ be a (codimension one) bifurcation surface composed of symplectic C^r-maps close to f_0 such that every map of $\mathcal{H}_s$ has a nontransversal homoclinic orbit close to Γ_0. Let f_ε be a family of symplectic C^r-maps that contains the map f_0 at $\varepsilon = 0$. We suppose that the family depends smoothly on parameters $\varepsilon = (\varepsilon_1, ..., \varepsilon_m)$ and satisfies the following condition:

C. The family f_ε is transverse to $\mathcal{H}_s$ at $\varepsilon = 0$.

Let U be a small neighborhood of $O \cup \Gamma_0$. It consists of a small disk U_0 containing O and a number of small disks surrounding those points of Γ_0 that do not lie in U_0 (see Figure 1).

Definition 1.1. *A periodic or homoclinic orbit entirely lying in U is called* p-round *if it has exactly p intersection points with any disk of the set $U \backslash U_0$.*

In this chapter we study bifurcations of *single-round ($p = 1$) periodic orbits* in the families f_ε. Note that every point of such an orbit can be considered as a fixed point of the corresponding *first return map*. Such a map is usually constructed as a

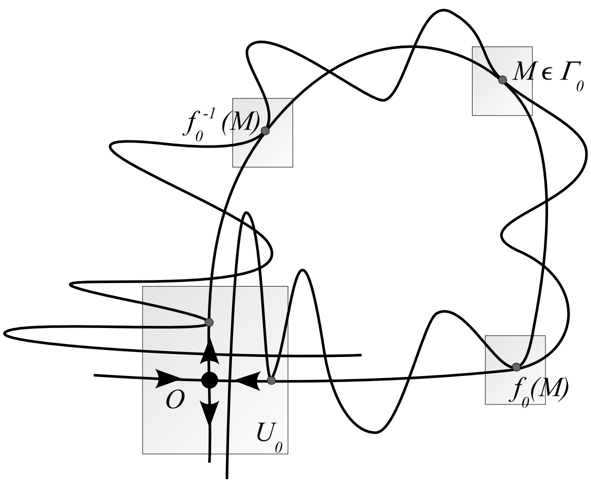

Figure 1.1: An example of area-preserving planar map having a quadratic homoclinic tangency at the points of a homoclinic orbit Γ_0. Some of these homoclinic points are shown as grey circles. Also a small neighborhood of the set $O \cup \Gamma_0$ is shown to be the union of the squares.

superposition $T_k = T_1 T_0^k$ of two maps $T_0 \equiv T_0(\varepsilon)$ and $T_1 \equiv T_1(\varepsilon)$, see Figure 1.2. The map T_0 is called *local map* and it is defined as the restriction of f_ε onto U_0, i.e. $T_0(\varepsilon) \equiv f_\varepsilon\big|_{U_0}$. The map T_1 is called *global map* and it is defined as $T_1 \equiv f_\varepsilon^q$ and acts from a small neighbourhood $\Pi^- \subset U_0$ of some point $M^- \in W^u_{loc}(O)$ of the orbit Γ_0 into a neighbourhood $\Pi^+ \subset U_0$ of another point $M^+ \in W^s_{loc}(O)$ of Γ_0, where q is an integer such that

$$f_0^q(M^-) = M^+. \tag{1.1}$$

Thus, any fixed point of T_k is a point of a single-round periodic orbit for f_ε with period $k+q$. We will study maps T_k for all any sufficiently large integer k. Therefore, it is very important to have "good" coordinate representations for both maps T_0 and T_1, especially it relates to the local map T_0 and its iterations T_0^k for large k.

It is well known that one can introduce such symplectic coordinates (x, y) in U_0 (with the origin at O) that the local map T_0 takes the following form near O:

$$\bar{x} = \lambda x + h_1(x,y,\varepsilon)x \ , \ \bar{y} = \lambda^{-1} y + h_2(x,y,\varepsilon)y \ , \tag{1.2}$$

where $h_i(0,0,\varepsilon) \equiv 0$, $i = 1, 2,$. Thus, in these coordinates, the equations of $W^s_{loc} \cap U_0$ and $W^u_{loc} \cap U_0$ are $y = 0$ and $x = 0$, respectively. However, form (1.2) is very inconvenient for calculations. The point is that the functions h_i can contain too much non-resonant terms that give a bad contribution into formulas for iterations of T_0. Therefore, we will use the so-called *finitely smooth normal forms* provided by the following lemma.

Lemma 1.1. *For any given integer n (such that $n < r/2$ or n is arbitrary for $r = \infty$ or $r = \omega$–the real analytic case), there is a canonical change of coordinates, of class C^r*

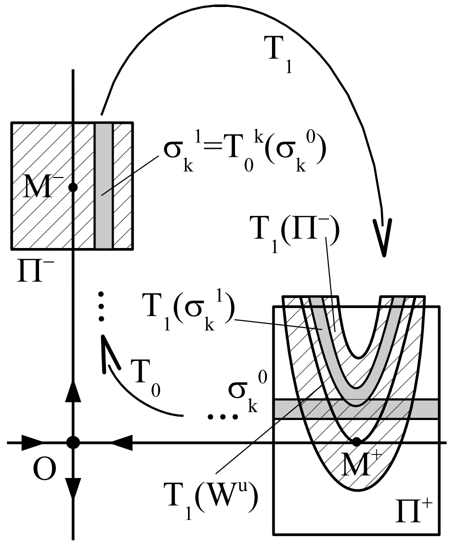

Figure 1.2:

for $n = 1$ or C^{r-2n} for $n \geq 2$, that brings T_0 to the following form

$$\begin{aligned} \bar{x} &= \lambda x \left(1 + \beta_1 \cdot xy + \cdots + \beta_n \cdot (xy)^n\right) + xO\left(|xy|^n(|x| + |y|)\right), \\ \bar{y} &= \lambda^{-1} y \left(1 + \hat{\beta}_1 \cdot xy + \cdots + \hat{\beta}_n \cdot (xy)^n\right) + yO\left(|xy|^n(|x| + |y|)\right). \end{aligned} \tag{1.3}$$

The smoothness of these coordinate changes with respect to parameters can be less on 2 than that by coordinates, i.e. C^{r-2} for $n = 1$ or C^{r-2n-2} for $n \geq 2$, respectively.

Remark 1.1. The normal form of the first order ($n = 1$) for T_0

$$\begin{aligned} \bar{x} &= \lambda x \left(1 + \beta_1 \cdot xy\right) + xO\left(|xy|(|x| + |y|)\right), \\ \bar{y} &= \lambda^{-1} y \left(1 + \hat{\beta}_1 \cdot xy\right) + yO\left(|xy|(|x| + |y|)\right) \end{aligned} \tag{1.4}$$

is well known from [GS90, MR97] where it was proved the existence of normalizing C^{r-1}-coordinates. The existence of C^r-smooth canonical changes of coordinates (which are C^{r-2}-smooth with respect to parameters) bringing a symplectic saddle map to form (1.4) was proved in [GST07].

The normal forms (1.3) are very suitable for effective calculation of maps T_0^k : $(x_0, y_0) \to (x_k, y_k)$ with sufficiently large integer k. So, the following result is valid.

Lemma 1.2. *Let T_0 be given by (1.3), then the map T_0^k can be written, for any integer k, as follows*

$$\begin{aligned} x_k &= \lambda^k x_0 \cdot R_n^{(k)}(x_0 y_k, \varepsilon) + \lambda^{(n+1)k} P_n^{(k)}(x_0, y_k, \varepsilon), \\ y_0 &= \lambda^k y_k \cdot R_n^{(k)}(x_0 y_k, \varepsilon) + \lambda^{(n+1)k} Q_n^{(k)}(x_0, y_k, \varepsilon), \end{aligned} \tag{1.5}$$

where

$$R_n^{(k)} \equiv 1 + \tilde\beta_1(k)\lambda^k x_0 y_k + \dots + \tilde\beta_n(k)\lambda^{nk}(x_0y_k)^n, \tag{1.6}$$

$\tilde\beta_i(k)$, $i = 1, \dots, n$, are some polynomials (of degree i) with respect to k with coefficients depending on $\beta_1, \dots, \beta_i$, and the functions $P_n^{(k)}, Q_n^{(k)} = o\,(x_0^n y_k^n)$ are uniformly bounded in k along with all derivatives by coordinates up to order either $(r-2)$ for $n = 1$ or $(r - 2n - 1)$ for $n \geq 2$.

The proof of Lemmas 1.1 and 1.2 is referred to section 4.2 of Chapter 4.

Note that form (1.3) can be considered as a certain finitely smooth approximation of the following analytical Moser normal form

$$\bar x = \lambda(\varepsilon)x \cdot B(xy, \varepsilon), \;\; \bar y = \lambda^{-1}(\varepsilon)y \cdot B^{-1}(xy, \varepsilon), \tag{1.7}$$

taking place for $\lambda > 0$ [Mos56], where $B(xy, \varepsilon) = 1 + \beta_1 \cdot xy + \dots + \beta_n \cdot (xy)^n + \dots$. We show that the approximations of form (1.3) take place also in the case $\lambda < 0$ (although, it does not imply formally the existence of analytical form (1.7) for the case $\lambda < 0$). Accordingly, relation (1.5) looks as a very good approximation for the corresponding formula in the analytical case, [GS97],

$$x_k = \lambda^k x_0 \cdot R^{(k)}(x_0y_k, \varepsilon), \;\; y_0 = \lambda^k y_k \cdot R^{(k)}(x_0y_k, \varepsilon), \tag{1.8}$$

where $R^{(k)} \equiv 1 + \tilde\beta_1(k)\lambda^k x_0 y_k + \dots + \tilde\beta_n(k)\lambda^{nk}(x_0y_k)^n + \dots$ and

$$\tilde\beta_1(k) = \beta_1 k, \;\; \tilde\beta_2(k) = \beta_2 k - \frac{1}{2}\beta_1^2 k^2, \dots. \tag{1.9}$$

In coordinates of Lemma 1.1, we have that $M^+ = (x^+, 0), M^- = (0, y^-)$. Without loss of generality, we assume that $x^+ > 0$ and $y^- > 0$. Let the neighborhoods Π^+ and Π^- of the homoclinic points M^+ and M^-, respectively, be sufficiently small such that $T_0(\Pi^+) \cap \Pi^+ = \emptyset$, $T_0^{-1}(\Pi^-) \cap \Pi^- = \emptyset$. Then, as usually (see e.g. [GS73, SSTC98]), the successor map from Π^+ into Π^- by orbits of T_0 is defined, for all sufficiently small ε, on the set consisting of infinitely many strips $\sigma_k^0 \equiv \Pi^+ \cap T_0^{-k}\Pi^-$, $k = \bar k, \bar k + 1, \dots$. The image of σ_k^0 under T_0^k is the strip $\sigma_k^1 = T_0^k(\sigma_k^0) \equiv \Pi^- \cap T_0^k\Pi^+$. As $k \to \infty$, the strips σ_k^0 and σ_k^1 accumulate on W^s_{loc} and W^u_{loc}, respectively.

We write the global map $T_1(\varepsilon) : \Pi^- \to \Pi^+$ as follows (in the coordinates of Lemma 1.1)

$$\overline{x} - x^+ = F(x, y - y^-, \varepsilon), \;\; \overline{y} = G(x, y - y^-, \varepsilon), \tag{1.10}$$

where $F(0) = 0, G(0) = 0$. Besides, we have that $G_y(0) = 0, G_{yy}(0) = 2d \neq 0$ which follows from the fact (condition B) that at $\varepsilon = 0$ the curve $T_1(W^u_{loc}) : \{\overline{x} - x^+ = F(0, y - y^-, 0), \overline{y} = G(0, y - y^-, 0)\}$ has a quadratic tangency with $W^s_{loc} : \{\overline{y} = 0\}$ at M^+. When parameters vary this tangency can split and, moreover, we can introduce the corresponding splitting parameter as $\mu \equiv G(0, 0, \varepsilon)$. By condition C, the parameter

μ must belong to the set of parameters ε; and, without loss of generality, we assume that $\varepsilon_1 \equiv \mu$. Accordingly, we can write the following Taylor expansions for F and G

$$\begin{array}{rcl} F(x, y-y^-, \varepsilon) & = & ax + b(y-y^-) + e_{20}x^2 + e_{11}x(y-y^-) + e_{02}(y-y^-)^2 + \ h.o.t., \\ G(x, y-y^-, \varepsilon) & = & \mu + cx + d(y-y^-)^2 + f_{20}x^2 + f_{11}x(y-y^-) + f_{30}x^3 \\ & & + f_{21}x^2(y-y^-) + f_{12}x(y-y^-)^2 + f_{03}(y-y^-)^3 + \ h.o.t., \end{array} \tag{1.11}$$

where the coefficients $a, b, \ldots, f_{03}$ (as well as x^+ and y^-) depend smoothly on ε. Note also that

$$\det \begin{pmatrix} F_x & F_y \\ G_x & G_y \end{pmatrix} \equiv 1 \tag{1.12}$$

since T_1 is the symplectic map. In particular, we have

$$\begin{array}{c} bc \ \equiv \ -1, \\ \tilde{R} = (2a + 2e_{02}/bd - bf_{11}/d) \ \equiv 0 \end{array} \tag{1.13}$$

Henceforth, the following coefficients will be very important for us:

$$c = G_x(0,0,\varepsilon), \quad d = \frac{G_{yy}(0,0,\varepsilon)}{2}, \tag{1.14}$$

since together with λ they define the character of geometry of the homoclinic tangencies.

It is easy to see from (1.10), (1.11) that μ is the parameter of splitting of manifolds $W^s(O_\varepsilon)$ and $W^u(O_\varepsilon)$ with respect to the homoclinic point M^+. Indeed, the curve $l_u = T_1(W^u_{loc} \cap \Pi^-)$ has the equation $l_u \ : \ \bar{y} = \mu + \dfrac{d}{b^2}(\bar{x} - x^+)^2(1 + O(\bar{x} - x^+))$. Since the equation of W^s_{loc} is $y = 0$ for all (small) ε, it implies that the manifolds $T_1(W^u_{loc})$ and W^s_{loc} do not intersect for $\mu d > 0$, intersect transversally at two points for $\mu d < 0$, and have a quadratic tangency (at M^+) for $\mu = 0$. In turn, since the strips σ^1_k accumulate on the segment $W^u_{loc} \cap \Pi^-$ as $k \to \infty$, it follows that $T_1(\sigma^1_k)$ has a horseshoe form and, moreover, these horseshoes accumulate to l_u as $k \to \infty$. Therefore, the first return maps $T_k = T_1 T_0^k : \sigma^0_k \to \sigma^0_k$ are, in fact, conservative horseshoe maps. When μ varies near zero value infinitely many bifurcations of horseshoes creation (destruction) occur. In this chapter we study these bifurcations and show that they include birth (disappearance) of *elliptic periodic points.*

However, we can also see these horseshoe bifurcations must have different scenarios depending on a type of the initial homoclinic tangency. Indeed, at $\mu = 0$ a character of reciprocal position of the strips σ^0_k and their horseshoes $T_1(\sigma^1_k)$ is essentially defined by the signs of the parameters λ, c and d. Moreover, by this peculiarity, we can select 6 different cases of symplectic maps with quadratic homoclinic tangencies. The corresponding examples are shown in Figures 1.3 and 1.4. Note that in the cases with $\lambda < 0$ we can always consider d to be positive: if d is negative for the given pair of homoclinic points, M^+ and M^-, we can take another pair of points, like $\{T_0(M^+), M^-\}$ or $\{M^+, T_0^{-1}(M^-)\}$, for which the corresponding d' becomes positive.

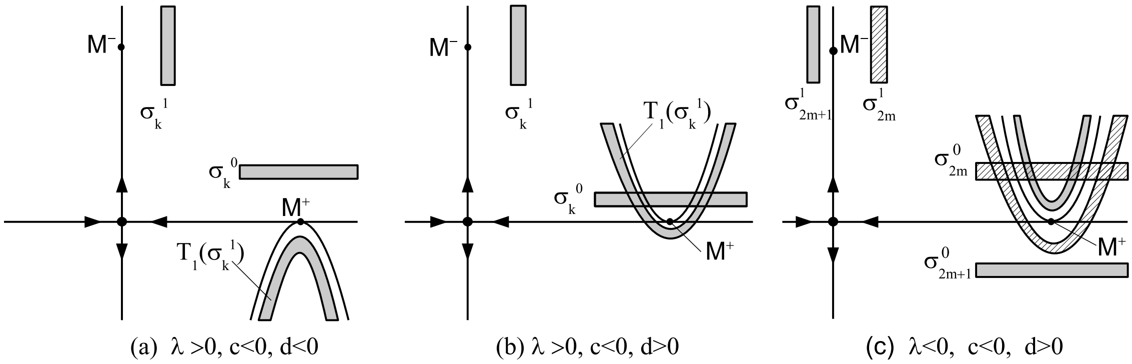

Figure 1.3: Symplectic maps with a homoclinic tangency for $c < 0$.

Note that in the cases with $c < 0$, see Figure 1.3, a reciprocal position of all the strips σ_j^0 and their horseshoes $T_1(\sigma_j^1)$ at $\mu = 0$ is defined quite simply: $\sigma_j^0 \cap T_1(\sigma_j^1) = \emptyset$ if $\lambda > 0, d < 0$; the strips σ_j^0 and horseshoes $T_1(\sigma_j^1)$ have regular intersections if $\lambda > 0, d > 0$; the corresponding intersections are either regular for even j or empty for odd j if $\lambda < 0, d > 0$. Recall that the regular intersection means (by [GS87] and [GST96b], see also Definition 1.2) that the set $\sigma_j^0 \cap T_1(\sigma_j^1)$ consists of two connected components and, moreover, the first return map $T_j \equiv T_1 T_0^j : \sigma_j^0 \mapsto \sigma_j^0$ is *the Smale horseshoe map*: its nonwandering set Ω_j is hyperbolic and $T_j\big|_{\Omega_j}$ is topologically conjugate to the Bernoulli shift with two symbols (for more details see [GST96b] and section 1.2). Therefore, we can say that every map f_0 in the case $c < 0, d > 0$ has infinitely many horseshoes Ω_j, where j runs all sufficiently large positive integers (respectively, even positive integers) in the case $\lambda > 0$ (respectively, in the case $\lambda < 0$). On the other hand, every map f_0 with $\lambda > 0, c < 0, d < 0$ has no horseshoes at all (in a small neighborhood U).

In the cases of homoclinic tangencies with $c > 0$, see Figure 1.4, a reciprocal position of the strips σ_j^0 and horseshoes $T_1(\sigma_j^1)$ depends also on other invariants of the homoclinic structure. The principal such invariant is

$$\tau = -\frac{1}{\ln|\lambda|} \ln \left| \frac{cx^+}{y^-} \right|. \tag{1.15}$$

Note that (see section 1.2 and [GS87], [GS01]) the sign of τ is very important here. For example, in Figure 1.5 it is shown a reciprocal position of the strips σ_j^0 and horseshoes $T_1(\sigma_j^1)$ (with sufficiently large j) for various values of τ for the case $\lambda > 0, c > 0, d > 0$. Thus, we can see that if $\tau > 0$, then f_0 has infinitely many horseshoes Ω_j; if $\tau < 0$, then there exists such a neighbourhood $U(O \cap \Gamma_0)$ in which dynamics of f_0 is trivial: only orbits O and Γ_0 do not leave U under iterations of f_0. The case $\tau = 0$ is "bifurcational", since infinitely many horseshoes appear (disappear) when varying τ near zero (even without splitting the initial tangency).

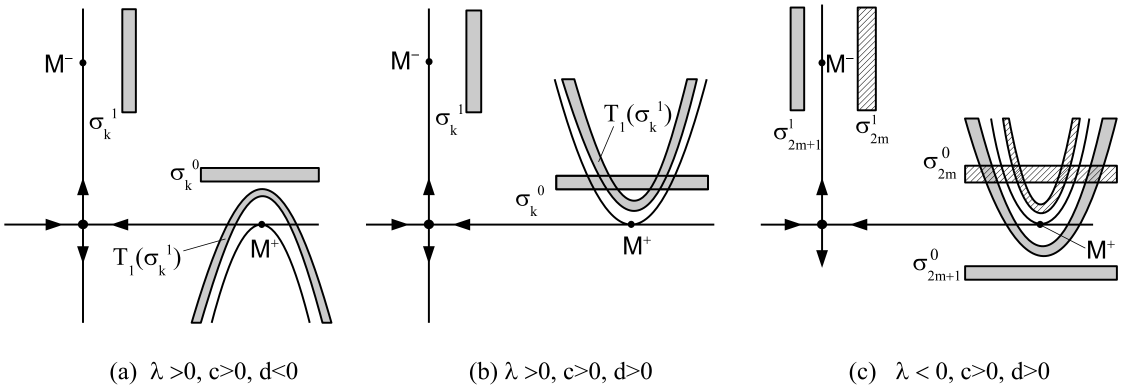

Figure 1.4: Symplectic maps with a homoclinic tangency for $c > 0$.

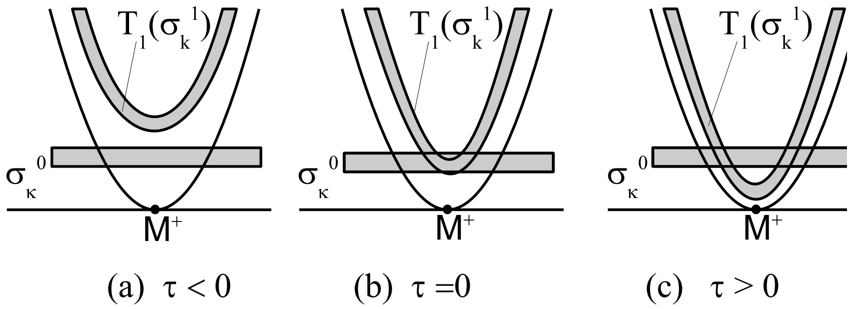

Figure 1.5: A horseshoe geometry of symplectic maps with a homoclinic tangency in the case $\lambda > 0, c > 0, d > 0$ for different τ.

Thus, we can draw the following conclusions: 1) the cases of homoclinic tangencies with $c < 0$ or with $\tau \neq 0$ at $c > 0$ are "ordinary" and it is sufficient to study bifurcations of single-round periodic orbits within framework of one parameter general families only (with the parameter μ); 2) the cases of homoclinic tangencies with $\tau = 0$ at $c > 0$ are "special" and it is necessary to consider at least two parameter general unfoldings (for example, with parameters μ and τ). In this chapter we adhere to this approach and present three following theorems as our main results in the symplectic case.

Theorem 1.1 (On one parameter cascades of elliptic points)**.** *Let f_0 be a symplectic map satisfying conditions **A** and **B** and let f_μ be a one parameter general unfolding under condition **C**. Then the following statements take place:*

1. *In any segment $[-\mu_0, \mu_0]$ of values of μ, there are infinitely many open intervals δ_k, $k = \bar{k}, \bar{k}+1, \ldots$ ($\bar{k}$ is some integer), such that $\delta_k \to 0$ as $k \to +\infty$ and the map f_μ has at $\mu \in \delta_k$ a single-round elliptic periodic orbit (of period $k + q$, where*

q is defined in (1.1));

2. *At the border points $\mu = \mu_k^+$ and $\mu = \mu_k^-$ of δ_k, f_μ has a single-round parabolic periodic orbit with double multipliers $+1$ or -1, respectively;*

3. *The elliptic orbit is generic for all values of μ from δ_k, except for exactly two values corresponding to the strong resonances when the multipliers are $e^{\pm i\pi/2}$ and $e^{\pm i2\pi/3}$;*

4. *In the cases $c < 0$ or $c > 0$ and $\tau \neq 0$ (c is given in (1.14)), the intervals δ_i and δ_j do not intersect when integers i and j are different and sufficiently large.*

Note that some analogs related to items 1, 2 and 3 of this theorem were proved in [Bir87], [BS89] and [MR97]. However, problems on the coexistence of single-round elliptic periodic orbits were not considered. The item 4 of Theorem 1.1 shows that, in general, such orbits of different and large periods can not coexist.[1] By "general" we mean that the case $\tau = 0$ is excluded. However, from the geometrical point of view, this case looks to be quite interesting. Indeed, as one can extract from Figures 1.5 that if τ varies near zero, the position of intervals δ_k can sharply change and, moreover, the intervals δ_i and δ_j with different i and j can intersect and they can be even "nested". Therefore, in order to understand the corresponding phenomena we must include τ into the set of parameters.

Theorem 1.2 (On two parameter cascades of elliptic points)**.** *Let f_0 be a symplectic map satisfying conditions **A** and **B** and $f_{\mu,\tau}$ be a two parameter family which unfolds generally, under condition **C**, the given homoclinic tangency with $\tau = 0$. Then the following statements take place:*

1. *In any neighborhood of the origin in the (τ, μ)-plane, there are infinitely many open domains Δ_k, for $k = \bar{k}, \bar{k}+1, \dots$ ($\bar{k}$ is some integer), such that the map $f_{\mu,\tau}$ has a single-round periodic (of period $k + q$, where q is defined in (1.1)) elliptic orbit at $(\tau, \mu) \in \Delta_k$;*

2. *The domains Δ_k accumulate to the axis $\mu = 0$ as $k \to \infty$;*

3. *The boundaries of Δ_k are two curves L_k^+ and L_k^- such that the map $f_{\mu,\tau}$ has a parabolic single-round periodic orbit with double multipliers either $+1$ if $(\tau, \mu) \in L_k^+$ or -1 if $(\tau, \mu) \in L_k^-$;*

4. *The elliptic orbit is generic for all values of $(\tau, \mu) \in \Delta_k$, except for those which belong to curves $L_k^{\pi/2}$ and $L_k^{2\pi/3}$ when the multipliers of the orbit are equal to $e^{\pm i\pi/2}$ and $e^{\pm i2\pi/3}$, respectively;*

[1] It is not the case for two-dimensional symplectic maps with nontransversal heteroclinic cycles: as it was shown in [GS97] and [GS00], maps having simultaneously infinitely many single-round (generic) elliptic periodic orbits are dense in a bifurcation codimension-one surface composed from maps with nontransversal heteroclinic cycles.

5. *In the case $c < 0$ (c is given in (1.14)), the domains Δ_i and Δ_j do not intersect for any sufficiently large and different integers i and j;*

6. *In the cases $c > 0$, the domains Δ_i and Δ_j are necessarily crossed and they intersect the axis $\mu = 0$; Moreover, if $-3 < s_0 < 1/4$, where*

$$s_0 = dx^+(ac + f_{20}x^+) - f_{11}x^+(1 - \frac{1}{4}f_{11}x^+), \tag{1.16}$$

where all the coefficients are given in (1.11) then all domains Δ_k with sufficiently large k contain the origin ($\tau = 0, \mu = 0$).

See Figure 1.6 for the illustration of the theorem. For example, you can see the non-intersecting (in Figure 1.6(a)–(c)) as well as the crossed and intersecting $\mu = 0$ (in Figure 1.6 (d)–(f)) domains Δ_i and Δ_j, $i \neq j$, in the case $c < 0$ and $c > 0$, respectively.

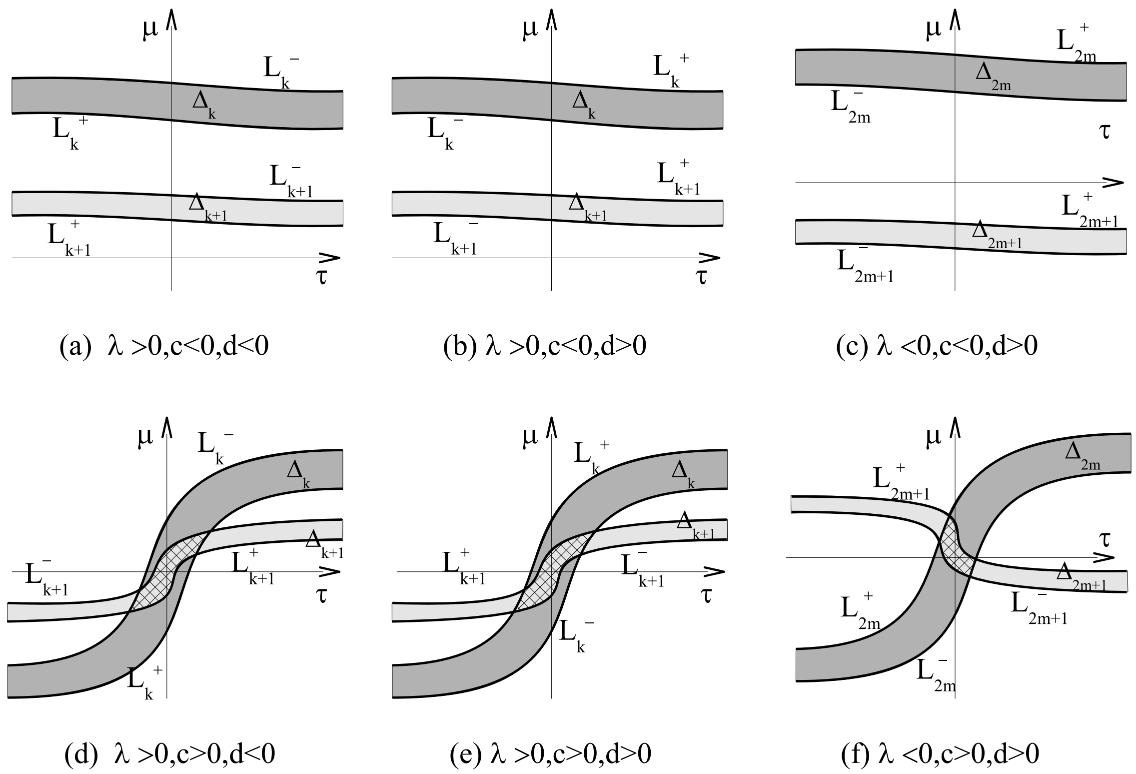

Figure 1.6:

Corollary 1.1. *Let the following relations* $c > 0$*,* $\tau = 0$*,* $-3 < s_0 < 1/4$ *and* $s_0 \neq \{0; -5/4\}$ *take place for the map* f_0*. Then there exists such* $k_0 > 0$ *that* f_0 *has a countable set of generic (KAM-stable) single-round elliptic periodic orbits of all successive periods beginning with* $k_0 + q$*.*

In the following theorem we clarify both a disposition of the intervals δ_k from Theorem 1.1 and a character of the corresponding bifurcations when μ varies inside δ_k.[2]

Theorem 1.3. *The intervals* δ_k *from Theorem* 1.1 *have form* $\delta_k = (\mu_k^+, \mu_k^-)$ *if* $d < 0$ *(d is given in (1.14)) and* $\delta_k = (\mu_k^-, \mu_k^+)$ *if* $d > 0$*. Bifurcations of fixed points in the first return map* $T_k(\mu)$ *follow, in general, the scenario observed in the conservative generalize Hénon map*

$$\bar{x} = y, \quad \bar{y} = M - x - y^2 + \nu_k y^3, \tag{1.17}$$

where

$$\begin{array}{l} \nu_k = \frac{f_{03}}{d^2}\lambda^k, \\ M = -d(1+\nu_k^1)\lambda^{-2k}\left(\mu + \lambda^k(cx^+ - y^-)(1 + k\beta_1\lambda^k x^+ y^-)\right) - s_0 + \nu_k^2 \end{array} \tag{1.18}$$

being f_{03} *the coefficient given in (1.11),* s_0 *the coefficient given by (1.16) and* $\nu_k^1 = O(\lambda^k)$*,* $\nu_k^2 = O(k\lambda^k)$ *some asymptotically small coefficients. See Figure 1.7 for* $d < 0$ *for general bifurcations of the map (1.17). The resonance* $1:3$ *is non-degenerate for all values of* ν_k*, while the resonance* $1:4$ *is non-degenerate for* $\nu_k \neq 0$ *(see Figure 1.7(b) for* $\nu_k > 0$ *and Figure 1.7(c) for* $\nu_k < 0$*).*

Note that if $f_{03} \neq 0$, then both scenarios of Figure 1.7 (b) and (c) take place for $\lambda < 0$: we have the case (b) for even k and the case (c) for odd k. Analogous phenomenon was observed in [BS89] in bifurcations of appearance (disappearance) of horseshoes in three-dimensional conservative systems with homoclinic loops to saddle-foci.

The content of the rest part of this chapter is the following. In section 1.2 we study the semi-local dynamics of symplectic maps with quadratic homoclinic tangencies: we select three classes of such maps and describe the structure of orbits entirely lying in a small neighbourhood $U(O \cup \Gamma_0)$. In section 1.3 we prove our main technical result, the Rescaling Lemma 1.4, which shows that one can reduce the study of bifurcations of the first return maps to the quite standard analysis of bifurcations of two-dimensional conservative Hénon-like maps. Section 1.4 contains the proofs of our main Theorems 1.1, 1.2 and 1.3.

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[2] Note that the intervals δ_k can be also viewed as intervals obtained on lines $\tau = const$ when intersection with domains Δ_k. Therefore, we can extract from Figure 1.6 a certain information on disposition of these intervals.

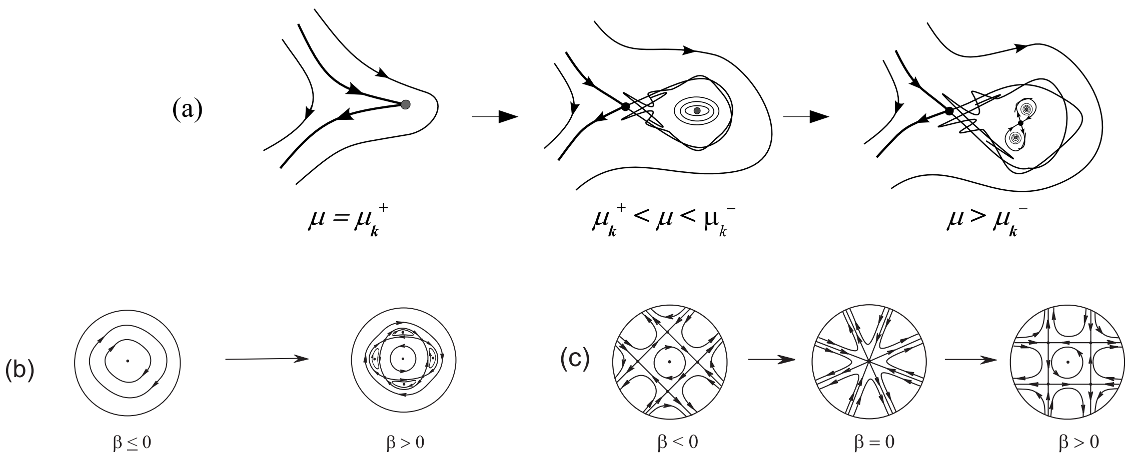


Figure 1.7: Bifurcations of fixed points in the first return map $T_k(\mu)$ for $d < 0$. (a) The main scenario, here $\mu_k^+ < \mu_k^-$ (if $d > 0$, then $\mu_k^+ > \mu_k^-$). (b)–(c) Bifurcations near resonance $1:4$ ($\beta = 0$ corresponds to $\mu = \mu_k^{\pi/2}$) for the cases (b) $\nu_k > 0$ (here the fixed point is always elliptic) and (c) $\nu_k < 0$ (for $\beta = 0$ the fixed point is a saddle with eight separatrices).

1.2 Three classes of symplectic maps with homoclinic tangencies.

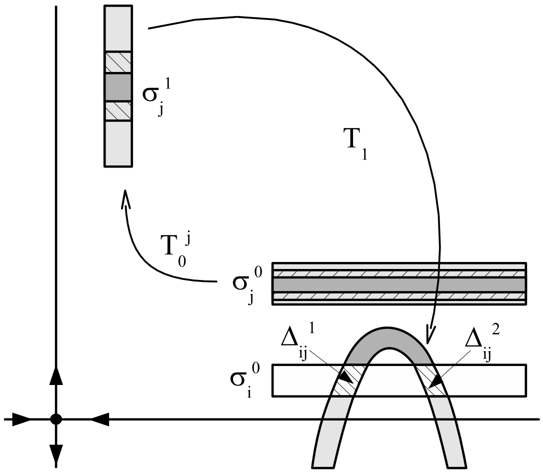
Let f_0 be a symplectic map satisfying conditions **A** and **B**. Evidently, any orbit of f_0 entirely lying in U, except for O, must visit both neighborhoods Π^- and Π^+ (otherwise, it wouldn't be close to $\overline{\Gamma}_0$). Moreover, such orbits must have points belonging to the intersections $T_1(\sigma_j^1) \cap \sigma_i^0$ for all possible integer i and j. Given a sufficiently large integer $\bar{k} > 0$, we assume that Π^+ and Π^- contain the strips σ_k^0 and σ_k^1, respectively, only with numbers $k \geq \bar{k}$. In other words, we will consider only such entirely lying in U orbits of f_0 whose points from Π^+ reach Π^- for a number of iterations that is not less than $\bar{k}$. Denote the set of such orbits as $\mathcal{N}_{\bar{k}} \equiv \mathcal{N}_{\bar{k}}(f_0)$.

In this section we study the structure of the set $\mathcal{N}_{\bar{k}}(f_0)$ and, thus, extend the results of [GS87], [GS01] and [GS03] to the symplectic case. Recall the following definition and result.

Definition 1.2. [GS87], [GST96b] We say that the horseshoe $T_1(\sigma_j^1)$ has a *regular intersection* with the strip σ_i^0 if (see Figure 1.2)

(i) the set $T_1(\sigma_j^1) \cap \sigma_i^0$ consists of two connected components Δ_{ij}^1 and Δ_{ij}^2 ;

(ii) the map $T_1 T_0^j$ restricted to the preimage $(T_1 T_0^j)^{-1} \Delta_{ij}^\alpha \subset \sigma_j^0$ of the component Δ_{ij}^α, where $\alpha = 1, 2$, is a saddle map (i.e., it is exponentially contracting along one of the coordinates and expanding along another).



Lemma 1.3. [GS87] *There are a constant $S_1 > 0$ and a sufficiently large integer $\bar{k}$ such that, for any $i, j \geq \bar{k}$, the following assertions are valid.*

(i) *If*

$$d(\lambda^i y^- - c\lambda^j x^+) > S_{ij}(\bar{k}) \ , \tag{1.19}$$

where $S_{ij} = S_1(|\lambda|^i + |\lambda|^j)|\lambda|^{\bar{k}/2}$, the horseshoe $T_1(\sigma_j^1)$ intersects regularly the strip σ_i^0.

ii) *If*

$$d(\lambda^i y^- - c\lambda^j x^+) < -S_{ij}(\bar{k}) \ , \tag{1.20}$$

then $T_1(\sigma_j^1) \cap \sigma_i^0 = \emptyset$.

The inequalities (1.19)–(1.20) have a rather simple geometrical sense. The strip σ_i^0 is a narrow horizontal rectangle in Π^+ having a central line $y = \lambda^i y^-$, while, the strip σ_j^1 is a narrow vertical rectangle in Π^- having a central line $x = \lambda^j x^+$. By (1.10), the strip σ_j^1 is mapped under T_1 into a horseshoe which contains a parabola $y = c\lambda^j x^+ + d(x - x^+)^2/b^2$. The inequality $d(\lambda^i y^- - c\lambda^j x^+) > 0$ means that the straight line $y = \lambda^i y^-$ and the parabola are crossed in two points, whereas, the inequality $d(\lambda^i y^- - c\lambda^j x^+) < 0$ implies that these curves do not intersect. By coefficient $S_{ij}(\bar{k})$ we take into account a non-zero thickness of the strips and horseshoes.

It is convenient to reformulate this lemma as follows: if the horseshoe $T_1(\sigma_j^1)$ has an irregular intersection with the strip σ_i^0, then the following inequalities must hold

$$|d||\lambda^i y^- - c\lambda^j x^+| \leq S_{ij}(\bar{k}) \ , \tag{1.21}$$

and if $T_1(\sigma_j^1) \cap \sigma_i^0 \neq \emptyset$, then

$$d(\lambda^i y^- - c\lambda^j x^+) \geq -S_{ij}(\bar{k}) \ , \tag{1.22}$$

It is evident that the character of integer solutions of inequalities (1.19)–(1.20) depends essentially on the signs of the quantities λ, c and d. In turn, this means that the structure of $\mathcal{N}_{\bar{k}}$ depends essentially on the type of homoclinic tangency. By this principle, the same as for the case of general diffeomorphisms (see [GS73]), we can subdivide quadratic homoclinic tangencies in the symplectic case into three big classes in the following way:

- *The first class* relates to the tangencies with $\lambda > 0$, $c < 0$ and $d < 0$ (see Figure 1.3a).
- *The second class* relates to the tangencies with $\lambda > 0$, $c < 0$ and $d > 0$ (see Figure 1.3b).
- The tangencies of all other types (with all other combinations of the signs of λ, c and d) are related to *the third class* (see Figures 1.3c and 1.4).

We will say also that a given symplectic map is of the first, second or third class, if it has the homoclinic tangency under consideration to be the first, second or third class, respectively.

1.2.1 Maps of the first and second classes.

In the case of a map of the first class, since $\lambda > 0$, $c < 0$ and $d < 0$, the inequality (1.20) holds for all $i, j \geq \bar{k}$. It follows, by Lemma 1.3, that $T_1(\sigma_j^1) \cap \sigma_i^0 = \emptyset$ for any $i, j \geq \bar{k}$ which implies

Proposition 1.1. [GS87] *Let f_0 be a map of the first class. Then the set $\mathcal{N}_{\bar{k}}$ has the trivial structure: $\mathcal{N}_{\bar{k}} = \{O, \Gamma_0\}$.*

For a map of the second class, since $\lambda > 0$, $c < 0$ and $d > 0$, we have now that inequality (1.19) holds for all $i, j \geq \bar{k}$. It means, in turn, that all the horseshoes $T_1(\sigma_j^1)$ and strips σ_i^0 (for any $i, j \geq \bar{k}$) have the regular intersections. Therefore the set $\mathcal{N}_{\bar{k}}$ has a non-uniformly hyperbolic structure and all orbits from $\mathcal{N}_{\bar{k}}$, except for Γ_0, are saddle (see also [GS87]). Moreover, we can give the exact description of the set $\mathcal{N}_{\bar{k}}$ in this case. Namely, let $B^3_{\bar{k}+q}$ be a subsystem of the topological Bernoulli scheme (shift) on three symbols $(0, 1, 2)$ consisting only of (bi-infinite) sequences of form

$$(..., 0, \alpha_{s-1}, \overbrace{0, ..., 0, \alpha_s}^{k_s+q}, \overbrace{0, ..., 0, \alpha_{s+1}}^{k_{s+1}+q}, 0, ...), \tag{1.23}$$

where $\alpha_s \in \{1, 2\}$, $k_s \geq \bar{k}$ for any s, and any sequence (1.23) does not contain two successive nonzero symbols. Let $\tilde{B}^3_{\bar{k}+q}$ be the factor-system that is resulted from $B^3_{\bar{k}+q}$ if to identify two homoclinic orbits $(..., 0, ..., 0, 1, 0, ..., 0, ...)$ and $(..., 0, ..., 0, 2, 0, ..., 0, ...)$. Denote this identified orbit as $\tilde{\omega}$.

Proposition 1.2. [GS87], [GS01] *Let f_0 be a map of the second class. Then the system $f_0\big|_{N_{\bar{k}}}$ is topologically conjugate to $\tilde{B}^3_{\bar{k}+q}$ and all orbits from $\mathcal{N}_{\bar{k}} \backslash \Gamma_0$ are saddle.*

1.2.2 Maps of the third class

We have formally 6 different combinations of the signs of coefficients λ, c and d related to the third class. However, if $\lambda < 0$ we can always choose such pairs of the homoclinic points M^+ and M^- for which d is positive. Besides, since f_0 is symplectic, there is no necessity to distinguish f_0 and f_0^{-1}. Therefore, the combinations $\lambda > 0, c > 0, d > 0$ and $\lambda > 0, c > 0, d < 0$ can be transformed one to another, if to consider f_0^{-1} instead of f_0.[3] Thus, we can reduce the number of different types of homoclinic tangencies of the third class to the following three ones.

1. maps with $\lambda > 0, c > 0, d > 0$ (see Figure 1.4b);

2. maps with $\lambda < 0, c > 0, d > 0$ (see Figure 1.4c);

3. maps with $\lambda < 0, c < 0, d > 0$ (see Figure 1.3c).

Denote by $H_3^i, i = 1, 2, 3$, codimension-one bifurcation surfaces, in the space of C^r-smooth symplectic maps, composed from maps with homoclinic tangencies of pointed out types.

It is typical for maps of the third class that the structure of the set $\mathcal{N}_{\bar{k}}$ depends essentially on the value of the invariant τ (defined in (1.15)). In particular, the following result takes place for maps on H_3^1 (it was announced in [GS01], we give here the proof).

Proposition 1.3. *Let $f_0 \in H_3^1$.*
1) If $\tau < 0$, then there exists such $\bar{k}_1 = \bar{k}_1(\tau) \to \infty$ as $\tau \to 0$ that the set $\mathcal{N}_{\bar{k}_1}$ has the trivial structure: $\mathcal{N}_{\bar{k}_1} = \{O, \Gamma_0\}$.
2) If $\tau > 0$, the set $\mathcal{N}_{\bar{k}}$, for any $\bar{k}$, contains nontrivial hyperbolic subsets.
3) If $\tau > 0$ and $\tau \notin \mathbb{Z}^+$ (where $\mathbb{Z}^+$ is the set of positive integer numbers), then there exists $\bar{k}_2 = \bar{k}_2(\tau) \to \infty$ as dist$\{\tau, \mathbb{Z}^+\} \to 0$ such that the set $\mathcal{N}_{\bar{k}_2}$ allows a complete description in terms of the symbolic dynamics and all orbits of $\mathcal{N}_{\bar{k}_2}$, except for Γ_0, are saddle.

Proof. 1) Taking logarithm of the both hands of (1.20) we obtain the inequality

$$j - i + \tau < -S|\lambda|^{\bar{k}/2}, \qquad (1.24)$$

where S is a positive constant (independent of i, j and $\bar{k}$). By lemma 1.3, if $i \geq \bar{k}$ and $j \geq \bar{k}$ satisfy (1.24), then the horseshoe $T_1(\sigma_j^1)$ has empty intersection with the strip σ_i^0. Note that in the case $\tau < 0$, the inequality (1.24) has solutions only of form $j > i$. In particular, it means that for all $i \geq \bar{k}$ the horseshoes $T_1\sigma_i^1$ lie above the *own* strips σ_i^0 (see Figure 1.5a) and, therefore, all orbits, except for O and Γ_0, leave U under positive iterations of f_0.

[3] It is easy to check that the following relations $\tilde{c} = c^{-1}$, $\tilde{d} = -d(cb^2)^{-1}$ take place for the map $\tilde{T}_1 = T_1^{-1}$.

2) The inequality (1.19) can be written in the form

$$j - i + \tau > S|\lambda|^{\bar{k}/2}. \tag{1.25}$$

If τ is positive and $\bar{k}$ is sufficiently large, inequality (1.25) has always infinitely many integer solutions of form $j \leq i$ including solutions $j = i$. The latter means, by lemma 1.3, that , for all sufficiently large i, the horseshoes $T_1\sigma_i^1$ have the regular intersections with the own strips σ_i^0, see Figure 1.5c. It implies that if $\tau > 0$, the corresponding map $f_0 \in H_3^1$ has infinitely many Smale horseshoes Ω_i (i.e. for every sufficiently large i, the first return map $T_i \equiv T_1 T_0^i : \sigma_i^0 \to \sigma_i^0$ has the nonwandering set which is conjugate to the Smale horseshoe).

3) Consider the inequality (1.21). After taking logarithm, it is rewritten as

$$|j - i + \tau| \leq S|\lambda|^{\bar{k}/2}. \tag{1.26}$$

If $\tau > 0$ is not integer, inequality (1.26) has no integer solutions when $\bar{k} = \bar{k}(\tau)$ is sufficiently large. Thus, all the strips and horseshoes have only either regular or empty intersections. It allows to give the complete description for $\mathcal{N}_{\bar{k}}$. Namely, let $\mathcal{B}_\tau(\bar{k})$ be a subsystem of $\tilde{B}^3_{\bar{k}+q}$ such that

(i) $\mathcal{B}_\tau(\bar{k})$ contains the orbits $(\dots, 0, \dots, 0, \dots)$ and $\tilde{\omega}$;

(ii) in any sequence (1.23) the lengths $(k_s + q)$ and $(k_{s+1} + q)$ of any two successive strings composed from zero symbols satisfy inequality (1.25) with $j = k_s, i = k_{s+1}$.

Then $f_0\big|_{N_{\bar{k}}}$ is conjugate to $\mathcal{B}_\tau(\bar{k})$. □

Now we consider the cases with $\lambda < 0$, i.e. $f_0 \in H_3^2$ and $f_0 \in H_3^3$ (see Figure 1.8).

Proposition 1.4. *Let $f_0 \in H_3^2 \cup H_3^3$. Then the set $\mathcal{N}_{\bar{k}}(f_0)$, for any $\bar{k}$, contains nontrivial hyperbolic subsets always, except maybe for the "global resonance" case $f_0 \in H_3^2$ with $\tau = 0$.*

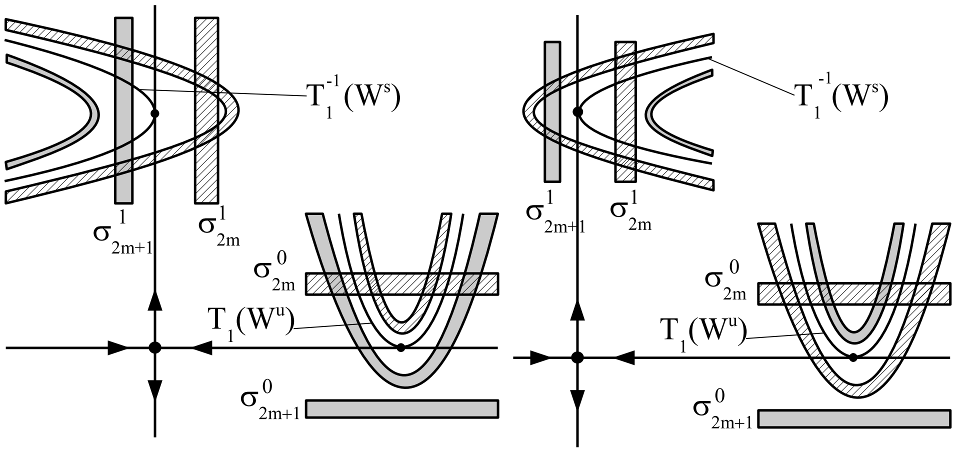
Proof. Let $f_0 \in H_3^2$. Since $\lambda < 0, c > 0, d > 0$, inequality (1.19) for even i and j, can be written as

$$j - i + \tau > S|\lambda|^{\bar{k}/2}, \text{ where } i, j \geq \bar{k} \text{ and } i, j = 0(\text{mod}2),$$

If $\tau > 0$, this inequality has infinitely many integer solutions of form $j \geq i$ including solutions $i = j$. It implies that, if $\tau > 0$, the map $f_0 \in H_3^2$ has infinitely many horseshoes Ω_i with even i.

On the other hand, inequality (1.19) for odd i and j is rewritten as

$$j - i + \tau < -S|\lambda|^{\bar{k}/2}, \text{ where } i, j \geq \bar{k} \text{ and } i, j = 1(\text{mod}2).$$



(a) $\lambda < 0$, $c > 0$, $d > 0$ (b) $\lambda < 0$, $c < 0$, $d > 0$

Figure 1.8:

If $\tau < 0$, this inequality has infinitely many integer solutions of form $i \geq j$. Thus, when $\tau < 0$ the map $f_0 \in H_3^2$ has infinitely many horseshoes Ω_i with odd i.

Let $f_0 \in H_3^3$. Since $\lambda < 0, c < 0, d > 0$, inequality (1.19) holds for any sufficiently large even i and j. It means that all horseshoes $T_1(\sigma_j^1)$ intersect regularly with all strips σ_i^0 when the numbers i and j are even (here, a certain analogy with maps of the second class is observed). In any case, map $f_0 \in H_3^3$ has infinitely many horseshoes Ω_i where i runs all sufficiently large even integers. □

Corollary 1.2. 1) *Let $f_0 \in H_3^2$. Then, if $\tau > 0$ (resp., $\tau < 0$), the map f_0 has infinitely many horseshoes Ω_i with even i (resp., odd i) and has no horseshoes with odd i (resp., with even i).*
2) *Let $f_0 \in H_3^3$. Then the map f_0 has infinitely many horseshoes Ω_i with even i and has no horseshoes with odd i.*

Proposition 1.5. *Let $f_0 \in H_3^2$ (resp., $f_0 \in H_3^3$). If $|\tau|$ is not even integer (resp., odd integer), then there exists such $\bar{k}_3$ that the set $\mathcal{N}_{\bar{k}_3}$ allows the complete descriptions in terms of the symbolic dynamics and all orbits of $\mathcal{N}_{\bar{k}}$, except for Γ_0, are saddle.*

Proof. Consider the case $f_0 \in H_3^2$. Suppose that $\tau \neq 0$. It is easy to see that, for sufficiently large $\bar{k} = \bar{k}(\tau)$ ($\bar{k} \to \infty$ as $\tau \to 0$), the set $\mathcal{N}_{\bar{k}}$ consists of orbits intersecting strips σ_j^0 either with only even numbers when $\tau > 0$ or with only odd numbers when $\tau < 0$. In the first case ($\tau > 0$), see Figure 1.9a, forward iterations of any point from σ_j^0 with odd j can not return on σ_j^0. Indeed, the horseshoe $T_1(\sigma_i^1)$ can intersect only strips with odd numbers such that $j - i + \tau < 0$, i.e. $i > j + \tau$; besides, any horseshoe $T_1(\sigma_l^1)$ with even l does not intersect the strips with odd numbers. In the case $\tau < 0$, see Figure 1.9b, any the horseshoe $T_1(\sigma_j^1)$ with even j can intersect only such strip σ_i^0 for which i is even and $j - i + \tau > 0$, i.e. $i < j$ because $\tau < 0$.

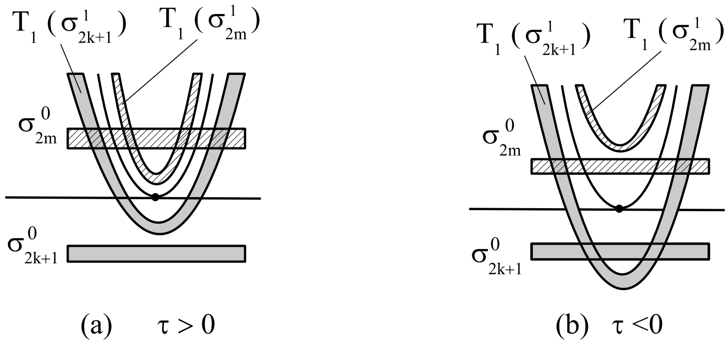


Figure 1.9:

Now we can give a complete description of the set $\mathcal{N}_{\bar{k}}$ for $f_0 \in H_3^2$ when $|\tau|$ is not even integer. We note only that numbers i and j such that $T_1(\sigma_j^1)$ has an irregular intersection with σ_i^0 must satisfy the inequality (1.26). If $|\tau|$ is not even integer, the inequality (1.26) has no integer solutions i and j of the same parity. Consider a subsystem $\mathcal{B}_\tau^{2+}$ of $\tilde{\mathcal{B}}_{\bar{k}+q}^3$ such that in any sequence (1.23) the numbers k_s are even for all s and satisfy inequality $k_s - k_{s+1} + \tau > 0$. Analogously, let $\mathcal{B}_\tau^{2-}$ be such a subsystem of $\tilde{\mathcal{B}}_{\bar{k}+q}^3$ that in any sequence (1.23) numbers k_s are odd for all s and satisfy inequality $k_s - k_{s+1} + \tau < 0$. Suppose that $|\tau|$ is not even integer. Then the system $f_0\big|_{N_{\bar{k}}}$ is conjugate either to $\mathcal{B}_\tau^{2+}$ in the case $\tau > 0$ or to $\mathcal{B}_\tau^{2-}$ in the case $\tau < 0$.

For $f_0 \in H_3^3$ we have, due to the geometry, that irregular intersections of the horseshoes $T_1(\sigma_j^1)$ and strips σ_i^0 can exist only in those cases where the numbers i and j have opposite parities. Moreover, the inequality (1.26) can have such solutions only for odd $|\tau|$. Consider a subsystem $\mathcal{B}_\tau^3$ of $\tilde{\mathcal{B}}_{\bar{k}+q}^3$ satisfying the following conditions:
(i) $\mathcal{B}_\tau^3$ contains all sequences of form (1.23) in which all numbers $k_s \geq \bar{k}(\tau)$ are even;
(ii) $\mathcal{B}_\tau^3$ do not contain the sequences with k_s and k_{s+1} to be both odd;
(iii) $\mathcal{B}_\tau^3$ contains all the sequences with even k_s and odd k_{s+1} such that $k_s - k_{s+1} + \tau < 0$;
(iv) $\mathcal{B}_\tau^3$ contains all the sequences with odd k_s and even k_{s+1} such that $k_s - k_{s+1} + \tau > 0$.
Then $f_0\big|_{N_{\bar{k}}}$ is conjugate to $\mathcal{B}_\tau^3$ when $|\tau|$ is not odd integer. □

1.3 General unfoldings and Rescaling Lemma

In this section we calculate the first return maps $T_k \equiv T_1 T_0^k : \sigma_k^0 \mapsto \sigma_k^0$ for all sufficiently large k and apply the results obtained (Lemma 1.4) for studying bifurcations of fixed points. Moreover, we consider in this section, and what follows, one and two parameter families and we take, as parameters, either $\varepsilon = \mu$ or $\varepsilon = (\mu, \tau)$, respectively. Recall that μ is the parameter of splitting of manifolds $W^u(O)$ and $W^s(O)$ with respect to the homoclinic point M^+ and τ is an invariant of the homoclinic structure given by (1.15).

The main technical result of this section is the following

Lemma 1.4. [Rescaling Lemma]
For every sufficiently large k the map $T_k : \sigma_k^0 \to \sigma_k^0$ can be brought, by a linear transformation of coordinates and parameters, to the following form

$$\begin{aligned} \bar{X} &= Y + k\lambda^{2k}\varepsilon_k^1, \\ \bar{Y} &= M - X - Y^2 + \frac{f_{03}}{d^2}\lambda^k Y^3 + k\lambda^{2k}\varepsilon_k^2 \ , \end{aligned} \tag{1.27}$$

where functions $\varepsilon_k^{1,2}(X,Y,M)$ are defined on a ball $\|(X,Y,M)\| \le R$ with arbitrary large R (when k are big) and are uniformly bounded in k along with all derivatives up to order $(r-3)$. Besides,

$$M = -d(1+\nu_k^1)\lambda^{-2k}\left(\mu + \lambda^k(cx^+ - y^-)(1 + k\beta_1\lambda^k x^+ y^-)\right) - s_0 + \nu_k^2$$

where s_0 is the coefficient given by (1.16) and $\nu_k^1 = O(\lambda^k)$, $\nu_k^2 = O(k\lambda^k)$ are some asymptotically small coefficients.

Proof. We will use the representation of T_0 in the "second normal form", i.e. in form (1.3) with $n = 2$. Then, by Lemma 1.2, the map $T_0^k : \ \sigma_k^0 \to \sigma_k^1$, for all sufficiently large k, can be written in the following form (taking into account relations (1.9))

$$x_k = \lambda^k x_0(1 + \beta_1 k\lambda^k x_0 y_k) + O(k^2\lambda^{3k}), \ \ y_0 = \lambda^k y_k(1 + \beta_1 k\lambda^k x_0 y_k) + O(k^2\lambda^{3k}). \tag{1.28}$$

We will use the notation $x = x_0, y = y_k$. Then, by virtue of (1.10), (1.11) and (1.28), we can write the first return map $T_k : \sigma_k^0 \to \sigma_k^0$ in the following form

$$\begin{aligned} &\bar{x} - x^+ = a\lambda^k x + b(y - y^-) + e_{02}(y - y^-)^2 + \\ &\qquad + O(k|\lambda|^{2k}|x| + |y - y^-|^3 + |\lambda|^k|x||y - y^-|), \\ &\lambda^k\bar{y}\left(1 + k\lambda^k\beta_1\bar{x}\bar{y}\right) + k\lambda^{3k}O(|\bar{x}| + |\bar{y}|) \ = \\ &\qquad = \ \mu + c\lambda^k x\left(1 + k\lambda^k\beta_1 xy\right) + d(y - y^-)^2 + \lambda^{2k} f_{02}\, x^2 + \\ &\qquad + \lambda^k f_{11}(1 + k\lambda^k\beta_1 xy)\, x(y - y^-) + \lambda^k f_{12}\, x(y - y^-)^2 + f_{03}(y - y^-)^3 + \\ &\qquad + O\left((y - y^-)^4 + \lambda^{2k}|x||y - y^-| + k|\lambda|^{3k}|x| + k\lambda^{2k}|x||y - y^-|^2\right). \end{aligned} \tag{1.29}$$

Below, we will denote by α_k^i, $i = 1, 2, ...$, some asymptotically small in k coefficients such that $\alpha_k^i = O(k\lambda^k)$. Now we shift the coordinates, $\eta = y - y^-$, $\xi = x - x^+ - \lambda^k x^+(a + \alpha_k^1)$, in order to nullify constant terms (independent of coordinates) in the first equation of (1.29). Thus, (1.29) recasts as follows

$$\bar{\xi} = a\lambda^k\xi + b\eta + e_{02}\eta^2 + O\left(k\lambda^{2k}|\xi| + |\eta|^3 + |\lambda|^k O(|\xi||\eta|)\right),$$

$$\begin{array}{rcl} \bar{\eta}(1+\alpha_k^2) + k\lambda^k O(|\bar{\xi}| + \bar{\eta}^2) + k\lambda^{2k}O(|\bar{\eta}|) & = & M_1\lambda^{-k} + \\ + c\xi(1+\alpha_k^3) + \lambda^{-k}\eta^2(d + \lambda^k f_{12}x^+) + \eta(f_{11}x^+ + \alpha_k^4) + f_{11}\xi\eta + \lambda^{-k}f_{03}\eta^3 + \\ + O\left(|\lambda|^{-k}\eta^4 + k|\lambda|^{2k}|\xi| + k\lambda^k(\xi^2+\eta^2) + |\xi|\eta^2\right), \end{array} \tag{1.30}$$

where $M_1 = \mu + \lambda^k(cx^+ - y^-)(1 + k\lambda^k\beta_1 x^+ y^-) + \lambda^{2k}x^+(ac + f_{02}x^+) + O(k\lambda^{3k})$

Now, we rescale the variables:

$$\xi = -\frac{b(1+\alpha_k^2)}{d + \lambda^k f_{12}x^+}\lambda^k u\ ,\ \eta = -\frac{1+\alpha_k^2}{d + \lambda^k f_{12}x^+}\lambda^k v. \tag{1.31}$$

Then system (1.30) in coordinates (u, v) is rewritten in the following form

$$\begin{array}{l} \bar{u} = v + a\lambda^k u - \dfrac{e_{02}}{bd}\lambda^k v^2 + O(k\lambda^{2k}), \\ \bar{v} \ = \ M_2\lambda^{-2k} - u(1+\alpha_k^5) - v^2 + \\ \qquad + v(f_{11}x^+ + \alpha_k^6) - \dfrac{f_{11}b}{d}\lambda^k uv + \dfrac{f_{03}}{d^2}\lambda^k v^3 + O(k\lambda^{2k}), \end{array} \tag{1.32}$$

where $M_2 = -(d + \lambda^k f_{12}x^+)(1+\alpha_k^2)^{-1}M_1$. The following shift of coordinates, $u_{new} = u - \frac{1}{2}(f_{11}x^+ + \alpha_k^6)$, $v_{new} = v - \frac{1}{2}(f_{11}x^+ + \alpha_k^6)$, brings map (1.32) to the following form

$$\begin{array}{l} \bar{u} = v + a\lambda^k u - \dfrac{e_{02}}{bd}\lambda^k v^2 + O(k\lambda^{2k}), \\ \bar{v} = M_3 - u - v^2 - \dfrac{f_{11}b}{d}\lambda^k uv + \dfrac{f_{03}}{d^2}\lambda^k v^3 + O(k\lambda^{2k}), \end{array} \tag{1.33}$$

where $M_3 = M_2\lambda^{-2k} - f_{11}x^+ + \dfrac{(f_{11}x^+)^2}{4}$.

Now, we make the following linear change of coordinates $x = u + \tilde{\nu}_k^1 v$, $y = v - \tilde{\nu}_k^2 u$, where $\tilde{\nu}_k^1 = -\dfrac{e_{02}}{bd}\lambda^k$, $\tilde{\nu}_k^2 = -\dfrac{e_{02}}{bd}\lambda^k - a\lambda^k$. Then system (1.33) is rewritten as

$$\begin{array}{l} \bar{x} = y + M_3\tilde{\nu}_k^1 + O(k\lambda^{2k}), \\ \bar{y} = M_3 - x - y^2 + a\lambda^k y - \tilde{R}\lambda^k xy + \dfrac{f_{03}}{d^2}\lambda^k y^3 + O(k\lambda^{2k}), \end{array} \tag{1.34}$$

where $\tilde{R} = (2a + 2e_{02}/bd - bf_{11}/d) \equiv 0$ by (1.13).

Finally, make one more shift of coordinates $X = x - \dfrac{1}{2}a\lambda^k - \tilde{\nu}_k^1 M_3$, $Y = y - \dfrac{1}{2}a\lambda^k$, in order to nullify the constant term in the first equation and the linear in y term in the second equation of (1.34). After this, we obtain the final form (1.27) of map T_k in the rescaled coordinates where formula (1.18) takes place for the parameter M. $\square$

Thus, the Rescaling Lemma shows that the unified limit form for the first return maps T_k is the conservative Hénon map

$$\bar{x} = y, \ \bar{y} = M - x - y^2. \tag{1.35}$$

Bifurcations of fixed points in the conservative Hénon family are well known. There exists a generic elliptic fixed point for every $M \in (-1;3)$ except for $M = 0$ when $\psi = \pi/2$ and $M = 5/4$ when $\psi = 2\pi/3$.[4] The latter cases correspond to the strong resonances 1 : 4 and 1 : 3, respectively. The conservative resonance 1 : 3 is non-degenerate, while 1 : 4 is degenerate (here the so-called case "$A = 1$" is realized). However, in the "refined" map

$$\bar{x} = y, \ \bar{y} = M - x - y^2 + \frac{f_{03}}{d^2}\lambda^k y^3, \tag{1.36}$$

where the cubic term is non-zero, i.e. if $f_{03} \neq 0$, the resonance 1 : 4 becomes non-degenerate [Bir87], [Gon05], see also Appendix A. The conservative Hénon map has also a fixed parabolic point: with multipliers $\nu_1 = \nu_2 = +1$) at $M = -1$ and with multipliers $\nu_1 = \nu_2 = -1$ at $M = 3$. It follows that the bifurcation scenario in map (1.27) looks like in Figure 1.7(a) accompanying with one of scenarios of of bifurcations near $\psi = \pi/2$ for $f_{03}\lambda^k > 0$ (Figure 1.7(b)) or $f_{03}\lambda^k < 0$ (Figure 1.7(c)).

1.4 Proofs of Theorems 1.1, 1.2 and 1.3

We can easily deduce the theorems from the Rescaling Lemma. Indeed, since bifurcations in the conservative Hénon maps (1.35) and (1.36) are known, we can use the corresponding facts directly for recovering bifurcations of single-round periodic orbits in the families f_μ and $f_{\mu,\tau}$.

Proof of Theorem 1.1. Rewrite (1.18) in the following way

$$\mu = -\lambda^k y^- \alpha(1 + k\beta_1\lambda^k x^+ y^-) - \frac{1}{d}(M + s_0 + \hat{\rho}_k)\lambda^{2k}, \tag{1.37}$$

where $\hat{\rho}_k = O(k\lambda^k)$ is some small coefficient and

$$\alpha = \frac{cx^+}{y^-} - 1. \tag{1.38}$$

Since the Hénon map (1.35) has fixed parabolic points at $M = -1$ (with double multiplier $+1$) and $M = 3$ (with double multiplier -1), we obtain, by (1.37), that the first return map T_k has a fixed point with multipliers $\nu_1 = \nu_2 = +1$ for

$$\mu = \mu_k^+ \equiv -\lambda^k y^- \alpha(1 + k\beta_1\lambda^k x^+ y^-) - \frac{1}{d}(s_0 - 1 + \hat{\rho}_k)\lambda^{2k}, \tag{1.39}$$

[4]Note that the fixed point of (1.35) with $\psi = \arccos(-1/4)$ is degenerate since it has local normal form in which the first Birkhoff coefficient vanishes, however, its second Birkhoff coefficient is nonzero [Bir87]. It means that this point is KAM-stable and, hence, generic.

and a fixed point with multipliers $\nu_1 = \nu_2 = -1$ for

$$\mu = \mu_k^- \equiv -\lambda^k y^- \alpha(1 + k\beta_1 \lambda^k x^+ y^-) - \frac{1}{d}(s_0 + 3 + \hat{\rho}_k)\lambda^{2k}. \tag{1.40}$$

As μ_k^+ and μ_k^- are border points of the interval δ_k, it follows that $\delta_k \to 0$ as $k \to \infty$. By (1.39) and (1.40), the intervals δ_k have length equal to $4|\lambda|^{2k}(1 + \dots)/|d|$; besides, intervals δ_k and δ_{k+1} and posed one from other in the distance of order $|\lambda|^k|\alpha|(1-|\lambda|)$, if $\alpha \neq 0$ and k is sufficiently large. Thus, if $\alpha \neq 0$, the intervals with different numbers do not intersect. Evidently, condition $\alpha = 0$ means, by (1.15), that $c > 0$ and $\tau = 0$. It completes the proof of Theorem 1.1. □

Proof of Theorem 1.2. By (1.15) and (1.38), we can express α in terms of τ; namely, $\alpha = -1 - |\lambda|^{-\tau}$ if $c < 0$ and $\alpha = -1 + |\lambda|^{-\tau}$ if $c > 0$. Then we can rewrite equations (1.39) and (1.40) as the corresponding equations of the bifurcation curves L_k^+ and L_k^- on the (τ, μ)-plane.

In the case $c < 0$ we have that $\alpha < -1$. It follows that the domains Δ_i and Δ_j (defined on the (τ, μ)-plane) do not crossed for sufficiently large $i \neq j$, see Figure 1.6 (a)–(c).

However, in the case $c > 0$, the value $\alpha(\tau)$ changes the sign: $\alpha(0) = 0$, $\alpha(\tau) > 0$ at $\tau < 0$ and $\alpha(\tau) < 0$ at $\tau > 0$. We can rewrite equations (1.39) and (1.40) as the corresponding equations of the bifurcation curves L_k^+ and L_k^- on the (τ, μ)-plane:

$$\begin{aligned} L_k^+ :\ & \mu = -\lambda^k y^-(|\lambda|^{-\tau} - 1)(1 + k\beta_1\lambda^k x^+ y^-) - d^{-1}(s_0 - 1 + \hat{\rho}_k)\lambda^{2k}, \\ L_k^- :\ & \mu = -\lambda^k y^-(|\lambda|^{-\tau} - 1)(1 + k\beta_1\lambda^k x^+ y^-) - d^{-1}(s_0 + 3 + \hat{\rho}_k)\lambda^{2k}. \end{aligned} \tag{1.41}$$

Evidently, these curves near the origin ($\tau = 0, \mu = 0$) have a form showing in Figure 1.6 (d)–(f). If we put $\tau = 0$ into (1.41), we obtain that curves L_k^+ and L_k^- intersect the axis μ in points $\mu = -\frac{1}{d}(s_0 - 1 + \hat{\rho}_k)\lambda^{2k}$ and $\mu = -\frac{1}{d}(s_0 + 3 + \hat{\rho}_k)\lambda^{2k}$, respectively. If $-3 < s_0 < 1$, the point $\mu = 0$ lies between these border points and, thus, the origin ($\tau = 0, \mu = 0$) belongs to all domains Δ_k with sufficiently large k. It completes the proof of Theorem 1.2 and gives Corollary 1.1. □

Proof of Theorem 1.3. The theorem follows from the Rescaling Lemma which gives the rescaled form (1.27) for first return maps. Since we know the bifurcation scenarios observed in (1.27), see [Bir87] and [Gon05], and due to relations (1.39) and (1.40) connecting M and μ, we can complete the theorem. □

1.5 Invariants of homoclinic tangencies in symplectic two-dimensional maps

We see that in the case of the global resonance $\tau = 0$ the dynamics of area-preserving maps of the third class (except for maps on H_3^5) depends, in fact, on only one quantity s_0. In this section we prove the invariance of both τ and s_0

First, we recall the result from [GS87] that τ is an invariant of two-dimensional diffeomorphisms with homoclinic tangencies to a saddle with $\sigma \equiv |\lambda\gamma| = 1$ and prove it in the case of area-preserving maps. Note that we prove the invariance of τ in those C^r-coordinate which conserve the first order normal form (1.4) of the saddle map T_0. However, as it was shown in [AY05], τ is also invariant under C^1-linearization coordinates that allows to say about the existence/absence of (topological, of course) Smale horseshoes near a homoclinic tangency in terms of τ.

After this, we prove the invariance of s_0. However, it will be invariant only in those C^{r-2}-coordinates which conserve the n-order normal form (1.3) for $n \geq 2$. Naturally, s_0 "disappears" when C^1-linearization is used, since it depends on the coefficients of the quadratic terms of T_1 which become indefinite at uncontrolled C^1-changes.

The following lemma is an area-preserving variant of the corresponding result from [GS87].

Lemma 1.5. *In coordinates (1.4), the value of τ does not depend on the choice (in W^u_{loc} and W^s_{loc}) of any pair of homoclinic points of the orbit Γ_0.*

Proof. We take a pair $M^{+\prime} = T_0(M^+)$ and M^- of points of Γ_0. Then, by (1.4), $x^{+\prime} = \lambda x^+$. The new global map T_1' is defined as $T_1' = T_0 T_1 : \Pi^- \to T_0(\Pi^+)$ and written as

$$\begin{aligned} \bar{x}' &= \lambda(x^+ + F(x, y - y^-) + O\left[(x^+ + F(x, y - y^-)^2 G(x, y - y^-)\right] = \\ &\qquad = \lambda x^+ + F'(x, y - y^-), \\ \bar{y}' &= \lambda^{-1} G(x, y - y^-) + O\left[(x^+ + F(x, y - y^-) G^2(x, y - y^-)\right] = G'(x, y - y^-). \end{aligned}$$

Since $F(0,0) = 0, G(0,0) = 0, G_y(0,0) = 0$, we obtain that $F'(0,0) = 0$ and

$$c' = G_x'(0,0) = \lambda^{-1} G_x(0,0) + O(G(x, y - y^-))_{|x=0, y=y^-} = \lambda^{-1} c$$

Finally, since also $y^{-\prime} = y^-$, we have

$$\tau' = \frac{1}{\ln|\lambda|} \ln \left| \frac{c' x^{+\prime}}{y^{-\prime}} \right| = \frac{1}{\ln|\lambda|} \ln \left| \frac{c \lambda^{-1} x^+ \lambda}{y^-} \right| = \tau.$$

Take now a pair $M^{+\prime} = M^+$ and $M^{-\prime} = T_0^{-1}(M^-)$. Then the new global map T_1' is defined as $T_1' = T_1 T_0 : T_0^{-1}(\Pi^-) \to \Pi^+$ and written as

$$\bar{x} = x^+ + F(\lambda x' + O(x'^2 y'), \lambda^{-1} y' + O(x' y'^2) - y^-), \bar{y} = G(\lambda x' + O(x'^2 y'), \lambda^{-1} y' + O(x' y'^2) - y^-).$$

Since $F(0,0) = 0$, we have $x^{+\prime} = x^+$; since W^u_{loc} has equation $x = 0$, we have $y^{-\prime} = \lambda y^-$; finally, since

$$G_x^{new}(0,0) = \lambda G_x(0,0) + \left[G_x \cdot O(x) + G_y \cdot O(y^2)\right]_{x=0, y=y^-}$$

and $G_x(0,0) = c, G_y(0,0) = 0$, we have $c' = \lambda c$. Thus,

$$\tau' = \frac{1}{\ln|\lambda|} \ln \left| \frac{c' x^{+\prime}}{y^{-\prime}} \right| = \frac{1}{\ln|\lambda|} \ln \left| \frac{\lambda c x^+}{\lambda y^-} \right| = \tau.$$

Any pair of points of Γ_0 one of which is in W^u_{loc} and the other is in W^s_{loc} can be evidently obtained by means of a finite series of elementary choices (like before) of neighboring homoclinic points. Every such choice does not change value of τ, it follows that the same relates to the resulting choice. □

Note that τ does not also depend also on smooth changes of coordinates preserving the main normal form (1.4) of T_0, see [GS87].

Lemma 1.6. *Let the local map T_0 be given in the second normal form, i.e in the form (1.3) for $n = 2$. If $\mu = 0$ and $\tau = 0$, then s_0 does not depend on choice of pairs of the points in W^s_{loc} and W^u_{loc} of orbit Γ_0.*

Proof. Take, first, a pair $M^{+\prime} = T_0(M^+)$ and M^- of points of Γ_0. The new global map T_1' is defined as $T_1' = T_0 T_1 : \Pi^- \to T_0(\Pi^+)$ and can be written in form

$$\bar{x}' = \lambda\bar{x}(1 + \beta_1\bar{x}\bar{y}) + O\left[\bar{x}^3\bar{y}^2\right], \quad \bar{y}' = \lambda^{-1}\bar{y}(1 - \beta_1\bar{x}\bar{y}) + O\left[\bar{x}^2\bar{y}^3\right], \tag{1.42}$$

where $\bar{x} = x^+ + F(x, y - y^-), \bar{y} = G(x, y - y^-)$. We will calculate the corresponding coefficients (that defines new s_0') at the homoclinic point $x = 0, y = y^-$ that gives also that $\bar{x} = x^+, \bar{y} = 0$. We use also that $G_y(0, 0) = 0$ and

$$\frac{\partial\bar{y}'}{\partial\bar{x}} = 0, \quad \frac{\partial\bar{x}'}{\partial\bar{x}} = \lambda, \quad \frac{\partial\bar{y}'}{\partial\bar{y}} = \lambda^{-1} \text{ at } \bar{x} = x^+, \bar{y} = 0.$$

It is evident also that O-terms in (1.42) vanish at $\bar{y} = 0$ together with all required derivatives (note that the second derivatives with respect to $\bar{y}'$ are only needed). Thus, we have

$$a' = \frac{\partial\bar{x}'}{\partial x} = \lambda\frac{\partial F}{\partial x} + \lambda\beta_1(\bar{x})^2\frac{\partial G}{\partial x} + O(\bar{y}), \quad c' = \frac{\partial\bar{y}'}{\partial x} = \lambda^{-1}\frac{\partial G}{\partial x} + O(\bar{y}),$$

$$d' = \frac{1}{2}\frac{\partial^2\bar{y}'}{\partial y^2} = \frac{1}{2}\lambda^{-1}\frac{\partial^2 G}{\partial y^2} + O(\bar{y}) + O(\partial\bar{y}/\partial y),$$

$$f_{02}' = \frac{1}{2}\frac{\partial^2\bar{y}'}{\partial x^2} = \frac{1}{2}\lambda^{-1}\left(\frac{\partial^2 G}{\partial x^2} - 2\beta_1\bar{x}\left(\frac{\partial G}{\partial x}\right)^2\right) + O(\bar{y}),$$

$$f_{11}' = \frac{\partial^2\bar{y}'}{\partial x\partial y} = \lambda^{-1}\frac{\partial^2 G}{\partial x\partial y} + O(\bar{y}) + O(\partial\bar{y}/\partial y)$$

and $x^{+\prime} = \lambda x^+$. Since we find these derivatives at a point where $x = 0, y = y^-$, $\bar{x} = x^+, \bar{y} = 0$ and $\partial\bar{y}/\partial y \equiv G_y = 0$, it gives

$$a' = \lambda a + \lambda(x^+)^2\beta_1 c, \quad c' = \lambda^{-1}c, \quad d' = \lambda^{-1}d, \quad f_{02}' = \lambda^{-1}f_{02} - \lambda^{-1}c^2\beta_1 x^+, \quad f_{11}' = \lambda^{-1}f_{11}. \tag{1.43}$$

Then we have

$$d'x^{+\prime} = dx^+, \ \ f'_{11}x^{+\prime} = f_{11}x^+,$$
$$a'c' = ac + (x^+)^2\beta_1 c^2 + f_{02}x^+ - c^2\beta_1(x^+)^2 = ac + f_{20}x^+.$$

Thus, by (1.16), $s_0 = s'_0$.

Take now a pair $M^{+\prime} = M^+$ and $M^{-\prime} = T_0^{-1}(M^-)$. Then the new global map T'_1 is defined as $T'_1 = T_1T_0 : T_0^{-1}(\Pi^-) \to \Pi^+$ and, thus, it can be written as

$$\bar{x} = x^+ + F(x', y' - y^-), \ \ \bar{y} = G(x', y' - y^-),$$

where $x' = \lambda x(1+\beta_1 xy) + O(x^3y^2)$, $y' = \lambda^{-1}y(1-\beta_1 xy) + O(x^2y^3)$ and $(x,y) \in T_0^{-1}\Pi^-$. Thus, we have that $x^{+\prime} = x^+, y^{-\prime} = \lambda y^-$. Then we calculate other coefficients as the corresponding derivatives at $x = 0, y = \lambda y^-$. We have

$$a' = \frac{\partial F}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial F}{\partial y'}\frac{\partial y'}{\partial x}, \quad c' = \frac{\partial G}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial G}{\partial y'}\frac{\partial y'}{\partial x},$$

$$f'_{20} = \frac{1}{2}\left(\frac{\partial^2 G}{(\partial x')^2}\left(\frac{\partial x'}{\partial x}\right)^2 + 2\frac{\partial^2 G}{\partial x'\partial y'}\frac{\partial y'}{\partial x}\frac{\partial x'}{\partial x} + \frac{\partial^2 G}{\partial y'^2}\left(\frac{\partial y'}{\partial x}\right)^2 + \frac{\partial G}{\partial x'}\frac{\partial^2 x'}{\partial x^2} + \frac{\partial G}{\partial y'}\frac{\partial^2 y'}{\partial x^2}\right),$$

$$f'_{11} = \frac{\partial^2 G}{(\partial x')^2}\frac{\partial x'}{\partial x}\frac{\partial x'}{\partial y} + \frac{\partial^2 G}{\partial x'\partial y'}\left(\frac{\partial y'}{\partial x}\frac{\partial x'}{\partial y} + \frac{\partial x'}{\partial x}\frac{\partial y'}{\partial y}\right) + \frac{\partial^2 G}{\partial y'^2}\frac{\partial y'}{\partial x}\frac{\partial y'}{\partial y} + \frac{\partial G}{\partial x'}\frac{\partial^2 x'}{\partial x\partial y} + \frac{\partial G}{\partial y'}\frac{\partial^2 y'}{\partial x\partial y},$$

$$d' = \frac{1}{2}\left(\frac{\partial^2 G}{(\partial x')^2}\left(\frac{\partial x'}{\partial y}\right)^2 + 2\frac{\partial^2 G}{\partial x'\partial y'}\frac{\partial y'}{\partial y}\frac{\partial x'}{\partial y} + \frac{\partial^2 G}{\partial y'^2}\left(\frac{\partial y'}{\partial y}\right)^2 + \frac{\partial G}{\partial x'}\frac{\partial^2 x'}{\partial y^2} + \frac{\partial G}{\partial y'}\frac{\partial^2 y'}{\partial y^2}\right).$$

Since for $x = 0, y = \lambda y^-$

$$\frac{\partial G}{\partial y'} = 0, \ \ \frac{\partial}{\partial y}\left(x', \frac{\partial x'}{\partial x}, \frac{\partial x'}{\partial y}\right) = 0, \ \ \frac{\partial x'}{\partial x} = \lambda, \ \ \frac{\partial y'}{\partial y} = \lambda^{-1}, \ \ \frac{\partial y'}{\partial x} = -\lambda\beta_1(y^-)^2,$$

we obtain that

$$a' = \lambda a - b\beta_1\lambda(y^-)^2, \ \ c' = \lambda c, \ \ f'_{11} = f_{11} - 2d\beta_1(y^-)^2, \ \ d' = d\lambda^{-2},$$
$$f'_{20} = f_{20}\lambda^2 - f_{11}\lambda^2\beta_1(y^-)^2 + d\lambda^2\beta_1^2(y^-)^4 + c\lambda^2\beta_1 y^-,$$

Thus, we have

$$d'x^{+\prime} = dx^+\lambda^{-2},$$
$$a'c' = \lambda^2 ac - bc\beta_1\lambda^2(y^-)^2 = \lambda^2 ac + \beta_1\lambda^2(y^-)^2 \ \ (\text{since } \ bc = -1,$$
$$f'_{11}x^{+\prime} = f_{11}x^+ - 2d\beta_1(y^-)^2x^+,$$
$$d' = dx^+\lambda^{-2},$$
$$f'_{20}x^{+\prime} = f_{20}\lambda^2x^+ - f_{11}\lambda^2\beta_1(y^-)^2x^+ + d\lambda^2\beta_1^2(y^-)^4x^+ + c\lambda^2\beta_1 y^-x^+.$$

We obtain, by (1.16),

$$\begin{aligned} s_0' = {} & dx^+ [ac + f_{20}x^+] - \beta_1 d(y^-)^2 x^+ - df_{11}\beta_1 (y^-)^2 (x^+)^2 + d^2\beta_1^2 (y^-)^4 (x^+)^2 \\ & + dc\beta_1 y^- (x^+)^2 + f_{11}x^+ - 2d\beta_1 (y^-)^2 x^+ - \tfrac{1}{4}(f_{11}x^+)^2 + d\beta_1 (y^-)^2 (x^+)^2 \\ & - d^2\beta_1^2 (y^-)^4 (x^+)^2 = s_0 + d\beta_1 x^+ y^- (cx^+ - y^-) \end{aligned}$$

Since $cx^+ = y^-$ for $\tau = 0$, this completes the proof. □

Chapter 2

Dynamics and bifurcations of non-orientable area-preserving maps with quadratic homoclinic tangencies

In this chapter we study bifurcations of area-preserving and non-orientable maps with quadratic homoclinic tangencies. It seems that it is a quite new topic in homoclinic bifurcations. Up to now, homoclinic tangencies of non-orientable maps were studied only for general (dissipative) systems, see e.g. [GS73, GS86, PT87, GS07]. However, this theme should be interesting for understanding dynamics of chaotic conservative maps such as non-orientable planar maps like the Hénon maps with the Jacobian -1 and symplectic maps on two-dimensional non-orientable closed manifolds (like Klein bottle).

2.1 Statement of the problem and preliminary constructions

In this chapter we consider non-orientable APMs close to the one with a quadratic homoclinic tangency. Let f_0 be such a map. As in Chapter 1 we denote by O the saddle fixed point of f_0, by U_0 a small neighbourhood of O and by Γ_0 the nontransversal to O homoclinic orbit. We embed f_0 into a parameter family f_ε. We assume that f_0 and f_ε satisfy the conditions **B** and **C** of Chapter 1 (Section 1.1).

However, due to non-orientability, instead of condition **A** we suppose the following condition:

A′. f_0 has a saddle fixed point O with multipliers λ and γ, where $0 < |\lambda| < 1 < |\gamma|$ and $|\lambda\gamma| = 1$. Moreover, we will consider two different cases:

A′.1 the saddle is *orientable*, i.e. $\lambda = \gamma^{-1}$;

A′.2 the saddle is *non-orientable*, i.e. $\lambda = -\gamma^{-1}$.

Note that in condition A′.1 we assume that the global map T_1 defined in Chapter 1, Section 1.1, is non-orientable, i.e. the Jacobian of T_1 is equal to -1. This behavior of orbits for return maps is typical for maps on two-dimensional non-orientable manifolds (Möbius strip, Klein bottle, projective plane etc.), see an example of such a map in Figure 2.1.

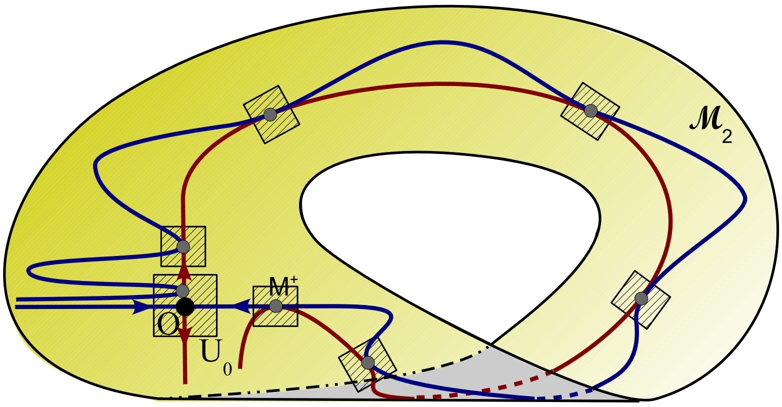


Figure 2.1: An example of non-orientable area-preserving map (on the Möbius band) having a quadratic homoclinic tangency at the points of a homoclinic orbit Γ_0. Some of these homoclinic points are shown as grey circles. Also a small neighborhood of the set $O \cup \Gamma_0$ is shown to be the union of a number of "squares".

The main goal is the same: to study bifurcations of *single-round periodic orbits* (see Def. inition 1.1) in the families f_ε. Every point of such an orbit is considered again as a fixed point of the corresponding first return map $T_k = T_1 T_0^k$, where $T_0 \equiv T_0(\varepsilon)$ is the local map and $T_1 \equiv T_1(\varepsilon)$ is the global map whose definition is similar to the symplectic case (see Chapter 1). In particular, the coordinate expression for T_1 is the same as in Chapter 1, formulas (1.10) and (1.11). Concerning the local map T_0, its normal form is symplectic (the same as in Lemma 1.1) in the case A′.1, while in the case A′.1 we provide non-orientable normal forms in the next section.

2.1.1 Finite-smooth normal forms for non-orientable saddle area-preserving maps

In the following lemma we give the main normal form (of the first order) for the local map $T_0(\varepsilon)$ in the non-orientable case.

Lemma 2.1. [GST07]. *Let $T_0(\varepsilon)$ be a C^r-smooth, $r \geq 3$, saddle area-preserving map that has a fixed point O with multipliers λ and $-\lambda^{-1}$, where $|\lambda| < 1$. Then there exists such C^r-smooth local canonical coordinate change, which is C^{r-2} with respect to the parameters, that the map T_0 takes the following form*

$$\bar{x} = \lambda(\varepsilon)x + o(x^2y), \quad \bar{y} = -\lambda^{-1}(\varepsilon)y + o(xy^2), \tag{2.1}$$

The following lemma relates to *the n-th order normal form.*

Lemma 2.2. *For any $p = r - 2n + 1$, where $n \geq 2$ is an integer such that $n < r/2$ (if $r = \infty$, then n is arbitrary), there exists such C^p-smooth local canonical coordinate change, which is C^{p-2} with respect to the parameters, that the map T_0 takes the following form*

$$\begin{aligned} \bar{x} &= \lambda(\varepsilon)x\left(1 + \sum_2^n \beta_i(\varepsilon)\cdot(xy)^i\right) + o(x^{n+1}y^n), \\ \bar{y} &= -\lambda^{-1}(\varepsilon)y\left(1 + \sum_2^n \tilde{\beta}_i(\varepsilon)\cdot(xy)^i\right) + o(x^n y^{n+1}), \end{aligned} \tag{2.2}$$

where the coefficients β_i and $\tilde{\beta}_i$ are invariants of smooth canonical changes of coordinates preserving form (2.2), and, moreover, $\beta_i(\varepsilon) = \tilde{\beta}_i(\varepsilon) \equiv 0$ for all odd $i \leq n$.

As in the symplectic case, see Lemma 1.2, the normal forms of Lemmas 2.1 and 2.2 allow to obtain a quite simple coordinate expression for iterations T_0^k for all integer k. Namely, let $(x_i, y_i) \in U_0$, $i = 0, \ldots, k-1$, be such points that $(x_{i+1}, y_{i+1}) = T_0(x_i, y_i)$, then the following results hold.

Lemma 2.3. 1) *If T_0 takes the first order normal form (2.1), then T_0^k can be written as follows*

$$x_k = \lambda^k x_0 + \lambda^{2k} P_1(x_0, y_k, \varepsilon), \quad y_0 = (-\lambda)^k y_k + \lambda^{2k} Q_1(x_0, y_k, \varepsilon), \tag{2.3}$$

where the functions P_1 and Q_1 are uniformly bounded along with all derivatives up to order $(r-2)$ and the following estimates takes place for the last derivatives

$$\|(x_k, y_0)\|_{C^{r-1}} = O(|\lambda|^k), \ \|(x_k, y_0)\|_{C^r} = o(1)_{k\to\infty}.$$

2) *If T_0 takes the n-th order normal form (2.2), then T_0^k can be written as*

$$\begin{aligned} x_k &= \lambda^k x_0 \cdot R_n^{(k)}(x_0 y_k, \varepsilon) + \lambda^{(n+1)k} P_n^{(k)}(x_0, y_k, \varepsilon), \\ y_0 &= (-\lambda)^k y_k \cdot R_n^{(k)}(x_0 y_k, \varepsilon) + \lambda^{(n+1)k} Q_n^{(k)}(x_0, y_k, \varepsilon)\ , \end{aligned} \tag{2.4}$$

where $R_n^{(k)}(x_0 y_k, \varepsilon)$ is given by formula (1.6) in which $\tilde{\beta}_1(k) = 0, \tilde{\beta}_2(k) = \beta_2\, k, \ldots$ The functions $P_n^{(k)} = o(x_0^{n+1} y_k^n), Q_n^{(k)} = o(x_0^n y_k^{n+1})$ are uniformly bounded in k along with all their derivatives with respect to x_0 and y_k up to the order $(r-2n-1)$, besides, $\|(x_k, y_0)\|_{C^{r-2n}} = O(|\lambda|^k)$, $\|(x_k, y_0)\|_{C^{r-2n+1}} = o(1)_{k\to\infty}$.

Remark 2.1. In principle, we will use Lemma 2.3 only for the cases where T_0 takes the first and second order normal forms. In the latter case (for the second order normal form) the iterations of T_0^k can be written as

$$x_k = \lambda^k x_0 + O(k\lambda^{3k}), \;\; y_0 = (-\lambda)^k y_k + O(k\lambda^{3k}), \tag{2.5}$$

We prove Lemma 2.2 in Chapter 4. The proof of Lemma 2.3 is the same as for Lemma 1.2 from Chapter 1 and, therefore, we omit it.

In what follows, we use in U_0 the local normal form coordinates (x, y) introduced in the Sections 1.1 and 2.1. In these coordinates both W^s_{loc} and W^u_{loc} are straightened and, hence, we can put $M^+ = (x^+, 0), M^- = (0, y^-)$, where $x^+ > 0$ and $y^- > 0$. Without loss of generality, we assume that $x^+ > 0$ and $y^- > 0$. Then the global map $T_1(\varepsilon) \equiv f^q(\varepsilon) : \Pi^- \to \Pi^+$ can be written in the form (1.10), where relations (1.11) hold.

In the area-preserving case, the Jacobian $J(T_1)$ of T_1 is equal to ± 1 identically for all values of parameters ε. We note that APMs with homoclinic tangencies have various numerical invariants. In particular, we introduce the following important quantities

$$\alpha = \frac{cx^+}{y^-} - 1 \;\text{ and }\; \hat{\alpha} = \frac{cx^+}{y^-} + 1 \tag{2.6}$$

which are some analogous of the invariant τ for the symplectic case (α is the same as in the symplectic case, see formula (1.38)).

2.1.2 Strips, horseshoes and return maps

We assume that the neighbourhoods Π^+ and Π^- are sufficiently small and fixed, so that $T_0(\varepsilon)(\Pi^+) \cap \Pi^+ = \emptyset$ and $T_0^{-1}(\varepsilon)(\Pi^-) \cap \Pi^- = \emptyset$ for all small ε. Then the domain of definition of the successor map from Π^+ to Π^- under iterations of $T_0(\varepsilon)$ consists of infinitely many nonintersecting strips σ_k^0 belonging to Π^+ and accumulating at $W^s_{loc} \cap \Pi^+$ as $k \to \infty$. Analogously, the range of this map consists of infinitely many (nonintersecting) strips $\sigma_k^1 = T_0^k(\sigma_k^0)$ belonging to Π^- and accumulating at $W^u_{loc} \cap \Pi^-$ as $k \to \infty$. See Figure 2.2 where a location of the strips is shown for various cases.

According to (1.10) and (1.11), the images $T_1(\sigma_j^1)$ of the strips σ_j^1 have a horse-shoe shape form and accumulate to the curve $l_u = T_1(W^u_{loc})$ as $j \to \infty$. Note that any orbit staying entirely in U must intersect both the neighborhoods Π^- and Π^+ (otherwise, it would not be close to $\overline{\Gamma}_0$). Thus, such orbits must have points belonging to intersections of the horseshoes $T_1(\sigma_j^1)$ and the strips σ_i^0 for all possible integer i and j.

When μ varies the location of the horseshoes $T_1(\sigma_j^1)$ is changed: they move together with $T_1(W^u_{loc})$. It implies that a character of mutual intersections of the strips and horseshoes can cardinally change. It relates, in particular, to the strips σ_i^0 and horseshoes $T_1(\sigma_i^1)$ with the same numbers i. Thus, at varying μ bifurcations of creation/destruction Smale horseshoes will occur in the first return maps $T_k(\mu)$. We will study the main accompanied bifurcations in Sections below.

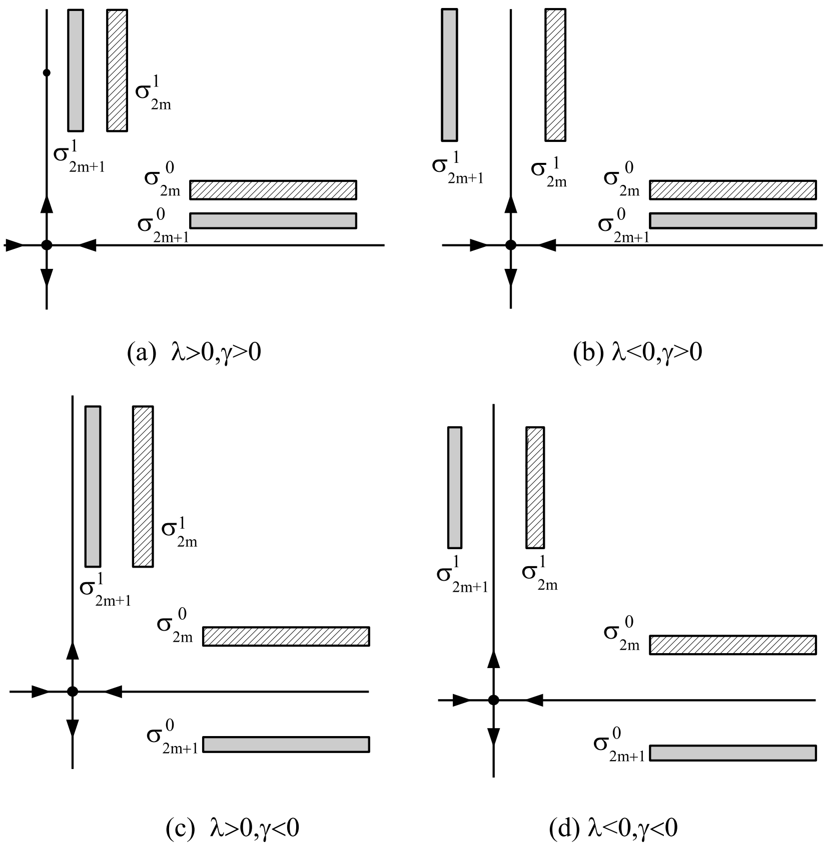


Figure 2.2: The strips σ_k^0 and σ_k^1 for λ and γ of various signs.

2.2 Main results: on cascades of elliptic periodic orbits

We divide non-orientable APMs under consideration into two groups:

(i) *the globally non-orientable APMs* when T_0 is orientable and T_1 is non-orientable ($\lambda\gamma = +1$ and $bc = +1$), i.e. the condition A′.1 holds;

(ii) *the locally non-orientable APMs* when T_0 is non-orientable ($\lambda\gamma = -1$), , i.e. the condition A′.2 holds.

Note that in the locally non-orientable case the Jacobian of T_1 equals $+1$ or -1 depending on choice of a pair of the homoclinic points. Indeed, if the Jacobian of T_1 is equal to -1 (i.e. $bc = +1$) for a given pair $M^+ \in W^s_{loc}$ and $M^- \in W^u_{loc}$ of the points, then, for the pair $\tilde{M}^+ = T_0(M^+)$ and M^- of the homoclinic points, the Jacobian of the new global map $\tilde{T}_1 = T_0 T_1$ will be equal $+1$, since $J(T_0) = -1$. Therefore, in the locally non-orientable case, we will assume, for more definiteness,

- that the homoclinic points $M^+ \in W^s_{loc}$ and $M^- \in W^u_{loc}$ are such that $J(T_1) = +1$.

Remark 2.2. It can show[1] that the quantities α and $\hat{\alpha}$ from (2.6) do not depend on choice of pairs of homoclinic points $M^+ \in W^s_{loc}$ and $M^- \in W^u_{loc}$; conditionally that, in the locally non-orientable case, the points are chosen in such a way that the sign of $J(T_1)$ is not changed.

Theorem 2.1 (One parameter cascades of elliptic points in APMs)**.** *Let f_0 be APM satisfying conditions A′ and B and f_μ be a one parameter family of close to f_0 APMs that unfolds generally (under condition* C*) at $\mu = 0$ the quadratic homoclinic tangency. For any interval $I = (-\mu_0, \mu_0)$ values of μ, there exists such integer and positive $\bar{k}$ that the following holds:*

1. (a) *In the globally non-orientable case, the maps f_μ have no single-round elliptic periodic orbits, while there exist intervals $\mathsf{e}^2_k \subset I$, $k = \bar{k}, \bar{k}+1, \dots,$ where f_μ has a double-round elliptic orbit, of period $2(k+q)$, which corresponds to a period two point of the first return map T_k.*

   (b) *In the locally non-orientable case, there exist intervals e_{2m} and e^2_{2m+1} in I for any integer m such that $2m \geq \bar{k}$, where the map f_μ has a single-round elliptic periodic of (period $2m+q$) orbit at $\mu \in \mathsf{e}_{2m}$ and has a double-round elliptic periodic orbit at $\mu \in \mathsf{e}^2_{2m+1}$.*

2. *The intervals e_k as well as e^2_k accumulate to $\mu = 0$ as $k \to \infty$ and do not intersect for sufficiently large and different integer k if $\alpha \neq 0$ in the globally non-orientable case as well as $\alpha \neq 0$ and $\hat{\alpha} \neq 0$ in the locally non-orientable case.*

3. *Any interval e_k has border points $\mu = \mu^+_k$ and $\mu = \mu^-_k$ such that the map f_μ has a single-round periodic orbit (of period $k+q$) with double multiplier $+1$ at $\mu = \mu^+_k$ and with double multiplier -1 at $\mu = \mu^-_k$. Any interval e^2_k has border points $\mu = \mu^{2+}_k$ and $\mu = \mu^{2-}_k$ such that the map f_μ has a single-round periodic orbit (of period $(k+q)$) with multipliers $+1$ and -1 at $\mu = \mu^{2+}_k$ and a double-round periodic orbit (of period $2(k+q)$) with double multiplier -1 at $\mu = \mu^{2-}_k$. See Figure 2.3.*

4. *The angular argument φ of the multipliers $e^{\pm i\varphi}$ of the elliptic points at $\mu \in \mathsf{e}_k$ or $\mu \in \mathsf{e}^2_k$ depends monotonically on μ and the elliptic point is generic (KAM-stable) for all such μ, except for those where $\varphi(\mu) = \frac{\pi}{2}, \frac{2\pi}{3}$.*

Note that Theorem 2.1 does not give answer on the question on a mutual position of the intervals e_k and e^2_k in the critical cases $\alpha = 0$ and $\hat{\alpha} = 0$. But this moment is quite important, since relates to the coexistence of elliptic orbits of different periods. The same as in the symplectic case, we consider this question by means of two parameter families.

We assume now that f_0 is a map satisfying conditions A′ and B with $\alpha = 0$ in the globally non-orientable case and with $\alpha = 0$ (when $c > 0$) or $\hat{\alpha} = 0$ (when $c < 0$) in the locally non-orientable case. Denote by $\mathcal{H}_{0,0}$ and $\hat{\mathcal{H}}_{0,0}$ codimension 2 bifurcation surfaces from the space of APMs consisting of maps close to f_0 and having a nontransversal

[1] the same as the invariance of τ in Lemma 1.5 from Chapter 1

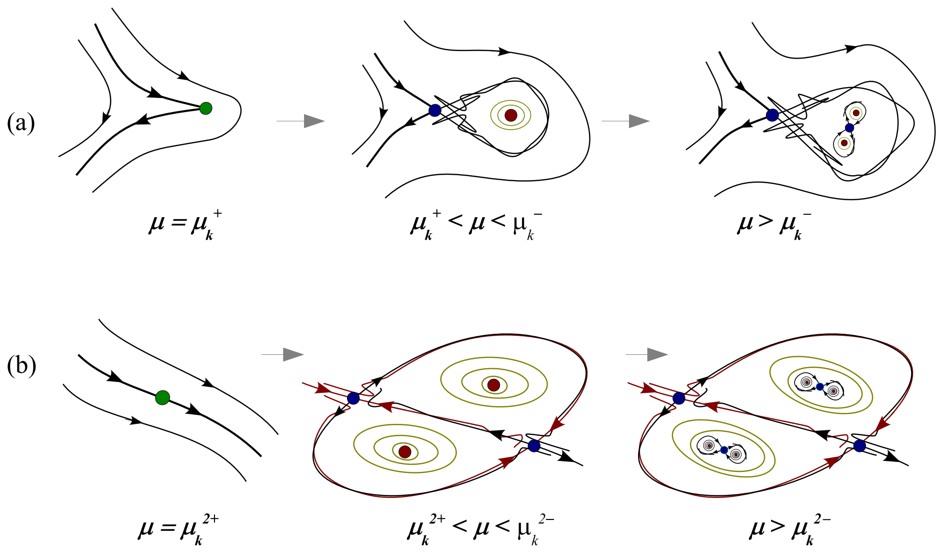


Figure 2.3: Bifurcation scenarios in the first return maps T_k according to item 3 of Theorem 2.1. We show here that the birth of the elliptic point is happened when increasing μ, while for some types of homoclinic tangencies it can occur at decreasing μ. (a) The map T_k is orientable, then the value $\mu = \mu_k^+$ corresponds to the appearance of a fixed point of T_k that is a non-degenerate parabolic fixed point with double multiplier +1. This point falls into two fixed points, saddle and elliptic ones, when $\mu \in \mathsf{e}_k$. The moment $\mu = \mu_k^-$ corresponds to the period doubling bifurcation with the elliptic fixed point. (b) If T_k is the non-orientable map, then the value $\mu = \mu_k^-$ corresponds to the appearance of a fixed point with multipliers +1 and −1. This point falls into four points, two saddle fixed ones and other two points compose an elliptic cycle of period 2, when $\mu \in \mathsf{e}_k^2$. The moment $\mu = \mu_k^{2-}$ corresponds to the period doubling bifurcation of this period 2 cycle.

homoclinic orbit close to Γ_0 and such that the condition $\alpha = 0$ and $\hat{\alpha} = 0$ holds, respectively. We will consider two parameter families $\{f_{\mu,\alpha}\}$ and $\{f_{\mu,\hat{\alpha}}\}$ of APMs which are transverse to $\mathcal{H}_{0,0}$ and $\hat{\mathcal{H}}_{0,0}$ at $\mu = 0, \alpha = 0$ and $\mu = 0, \hat{\alpha} = 0$, respectively.

Let D_ϵ and $\hat{D}_\epsilon$ be sufficiently small neighbourhoods (of diameter $\epsilon > 0$) of the origin in the parameter planes (μ, α) and $(\mu, \hat{\alpha})$.

It follows from Theorem 2.1 that infinitely many such open domains, E_k^2 for the globally non-orientable case and E_{2m} and E_{2m+1}^2 for the locally non-orientable case, exist in D_ϵ and $\hat{D}_\epsilon$ that the following holds.

- If $(\mu, \alpha) \in E_k$, then the map $f_{\mu,\alpha}$ or $f_{\mu,\hat{\alpha}}$ has a single-round elliptic orbit of period $(k + q)$ and if $(\mu, \alpha) \in E_k^2$, then the map $f_{\mu,\alpha}$ or $f_{\mu,\hat{\alpha}}$ has a double-round elliptic orbit of period $2(k + q)$.

- The domains E_k and E_k^2 accumulate to the axis $\mu = 0$ as $k \to \infty$.
- Any domain E_k has two boundaries, bifurcation curves L_k^+ and L_k^-, correspond to the existence of a single-round nondegenerate periodic orbit with double multipliers $+1$ and -1, respectively.
- Any domain E_k^2 has two boundaries, bifurcation curves L_k^{2+} and L_k^{2-}, which correspond, respectively, to the existence of a single-round nondegenerate periodic orbit with double multipliers $+1$ and a double-round nondegenerate periodic orbit with double multipliers -1.

In the following theorem we state that the domains E_k or E_k^2 can mutually intersect for different large enough k.

Theorem 2.2. 1) *In the globally non-orientable case, there exist domains E_k^2 either in D_ϵ for $c > 0$ or in $\hat{D}_\epsilon$ for $c < 0$. In D_ϵ the domains E_k^2 intersect the axis μ and all of them are mutually crossed for different and sufficiently large k. In $\hat{D}_\epsilon$ the domains E_k^2 do not intersect the axis μ and are not crossed.*

2) *In the locally non-orientable case, there exist domains E_{2m} and E_{2m+1}^2 both in D_ϵ and $\hat{D}_\epsilon$. In the case $c > 0$, the domains E_{2m} are crossed in D_ϵ and are not crossed in $\hat{D}_\epsilon$; the domains E_{2m+1}^2 are crossed in $\hat{D}_\epsilon$ and are not crossed in D_ϵ. In the case $c < 0$, the domains E_{2m} are crossed in $\hat{D}_\epsilon$ and are not crossed in D_ϵ; the domains E_{2m+1}^2 are crossed in D_ϵ and do not crossed in $\hat{D}_\epsilon$.*

In the case $c > 0$, the domain E_{2m} intersects in D_ϵ all domains E_{2j+1}^2 with $j \geq m$; in the case $c < 0$, the domain E_{2m+1}^2 intersects in $\hat{D}_\epsilon$ all domains E_{2j} with $j > m$.

In Figure 2.7 some qualitative illustrations to Theorem 2.2 are shown for the cases where $\lambda\gamma = +1$ and the map T_1 is non-orientable, (a) and (b), and $\lambda\gamma = -1$, (c) and (d).

We introduce now the following quantities

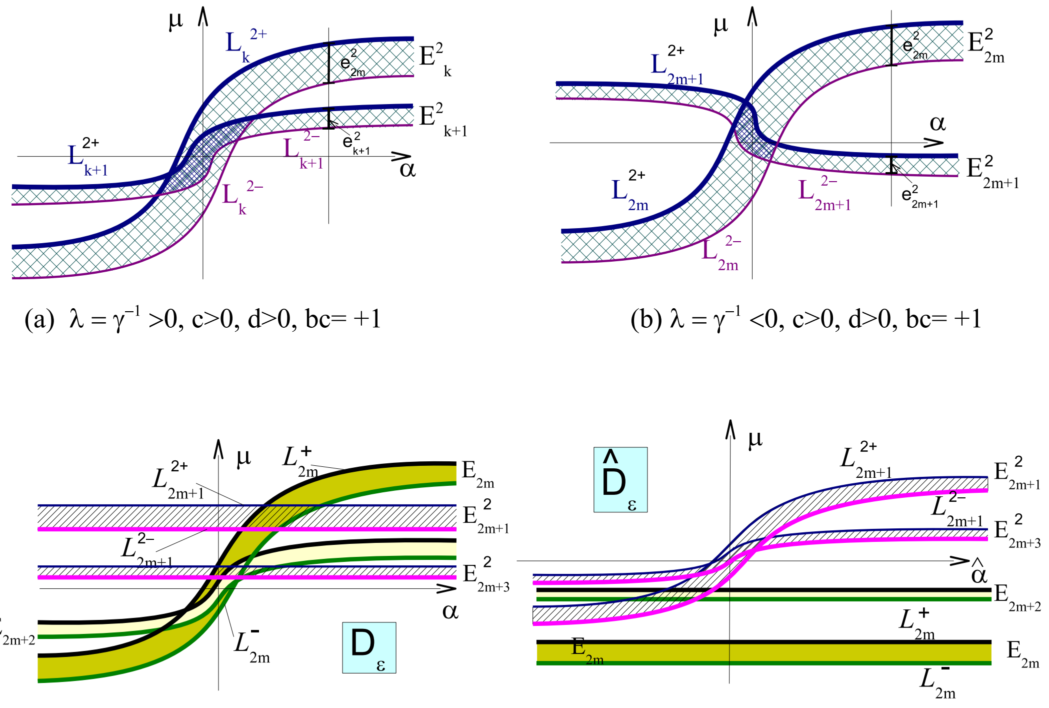
$$s_0^{or} = dx^+(ac + f_{20}x^+) + ef_{11}x^+(1 - \frac{1}{4}f_{11}x^+) \ , \tag{2.7}$$

and

$$s_0^{nor} = dx^+(ac + f_{20}x^+) - \frac{1}{4}(f_{11}x^+)^2 \ , \tag{2.8}$$

which are calculated by coefficients of the global map T_1, see formula (1.11), and play an important role in global dynamics of the map f_0 with $\alpha = 0$ or $\hat{\alpha} = 0$.

Theorems 2.1 and 2.2 show that elliptic single-round or double-round periodic orbits of different periods can coexist when values of μ and α or μ and $\hat{\alpha}$ close to zero. Moreover, infinitely many such orbits can coexist, in principle, at the global resonance $\mu = 0, \alpha = 0$ or $\mu = 0, \hat{\alpha} = 0$. The following theorem give us sufficient conditions for this event.



(a) λ = γ⁻¹ >0, c>0, d>0, bc= +1 (b) λ = γ⁻¹ <0, c>0, d>0, bc= +1

(c) λ = −γ⁻¹ < 0, c>0,d>0 and bc= −1 (d) λ = −γ⁻¹ < 0, c<0,d>0 and bc= −1

Figure 2.4: Elements of the bifurcation diagrams for the families $f_{\mu,\alpha}$ in the cases with $\lambda\gamma = +1$, where T_1 is orientable, (a) and (b), and T_1 is non-orientable, (c) and (d).

Theorem 2.3 (On infinitely many elliptic points in non-orientable APMs with homoclinic tangency)**.** *Let f_0 be a non-orientable APM with a quadratic homoclinic tangency (i.e., conditions A′ and B hold). We assume also that the resonant condition $\alpha = 0$ or $\hat{\alpha} = 0$ takes place for f_0.*

1) Let f_0 be a globally non-orientable map and $\alpha = 0$. Then, if $s = s_{nor}$ (see formula (2.8)) is such that $-1 < s_{nor} < 0$, then f_0 has infinitely many double-round elliptic periodic orbits of all successive periods $2(k+q)$ (where $k \geq \bar{k}$).

2) Let f_0 be a locally non-orientable map and the homoclinic points M^+ and M^- be chosen such that T_1 is orientable (by our arrangement), i.e. $bc = -1$. Then, if $c > 0$, $\alpha = 0$ and $-1 < s_{or} < 3$, the map f_0 has infinitely many single-round elliptic periodic orbits of all successive periods $2m + q$ (where $2m \geq \bar{k}$). If $c < 0$, $\hat{\alpha} = 0$ and $-1 < s_{nor} < 0$, the map f_0 has infinitely many double-round elliptic periodic orbits of all successive periods of form $2(2m + 1 + q)$ (where $2m \geq \bar{k}$).

3) If $s_{or} \neq 0; \frac{5}{4}$ or $s_{nor} \neq -\frac{1}{2}; -\frac{1}{\sqrt{2}}; -\frac{5}{8}$, then all these elliptic orbits of sufficiently large periods are generic (KAM-stable).

In the rest part of this chapter we prove these and related results.

2.3 The rescaling lemmas in the non-orientable case

We embed f_0 into a parametric family f_ε which unfolds generally the initial homoclinic tangency. Moreover, our main attention will be paid to one and two parameter families when the parameters $\varepsilon = \mu$, $\varepsilon = (\mu, \alpha)$ or $\varepsilon = (\mu, \hat{\alpha})$, respectively, are considered as governing ones. Recall that μ is the parameter of splitting manifolds $W^u(O)$ and $W^s(O)$ with respect to the homoclinic point M^+ (see formula (1.11)), and α is the parameter given by formula (2.6).

The main goal of the rest part of the chapter is the study of bifurcations of *single-round periodic orbits.* By definition (see Definition 1.1), every such an orbit has only one intersection point with Π^+ (or with Π^-). In turn, this point can be considered as a fixed point of the corresponding *first return map* $T_k \equiv T_1 T_0^k : \sigma_k^0 \mapsto \sigma_k^0$ with an appropriate integer $k \geq \bar{k}$. Note that the integers k can run all values from the set $\{\bar{k}, \bar{k}+1, \dots\}$ where $\bar{k}$ is some sufficiently large positive integer.

In principle, to study bifurcations, we can write the first return maps T_k in the initial coordinates and with the initial parameters ε, using formula (1.11) for the global map T_1 and the corresponding formulas from Lemma 2.3 for maps T_0^k. However, there is a more effective way for studying bifurcations. Namely, we can bring maps T_k to some unified form for all large k using the so-called rescaling method as it has been done in many papers.[2] After this, we can study (one time) bifurcations in the unified map and "project" obtained results onto the first return maps T_k for various k.

First we consider the globally non-orientable maps, i.e. we assume that the map T_0 is symlectic ($\lambda = \gamma^{-1}$) and the Jacobian of the global map T_1 equals to -1 (i.e. $bc = +1$). Then the following result holds.

Lemma 2.4. [Rescaling lemma in the globally non-orientable case]
In the globally non-orientable case, for every sufficiently large k, the first return map $T_k : \sigma_k^0 \to \sigma_k^0$ can be brought, by a linear transformation of coordinates and parameters, to the following form

$$\begin{aligned} \bar{X} &= Y + k\lambda^{2k}\varepsilon_k^1, \\ \bar{Y} &= M + X - Y^2 + \frac{f_{03}}{d^2}\lambda^k Y^3 + k\lambda^{2k}\varepsilon_k^2 \,, \end{aligned} \tag{2.9}$$

where functions $\varepsilon_k^{1,2}(X,Y,M)$ are defined on some asymptotically big domain covering in the limit $k \to +\infty$ all finite values of X, Y and M, and these functions are uniformly bounded in k along with all derivatives up to order $(r-4)$; and the following formula takes place for M:

$$M = -d(1+\nu_k^1)\lambda^{-2k}\left(\mu + \lambda^k(cx^+ - y^-)(1 + k\beta_1\lambda^k x^+ y^-)\right) - s_0^{nor} + \nu_k^2 \,; \tag{2.10}$$

[2] see, for example, papers [BS89, MR97, GS97, LS04, DGGLO13] in which the rescaling method was applied for conservative and reversible cases

where formulas (2.8) valid for s_0^{nor} *and* $\nu_k^{1,3} = O(\lambda^k), \nu_k^{2,4} = O(k\lambda^k)$ *are some asymptotically small coefficients.*

Proof. We will use the representation of the symplectic map T_0 in the "second normal form", i.e. in form (2.2) for $n = 2$.[3] Then the map $T_0^k :\ \sigma_k^0 \to \sigma_k^1$, for all sufficiently large k, can be written in form (1.28).

Then, using formulae (1.11) and (1.28), we can write the first return map $T_k : \sigma_k^0 \to \sigma_k^0$ in the following form

$$\begin{aligned}
&\bar{x} - x^+ = a\lambda^k x + b(y-y^-) + e_{02}(y-y^-)^2 + \\
&\qquad +O(k|\lambda|^{2k}|x| + |y-y^-|^3 + |\lambda|^k|x||y-y^-|), \\
&\lambda^k \bar{y}\left(1 + k\lambda^k\beta_1\bar{x}\bar{y}\right) + k\lambda^{3k}O(|\bar{x}| + |\bar{y}|) \ = \\
&\qquad = \ \mu + c\lambda^k x\left(1 + k\lambda^k\beta_1 xy\right) + d(y-y^-)^2 + \lambda^{2k} f_{02}\, x^2 + \\
&\qquad +\lambda^k f_{11}(1 + k\lambda^k\beta_1 xy)\, x(y-y^-) + \lambda^k f_{12}\, x(y-y^-)^2 + f_{03}(y-y^-)^3 + \\
&\qquad +O\left((y-y^-)^4 + \lambda^{2k}|x||y-y^-| + k|\lambda|^{3k}|x| + k\lambda^{2k}|x||y-y^-|^2\right),
\end{aligned} \tag{2.11}$$

where $x = x_0, y = y_k$.

Below, we will denote by α_k^i, $i = 1, 2, ...$, some asymptotically small in k coefficients such that $\alpha_k^i = O(k\lambda^k)$. Now we shift the coordinates

$$\eta = y - y^-,\ \xi = x - x^+ - \lambda^k x^+(a + \alpha_k^1),$$

in order to nullify the constant term (independent of coordinates) in the first equation of (2.11). Thus, (2.11) is recast as follows

$$\begin{aligned}
&\bar{\xi} = a\lambda^k\xi + b\eta + e_{02}\eta^2 + O\left(k\lambda^{2k}|\xi| + |\eta|^3 + |\lambda|^k O(|\xi||\eta|)\right), \\
&\lambda^k\bar{\eta}(1+\alpha_k^2) + k\lambda^{2k}O(|\bar{\xi}| + \bar{\eta}^2) + k\lambda^{3k}O(|\bar{\eta}|) \ = \\
&\quad = \ M_1 + c\lambda^k\xi(1+\alpha_k^3) + \eta^2(d + \lambda^k f_{12}x^+) + \lambda^k\eta(f_{11}x^+ + \alpha_k^4) + \lambda^k f_{11}\xi\eta + f_{03}\eta^3 + \\
&\qquad\qquad +O\left(\eta^4 + k|\lambda|^{3k}|\xi| + k\lambda^{2k}(\xi^2+\eta^2) + \lambda^k|\xi|\eta^2\right),
\end{aligned} \tag{2.12}$$

where

$$M_1 = \mu + \lambda^k(cx^+ - y^-)(1 + k\lambda^k\beta_1 x^+y^-) + \lambda^{2k}x^+(ac + f_{02}x^+) + O(k\lambda^{3k}). \tag{2.13}$$

Now, we rescale the variables:

$$\xi = -\frac{b(1+\alpha_k^2)}{d + \lambda^k f_{12}x^+}\lambda^k u\ ,\ \eta = -\frac{1+\alpha_k^2}{d + \lambda^k f_{12}x^+}\lambda^k v. \tag{2.14}$$

[3] Of course, we lose, a little, in a smoothness, since the second order normal form is C^{r-2} only, see Lemma 2.2. However, we get more principally important information on form of the first return maps. On the other hand, our considerations cover also C^∞ and real analytical cases.

System (2.12) in coordinates (u, v) is rewritten in the following form

$$\begin{aligned} \bar{u} &= v + a\lambda^k u - \frac{e_{02}}{bd}\lambda^k v^2 + O(k\lambda^{2k}), \\ \bar{v} &= M_2 + u(1+\alpha_k^5) - v^2 + \\ &\quad + v(f_{11}x^+ + \alpha_k^6) - \frac{f_{11}b}{d}\lambda^k uv + \frac{f_{03}}{d^2}\lambda^k v^3 + O(k\lambda^{2k}) \,, \end{aligned} \tag{2.15}$$

where

$$M_2 = -\frac{d + \lambda^k f_{12}x^+}{1+\alpha_k^2}\lambda^{-2k} M_1.$$

The following shift of coordinates (we remove the linear in v terms from the second equation)

$$u_{new} = u - \frac{1}{2}(f_{11}x^+ + \alpha_k^6), \ v_{new} = v - \frac{1}{2}(f_{11}x^+ + \alpha_k^6),$$

brings map (2.15) to the following form

$$\begin{aligned} \bar{u} &= v + a\lambda^k u - \frac{e_{02}}{bd}\lambda^k v^2 + O(k\lambda^{2k}), \\ \bar{v} &= M_3 + u - v^2 - \frac{f_{11}b}{d}\lambda^k uv + \frac{f_{03}}{d^2}\lambda^k v^3 + O(k\lambda^{2k}) \,, \end{aligned} \tag{2.16}$$

where

$$M_3 = M_2 + \frac{(f_{11}x^+)^2}{4}.$$

We make the following linear change of coordinates

$$x \ = \ u + \tilde{\nu}_k^1 v \ \ , \ \ y \ = \ v - \tilde{\nu}_k^2 u \ , \tag{2.17}$$

where

$$\tilde{\nu}_k^1 = -\frac{e_{02}}{bd}\lambda^k, \ \tilde{\nu}_k^2 = -\frac{e_{02}}{bd}\lambda^k - a\lambda^k. \tag{2.18}$$

Then, system (2.16) is rewritten as

$$\begin{aligned} \bar{x} &= y + M_3\tilde{\nu}_k^1 + O(k\lambda^{2k}), \\ \bar{y} &= M_3 + x - y^2 + a\lambda^k y - \tilde{R}\lambda^k xy + \frac{f_{03}}{d^2}\lambda^k y^3 + O(k\lambda^{2k}) \,, \end{aligned} \tag{2.19}$$

where $\tilde{R} = (2a + 2e_{02}/bd - bf_{11}/d) \equiv 0$ by (1.13). Hence, map (2.19) has the following form

$$\begin{aligned} \bar{x} &= y + M_3\tilde{\nu}_k^1 + O(k\lambda^{2k}), \\ \bar{y} &= M_3 + x - y^2 + a\lambda^k y + \frac{f_{03}}{d^2}\lambda^k y^3 + O(k\lambda^{2k}) \,, \end{aligned} \tag{2.20}$$

Finally, make one more shift of coordinates

$$X = x - \frac{1}{2}a\lambda^k - \tilde{\nu}_k^1 M_3, \quad Y = y - \frac{1}{2}a\lambda^k,$$

in order to nullify the constant term in the first equation and the linear in y term in the second equation of (2.20). After this, we obtain the final form (2.9) of map T_k in the rescale coordinates where formula (2.10) takes place for the parameter M. □

Now we consider the locally non-orientable case. Thus, we assume that $\lambda = -\gamma^{-1}$. Moreover, as our agreement, we take the pairs M^+ and M^- of the homoclinic points in such a way that the corresponding global map T_1 becomes orientable ($J(T_1) = +1$), i.e. $bc = -1$.

Lemma 2.5. [Rescaling lemma in the locally non-orientable case]
1) *Let a sufficiently large k be even. Then the first return map $T_k : \sigma_k^0 \to \sigma_k^0$ can be brought, by a linear transformation of coordinates and parameters, to the following form*

$$\begin{aligned} \bar{X} &= Y + k\lambda^{2k}\varepsilon_k^1, \\ \bar{Y} &= M - X - Y^2 + \frac{f_{03}}{d^2}\lambda^k Y^3 + k\lambda^{2k}\varepsilon_k^2 \,, \end{aligned} \tag{2.21}$$

where

$$M = -d(1+\nu_k^1)\lambda^{-2k}\left(\mu + \lambda^k(cx^+ - y^-)\right) - s_0^{or} + \nu_k^2 \,; \tag{2.22}$$

2) *Let a sufficiently large k be odd. Then the first return map $T_k : \sigma_k^0 \to \sigma_k^0$ can be brought, by a linear transformation of coordinates and parameters, to the following form*

$$\begin{aligned} \bar{X} &= Y + k\lambda^{2k}\varepsilon_k^1, \\ \bar{Y} &= M + X - Y^2 - \frac{f_{03}}{d^2}\lambda^k Y^3 + k\lambda^{2k}\varepsilon_k^2 \,, \end{aligned} \tag{2.23}$$

where

$$M = -d(1+\nu_k^1)\lambda^{-2k}\left(\mu + \lambda^k(cx^+ - y^-)\right) - s_0^{nor} + \nu_k^2 \,; \tag{2.24}$$

3) *Here functions $\varepsilon_k^{1,2}(X,Y,M)$ are defined on some asymptotically big domain covering in the limit $k \to +\infty$ all finite values of X, Y and M, and these functions are uniformly bounded in k along with all derivatives up to order $(r-4)$); $\nu_k^{1,3} = O(\lambda^k), \nu_k^{2,4} = O(k\lambda^k)$ are some asymptotically small coefficients.*

Proof. We will again use the representation of the map T_0 in the "second normal form", now in form (2.2). Then the map $T_0^k : \sigma_k^0 \to \sigma_k^1$, for all sufficiently large k, can be written in form (2.5).

Evidently, in the case 1) of the lemma, the proof does not differ from the symplectic case, only $\beta_1 \equiv 0$ here, which gives us the sought result.

In the case 2) of the lemma, we proceed in the same way as at the beginning of Lemma 2.4 and since $\gamma^{-k} = -\lambda^k$, obtain the following formula, analogous to (2.12),

$$\begin{array}{l} \bar{\xi} = a\lambda^k \xi + b\eta + e_{02}\eta^2 + O\left(k\lambda^{2k}|\xi| + |\eta|^3 + |\lambda|^k O(|\xi||\eta|)\right), \\ -\lambda^k\bar{\eta}(1+\alpha_k^2) + k\lambda^{2k}O(|\bar{\xi}| + \bar{\eta}^2) + k\lambda^{3k}O(|\bar{\eta}|) \ = \\ \quad = \ M_1 + c\lambda^k\xi(1+\alpha_k^3) + \eta^2(d + \lambda^k f_{12}x^+) + \lambda^k\eta(f_{11}x^+ + \alpha_k^4) + \lambda^k f_{11}\xi\eta + f_{03}\eta^3 + \\ \qquad + O\left(\eta^4 + k|\lambda|^{3k}|\xi| + k\lambda^{2k}(\xi^2+\eta^2) + \lambda^k|\xi|\eta^2\right), \end{array} \tag{2.25}$$

where

$$M_1 = \mu + \lambda^k(cx^+ + y^-)(1 + k\lambda^k\beta_1 x^+ y^-) + \lambda^{2k}x^+(ac + f_{02}x^+) + O(k\lambda^{3k}). \tag{2.26}$$

(the difference is only that the factor $-\lambda^k$ stands in the left side of the second equation and $(cx^+ + y^-)$ is in formula for M_1).

Now, we rescale the variables:

$$\xi = \frac{b(1+\alpha_k^2)}{d + \lambda^k f_{12}x^+}\lambda^k u \ , \ \eta = \frac{1+\alpha_k^2}{d + \lambda^k f_{12}x^+}\lambda^k v. \tag{2.27}$$

System (2.25) in coordinates (u, v) is rewritten in the following form

$$\begin{array}{l} \bar{u} = v + a\lambda^k u + \dfrac{e_{02}}{bd}\lambda^k v^2 + O(k\lambda^{2k}), \\ \bar{v} \ = \ M_2 + u(1+\alpha_k^5) - v^2 + \\ \qquad -v(f_{11}x^+ + \alpha_k^6) - \dfrac{f_{11}b}{d}\lambda^k uv - \dfrac{f_{03}}{d^2}\lambda^k v^3 + O(k\lambda^{2k}) \ , \end{array} \tag{2.28}$$

where

$$M_2 = -\frac{d + \lambda^k f_{12}x^+}{1+\alpha_k^2}\lambda^{-2k}M_1.$$

The following shift of coordinates (we remove the linear in v terms from the second equation)

$$u_{new} = u + \frac{1}{2}(f_{11}x^+ + \alpha_k^6), \ v_{new} = v + \frac{1}{2}(f_{11}x^+ + \alpha_k^6),$$

brings map (2.28) to the following form

$$\begin{array}{l} \bar{u} = v + a\lambda^k u + \dfrac{e_{02}}{bd}\lambda^k v^2 + O(k\lambda^{2k}), \\ \bar{v} = M_3 + u - v^2 - \dfrac{f_{11}b}{d}\lambda^k uv - \dfrac{f_{03}}{d^2}\lambda^k v^3 + O(k\lambda^{2k}) \ , \end{array} \tag{2.29}$$

where

$$M_3 = M_2 + \frac{(f_{11}x^+)^2}{4}.$$

We make the following linear change of coordinates

$$x = u + \tilde{\nu}_k^1 v \ , \ y = v - \tilde{\nu}_k^2 u \ , \tag{2.30}$$

where

$$\tilde{\nu}_k^1 = \frac{e_{02}}{bd}\lambda^k, \ \tilde{\nu}_k^2 = \frac{e_{02}}{bd}\lambda^k + a\lambda^k. \tag{2.31}$$

Then, system (2.29) is rewritten as

$$\begin{aligned} \bar{x} &= y - M_3\tilde{\nu}_k^1 + O(k\lambda^{2k}), \\ \bar{y} &= M_3 + x - y^2 - a\lambda^k y + \hat{R}\lambda^k xy - \frac{f_{03}}{d^2}\lambda^k y^3 + O(k\lambda^{2k}) \ , \end{aligned} \tag{2.32}$$

where $\hat{R} = (2a - 2e_{02}/bd - bf_{11}/d) \equiv 0$.

Finally, make one more shift of coordinates

$$X = x + \frac{1}{2}a\lambda^k + \tilde{\nu}_k^1 M_3, \ \ Y = y + \frac{1}{2}a\lambda^k,$$

in order to nullify the constant term in the first equation and the linear in y term in the second equation of (2.32). After this, we obtain the final form (2.23) of map T_k in the rescaled coordinates where formula (2.24) takes place for the parameter M. □

2.4 Proof of the main results

Theorems 2.1, 2.2 and 2.3 are proved translating the results on bifurcations of fixed points of the first return maps T_k to single-round periodic orbits of f_μ, $f_{\mu,\alpha}$ or $f_{\mu,\hat{\alpha}}$. Bifurcations in first return maps T_k can be studied with using their normal forms deduced by the rescaling lemmas 2.4 and 2.5. Since these normal forms coincide up to asymptotically small as $k \to \infty$ terms with the non-orientable conservative Hénon map, we recall in the next section some necessary results on bifurcations of fixed points in one parameter families of conservative Hénon map in non-orientable case.

2.4.1 On bifurcations of fixed points in the conservative Hénon maps

Thus, the Rescaling Lemma 2.4 and 2.5 show that the unified limit form for the first return maps T_k is the conservative Hénon map

$$\bar{x} = y, \ \bar{y} = M + \nu x - y^2. \tag{2.33}$$

(orientable, if $\nu = -1$, and non-orientable, if $\nu = +1$). Bifurcations of fixed points in the conservative Hénon family are well known.

In the case $\nu = -1$, the Hénon map has a generic elliptic fixed point for every $M \in (-1; 3)$ except for $s_0 = 0$ when $\psi = \pi/2$ and $s_0 = 5/4$ when $\psi = 2\pi/3$. These cases corresponds to the strong resonances 1 : 4 and 1 : 3, respectively. The conservative resonance 1 : 3 is non-degenerate, while 1 : 4 is degenerate: the so-called case "$A = 1$", [Arn96, AAIS86], is realized here. However, it is important that in the first return map (2.9), when the coefficients f_{03} is non-zero, the resonance 1 : 4 becomes non-degenerate [Bir87, Gon05], see also Appendix A. The conservative Hénon map has also fixed parabolic points, at $M = -1$ with multipliers $\nu_1 = \nu_2 = +1$ and at $M = 3$ with multipliers $\nu_1 = \nu_2 = -1$. The corresponding bifurcations are nondegenerate. See Figure 2.5 for an illustration.

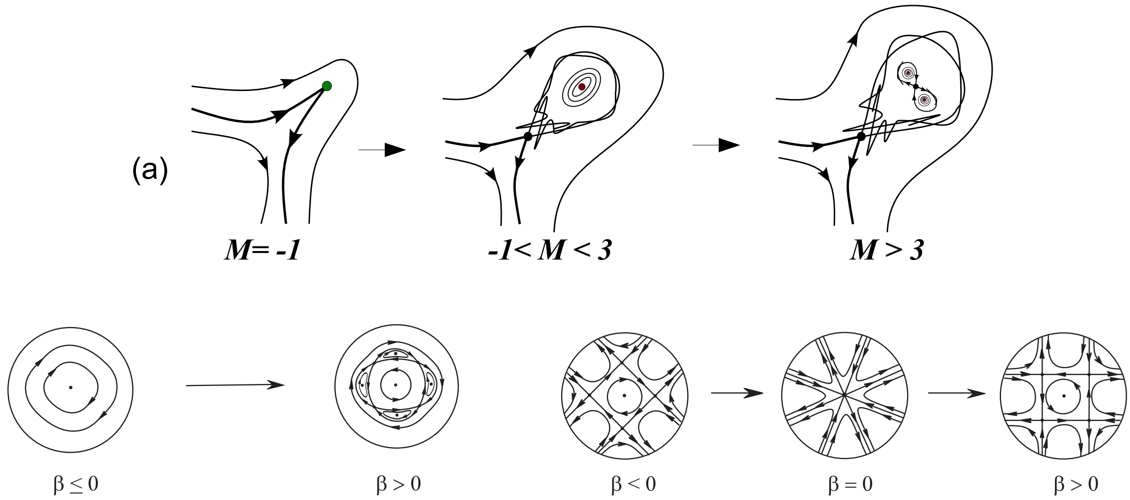


Figure 2.5: Bifurcations of fixed points in the Hénon map: (a) the main scenario, here $\mu_k^+ < \mu_k^-$ (if $d > 0$, then $\mu_k^+ > \mu_k^-$); (b)–(c) bifurcations near resonance 1 : 4 in map (2.9) for the cases (b) $\nu_k = f_{03}d^{-2}\lambda^k > 0$ (here the fixed point is always elliptic) and (c) $\nu_k < 0$ (for $\beta = 0$ the fixed point is a saddle with eight separatrices); here β is a parameter characterizing a deviation of ψ from $\pi/2$.

In the case $\nu = +1$, the Hénon map is non-orientable and, thus it does not have elliptic fixed points. However, elliptic points of period 2 exist for $M \in (0, 1)$. The map has no fixed points for $M < 0$, it has one fixed point $\bar{O}(0, 0)$ with multipliers $\nu_1 = +1, \nu_2 = -1$ at $M = 0$ and two saddle fixed points $(\bar{O}_1(-\sqrt{M}, -\sqrt{M})$ and $\bar{O}_2(\sqrt{M}, \sqrt{M}))$ at $M > 0$. Besides, an elliptic orbit of period 2 exists for $0 < M < 1$, it consists of two points $(p_1(-\sqrt{M}, \sqrt{M})$ and $p_2(\sqrt{M}, -\sqrt{M}))$; the value $M = +1$ corresponds to the period doubling bifurcation of this orbit. See Figure 2.6 for an illustration. Note that the elliptic orbit of period 2 is generic for all $M \in (0, 1)$ except for $M = \frac{1}{2}$, $M = \frac{1}{\sqrt{2}}$ which correspond to the strong resonances 1 : 4 and 1 : 3, respectively, and $M = \frac{5}{8}$ which corresponds to the zero first Birkhoff coefficient at the cycle $\{p_1, p_2\}$, see [DGGLO13].

It is also known (see, e.g., [DN78, AS82]) that, if $M > 5 + 2\sqrt{5}$ (this is only

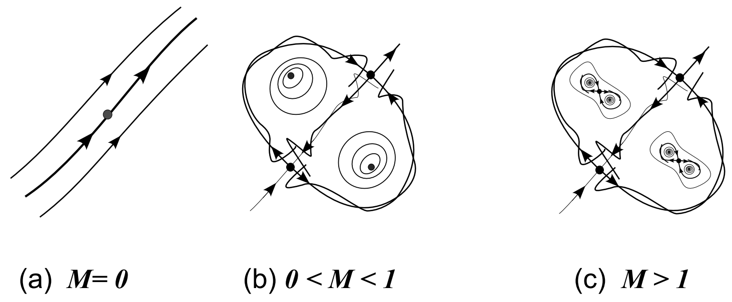


Figure 2.6: The main bifurcation scenario in the non-orientable conservative Hénon map.

a sufficient condition), then the nonwandering set of map (2.33) is Smale horseshoe which is orientable for $\nu = -1$ and non-orientable for $\nu = +1$.

2.4.2 Proof of Theorem 2.1

The proof is deduced from the rescaling lemmas 2.4 and 2.5. Indeed, since bifurcations of fixed points of the Hénon map are known, we can use this information directly for recovering bifurcations of single-round periodic orbits in the family f_μ. We need only to know relations between the parameters of the rescaled map (2.9) and the initial parameters (i.e., in fact, between M and μ).

In the globally non-orientable case, the relations between M and μ are given by formula (2.10) from which we find μ as follows

$$\mu = -\lambda^k y^- \alpha(1 + k\beta_1 \lambda^k x^+ y^-) - \frac{1}{d}(M + s_0^{nor} + \hat{\rho}_k^1)\lambda^{2k}, \tag{2.34}$$

where $\hat{\rho}_k^1 = O(k\lambda^k)$ is some small coefficient and $\alpha = \dfrac{cx^+}{y^-} - 1$ (see formula (2.6)).

As it follows from Lemma 2.4, the conservative non-orientable Hénon map $\bar{x} = y, \ \bar{y} = M + x - y^2$, where M satisfies (2.10), is normal (rescaled) form for the first return maps T_k with all sufficiently large k. This Hénon map has no elliptic fixed points, however, period 2 elliptic points exists for $0 < M < 1$. Thus, we obtain, by (2.10), that the first return map T_k has a fixed point with multipliers $\nu_1 = +1, \nu_2 = -1$ (i.e. when $M = 0$) if

$$\mu = \mu_k^{\pm} = -\lambda^k y^- \alpha(1 + k\beta_1 \lambda^k x^+ y^-) - \frac{1}{d}(s_0^{nor} + \hat{\rho}_k)\lambda^{2k}, \tag{2.35}$$

and a period 2 point with multipliers $\nu_1 = \nu_2 = -1$ (i.e. when $M = 1$) if

$$\mu = \mu_k^{2-} = -\lambda^k y^- \alpha(1 + k\beta_1 \lambda^k x^+ y^-) - \frac{1}{d}(s_0^{nor} + 1 + \hat{\rho}_k)\lambda^{2k}, \tag{2.36}$$

Thus, the first return map T_k has in this case a period two elliptic periodic point when $\mu \in \mathsf{e}_k^2$, where e_k^2 is the interval of values of μ with the border points $\mu = \mu_k^{\pm}$ and $\mu = \mu_k^{2-}$. Evidently, if $\alpha \neq 0$, the intervals e_k^2 with sufficiently large and different k do not intersect.

In the locally non-orientable case, as our agreement, we take such a pair of homoclinic points M^+ and M^- that the global map T_1 is orientable (i.e. $J(T_1) = 1$). Then, evidently, the first return maps $T_k \equiv T_1 T_0^k$ will be orientable for even k and non-orientable for odd k (i.e. $J(T_k) = (-1)^k$).

Consider first the case of even k. By Lemma 2.5, the normal rescaled form for T_k is the Hénon map $\bar{X} = Y$, $\bar{Y} = M - X - Y^2$, where M satisfies (2.22). We find from here that μ is given by the relation

$$\mu = -\lambda^k y^- \alpha - \frac{1}{d}(M + s_0^{or} + \hat{\rho}_k^2)\lambda^{2k}, \tag{2.37}$$

for even k, where $\hat{\rho}_k^1 = O(k\lambda^k)$, α is given by (2.6) and s_0^{or} by (2.7). In this case, since the Hénon map has parabolic fixed point for $M = -1$ and $M = 3$, we obtain that the interval e_k with even k has border points $\mu = \mu_k^+$ and $\mu = \mu_k^-$, where

$$\mu_k^+ \equiv \lambda^k y^- \alpha - \frac{1}{d}(s_0^{or} - 1 + \hat{\rho}_k^2)\lambda^{2k}, \tag{2.38}$$

$$\mu_k^- \equiv \lambda^k y^- \alpha - \frac{1}{d}(s_0^{or} + 3 + \hat{\rho}_k^2)\lambda^{2k}. \tag{2.39}$$

Here, the map T_k has a fixed point which is parabolic with multipliers $\nu_1 = \nu_2 = +1$ for $\mu = \mu_k^+$ and with multipliers $\nu_1 = \nu_2 = -1$ for $\mu = \mu_k^-$ and elliptic for $\mu \in \mathsf{e}_k$. Note that here k runs all sufficiently large *even* integers and, evidently, if $\alpha \neq 0$, the intervals e_k with different sufficiently large even k do not intersect.

Consider now the case of odd k. Then the map T_k is non-orientable and its normal rescaled form is the non-orientable conservative Hénon map $\bar{x} = y$, $\bar{y} = M + x - y^2$, where M satisfies (2.24). Then we find that the interval e_k^2 has border points $\mu = \mu_k^-$ and $\mu = \mu_k^{2-}$, where

$$\mu_k^{\pm} \equiv -\lambda^k \hat{\alpha} - \frac{1}{d}(s_0^{nor} + \hat{\rho}_k^3)\lambda^{2k}, \tag{2.40}$$

$$\mu_k^{2-} \equiv -\lambda^k y^- \hat{\alpha} - \frac{1}{d}(s_0^{nor} + 1 + \hat{\rho}_k^3)\lambda^{2k}, \tag{2.41}$$

where $\hat{\alpha} = cx^+/y^- + 1$ and s_0^{nor} is given by (2.8). If $\hat{\alpha} \neq 0$, the intervals e_k^2 with sufficiently large and different (odd) numbers k do not intersect. It completes the proof of Theorem 2.1.

2.4.3 Proof of Theorems 2.2 and 2.3

Proof of Theorem 2.2. 1) In the *globally non-orientable case*, by (2.35) and (2.36) the equations of bifurcation curves L_k^{2+} and L_k^{2-}, which are boundaries of the domain E_k^2, can be written as follows

$$L_k^{2+}: \quad \mu = -\lambda^k y^- \left(\frac{cx^+}{y^-} - 1\right)(1 + k\beta_1\lambda^k x^+ y^-) - \frac{s_0^{nor} + \cdots}{d}\lambda^{2k}, \tag{2.42}$$

$$L_k^{2-}: \quad \mu - \lambda^k y^- \left(\frac{cx^+}{y^-} - 1\right)(1 + k\beta_1\lambda^k x^+ y^-) - \frac{1 + s_0^{nor} + \cdots}{d}\lambda^{2k}. \tag{2.43}$$

Since $\lambda^{2k} \ll \lambda^k$, it means that the domains E_k^2 with sufficiently large k are not mutually crossed and do not intersect the axis $\mu = 0$, if $cx^+ \neq y^-$. Thus, the domains do not intersect always in the cases with $c < 0$ ((as in Figure 2.7 (a)–(c)). However, at the global resonance $\alpha = (cx^+/y^- - 1) = 0$, as it follows from (2.42) and (2.43), all domains E_k^2 with sufficiently large k are mutually crossed and all of them intersect the axis $\mu = 0$ (as in Figure 2.7 (d) and (e)).

2) In the *locally non-orientable case* we have, by (2.38) and (2.39), that the domains E_{2m} have boundaries

$$L_{2m}^{+}: \quad \mu = -\lambda^{2m} y^- \left(\frac{cx^+}{y^-} - 1\right) + \frac{1 - s_0^{or} + \cdots}{d}\lambda^{4m}, \tag{2.44}$$

$$L_{2m}^{-}: \quad \mu = -\lambda^{2m} y^- \left(\frac{cx^+}{y^-} - 1\right) - \frac{3 + s_0^{or} + \cdots}{d}\lambda^{4m}, \tag{2.45}$$

corresponding to the existence of a parabolic single-round periodic orbit with double multiplier $+1$ at $(\mu, \alpha) \in L_{2m}^+$ or with double multiplier -1 at $(\mu, \alpha) \in L_{2m}^-$. At the same time, by (2.40) and (2.41), the domains E_{2m+1}^2 have boundaries

$$L_{2m+1}^{2+}: \quad \mu = -\lambda^{2m+1} y^- \left(\frac{cx^+}{y^-} + 1\right) - \frac{s_0^{nor} + \cdots}{d}\lambda^{4m+2}, \tag{2.46}$$

$$L_{2m+1}^{2-}: \quad \mu = -\lambda^{2m+1} y^- \left(\frac{cx^+}{y^-} + 1\right) - \frac{1 + s_0^{nor} + \cdots}{d}\lambda^{4m+2}, \tag{2.47}$$

corresponding to the existence of a single-round periodic orbit with multipliers $+1$ and -1 at $(\mu, \hat{\alpha}) \in L_{2m+1}^{2+}$ or a double-round periodic orbit with double multiplier -1 at $(\mu, \alpha) \in L_{2m+1}^{2-}$.

Thus, for the diffeomorphisms under consideration with $c > 0$, the global resonance occurs at $\alpha = 0$. It corresponds to such situation when all the domains E_{2m} (with sufficiently large m) mutually intersect and intersect the axis $\mu = 0$ near the origin $\alpha = 0, \mu = 0$, whereas, the domains E_{2m+1}^2 do not intersect for different and sufficiently large m, accumulate to the axis $\mu = 0$ as $m \to \infty$ but do not intersect it, see Figure 2.8 (a). Otherwise, for the case $c < 0$, the global resonance occurs at $\hat{\alpha} = cx^+/y^- + 1 =$

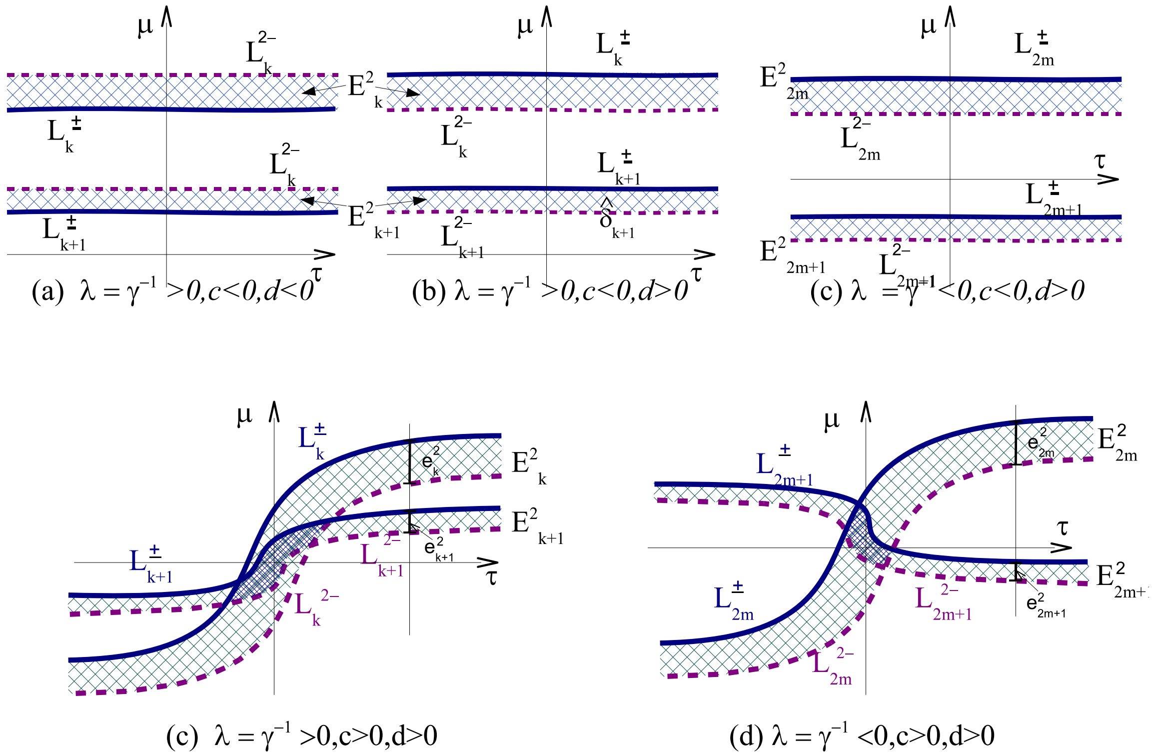


Figure 2.7: Elements of bifurcation diagrams for families $f_{\mu,\alpha}$ in the globally non-orientable case.

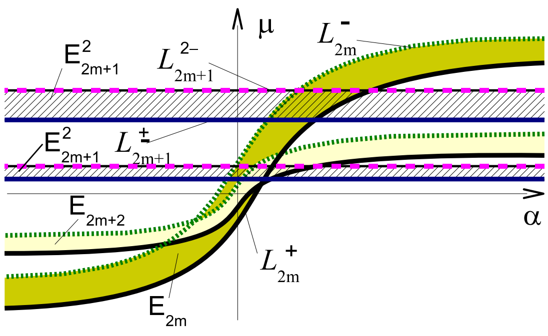
0 and correspond to the situation when the domains E^2_{2m+1} mutually intersect and intersect the axis $\mu = 0$ (near the origin $\hat{\alpha} = 0, \mu = 0$), whereas, the domains E_{2m} do not intersect for different and sufficiently large m, see Figure 2.8 (b). It completes the proof. □

Proof of Theorem 2.3. Assume, for more definiteness, that $d > 0$ for all cases under consideration. The case $d < 0$ is treated in the same way.

1) Let f_0 be a globally non-orientable map with $\alpha = 0$. Then, for the one parameter family f_μ with fixed $\alpha = 0$, the intervals e^2_k have, by (2.35)–(2.36), a form

$$\mathsf{e}^2_k = (-1 - s_0^{nor}, -s_0^{nor})\frac{\lambda^{2k}}{d}.$$

Evidently, if $-1 < s_0^{nor} < 0$, these intervals will be nested and containing $\mu = 0$. It implies that the diffeomorphism f_0 has infinitely many double-round elliptic periodic orbits.



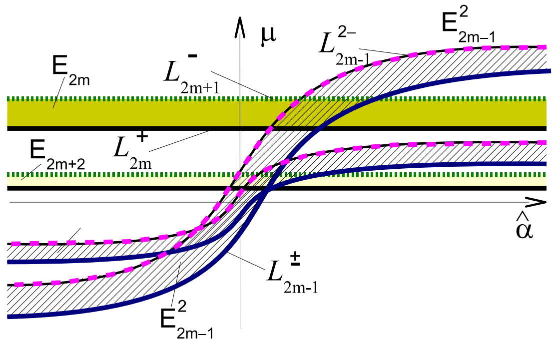


Figure 2.8: Elements of bifurcation diagrams in the locally non-orientable case: (a) for a family $f_{\mu,\alpha}$; (b) for a family $f_{\mu,\hat\alpha}$.

2) In the *locally non-orientable case*, let f_0 have $cx^+ = y^-$ and $bc = -1$, i.e. $\alpha = 0$ and the global map T_1 is orientable. In the family f_μ with fixed $\alpha = 0$, by (2.44)–(2.45), the intervals e_{2m} have a form

$$\mathsf{e}_{2m} = (-3 - s_{or}\ ,\ 1 - s_{or})\,\frac{\lambda^{4m}}{d}.$$

These intervals of values of μ will be nested and all contain $\mu = 0$ for all sufficiently large m if $-3 < s_{or} < 1$. It implies that the diffeomorphism f_0 has infinitely many single-round elliptic periodic orbits.

In the *locally non-orientable case*, let now f_0 be such a map that $cx^+ = -y^-$ and $bc = -1$, i.e. $\hat\alpha = 0$ and the global map T_1 is orientable again. Consider the one parameter family f_μ with fixed $\hat\alpha = 0$. Then, by (2.46)–(2.47), the intervals $\mathsf{e}^2_{2m+1}+$ have a form

$$\mathsf{e}^2_{2m+1} = (-1 - s_{or}\ ,\ -s_{or})\,\frac{\lambda^{4m+2}}{d}.$$

If $-1 < s_{nor} < 0$, these intervals are nested and all contain $\mu = 0$. It implies that the diffeomorphism f_0 has infinitely many double-round elliptic periodic orbits.

3) For the globally non-orientable case with $\alpha = 0$, as follows from Lemma 2.4, all the first return maps T_k (with sufficiently large k) are reduced to the same rescaled normal form – the non-orientable Hénon map $\bar x = y,\ \bar y = -s_0^{nor} + x - y^2$. It is well known that, for $-1 < s_{nor} < 0$, the period 2 elliptic point of this map is generic if $s_{nor} \neq -\frac{1}{2}; -\frac{1}{\sqrt{2}}; -\frac{5}{8}$. The exceptional cases relates, respectively, to resonances 1 : 4, 1 : 3 and such elliptic point whose first Birkhoff coefficient is zero.

For the locally non-orientable case with $\alpha = 0$, as follows from Lemma 2.4, all maps T_{2m} are reduced to the same rescaled normal form – the orientable Hénon map $\bar x = y,\ \bar y = -s_{or} - x - y^2$. It is well known that, for $-3 < s_{nor} < 1$, this map has a fixed elliptic point which is generic (KAM-stable) if $s_{or} \neq \{0; -\frac{5}{\sqrt{4}}\}$. The exceptional cases relates, respectively, to the strong resonances $\psi = \pi/2$ and $\psi = 2\pi/3$.

For the locally non-orientable case with $\hat{\alpha} = 0$, all the first return maps T_{2m+1} are reduced, by Lemma 2.5, to the the non-orientable Hénon map $\bar{x} = y,\ \bar{y} = -s_0^{nor} + x - y^2$ and, thus, if $s_{nor} \neq -\frac{1}{2}; -\frac{1}{\sqrt{2}}; -\frac{5}{8}$, the corresponding double round elliptic periodic orbits are generic. It completes the proof. $\square$

Chapter 3

Bifurcations of cubic homoclinic tangencies in area-preserving maps

In this chapter we study bifurcations of cubic homoclinic tangencies in two-dimensional symplectic maps. We distinguish two types of cubic homoclinic tangencies, and each type gives different first return maps derived to diverse cubic conservative Hénon maps with quite different bifurcation diagrams. In this way, we establish the structure of bifurcations of single-round periodic orbits in two parameter general unfoldings. Note that the results of this chapter generalize to the conservative case the results of [Gon85] obtained for the dissipative case.

3.1 Preambles

Bifurcations of cubic homoclinic tangencies in general setting were studied in [Gon85], see also [GK88, Tat91, GST96a]. More precisely, in [Gon85] there were studied bifurcations of a $(m+2)$-dimensional C^r-smooth flow X_0, where $m \geq 1$ and $r \geq 6$, satisfying the following conditions:

- X_0 has a saddle periodic orbit L_0 with multipliers $\lambda_1, \ldots, \lambda_m$ and γ such that $|\lambda_m| \leq \cdots \leq |\lambda_1| < 1 < |\gamma|$ and either

  λ_1 is real and $|\lambda_1| > |\lambda_2|$; or

  $\lambda_1 = \bar{\lambda_2} = \rho e^{i\varphi}$, $\varphi \neq 0, \pi$, $|\lambda_1| > |\lambda_3|$;

- the saddle value $\sigma = |\lambda_1||\gamma| < 1$;

- the stable W_0^s and unstable W_0^u invariant manifolds of L_0 have a tangency of the second order (i.e. the cubic homoclinic tangency) along a homoclinic curve Γ_0.

In this chapter we consider the case $\sigma = 1$. Moreover, we restrict ourself to two-dimensional symplectic diffeomorphisms whose bifurcations of cubic homoclinic tangencies will be studied.

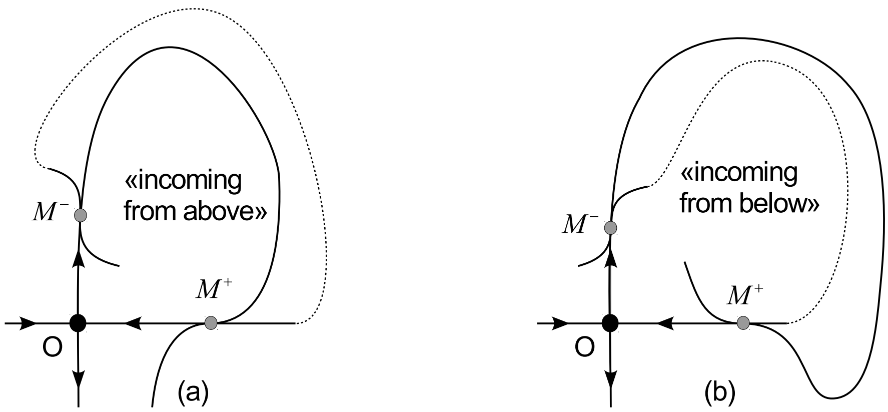


Figure 3.1: Two types of cubic homoclinic tangencies: (a) "incoming from above" and (b) "incoming from below".

Let f_0 be a C^r-smooth, $r \geq 5$, two-dimensional symplectic diffeomorphism which satisfies the following conditions:

(A) f_0 has a saddle fixed point O with multipliers λ and λ^{-1}, where $|\lambda| < 1$;

(B) the invariant manifolds $W^u(O)$ and $W^s(O)$ have a cubic homoclinic tangency at the points of some homoclinic orbit Γ_0.

We distinguish two types of cubic tangencies of $W^u(O)$ and $W^s(O)$ at a homoclinic point: the tangency of the first type or "incoming from above" and the tangency of the second type or "incoming from below". Both these types are shown in Figure 3.1. When the multiplier λ is positive the type of tangency will be the same for all points of the homoclinic orbit. However, if λ is negative the tangencies "incoming from above" and "incoming from below" will be alternated from point to point. Note also that bifurcations in these two cases are quite different, compare bifurcation diagrams of Figures 3.4 and 3.5 that are be typical for first return maps, respectively, for the case of "incoming from above" and "incoming from below" homoclinic tangencies.

Let $\mathcal{H}_2^r$ be a codimension two bifurcation surface composed of symplectic C^r-maps close to f_0 and such that every map of $\mathcal{H}_2^r$ has a nontransversal – cubic tangency – homoclinic orbit close to Γ_0. Let f_ε be a family of symplectic C^r-maps that contains the map f_0 at $\varepsilon = 0$. We suppose that the family depends smoothly on the parameters $\varepsilon = (\varepsilon_1, ..., \varepsilon_m)$ and satisfies the following condition:

(C) the family f_ε is transverse to $\mathcal{H}_2^r$ at $\varepsilon = 0$.

As in Chapter 1, let U be a small neighborhood of the contour $\{O, \Gamma_0\}$. It consists of a small disk U_0 containing O and a number of small disks surrounding those points of Γ_0 that do not lie in U_0, see Figure 3.2.

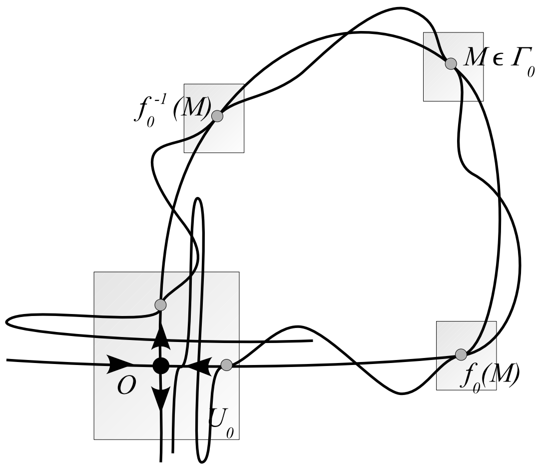


Figure 3.2: An example of planar map having a cubic homoclinic tangency along a homoclinic orbit Γ_0.

In this chapter we study bifurcations of *single-round periodic orbits* within the families f_ε. Note that every point of such an orbit can be considered as a fixed point of the corresponding first return map $T_k = T_1 T_0^k$, where $T_0 \equiv T_0(\varepsilon)$ is local map and $T_1 \equiv T_1(\varepsilon)$ is the global map that can be introduced in a similar way as in Chapter 1. For analytical constructions of T_0 and T_1, we use the local coordinates $(x, y) \in U_0$ in which T_0 takes the first order normal form given by Lemma 1.1 (for $n = 1$). We choose a pair of points $M^+ = (x^+, 0) \in W^s_{loc}(O)$ and $M^- = (0, y^-) \in W^u_{loc}(O)$ of the homoclinic orbit Γ_0 (we assume that $x^+ > 0$ and $y^- > 0$) and their sufficiently small neighbourhoods Π^+ and Π^-, respectively. We suppose that $f_0^q(M^-) = M^+$ for some integer q. Then the global map $T_1(\varepsilon) \equiv f_\varepsilon^q : \Pi^- \to \Pi^+$ is written as follows

$$\overline{x} - x^+ = F(x, y - y^-, \varepsilon), \;\; \overline{y} = G(x, y - y^-, \varepsilon), \tag{3.1}$$

where $F(0,0,0) = 0, G(0,0,0) = 0$ and the following relations

$$\frac{\partial G(0)}{\partial y} = 0, \;\; \frac{\partial^2 G(0)}{\partial y^2} = 0, \;\; \frac{\partial^3 G(0)}{\partial y^3} = 6d \neq 0 \tag{3.2}$$

hold due to the condition B which means that at $\varepsilon = 0$ the curve $T_1(W^u_{loc}) : \{\overline{x} - x^+ = F(0, y - y^-, 0), \overline{y} = G(0, y - y^-, 0)\}$ has a cubic tangency with $W^s_{loc} : \{\overline{y} = 0\}$ at M^+. When parameters vary this tangency can split and, by condition C, the family (3.1) unfolds generally the initial cubic tangency. In this case the global map T_1 can be written in a certain normal form that the following lemma shows.

Lemma 3.1. *If the condition* C *holds, the map* $T_1(\varepsilon)$ *can be written in the form*

$$
\begin{array}{l}
\overline{x} - x^+ = ax + b(y - y^-) + O\left(x^2 + (y - y^-)^2\right), \\
\overline{y} = \mu_1 + \mu_2(y - y^-) + cx + d(y - y^-)^3 + O\left(x^2 + |x||y - y^-| + (y - y^-)^4\right),
\end{array}
\tag{3.3}
$$

where the coefficients a, b, c, d *as well as* x^+, y^- *depend smoothly (the smoothness is the same as for the initial map (3.1)) on the new parameters* ε *such that* $\mu_1 = \varepsilon_1$ *and* $\mu_2 = \varepsilon_2$.

Proof. By virtue of (3.2), the equation $\partial^2 G(x, y - y^-, \varepsilon)/\partial y^2 = 0$ for small x and ε can be resolved with respect to y. This solution has the form $y - y^-(\varepsilon) = \varphi(x, \varepsilon)$ where $\varphi(0, \varepsilon) \equiv 0$. Then, we can write the following Taylor expansion with the remainder term for the function G near the curve $y - y^-(\varepsilon) = \varphi(x, \varepsilon)$:

$$
G \equiv G(x, 0, \varepsilon) + \frac{\partial G(x, 0, \varepsilon)}{\partial y}(y - y^- - \varphi) + \frac{\partial^3 G(x, 0, \varepsilon)}{\partial y^3}(y - y^- - \varphi)^3 + O\left((y - y^- - \varphi)^4\right).
$$

Since $\varphi \equiv \varphi(x, \varepsilon) = O(x)$, we can write

$$
\begin{array}{rl}
G \equiv & E_1(\varepsilon) + cx + E_2(\varepsilon)(y - y^-) + d(y - y^-)^3 + \\
& + O\left(x^2 + |x||y - y^-| + (y - y^-)^4\right),
\end{array}
$$

where $E_1(\varepsilon) \equiv G(0, 0, \varepsilon)$, $E_2(\varepsilon) \equiv G_y(0, 0, \varepsilon)$ and, hence, $E_i(0) = 0$, $i = 1, 2$. Then the map $T_1(\varepsilon)$ can be written in the following form

$$
\begin{array}{l}
\overline{x} - x^+ = ax + b(y - y^-) + O\left(x^2 + (y - y^-)^2\right), \\
\overline{y} = E_1(\varepsilon) + cx + E_2(\varepsilon)(y - y^-) + d(y - y^-)^3 + \\
\qquad + O\left(x^2 + |x||y - y^-| + (y - y^-)^4\right).
\end{array}
\tag{3.4}
$$

Putting $x = 0$ in (3.4) we find the equation of the curve $T_1(W^u_{loc}) \subset \Pi^+$ in the following parametric form

$$
\begin{array}{l}
\overline{x} - x^+ = b(y - y^-) + O\left((y - y^-)^2\right), \\
\overline{y} = E_1(\varepsilon) + E_2(\varepsilon)(y - y^-) + d(y - y^-)^3 + O\left((y - y^-)^4\right),
\end{array}
\tag{3.5}
$$

where $(y - y^-)$ is the parameter now. Since $b \neq 0$, we find from the first equation of (3.5) that $y - y^- = (\bar{x} - x^+)/b + O\left((\bar{x} - x^+)^2\right)$ and, thus, we can write the explicit equation of the curve $T_1(W^u_{loc}) \subset \Pi^+$ as follows

$$
y = E_1(\varepsilon) + \frac{E_2(\varepsilon)}{b}(x - x^+)\left(1 + O\left(x - x^+\right)\right) + \frac{d}{b^3}(x - x^+)^3 + O\left((x - x^+)^4\right), \tag{3.6}
$$

Condition C means that $E_i(0) = 0, E_i'(0) \neq 0$ and the coefficients $E_1(\varepsilon)$ and $E_2(\varepsilon)$ can take any values from the ball $\|\varepsilon\| \leq \delta_0$, where $\delta_0 > 0$ is a small constant. Thus, the system $\mu_1 = E_1(\varepsilon), \mu_2 = E_2(\varepsilon)$ has always a solution and we can consider μ_1 and μ_2 as new parameters. □

3.2 On bifurcations of periodic orbits

Due to Lemma 3.1, we can consider directly two parameter families f_{μ_1,μ_2} of symplectic maps. By our construction, μ_1 and μ_2 are the parameters of splitting of the invariant manifolds $W^s(O_\varepsilon)$ and $W^u(O_\varepsilon)$ with respect to the homoclinic point M^+. Indeed, by (3.6), the curve $l_u = T_1(W^u_{loc} \cap \Pi^-)$ has the equation

$$l_u \ : \ y = \mu_1 + \frac{\mu_2}{b}(x - x^+)\left(1 + O(x - x^+)\right) + \frac{d}{b^3}(x - x^+)^3 + O\left((x - x^+)^4\right). \quad (3.7)$$

Thus, the family f_{μ_1,μ_2} is a general two parameter unfolding of the initial cubic tangency which takes place at $\mu_1 = \mu_2 = 0$. For any such unfolding, the curves W^s_{loc} and $T_1(W^u_{loc})$ must have a quadratic tangency for certain values of μ_1 and μ_2. It is true for our family for which the following result holds.

Theorem 3.1. *On the (μ_1, μ_2)-parameter plane there exists a bifurcation curve B_0:*

$$\mu_1 = \pm 2d \left[-\frac{\mu_2}{3d}(1 + O\left(\sqrt{|\mu_2|}\right))\right]^{3/2}$$

(see Figure 3.3) such that at $\mu \in B_0$ the map f_μ has a close to Γ_0 homoclinic orbit of a quadratic tangency of the manifolds $W^u(O_\mu)$ and $W^s(O_\mu)$.

Proof. Consider the curve l_u given by the equation (3.7). If this curve has a tangency with the line $y = 0$, the following system has solutions

$$\begin{aligned} &\mu_1 + \frac{\mu_2}{b}\xi(1 + O(\xi)) + \frac{d}{b^3}\xi^3 + O\left(\xi^4\right) = 0, \\ &\mu_2(1 + O(\xi)) + \frac{3d}{b^2}\xi^2 + O\left(\xi^3\right) = 0, \end{aligned}$$

where $\xi = x - x^+$. We solve the second equation for ξ as follows

$$\xi = \pm b\sqrt{\frac{-\mu_2(1 + O(\sqrt{|\mu_2|}))}{3d}}$$

Putting this value into the first equation we find the equation of the curve B_0. □

Now we start studying bifurcations of single-round periodic orbits in the family f_{μ_1,μ_2}. As well-known it is the same as to study bifurcations of fixed points in the first return maps $T_k = T_1 T_0^k$ for every sufficiently large integer k $(k = k_0, k_0 + 1, ...)$. We again apply the rescaling method to find normal rescale forms for these maps. The result is the following

Lemma 3.2. [Rescaling Lemma for cubic homoclinic tangencies]
For every sufficiently large k the first return map T_k can be brought, by a linear transformation of coordinates and parameters, to the following form

$$\begin{aligned} \bar{x} &= y + O(\lambda^k), \\ \bar{y} &= M_1 - x + M_2 y + \nu y^3 + O(\lambda^k), \end{aligned} \quad (3.8)$$

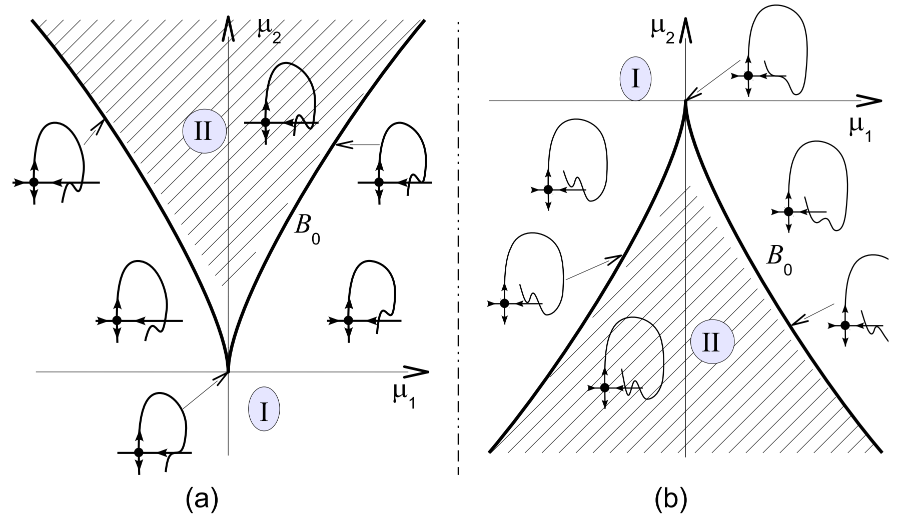


Figure 3.3: The bifurcation curve B_0 in (a) "incoming from above" ($d > 0$) and (b) "incoming from below" ($d < 0$) cases. For $\mu \in I$ the map f_μ has only one (single-round and transverse) homoclinic orbit close to Γ_0, whereas, three such orbits exist when $\mu \in II$.

where

$$\nu \;=\; \mathsf{sign}\ (d\lambda^k), \tag{3.9}$$

$$\begin{array}{l} M_1 = \sqrt{|d|}\lambda^{-3k/2}\left(\mu_1 - \lambda^k(y^- - cx^+) + O\left(k\lambda^{2k}\right)\right), \\ M_2 = \lambda^{-k}\left(\mu_2 + f_{11}\lambda^k x^+ + O\left(k\lambda^{2k}\right)\right) \end{array} \tag{3.10}$$

and $f_{11} = G_{xy}(0)$.

Proof. By Lemma 1.2 and (3.3), the first return map $T_k = T_1 T_0^k$ can be written as follows

$$\begin{array}{rcl} \bar{x} - x^+ &=& a\lambda_1^k x + b(y - y^-) + O\left(k\lambda^{2k}|x| + |\lambda|^k|x||y - y^-| + (y - y^-)^2\right), \\ \lambda^k\bar{y}(1 + k\lambda^k O(\bar{x}, \bar{y})) &=& \mu_1 + \mu_2(y_1 - y^-) + d(y_1 - y^-)^3 + c\lambda^k x + \\ && + O\left(k\lambda^{2k}|x| + |\lambda|^k|x||y - y^-|\right) + O\left((y - y^-)^4\right) \end{array} \tag{3.11}$$

We shift the coordinates, $\xi = x - x^+ + \alpha_k^1, \eta = y - y^- + \alpha_k^2$, where $\alpha_k^1 = -a\lambda^k x^+ + O(k\lambda^{2k})$, $\alpha_k^2 = -\dfrac{f_{12}}{3d}\lambda^k x^+ + O(k\lambda^{2k})$ and $f_{12} = \dfrac{1}{2}G_{xy^2}(0)$. Then the system (3.11) is rewritten as follows

$$\begin{array}{rcl} \bar{\xi} &=& a\lambda_1^k\xi + b\eta + O\left(k\lambda^{2k}|\xi| + |\lambda|^k|\xi||\eta| + \eta^2\right), \\ \bar{\eta} &=& \lambda^{-k}\left(\mu_1 - \lambda^k(y^- - cx^+) + O(k\lambda^{2k})\right) + \lambda^{-k}\left(\mu_2 + f_{11}\lambda^k x^+ + O\left(k\lambda^{2k}\right)\right)\eta + \\ && + d\lambda^{-k}\eta^3 + cx + O\left(k\lambda^k|\xi| + |\xi||\eta|\right) + \lambda^{-k}o\left(\eta^3\right). \end{array} \tag{3.12}$$

Here the first equation of (3.12) does not contain constant terms and the quadratic term η^2 vanishes in the second equation.

Finally, by means of the coordinate rescaling

$$\xi = b\frac{\lambda^{k/2}}{\sqrt{|d|}}\ x, \quad \eta = \frac{\lambda^{k/2}}{\sqrt{|d|}}\ y$$

we bring the map (3.12) to the form (3.8). □

3.2.1 The description of bifurcations of fixed points in the cubic Hénon maps

By Rescaling Lemma 3.2, the following conservative cubic Hénon maps

$$\bar{x} = y, \quad \bar{y} = M_1 - x + M_2 y + y^3 \tag{3.13}$$

and

$$\bar{x} = y, \quad \bar{y} = M_1 - x + M_2 y - y^3 \tag{3.14}$$

have to be considered as certain normal forms for the first return maps in the cases $d\lambda^k > 0$ and $d\lambda^k < 0$, respectively. Thus, if $\lambda > 0$, the map (3.13) relates to the cubic homoclinic tangency with $d > 0$, i.e. the tangency "incoming from below" (see Figure 3.1 (b)), while the map (3.13) relates to the cubic homoclinic tangency with $d < 0$, i.e. the tangency "incoming from above" (see Figure 3.1 (a)). In Figures 3.4 and 3.5, the main elements of the bifurcation diagrams for these cubic maps are presented.

The bifurcation curves in these figures are found analytically (see e.g. [GK88]) and their equations are as follows ($\nu = +1$ and $\nu = -1$ relate to the map (3.13) and (3.14), respectively):

$$\begin{aligned}
&L^+ : \quad M_1 = \pm 2\left(\frac{2 - M_2}{3\nu}\right)^{3/2} \\
&L^- : \quad M_1 = \pm\frac{2}{3}\left(\frac{-2 - M_2}{3\nu}\right)^{1/2}(4 - M_2) \\
&C^+_{1,2} : \quad M_1 = \pm 2\left(\frac{-M_2 - 4}{3}\right)^{3/2} \quad \text{in the case } \nu = +1 \\
&C^+_{1,2} : \quad M_1 = \pm 2\left(\frac{M_2 + 4}{3}\right)^{3/2}, \quad M_2 \geq -\frac{4}{3} \quad \text{in the case } \nu = -1 \\
&C^-_{1,2} : \quad M_1^2 = \frac{1}{216\nu}\left[12 + M_2 \pm S\right]^2\left[-5M_2 - 12 \pm S\right], \quad \text{where} \\
&\qquad\qquad S = \sqrt{9M_2^2 + 24M_2}.
\end{aligned} \tag{3.15}$$

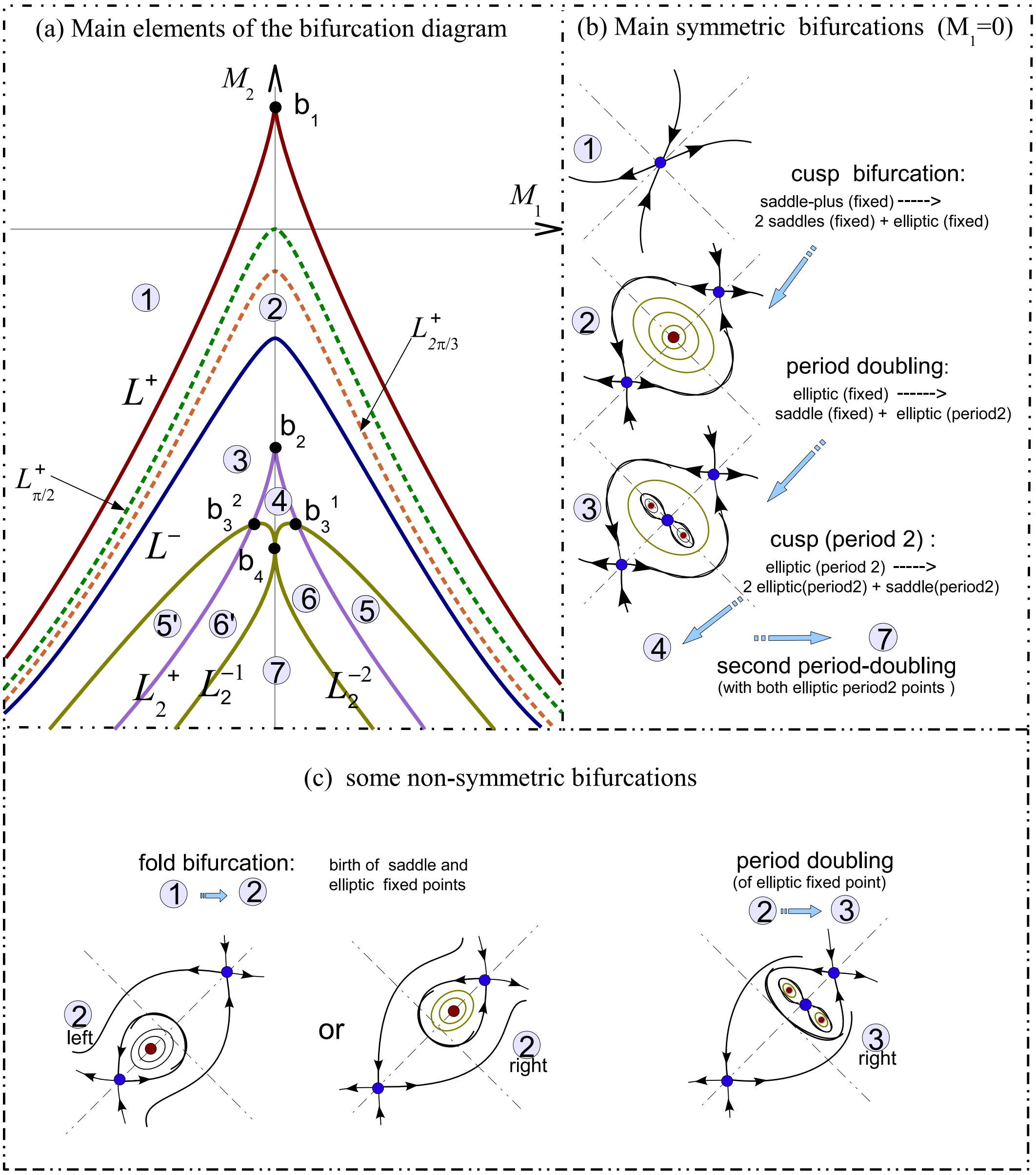


Figure 3.4: (a) The main elements of the bifurcation diagram for the map (3.13). The codimension 2 bifurcation points b_i are as follows: b_1 — a nonhyperbolic saddle fixed point with double multiplier +1 exists; b_2 — two period 2 cycles with double multiplier +1 exist; $b_3^{1,2}$ – two parabolic period 2 cycles with double multipliers −1 and +1, respectively, coexist; b_4 – two period 2 cycles with double multiplier −1 coexist. Examples of (b) symmetric ($M_1 = 0$) and (c) asymmetric ($M_1 \neq 0$) bifurcations are shown.

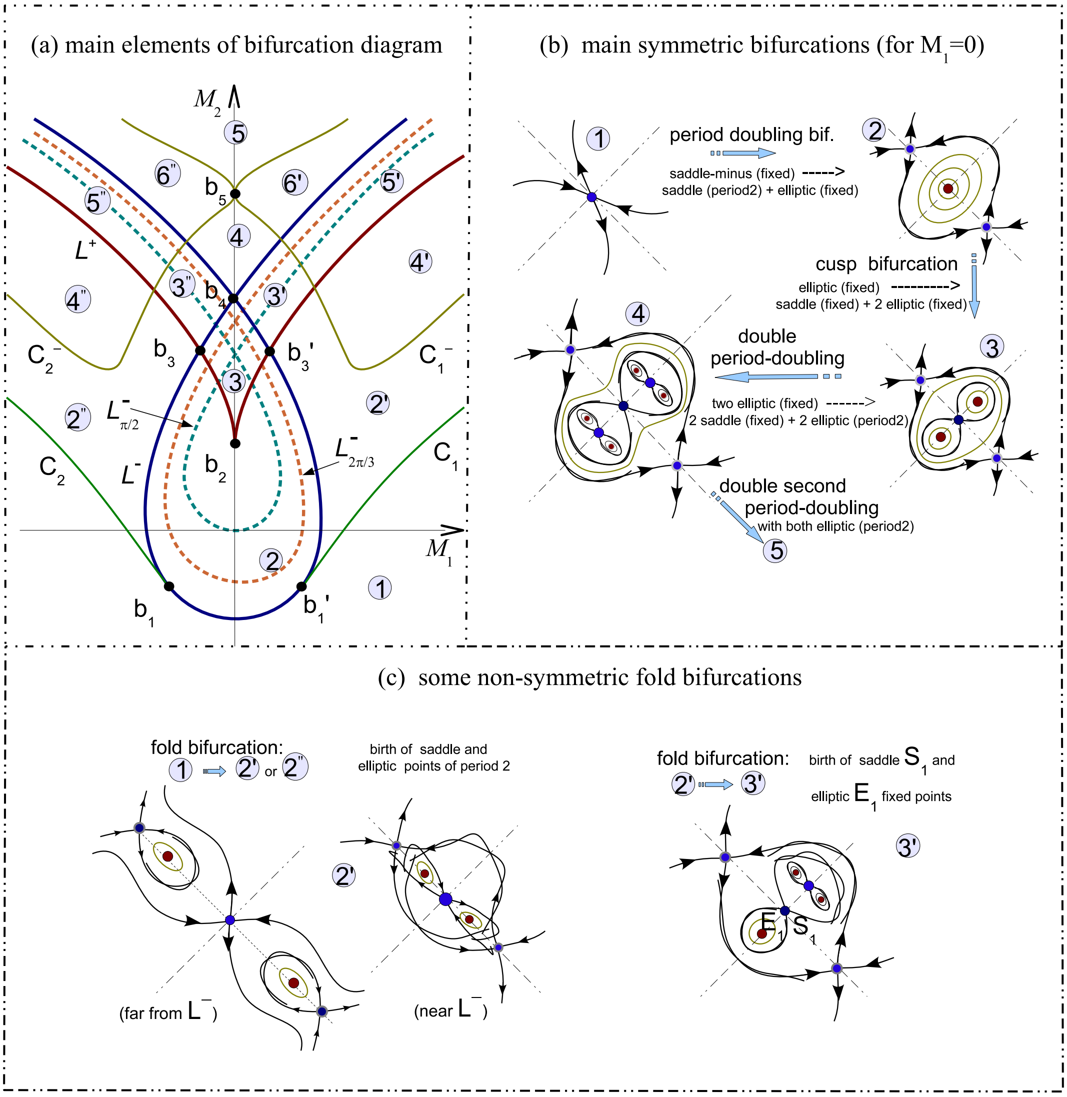


Figure 3.5: (a) The main elements of the bifurcation diagram for the the map (3.13). The codimension 2 bifurcation points b_i are as follows: b_1 and b_1' correspond to the existence of a fixed point with double multiplier -1, zero first Lyapunov value and nonzero second one; b_2 – a triple (stable) fixed point exists ; b_3 and b_3' – two fixed points with multipliers $(-1,-1)$ and $(+1,+1)$ coexist; b_4 – two fixed points with multipliers $(-1,-1)$ coexist; b_5 – two period 2 points with multipliers $(-1,-1)$ coexist.

Here the following bifurcation curves are indicated: L^+ is the line of conservative fold bifurcation (the birth of a parabolic fixed point with double multiplier $+1$); L^- is the line of conservative period doubling (connected with the appearance of a fixed point with double multiplier -1); $C_{1,2}^+$ is the line of conservative fold bifurcation for period 2 points; $C_{1,2}^-$ is the line of conservative period doubling bifurcation for period 2 points (second period doubling).

For cubic Hénon maps of form $\bar{x} = y, \;\; \bar{y} = M_1 - bx + M_2 y \pm y^3$, main bifurcations were studied in [GK88]. However, the main attention in [GK88] was paid to the dissipative case $|b| < 1$. Nevertheless, one can show that the bifurcation scenarios in conservative case ($b = 1$), i.e. for maps (3.13) and (3.14), are quite similar, in many aspects, to the case $0 < b < 1$. But we need to remember that the maps under consideration are area-preserving and, hence, a big specific presents here.[1] Then the main bifurcations (bifurcations related to fixed points) are as follows.

Bifurcation scenario in the map (3.13), see Figure 3.4. For $(M_1, M_2) \in \mathbf{1}$ the map (3.13) has only one fixed point p_1 which is a saddle-plus (with multipliers λ and λ^{-1}, where $0 < \lambda < 1$). The transition $\mathbf{1} \Rightarrow \mathbf{2}$ corresponds to the birth of a pair (saddle and elliptic) of fixed points. When $(M_1, M_2) = b_1$, the fixed point p_1 is a non-hyperbolic saddle with double multiplier $+1$, and this point falls in $\mathbf{2}$ onto 3 fixed points (two saddle and one elliptic) under a conservative cusp-bifurcation. The transition $\mathbf{2} \Rightarrow \mathbf{3}$ corresponds to a nondegenerate period-doubling bifurcation of the elliptic fixed point. Thus, for region $\mathbf{3}$, the map (3.13) has 3 saddle fixed points (two saddle-plus and one saddle-minus) and one period two elliptic orbit. Further primary bifurcations, when crossing the curves L_2^+ and L_2^-, relate to points of period 2 and more and, therefore, we do not observe them (see e.g. [GK88]).

Bifurcation scenario in the map (3.14), see Figure 3.5. For $(M_1, M_2) \in \mathbf{1}$ the map (3.14) has only one fixed point $\hat{p}_1$ which is a saddle-minus (with multipliers λ and λ^{-1}, where $-1 < \lambda < 0$). The transition $\mathbf{1} \Rightarrow \mathbf{2}$ through the segment (b_1, b_1') of the curve L^- corresponds to the period-doubling bifurcation of the saddle point $\hat{p}_1$: the point becomes elliptic fixed one and a saddle period two orbit is born in its neighbourhood. A transition $\mathbf{1} \Rightarrow \mathbf{2'}$ (as well as $\mathbf{1} \Rightarrow \mathbf{2}$") implies the birth (under conservative fold bifurcation) a pair of saddle and elliptic period two points. A transition $\mathbf{2'} \Rightarrow \mathbf{2}$ corresponds to a period-doubling bifurcation under which the period 2 elliptic orbit merged with a saddle fixed point and becomes a fixed elliptic point. Thus, in the region $\mathbf{2}$ the map (3.14) has an elliptic fixed point and a saddle period two orbit. We also illustrate in Figure 3.6 bifurcations happened when a passage of (M_1, M_2)-values around the point b_1. Transitions cross the curve L^+ (such as $\mathbf{2} \Rightarrow \mathbf{3}$, $\mathbf{2'} \Rightarrow \mathbf{3'}$ etc) correspond to the appearance of two new fixed points, saddle and elliptic ones, under a conservative fold bifurcation. The elliptic fixed points undergo period-doubling bifurcation at transitions $\mathbf{3} \Rightarrow \mathbf{3'}$, $\mathbf{3} \Rightarrow \mathbf{3}$", $\mathbf{3'} \Rightarrow \mathbf{4}$ etc. We note that at $(M_1, M_2) = b_2$ a triple (stable) fixed point exists which falls in $\mathbf{3}$ onto 3 fixed points (two elliptic and

[1] for example, the presence of homoclinic and heteroclinic structures is quite usual phenomenon in conservative dynamics even in the case of simple bifurcations...

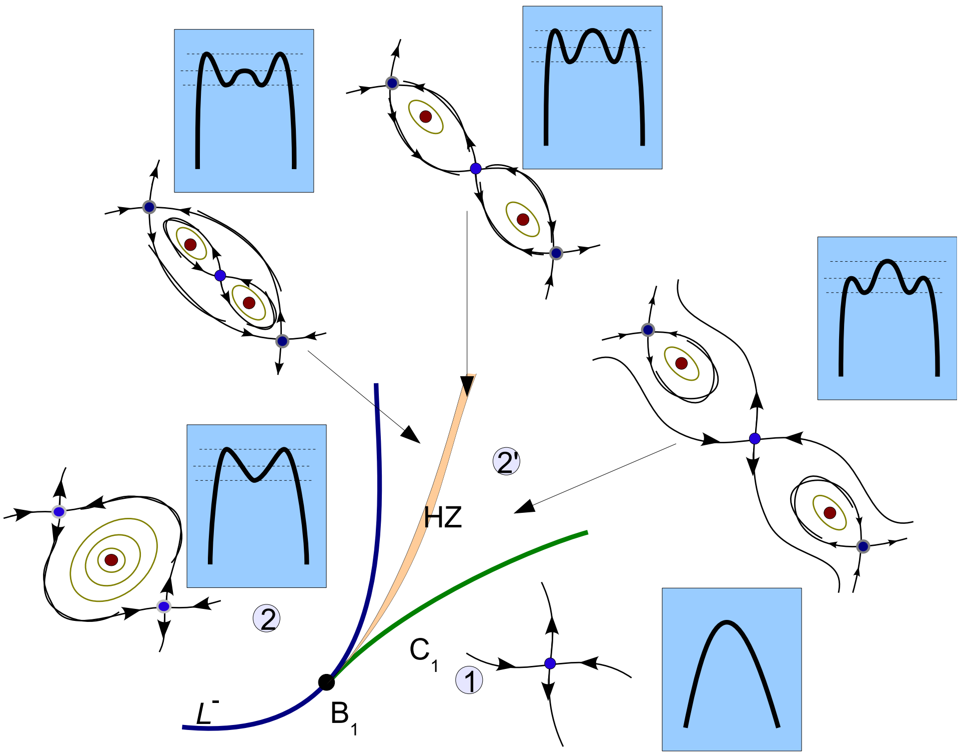


Figure 3.6: Bifurcations around the point b_1 of the bifurcation diagram of Figure 3.5. Region HZ (homo-heteroclinic zone) corresponds those values of (M_1, M_2) at which invariant manifolds of all saddles are intersected. In a rough approximation, these bifurcations are similar bifurcations of two dimensional Hamiltonian system whose potential function is symmetric and changes as in the figure.

one saddle).

3.2.2 Bifurcation Theorem

Due to the Rescaling Lemma 3.2, we can recover bifurcations of single-round periodic orbits in the initial family f_{μ_1,μ_2} of symplectic maps. As result, we obtain the following

Theorem 3.2. 1) *In any neighbourhood of the origin in the (μ_1, μ_2)-plane, there exist infinitely many bifurcation curves L_k^+ and L_k^- as well as $C_{1,2}^{k+}$ and $C_{1,2}^{k-}$ (see formulas (3.16)) which accumulate at the curve B_0 from Th. 3.1 as $k \to \infty$. The map $T_k(\mu)$ has a parabolic fixed point with multipliers $\nu_1 = \nu_2 = +1$ at $\mu \in L_k^+$ and a fixed point with multipliers $\nu_1 = \nu_2 = -1$ at $\mu \in L_k^-$. If $\mu \in C_{1,2}^{k+}$ (resp., $\mu \in C_{1,2}^{k-}$), then the map $T_k(\mu)$ has a period two point with multipliers $\nu_1 = \nu_2 = +1$ (resp., $\nu_1 = \nu_2 = -1$).*

2) *For any sufficiently large k, in the (μ_1, μ_2)-plane there is a domain E_k between the curves L_k^+ and L_k^- (see Figure 3.8) such that the map $T_k(\mu)$ has a fixed elliptic point at $\mu \in E_k$. This point is generic for all such μ except those ones which belong to the curves of strong resonances when the multipliers are $\nu_{1,2} = e^{\pm i\pi/2}$ or $\nu_{1,2} = e^{\pm i2\pi/3}$.*

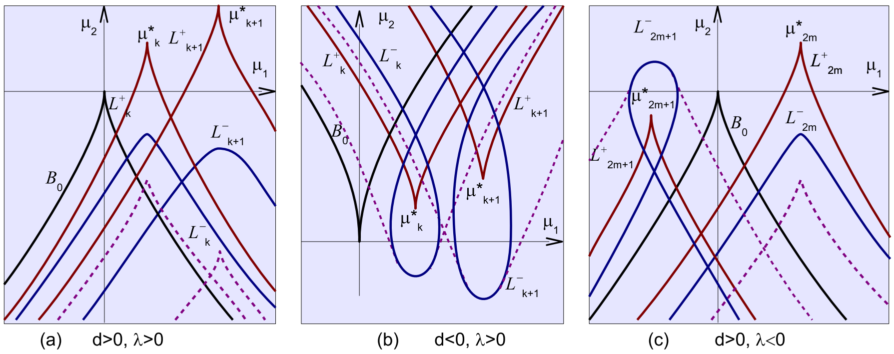


Figure 3.7: Main elements of bifurcation diagram for the families f_{μ_1,μ_2} in different cases.

Proof. This theorem is a corollary of the Rescaling Lemma 3.2 and the translation of the bifurcation scenarios for fixed points of first return maps (the conservative cubic Hénon maps, see Section3.2.1) to the corresponding single-round periodic orbits of f_{μ_1,μ_2}.

We find the equations of the curves L_k^+, L_k^- and $C_{1,2}^{k+}$ on the (μ_1, μ_2) -plane using formulas (3.15) and (3.10). We obtain the following formulae:

$$
\begin{aligned}
L_k^+ :\quad & \mu_1 = \lambda^k(y^- - cx^+ + \dots) \pm \frac{2}{\sqrt{|d|}} \left(\frac{(2 - f_{11}x^+)\lambda^k - \mu_2}{3\tilde{\nu}} \right)^{3/2} (1 + \dots), \\
L_k^- :\quad & \mu_1 = \lambda^k(y^- - cx^+ + \dots) \pm \frac{2}{3\sqrt{|d|}} \left(\frac{-(2 + f_{11}x^+)\lambda^k - \mu_2}{3\tilde{\nu}} \right)^{1/2} \times \\
& \quad \times \left((4 - f_{11}x^+)\lambda^k - \mu_2 \right) (1 + \dots), \\
C_{1,2}^{k+} :\quad & \mu_1 = \lambda^k(y^- - cx^+ \dots) \pm \frac{2}{\sqrt{|d|}} \left(\frac{-(4 + f_{11}x_1^+)\lambda^k - \mu_2}{3} \right)^{3/2} (1 + \dots) \\
& \quad \text{in the case} \quad d\lambda^k > 0, \\
C_{1,2}^{k+} :\quad & \mu_1 = \lambda^k(y^- - cx^+ \dots) \pm \frac{2}{\sqrt{|d|}} \left(\frac{4 + f_{11}x^+\lambda^k + \mu_2}{3} \right)^{3/2} (1 + \dots), \\
& \quad \mu_2\lambda^{-k}(1 + \dots) \geq -\frac{4}{3} \quad \text{in the case} \quad d\lambda^k < 0,
\end{aligned}
\tag{3.16}
$$

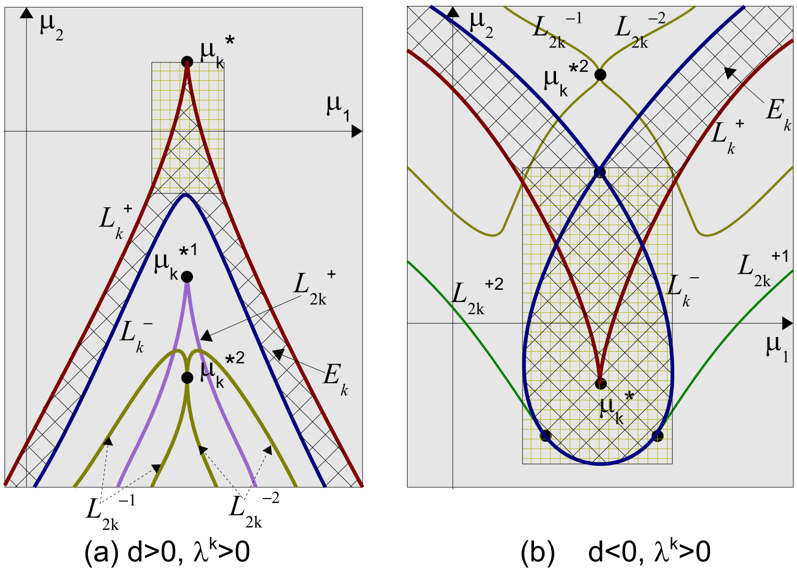


Figure 3.8: Bifurcation curves of the map $T_k(\mu)$: a) the case $d > 0, \lambda^k > 0$, b) the case $d < 0, \lambda^k > 0$. The shaded region E_k with boundaries L_k^+ and L_k^- corresponds to those values of μ at which the map $T_k(\mu)$ has an elliptic fixed point. The sizes of a specific "spring-area" zone in E_k has an order $\lambda^{3k/2} \times \lambda^k$ in $\mu_1 \times \mu_2$.

where $\hat{\nu} = \mathsf{sign}(d\lambda^k)$. We do not write a formula for $C_{1,2}^{k-}$ because of its largeness. □

Some elements of the bifurcation diagram for the family f_{μ_1,μ_2} are shown in Figure 3.7 for different cases. The domains E_k of stability for single-round elliptic periodic orbits, with boundaries L_k^+ and L_k^-, are illustrated in Figure 3.8. Note that typical sizes for stability "spring-area zones" (near the bifurcation point μ_k^*) have order $\lambda^{3k/2} \times \lambda^k$ and can be observed in numerical experiments, [GSV13], (whereas, such zones near quadratic homoclinic tangencies are very narrow, with width $\sim \lambda^{2k}$).

Note also that local bifurcations at strong resonances (for fixed points of T_k with multipliers $\nu_{1,2} = e^{\pm i\pi/2}$ or $\nu_{1,2} = e^{\pm i2\pi/3}$) are not degenerate. Such bifurcations were studied in [Gon05] for cubic Hénon maps and we can apply the corresponding results to our case. See also Appendix A.

Chapter 4

Finitely smooth normal forms for saddle area-preserving maps

4.1 Preambles

In this chapter we prove the main technical results, Lemmas 1.1, 1.2 and 2.2. Note that Lemma 2.3 is proved in the same way as Lemma 1.2 and we omit its proof. The proofs of Lemmas 1.1 and 2.2 differ only in some details. However, due to the importance of these results, we prove these lemmas independently.

Before proving the lemmas, we recall some necessary facts.

1) Consider a change of coordinates $(x, y) \mapsto (\xi, \eta)$ of the following form

$$\xi = \frac{\partial V(x, \eta, \varepsilon)}{\partial \eta} \ , \ \ y = \frac{\partial V(x, \eta, \varepsilon)}{\partial x}$$

where the function $V(x, \eta, \varepsilon)$ is some sufficiently smooth function of variables x, η and parameters ε satisfying conditions

$$V(0, 0, 0) = 0, \ \ \frac{\partial^2 V(0, 0, 0)}{\partial x \partial \eta} \neq 0.$$

It is well-known that such a change is an area-preserving map (when x, η and ε are small and V is sufficiently smooth, C^2 at least) . It is called the *canonical change of coordinates* and the function V is called the *generating function.*

In what follows, we will make only canonical changes of coordinates with canonical functions of the form $V = x\eta(1 + O(|x| + |y|)$. Thus, in fact, we consider close to identical and symplectic changes, independently whether the orientation is preserved or not.

2) Let F_ε be a parameter family of two-dimensional area-preserving maps which is C^r-smooth in both variables and coordinates. Let every map F_ε have a saddle fixed point O_ε with eigenvalues $\lambda(\varepsilon)$ and $\gamma(\varepsilon)$ such that where $|\lambda| < 1$ and $|\lambda\gamma| = 1$. We

can always assume that, for all sufficiently small ε , the fixed point O_ε is in the origin and that the coordinates, x and y , are such that the axes x and y correspond to the proper subspaces for eigenvalues λ and γ, respectively. Then the local map $T_\varepsilon \equiv F_{\varepsilon|U}$, where U is a small fixed neighbourhood of the origin, can be written in the form

$$\bar{x} = \lambda(\varepsilon)x + \phi(x,y,\varepsilon) \ , \ \bar{y} = \gamma(\varepsilon)y + \psi(x,y,\varepsilon) \tag{4.1}$$

where functions ϕ and ψ and their first derivatives in coordinates vanish at $x = y = 0$ for all small ε. In this case the equations of the local stable and local unstable manifolds can be written as $y = h_s(x,\varepsilon)$ and $x = h_u(y,\varepsilon)$, respectively, where h_s and h_u are C^r and such that

$$h_s(0,\varepsilon) = \frac{\partial h_s(0,\varepsilon)}{\partial x} = 0, \ \ h_u(0,\varepsilon) = \frac{\partial h_u(0,\varepsilon)}{\partial y} = 0.$$

If to make two consecutive changes of variables of the form $\xi = x - h_u(y,\varepsilon)$, $\eta = y$ and $\xi = x$, $\eta = y - h_s(x,\varepsilon)$, then the map T_ε is brought to the following form

$$\bar{x} = \lambda(\varepsilon)x + f(x,y,\varepsilon)x \ , \ \bar{y} = \gamma(\varepsilon)^{-1}y + g(x,y,\varepsilon)y \tag{4.2}$$

where $f(0,0,\varepsilon) = 0$, $g_s(0,0,\varepsilon) = 0$. Form (4.2) corresponds to the case where both the local stable and local unstable invariant manifolds of the point O_ε are straightened: the equation of $W^s_{loc}(O_s)$ and $W^u_{loc}(O_s)$ are $y = 0$ and $x = 0$, respectively (for all sufficiently small ε). Note that both the changes are C^r-smooth and canonical with generating functions $V = x\eta - \int h_u(\eta,\varepsilon)$ and $V = x\eta + \int h_s(x,\varepsilon)$, respectively.

3) Form (4.2) of the map T_ε is more convenient than (4.1) but its use gives some technical difficulties. This is connected, in particular, with the fact that "too much" resonance terms are in the right side of (4.2). Thus, there is very important the question on a reduction of the map (4.2) to a more simple form by means sufficiently smooth and area-preserving changes of coordinates. It is clear that the simplest form is the linear form of T_ε. But only C^1-linearization is possible here.

On the other hand, for the analytical case, J.Moser [Mos56] has been established that the map T_0 may be reduced to the following normal form

$$\bar{x} = \lambda B(xy)x \ , \ \bar{y} = \lambda^{-1}B^{-1}(xy)y \ , \tag{4.3}$$

where

$$B(xy) \equiv 1 + \beta_1 \cdot xy + \beta_2 \cdot (xy)^2 + ... + \beta_n \cdot (xy)^n + ...$$

and

$$B^{-1}(xy) \equiv 1 + \tilde{\beta}_1 \cdot xy + \tilde{\beta}_2 \cdot (xy)^2 + ... + \tilde{\beta}_n \cdot (xy)^n + ...$$

are series converging in some neighborhood of the origin. The Jacobian of (4.3) is equal to one. Thus, coefficients β_i and $\tilde{\beta}_i$ are connected by some relations. For example, $\beta_1 = -\tilde{\beta}_1$, $\tilde{\beta}_2 = \beta_1^2 - \beta_2$ etc. Moreover, let

$$\begin{aligned} B_n(xy) &\equiv 1 + \beta_1 \cdot xy + ... + \beta_n \cdot (xy)^n \ , \\ B_n^{-1}(xy) &\equiv 1 + \tilde{\beta}_1 \cdot xy + ... + \tilde{\beta}_n \cdot (xy)^n \end{aligned} \tag{4.4}$$

be the segments of the series B and B^{-1} , respectively. Then

$$\frac{\partial B_n}{\partial x} \cdot \frac{\partial B_n^{-1}}{\partial y} - \frac{\partial B_n}{\partial y} \cdot \frac{\partial B_n^{-1}}{\partial x} = 1 + O((xy)^{n+1}) \tag{4.5}$$

We will use these properties of the Birkhoff-Moser normal form.

4) Lemma 2.1 for $n = 1$ has been proved in [GST07] where the existence of the corresponding C^r-smooth canonical changes were derived, variants with C^{r-1}-changes were established in [GS90, MR97, GS00]. Finite smooth normal forms, as the ones from Lemma 2.2, near saddle equilibria of two-dimensional flows were found by E.A.Leontovich [Leo51, Leo88]. Here we use, in fact, the Leontovich method adapting to the discrete case. However, we combine the Leotovich method with the so-called "Afraimovich method", [Afr84], when the existence of the appropriate generating functions is proved with using the normal hyperbolicity theory [HPS77], i.e. we find this function as an equation of some strong stable (unstable) invariant manifold. In [Leo88] these functions are found as solutions of some homological equations.

4.2 Finitely smooth normal forms for symplectic saddle maps: the proof of Lemmas 1.1 and 1.2

Results of this section can be treated as an extension to the finitely smooth case of the classical Moser's theorems [Mos56] on the existence of analytical normal forms for area-preserving saddle maps.

4.2.1 Proof of Lemma 1.1

Note first that the main normal form ($n = 1$) of Lemma 1.1 was found earlier: the existence of a C^{r-1}-smooth normalized canonical change was proved in [GS00] and such type C^r-change was found in [GST07]. Therefore we need to prove the existence of normal forms with $n \geq 2$. However, we start with the map T_0 in the initial form (1.2). This map is C^r and can be represented in the following "n-th order extended form"

$$\begin{aligned} \bar{x} = \lambda(\varepsilon)x\{1 + \alpha_0^{(1)}(x,y,\varepsilon) + [\beta_1^{(1)} + \alpha_1^{(1)}(x,y,\varepsilon)] \cdot xy + \dots \\ +[\beta_n^{(1)} + \alpha_n^{(1)}(x,y,\varepsilon)] \cdot (xy)^n\} + O(x^{n+2}y^{n+1}), \\ \bar{y} = \lambda^{-1}(\varepsilon)y\{1 + \alpha_0^{(2)}(x,y,\varepsilon) + [\beta_1^{(2)} + \alpha_1^{(2)}(x,y,\varepsilon)] \cdot xy + \dots \\ +[\beta_n^{(2)} + \alpha_n^{(2)}(x,y,\varepsilon)] \cdot (xy)^n\} + O(x^{n+1}y^{n+2}), \end{aligned} \tag{4.6}$$

where $\beta_1^{(\nu)}, \dots, \beta_n^{(\nu)}$ are some coefficients, $\alpha_i^{(\nu)} \equiv [\varphi_i^{(\nu)}(x,\varepsilon) + \psi_i^{(\nu)}(y,\varepsilon)]$, $i = 0,\dots,n$, $\nu = 1,2$, are functions such that $\varphi_i^{(\nu)}(0,\varepsilon) \equiv 0, \psi_i^{(\nu)}(0,\varepsilon) \equiv 0$. Since $T_0 \in C^r$, it follows from (4.6) that $\alpha_i^{(\nu)} \in C^{r-2i-1}$.

The lemma states that there exist canonical changes which annihilate functions $\alpha_i^{(1,2)}$ for $i = 0,1,\dots,n$. We will make these changes consequently, step by step. Then we will see that the change vanishing the term $\alpha_i^{(1,2)}$ is C^{r-2i-2} , while the next term α_{i+1} is $C^{r-2(i+1)-1} = C^{r-2i-3}$. That is, such a change does not vary the smoothness of the high order terms (in the sense of the decomposition in (4.6)). Thus, the final smoothness will be equal to the smoothness of the last change (i.e. $C^{r-2(n-1)-2} = C^{r-2n}$).

Suppose that for some $i \leq n$ the map T_0 is brought to the following form (compare with (1.3))

$$\begin{aligned}
&\bar{x} = \lambda x\{1 + \beta_1 \cdot xy + \beta_2 \cdot (xy)^2 + ... + \beta_{i-1} \cdot (xy)^{i-1} + \\
&+[\beta_i^{(1)} + \varphi_i^{(1)}(x,\varepsilon) + \psi_i^{(1)}(y,\varepsilon)] \cdot (xy)^i\} + O(x^{i+2}y^{i+1}) \ , \\
&\bar{y} = \lambda^{-1} y\{1 + \hat{\beta}_1 \cdot xy + \hat{\beta}_2 \cdot (xy)^2 + ... + \hat{\beta}_{i-1} \cdot (xy)^{i-1} + \\
&+[\beta_i^{(2)} + \varphi_i^{(2)}(x,\varepsilon) + \psi_i^{(2)}(y,\varepsilon)] \cdot (xy)^i\} + O(x^{i+1}y^{i+2}),
\end{aligned} \tag{4.7}$$

where β_i (and $\hat{\beta}_i$) are the Birkhoff coefficients. Note that all coefficients and functions depend on ε, but we do not express explicitly this dependence in many places.

Let us show that there exists a canonical C^{r-2i-2}-change of coordinates that vanishes the functions $\alpha_i^{(1)}$ and $\alpha_i^{(2)}$. Since the number "i" is arbitrary, it will imply the lemma.

Make two consecutive canonical changes with the following generating functions

$$V_1^{(i)} = x\eta + (x\eta)^{i+1} v_1^{(i)}(x,\varepsilon) \ \text{ and } \ V_2^{(i)} = x\eta + (x\eta)^{i+1} v_2^{(i)}(\eta,\varepsilon)$$

where $v^{(i)}(0,\varepsilon) = 0$. By means of these changes, we vanish the functions $\varphi_i^{(1)}$ and $\psi_i^{(2)}$ in (4.7). After this, we show that new functions $\tilde{\varphi}_i^{(2)}$ and $\tilde{\psi}_i^{(1)}$ vanish automatically since the Jacobian $J(T_0)$ of the map T_0 is equal to 1 identically.

First, we make the canonical change with the function $V_1^{(i)}$. Thus, this change is

$$\xi = x + (i+1)x^{i+1}\eta^i v_1^{(i)}(x,\varepsilon) \ \ , \ \ y = \eta + x^i \eta^{i+1} \tilde{v}_1^{(i)}(x,\varepsilon) \tag{4.8}$$

where $\tilde{v}_1^{(i)}(x,\varepsilon) = (i+1)v_1^{(i)}(x,\varepsilon) + x \cdot \partial v_1^{(i)}/\partial x$ and $\tilde{v}_1^{(i)}(0,\varepsilon) \equiv 0$.

The first equation of (4.7) is transformed as

$$\begin{aligned}
&\bar{\xi} = \bar{x} + (i+1)\bar{x}^{i+1}\bar{\eta}^i v_1^{(i)}(\bar{x},\varepsilon) = \lambda x\{1 + \beta_1 \cdot xy + \beta_2 \cdot (xy)^2 + ... \\
&+\beta_{i-1} \cdot (xy)^{i-1} + \beta_i^{(1)} \cdot (xy)^i + \varphi_i^{(1)}(x,\varepsilon) \cdot (xy)^i + \psi_i^{(1)}(y,\varepsilon) \cdot (xy)^i\} + \\
&+(i+1)\lambda^{i+1}x^{i+1}\lambda^{-i}y^i v_1^{(i)}(\lambda x,\varepsilon) + O(\xi^{i+2}\eta^{i+1}) = \\
&= \lambda\xi\{\beta_1 \cdot \xi\eta + \beta_2 \cdot (\xi\eta)^2 + ... + \beta_{i-1} \cdot (\xi\eta)^{i-1} + \beta_i^{(1)} \cdot (\xi\eta)^i\} + \\
&+x^{i+1}y^i \left[-(i+1)\lambda v_1^{(i)}(x,\varepsilon)) + (i+1)\lambda v_1^{(i)}(\lambda x,\varepsilon) + \lambda\varphi_i^{(1)}(x,\varepsilon)\right] + \\
&+\lambda\psi_i^{(1)}(\eta,\varepsilon) \cdot \xi(\xi\eta)^i + O(\xi^{i+2}\eta^{i+1}).
\end{aligned} \tag{4.9}$$

We take the function $v_1^{(i)}(x,\varepsilon)$ to vanish the expression in the square brackets in (4.9), i.e.,

$$v_1^{(i)}(\lambda x,\varepsilon) = v_1^{(i)}(x,\varepsilon) - \frac{1}{i+1}\varphi_i^{(1)}(x,\varepsilon) \tag{4.10}$$

Equation (4.10) has a solution in the class of functions whose smoothness coincides with the smoothness of function $\varphi_1^{(i)}(x,\varepsilon)$ (namely $\varphi_1^{(i)} \in C^{r-2i-1}$) . Such a solution is found as the graph of the strong stable invariant manifold containing a fixed point $(0,0)$ of the following map

$$\bar{u} = u + \frac{1}{i+1}\varphi_i^{(1)}(x,\varepsilon) \ , \ \ \bar{x} = \lambda(\varepsilon)x$$

Note also, that the manifold is C^{r-2i-1} and, thus, the change (4.8) is C^{r-2i-2}.

We can see from (4.8) that $x = \xi + O(\xi^2(\xi\eta)^i)$, $y = \eta + O((\xi\eta)^{i+1})$. Then, in the second equation of (4.7) such coordinate transformation does not change the function $\psi_i^{(2)}$ and other functions which do not enter to O-terms, except for the function $\varphi_i^{(2)}(x,\varepsilon)$; let the latter be transformed as $\varphi_2^{(i)} \Rightarrow \tilde{\varphi}_2^{(i)}$. So, after change (4.8), the map T_ε has form (4.7) where

$$\varphi_i^{(1)}(x,\varepsilon) \equiv 0 \ , \ \varphi_i^{(2)} \Rightarrow \tilde{\varphi}_i^{(2)} \tag{4.11}$$

and the other explicitly given functions are the same.

We make now the change with the second generating function $V_2^{(i)} = x\eta + (x\eta)^{i+1}v_2^{(i)}(\eta,\varepsilon)$ where $v_2^{(i)}(0,\varepsilon) = 0$. Thus, this change is

$$\begin{aligned} &\xi = x + (i+1)x^{i+1}\eta^i v_2^{(i)}(\eta,\varepsilon) + (x\eta)^{i+1}\frac{\partial v_2^{(i)}}{\partial \eta} = x + O((x\eta)^{i+1}) \ , \\ &y = \eta + (i+1)x^i\eta^{i+1}v_2^{(i)}(\eta,\varepsilon) \end{aligned} \tag{4.12}$$

The second equation of (4.7) (taking into account (4.11)) is transformed under change (4.12) as

$$\begin{aligned} &\bar{\eta} = \bar{y} - (i+1)\bar{x}^i\bar{\eta}^{i+1}v_2^{(i)}(\bar{\eta},\varepsilon) = \lambda^{-1}y\{1 + \hat{\beta}_1\cdot xy + \hat{\beta}_2\cdot(xy)^2 + ... \\ &+\hat{\beta}_{i-1}\cdot(xy)^{i-1} + \beta_i^{(2)}\cdot(xy)^i + \tilde{\varphi}_i^{(2)}(x,\varepsilon)\cdot(xy)^i + \\ &+\psi_i^{(2)}(y,\varepsilon)\cdot(xy)^i\} - (i+1)\lambda^i x^i\lambda^{-i-1}y^{i+1}v_2^{(i)}(\lambda^{-1}y,\varepsilon) + O(\xi^{i+1}\eta^{i+2}) = \\ &= \lambda^{-1}\eta(1 + \hat{\beta}_1\cdot\xi\eta + \hat{\beta}_2\cdot(\xi\eta)^2 + ... + \hat{\beta}_{i-1}\cdot(\xi\eta)^{i-1} + \beta_i^{(2)}\cdot(\xi\eta)^i) + \\ &+x^iy^{i+1}\left[\lambda^{-1}(i+1)v_2^{(i)}(y,\varepsilon) + \lambda^{-1}\psi_i^{(2)}(y,\varepsilon) - \lambda^{-1}(i+1)v_2^{(i)}(\lambda^{-1}y,\varepsilon)\right] + \\ &+\lambda^{-1}\tilde{\varphi}_i^{(2)}(x,\varepsilon)x^iy^{i+1} + O(\xi^{i+1}\eta^{i+2}) \end{aligned} \tag{4.13}$$

We find the function $v_2^{(i)}(y,\varepsilon)$ to vanish the expression in the square brackets in (4.13), i.e.,

$$v_2^{(i)}(\lambda^{-1}y,\varepsilon) = v^{(i)}(x,\varepsilon)) - \tfrac{1}{i+1}\psi_i^{(2)}(y,\varepsilon) \tag{4.14}$$

This equation can be considered as an equation for the strong unstable invariant manifold of the following map of the plane

$$\bar{u} = u + \frac{1}{i+1}\psi_i^{(2)}(y,\varepsilon) \ , \ \ \bar{y} = \lambda^{-1}(\varepsilon)y.$$

The sought manifold exists and is C^{r-2i-1}. It follows that change (4.12) is C^{r-2i-2}.

So, after the canonical changes (4.8) and (4.12), the map T_ε takes the following form

$$\begin{array}{l} \bar{x} = \lambda(\varepsilon)x\{1+\beta_1(\varepsilon)\cdot xy+...+\beta_i(\varepsilon)\cdot(xy)^i\} + \tilde{\psi}_i^{(1)}(y,\varepsilon)\cdot x^{i+1}y^i + O(x^{i+2}y^{i+1})\ , \\ \bar{y} = \lambda^{-1}(\varepsilon)y\{1+\hat{\beta}_1(\varepsilon)\cdot xy+...+\hat{\beta}_i(\varepsilon)\cdot(xy)^i\} + \tilde{\varphi}_i^{(2)}(x,\varepsilon)\cdot x^iy^{i+1} + O(x^{i+1}y^{i+2}) \end{array} \tag{4.15}$$

Let us show that the equality $J(T_0)\equiv 1$ implies $\tilde{\psi}_1^{(i)}\equiv 0$ and $\tilde{\varphi}_2^{(i)}\equiv 0$. Indeed, we can represent map (4.15) as follows

$$\begin{array}{l} \bar{x} = \lambda(\varepsilon)xB_i(xy) + \tilde{\psi}_1^{(i)}(y,\varepsilon)\cdot x^{i+1}y^i + O(x^{i+2}y^{i+1})\ , \\ \bar{y} = \lambda^{-1}(\varepsilon)yB_i^{-1}(xy) + \tilde{\varphi}_2^{(i)}(x,\varepsilon)\cdot x^iy^{i+1} + O(x^{i+1}y^{i+2}), \end{array} \tag{4.16}$$

where $B_i = 1+\beta_1 xy+\cdots+\beta_i(xy)^i$. Then, by the properties of the Bikhoff-Moser normal form (1.7), we obtain that the Jacobian of (4.16) is as follows

$$J = 1+(i+1)\left(\lambda\tilde{\varphi}_2^{(i)}(x,\varepsilon)+\lambda^{-1}\tilde{\psi}_1^{(i)}(y,\varepsilon)\right)\cdot x^iy^i + O((xy)^{i+1}).$$

Since $J\equiv 1$, it follows that $\tilde{\varphi}_2^{(i)}\equiv 0$ and $\tilde{\psi}_1^{(i)}\equiv 0$. This completes the proof of Lemma 1.1. □

4.2.2 Proof of Lemma 1.2

By Lemma 1.1, the map T_0 (C^{r-2n-2}-smooth) can be written in the form

$$\begin{array}{rcl} \bar{x} &=& \lambda(\varepsilon)x+\tilde{f}(x,y,\varepsilon)\equiv\lambda(\varepsilon)xB(xy,\varepsilon)+F(x,y,\varepsilon)x(xy)^{n+1}\ , \\ \bar{y} &=& \lambda(\varepsilon)^{-1}y+\tilde{g}(x,y,\varepsilon)\equiv\lambda(\varepsilon)^{-1}yB^{-1}(xy,\varepsilon)+G(x,y,\varepsilon)y(xy)^{n+1}, \end{array} \tag{4.17}$$

where $F(0,0,\varepsilon)\equiv 0, G(0,0,\varepsilon)\equiv 0$. Note that if F and G are identical zeros, then form (4.17) becomes exactly the Birkhoff-Moser normal form (1.7).

Consider the following operator $\mathbf{\Phi}$:

$$\begin{array}{rcl} \bar{x}_j &=& \lambda(\varepsilon)^jx_0+\sum\limits_{s=0}^{j-1}\lambda(\varepsilon)^{j-s-1}\tilde{f}(x_s,y_s,\varepsilon), \\ \bar{y}_j &=& \lambda(\varepsilon)^{k-j}y_k-\sum\limits_{s=j}^{k-1}\lambda(\varepsilon)^{s-j+1}\tilde{g}(x_s,y_s,\varepsilon), \end{array} \tag{4.18}$$

($j=0,1,\ldots,k$) defined on the set $R(\delta) = \{z=[(x_j,y_j)]_{j=0}^k,\ \|z\|\le\delta\}$, where $\|z\|$ means the maximum of the absolute value of components x_j,y_j of vector z and δ is a positive small quantity. If $z_0 = [(x_j^0,y_j^0)]_{j=0}^k$ is a fixed point of Φ , then the following diagram takes place

$$(x_0^0,y_0^0)\xrightarrow{T_0}(x_1^0,y_1^0)\xrightarrow{T_0}\cdots\xrightarrow{T_0}(x_k^0,y_k^0)$$

It was proved in [AS73] that, for all sufficiently small ε and $\delta = \delta_0$ $\|x_0\| \leq \delta_0/2$, $|y_k| \leq \delta_0/2$, the operator $\mathbf{\Phi}$ maps region $R(\delta_0)$ into itself and is contracting. Thus, the map (4.18) has a unique fixed point $z_0 = [(x_j^0(x_0, y_k), y_j^0(x_0, y_k)]_{j=0}^k$. Due to contractibility, its coordinates x_j^0 and y_j^0 can be found, for example, by the method of successive approximations. However, we know the solution in the case $F \equiv 0$ and $G \equiv 0$ (it is given by (1.8)). Thus, we can only check corrections to this solution due to the presence of F- and G- terms in (4.17). Taking into account that $x_s \sim \lambda^s x_0, y_s \sim \lambda^{k-s} y_k$ we can easily estimate contributions $\tilde{\xi}_k$ and $\tilde{\eta}_k$ of F- and G-terms into the final formulas for x_k^0 and y_0^0. Thus, we obtain for $\tilde{\xi}_k$ from (4.18) that

$$\tilde{\xi}_k \sim \sum_{s=0}^{k-1} \lambda^{k-s-1} \left(x_s^{n+3} y_s^{n+1} + x_s^{n+2} y_s^{n+2}\right) \sim$$
$$\sim x_0^{n+2} y_k^{n+1} \lambda^{k-1} \sum_{s=0}^{k-1} \lambda^{-s} \left(\lambda^{s(n+3)} \lambda^{(k-s)(n+1)} x_0 + \lambda^{s(n+2)} \lambda^{(k-s)(n+2)} y_k\right) = O\left(\lambda^{k(n+2)}\right).$$

The estimate $\tilde{\eta}_k = O\left(\lambda^{k(n+2)}\right)$ is deduced in the same way. It completes the proof. □

4.3 Finitely smooth normal forms for non-orientable area preserving saddle maps

Proof of lemma 2.2. We start from the well-known fact that the local stable and unstable manifolds of O can be straightened by means of a certain C^r-symplectic change of coordinates,[1] i.e. the map T_0 can be written in the following form

$$\bar{x} = \lambda(\varepsilon)x + f(x, y, \varepsilon)x \ , \ \bar{y} = \gamma(\varepsilon)y + g(x, y, \varepsilon)y \ , \qquad (4.19)$$

where $f(0, 0, \varepsilon) \equiv 0, g(0, 0, \varepsilon) \equiv 0$. $f(x, y, \varepsilon) \in C^{r-1}$, $g(x, y, \varepsilon) \in C^{r-1}$. In these coordinates, the fixed point O_ε is in the origin and the equations of W^s_{loc} and W^u_{loc} are $y = 0$ and $x = 0$, respectively, for all sufficiently small ε.

Consider the map T_ε in the initial form (4.19). This map is C^r and can be repre-

[1] Let us recall some details of this. We can always write the local map in the form $\bar{x} = \lambda(\varepsilon)x + h_1(x, y, \varepsilon)$, $\bar{y} = \gamma(\varepsilon)y + h_2(x, y, \varepsilon)$, where $|\lambda\gamma| = 1$, $h_i(0, 0, \varepsilon) = 0, h_i'(0, 0, \varepsilon) = 0$. Let $y = \varphi(x, \varepsilon)$ be the equation of W^s_{loc}. Then, by the change $\xi = x, \eta = y - \varphi(x, \varepsilon)$, we straighten W^s_{loc}. Moreover, this change is symplectic, since it is produced by the generating function $V(x, \eta, \varepsilon) = x\eta + \int \varphi(x, \varepsilon)$. The manifold W^u_{loc} is straightened analogously.

sented in the following "n-th order extended form"

$$\begin{array}{l}
\bar{x} = \lambda(\varepsilon)x\{1 + [\varphi_1^{(0)}(x,\varepsilon) + \psi_1^{(0)}(y,\varepsilon)] + [\beta_1^{(1)} + \varphi_1^{(1)}(x,\varepsilon) + \psi_1^{(1)}(y,\varepsilon)] \cdot xy + \\
+[\beta_1^{(2)} + \varphi_1^{(2)}(x,\varepsilon) + \psi_1^{(2)}(y,\varepsilon)] \cdot (xy)^2 + ... + \\
+[\beta_1^{(n)} + \varphi_1^{(n)}(x,\varepsilon) + \psi_1^{(2)}(y,\varepsilon)] \cdot (xy)^n\} + O(x^{n+2}y^{n+1}) , \\
\bar{y} = \gamma(\varepsilon)y\{1 + [\varphi_2^{(0)}(x,\varepsilon) + \psi_2^{(0)}(y,\varepsilon)] + \\
+[\beta_2^{(1)} + \varphi_2^{(1)}(x,\varepsilon) + \psi_2^{(1)}(y,\varepsilon)] \cdot xy + [\beta_2^{(2)} + \varphi_2^{(2)}(x,\varepsilon) + \psi_2^{(2)}(y,\varepsilon)] \cdot (xy)^2 + ... + \\
+[\beta_2^{(n)} + \varphi_2^{(n)}(x,\varepsilon) + \psi_2^{(2)}(y,\varepsilon)] \cdot (xy)^n\} + O(x^{n+1}y^{n+2})
\end{array} \tag{4.20}$$

where $|\lambda\gamma| = 1$, $\beta_1^{(i)}$ and $\beta_2^{(i)}$ are number coefficients, $i = 1, ..., n$, $\varphi_k^{(i)}(0,\varepsilon) = \psi_k^{(i)}(0,\varepsilon) \equiv 0$, $k = 1,2$. Denote $\alpha_{ki} \equiv [\varphi_k^{(i)}(x,\varepsilon) + \psi_k^{(i)}(y,\varepsilon)]$. Since $T_\varepsilon \in C^r$, we have, due to the decomposition in (4.20), that $\alpha_{ki} \in C^{r-2i-1}$.

The lemma states that there exist canonical changes which annihilate functions α_{ki} and transform constants β_1^i and β_2^i into "Birkgoff-Moser coefficients" β_i and $\tilde{\beta}_i$, respectively. If to make these changes consequently, then one can see that the change annihilating the term α_{ki} is C^{r-2i-2} , while the next term $\alpha_{k,i+1}$ is $C^{r-2(i+1)-1} = C^{r-2i-3}$. That is, such a change does not change the smoothness of the high order terms (in the sense of the decomposition in (4.20). Thus, the final smoothness will be equal to the smoothness of the last coordinate transformation.

Now we prove the lemma by induction on i . Note that Lemma 2.1 can be considered here as "the first step of induction".

Suppose that for some $i \leq n$ we have brought the map T_ε to the form

$$\begin{array}{l}
\bar{x} = \lambda(\varepsilon)x\{1 + \beta_1(\varepsilon) \cdot xy + \beta_2(\varepsilon) \cdot (xy)^2 + ... + \beta_{i-1}(\varepsilon) \cdot (xy)^{i-1} + \\
+\beta_1^{(i)} + [\varphi_1^{(i)}(x,\varepsilon) + \psi_1^{(i)}(y,\varepsilon)] \cdot (xy)^i\} + O(x^{i+2}y^{i+1}) , \\
\bar{y} = \gamma(\varepsilon)y\{1 + \tilde{\beta}_1(\varepsilon) \cdot xy + \tilde{\beta}_2(\varepsilon) \cdot (xy)^2 + ... + \tilde{\beta}_{i-1}(\varepsilon) \cdot (xy)^{i-1} + \\
+\tilde{\beta}_2^{(i)} + [\varphi_2^{(i)}(x,\varepsilon) + \psi_2^{(i)}(y,\varepsilon)] \cdot (xy)^i\} + O(x^{i+1}y^{i+2})
\end{array} \tag{4.21}$$

Let us show that there exist a canonical change annihilating the terms α_{1i} and α_{2i} and that smoothness of such a change is equal to the smoothness of functions $\alpha_{k,i}$ minus one. Then, the lemma will be proven.

For this goal we make two consecutive canonical changes with the following generating functions

$$V_1^{(i)} = x\eta + (x\eta)^{i+1} v_1^{(i)}(x,\varepsilon) \;\text{ and }\; V_2^{(i)} = x\eta + (x\eta)^{i+1} v_2^{(i)}(\eta,\varepsilon), \tag{4.22}$$

where $v_k^{(i)}(0,\varepsilon) = 0$, $k = 1,2$. By means of these changes one can vanish functions $\varphi_1^{(i)}$ and $\psi_2^{(i)}$ in (4.21), respectively. After this, we show that new functions $\tilde{\varphi}_2^{(i)}$ and $\tilde{\psi}_1^{(i)}$ vanish due to equality to one of $|J(T_\varepsilon)|$.

First, we make the change by means of the generating function $V_1^{(i)}$ where $v_1^{(i)}(0,\varepsilon) = 0$. Thus, this change is

$$\xi = x + (i+1)x^{i+1}\eta^i v_1^{(i)}(x,\varepsilon) \;\;,\;\; y = \eta + x^i\eta^{i+1}\tilde{v}_1^{(i)}(x,\varepsilon) \tag{4.23}$$

where $\tilde{v}_1^{(i)}(x,\varepsilon) = (i+1)v_1^{(i)}(x,\varepsilon) + x\cdot \partial v_1^{(i)}/\partial x$ and $\tilde{v}_1^{(i)}(0,\varepsilon)\equiv 0$.

The first equation of (4.21) is transformed as

$$\begin{array}{l} \bar{\xi} = \bar{x} + (i+1)\bar{x}^{i+1}\bar{\eta}^i v_1^{(i)}(\bar{x},\varepsilon) = \lambda x\{1+\beta_1\cdot xy + \beta_2\cdot (xy)^2 + ... \\ +\beta_{i-1}\cdot (xy)^{i-1} + \beta_1^{(i)}\cdot (xy)^i + \varphi_1^{(i)}(x,\varepsilon)\cdot (xy)^i + \\ +\psi_1^{(i)}(y,\varepsilon)\cdot (xy)^i\} + (i+1)\lambda^{i+1}x^{i+1}\gamma^i y^i v_1^{(i)}(\lambda x,\varepsilon) + O(\xi^{i+2}\eta^{i+1}) = \lambda\xi + \\ +x^{i+1}y^i\left[-(i+1)\lambda v_1^{(i)}(x,\varepsilon)) + (i+1)\lambda\delta_i v_1^{(i)}(\lambda x,\varepsilon) + \lambda\varphi_1^{(i)}(x,\varepsilon)\right] + \\ +\lambda\xi\{\beta_1\cdot\xi\eta + \beta_2\cdot(\xi\eta)^2 + ... + \beta_{i-1}\cdot(\xi\eta)^{i-1} + \beta_1^{(i)}\cdot(\xi\eta)^i) + \\ +\psi_1^{(i)}(\eta,\varepsilon)\cdot\xi(\xi\eta)^i\} + O(\xi^{i+2}\eta^{i+1}), \end{array} \tag{4.24}$$

where $\delta_i = \text{sign }(\lambda\gamma)^i$. Now we take a function $v_1^{(i)}(x,\varepsilon)$ to vanish the expression from the square brackets in (4.24), i.e.,

$$v_1^{(i)}(\lambda x,\varepsilon) = \delta_i v_1^{(i)}(x,\varepsilon)) - \tfrac{1}{i+1}\varphi_1^{(i)}(x,\varepsilon) \tag{4.25}$$

Note that this equation has a solution in the class of functions (of variable x) whose smoothness coincides with the smoothness of the function $\varphi_1^{(i)}(x,\varepsilon)$ (recall that $\varphi_1^{(i)}\in C^{r-2i-1}$). The sought solution, $u = v_1^{(i)}(x,\varepsilon)$, can be viewed as the equation of the strong stable invariant manifold W_i^{ss} containing the point $(0,0)$ of the following planar map

$$\bar{u} = \delta_i u - \tfrac{1}{i+1}\varphi_1^{(i)}(x,\varepsilon)\ \ ,\ \ \bar{x} = \lambda(\varepsilon)x \tag{4.26}$$

(since W^{ss} is invariant, its equation $u = \phi_{ss}(x,\varepsilon)$ has to satisfy the following homological equation: $\phi_{ss}(\lambda x,\varepsilon) = \delta_i\phi_{ss}(x,\varepsilon) - \frac{1}{i+1}\varphi_1^{(i)}(x,\varepsilon)$ that is (4.25, in fact). Since $\delta_i = \pm 1$, such a manifold exists, it is C^{r-2i-1} and, thus, the change (4.25) is C^{r-2i-2}.

We can see from (4.23) that the sought change is of form

$$x = \xi + O((\xi\eta)^{i+1})\ ,\ y = \eta + O((\xi\eta)^{i+1})$$

This means that, in the second equation of (4.21), such a change can vary only the function $\lambda^{-1}\varphi_2^{(i)}(x,\varepsilon)x^i y^{i+1}$ from the explicitly shown ones in (4.21): $\varphi_2^{(i)}\Rightarrow\tilde{\varphi}_2^{(i)}$.

Thus, after change (4.23), the map T_ε have form (4.21) where

$$\varphi_1^{(i)}(x,\varepsilon)\equiv 0\ ,\ \varphi_2^{(i)}\equiv\tilde{\varphi}_2^{(i)} \tag{4.27}$$

and the other explicitly given functions are the same. Note especially, that function $\psi_2^{(i)}(y,\varepsilon)$ does not vary.

It is evident that the second coordinate transformation, by means of the second generating function $V_2^{(i)} = x\eta + (x\eta)^{i+1}v_2^{(i)}(\eta,\varepsilon)$ with $v_2^{(i)}(0,\varepsilon) = 0$, is conducted quite symmetrically, due to the condition $|\lambda\gamma|\equiv 1$.[2]

[2]See also the paper [GG09] in which this change is explicitly conducted for the sympletic case.

Thus, after the canonical changes with the generating functions $V_1^{(i)}$ and $V_2^{(i)}$ from (4.22), the map T_ε takes the following form

$$\begin{array}{l} \bar{x} = \lambda(\varepsilon)x\{1+\beta_1(\varepsilon)\cdot xy+...+\beta_i(\varepsilon)\cdot(xy)^i\}+ \\ +\tilde{\psi}_1^{(i)}(y,\varepsilon)\cdot x^{i+1}y^i + O(x^{i+2}y^{i+1})\ , \\ \bar{y} = \gamma(\varepsilon)y\{1+\tilde{\beta}_1(\varepsilon)\cdot xy+...+\tilde{\beta}_i(\varepsilon)\cdot(xy)^i\}+ \\ +\tilde{\varphi}_2^{(i)}(x,\varepsilon)\cdot x^iy^{i+1}+O(x^{i+1}y^{i+2}) \end{array} \tag{4.28}$$

Let us show that the equality $J(T_\varepsilon)\equiv 1$ implies $\tilde{\psi}_1^{(i)}\equiv 0$ and $\tilde{\varphi}_2^{(i)}\equiv 0$. Really, we may represent the map (4.28) as

$$\begin{array}{l} \bar{x} = \lambda(\varepsilon)xB_i(xy)+\tilde{\psi}_1^{(i)}(y,\varepsilon)\cdot x^{i+1}y^i+O(x^{i+2}y^{i+1})\ , \\ \bar{y} = \gamma(\varepsilon)yB_i^{-1}(xy)\tilde{\varphi}_2^{(i)}(x,\varepsilon)\cdot x^iy^{i+1}+O(x^{i+1}y^{i+2}) \end{array} \tag{4.29}$$

where B_i and B_i^{-1} are the segments of the Bikhoff-Moser normal form. Taking into account the property (4.5), one can write Jacobian of (4.29) in the form

$$J = \pm 1 + (i+1)(\lambda\tilde{\varphi}_2^{(i)}(x,\varepsilon)+\gamma\tilde{\psi}_1^{(i)}(y,\varepsilon))\cdot x^iy^i + O((xy)^{i+1})$$

It follows from here that $\tilde{\varphi}_2^{(i)}\equiv 0$ and $\tilde{\psi}_1^{(i)}\equiv 0$.

In the nonorientable case $\lambda\gamma=-1$, the monomials of (2.2) with $\beta_i,\tilde{\beta}_i\equiv 0$ for odd i are not resonant. Therefore, they can be killed (inside of the every corresponding step of the proof) by the canonical polynomial coordinate transformations with the generating functions $\tilde{V}_i = x\eta+\nu_i(x\eta)^{i+1}$. One can check that if in (4.4) all terms β_i and $\tilde{\beta}_i$ vanish for odd i, except for the last ones β_n and $\tilde{\beta}_n$ for odd n, then $\beta_n=-\tilde{\beta}_n$. Then the change with the generating functions $\tilde{V}_n$ kills both these terms simultaneously.

This completes the proof of the lemma. □

Appendix A

On structure of 1:4 resonances in conservative Hénon-like maps

We study bifurcations of fixed points with multipliers $e^{\pm i\pi/2}$ (the main 1:4 resonances) in some conservative Hénon-like maps. We analyze the 1:4 resonance in the cases of conservative generalized Hénon maps (GHMs) and cubic Hénon maps. We find conditions of nondegeneracy of the corresponding resonances and give a description of accompanying bifurcations.

Introduction

The Hénon map [Hen76]

$$\bar{x} = y,\ \bar{y} = 1 - bx + ay^2, \tag{A.1}$$

is one of the most popular artificial maps demonstrating a complicated chaotic dynamics. In the coordinates $x_{new} = -ax, y_{new} = -ay$ map (A.1) is written in the so-called *standard* form

$$\bar{x} = y,\ \bar{y} = M_1 - M_2x - y^2, \tag{A.2}$$

where $M_1 = -a$ and $M_2 = b$ are new parameters. Both maps (A.1) and (A.2) have the constant Jacobian, $J = b$, and, therefore, they are degenerate with respect to (Andronov-Hopf) bifurcations of birth of invariant circles. Moreover, if we restrict ourselves to the conservative case ($J \equiv 1$), then maps (A.1) and (A.2) are degenerate again with respect to bifurcations of fixed points with multipliers $e^{\pm i\pi/2}$. However, it is well known that the standard Hénon map has also "homoclinic origination". It appears as a model map for rescaled first return maps near quadratic homoclinic tangencies [GS73, GG04]. Thus, the degeneracy shows that map (A.2) is only "first approximation" of the return map. The corresponding "second approximations", so-called generalized Hénon maps of form

$$\bar{x} = y,\ \bar{y} = M_1 - M_2x - y^2 + Rxy + Sy^3, \tag{A.3}$$

where R and S are some coefficients, were derived in [GG00, GG04, GGT02, GSS02, GST02] for various situations with quadratic homoclinic and heteroclinic tangencies. Usually, the coefficients R and S (some invariants of the homoclinic or heteroclinic structure) are small and depend on "time of the return" k : if k is the number of iterations of the diffeomorphism such that the image of an initial point is in its some small neighbourhood, then $R = R_k$ and $S = S_k$ tend to 0 as $k \to \infty$.

When the initial homoclinic tangency is cubic, the cubic Hénon maps of form

$$\bar{x} = y,\ \bar{y} = M_1 + M_2 y - Bx \pm y^3 \qquad \text{(A.4)}$$

naturally appear [GST96a] as normal forms of rescaled first return maps. The signs "+" and "−" correspond to different maps which appear, in turn, near cubic tangencies of different types (see Figure 3.1). Besides, the maps with " + " and " − " have a rather different structure of bifurcations [Gon85].

Main bifurcations of GHMs were studied in [GG00, GG04, Gon02, GKM05] and bifurcations of the cubic Hénon maps were considered in [Gon85, GK88]. In this appendix we pay attention only to bifurcations of fixed points with multipliers $e^{\pm i\pi/2}$. We explain a character of the conservative bifurcation in cases of the GHMs with $M_2 = 1, R = 0$ and the cubic Hénon maps with $B = 1$.

A.1 The resonance $1:4$ in area-preserving maps

Let a planar area-preserving map have a fixed point with multipliers $e^{\pm i\pi/2}$. Then, it is well known [Arn96], that the corresponding complex local normal form is written as

$$\bar{\zeta} = i(1+\beta)\zeta + D'_{21}\zeta^2\zeta^* + D'_{03}\zeta^{*3} + O(|\zeta|^4), \qquad \text{(A.5)}$$

where β is a parameter which characterizes a deviation of the angle argument φ of multipliers of the fixed points from $\pi/2$ ($\varphi > \pi/2$ at $\beta > 0$ and $\varphi < \pi/2$ at $\beta < 0$), the coefficients D'_{21} and D'_{03} (depending on β) are real and, in general,

$$|D'_{21}| + |D'_{03}| \neq 0. \qquad \text{(A.6)}$$

This condition implies that $O(|\zeta|^4)$ terms in (A.5) do not influence on a character of the local bifurcations. In this case, main details of reconstructions of phase portraits can be described by means of the analysis of bifurcations in the following flow normal form

$$\dot{\zeta} = 4i\beta\zeta + A\zeta|\zeta|^2 + \zeta^{*3}, \qquad \text{(A.7)}$$

where

$$A = -i\frac{D'_{21}}{|D'_{03}|}. \qquad \text{(A.8)}$$

Form (A.7) is a result of embedding fourth degree of map (A.5) into flow up to terms of order $O(|\zeta|^4)$. The nondegeneracy condition for the conservative $1:4$ resonance is $|A| \neq 1$. In this case, bifurcations of the trivial equilibrium of the Hamiltonian flow (A.7) under changing β look as in Figures 2 and 3 in the cases $|A| > 1$ and $|A| < 1$, respectively.[1] When $|A| = 1$ in the critical moment, possible bifurcations can be described within two-parameter families

$$\dot{\zeta} = 4i\beta\zeta + i(A+\mu)\zeta|\zeta|^2 + \zeta^{*3} \tag{A.9}$$

(where $|A| \equiv 1$) with parameters β and μ. The corresponding bifurcational diagram for flow (A.9) is shown in Figure 4.

Thus, returning to the case of map (A.5), we can describe main reconstructions of phase portrait in the following way:

1) In case $|A| > 1$, the fixed point O is always elliptic, but when $\beta > 0$ two period 4 cycles appear in its neighborhood: one cycle is saddle and the other is elliptic (see Figure 2).

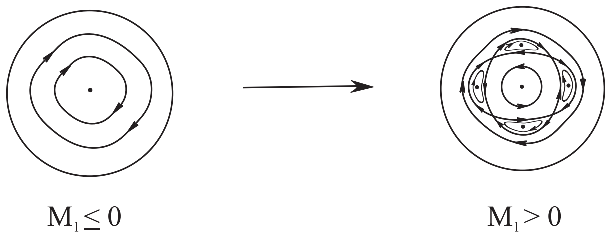


Figure A.1: Bifurcation of the trivial equilibrium for $|A| > 1$.

2) In case $|A| < 1$, the point O is a saddle with 8 separatrices when $\beta = 0$. Main bifurcations are connected here with a reconstruction of period 4 saddle cycles (see Figure 3).

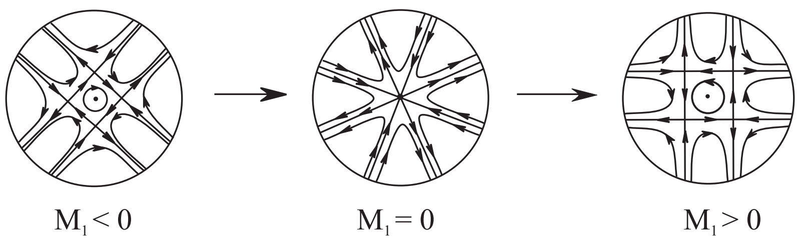


Figure A.2: Bifurcations of the trivial equilibrium for $|A| < 1$.

3) In case $|A| = 1$ there exist three bifurcation curves L_1, L_2 and L_3 which divide a neighbourhood of the origin of the parameter plane (β, μ) onto three regions with different local phase portraits (see Fifure 4). Note that curve L_3 corresponds to the existence of period 4 parabolic point.

[1] Note that case $D'_{03} = 0$ is not special. In this case (if also $D'_{21} \neq 0$), one can consider, instead of (A.7), the flow of form $\dot{\zeta} = 4i\beta\zeta + i\zeta|\zeta|^2 + \varepsilon(\beta)\zeta^{*3}$ (where $\varepsilon(0) = 0$) whose bifurcations of the zero equilibrium are the same as in Figure 2 (case $|A| > 1$).

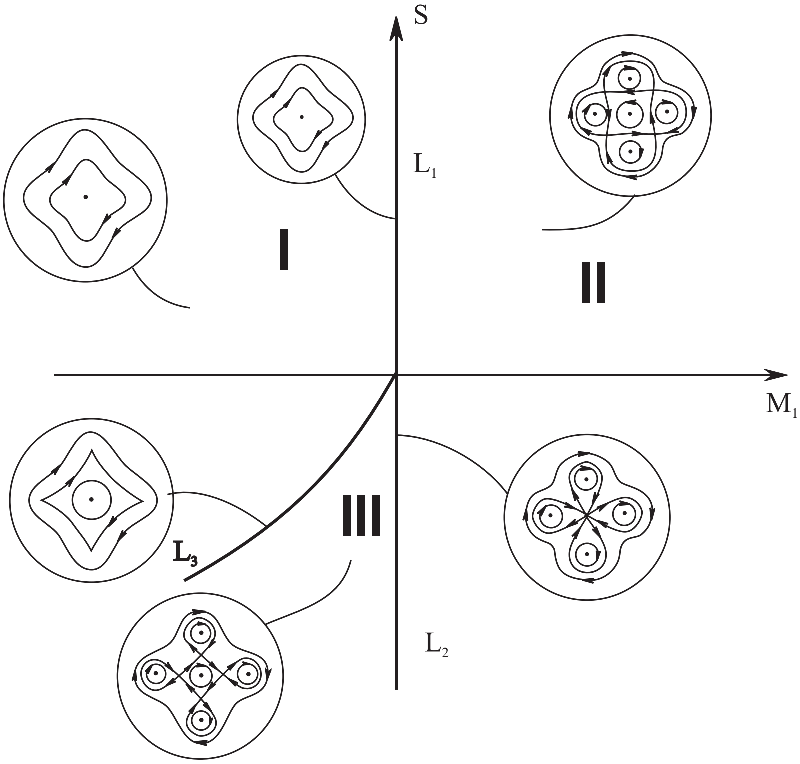


Figure A.3: Bifurcation diagram in case $|A| = 1$.

We can apply these theoretical results to our concrete conservative maps: the generalized Hénon map and cubic Hénon maps.

A.2 Conservative generalized Hénon maps

In the case of the conservative generalized Hénon map

$$\bar{x} = y; \quad , \quad \bar{y} = M_1 - x - y^2 + Sy^3. \tag{A.10}$$

we find that it has a fixed point with multipliers $e^{\pm i\pi/2}$ at $M_1 = 0$. This point is in the origin and the corresponding complex local normal form (A.5) has the coefficients (see [GKM05]): $8D'_{21} = 1 + 3S$, $8D'_{03} = -1 + S$ and, thus,

$$A = -i\frac{1+3S}{1-S} \quad \text{and} \quad |A| = 1 + \frac{4S}{1-S}.$$

Therefore, $|A| > 1$ if $S > 0$ and $|A| < 1$ if $S < 0$ (recall that we consider case of small S). In this case, local bifurcations can be described by means of flow normal forms (A.7) or (A.9) where $\beta = M_1/2 + O(M_1^2)$ and $\mu = 4S + O(S^2)$.

A.3 Conservative cubic Hénon maps

Consider now the conservative cubic Hénon map of form

$$\bar{x} = y,\ \bar{y} = M_1 + M_2 y - x + y^3 \tag{A.11}$$

It has one fixed point M^* with multipliers $e^{\pm i\pi/2}$ at values of parameters M_1 and M_2 belonging to the curve $L^+_{\pi/2}$ whose equation is

$$M_1 = \pm 2\sqrt{-\frac{M_2}{3}}\left(1 - \frac{1}{3}M_2\right). \tag{A.12}$$

The coefficients of the local complex normal form (A.5) are

$$8D'_{21} = 3 - 3M_2, \quad 8D'_{03} = 1 + 3M_2$$

and, thus,

$$A = -i\frac{3 - 3M_2}{|1 + 3M_2|}.$$

Since $M_2 \leq 0$, it implies that $|A|$ is always greater than 1 here. The main local bifurcations which occur here at transition of the parameters cross the curve $L^+_{\pi/2}$ are shown in Figure 5.

Consider now the following cubic Hénon map

$$\bar{x} = y, \qquad \bar{y} = M_1 + M_2 y - x - y^3. \tag{A.13}$$

It has a fixed point M^{**} with multipliers $e^{\pm i\pi/2}$ when the parameters M_1, M_2 belong to the following curve $L^-_{\pi/2}$

$$M_1 = \pm 2\sqrt{\frac{M_2}{3}}\left(1 - \frac{M_2}{3}\right). \tag{A.14}$$

Note that curve $L^-_{\pi/2}$ has a self-intersection point ($M_1 = 0, M_2 = 3$), and only in this moment the map has simultaneously two fixed points with multipliers $e^{\pm i\pi/2}$. The coefficients of the local complex normal form (A.5) are

$$8D'_{21} = -3 + 3M_2,\ 8D'_{03} = -1 - 3M_2$$

and, thus

$$A = -i\frac{3 - 3M_2}{1 + 3M_2}. \tag{A.15}$$

It implies that there are two points P^+ and P^- on $L^-_{\pi/2}$ (with $M_2 = \frac{1}{3}$ and $M_1 = 16/27$ and $M_1 = -16/27$, respectively,) where $|A| = 1$. Moreover, $|A| < 1$ if $M_2 > \dfrac{1}{3}$ and $|A| > 1$ if $0 \leq M_2 < \dfrac{1}{3}$.

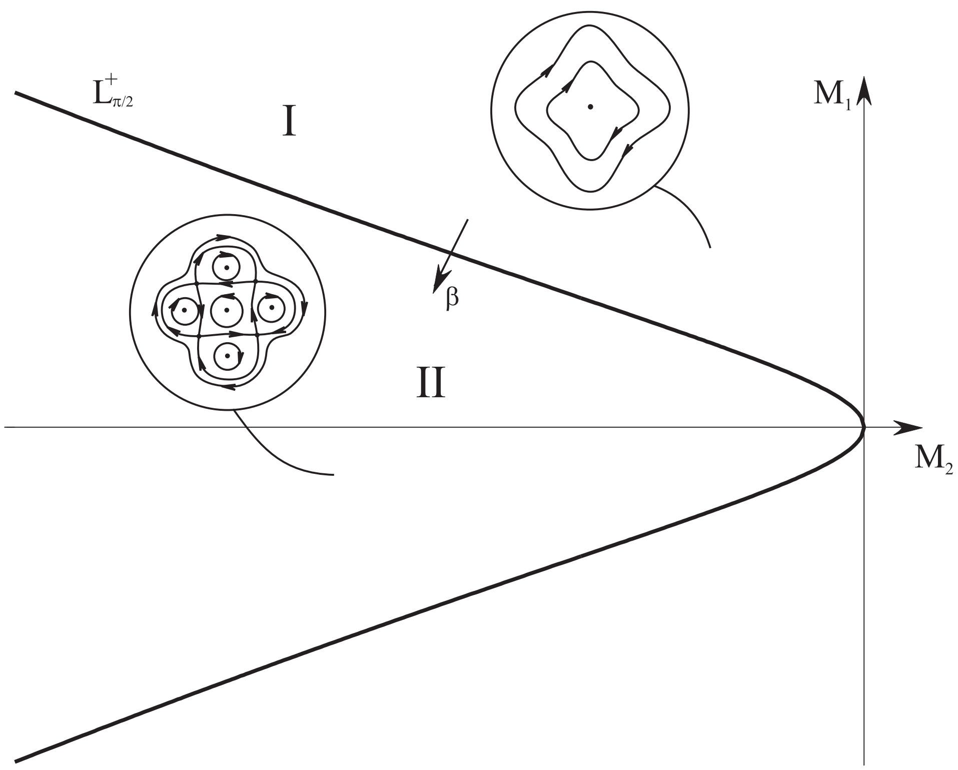


Figure A.4: Local bifurcations at transition of the parameters cross the curve $L^+_{\pi/2}$ in the case of map (A.11).

Then, in accordance to the observation above of conservative bifurcations at 1:4 resonance, we can describe a character of these bifurcations in the case under consideration. We explain this with the help of Figure 8 where three bifurcational curves $L^-_{\pi/2}$, L_3 and L_{+1} are shown. The curve L_3 corresponds to the existence of a period 4 parabolic point near the fixed point. The curve L_{+1} corresponds to the appearance of a parabolic fixed point, its equation is

$$M_1 = \pm\frac{2}{3}\left(\frac{M_2-2}{3}\right)^{3/2}.$$

Note that we restrict ourself by the consideration of a small neighbourhood of the curve $L^-_{\pi/2}$, This neighbourhood is divided by the curves $L^-_{\pi/2}$, L_3 and L_1 into 16 domains of values of parameters M_1 and M_2. We show in Figure 8 the corresponding phase portraits.

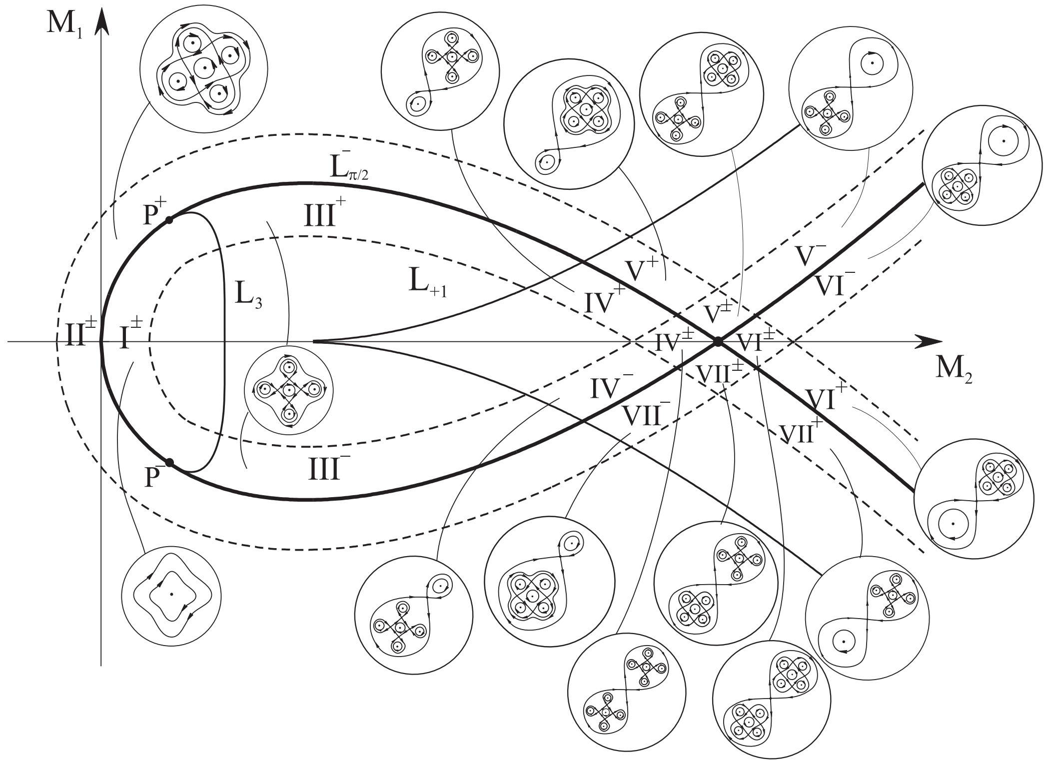


Figure A.5: Local bifurcations at transition of the parameters cross the curve $L^+_{\pi/2}$ in the case of map (A.11).

Part II

Exponentially small splitting of separatrices for whiskered tori in Hamiltonian systems

Chapter 5

Setup

In this chapter we describe the system under consideration. We study a singular or weakly hyperbolic Hamiltonian with $n+1$ degrees of freedom possessing an n-dimensional whiskered tori with fast frequencies. This is a generalization of the Arnold example that can be considered as a model for behavior of a near-integrable Hamiltonian system near a single resonance. In Chapters 6 and 7 we study the splitting of separatrices in the cases $n = 2$ and $n = 3$, respectively.

5.1 A singular Hamiltonian with $n + 1$ degrees of freedom

We consider a Hamiltonian system with $n+1$ ($n = 2$ or $n = 3$) degrees of freedom that is a perturbation of an integrable one. In the canonical coordinates $(x, y, \varphi, I) \in \mathbb{T}\times\mathbb{R}\times\mathbb{T}^n\times\mathbb{R}^n$, with the symplectic form $dx\wedge dy + d\varphi\wedge dI$, the Hamiltonian is defined by

$$H(x, y, \varphi, I) = H_0(x, y, I) + \mu H_1(x, \varphi), \tag{5.1}$$

$$H_0(x, y, I) = \langle\omega_\varepsilon, I\rangle + \frac{1}{2}\langle\Lambda I, I\rangle + \frac{y^2}{2} + \cos x - 1, \tag{5.2}$$

$$H_1(x, \varphi) = h(x)f(\varphi). \tag{5.3}$$

Our system has two perturbation parameters ε and μ and we assume that $\varepsilon > 0$ and $\mu > 0$ with no loss of generality.

The vector $\omega_\varepsilon = \omega/\sqrt{\varepsilon}$ in (5.2) is the vector of *fast frequencies* given by an n-dimensional frequency vector ω satisfying a *Diophantine condition* of constant type

$$|\langle k, \omega\rangle| \geq \frac{\gamma}{|k|^\tau}, \quad \forall k \in \mathbb{Z}^n\backslash\{0\} \tag{5.4}$$

with some $\gamma > 0$ and the exponent $\tau \geq n$.

We also consider in (5.2) a symmetric $n \times n$ matrix Λ such that H_0 satisfies the condition of *isoenergetic nondegeneracy*

$$\det \begin{pmatrix} \Lambda & \omega \\ \omega^\top & 0 \end{pmatrix} \neq 0. \tag{5.5}$$

For the perturbation (5.3), we deal with the following analytic periodic functions

$$h(x) = \cos x - \nu, \quad \text{with } \nu = 0 \text{ or } \nu = 1 \tag{5.6}$$

$$f(\varphi) = \sum_{k \in \mathcal{Z}} e^{-\rho|k|} \cos(\langle k, \varphi\rangle - \sigma_k), \text{ with } \sigma_k \in \mathbb{T}, \tag{5.7}$$

where we introduce the set $\mathcal{Z}$ in order to avoid repetitions in the Fourier series:

$$\begin{array}{rcll} \mathcal{Z} & = & \{k = (k_1, k_2) \in \mathbb{Z}^2 : k_2 > 0 \text{ or } (k_2 = 0, k_1 \geq 0)\}, & \text{if } n = 2, \\ \text{or} & & & \\ \mathcal{Z} & = & \{k = (k_1, k_2, k_3) \in \mathbb{Z}^3 : (k_3 > 0, k_1, k_2 \in \mathbb{Z}) \text{ or} & \\ & & (k_3 = 0, k_2 > 0, k_1 \in \mathbb{Z}) \text{ or } (k_2 = k_3 = 0, k_1 \geq 0)\}, & \text{if } n = 3. \end{array} \tag{5.8}$$

In the Fourier expansion of $f(\varphi)$ the constant $\rho > 0$ gives the complex width of analyticity of $f(\varphi)$. In principle, the phases σ_k can be chosen arbitrarily, although some quite general condition on these phases will have to be fulfilled for the validity of our results.

The two parameters ε and μ are not independent, but they are linked by a relation of the type $\mu = \varepsilon^p$ (the smaller p the better), i.e we consider a *singular problem* for $\varepsilon \to 0$. See [DG01] for discussion about singular and regular problems.

Notice that the unperturbed system H_0 (that corresponds to $\mu = 0$) consists of the pendulum given by $P(x, y) = \frac{y^2}{2} + \cos x - 1$ and n rotors with fast frequencies: $\dot{I} = 0, \dot{\varphi} = \omega_\varepsilon + \Lambda I$. The pendulum has a hyperbolic equilibrium at the origin. The hyperbolic point has separatrices that correspond to curves where $P(x, y) = 0$. We parameterize the upper separatrix as $(x_0(s), y_0(s))$, $s \in \mathbb{R}$, where

$$x_0(s) = 4 \arctan e^s, \quad y_0(s) = \frac{2}{\cosh s}. \tag{5.9}$$

The lower separatrix has the parametrization $(x_0(-s), -y_0(-s))$. The rotors system (φ, I) has the solution $I = I_0$, $\varphi = (\omega_\varepsilon + \Lambda I_0)t + \varphi_0$.

Then the Hamiltonian H_0 has a family of n-dimensional whiskered tori given by $I = I_0 = const, x = y = 0$

$$\Lambda_{I_0} = \{(x = 0, y = 0, I = I_0, \varphi) : \varphi \in \mathbb{T}^n\}$$

where the dynamics is $\dot{\varphi} = \omega_\varepsilon + \Lambda I$ The collection of the whiskered tori at each value of I gives a $2n$-dimensional *normally hyperbolic invariant manifold* (NHIM)

$$\Lambda_I = \{(x = 0, y = 0, I, \varphi) : I \in \mathbb{R}^n, \varphi \in \mathbb{T}^n\}.$$

The stable and unstable manifolds (whiskers) of the NHIM that coincide in the unperturbed system, are

$$W^s\Lambda_I = W^u\Lambda_I = \{(x_0(s), y_0(s), I, \varphi) : P(x_0(s), y_0(s)) = 0, s \in \mathbb{R}, I \in \mathbb{R}^n, \varphi \in \mathbb{T}^n\},$$

they are of dimension $2n+1$ and with the inner flow $\dot\varphi = \omega_\varepsilon + \Lambda I, \dot I = 0, \dot s = 1$.

Without loss of generality we consider the torus located at $I_0 = 0$

$$\Lambda_0 = \{(x = 0, y = 0, I = 0, \varphi) : \varphi \in \mathbb{T}^n\}$$

with the dynamics $\dot\varphi = \omega_\varepsilon$ and its coincident $n+1$-dimensional whiskers

$$W^s\Lambda_0 = W^u\Lambda_0 = W_0 = \{(x_0(s), y_0(s), I = 0, \varphi) : P(x_0(s), y_0(s)) = 0, s \in \mathbb{R}, \varphi \in \mathbb{T}^n\} \tag{5.10}$$

We can check that the hypotheses (H1)-(H4) of [DGS04] hold in our case:

(H1) the isoenergetic condition (5.5);

(H2) the Diophantine condition (5.4);

(H3) the function $h(x)$ is a trigonometric polynomial of degree $l = 1$;

(H4) the function $f(\varphi)$ is analytic in a complex strip $|\mathtt{Im}\varphi| < \rho$ and there exists $\alpha \geq 0$ and a constant $c > 0$ such that, for any $0 < \delta < \rho$, $\|f\|_{\rho-\delta} \leq c/\delta^\alpha$. (We denote $\|f\|_{\rho-\delta}$ the norm of $f(\varphi)$ on the complex strip $|\mathtt{Im}\varphi| \leq \rho - \delta$). In our cases, we have: $\alpha = 2$ if $n = 2$ and $\alpha = 3$ if $n = 3$.

Thus, when perturbing the problem ($\mu \neq 0$), the *hyperbolic KAM theorem* [Eli94],[DGS04] implies that, under non-degeneracy and non-resonance conditions (5.5) and (5.4), for μ small enough, the whiskered torus Λ_0 as well as its local whiskers persist. We explain the difference between two values of ν in (5.6): the point is that in the case $\nu = 0$ the whiskered torus persists with some shift and deformations, whereas in the case $\nu = 1$ it remains fixed under the perturbation, though the whiskers deform. The Lyapunov exponent of the torus, which initially is 1, becomes a close amount b. Also the frequency vector ω_ε becomes perturbed and proportional vector:

$$\tilde\omega_\varepsilon = b'\omega_\varepsilon = \frac{b'\omega}{\sqrt{\varepsilon}}.$$

The amounts b and b' tend to 1 as $\mu \to 0$, and $b' = 1$ in the case $\nu = 1$. See a precise statement of the hyperbolic KAM theorem with the corresponding estimates in [DGS04, Th. 1].

5.2 The Poincaré-Melnikov method

When the local whiskers are extended to global ones, one expects in general the existence of splitting between the stable and unstable whiskers (denoted $\mathcal{W}^+$ and $\mathcal{W}^-$), since they do not coincide anymore. To study the splitting, the so-called *flow-box coordinates* are introduced in [DGS04], in a neighbourhood containing a piece of both whiskers and excluding the torus where such coordinates are not valid. Those coordinates can be constructed in such a way that the stable whisker is given by a coordinate plane

$$\mathcal{W}^+ : (s, 0, \theta, 0), \ \ |s| \leq s^*, \theta \in \mathbb{T}^n,$$

where the parameters (s, θ) are inherited from (5.10). Then the unstable whisker is parameterized as

$$\mathcal{W}^- : (s, \mathcal{E}(s, \theta), \theta, \mathcal{M}(s, \theta)) \ \ |s| \leq s^*, \theta \in \mathbb{T}^n.$$

The inner flow on both whiskers in $\dot{s} = b, \dot{\theta} = \tilde{\omega}_\varepsilon$. To study the splitting, it is sufficient to consider the vector function $\mathcal{M}$, called the *splitting function*, that measures the distance between the stable and unstable whiskers (the function $\mathcal{E}$ is directly related to $\mathcal{M}$ by the energy conservation).

The function $\mathcal{M}$ has two important properties that we will use. The first one is that $\mathcal{M}$ is $\hat{\omega}_\varepsilon$-*quasiperiodic*

$$\mathcal{M}(s, \theta) = \mathcal{M}(0, \theta - s\hat{\omega}_\varepsilon), \text{ where } \hat{\omega}_\varepsilon := \frac{\tilde{\omega}_\varepsilon}{b} = \frac{b'\omega}{b\sqrt{\varepsilon}}. \tag{5.11}$$

The second property of $\mathcal{M}$ is that it is the gradient of a scalar function $\mathcal{L}$, called the *splitting potential*

$$\mathcal{M}(s, \theta) = \partial_\theta \mathcal{L}(s, \theta).$$

Then we can consider the section $s = 0$ and the simple zeros of $\mathcal{M}(0, \theta)$ which give rise to transverse homoclinic orbits are given by nondegenerate critical points by $\mathcal{L}(0, \theta)$.

The Poincaré-Melnikov method gives us a first order approximation in μ for the splitting in terms of the Melnikov potential L and the Melnikov function M defined as follows

$$\begin{gathered} L(s, \theta) = -\int_{-\infty}^{\infty} [h(x_0(s + bt)) - h(0)] f(\theta + \tilde{\omega}_\varepsilon t) dt + const, \\ M(s, \theta) = \partial_\theta L(\theta). \end{gathered} \tag{5.12}$$

These functions are also $\hat{\omega}_\varepsilon$-quasiperiodic, since they are defined in terms of the perturbed Lyapunov exponent b and the perturbed frequencies $\tilde{\omega}_\varepsilon$. Then, the *error term* defined as

$$\mathcal{R}(s, \theta) = \mathcal{M}(s, \theta) - \mu M(s + s^{(0)}, \theta) \tag{5.13}$$

is also $\hat{\omega}_\varepsilon$-quasiperiodic. The amount $s^{(0)}$, not relevant, compensates a translation of the parametrization of the perturbed whiskers $\mathcal{W}^-$ and $\mathcal{W}^+$: with respect to the initial parametrization of $\mathcal{W}_0$.

However, the approximations (5.12), given by the Poincaré-Melnikov approach, are exponentially small in ε. Then in order to have the error term to be small, it is necessary μ to be exponentially small (this condition was imposed in [Arn64] and [DG00]) that it is not true in our case $\mu = \varepsilon^p$ $(p > 0)$. In the last case, the problem becomes much more complicated, since we may not guarantee that the main term (5.12) dominates the error term and, thus, we must provide estimates to ensure it. The idea is to present the Melnikov potential L in the Fourier series and then to find the dominant harmonics L_k and give estimates to show that the selected harmonics dominate the error term. These estimates have to be big enough to be valid for the dominant harmonics of $\mathcal{L}$. Then it is possible to prove the non-degeneracy of the critical points of $\mathcal{L}$ and, thus, the simplicity of the zeros of $\mathcal{M}$.

To obtain exponentially small estimates of the functions we go to the complex plane and use the quasi-periodic properties of the splitting. We define the complex domain

$$\mathcal{P}_{\kappa,\nu,\rho} = \{(s,\theta) : |Res| < \kappa, |Ims| < \nu, Re\theta \in \mathbb{T}^n, |Im\theta| < \rho\}.$$

Initially, the whiskers can be defined in the complex domain $|Ims| < \frac{\pi}{2}$, $|Im\theta| < \rho$. This domain is restricted by the singularity of the pendulum separatrix parametrization (5.9) at $s = \pm i\pi/2$ and the width of analyticity of the function f in (5.7), and it is reduced along the successive steps leading to define the splitting function and potential. In [DGS04] flow-box coordinates are constructed at which the loss of the complex domain is controlled by a small parameter δ, $\delta \ll \frac{\pi}{2}$ and $\delta \ll \rho$. Choosing $\delta = \varepsilon^a$, $a > 0$, and using that our functions are analytic, quasiperiodic, and with the zero average, one can get exponentially small estimates (see [DG03, DG04, DGS04] for more details). Therefore, it is possible to obtain exponential small estimates for splitting function $\mathcal{M}(s,\theta)$ in the singular case $\mu = \varepsilon^p$ with some restriction $p > p^*$.

We give the results of [DGS04] (Theorems 1 and 10) applied to our cases: $n = 2, \tau = 1, \alpha = 2$ and $n = 3, \tau = 2, \alpha = 3$, to have an upper bound for the error term $\mathcal{R}$. We also provide bounds for $|b-1|$ and $|b'-1|$. The exponents q_1, q_2, q_3, q_4 are easily computed by expressions given in [DGS04].

Throughout Part II we use the notation $|f| \preceq |g|$ if we can bound $|f| \le c|g|$ with some positive constant c not depending on ε, μ. Also we write $f \sim g$ if $|g| \preceq |f| \preceq |g|$.

Theorem 5.1. *For a given $\delta > 0$, assuming*

$$\varepsilon \preceq 1, \quad \mu \preceq \delta^{q_1}, \quad \mu \preceq \delta^{q_2+1}\sqrt{\varepsilon},$$

the splitting function $\mathcal{M}(\theta)$ is analytic on $\mathcal{P}_{\kappa,\pi/2-\delta,\rho-\delta}(\kappa > 0)$, and $\hat{\omega}_\varepsilon$-quasiperiodic. For the amounts b and b', one has bounds:

$$|b-1|, |b'-1| \preceq \frac{\mu}{\delta^{q_2}}.$$

The error term $\mathcal{R}(s,\theta)$ is also $\hat{\omega}_\varepsilon$-quasiperiodic and satisfies the bound:

$$|\mathcal{R}|_{\kappa,\pi/2-\delta,\rho-\delta} \preceq \frac{\mu^2}{\delta^{q_3}} + \frac{\mu^2}{\delta^{q_4}}. \tag{5.14}$$

The exponents q_1, q_2, q_3, q_4 are given by:
in the case $n = 2$

$$\begin{array}{ll} q_1 = 8, q_2 = 4, q_3 = 14, q_4 = 12, & \textit{if } \nu = 0, \\ q_1 = 6, q_2 = 2, q_3 = 10, q_4 = 8, & \textit{if } \nu = 1, \end{array}$$

and in the case $n = 3$

$$\begin{array}{lllll} \textit{if } \nu = 0: & q_1 = 12, & q_2 = 6, & q_3 = 20, & q_4 = 17; \\ \textit{if } \nu = 1: & q_1 = 8, & q_2 = 3, & q_3 = 14, & q_4 = 11; \end{array}$$

Note that in [DG04, DG03] a similar Hamiltonian with 3 degrees of freedom was considered. The most accurate results were obtained for $\omega = (1, \Omega)$ with $\Omega = (\sqrt{5} - 1)/2$, the so-called *quadratic golden* number. and the existence of 4 transverse homoclinic points to the whiskered torus was proved for all values $\varepsilon \to 0$.

Due to the quasiperiodicity (5.11) of $\mathcal{M}(s, \theta)$ and the other functions involved we can restrict ourselves into the section $s = 0$ and redefine the functions as:

$$\mathcal{M}(\theta) := \mathcal{M}(0, \theta), \quad \mathcal{L}(\theta) := \mathcal{L}(0, \theta), \quad M(\theta) := M(0, \theta), \quad L(\theta) := L(0, \theta),$$

and then extend the results obtained for $s = 0$ to any $s \in \mathbb{R}$.

Chapter 6

Exponentially small splitting of separatrices for whiskered tori with quadratic frequencies

In this chapter we study the splitting of invariant manifolds of whiskered tori in a nearly-integrable Hamiltonian system with 3 degrees of freedom. We consider two-dimensional tori whose frequency ratios are quadratic irrational numbers. We deal with numbers whose continued fractions satisfy certain arithmetic properties which give us 24 cases for consideration. We show that the Poincaré-Melnikov method can be applied to establish the existence of 4 homoclinic orbits to the whiskered tori and prove that these homoclinic orbits are transverse. We also prove the continuation of these homoclinic orbits for the silver number $\sqrt{2}-1$.

We consider the Hamiltonian system (5.1-5.3) for $n=2$. Here the frequency vector ω is given by a *quadratic vector*

$$\omega = (1, \Omega), \tag{6.1}$$

where the frequency ratio Ω is a quadratic irrational number, i.e. an irrational real root of a quadratic polynomial with integer coefficients. It is well known that quadratic frequency vectors satisfy the *Diophantine condition* (5.4) with the exponent $\tau = 1$.

6.1 Quadratic frequencies

6.1.1 Continued fractions of quadratic numbers

It is well known that all the quadratic irrational numbers $\Omega \in (0,1)$, i.e. the real roots of quadratic polynomials with rational coefficients, have the continued fractions

$$\Omega = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \dots}}} = [a_1, a_2, a_3, \dots] \tag{6.2}$$

that are periodic starting with some element a_i. We consider only the numbers with the purely periodic continued fractions $[\overline{a_1, a_2, a_3, \ldots, a_m}]$ and denote them according to their periodic part by $\Omega_{a_1,a_2,a_3,\ldots,a_m}$, where $a_1, a_2, a_3, \ldots, a_m$ is the corresponding periodic part of the continued fraction, and we say that this continued fraction is m-periodic. For example, the famous *golden* number is $\Omega_1 = [\overline{1}] = (\sqrt{5}-1)/2$, the *silver* number $\Omega_2 = [\overline{2}] = \sqrt{2}-1$. Another interesting case is that of 2-periodic continued fractions, as for example: $\Omega_{1,2} = [\overline{1,2}] = \sqrt{3}-1$.

Remark 6.1. The same results apply for periodic but not purely periodic continued fractions, say $\widehat{\Omega} = [b_1, \ldots, b_n, \overline{a_1, \ldots, a_m}]$, since for small enough ε, we only need to consider the periodic part of the continued fraction of $\widehat{\Omega}$. We will call these continued fractions with the same periodic part as *equivalent* continued fractions. These numbers have a relation of type

$$\widehat{\Omega} = \frac{a + b\Omega_{a_1,\ldots,a_m}}{c + d\Omega_{a_1,\ldots,a_m}} \quad \text{with} \quad \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \pm 1.$$

6.1.2 Arithmetic properties

From the Diophantine condition (5.4) we define the quantity $\gamma_k = |\langle k, \omega\rangle||k|$. We aim to find two-dimensional non-zero integer vectors k which give the smallest values γ_k, we call these vectors k as *primary resonances*, and to study their separation from the other vectors, *secondary resonances*.

We say that the integer vector k is *admissible* if $|\langle k, \omega\rangle| < 1/2$ and denote by $\mathcal{A}$ the set of admissible vectors. We restrict ourselves to the set $\mathcal{A}$, since for any $k \notin \mathcal{A}$ we have $|\langle k, \omega\rangle| > 1/2$ and $\gamma_k \geq |k|/2$.

It is a well known fact that for frequency vectors there exists a unimodular matrix T (with integer entries and determinant ± 1) having the frequency vector as an eigenvector with the associated eigenvalue λ of modulus greater than 1. In the two-dimensional case, such matrix T can be derived from the continued fraction of the number Ω. For instance, for the silver number Ω_2 the matrix T is

$$T = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$$

(see Section 6.3).

We define the matrix $U = (T^{-1})^\top$ that satisfies the following equality

$$\langle Uk, \omega\rangle = \langle k, U^\top \omega\rangle = \frac{1}{\lambda}\langle k, \omega\rangle. \tag{6.3}$$

Thus, if $k \in \mathcal{A}$, then also $Uk \in \mathcal{A}$. We say that k is *primitive* if $k \in \mathcal{A}$ but $U^{-1}k \notin \mathcal{A}$. From (6.3) we deduce that k is primitive if and only if

$$\frac{1}{2|\lambda|} < |\langle k, \omega\rangle| < \frac{1}{2}.$$

Admissible vectors can be presented in form

$$k^0(j) = (-\mathrm{rint}\,(j\Omega), j),$$

where $j = \mathbb{Z}\backslash\{0\}$ and $\mathrm{rint}\,(a)$ is the closest integer to a. We say that an integer j is *primitive* if $k^0(j)$ is primitive. Let $\mathcal{P}$ be the set of primitive integers j.

For each primitive j, we define the following *resonant sequences* of integer vectors:

$$s(j,n) = U^{n-1}k^0(j), \quad n \geq 1. \tag{6.4}$$

It turns out that such resonant sequences cover the whole set of admissible vectors.

Proposition 6.1 (DG03). *For any primitive j, there exists the limit*

$$\gamma_j^* = \lim_{n\to\infty} \gamma_{s(j,n)} = |\langle k^0(j), \omega\rangle| K(j), \quad K(j) := |k^0(j) - \frac{\langle k^0(j), \omega\rangle}{\langle u, \omega\rangle} u|,$$

and one has

(a) $\gamma_{s(j,n)} = \gamma_{j^*} + \mathcal{O}(\lambda^{-2n}), n \geq 1$;

(b) $|s(j,n)| = K(j)|\lambda|^{n-1} + \mathcal{O}(|\lambda|^{-n}), n \geq 1$;

(c) $\dfrac{(1+\Omega)|j| - a}{2|\lambda|} < \gamma_j^* < \dfrac{(1+\Omega)|j| + a}{2|\lambda|}, a = \dfrac{1}{2}\left(1 + \dfrac{|u|}{|\langle u, \omega\rangle|}\right)$

Provided the bounds (c) of γ_j^* for each primitive j, we can select the minimal of them, say $\gamma_{j_0}^*$. We get

$$\gamma^* = \liminf_{|k|\to\infty} \gamma_k = \min_{j\in\mathcal{P}} \gamma_j^* = \gamma_{j_0}^*. \tag{6.5}$$

The corresponding sequence $s(j_0, n)$ gives us *the primary resonances*. Denote $s_0(n) = s(j_0, n)$. We call *secondary resonances* integer vectors belonging to any of the remaining sequences $s(j,n)$, $j \neq j_0$. We normalize the limits γ_j^* dividing by $\gamma_{j_0}^*$

$$\tilde{\gamma}_{j_0}^* = 1, \quad \tilde{\gamma}_j^* = \frac{\gamma_j^*}{\gamma_{j_0}^*}. \tag{6.6}$$

and define a parameter $\tilde{\gamma}^{**}$ measuring the *separation* between the primary and secondary resonances

$$\tilde{\gamma}^{**} = \min_{j\in\mathcal{P}\backslash\{j_0\}} \tilde{\gamma}_j^*$$

Remark 6.2. We implicitly assume the hypothesis that the primitive j_0 is unique, and hence $\tilde{\gamma}^{**} > 1$. In fact, this happens for all the cases we have explored, provided we choose the matrix T suitably.

6.2 Asymptotic estimates

We need to show that the Poincaré-Melnikov method is applied in our singular case $\mu = \varepsilon^p$ $(p > 0)$. To do this we provide asymptotic estimates (or lower bounds) for the dominant harmonics of the Melnikov potential $L(\theta)$ and prove that the corresponding dominant harmonics of the splitting potential $\mathcal{L}(\theta)$ overcome the remaining terms as well as the error term of the Poicaré-Melnikov approach. It is convenient for us to work with the scalar functions L and $\mathcal{L}$, but we state our main results in terms of the splitting function $\mathcal{M}$ (recall $\mathcal{M}(\theta) = \partial_\theta \mathcal{L}(\theta)$) whose values coincide with the distance between the invariant manifolds of the whiskered torus.

We put our functions f and h defined in (5.6) and (5.7) into the integral (5.12) and get the Fourier expansion of the Melnikov potential

$$L(\theta) = \sum_{k \in \mathcal{Z}\setminus\{0\}} L_k \cos(\langle k, \theta\rangle - \sigma_k)$$

with the Fourier coefficients

$$L_k = \frac{2\pi |\langle k, \hat{\omega}_\varepsilon\rangle| e^{-\rho|k|}}{b \sinh |\frac{\pi}{2}\langle k, \hat{\omega}_\varepsilon\rangle|}. \tag{6.7}$$

Remark 6.3. Note that due to the presence of $\sinh|\cdot|$ in (6.7) coefficients L_k turn out to be exponentially small in ε.

Recall that

$$|\langle k, \hat{\omega}_\varepsilon\rangle| = |\langle k, \frac{b'\omega}{b\sqrt{\varepsilon}}\rangle| = \frac{b'\gamma_k}{b|k|\sqrt{\varepsilon}}.$$

We present the coefficients L_k in the exponential form

$$L_k = \alpha_k e^{-\beta_k}, \; k \in \mathbb{Z}^2\setminus\{0\}, \tag{6.8}$$

where

$$\alpha_k = \frac{4\pi b'\gamma_k}{b^2|k|\sqrt{\varepsilon}(1 - e^{-\{\pi \frac{b'\gamma_k}{b|k|\sqrt{\varepsilon}}\}})}, \quad \beta_k = \rho|k| + \frac{\pi b'\gamma_k}{2b|k|\sqrt{\varepsilon}}. \tag{6.9}$$

Thus, the largest coefficients L_k are given by the smallest exponents β_k. We can present β_k in more convenient form

$$\beta_k = \frac{C_\mu}{\varepsilon^{1/4}} g_k(\varepsilon), \tag{6.10}$$

where we write the functions g_k in the form

$$g_k(\varepsilon) = \frac{\sqrt{\tilde{\gamma}_k}}{2}\left[\left(\frac{\varepsilon}{\varepsilon_k}\right)^{1/4} + \left(\frac{\varepsilon_k}{\varepsilon}\right)^{1/4}\right], \; \varepsilon_k^{1/4} = \frac{C_\mu}{2\rho}\frac{\sqrt{\tilde{\gamma}_k}}{|k|}, \; C_\mu = \sqrt{\frac{2\pi b'}{b}\rho\gamma_{j_0}^*}, \; \tilde{\gamma}_k = \frac{\gamma_k}{\gamma_{j_0}^*}. \tag{6.11}$$

Here we denote C_μ to depend implicitly on μ, since b and b' are μ-close to 1. Indeed, C_μ is μ-close to the constant

$$C_0 = \sqrt{2\pi\rho\gamma^*_{j0}} \tag{6.12}$$

with γ^*_{j0} given in (6.5). Notice that the functions g_k contain the main information on the size of β_k.

For any ε fixed we have to find the dominant terms L_k and the corresponding vectors k. Since L_k are exponentially small in ε, it is more convenient to work with the functions g_k whose smallest values correspond to to the largest L_k. To this aim we represent the functions g_k, $k \in \mathbb{Z}^2\backslash\{0\}$, in a figure (see, for example, the Figure 6.1 for Ω_2) and for every ε fixed we find the vectors $S_i = S_i(\varepsilon)$, $i = 1, 2, \ldots$ minimizing the functions such that

$$g_{S_1}(\varepsilon) \leq g_{S_2}(\varepsilon) \leq g_{S_3}(\varepsilon) \leq \ldots$$

Hence the dominant harmonics of $L(\theta)$ will be L_{S_1}, L_{S_2}, L_{S_3}, etc.

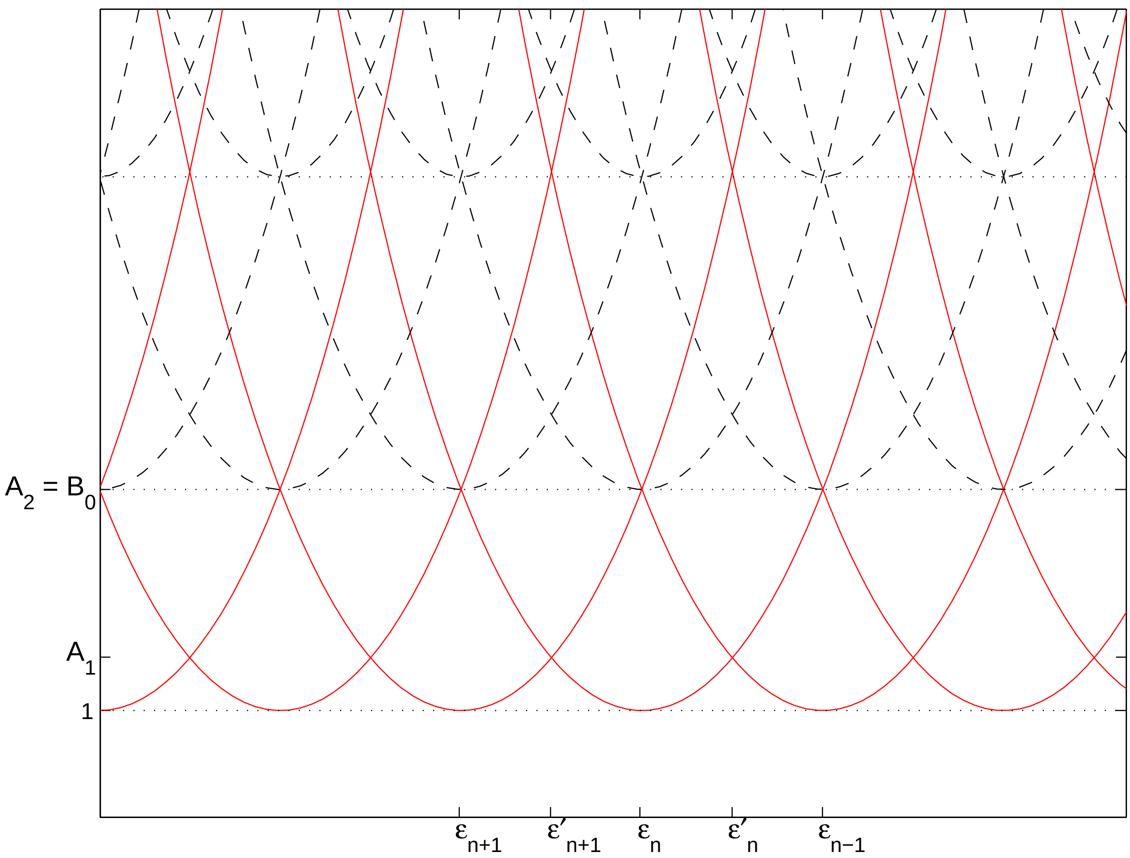


Figure 6.1: Graphs of the functions $g_k(\varepsilon)$, $k \in \mathbb{Z}^2\backslash\{0\}$, for Ω_2 using a logarithmic scale for ε. The ones with solid lines are primary functions $\widehat{g}_n(\varepsilon)$

Note that the functions g_k have their minimum at $\varepsilon = \varepsilon_k$ and the corresponding minimal values are $g_k(\varepsilon_k) = \sqrt{\tilde{\gamma}_k}$. Recall (see the Section 6.1) that all the admissible k can be subdivided into the sequences of vectors $s(j,n)$ defined in (6.4) according to their limit values $\tilde{\gamma}_j^*$. The primary integer vectors k belonging to the sequence $s_0(n) = s(j_0, n)$ play an important rôle here, since $\tilde{\gamma}_{j_0}^* = 1$, and, thus, they give the smallest g_k (at least near the minimum points $\varepsilon_{s_0(n)}$). Denote the primary functions by

$$\widehat{g}_n(\varepsilon) := g_{s_0(n)}(\varepsilon)$$

and also the minimum points of $\widehat{g}_n$ by ε_n and the intersection points between $\widehat{g}_n$ and $\widehat{g}_{n-1}$ by ε'_n. Hence, we get the geometric sequences for the points

$$\varepsilon_n := \varepsilon_{s_0(n)} = \left(\frac{C_0}{2\rho K(j_0)|\lambda|^{n-1}}\right)^4 = \frac{\varepsilon_1}{\lambda^{4n}}, \quad \varepsilon'_n := \sqrt{\varepsilon_n \varepsilon_{n-1}} = \frac{\varepsilon_1}{\lambda^{4n-2}}. \tag{6.13}$$

Notice that the following scaling property is fulfilled:

$$\widehat{g}_n(\varepsilon) = \widehat{g}_{n-1}(\lambda^4 \varepsilon) = \widehat{g}_0(\lambda^{4n}\varepsilon).$$

This implies that as a function of $\ln \varepsilon$, the graph of g_n is simply the graph of g_0 translated a distance $4n \ln|\lambda|$, thus, the representation in Figure 6.1 (that uses a logarithmic scale for ε) is $4\ln|\lambda|$-periodic. Thus, it is sufficient to draw figures for a width $4\ln|\lambda|$ as in Figures 6.2-6.2.2.

We define the constants (the so-called *levels*)

$$A_i = \frac{1}{2}(|\lambda|^{i/2} + |\lambda|^{-i/2}).$$

Note that $A_0 = \widehat{g}_n(\varepsilon_n) = 1$ is the minimal value of $\widehat{g}_n$ and, for $i \geq 1$, A_i is the value of $\widehat{g}_n$ at the intersections points of $\widehat{g}_n$ and $\widehat{g}_{n+i}$, for example $A_1 = \widehat{g}_n(\varepsilon'_n)$. Denote by B_0 the minimal value of the secondary functions $g_{s(j,n)}$, $j \neq j_0$. It is clear that $B_0 = \tilde{\gamma}^{**} > 1$. We are interested in the frequencies Ω satisfying the condition

$$B_0 \geq A_1, \tag{6.14}$$

that ensures that the most dominant harmonic for all ε is found among the primary resonances. Numerical explorations indicate that the condition (6.14) is satisfied only by a finite number of 1 or 2-periodical continued fractions, namely, for the 24 quadratic irrational numbers

$$\Omega_1, \ldots, \Omega_{13}, \Omega_{1,2}, \ldots, \Omega_{1,12}. \tag{6.15}$$

Remark 6.4. It is clear that, for example, $\Omega_{2,1}$ also satisfies the condition (6.14), but we can present it by means $\Omega_{1,2}$ as

$$\Omega_{2,1} = [\overline{2,1}] = [2, \overline{1,2}] = \frac{\Omega_{1,2}}{1 + 2\Omega_{1,2}}.$$

Thus, they are equivalent (see Remark 6.1) and, for ε small enough, the graphs of $g_k(\varepsilon)$ in the case of $\Omega_{2,1}$ will be similar to the ones of $\Omega_{1,2}$.

We introduce the parameter *ind* satisfying the equation

$$\frac{1}{2}(|\lambda|^{ind/2} + |\lambda|^{-ind/2}) = B_0. \tag{6.16}$$

Then the condition (6.14) may be expressed as $ind \geq 1$.

6.2.1 Asymptotic estimates for the splitting distance

Under the condition (6.14), we can ensure that the function giving the values of the minimum

$$h_1(\varepsilon) = \min_k g_k(\varepsilon) = g_{S_1}(\varepsilon)$$

is given by the primary vectors $S_1 = s(j_0, \cdot)$ (see, for example, Figure 6.2 for Ω_2) and we find that $j_0 = 1$ for the 24 numbers of (6.15). We can rewrite the function as (note that $\tilde{\gamma}^*_{j_0} = 1$)

$$h_1(\varepsilon) = \widehat{g}_n(\varepsilon) = \frac{1}{2}\left(\left(\frac{\varepsilon}{\varepsilon_n}\right)^{1/4} + \left(\frac{\varepsilon_n}{\varepsilon}\right)^{1/4}\right), \quad \varepsilon \in [\varepsilon'_{n+1}, \varepsilon'_n], n \geq 1, \tag{6.17}$$

extended as a $4\ln|\lambda|$-periodic function of $\ln\varepsilon$. This function is continuous for all $0 < \varepsilon < \varepsilon'_1$ and $\min h_1(\varepsilon) = h_1(\varepsilon_n) = 1$ and $\max h_1(\varepsilon) = h_1(\varepsilon'_n) = A_1 > 1$. Note that the vector S_1 changes at the points ε'_n from $s(1, n+1)$ to $s(1, n)$.

In the following theorem we provide an estimate for the maximal distance between the stable and unstable invariant manifolds in terms of the maximum value of the splitting function $\mathcal{M}(\theta)$. The maximum value of $\mathcal{M}$ is given by the most dominant harmonic having the function $h_1(\varepsilon)$ as exponent. The estimate obtained shows that the splitting of separatrices exists. Note that we have introduced the notation of '$\sim$' just before Theorem 5.1 in Chapter 5.

Theorem 6.1 ((Maximal) splitting distance)**.** *For the Hamiltonian system (5.1-5.7) with $n = 2$, assume that $\varepsilon \preceq 1$ and $\mu = \varepsilon^p$, $p > p^*$ with $p^* = 2$ if $\nu = 1$ and $p^* = 3$ if $\nu = 0$, then in the 24 quadratic numbers (6.15), the following estimate holds*

$$\max_{\theta\in\mathbb{T}^2} |\mathcal{M}(\theta)| \sim \frac{\mu}{\sqrt{\varepsilon}} \exp\left\{-\frac{C_0 h_1(\varepsilon)}{\varepsilon^{1/4}}\right\}$$

where the constant C_0 is defined in (6.12) and the function $h_1(\varepsilon)$ is the periodic function in $\ln\varepsilon$ defined in (6.17) which satisfies $\min h_1(\varepsilon) = 1$ and $\max h_1(\varepsilon) = A_1 > 1$.

6.2.2 Asymptotic estimates for the transversality of the splitting

In order to show that $\mathcal{L}(\theta)$ has nondegenerate critical points, we need to consider at least 2 dominant harmonics of its Fourier expansion. However, for some values of ε it

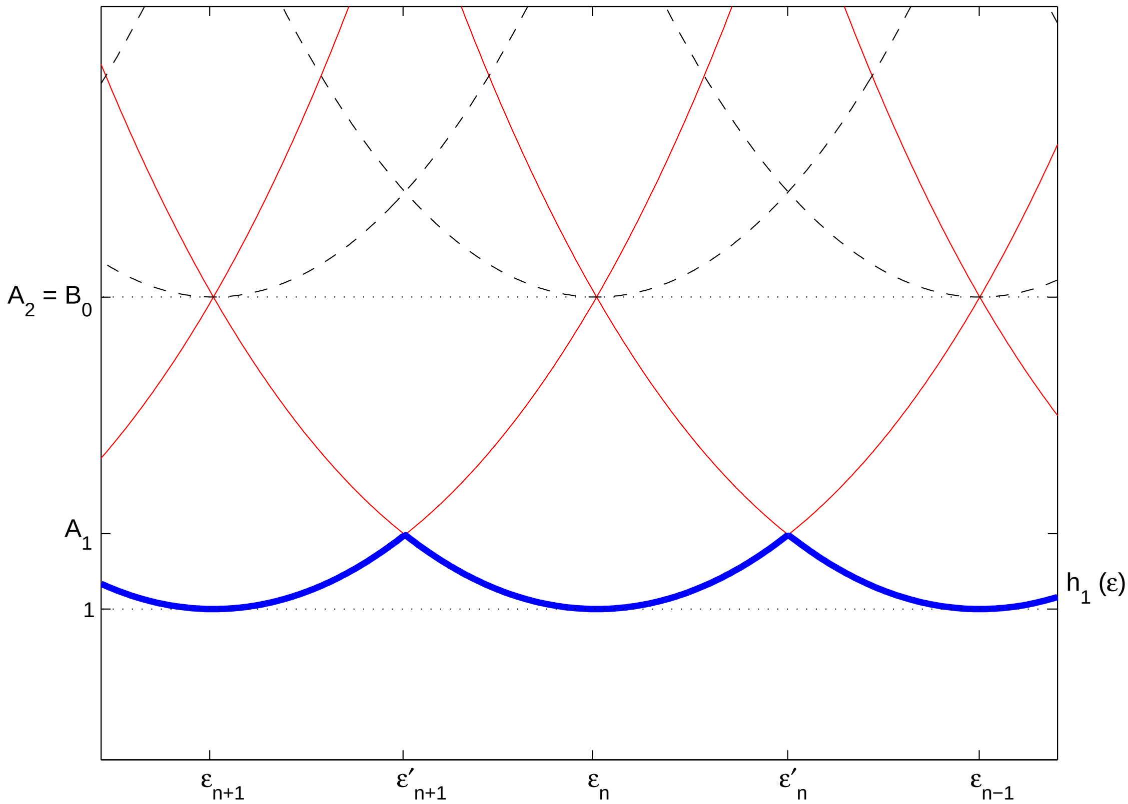


Figure 6.2: Graphs of the functions $g_k(\varepsilon)$ and $h_1(\varepsilon)$ defined in (6.17) for Ω_2 ($ind = 2$ where parameter ind is introduced in (6.16)).

is necessary to consider 3 and even more dominant harmonics if the second and some consecutive harmonics are of the same magnitude. Also it can happen that the corresponding vectors S_1 and S_2 of the 2 dominant harmonics are linearly dependent, thus, to prove the nondegeneracity of the critical points we have to consider enough consecutive dominant terms $L_{S_1}, L_{S_2}, \ldots, L_{S_m}, L_{S'}$ to have 2 linearly independent vectors S_1 and S', while $S_2, \ldots, S_m$ are dependent with S_1 (the number $m \geq 1$ depends on ε).

Definition 6.1. We will call L_{S_1} and $L_{S'}$ *essential* dominant harmonics if they satisfy:
(i) L_{S_1} is the most dominant harmonic,
(ii) S_1 and S' are independent,
(iii) if there are harmonics $L_{S_2}, \ldots, L_{S_m}$ such that $L_{S_1} \leq L_{S_2} \leq \ldots \leq L_{S_m} \leq L_{S'}$, then

$$S_i = c_i S_1, i = 2, \ldots, m \tag{6.18}$$

with constants $c_i > 1$.

We will define the number m (*index of non-essentiality*) as

$$m = \begin{cases} 1, & \text{if there are no non-essential harmonics between } L_{S_1} \text{ and } L_{S'} \\ l+1, & \text{if there are } l \text{ non-essential harmonics between } L_{S_1} \text{ and } L_{S'} \end{cases} \tag{6.19}$$

In the worst cases of 24 numbers (6.15), Ω_{13}, $\Omega_{1,11}$, $\Omega_{1,12}$, $m = 6$ and $c_i = i$ for $i = 2, \ldots, 6$. As we show later, the terms $L_{S_2}, \ldots, L_{S_m}$ are not relevant for the transversality.

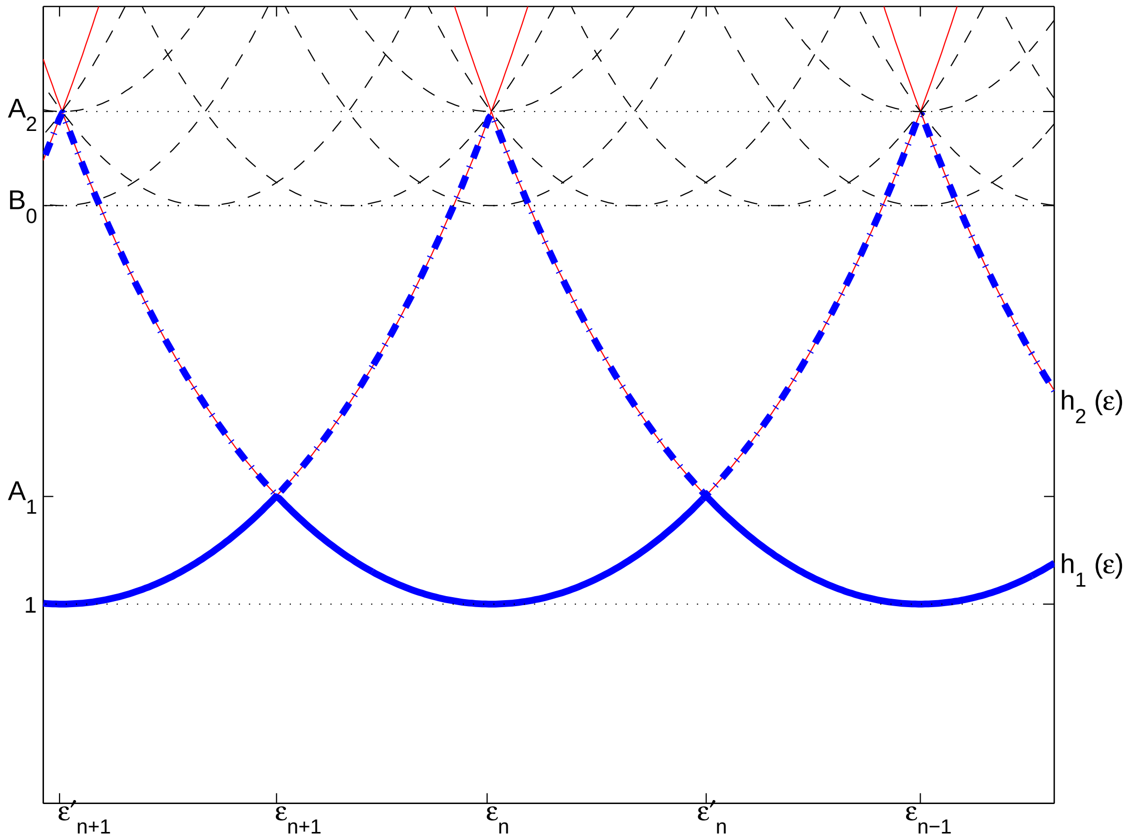


Figure 6.3: Functions h_1 and h_2 for Ω_4 ($ind = 1.8245$) in the logarithmic scale of ε

Therefore, we define the function

$$h_2(\varepsilon) = g_{S'}(\varepsilon),$$

where S' is the first vector linearly independent with S_1 minimizing the functions g_k. Thus, $h_2(\varepsilon)$ is defined by 2 vectors S_1 and S'. Note that $S' = S'(\varepsilon)$ changes if ε varies, and later we will discuss these critical values of ε at which S' changes.

We stress that S_1 is always a primary vector, whereas S' can be either a primary vector or a secondary one. Such situations can be checked from the graphics corresponding to the 24 numbers (6.15). We have found two different situations:

(a) $\Omega_1, \Omega_2, \ldots, \Omega_{13}$ – for all ε the vector S' is also a primary vector. More precisely, $S' = s(1, n \pm 1)$, and the critical values of ε when S' changes are $\varepsilon'_{n+1}, \varepsilon_n, \varepsilon'_n$ (see,

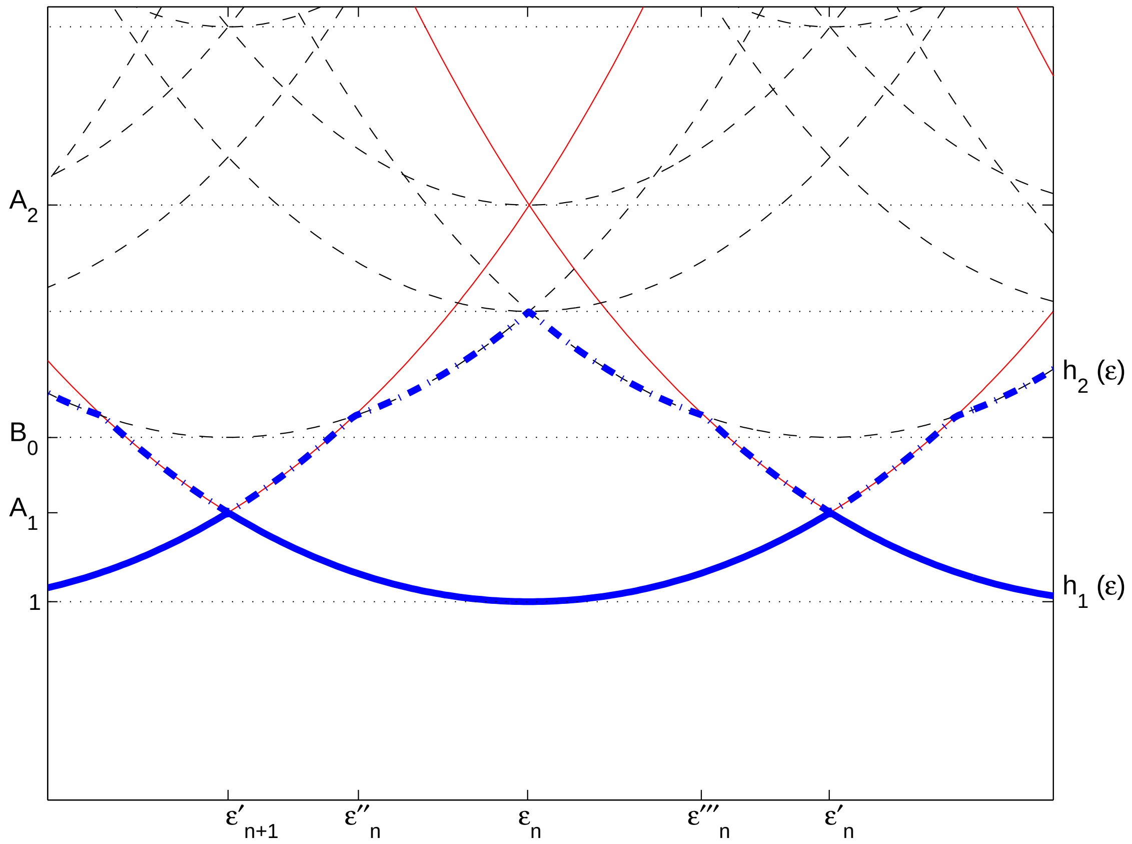


Figure 6.4: Graphs of the functions h_1 and h_2 for $\Omega_{1,2}$ ($ind = 1.3385$).

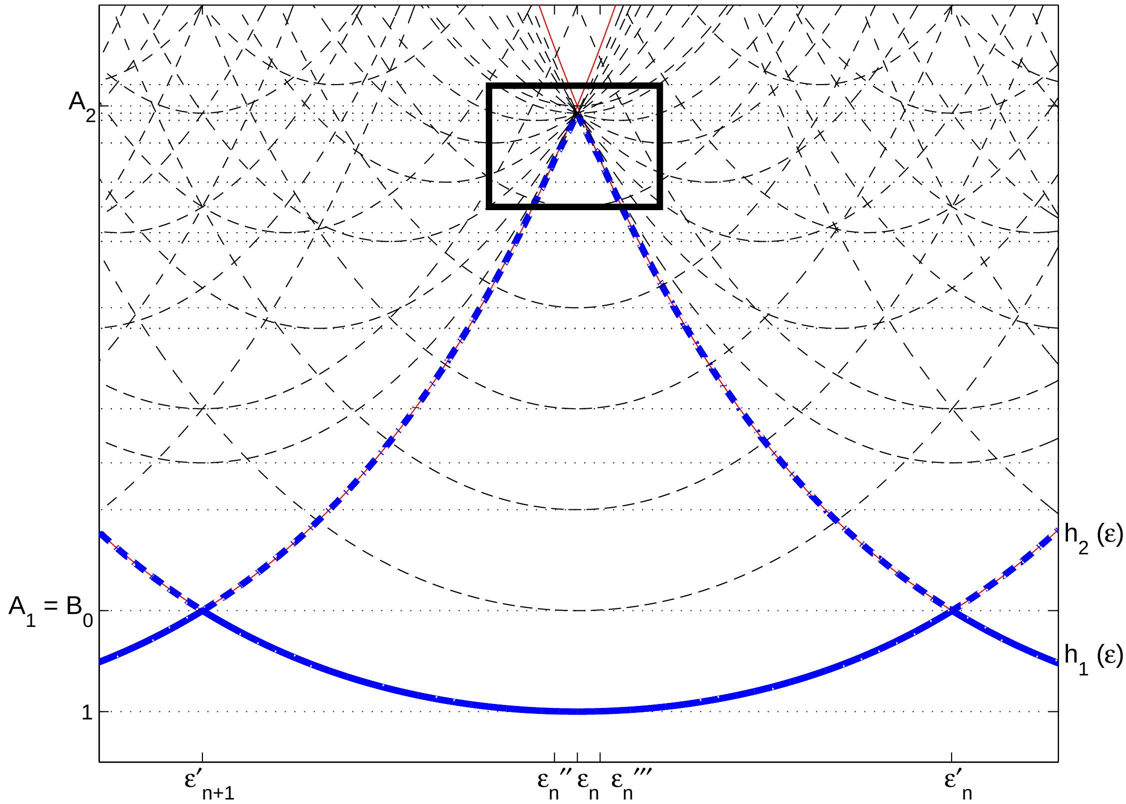
for example, the Figure 6.3 for the number Ω_4). We can write $h_2(\varepsilon)$ in the form

$$h_2(\varepsilon) = \begin{cases} \widehat{g}_{n+1}(\varepsilon), & \varepsilon \in [\varepsilon'_{n+1}, \varepsilon_n] \\ \widehat{g}_{n-1}(\varepsilon), & \varepsilon \in [\varepsilon_n, \varepsilon'_n] \end{cases} \tag{6.20}$$

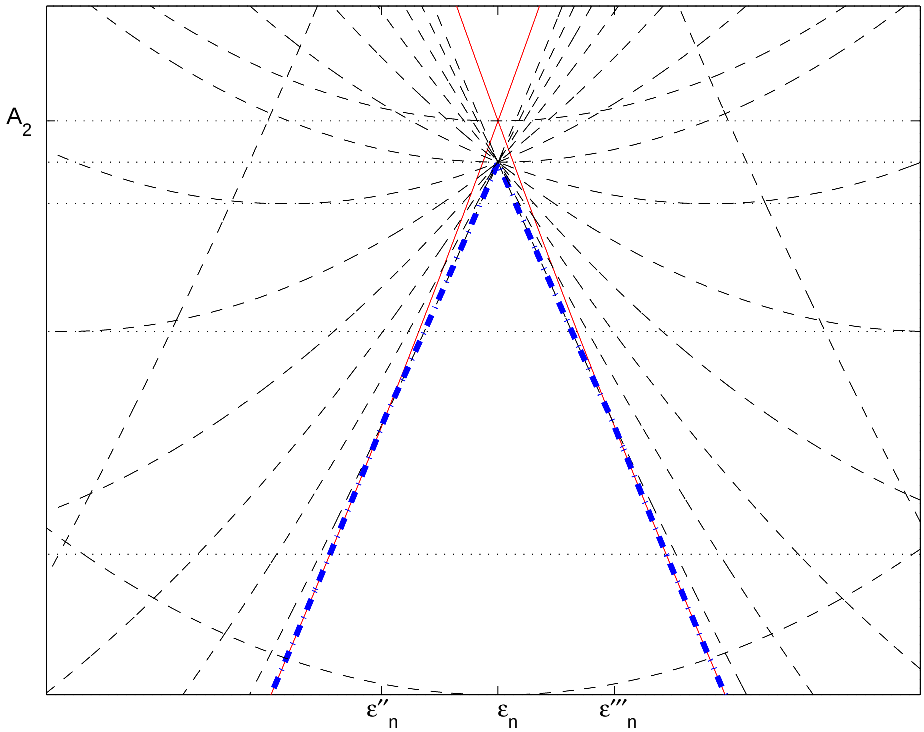
Note that this function is continuous, $4\ln|\lambda|$-periodic in $\ln\varepsilon$ and satisfies that $\min h_2(\varepsilon) = h_2(\varepsilon'_n) = A_1$ and $\max h_2(\varepsilon) = h_2(\varepsilon_n) = A_2$.

(b) $\Omega_{1,2}, \ldots, \Omega_{1,12}$ – for some values of ε, $S' = s(1, n\pm 1)$ is one of the primary vectors, while for other ε the vector $S' = s(j', n)$ is a secondary resonance(see, for example, the Figure 6.4 for the number $\Omega_{1,2}$). We denote by ε''_n and ε'''_n the points at which S' changes from a primary vector to a secondary one and viceversa

$$\begin{aligned} \varepsilon''_n &= \left(\frac{\sqrt{\tilde{\gamma}_{j'}}\varepsilon^{1/4}_{s(j',n)} - \varepsilon^{1/4}_{n+1}}{\varepsilon^{1/4}_{s(j',n)} - \sqrt{\tilde{\gamma}_{j'}}\varepsilon^{1/4}_{n+1}} \right)^2 \varepsilon^{1/2}_{n+1}\varepsilon^{1/2}_{s(j',n)}, \\ \varepsilon'''_n &= \left(\frac{\sqrt{\tilde{\gamma}_{j'}}\varepsilon^{1/4}_{s(j',n-1)} - \varepsilon^{1/4}_{n-1}}{\varepsilon^{1/4}_{s(j',n-1)} - \sqrt{\tilde{\gamma}_{j'}}\varepsilon^{1/4}_{n-1}} \right)^2 \varepsilon^{1/2}_{n-1}\varepsilon^{1/2}_{s(j',n-1)}. \end{aligned} \tag{6.21}$$



(a)



(b)

Figure 6.5: Graphs of functions h_1 and h_2 for $\Omega_{1,12}$ ($ind = 1$).

Note that $\varepsilon_{s(j',n)}$ is the minimum point of the function $g_{s(j',n)}$, one can obtain their formulae from (6.11), putting $k = s(j',n)$. Also S' changes from a secondary vector $s(j',n)$ to another secondary vector $s(j',n-1)$ at ε_n, the minimum point of the primary function $\widehat{g}_n$ given in (6.13). Hence, the critical values of ε are $\varepsilon'_{n+1}, \varepsilon''_n, \varepsilon_n, \varepsilon'''_n, \varepsilon'_n$. In the case of these numbers $h_2(\varepsilon)$ takes form

$$h_2(\varepsilon) = \begin{cases} \widehat{g}_{n+1}(\varepsilon), & \varepsilon \in [\varepsilon'_{n+1}, \varepsilon''_n] \\ g_{s(j',n)}(\varepsilon), & \varepsilon \in [\varepsilon''_n, \varepsilon_n] \\ g_{s(j',n-1)}(\varepsilon), & \varepsilon \in [\varepsilon_n, \varepsilon'''_n] \\ \widehat{g}_{n-1}(\varepsilon), & \varepsilon \in [\varepsilon'''_n, \varepsilon'_n] \end{cases} \tag{6.22}$$

extended to all ε as a $4\ln|\lambda|$-periodic of $\ln\varepsilon$. It is also continuous and satisfies $\min h_2(\varepsilon) = h_2(\varepsilon'_n) = A_1 > 1$ and $B_0 < \max h_2(\varepsilon) = h_2(\varepsilon_n) < A_2$.

The function $h_2(\varepsilon)$ is relevant to establish the transversality of homoclinic orbits associated to the whiskered torus considered. In the following theorem we prove the existence of 4 transverse homoclinic orbits which correspond to simple zeros of the splitting function $\mathcal{M}(\theta)$. We also give an estimate for the minimal eigenvalue (in modulus) of the splitting matrix $\partial_\theta\mathcal{M}$ at each zero in terms of the function $h_2(\varepsilon)$. This eigenvalue provides a measure of the transversality of the homoclinic orbits. Note that the notation of '$\sim$' has been introduced just before Theorem 5.1 in Chapter 5.

Theorem 6.2 (Transversality of the splitting)**.** *For the Hamiltonian system (5.1-5.7) with $n = 2$, assume that $0 < \varepsilon \ll 1$ and $\mu = \varepsilon^p$, $p > p^*$ with $p^* = 2$ if $\nu = 1$ and $p^* = 3$ if $\nu = 0$, then one has:*

- *the Melnikov function $\mathcal{M}(\theta)$ has exactly 4 zeros θ_*, all simple, for all ε except for some small neighbourhood of some geometric sequences of ε;*
- *The minimal eigenvalue of $\partial_\theta\mathcal{M}(\theta_*)$ satisfies*

$$m_* \sim \mu\varepsilon^{1/4} \exp\left\{-\frac{C_0 h_2(\varepsilon)}{\varepsilon^{1/4}}\right\}$$

where the constant C_0 is defined in (6.12) and $h_2(\varepsilon)$ is the positive periodic in $\ln\varepsilon$ function defined in (6.20) for the numbers $\Omega_1, \Omega_2, \ldots, \Omega_{13}$ and in (6.22) for the numbers $\Omega_{1,2}, \Omega_{1,3}, \ldots, \Omega_{1,12}$.

Remark 6.5. The geometric sequences mentioned in the Theorem 6.2 are ε_n (given in (6.13)), ε''_n and ε'''_n (given in (6.21)) where the Melnikov function has more than 2 essential dominant harmonics because the second essential harmonic coincides with the third one, and this requires a special study (as an illustration, we carry out this study for $\Omega_2 = \sqrt{2} - 1$ in Section 6.3). We can conjecture that depending on the type of perturbation (function f) some bifurcations are possible to occur at these points.

In the rest part of this section we give estimates for the dominant harmonics of the Melnikov potential L and show that the sum of the remaining terms is much smaller (Lemma 6.1, p. 132), then we translate these estimates to the splitting potential $\mathcal{L}$ (Lemma 6.2, p. 138), and, finally, we find 4 nondegenerate critical points of $\mathcal{L}$. All this will allow us to prove our main results Theorems 6.1 and 6.2, taking into account that $\mathcal{M} = \partial_\theta \mathcal{L}$.

6.2.3 Dominant harmonics of the Melnikov potential

To show that the Melnikov potential L has nondegenerate critical points, we have to consider at least the 2 essential dominant harmonics L_{S_1} and $L_{S'}$ (see Definition 6.1 of essential dominant harmonics) in its Fourier expansion and give estimates for them to prove that they overcome the sum of the remaining terms. Also we have to take into account that, for some intervals of ε, we can have non-essential dominant terms $L_{S_2}, \dots, L_{S_m}$ between L_{S_1} and $L_{S'}$ such that $L_{S_1} \geq L_{S_2} \geq \dots \geq L_{S_m} \geq L_{S'}$ and $S_2, \dots, S_m$ (the number $m \geq 1$ depends on ε) are dependent with S_1 (recall that S_1 and S' are independent). Since these vectors are linearly dependent with S_1, they satisfy the relations (6.18).

We fix a point ε, for any primitive j and $N = N(j, \varepsilon) \geq 1$ let $\varepsilon_{s(j,N)}$ defined in (6.11) be the nearest point to ε. As a consequence of Proposition 6.1 we have estimates

$$|s(j, N)| \sim K(j)|\lambda|^N,$$

and applying the definition of ε_k for $k = s(j, N)$ in (6.11), one gets

$$\varepsilon \approx \varepsilon_{s(j,N)} \sim \left(\frac{\sqrt{\tilde{\gamma}_j^*}}{|\lambda|^N K(j)} \right)^4 .$$

From this, one concludes

$$|s(j, N)| \sim \sqrt{\tilde{\gamma}_j^*} \varepsilon^{-1/4}.$$

Note that the vectors S_1, $S_2, \dots, S_m$ and S', if secondary, are dominant in their sequences, i.e. $S_1 = s(j_0, N)$, $S_i = s(j_i, N), i = 2, \dots, m$, $S' = s(j', N)$ if $j' \neq j_0$. If S' is primary, then it is one of the vectors $s(j_0, N \pm 1)$ depending on the side of $\varepsilon_{s(j_0,N)}$ to which ε belongs: $S' = s(j_0, N - 1)$ if $\varepsilon > \varepsilon_{s(j_0,N)}$ and $S' = s(j_0, N + 1)$ if $\varepsilon < \varepsilon_{s(j_0,N)}$. Then having that for primary vectors $\tilde{\gamma}_{j_0}^* = 1$ and choosing, if necessary, from a finite number of $\tilde{\gamma}_{j_2}^*, \dots, \tilde{\gamma}_{j_m}^*, \tilde{\gamma}_{j'}^*$, we obtain the following estimates:

$$|S_1| \sim \varepsilon^{-1/4}, \quad |S'| \sim \varepsilon^{-1/4}, \quad |S_i| = c_i |S_1| \sim \varepsilon^{-1/4} \tag{6.23}$$

In the next lemma we give estimates for $L_{S_i}, i = 1, \dots, m$, and $L_{S'}$. Besides, we provide an estimate for the sum of all the remaining terms L_k in terms of the first neglected harmonics. Since we are interested in some derivatives of the Melnikov potential, we consider the sum of (positive) amounts of the type $|k|^l L_k$

Lemma 6.1. *Assuming $\varepsilon \preceq 1$, one has:*

(a) $L_{S_1} \sim \frac{1}{\varepsilon^{1/4}} \exp\left\{-\frac{C_0 h_1(\varepsilon)}{\varepsilon^{1/4}}\right\}$,
$L_{S'} \sim \frac{1}{\varepsilon^{1/4}} \exp\left\{-\frac{C_0 h_2(\varepsilon)}{\varepsilon^{1/4}}\right\}$,
$L_{S_i} \sim \frac{c_i}{\varepsilon^{1/4}} \exp\left\{-c_i\frac{C_0 h_1(\varepsilon)}{\varepsilon^{1/4}}\right\} \ll L_{S_1}, i = 2, \dots, m;$

(b) For any primitive j, if $\varepsilon \approx \varepsilon_{s(j,N)}, N = N(j,\varepsilon)$, then

$$\sum_{n\geq 1} |s(j,n)|^l L_{s(j,n)} \preceq |s(j,N)|^l L_{s(j,N)};$$

(c) $\sum_{k\neq S_1, S', S_i} |k|^l L_k \preceq \frac{1}{\varepsilon^{l/4}} L_{S_{m+2}}, \quad i = 1, \dots, m, l \geq 0.$

Proof. According to (6.8)-(6.9), the largest coefficients L_k are given essentially by the smallest exponents β_k. Due to (6.10) and to the fact the smallest value of g_k for any ε fixed is given by $k = S_1$ and that this smallest value coincides with the function $h_1(\varepsilon)$, we get that

$$\beta_{S_1} = \frac{C_0 h_1(\varepsilon)}{\varepsilon^{1/4}}.$$

Analogously, by definition of the function $h_2(\varepsilon)$, we obtain

$$\beta_{S'} = \frac{C_0 h_2(\varepsilon)}{\varepsilon^{1/4}}.$$

For the non-essential dominant terms with the vectors $S_2, \dots, S_m$, due to the relations (6.18), we have

$$\gamma_{S_i} = c_i^2 \gamma_{S_1}, \quad \varepsilon_{S_i} = \varepsilon_{S_1}, \quad g_{S_i}(\varepsilon) = c_i g_{S_1}(\varepsilon), \quad \beta_{S_i} = c_i \beta_{S_1} = c_i \frac{C_0 h_1(\varepsilon)}{\varepsilon^{1/4}}.$$

Once we have found the smallest exponents β_k, we show that the dominance in L_k is not affected by the multiplicative term α_k defined in (6.9). Indeed, in general if the denominator $[1 - \exp\{\dots\}] \sim 1$, we deduce

$$\alpha_k \sim \frac{\gamma_k}{|k|\sqrt{\varepsilon}} = \frac{2b}{\pi b'}\left(\rho|k| + \frac{\pi b' \gamma_k}{2b|k|\sqrt{\varepsilon}}\right) - \frac{2b}{\pi b'}\rho|k| \preceq \beta_k,$$

and hence $|\ln \alpha_k| \ll \beta_k$. The only exception can occur if $[1 - \exp\{\dots\}]$ is too small, that only happens if $|k| \succeq \frac{\gamma_k}{\sqrt{\varepsilon}}$, and since $\frac{x}{1-exp\{-x\}} \to 1$ as $x \to 0$, one can obtain that $\alpha_k \preceq 1$ in this case. For the dominant harmonics $L_{S_1}, L_{S'}, L_{S_i}, i = 2, \dots, m$, we see that $[1 - \exp\{\dots\}] \sim 1$. Besides, S_1 is always primary with $\tilde{\gamma}_{j_0} = 1$ and, thus, using (6.23), we deduce the following estimate: $\alpha_{S_1} \sim \varepsilon^{-1/4}$. If S' is primary (that it is true for $\Omega_1, \ldots, \Omega_{13}$), then the estimates are similar to the last one. In the case if S' is a secondary vector (it can be for some ε in the case $\Omega_{1,2}, \ldots, \Omega_{1,12}$), we bound $\gamma^*_{j'}$ by its maximum value among all these 11 numbers and, hence, $\alpha_{S'} \sim \varepsilon^{-1/4}$. Concerning to the non-essential vectors, we get $\gamma_{S_i} = c_i^2 \gamma_{S_1}$, $|S_i| = c_i |S_1|$, and, thus, $\alpha_{S_i} \sim c_i \varepsilon^{-1/4}$, $i = 2, \ldots, m$. These estimates imply part (a).

To prove (b), first we bound the sum of the terms L_k for k belonging to any fixed resonant sequence $s(j,\cdot)$ (we fix j) by its dominant term $L_{s(j,N)}$ (recall that we have fixed ε near to $\varepsilon_{s(j,N)}$). Consider other vectors $s(j,n)$ of the same sequence with $n \neq N$. For $n > N$, we have an infinite number of points $\varepsilon_{s(j,n)} < \varepsilon_{s(j,N)}$ and we calculate (using $\varepsilon_{s(j,n+1)} = |\lambda|^{-4} \varepsilon_{s(j,n)}$)

$$\begin{aligned} g_{s(j,n+1)} - g_{s(j,n)} &= \frac{\sqrt{\tilde{\gamma}^*_j}}{2}(|\lambda|-1)\left[\left(\frac{\varepsilon}{\varepsilon_{s(j,n)}}\right)^{1/4} - \frac{1}{|\lambda|}\left(\frac{\varepsilon_{s(j,n)}}{\varepsilon}\right)^{1/4}\right] \\ &\approx \frac{\sqrt{\tilde{\gamma}^*_j}}{2}(|\lambda|-1)\left[|\lambda|^{n-N} - \frac{1}{|\lambda|}|\lambda|^{N-n}\right] \\ &\geq \begin{cases} \frac{\sqrt{\tilde{\gamma}^*_j}}{2}(|\lambda|-1)(1-\frac{1}{|\lambda|}) & \text{if } \varepsilon > \varepsilon_{s(j,N)} \\ \frac{\sqrt{\tilde{\gamma}^*_j}}{2}(|\lambda|-1)(|\lambda|-\frac{1}{\lambda^2}) & \text{if } \varepsilon < \varepsilon_{s(j,N)} \end{cases} \\ &\geq \sqrt{\tilde{\gamma}^*_j} C_\lambda, \quad C_\lambda = \frac{(|\lambda|-1)^2}{2|\lambda|}, \end{aligned}$$

since the difference is growing and, thus, is greater than the minimal difference at $n - N = 0$ if $\varepsilon > \varepsilon_{s(j,N)}$ or $n - N = 1$ if $\varepsilon < \varepsilon_{s(j,N)}$; also $1 - \frac{1}{|\lambda|} < |\lambda| - \frac{1}{\lambda^2}$ for $|\lambda| > 1$. Thus, we get that

$$g_{s(j,N+l_1)}(\varepsilon) \geq g_{s(j,N)}(\varepsilon) + \sqrt{\tilde{\gamma}^*_j} C_\lambda l_1, \tag{6.24}$$

where $l_1 \geq 0$ if $\varepsilon > \varepsilon_{s(j,N)}$ or $l_1 \geq 1$ if $\varepsilon < \varepsilon_{s(j,N)}$.

For $n < N$, we have $\varepsilon_{s(j,n)} > \varepsilon_{s(j,N)}$ that are a finite number of points, and we get

$$\begin{aligned} g_{s(j,n)} - g_{s(j,n+1)} &\approx \frac{\sqrt{\tilde{\gamma}^*_j}}{2}(|\lambda|-1)\left[\frac{1}{|\lambda|}|\lambda|^{N-n} - |\lambda|^{n-N}\right] \\ &\geq \begin{cases} \frac{\sqrt{\tilde{\gamma}^*_j}}{2}(|\lambda|-1)\left[1-\frac{1}{|\lambda|}\right] & \text{if } \varepsilon < \varepsilon_{s(j,N)} \\ \frac{\sqrt{\tilde{\gamma}^*_j}}{2}(|\lambda|-1)(|\lambda|-\frac{1}{\lambda^2}) & \text{if } \varepsilon > \varepsilon_{s(j,N)} \end{cases} \\ &\geq \sqrt{\tilde{\gamma}^*_j} C_\lambda, \end{aligned}$$

and, therefore, we have

$$g_{s(j,N-l_2)}(\varepsilon) \geq g_{s(j,N)}(\varepsilon) + \sqrt{\tilde{\gamma}^*_j} C_\lambda l_2, \tag{6.25}$$

where $l_2 = 0, \ldots, N-1$ if $\varepsilon < \varepsilon_{s(j,N)}$ or $l_2 = 1, \ldots, N-1$ if $\varepsilon > \varepsilon_{s(j,N)}$.

Recall ε is close to $\varepsilon_{s(j,N)}$, and assume that $\varepsilon > \varepsilon_{s(j,N)}$ (ε is on the right of $\varepsilon_{s(j,N)}$; for $\varepsilon < \varepsilon_{s(j,N)}$ the proof is analogous). We divide the sum of the harmonics $L_{s(j,n)}$ into the ones previous to $s(j,N)$ $(n \geq N)$ and the ones after $s(j,N)$ $(n < N)$.

$$\sum_{k\in\{s(j,n)\}} |k|^l L_k = \sum_{l_1\geq 0} |s(j,N+l_1)|^l L_{s(j,N+l_1)} + \sum_{l_2=1}^{N-1} |s(j,N-l_2)|^l L_{s(j,N-l_2)}.$$

The first sum is

$$\begin{aligned}\Sigma_1 &= \sum_{l_1\geq 0} |s(j,N+l_1)|^l L_{s(j,N+l_1)} = \sum_{l_1\geq 1} |s(j,N+l_1)|^l \alpha_{s(j,N+l_1)} \exp\left\{-\tfrac{C_0 g_{s(j,N+l_1)}}{\varepsilon^{1/4}}\right\}\\ &\preceq |s(j,N)|^l \exp\left\{-\tfrac{C_0}{\varepsilon^{1/4}} g_{s(j,N)}\right\} \sum_{l_1\geq 1} |\lambda|^{l_1 l} \alpha_{s(j,N+l_1)} \exp\left\{-\tfrac{C_0 C_\lambda}{\varepsilon^{1/4}} \sqrt{\tilde\gamma_j^*} l_1\right\}\end{aligned}$$

where we used the exponential form of $L_{s(j,n)}$ (6.8), the inequality (6.24) and the fact that $|s(j,N+l_1)| \sim |\lambda|^{l_1}|s(j,n)|$. Moreover, due to (6.9) we have

$$\alpha_{s(j,N+l_1)} \sim \frac{\tilde\gamma_j^*}{|\lambda|^{l_1}|s(j,N)|\sqrt{\varepsilon}(1-\exp\{-\frac{C\tilde\gamma_j^*}{|\lambda|^{l_1}|s(j,N)|\sqrt{\varepsilon}}\})}.$$

If we consider the sequence $a_{l_1} = |\lambda|^{l_1 l}\alpha_{s(j,N+l_1)}$, we obtain that

$$\frac{a_{l_1+1}}{a_{l_1}} = \frac{|\lambda|^{(l_1+1)l}\alpha_{s(j,N+(l_1+1))}}{|\lambda|^{l_1 l}\alpha_{s(j,N+l_1)}} \sim \frac{|\lambda|^{(l_1+1)l}}{|\lambda|^{l_1 l}} \frac{|\lambda|^{l_1}(1-\exp\{-\frac{C\tilde\gamma_j^*}{|\lambda|^{l_1}|s(j,N)|\sqrt{\varepsilon}}\})}{|\lambda|^{l_1+1}(1-\exp\{-\frac{C\tilde\gamma_j^*}{|\lambda|^{l_1+1}|s(j,N)|\sqrt{\varepsilon}}\})} \leq |\lambda|^l,$$

since $\frac{1-\exp\{-x\}}{1-\exp\{-x/|\lambda|\}} \leq |\lambda|$ for $x > 0$. Therefore, $a_{l_1+1} \leq |\lambda|^l a_{l_1}$ (or $a_{l_1} \leq |\lambda|^{l_1 l} a_0$, $a_0 = \alpha_{s(j,N)}$) and the sum is bounded above by a geometric series that can be estimated by the first term

$$\begin{aligned}\Sigma_1 &\preceq |s(j,N)|^l \alpha_{s(j,N)} \exp\left\{-\frac{C_0 g_{s(j,N)}}{\varepsilon^{1/4}}\right\} \sum_{l_1\geq 0} |\lambda|^{l_1 l} \exp\left\{-\frac{C_0 C_\lambda \sqrt{\tilde\gamma_j^*}}{2\varepsilon^{1/4}} l_1\right\}\\ &= |s(j,N)|^l L_{s(j,n)} \left(1-|\lambda|^l \exp\left\{-\frac{C_0 C_\lambda\sqrt{\tilde\gamma_j^*}}{2\varepsilon^{1/4}}\right\}\right)^{-1} \leq 2|s(j,N)|^l L_{s(j,N)}.\end{aligned}$$

The inequality is true for ε small enough ($\varepsilon^{1/4} < C_0 C_\lambda \sqrt{\tilde\gamma_j^*}\ln(2|\lambda|^l)/2$).

For the second sum we proceed analogously, using (6.25) and $|s(j,N-l_2)| \sim |\lambda|^{-l_2}|s(j,N)|$

$$\begin{aligned}\Sigma_2 &= \sum_{l_2=1}^{N-1} |s(j,N-l_2)|^l L_{s(j,N-l_2)}\\ &\preceq |s(j,N)|^l \exp\left\{-\tfrac{C_0}{\varepsilon^{1/4}} g_{s(j,N)}\right\} \sum_{l_2=0}^{N-1} |\lambda|^{-l_2 l} \alpha_{s(j,N-l_2)} \exp\left\{-\tfrac{C_0 C_\lambda}{\varepsilon^{1/4}} \sqrt{\tilde\gamma_j^*} l_2\right\},\end{aligned}$$

where

$$\alpha_{s(j,N-l_2)} \sim \frac{\tilde\gamma_j^* |\lambda|^{l_2}}{|s(j,N)|\sqrt{\varepsilon}\left(1-\exp\left\{-\frac{C\tilde\gamma_j^*}{|\lambda|^{-l_2}|s(j,N)|\sqrt{\varepsilon}}\right\}\right)}.$$

Considering $a_{l_2} = |\lambda|^{-l_2 l}\alpha_{s(j,N-l_2)}$ and having

$$\begin{aligned}\frac{a_{l_2+1}}{a_{l_2}} &= |\lambda|^{-l}\frac{\alpha_{s(j,N-(l_2+1))}}{\alpha_{s(j,N-l_2)}} \sim |\lambda|^{-l}\frac{|\lambda|^{l_2+1}\left(1-\exp\left\{-\frac{C\tilde{\gamma}_j^*|\lambda|^{l_2}}{|s(j,N)|\sqrt{\varepsilon}}\right\}\right)}{|\lambda|^{l_2}\left(1-\exp\left\{-\frac{C\tilde{\gamma}_j^*|\lambda|^{l_2+1}}{|s(j,N)|\sqrt{\varepsilon}}\right\}\right)} \le |\lambda|^{-l}|\lambda| \\ &= |\lambda|^{1-l} \le |\lambda|,\end{aligned}$$

since $\frac{1-\exp\{-x\}}{1-\exp\{-|\lambda|x\}} \le 1$ for $|\lambda| > 1$, we deduce that $a_{l_2+1} \le |\lambda|^{l_2}a_0 = |\lambda|^{l_2}\alpha_{s(j,N)}$.

As before, we obtain

$$\begin{aligned}\Sigma_2 &\preceq |s(j,N)|^l L_{s(j,N)} \sum_{l_2=0}^{N-1} |\lambda|^{l_2} \exp\left\{-\tfrac{C_0 C_\lambda}{\varepsilon^{1/4}}\sqrt{\tilde{\gamma}_j^*}\,l_2\right\} \\ &= |s(j,N)|^l L_{s(j,n)} \frac{1}{1-|\lambda|\exp\left\{-C_0 C_\lambda\sqrt{\tilde{\gamma}_j^*}/(2\varepsilon^{1/4})\right\}} \le 2|s(j,N)|^l L_{s(j,N)}.\end{aligned}$$

The both sums complete the proof of (b).

To prove (c), we recall that for the 24 numbers from (6.15) the two essential dominant terms are L_{S_1} and $L_{S'}$ where S_1 is primary while S' can be primary or secondary. Also there can be non-essential terms L_{S_i} between L_{S_1} and $L_{S'}$. We consider 4 cases: (1) S' is primary and no non-essential terms (numbers $\Omega_1, \Omega_2, \Omega_3$); (2) S' is primary and there are non-essential terms (numbers $\Omega_4, \ldots, \Omega_{13}$); (3) S' is secondary and no non-essential terms (number $\Omega_{1,2}$); (4) S' is secondary and there are non-essential terms (numbers $\Omega_{1,3}, \ldots, \Omega_{1,12}$

Then in every case we consider the sums of $L_{s(j,n)}$ for the sequences to which S_1, S' and (in the case 2 and 4) S_i belong to, i.e. $s(j_0,n), s(j',n), s(j_i,n)$, $n \ge 1$. According to (b) these sums are bounded by the dominant term $L_{s(j,N)}$ where $s(j,N)$ is S_1 if $j = j_0$, S' if $j = j'$ (cases 3 and 4), S_i if $j = j_i$ (cases 2 and 4), respectively. Then excluding these dominant terms from the sums, the corresponding upper bounds are estimated by the next dominant harmonic which is one of $L_{s(j,N-1)}$ or $L_{s(j,N+1)}$

$$\sum_{n\ge 1, n\ne N} |s(j_0,n)|^l L_{s(j_0,n)} \preceq |s(j_0,N\pm 1)|^l L_{s(j_0,N\pm 1)}$$
$$\sum_{n\ge 1, n\ne N} |s(j',n)|^l L_{s(j',n)} \preceq |s(j',N\pm 1)|^l L_{s(j',N\pm 1)}$$
$$\sum_{n\ge 1, n\ne N} |s(j_i,n)|^l L_{s(j_i,n)} \preceq |s(j_i,N\pm 1)|^l L_{s(j_i,N\pm 1)}$$

One of these harmonics coincides with $L_{S_{m+2}}$, while the other ones are smaller, and, hence,

$$\sum_{\substack{k\in\{s(j,n)\}\\ j=j_0,j',j_i\\ k\ne S_1,S',S_i,i=1,\ldots,m}} |s(j,n)|^l L_{s(j_0,n)} \sim \frac{1}{\varepsilon^{l/4}} L_{S_{m+2}} \tag{6.26}$$

Further, we calculate the sum of the remaining sequences $s(j,n)$, $j \ne j_0, j', j_i; n \ne 1$. Due to (b), for each j the sum of the coefficients $L_{s(j,n)}$ of the same sequence is estimated essentially by $L_{s(j,N)}$ (note that $N = N(j,\varepsilon)$, and for different j the number N is different).

$$\begin{aligned}\Sigma_3 &= \sum_{\substack{j\in\mathcal{P}\\ j\neq j_0,j',j_i\\ n\geq 1}} |s(j,n)|^l L_{s(j,n)} \preceq \sum_{\substack{j\in\mathcal{P}\\ j\neq j_0,j',j_i}} |s(j,N)|^l L_{s(j,N)} \\ &= \sum_{\substack{j\in\mathcal{P}\\ j\neq j_0,j',j_i}} |s(j,N)|^l \alpha_{s(j,N)} \exp\left\{-\frac{C_0 g_{s(j,N)}(\varepsilon)}{\varepsilon^{1/4}}\right\}.\end{aligned}$$

Recall that the set $\mathcal{P}$ is the set of primitive numbers j defined in Section 6.1. We use the estimates $|s(j,N)| \sim \sqrt{\tilde\gamma_j^*}\varepsilon^{-1/4}$, $g_{s(j,N)} \sim \sqrt{\tilde\gamma_j^*}$ for $\varepsilon \approx \varepsilon_{s(j,N)}$, and, for ε small enough ($\varepsilon^{1/4} \leq C\sqrt{\tilde\gamma_j^*}/\ln 2$), $\alpha_{s(j,N)} \leq \frac{\sqrt{\tilde\gamma_j^*}}{2\varepsilon^{1/4}}$. Also for primitive j we apply the bound (6.1) for γ_j, we will have for $\tilde\gamma_j^*$:

$$(A|j|-B)/|\lambda| \leq \tilde\gamma_j^* \leq A|j|+B, \quad \text{where } A = \frac{1+\Omega}{2\gamma_{j_0}^*},\ B = \frac{a}{2\gamma_{j_0}^*}.$$

Hence, we get

$$\begin{aligned}\Sigma_3 &\preceq \varepsilon^{(l-1)/4} \sum_{\substack{j\in\mathcal{P}\\ j\neq j_0,j',j_i}} \left(\sqrt{\tilde\gamma_j^*}\right)^{l+1} \exp\left\{-\frac{C_0\sqrt{\tilde\gamma_j^*}}{\varepsilon^{1/4}}\right\} \\ &\preceq \varepsilon^{(l-1)/4} \sum_{|j|\geq\bar\jmath} (\sqrt{A|j|+B})^{l+1} \exp\left\{-\frac{C_0\sqrt{A|j|-B}}{\sqrt{|\lambda|}\varepsilon^{1/4}}\right\}.\end{aligned}$$

We denote $\bar\jmath = \min\limits_{\substack{j\in\mathcal{P}\\ j\neq j_0,j',j_i}} |j|$. It turns out that it satisfy $\bar\jmath \leq 2$ (and $\bar\jmath > 2$ for $\Omega \neq \Omega_3, \Omega_{1,2}$)
We can bound $A|j|+B \leq \chi(A|j|-B)$ with a constant χ satisfying $1 < \chi \leq \frac{A\bar\jmath-A+B}{A\bar\jmath-A-B}$ for $|j| \geq \bar\jmath - 1$, and, hence, the sum is estimated by the corresponding integral

$$\begin{aligned}\Sigma_3 &\preceq \varepsilon^{(l-1)/4} \int\limits_{\bar\jmath-1}^{\infty} (\sqrt{Ax-B})^{l+1} \exp\left\{-\frac{C_0\sqrt{Ax-B}}{\sqrt{|\lambda|}\varepsilon^{1/4}}\right\} dx \\ &= \frac{1}{A}\varepsilon^{(l-1)/4} \int\limits_{A\bar\jmath-A-B}^{\infty} (\sqrt{u})^{l+1} \exp\left\{-\frac{C_0\sqrt{u}}{\sqrt{|\lambda|}\varepsilon^{1/4}}\right\} du \\ &= \frac{2(\sqrt{|\lambda|})^{l+3}}{AC_0^{l+3}}\varepsilon^{(l+1)/2} \int\limits_{\frac{C_0\sqrt{A\bar\jmath-A-B}}{\sqrt{|\lambda|}}\varepsilon^{-1/4}}^{\infty} (z)^{l+2}\exp\{-z\} du\end{aligned}$$

Thus, we get the following bounds

$$
\begin{aligned}
&l=0 \quad \Sigma_3 \preceq (2\varepsilon^{1/2}+2\varepsilon^{1/4}+1)\exp\left\{\frac{C_0\sqrt{A\bar{\jmath}-A-B}}{\sqrt{|\lambda|}}\varepsilon^{-1/4}\right\}\\
&l=1 \quad \Sigma_3 \preceq (6\varepsilon+5\varepsilon^{3/4}+3\varepsilon^{1/2}+\varepsilon^{1/4})\exp\left\{\frac{C_0\sqrt{A\bar{\jmath}-A-B}}{\sqrt{|\lambda|}}\varepsilon^{-1/4}\right\}\\
&l=2 \quad \Sigma_3 \preceq (24\varepsilon^{3/2}+24\varepsilon^{5/4}+12\varepsilon+4\varepsilon^{3/4}+\varepsilon^{1/2})\exp\left\{\frac{C_0\sqrt{A\bar{\jmath}-A-B}}{\sqrt{|\lambda|}}\varepsilon^{-1/4}\right\}\\
&\dots
\end{aligned}
$$

These decreasing bounds are smaller than (6.26).

Finally, we consider non-admissible vectors k for which $\gamma_k \geq |k|/2$. Also we have $\gamma_k = |\langle k,\omega\rangle||k| \leq |k|^2|\omega|$. From (6.9), we obtain

$$
\alpha_k \leq \frac{|k||\omega|}{\sqrt{\varepsilon}(1-\exp\{-C\varepsilon^{-1/2}\})}, \quad \beta_k \geq \rho|k| + \frac{C}{4\varepsilon^{1/2}}
$$

Therefore,

$$
\sum_{k\notin\mathcal{A}} |k|^l L_k \leq \frac{\exp\{-C\varepsilon^{-1/2}/4\}}{\sqrt{\varepsilon}(1-\exp\{-C\varepsilon^{-1/2}\})}\sum_{k\notin\mathcal{A}} |k|^{l+1}\exp\{-\rho|k|\}.
$$

This sum can be bounded from above by the sum of a geometric series which is smaller than (6.26). This completes the proof of Lemma. □

6.2.4 Dominant harmonics of the splitting potential

To study the nondegenerate critical points of the whole splitting potential $\mathcal{L}(\theta)$ (the simple zeros of the splitting function $\mathcal{M}(\theta)$) we can use as a first approximation the estimates obtained in Lemma 6.1. We prove that assuming $\mu = \varepsilon^p$, for a suitable $p > 0$, the dominant harmonics don't change essentially in $\mathcal{L}$ if we add the error term of order $\mathcal{O}(\mu^2)$.

Recall that $\mathcal{M} = \partial_\theta \mathcal{L}$ and take into account that $\mathcal{L}$ is $\hat{\omega}_\varepsilon$-quasiperiodic, we can consider the following Fourier expansion:

$$
\mathcal{L}(s,\theta) = \sum_{k\in\mathbb{Z}^2} \mathcal{L}_k^* e^{i\langle k,\theta-\hat{\omega}_\varepsilon s\rangle} = \sum_{k\in\mathcal{Z}} \mathcal{L}_k \cos(\langle k,\theta-\hat{\omega}_\varepsilon s\rangle - \tau_k),
$$

where $\mathcal{L}_k, \tau_k$ are real, $\mathcal{L}_k \geq 0$ and $\mathcal{Z}$ is defined in (5.8). For every $k \in \mathcal{Z}$, the exponential and the trigonometric forms are related by $\mathcal{L}_k^* = \frac{1}{2}\mathcal{L}_k e^{-i\tau_k}$. Then the corresponding Fourier coefficients of the splitting function $\mathcal{M}(s,\theta)$ and the Melnikov function $M(s,\theta)$ are (in the exponential form) $\mathcal{M}_k^* = ik\mathcal{L}_k^*$ and $M_k^* = ikL_k^*$, respectively.

Lemma 6.2. *Assuming $\varepsilon \preceq 1$, $\mu = \varepsilon^p$ with $p > p^*$, with $p^* = 2$ if $\nu = 1$ and $p^* = 3$ if $\nu = 0$, one has:*

(a) $\mathcal{L}_{S_1} \sim \frac{\mu}{\varepsilon^{1/4}} \exp\{-\frac{C_0 h_1(\varepsilon)}{\varepsilon^{1/4}}\}$,
$\mathcal{L}_{S'} \sim \frac{\mu}{\varepsilon^{1/4}} \exp\{-\frac{C_0 h_2(\varepsilon)}{\varepsilon^{1/4}}\}$,
$\mathcal{L}_{S_i} \sim \frac{c_i\mu}{\varepsilon^{1/4}} \exp\{-c_i\frac{C_0 h_1(\varepsilon)}{\varepsilon^{1/4}}\}, \quad i = 2, \dots, m;$

(b) $|\tau_k - \sigma_k - s^{(0)}\langle k, \hat{\omega}_\varepsilon\rangle| \preceq \frac{\mu}{\varepsilon^{p^*}}, k = S_1, S_2, \dots, S_m, S'$;

(c) $\sum\limits_{k \neq S_1, S', S_i} |k|^l \mathcal{L}_k \preceq \frac{1}{\varepsilon^{l/4}} \mathcal{L}_{S_{m+2}}, \quad i = 1, \dots, m, l \geq 0.$

Proof. The proof is similar to the one of Lemma 5 in [DG04], here we just adapt it to the quadratic numbers from (6.15) The splitting function $\mathcal{M}(s,\theta)$ can be defined on a complex domain: $|\text{Im } s| < \frac{\pi}{2} - \delta$, $|\text{Im } \theta| < \rho$, where δ is a small reduction (to be chosen). On this domain, the upper bound (5.14), pointed out in Theorem 5.1 (for the two-dimensional case), can be applied to the error term (5.13). Also we can deduce from (5.13) that the Fourier coefficients for the error term are $\mathcal{R}_k^* = ik(\mathcal{L}_k^* - \mu L_k^* e^{-is^{(0)}\langle k, \hat{\omega}_\varepsilon\rangle})$, $k \neq 0$. Taking modulus and argument, we get

$$|\mathcal{L}_k - \mu L_k| \preceq \frac{|\mathcal{R}_k^*|}{|k|}, \quad |\tau_k - \sigma_k - s^{(0)}\langle k, \hat{\omega}_\varepsilon\rangle| \preceq \frac{|\mathcal{R}_k^*|}{|k|\mu L_k}.$$

Since $\mathcal{R}$ is $\hat{\omega}_\varepsilon$-quasiperiodic, a standard result (Lemma 11 of [DGS04]) can be applied to it to get bounds for its Fourier coefficients:

$$|\mathcal{R}_k^*| \preceq \left(\frac{\mu^2}{\delta^{q_3}} + \frac{\mu^2}{\delta^{q_4}\sqrt{\varepsilon}}\right) e^{-\tilde{\beta}_k(\varepsilon)}, \quad \tilde{\beta}_k(\varepsilon) = (\rho - \delta)|k| + \frac{(\pi/2 - \delta)b'\gamma_k}{b|k|\sqrt{\varepsilon}}.$$

We present the function $\tilde{\beta}_k$ as

$$\tilde{\beta}_k(\varepsilon) = \frac{C_{\mu,\delta}\sqrt{\tilde{\gamma}_k}}{2\varepsilon^{1/4}} \left[\left(\frac{\varepsilon}{\varepsilon_k}\right)^{1/4} + \left(\frac{\varepsilon_k}{\varepsilon}\right)^{1/4}\right],$$

where

$$\varepsilon_k^{1/4} = \frac{C_{\mu,\delta}\sqrt{\tilde{\gamma}_k}}{2(\rho - \delta)|k|}, \quad C_{\mu,\delta} = \sqrt{4\frac{(\pi/2 - \delta)b'}{b}(\rho - \delta)\gamma_{j_0}^*} = C_0 + \mathcal{O}(\mu\delta^{-q_2}, \delta).$$

The difference with (6.10) is that for $\tilde{\beta}_k$ we write $\pi/2 - \delta$ and $\rho - \delta$ instead of $\pi/2$ and ρ in $\beta_k(\varepsilon)$. In fact, we consider $\tilde{\beta}_k$ as a perturbation of β_k. Indeed, proceeding as in the proof of Lemma 6.1, we get for the most dominant term:

$$\tilde{\beta}_{S_1} = \frac{C_0 h_1(\varepsilon) + \mathcal{O}(\sqrt{\varepsilon}, \mu\delta^{-q_2}, \delta)}{\varepsilon^{1/4}}.$$

We can neglect the perturbation term if $\mu\delta^{-q_2} \preceq \varepsilon^{1/4}$, $\delta \preceq \varepsilon^1/4$. So we choose $\delta = \varepsilon^{1/4}$. The smallness conditions on μ become $\mu \preceq \varepsilon^{q_1/4}$ (the condition containing the exponent q_2 can be ignored, since $q_1 \geq q_2 + 3$). Then using (6.23), we conclude

$$|\mathcal{L}_{S_1} - \mu L_{S_1}| \preceq \frac{\mu^2}{\varepsilon^{(q_3-1)/4}} \exp\left\{-\frac{C_0 h_1(\varepsilon)}{\varepsilon^{1/4}}\right\}.$$

Therefore, the term $|\mu L_{S_1}| \sim \frac{\mu}{\varepsilon^{1/4}} \exp\{-\frac{C_0 h_1(\varepsilon)}{\varepsilon^{1/4}}\}$ (estimated in Lemma 6.1) dominates if

$$\frac{\mu}{\varepsilon^{1/4}} \succeq \frac{\mu^2}{\varepsilon^{(q_3-1)/4}}.$$

If one takes $\mu = \varepsilon^p$, the last condition is fulfilled at $p > (q_3 - 2)/4$. We get $p^* = \max\{(q_3 - 2)/4, q_1/4\} = (q_3 - 2)/4$, since $q_3 - 2 \geq q_1$. In fact, for $\nu = 0$ we have $q_3 = 14$ and hence $p^* = 3$, and for $\nu = 1$ we have $q_3 = 10$ and $p^* = 2$. The remaining statements of (a) for the other dominant terms S_i, $i = 2, \ldots, m$ and S' as well as the part (b) are proved in a similar way.

To prove (c), we bound the sum of $|k|^l \mathcal{L}_k$, $k \neq S_i, S'$, $i = 1, \ldots, m$, by the sum of a geometric series analogously as it was done in the proof of Lemma 6.1. □

6.2.5 Nondegenerate critical points of $\mathcal{L}$

To prove the nondegenericity of the critical points of $\mathcal{L}(\theta)$ we need to consider at least 2 essential dominant harmonics $\mathcal{L}_{S_1}$ and $\mathcal{L}_{S'}$ (see Definition 6.1 of essential dominant harmonics), also we have to take into account all the non-essential harmonics $\mathcal{L}_{S_2}, \ldots, \mathcal{L}_{S_m}$, if necessary, that $\mathcal{L}_{S_1} \geq \mathcal{L}_{S_2} \geq \ldots \geq \mathcal{L}_{S_m} \geq \mathcal{L}_{S'}$ and the corresponding vectors $S_2, \ldots, S_m$ are linearly dependent with S_1 (recall that S_1 and S' are independent). Since these vectors are linearly dependent with S_1, they satisfy the relations (6.18).

First we consider the approximation of $\mathcal{L}$ by these $m + 1$ harmonics

$$\mathcal{L}^{(m+1)}(\theta) = \sum_{i=1}^{m} \mathcal{L}_{S_i} \cos(\langle S_i, \theta\rangle - \sigma_{S_i}) + \mathcal{L}_{S'} \cos(\langle S', \theta\rangle - \sigma_{S'}).$$

After the linear change

$$\psi_1 = \langle S_1, \theta\rangle - \sigma_{S_1}, \quad \psi_2 = \langle S', \theta\rangle - \sigma_{S'} \tag{6.27}$$

which can be written as

$$\psi = \mathcal{A}_n\theta - b, \quad \text{where } \mathcal{A}_n = \begin{pmatrix} S_1^\top \\ (S')^\top \end{pmatrix}, b = \begin{pmatrix} \sigma_{S_1} \\ \sigma_{S'} \end{pmatrix},$$

the function $\mathcal{L}^{(m+1)}$ becomes

$$K^{(m+1)}(\psi) = \mathcal{L}_{S_1} \cos\psi_1 + \sum_{i=2}^{m} \mathcal{L}_{S_i} \cos(c_i\psi_1 + \triangle\tau_i) + \mathcal{L}_{S'} \cos\psi_2,$$

where $\triangle\tau_i = \sigma_{S_i} - c_i\sigma_{S_1}$, $i = 2, \ldots, m$.

Lemma 6.3. *1.* $K^{(m+1)}(\psi)$ *has 4 nondegenerate critical points* $\psi^{(1)} = (\mathcal{O}(\eta), 0), \psi^{(2)} = (\mathcal{O}(\eta), \pi), \psi^{(3)} = (\pi + \mathcal{O}(\eta), 0), \psi^{(4)} = (\pi + \mathcal{O}(\eta), \pi)$ *, where*

$$\eta = \begin{cases} 0, & \text{if } m = 1, \\ \mathcal{L}_{S_2}/\mathcal{L}_{S_1}, & \text{if } m > 1 \end{cases} \tag{6.28}$$

and m *is defined in (6.19).*

2. $\det D^2 K^{(m+1)}(\psi^{(j)}) = \mathcal{L}_{S_1}\mathcal{L}_{S'}(1 + \mathcal{O}(\eta))$.

Proof. The critical points of $K^{(m+1)}$ are the solutions of the system:

$$\mathcal{L}_{S_1} \sin\psi_1 + \sum_{i=2}^{m} c_i \mathcal{L}_{S_i} \sin(c_i\psi_1 + \triangle\tau_i) = 0, \quad \mathcal{L}_{S'} \sin\psi_2 = 0.$$

From the second equation we get $\psi_2 = 0$ and $\psi_2 = \pi$, while the first equation can be written in the form

$$\sin\psi_1 = -\sum_{i=2}^{m} c_i \frac{\mathcal{L}_{S_i}}{\mathcal{L}_{S_1}} \sin(c_i\psi_1 + \triangle\tau_i) = \eta f(\psi_1),$$

where $f(\psi_1)$ is bounded with its derivative

$$|f(\psi_1)| \le M = \sum_{i=2}^{m} c_i \mathcal{L}_{S_i}/\mathcal{L}_{S_2} \preceq 1, |f'(\psi_1)| \le N = \sum_{i=2}^{m} c_i^2 \mathcal{L}_{S_i}/\mathcal{L}_{S_2} \preceq 1. \tag{6.29}$$

Hence, for $\eta < \frac{1}{\sqrt{M^2+N^2}}$ small enough the condition $\eta^2(f')^2 + \eta^2 f^2 < 1$ is satisfied and, therefore, by Lemma B.1 there exist two simple solutions of the equation near to $\psi_1 = 0$ and $\psi_1 = \pi$.

These solutions give rise to the 4 critical points $\psi^{(1)}, \psi^{(2)}, \psi^{(3)}, \psi^{(4)}$ of $K^{(m+1)}$.

The determinant is easily computed. We have

$$\det D^2 K^{(m+1)}(\psi) = \mathcal{L}_{S_1}\mathcal{L}_{S'}(\cos\psi_1 - \eta f'(\psi_1))\cos\psi_2$$

for any $\psi \in \mathbb{T}^2$. Hence, at the critical points we get

$$|\det D^2 K^{(m+1)}(\psi^{(i)})| = \mathcal{L}_{S_1}\mathcal{L}_{S'}(1 + \mathcal{O}(\eta)).$$

□

Remark 6.6. Note that if there are no non-essential harmonics between $\mathcal{L}_{S_1}$ and $\mathcal{L}_{S'}$, i.e. $m = 1$ and $S' = S_2$, then we put $\eta = 0$ and the critical points are $\psi^{(1)} = (0,0), \psi^{(2)} = (0,\pi), \psi^{(3)} = (\pi, 0), \psi^{(4)} = (\pi,\pi)$.

After having studied the critical points of the approximation $K^{(m+1)}$ we have to study their persistence if we add the remainder. Applying the linear change (6.27) to the whole splitting potential $\mathcal{L}(\theta)$, we get

$$K(\psi) = K^{(m+1)} + \mathcal{L}_{S_{m+2}} G(\psi_1, \psi_2),$$

where the term $\mathcal{L}_{S_{m+2}} G(\psi_1, \psi_2)$ corresponds to the sum of all the non-dominant terms of $\mathcal{L}$ (it is bounded according to part (b) of Lemma 6.2) and $\mathcal{L}_{S_{m+2}}$ (the next after $\mathcal{L}_{S'}$) is the maximum of non-dominant harmonics. Note that function G is obtained via the linear change (6.27) applied to the remaining terms of $\mathcal{L}$. Taking into account the bound (c) of Lemma 6.2 for the non-dominant terms, one gets the following bounds for G:

$$\begin{array}{lll} |G| \preceq 1, & |\partial_{\psi_1} G| \preceq \varepsilon^{-1/2}, & |\partial_{\psi_2} G| \preceq \varepsilon^{-1/2}, \\ |\partial^2_{\psi_1\psi_1} G| \preceq \varepsilon^{-1}, & |\partial^2_{\psi_1\psi_2} G| \preceq \varepsilon^{-1}, & |\partial^2_{\psi_2\psi_2} G| \preceq \varepsilon^{-1}. \end{array} \tag{6.30}$$

Lemma 6.4. *If* $\overline{\eta} = \max\left\{\eta, \mathcal{L}_{S_{m+2}}/\mathcal{L}_{S'}\right\} \preceq \varepsilon^2$*,* η *is given in (6.28), then the function* $K(\psi)$ *has 4 critical points, all nondegenerate:* $\psi_*^{(j)} = \psi_{(j),0} + \mathcal{O}(\overline{\eta})$*,* $j = 1, 2, 3, 4$*, where* $\psi_{(1),0} = (0, 0)$*,* $\psi_{(2),0} = (\pi, 0)$*,* $\psi_{(3),0} = (0, \pi)$*,* $\psi_{(4),0} = (\pi, \pi)$*. At the critical points,*

$$|\det D^2 K(\psi_*^{(j)})| = \mathcal{L}_{S_1} \mathcal{L}_{S'} \left(1 + \mathcal{O}\left(\frac{\overline{\eta}}{\varepsilon}\right)\right).$$

Proof. The critical points of K are the solutions of

$$\sin\psi_1 = \eta f(\psi_1) + \eta' g_1(\psi_1, \psi_2), \quad \sin\psi_2 = \eta'' g_2(\psi_1, \psi_2), \tag{6.31}$$

where ηf corresponds to non-essential dominant terms (the same as in the proof of Lemma 6.3), $\eta' = \frac{\mathcal{L}_{S_{m+2}}}{\mathcal{L}_{S_1}}$, $\eta'' = \frac{\mathcal{L}_{S_{m+2}}}{\mathcal{L}_{S'}}$ are exponentially small (note that $\eta' < \eta'' \ll 1$) and the functions $g_1(\psi) = \partial_{\psi_1} G(\psi)$ and $g_2(\psi) = \partial_{\psi_2} G(\psi)$ have bounds obtained from (6.30)

$$\begin{array}{lll} |g_1| \sim \varepsilon^{-1/2}, & |\partial_{\psi_1} g_1| \sim \varepsilon^{-1}, & |\partial_{\psi_2} g_1| \sim \varepsilon^{-1}, \\ |g_2| \sim \varepsilon^{-1/2}, & |\partial_{\psi_1} g_2| \sim \varepsilon^{-1}, & |\partial_{\psi_2} g_2| \sim \varepsilon^{-1}. \end{array}$$

Thus, $\overline{\eta} = \max\{\eta, \eta''\}$ and the conditions of Lemma B.2 becomes

$$\begin{array}{l} (\eta f(\psi_1) + \eta' g_1(\psi))^2 + (|\eta f'(\psi_1) + \eta' \partial_{\psi_1} g_1(\psi)| + |\eta' \partial_{\psi_2} g_1(\psi)|)^2 < 1 \\ (\eta f(\psi_1) + \eta'' g_2(\psi))^2 + (|\eta f'(\psi_1) + \eta'' \partial_{\psi_1} g_2(\psi)| + |\eta'' \partial_{\psi_2} g_2(\psi)|)^2 < 1 \end{array}$$

Due to $\eta' < \eta''$, the second inequality implies the first one. We recall also that $|f| \leq M$ and $|f'| \leq N$, M and N are the constants defined in (6.29). Then the conditions are satisfied if

$$\overline{\eta} < \frac{\varepsilon}{\sqrt{\varepsilon(M\varepsilon^{1/2} + 1)^2 + (N\varepsilon + 2)^2}},$$

and this inequality is true for $\overline{\eta} \preceq \varepsilon$, thus, $K(\psi)$ has 4 nondegenerate critical points. Moreover, since the left parts of (6.31) are small in $\overline{\eta}$, the critical points are a perturbation of the points $\psi_{(j),0}$, $j = 1, 2, 3, 4$.

The determinant is

$$\begin{aligned}\det(D^2K(\psi)) &= (-\mathcal{L}_{S_1}\cos\psi_1 + \eta\mathcal{L}_{S_1}f' + \eta'\mathcal{L}_{S_1}\partial_{\psi_1}g_1)(-\mathcal{L}_{S'}\cos\psi_2 + \eta''\mathcal{L}_{S'}\partial_{\psi_2}g_2)\\ &\quad -\eta'\eta''\mathcal{L}_{S_1}\mathcal{L}_{S'}\partial_{\psi_2}g_1\partial_{\psi_1}g_2 = \mathcal{L}_{S_1}\mathcal{L}_{S'}\left(\cos\psi_1\cos\psi_2 + \mathcal{O}\left(\overline{\eta}/\varepsilon\right)\right).\end{aligned} \tag{6.32}$$

If we choose $\overline{\eta} \preceq \varepsilon^2$, we have a small term $\mathcal{O}\left(\overline{\eta}/\varepsilon\right)$ in (6.32). Thus, substituting the critical points $\psi_*^{(j)}$, $j = 1, 2, 3, 4$, we will get the expected estimates and complete the proof of Lemma. □

Remark 6.7. In (6.32), we can choose $\eta \preceq \varepsilon^a$ with any $a > 1$ (for instance, $\eta \preceq \varepsilon^{1.01}$ or $\eta \preceq \varepsilon^{2013}$) for the validity of Lemma 6.4. Just for simplicity, we chose $\eta \preceq \varepsilon^2$. Note that $\overline{\eta}$ is exponentially small in ε and, hence, can be bounded by a power of ε.

We translate the results of Lemma 6.4 from the function $K(\psi)$ to the splitting potential $\mathcal{L}(\theta)$. It is well known that each critical point of $K(\psi)$ gives rise to κ critical points of $\mathcal{L}(\theta)$, where $\kappa := |\det\mathcal{A}_n|$. We have checked that for all the frequencies (6.15), $\kappa = 1$. Thus, the linear change (6.27) is one-to-one, and applying the inverse change of it, we obtain 4 nondegenerate critical points of $\mathcal{L}^{(m+1)}$

$$\theta_*^{(j)} = \mathcal{A}_n^{-1}(\psi_*^{(j)} + b), \quad j = 1, 2, 3, 4. \tag{6.33}$$

We also find an estimate for the minimal eigenvalue (in modulus) $m_{(j)}$ of $D^2\mathcal{L}^{(m+1)}$ at each critical point.

Lemma 6.5. *Assume $\overline{\eta} = \max\left\{\eta, \mathcal{L}_{S_{m+2}}/\mathcal{L}_{S'}\right\} \preceq \varepsilon^2$, η is given in (6.28). Then the splitting potential $\mathcal{L}$ has exactly 4 critical points $\theta_*^{(j)}$, given by (6.33), all nondegenerate and satisfying $m_{(j)} \sim \sqrt{\varepsilon}\mathcal{L}_{S'}$.*

Proof. We can present the minimal (in modulus) eigenvalue of $D^2\mathcal{L}(\theta_*^{(j)})$ in form

$$m_{(j)} = \frac{2|D|}{|T| + \sqrt{T^2 - 4D}},$$

where $D = \det D^2\mathcal{L}(\theta_*^{(j)})$ and $T = trD^2\mathcal{L}(\theta_*^{(j)})$. Thus, we need to find estimates for D and T.

One can prove that $D^2\mathcal{L}(\theta) = (\mathcal{A}_n)^\top D^2K(\psi)\mathcal{A}_n$. Therefore, if $D^2K = \begin{pmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{pmatrix}$, we see that

$$D^2\mathcal{L} = k_{11}S_1 \cdot S_1^\top + k_{12}[S_1 \cdot (S')^\top + S' \cdot S_1^\top] + k_{22}S' \cdot (S')^\top,$$

where

$$\begin{aligned}k_{11} &= \mathcal{L}_{S_1}(-\cos\psi_1 + \eta f' + \eta'\partial_{\psi_1}g_1),\\ k_{12} &= \mathcal{L}_{S_{m+2}}\partial^2_{\psi_1\psi_2}G = \eta'\mathcal{L}_{S_1}\partial_{\psi_2}g_1 = \eta''\mathcal{L}_{S'}\partial_{\psi_1}g_2,\\ k_{22} &= \mathcal{L}_{S'}(-\cos\psi_2 + \eta''\partial_{\psi_2}g_2).\end{aligned}$$

At the critical points $\psi_*^{(j)}$, $|k_{11}| \sim \mathcal{L}_{S_1}$, $|k_{12}| \sim \mathcal{L}_{S_{m+2}} \preceq \mathcal{L}_{S'}$, $k_{22} = \mathcal{L}_{S'} \preceq \mathcal{L}_{S_1}$. We get that

$$T = k_{11}|S_1|_2^2 + 2k_{12}\langle S_1, S'\rangle + k_{22}|S'|_2^2,$$

here we use the 2-norm $|x|_2 = \sqrt{(x_1)^2 + (x_2)^2}$, but since the 1-norm and 2-norm are equivalent ($|x|_1/\sqrt{2} \le |x|_2 \le |x|_1$), we can apply the estimates (6.23) to obtain

$$|T| \sim \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_{S_1}.$$

Moreover, since $|\det \mathcal{A}_n| = 1$, we get from Lemma 6.4

$$|D| = |\det D^2K(\psi_*^{(j)})| = \mathcal{L}_{S_1}\mathcal{L}_{S'}\left(1 + \mathcal{O}\left(\overline{\eta}/\varepsilon\right)\right) \sim \mathcal{L}_{S_1}\mathcal{L}_{S'}.$$

Thus, having $|D| \ll T^2$, we conclude

$$m_{(j)} \sim \frac{|D|}{|T|} \sim \sqrt{\varepsilon}\mathcal{L}_{S'}.$$

□

6.2.6 Proof of Theorems 6.1 and 6.2

Theorem 6.1, p. 125, is a consequence of the fact that $\mathcal{M}(\theta) = \partial_\theta\mathcal{L}(\theta)$ and the estimate for the most dominant harmonic $\mathcal{L}_{S_1}$ given in Lemma 6.2, p. 138. We consider the approximation $\mathcal{L}^{(m+1)}$ by 2 essential dominant harmonics and deduce the following estimates:

$$|\partial_\theta\mathcal{L}^{(m+1)}| \sim |S_1|\mathcal{L}_{S_1} \sim \frac{1}{\varepsilon^{1/4}}\mathcal{L}_{S_1}, \quad |\partial\mathcal{L} - \partial_\theta\mathcal{L}^{(m+1)}| \sim \frac{1}{\varepsilon^{1/4}}\mathcal{L}_{S_{m+2}}.$$

Since $\mathcal{L}_{S_{m+2}} \ll \mathcal{L}_{S_1}$, we get the estimate

$$|\mathcal{M}| = |\partial_\theta\mathcal{L}| \sim \frac{\mu}{\sqrt{\varepsilon}}\exp\left\{-\frac{C_0h_1(\varepsilon)}{\varepsilon^{1/4}}\right\}.$$

Theorem 6.2, p. 130, follows from Lemma 6.5 and that the nondegenerate critical points of $\mathcal{L}(\theta)$ correspond to simple zeros of $\mathcal{M}(\theta)$. Applying the estimate for the second essential dominant harmonics $\mathcal{L}_{S'}$ from Lemma 6.2, we obtain the expected estimate for the minimal (in modulus) eigenvalue of the splitting matrix $\partial_\theta\mathcal{M} = D^2\mathcal{L}$:

$$m_* \sim \sqrt{\varepsilon}\mathcal{L}_{S'}.$$

Note that the result of Lemma 6.5 applies only for $\overline{\eta} = \max\left\{\mathcal{L}_{S_2}/\mathcal{L}_{S_1}, \mathcal{L}_{S_{m+2}}/\mathcal{L}_{S'}\right\} \preceq \varepsilon$, and this condition excludes from consideration some (decreasing as $n \to \infty$) neighborhoods of the points ε_n (given in (6.13)), ε''_n and ε'''_n (given in (6.21)), where the second and the third essential dominant harmonics of $\mathcal{M}$ are of the same magnitude.

6.3 Continuation of transverse homoclinic orbits in the case Ω_2

In this section we focus our attention on the case of the silver number $\Omega_2 = \sqrt{2} - 1$. Recall that in Theorem 6.1, p. 125, an asymptotic estimate of the maximum distance between the stable and unstable invariant manifold in term of the maximum value (in modulus) of $\mathcal{M}(\theta)$ has been given and, thus, the splitting has been established for all $\varepsilon \to 0$, while in Theorem 6.2, p. 130, the nondegeneracity of the critical points of the splitting potential $\mathcal{L}(\theta)$ and the transversality of the corresponding homoclinic orbits of (5.1-5.7) with $n = 2$ has been proven for most values of ε where there are only two essential dominant harmonics $\mathcal{L}_{S_1}$ and $\mathcal{L}_{S'}$. Here we study the critical points of $\mathcal{L}(0, \theta)$ for ε close to a critical value ε_n at which the second dominant harmonic coincides with some other consecutive dominant harmonics. Notice that in the case Ω_2 there is no non-essential harmonics, and, thus, there is no points ε''_n and ε'''_n, defined in (6.21). Indeed, we can see in Figure 6.1 that at $\varepsilon = \varepsilon_n$ the primary function $\widehat{g}_n(\varepsilon)$ is the smallest, while the primary functions $\widehat{g}_{n-1}(\varepsilon)$, $\widehat{g}_{n+1}(\varepsilon)$ and the secondary function $g_{s(3,n-1)}(\varepsilon)$ take the same value. This means that for ε near to ε_n the most dominant term of $\mathcal{L}(\theta)$ is $\mathcal{L}_{s_0(n)}$, and the next dominant terms $\mathcal{L}_{s_0(n-1)}$, $\mathcal{L}_{s_0(n+1)}$, $\mathcal{L}_{s(3,n-1)}$ are of the same magnitude (recall that we denoted by $s_0(\cdot)$ the sequence of primary resonances (see Section 6.1)). Thus, the point ε_n requires the consideration of 4 dominant harmonics.

For the silver number Ω_2 we have the following data of Section 6.1:

$$T = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}, \quad U = \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix}$$

$$\lambda = 1 + \sqrt{2}, \quad \gamma^* = \gamma_1^* = 0.5, \quad \tilde{\gamma}^{**} = 2$$

$$\begin{array}{lll} j_0 = 1, & k^0(1) = [0, 1], & \tilde{\gamma}_1^* = 1; \\ j = 3, & k^0(3) = [-1, 3], & \tilde{\gamma}_3^* = 2; \\ j = 4, & k^0(4) = [-2, 4], & \tilde{\gamma}_4^* = 4; \\ j \geq 6 & & \tilde{\gamma}_j^* > 6.5723 \end{array}$$

We denote by $s_1(n)$ the sequence of secondary vectors generated by primitive $k^0(3) = [-1, 3]$. These are secondary vectors whose limit $\tilde{\gamma}_3^*$ is the smallest among the secondary $\tilde{\gamma}_j^*, j \neq j_0$, and coincides with the separation parameter $\tilde{\gamma}^{**} = 2$ ($\tilde{\gamma}^{**}$ is defined at the end of Section 6.1).

We have the relations for our vectors

$$\begin{array}{rcl} s_0(n+1) & = & s_0(n-1) - 2s_0(n), \\ s_1(n-1) & = & s_0(n-1) - s_0(n). \end{array} \tag{6.34}$$

To prove the nondegenericity of the critical points of $\mathcal{L}$ near $\varepsilon \approx \varepsilon_n$, we consider the approximation by its 4 dominant terms

$$\begin{array}{rcl} \mathcal{L}^{(4)}(\theta) & = & \mathcal{L}_{s_0(n)} \cos(\langle s_0(n), \theta\rangle - \sigma_{s_0(n)}) + \mathcal{L}_{s_0(n-1)} \cos(\langle s_0(n-1), \theta\rangle - \sigma_{s_0(n-1)}) \\ & + & \mathcal{L}_{s_0(n+1)} \cos(\langle s_0(n+1), \theta\rangle - \sigma_{s_0(n+1)}) + \mathcal{L}_{s_1(n-1)} \cos(\langle s_1(n-1), \theta\rangle - \sigma_{s_1(n-1)}) \end{array}$$

By a linear change

$$\psi_1 = \langle s_0(n-1), \theta\rangle - \sigma_{s_0(n-1)}, \quad \psi_2 = \langle s_0(n), \theta\rangle - \sigma_{s_0(n)}, \tag{6.35}$$

that could be written as

$$\psi = \mathcal{A}\theta - b, \quad \text{where } \mathcal{A} = \begin{pmatrix} s_0(n-1)^\top \\ s_0(n)^\top \end{pmatrix}, b = \begin{pmatrix} \sigma_{s_0(n-1)} \\ \sigma_{s_0(n)} \end{pmatrix},$$

the function $\mathcal{L}^{(4)}$ is transformed into (taking into account (6.34))

$$K^{(4)} = B\cos\psi_2 + B\eta(1-Q)\cos\psi_1 + B\eta Q\cos(\psi_1 - 2\psi_2 - \triangle\tau) + B\eta\tilde{Q}\cos(\psi_1 - \psi_2 - \triangle\tau_1),$$

where

$$B = \mathcal{L}_{s_0(n)}, \quad \triangle\tau = \sigma_{s_0(n+1)} - \sigma_{s_0(n-1)} + 2\sigma_{s_0(n)}, \quad \triangle\tau_1 = \sigma_{s_1(n-1)} - \sigma_{s_0(n-1)} + \sigma_{s_0(n)},$$
$$Q = \frac{\mathcal{L}_{s_0(n+1)}}{\mathcal{L}_{s_0(n-1)} + \mathcal{L}_{s_0(n+1)}}, \quad \tilde{Q} = \frac{\mathcal{L}_{s_1(n-1)}}{\mathcal{L}_{s_0(n-1)} + \mathcal{L}_{s_0(n+1)}}, \quad \eta = \frac{\mathcal{L}_{s_0(n-1)} + \mathcal{L}_{s_0(n+1)}}{\mathcal{L}_{s_0(n)}}.$$

Note that here η is exponentially small, $0 < Q < 1$ and $0 < \tilde{Q} \le 1/2$. At $Q = \tilde{Q} = \frac{1}{2}$, the harmonics $\mathcal{L}_{s_0(n-1)}, \mathcal{L}_{s_0(n+1)}$ and $\mathcal{L}_{s_1(n-1)}$ coincide.

We introduce the following important quantity

$$\begin{aligned} &E^* = \min(E^{(+)}, E^{(-)}), \text{ where} \\ &E^{(\pm)} = \sqrt{\left[1 - Q + Q\cos\triangle\tau \pm \tilde{Q}\cos\triangle\tau_1\right]^2 + \left[Q\sin\triangle\tau \pm \tilde{Q}\sin\triangle\tau_1\right]^2}. \end{aligned} \tag{6.36}$$

In the next lemma we prove the existence of 4 critical points of $K^{(4)}$ for η small enough, provided $E^* > 0$.

Lemma 6.6. *Assume $E^* > 0$ in (6.36). If $0 < \eta \preceq E^*$, the function $K^{(4)}(\psi)$ has 4 nondegenerate critical points: $\psi^{(1)} = (\alpha^{(+)}, 0) + \mathcal{O}(\eta), \psi^{(2)} = (\alpha^{(+)} + \pi, 0) + \mathcal{O}(\eta), \psi^{(3)} = (\alpha^{(-)}, \pi) + \mathcal{O}(\eta), \psi^{(4)} = (\alpha^{(-)} + \pi, \pi) + \mathcal{O}(\eta)$, where*

$$\cos\alpha^{(\pm)} = \frac{1 - Q + Q\cos\triangle\tau \pm \tilde{Q}\cos\triangle\tau_1}{E^{(\pm)}}, \ \sin\alpha^{(\pm)} = \frac{Q\sin\triangle\tau \pm \tilde{Q}\sin\triangle\tau_1}{E^{(\pm)}}. \tag{6.37}$$

At the critical points,

$$|\det D^2 K^{(4)}(\psi^{(1,2)})| = B^2(\eta E^{(+)} + \mathcal{O}(\eta^2)),$$

$$|\det D^2 K^{(4)}(\psi^{(3,4)})| = B^2(\eta E^{(-)} + \mathcal{O}(\eta^2)).$$

Proof. In order to find the critical points of $K^{(4)}(\psi)$, we study the equations

$$\begin{aligned} &(1-Q)\sin\psi_1 + Q\sin(\psi_1 - 2\psi_2 - \triangle\tau) + \tilde{Q}\sin(\psi_1 - \psi_2 - \triangle\tau_1) = 0 \\ &-\sin\psi_2 + 2\eta Q\sin(\psi_1 - 2\psi_2 - \triangle\tau) + \eta\tilde{Q}\sin(\psi_1 - \psi_2 - \triangle\tau_1) = 0. \end{aligned} \tag{6.38}$$

We want to deduce these equations to $\sin\psi_1 = G_1(\psi_1,\psi_2)$, $\sin\psi_2 = G_2(\psi_1,\psi_2)$, then apply Lemma B.2 (the Fixed Point theorem, see Appendix B) to obtain 4 simple solutions.

Indeed, from the second equation we conclude the equation with a small in η left part

$$\sin\psi_2 = \eta f(\psi_1,\psi_2) = O(\eta),$$

where

$$f(\psi_1,\psi_2) = 2Q\sin(\psi_1 - 2\psi_2 - \triangle\tau) + \tilde{Q}\sin(\psi_1 - \psi_2 - \triangle\tau_1). \tag{6.39}$$

The function $f(\psi_1,\psi_2)$ and its derivatives are bounded: $|f| \le 2Q + \tilde{Q} = \bar{M} < 5/2$, $|\partial f/\partial\psi_1| \le \bar{M}$, $|\partial f/\partial\psi_2| \le 4Q + \tilde{Q} = \bar{N} < 9/2$. Thus, by Lemma B.2 the equation has solution if $\eta^2(\bar{M}^2 + (\bar{M}+\bar{N})^2) < \eta[(5/2)^2 + (14/2)^2] < 1$ or $\eta < \frac{2}{\sqrt{221}}$. We will find first the solution $\psi_2^{(1)} \in [-\frac{\pi}{2},\frac{\pi}{2}]$, and then $\psi_2^{(2)} \in [\frac{\pi}{2},\frac{3\pi}{2}]$.

For $\psi_2^{(1)} \in [-\frac{\pi}{2},\frac{\pi}{2}]$, we have $\sin\psi_2 = \eta f(\psi_1,\psi_2)$ and $\cos\psi_2 = \sqrt{1-\eta^2 f(\psi_1,\psi_2)} = 1 + \eta^2 f_1(\psi_1,\psi_2;\eta)$, where for η small enough, for example $\eta < \frac{2}{\sqrt{221}}$, f_1 is some function bounded with its derivatives: $|f_1| \le \bar{M}^2$, $|\partial f_1/\partial\psi_1| < 3\bar{M}^2$, $|\partial f_1/\partial\psi_2| \le 3\bar{M}\bar{N}$. Putting this into the first equation of (6.38), we get the equation $F_\eta^{(+)}(\psi_1,\psi_2) = 0$, with the function

$$\begin{aligned} F_\eta^{(+)} &= (1-Q)\sin\psi_1 + Q\sin(\psi_1 - \triangle\tau) + \tilde{Q}\sin(\psi_1 - \triangle\tau_1) - \eta f_2(\psi_1,\psi_2;\eta), \\ &= \Big[1 - Q + Q\cos\triangle\tau + \tilde{Q}\cos\triangle\tau_1\Big]\sin\psi_1 + \Big[Q\sin\triangle\tau + \tilde{Q}\sin\triangle\tau_1\Big]\cos\psi_1 \\ &\quad -\eta f_2(\psi_1,\psi_2;\eta), = E^{(+)}\sin(\psi_1 - \alpha^{(+)}) - \eta f_2(\psi_1,\psi_2;\eta), \end{aligned}$$

where $E^{(+)}$ and $\alpha^{(+)}$ are the constants defined in (6.36) and (6.37), respectively, and function f_2 is also bounded with its derivatives: $|f_2| \le M_1 < 9$, $|\partial_{\psi_1} f_2| \le M_2 22.1$, $|\partial_{\psi_2} f_2| \le M_3 < 30.3$. Thus, if $E^{(+)} \ne 0$ or provided one of the conditions

$$1 - Q + Q\cos\triangle\tau + \tilde{Q}\cos\triangle\tau_1 \ne 0 \quad \text{or} \quad Q\sin\triangle\tau + \tilde{Q}\sin\triangle\tau_1 \ne 0 \tag{6.40}$$

is satisfied, the equation $F_\eta^{(+)} = 0$ is derived to

$$\sin(\psi_1 - \alpha^{(+)}) = \eta f_2/E^{(+)}.$$

By Lemma B.2 (the Fixed Point theorem in Appendix B), if

$$\Big(\frac{\eta}{E^{(+)}}\Big)^2 (M_1^2 + (M_2 + M_3)^2) < \frac{\eta^2}{(E^*)^2} 2827 < 1 \Rightarrow \eta \preceq E^*, \tag{6.41}$$

the equation has 2 solutions $\psi_1^{(1)} - \alpha^{(+)} \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ and $\psi_1^{(2)} - \alpha^{(+)} \in [\frac{\pi}{2}, \frac{3\pi}{2}]$.

We proceed analogously for $\psi_2^{(2)} \in [\frac{\pi}{2}, \frac{3\pi}{2}]$. Substitute $\sin\psi_2 = \eta f(\psi_1, \psi_2)$ and $\cos\psi_2 = -1 - \eta^2 f_1(\psi_1, \psi_2)$ into the first equation of (6.38), we find

$$F^{(-)} = \Big[1 - Q + Q\cos\triangle\tau - \tilde{Q}\cos\triangle\tau_1\Big]\sin\psi_1 - \Big[Q\sin\triangle\tau - \tilde{Q}\sin\triangle\tau_1\Big]\cos\psi_1 \\ -\eta f_3(\psi_1, \psi_2) = E^{(-)}\sin(\psi_1 - \alpha^{(+)}) - \eta f_3(\psi_1, \psi_2) = 0,$$

where $|f_3| \leq M_1$, $|\partial f_3/\partial\psi_1| \leq M_2$, $|\partial f_3/\partial\psi_2| \leq M_3$. This equation can have solution only if $E^{(-)} \neq 0$, i.e. if any of the conditions

$$1 - Q + Q\cos\triangle\tau - \tilde{Q}\cos\triangle\tau_1 \neq 0, \quad \text{or} \quad Q\sin\triangle\tau - \tilde{Q}\sin\triangle\tau_1 \neq 0 \qquad (6.42)$$

holds. Under this condition, the equation is

$$\sin(\psi_1 - \alpha^{(-)}) = \eta f_3/E^{(-)}.$$

which gives two solutions $\psi_1^{(3)} - \alpha^{(-)} \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ and $\psi_1^{(4)} - \alpha^{(-)} \in [\frac{\pi}{2}, \frac{3\pi}{2}]$ if additionally the condition (6.41) is true.

To compute the determinant, we have

$$\det D^2K^{(4)}(\psi) = B^2(\eta\cos\psi_2((1-Q)\sin\psi_1 + Q\sin(\psi_1 - 2\psi_2 - \triangle\tau) \\ +\tilde{Q}\sin(\psi_1 - \psi_2 - \triangle\tau_1)) \qquad + \mathcal{O}(\eta^2))$$

for any $\psi \in \mathbb{T}^2$. At $\psi^{(1)}$, for example, we have

$$\det D^2K^{(4)}(\psi^{(1)}) = B^2\left(\eta \left.\frac{\partial F_\eta^{(+)}}{\partial\psi_1}\right|_{\psi^{(1)}} \cos\psi_2^{(1)} + \mathcal{O}(\eta^2)\right) = B^2(\eta E^{(+)} + \mathcal{O}(\eta^2)),$$

and similarly with $\psi^{(2)}$, $\psi^{(3)}$, $\psi^{(4)}$. □

To find values of the parameters at which the conditions (6.40) and (6.42) fail, i.e. the following equalities hold:

$$E^{(+)} = 0: \quad 1 - Q + Q\cos\triangle\tau = -\tilde{Q}\cos\triangle\tau_1 \quad Q\sin\triangle\tau = -\tilde{Q}\sin\triangle\tau_1 \\ E^{(-)} = 0: \quad 1 - Q + Q\cos\triangle\tau = \tilde{Q}\cos\triangle\tau_1 \quad Q\sin\triangle\tau = \tilde{Q}\sin\triangle\tau_1,$$

we consider the points $P_1 = (1 - Q + Q\cos\triangle\tau, Q\sin\triangle\tau)$, $P_2 = (\tilde{Q}\cos\triangle\tau_1, \tilde{Q}\sin\triangle\tau_1)$ and $P_3 = (-\tilde{Q}\cos\triangle\tau_1, -\tilde{Q}\sin\triangle\tau_1)$. These points lie on the circles represented in Figure 6.6. Hence, $E^{(+)}$ is the distance P_1P_3, while $E^{(-)}$ is P_1P_2. Checking all the possible cases of the mutual location of the circles, we can see when the points P_1 coincides with P_2 and P_3 (and, thus, $E^{(-)} = 0$ and $E^{(+)} = 0$, respectively). From Figure 6.7 we get for $E^{(-)}$:

- if $1 - 2Q > \tilde{Q}$, $E^{(-)} > 0 \quad \forall\triangle\tau, \triangle\tau_1$,
- if $1 - 2Q = \tilde{Q}$, $E^{(-)} = 0$ when $\triangle\tau = \pi, \triangle\tau_1 = 0$,
- if $|1 - 2Q| < \tilde{Q}$, $E^{(-)} = 0$ when $\cos(\triangle\tau) = -\frac{1-2Q+2Q^2-\tilde{Q}^2}{2(1-Q)Q}, \cos\triangle\tau_1 = -\frac{\tilde{Q}^2+1-2Q}{2(1-Q)\tilde{Q}}$,
- if $1 - 2Q = -\tilde{Q}$, $E^{(-)} = 0$ when $\triangle\tau = \pi, \triangle\tau_1 = \pi$,
- if $1 - 2Q < -\tilde{Q}$, $E^{(-)} > 0 \quad \forall\triangle\tau, \triangle\tau_1$,

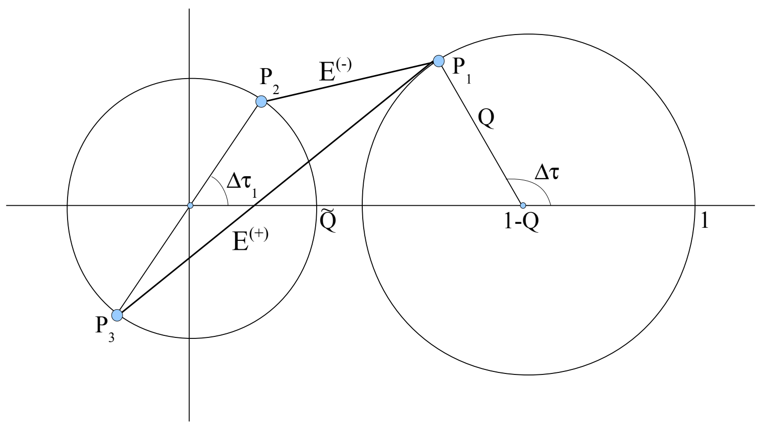


Figure 6.6: Geometrical representation of $E^{(+)}$ and $E^{(-)}$.

while for $E^{(+)}$ we have

$$
\begin{array}{lll}
\text{if } 1-2Q > \tilde{Q}, & E^{(+)} > 0 & \forall \triangle\tau, \triangle\tau_1, \\
\text{if } 1-2Q = \tilde{Q}, & E^{(+)} = 0 & \text{when } \triangle\tau = \pi, \triangle\tau_1 = \pi, \\
\text{if } |1-2Q| < \tilde{Q}, & E^{(+)} = 0 & \text{when } \cos(\triangle\tau) = -\frac{1-2Q+2Q^2-\tilde{Q}^2}{2(1-Q)Q}, \cos(\triangle\tau_1) = \frac{\tilde{Q}^2+1-2Q}{2(1-Q)\tilde{Q}}, \\
\text{if } 1-2Q = -\tilde{Q}, & E^{(+)} = 0 & \text{when } \triangle\tau = \pi, \triangle\tau_1 = 0, \\
\text{if } 1-2Q < -\tilde{Q}, & E^{(+)} > 0 & \forall \triangle\tau, \triangle\tau_1,
\end{array}
$$

Also we can provide a sufficient condition for $E^* \neq 0$ if we take into account that $0 < \tilde{Q} \leq 1/2$. Thus, the critical value for $\tilde{Q}$ and the radius of the circle centered at the origin is 1/2. In Figure 6.8 we consider two circles: C_1 is with radius 1/2 and C_2 is the unit circle. The map $Q \to (1-Q+Q\cos\triangle\tau, Q\sin\triangle\tau)$, for $0 \leq Q \leq 1$, gives us a family of straight lines connecting points $(1,0)$, at $Q=0$, and $(\cos\triangle\tau, \sin\triangle\tau) \in C_2$, at $Q=1$. We can see in Figure 6.8 that the straight lines don't intersect the circle C_1, i.e. $E^* > 0$, for the following values of the parameters:

$$|\triangle\tau| < \tau^* = \frac{2\pi}{3}, \quad \forall \triangle\tau_1 \in \mathbb{T}, 0 < Q1, 0 < \tilde{Q} \leq 1/2. \tag{6.43}$$

It turns out that both conditions (6.40) and (6.42) are true for $\triangle\tau = \triangle\tau_1 = 0$, since $1+\tilde{Q} \neq 0$ and $1-\tilde{Q} \neq 0$. Thus, $E^{(\pm)} = 1 \pm \tilde{Q}, \alpha^{(\pm)} = 0$. We get 4 critical points $\psi^{(1)} = (0,0) + \mathcal{O}(\eta)$, $\psi^{(2)} = (\pi,0) + \mathcal{O}(\eta)$, $\psi^{(3)} = (0,\pi) + \mathcal{O}(\eta)$, $\psi^{(4)} = (\pi,\pi) + \mathcal{O}(\eta)$).

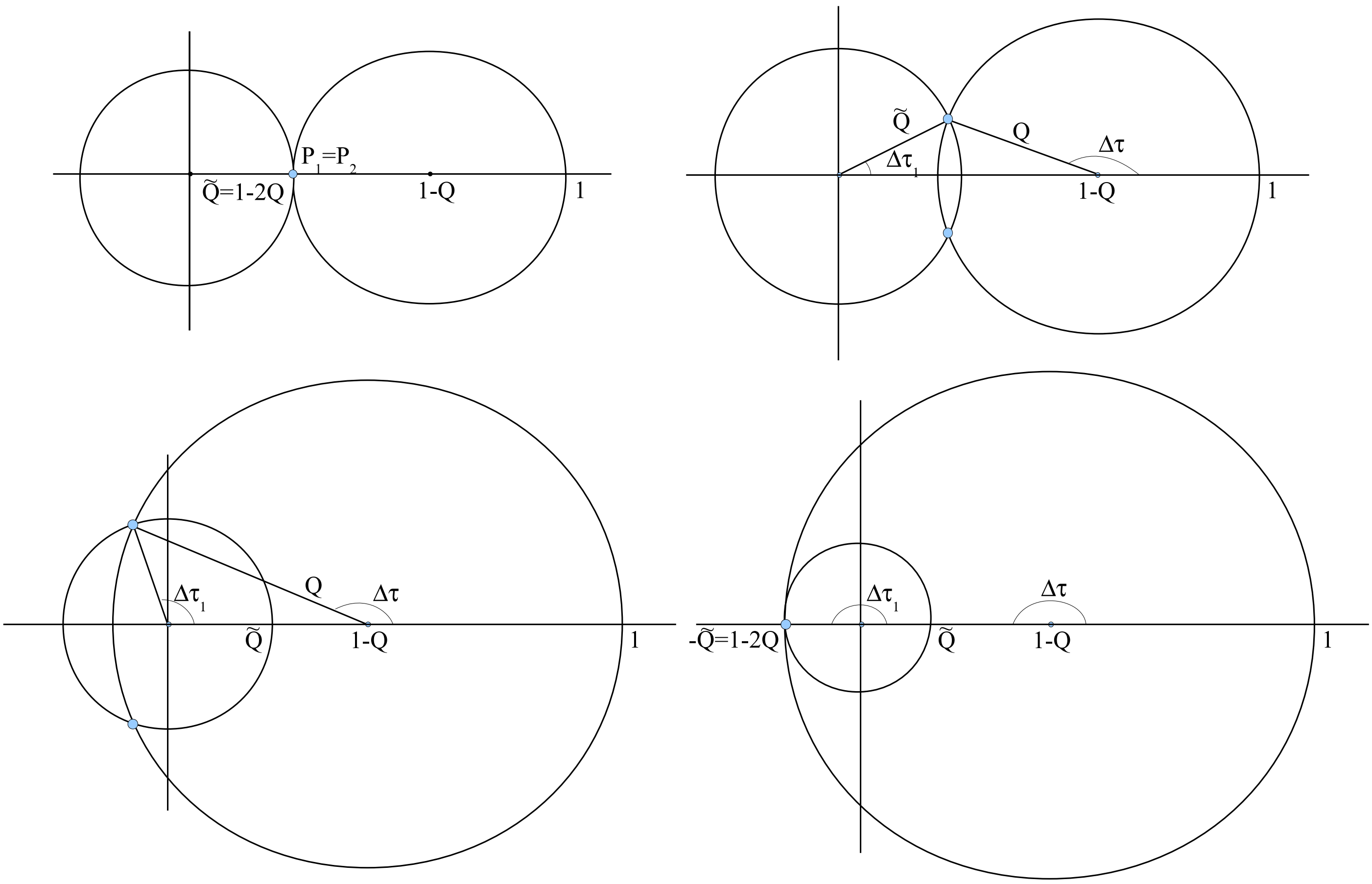


Figure 6.7: Cases of $E^{(\pm)} = 0$.

We prove the persistence of the critical points $\psi^{(j)}$, $j = 1, 2, 3, 4$, of the approximation $K^{(4)}(\psi)$ in the whole function $K(\psi)$ if we consider also non-dominant terms:

$$K(\psi) = K^{(4)}(\psi) + B\eta\eta'\tilde{G}(\psi),$$

where $B\eta\eta'\tilde{G}(\psi)$ correspond to the sum of all non-dominant harmonics and $\mathcal{L}_{S_5} = B\eta\eta'$ is the largest among them. Thus, we have

$$\eta' = \frac{\mathcal{L}_{S_5}}{\mathcal{L}_{s_0(n-1)} + \mathcal{L}_{s_0(n+1)}} \ll Q, \tilde{Q}.$$

From Lemma 6.2 (c), one gets the following bounds for function $G(\psi)$:

$$\begin{array}{lll} |G| \preceq 1, & |\partial_{\psi_1} G| \preceq \varepsilon^{-1/2}, & |\partial_{\psi_2} G| \preceq \varepsilon^{-1/2}, \\ |\partial^2_{\psi_1\psi_1} G| \preceq \varepsilon^{-1} & |\partial^2_{\psi_1\psi_2} G| \preceq \varepsilon^{-1} & |\partial^2_{\psi_2\psi_2} G| \preceq \varepsilon^{-1} \end{array} \tag{6.44}$$

Lemma 6.7. *Assume $E^* > 0$ in (6.36). If $\bar{\eta} = \max(\eta, \eta') \preceq E^*\varepsilon^2$, then the function $K(\psi)$ has 4 critical points, all nondegenerate: $\psi_*^{(j)} = \psi_{(j),0} + \mathcal{O}(\bar{\eta})$, $j = 1, 2, 3, 4$, where*

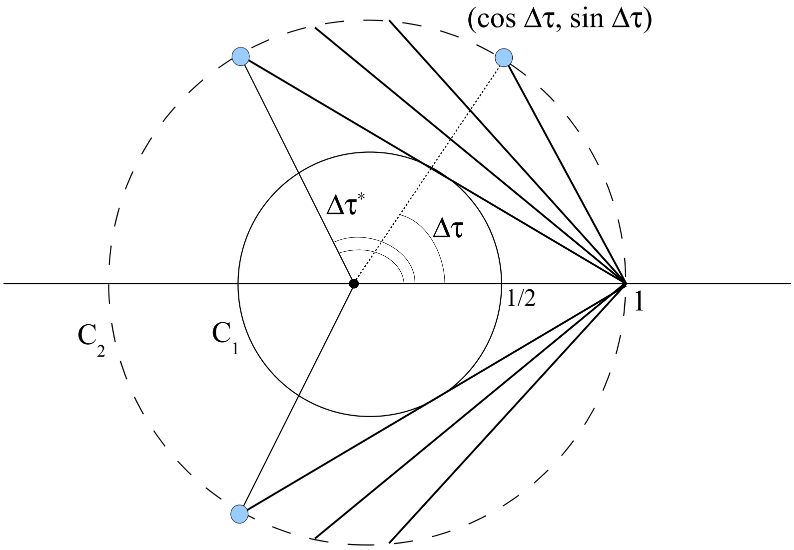


Figure 6.8: $E^{(\pm)} > 0$ (the straight lines don't intersect the circle C_1) if $|\triangle\tau| < \frac{2\pi}{3}$.

$\psi_{(1),0} = (\alpha^{(+)}, 0)$, $\psi_{(2),0} = (\alpha^{(+)} + \pi, 0)$, $\psi_{(3),0} = (\alpha^{(-)}, \pi)$, $\psi_{(4),0} = (\alpha^{(-)} + \pi, \pi)$. *At the critical points,*

$$|\det D^2 K^{(4)}(\psi^{(1,2)})| = B^2 \eta (E^{(+)} + \mathcal{O}(\bar{\eta}/\varepsilon)),$$
$$|\det D^2 K^{(4)}(\psi^{(3,4)})| = B^2 \eta (E^{(-)} + \mathcal{O}(\bar{\eta}/\varepsilon)).$$

Proof. We proceed analogously as in the proof of Lemma 6.6 and get that the critical points of $K(\psi)$ are the solution of

$$\begin{array}{l} F^{(+)}_{\eta,\eta'}(\psi) = 0 \\ \sin\psi_2 = \eta f(\psi_1,\psi_2) + \eta\eta' \tilde{g}_2(\psi_1,\psi_2), \text{ if } \psi_2 \in [-\frac{\pi}{2}, \frac{\pi}{2}] \end{array}$$

and the solution of

$$\begin{array}{l} F^{(-)}_{\eta,\eta'}(\psi) = 0 \\ \sin\psi_2 = \eta f(\psi_1,\psi_2) + \eta\eta' \tilde{g}_2(\psi_1,\psi_2), if \psi_2 \in [-\frac{\pi}{2}, \frac{\pi}{2}] \end{array} \tag{6.45}$$

with

$$\begin{array}{l} F^{(+)}_{\eta,\eta'}(\psi) = E^{(+)} \sin(\psi_1 - \alpha^{(+)}) - \eta f_2(\psi_1,\psi_2;\eta,\eta') + \eta' g_1(\psi_1,\psi_2) \\ F^{(-)}_{\eta,\eta'}(\psi) = E^{(-)} \sin(\psi_1 - \alpha^{(-)}) - \eta f_3(\psi_1,\psi_2;\eta,\eta') + \eta' g_1(\psi_1,\psi_2) \end{array} \tag{6.46}$$

In the last equations, $E^{(\pm)}$ and $\alpha^{(\pm)}$ are the constants defined in (6.36) and (6.37), respectively, the function f is defined in (6.39) and it is bounded $|f| \le \bar{M}$, $|\partial_{\psi_1} f| \le \bar{M}$,

$|\partial_{\psi_2} f| \le \bar{N}$, the functions f_2, f_3 are bounded: $|f_2| \le \tilde{M}_1$, $|\partial f_2/\partial\psi_1| \le \tilde{M}_2$, $|\partial f_2/\partial\psi_2| \le \tilde{M}_3$, $|f_3| \le \tilde{M}_1$, $|\partial f_3/\partial\psi_1| \le \tilde{M}_2$, $|\partial f_3/\partial\psi_2| \le \tilde{M}_3$ and the functions $\tilde{g}_1 = \partial_{\psi_1}\tilde{G}$ and $\tilde{g}_2 = \partial_{\psi_2}\tilde{G}$ have bounds according to Lemma 6.2(c)

$$\begin{array}{lll} |\tilde{g}_1| \sim \varepsilon^{-1/2}, & |\partial_{\psi_1}\tilde{g}_1| \sim \varepsilon^{-1}, & |\partial_{\psi_2}\tilde{g}_1 2| \sim \varepsilon^{-1}, \\ |\tilde{g}_2| \sim \varepsilon^{-1/2}, & |\partial_{\psi_1}\tilde{g}_2| \sim \varepsilon^{-1}, & |\partial_{\psi_2}\tilde{g}_2| \sim \varepsilon^{-1}. \end{array}$$

Therefore, the conditions (6.41) are refined as

$$\begin{array}{l} \left(\frac{\bar{\eta}}{E^{(\pm)}}\right)^2((\tilde{M}_1+\varepsilon^{-1/2})^2+(\tilde{M}_2+\tilde{M}_3+2\varepsilon^{-1})^2)<1 \\ \eta^2((\bar{M}+\eta'\varepsilon^{-1/2})^2+(\bar{M}+\bar{N}+2\eta'\varepsilon^{-1})^2)<1 \end{array}$$

These conditions are satisfied if the following inequalities are true, respectively,

$$\frac{\bar{\eta}^2}{(E^*)^2}\cdot\frac{1}{\varepsilon^2}\left[\varepsilon(\tilde{M}_1\varepsilon^{1/2}+1)^2+((\tilde{M}_2+\tilde{M}_3)\varepsilon+2)^2\right]<1$$

(this inequality is true for $\bar{\eta} \preceq E^*\varepsilon$), and

$$\frac{\eta^2}{\varepsilon^2}\cdot\left[\varepsilon(\bar{M}\varepsilon^{1/2}+\eta')^2+((\bar{M}+\bar{N})\varepsilon+2\eta')^2\right]<1$$

(this inequality is fulfilled for $\eta\eta' \le \bar{\eta}^2 \preceq \varepsilon$) We choose $\bar{\eta} \preceq \min\{E^*\varepsilon, \varepsilon^{1/2}\} = E^*\varepsilon$, and, under this condition, according to Lemma B.2 (the Fixed Point theorem in Appendix B), the equations (6.45-6.46) have 4 simple solutions $\psi_*^{(j)}, j = 1,2,3,4$.

The determinant is

$$\begin{array}{rcl} \det D^2K^{(4)}(\psi) & = & B^2\eta(\cos\psi_2((1-Q)\sin\psi_1+Q\sin(\psi_1-2\psi_2-\triangle\tau) \\ & & +\tilde{Q}\sin(\psi_1-\psi_2-\triangle\tau_1))+\mathcal{O}(\bar{\eta}/\varepsilon)) \end{array}$$

for any $\psi \in \mathbb{T}^2$. At the point $\psi_*^{(1)} = (\alpha^{(+)}, 0) + \mathcal{O}(\bar{\eta})$, we have

$$\det D^2K^{(4)}(\psi_*^{(1)}) = B^2\eta\left(\left.\frac{\partial F^{(+)}_{\eta,\eta'}}{\partial\psi_1}\right|_{\psi_*^{(1)}}\cos\psi_{*2}^{(1)}+\mathcal{O}(\frac{\bar{\eta}}{\varepsilon})\right) = B^2\eta(E^{(+)}+\mathcal{O}(\bar{\eta}/\varepsilon)),$$

and the same with $\psi_*^{(2)}$, $\psi_*^{(3)}$, $\psi_*^{(4)}$. □

Applying the inverse (one-to-one) linear change of (6.35), the critical points $\psi^{(j)}, j = 1,2,3,4$, of $K^{(4)}$ give rise to 4 nondegenerate critical points of $\mathcal{L}^{(4)}$:

$$\theta_*^{(j)} = \mathcal{A}^{-1}(\psi_*^{(j)}+b), \quad j = 1,2,3,4. \tag{6.47}$$

Lemma 6.8. *Assume $E^* > 0$ in (6.36). If $\bar{\eta} = \max(\eta,\eta') \preceq E^*\varepsilon^2$, the splitting potential $\mathcal{L}$ has exactly 4 critical points $\theta_*^{(j)}$, given by (6.33), all nondegenerate, and the minimal eigenvalue (in modulus) $m_{(j)}$ of $D^2\mathcal{L}(\theta_*^{(j)})$ satisfies*

$$E^*\sqrt{\varepsilon}\mathcal{L}_{S_2} \preceq m_{(j)} \preceq \sqrt{\varepsilon}\mathcal{L}_{S_2}, \quad j = 1,2,3,4. \tag{6.48}$$

Proof. The proof is similar to the one of Lemma 6.5 and, thus, we give here only a sketch of the proof. We have the following expression for the minimal (in modulus) eigenvalue:

$$m_{(j)} = \frac{2|D|}{|T| + \sqrt{T^2 - 4D}},$$

where $D = \det D^2\mathcal{L}(\theta_*^{(j)})$ and $T = trD^2\mathcal{L}(\theta_*^{(j)})$. Thus, we need to find estimates for D and T.

It is clear that

$$D = \det D^2\mathcal{L}(\theta) = \det((\mathcal{A})^\top D^2K(\psi)\mathcal{A}) = \det D^2K(\psi),$$

since $|\det\mathcal{A}| = 1$. Then

$$\begin{aligned} |D| &= B^2\eta(E^{(\pm)} + \mathcal{O}(\bar\eta/\varepsilon)) \sim \mathcal{L}^2_{s_0(n)}\eta E^{(\pm)} \sim E^{(\pm)}\mathcal{L}_{s_0(n)}(\mathcal{L}_{s_0(n-1)} + \mathcal{L}_{s_0(n+1)}) \\ &\succeq E^*\mathcal{L}_{s_0(n)}(\mathcal{L}_{s_0(n-1)} + \mathcal{L}_{s_0(n+1)}) \sim E^*\mathcal{L}_{S_1}\mathcal{L}_{S_2}, \end{aligned}$$

taking into account the definition of η and the facts that $B = \mathcal{L}_{S_1}$, $\mathcal{L}_{s_0(n-1)} \sim \mathcal{L}_{s_0(n+1)} \sim \mathcal{L}_{S_2}$, $E^{(\pm)} \geq E^*$.

If $D^2K = \begin{pmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{pmatrix}$, then

$$D^2\mathcal{L} = k_{11}s_0(n-1)\cdot s_0(n-1)^\top + k_{12}[s_0(n-1)\cdot s_0(n)^\top + S'\cdot s_0(n-1)^\top] + k_{22}s_0(n)\cdot s_0(n)^\top,$$

where the components of D^2K at the critical points $\psi_*^{(j)}$ are

$$\begin{aligned} |k_{11}| &= B\eta(E^{(\pm)} + \mathcal{O}(\bar\eta/\varepsilon)) \preceq B\eta \sim \mathcal{L}_{S_2}, \\ |k_{12}| &= B\eta(\tfrac{\partial f}{\partial\psi_1} + \mathcal{O}(\bar\eta/\varepsilon)) \preceq B\eta \sim \mathcal{L}_{S_2}, \\ |k_{22}| &= B + B\eta(\tfrac{\partial f}{\partial\psi_2} + \mathcal{O}(\bar\eta/\varepsilon)) \sim \mathcal{L}_{S_1}. \end{aligned}$$

We get that

$$T = k_{11}|s_0(n-1)|_2^2 + 2k_{12}\langle s_0(n-1), s_0(n)\rangle + k_{22}|s_0(n)|_2^2,$$

here we use the 2-norm $|x|_2 = \sqrt{(x_1)^2 + (x_2)^2}$, but since the 1-norm and 2-norm are equivalent ($|x|_1/\sqrt{2} \leq |x|_2 \leq |x|_1$), we can apply the estimates (6.23) to obtain

$$|T| \sim \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_{S_1}.$$

Since $|D| \ll T^2$, we can deduce that $m_{(j)} = \frac{|D|}{|T|}$, and we obtain the expected estimate. □

Therefore, recalling $\mathcal{M}(\theta) = \partial_\theta \mathcal{L}(\theta)$, it follows from Lemma 6.8 that the splitting function $\mathcal{M}(\theta)$ has 4 simple zeros θ_*, defined in (6.47), and the minimal eigenvalue of the splitting matrix $\partial\mathcal{M}(\theta_*)$ satisfies the estimate given in (6.48).

Recall also that $E^* > 0$ is fulfilled under the condition (6.43), i.e. when

$$|\sigma_{s_0(n+1)} - \sigma_{s_0(n-1)} + 2\sigma_{s_0(n)}| < \frac{2\pi}{3}$$

holds. Thus, imposing the condition $\sigma_k = 0$ on the phases of f in (5.7), we prove the continuation (without bifurcations) of the 4 transverse homoclinic orbits for all $\varepsilon \to 0$ in case of the silver frequency $\Omega = \Omega_2 = \sqrt{2} - 1$ in (3). Note that we have introduced the notation of '$\sim$' just before Theorem 5.1 in Chapter 5.

Theorem 6.3 (Transversality of the splitting for Ω_2)**.** *For the Hamiltonian system (5.1-5.7) with* $n = 2$ *and* $\Omega = \Omega_2$ *in (3), assume that* $0 < \varepsilon \ll 1$ *and* $\mu = \varepsilon^p$*,* $p > p^*$ *with* $p^* = 2$ *if* $\nu = 1$ *and* $p^* = 3$ *if* $\nu = 0$*, then if* $\sigma_k = 0$ *for all* $k \in \mathbb{Z}^2\backslash\{0\}$*, one has:*

- *the Melnikov function* $\mathcal{M}(\theta)$ *has exactly 4 zeros* θ_**, all simple, for all* ε*;*
- *The minimal eigenvalue of* $\partial_\theta\mathcal{M}(\theta_*)$ *satisfies*

$$m_* \sim \mu\varepsilon^{1/4} \exp\left\{-\frac{C_0 h_2(\varepsilon)}{\varepsilon^{1/4}}\right\}$$

where the constant C_0 *is defined in (6.12) and* $h_2(\varepsilon)$ *is the positive periodic in* $\ln\varepsilon$ *function defined in (6.20).*

Remark 6.8. The continuation of the homoclinic orbits was also proved in the case of the golden frequencies $\Omega = \Omega_1 = (\sqrt{5} - 1)/2$ in [DG03, DG04].

Chapter 7

Exponentially small splitting of separatrices for whiskered tori with cubic frequencies

In this chapter we study splitting of separatrices for a three-dimensional whiskered torus with cubic frequencies, paying special attention to the frequencies given by the *cubic golden number*. We prove that the Poincaré-Melnikov approach can be applied and establish the existence of 8 transverse homoclinic orbits associated to the whiskered torus.

We consider the Hamiltonian system, defined in (5.1-5.3), for $n = 3$ in which the frequency vector ω is given by

$$\omega = (1, \Omega, \Omega^2), \tag{7.1}$$

where Ω is an irrational cubic number (a real root of a polynomial of degree 3 with integer coefficients). In this chapter we put special emphasis to the concrete cubic number that is the real root of $\Omega^3 + \Omega - 1 = 0$ ($\Omega \approx 0.6823$), the so-called *cubic golden number*. It is well known that ω (7.1) satisfies a *Diophantine condition* of constant type

$$|\langle k, \omega \rangle| \geq \frac{\gamma}{|k|^\tau}, \quad \forall k \in \mathbb{Z}^3 \backslash \{0\} \tag{7.2}$$

with $\tau = 2$ (this is the least possible exponent for three-dimensional integer vectors, see for instance [Cas65]) and some $\gamma > 0$.

By the Poincaré-Melnikov method the splitting function $\mathcal{M}(\theta)$ and the splitting potential $\mathcal{L}(\theta)$ (recall $\mathcal{M}(\theta) = \partial_\theta \mathcal{L}(\theta)$) are approximated in first order in μ by the Melnikov function $M(\theta)$ and the Melnikov potential $L(\theta)$, given in (5.12), see details in Chapter 5). In the case $\mu = \varepsilon^p$, we have to justify the approach and ensure that the Poincaré-Melnikov approximation overcomes the error term (5.13).

In this chapter we generalize the technic developed in [DG04, DG03] as well as in Chapter 6 for two-dimensional quadratic frequency vectors to three-dimensional vectors $\omega = (1, \Omega, \Omega^2)$. It is worth mentioning that there is no standard theory of continued fractions in the case of three or more frequency vectors. This is the reason to consider the particular cases of a cubic frequency vector, just to be able to provide some results on exponentially small splitting of separatrices for 3 frequencies.

7.1 Cubic frequencies

Given a cubic frequency vector of form (7.1) with a cubic irrational number Ω. We want to give a classification of all three-dimensional non-zero integer vectors k according to the size of

$$\gamma_k = |\langle k, \omega\rangle||k|^2, \quad \forall k \in \mathcal{Z}^3 \setminus \{0\},$$

where we use the 2-norm $|\cdot| = |\cdot|_2$. The goal is to construct the resonant sequences, find the *primary resonances* (i.e. k's for which γ_k is smaller and, hence, they fit better the Diophantine condition (7.2)) and study their separation with respect to the secondary ones. Delshams and Gutiérrez [DG03, DG04] gave a technic to do this for the two-dimensional case with the quadratic golden number and for some other quadratic frequencies, here we aim to generalize their results for three-dimensional cubic frequencies. Note that the definition of γ_k is motivated by the fact that any cubic frequency is Diophantine with $\tau = 2$ [Cas65] satisfying (7.2).

We say that the integer vector k is admissible if $|\langle k, \omega\rangle| < 1/2$ and denote by $\mathcal{A}$ the set of admissible vectors.

The components of ω are a basis of the cubic field $\mathbb{Q}(\Omega)$ by Koch [Koc99], there exists a unimodular matrix (a square matrix with integer entries and determinant ± 1) T having eigenvector ω with associated eigenvalue λ ($|\lambda| > 1$, other eigenvalues are less than 1 in modulus). Let the eigenvalues of T be $\lambda_1^T = \lambda$, λ_2^T, λ_3^T and their associated eigenvectors be $v_1 = \omega$, v_2, v_3. We know that $\lambda_1^T \cdot \lambda_2^T \cdot \lambda_3^T = 1$. We distinguish two possible cases of the eigenvalues of T:

1. *the complex case*: the only one eigenvalue of T is real (that is λ) and the other two ones are a pair of complex conjugate numbers;

2. *the real case*: all the eigenvalues of T are real.

In this section we consider cubic numbers in the complex case.

Remark 7.1. 1. The matrix T satisfying the conditions above is not unique;

2. We can assume that $\det T = 1$, since if $\det T = -1$, we can consider matrix $-T$ instead (hence, in the complex case we assume that $\lambda > 0$)

3. To study the *real case* requires a different approach. In this case, the behavior of the associated small divisors seems to be different to the complex case considered here, and will require intensive numerical high-precision simulations in order to establish the properties of such vectors, and then try to obtain rigorous asymptotic estimates for the splitting.

Consider the matrix $U = (T^{-1})^{\top}$. Its eigenvalues are $\lambda_1^U = \lambda^{-1}$, $\lambda_2^U = (\lambda_2^T)^{-1}$, $\lambda_3^U = (\lambda_3^T)^{-1}$ and denote their associated eigenvectors by u_1, u_2, u_3.

In the complex case λ_2^T, λ_3^T as well as λ_2^U, λ_3^U are complex, and the same are vectors v_2, $v_3 = \bar{v}_2$, u_2, $u_3 = \bar{u}_2$. To avoid working with the complex vectors we consider $z_1 = Re(v_2)$, $z_2 = Im(v_2)$, $w_1 = Re(u_2)$, $w_2 = Im(u_2)$ as well as $a = Re(\lambda_2^T)$, $b = Im(\lambda_2^T)$. Let φ and r be the angle argument and modulus of λ_2^T, i.e. $\lambda_2^T = re^{i\varphi}$ (we have that $r^2 = 1/\lambda$ due to $\lambda\lambda_2^T\lambda_3^T = \lambda\lambda_2^T\bar{\lambda}_2^T = 1$).

Proposition 7.1. *If $\lambda_2^T = a + ib$, the following is true:*

(i) $\lambda_2^U = \lambda a - i\lambda b$;

(ii) $Tz_1 = az_1 - bz_2, \quad Tz_2 = bz_1 + az_2$;

(iii) $Uw_1 = \lambda a w_1 + \lambda b w_2, \quad Uw_2 = \lambda a w_2 - \lambda b w_1$;

(iv) $U^n w_1 = r^{-n}[w_1 \cos n\varphi + w_2 \sin n\varphi], U^n w_2 = r^{-n}[w_2 \cos n\varphi - w_1 \sin n\varphi]$;

(v) $U^{-1}w_1 = aw_1 - bw_2, \quad U^{-1}w_2 = bw_1 + aw_2$;

(vi) $\langle z_1, w_2\rangle = \langle z_2, w_1\rangle, \quad \langle z_1, w_1\rangle = -\langle z_2, w_2\rangle$;

(vii) $\langle \omega, w_1\rangle = \langle \omega, w_2\rangle = 0$.

Proof. (i) Since $\lambda\lambda_2^T\lambda_3^T = \lambda|\lambda_2^T|^2 = 1$, we get

$$\lambda_2^U = \frac{1}{\lambda_2^T} = \frac{a - ib}{a^2 + b^2} = \frac{a - ib}{|\lambda_2^T|^2} = \lambda(a - ib);$$

(ii) $Tv_2 = \lambda_2^T v_2$, thus, $Tz_1 + iTz_2 = (a + ib)(z_1 + iz_2) = (az_1 - bz_2) + i(az_2 + bz_1)$;

(iii) Analogously to (ii), taking into account (i);

(iv) $U^n u_2 = (\lambda_2^U)^n u_2 = (\lambda_2^T)^{-n} u_2 = r^{-n} e^{-in\varphi} u_2 = r^{-n}(\cos n\varphi - i \sin n\varphi)(w_1 + iw_2) = U^n(w_1 + iw_2)$.

(v) It follows from $U^{-1}\underbrace{(w_1 + iw_2)}_{u_2} = \underbrace{(\lambda_2^U)^{-1}}_{\lambda_2^T} u_2 = (a + ib)(w_1 + iw_2)$;

(vi) $\langle Tz_1, w_1\rangle = a\langle z_1, w_1\rangle - b\langle z_2, w_1\rangle$, on the other hand, $\langle Tz_1, w_1\rangle = \langle z_1, U^{-1}w_1\rangle = a\langle z_1, w_1\rangle - b\langle z_1, w_2\rangle$.
$\langle Tz_1, w_2\rangle = a\langle z_1, w_2\rangle - b\langle z_2, w_2\rangle$, and $\langle Tz_1, w_2\rangle = \langle z_1, U^{-1}w_2\rangle = a\langle z_1, w_2\rangle + b\langle z_1, w_1\rangle$. From this we conclude (vi).

(vii) $\langle T\omega, w_1\rangle = \lambda\langle\omega, w_1\rangle = \langle\omega, U^{-1}w_1\rangle = \lambda a\langle\omega, w_1\rangle + \lambda b\langle\omega, w_2\rangle$
$\langle T\omega, w_2\rangle = \lambda\langle\omega, w_2\rangle = \langle\omega, U^{-1}w_2\rangle = \lambda a\langle\omega, w_2\rangle - \lambda b\langle\omega, w_1\rangle$
Thus, we get equations

$$(1-a)\langle\omega, w_1\rangle + b\langle\omega, w_2\rangle = 0, \quad -b\langle\omega, w_1\rangle + (1-a)\langle\omega, w_2\rangle = 0$$

that have a unique solution ($\langle\omega, w_1\rangle = \langle\omega, w_2\rangle = 0$) if $(1-a)^2 + b^2 \neq 0$ that is true since $\lambda_2^T \neq 1$.

□

The following is also true:

$$\langle Uk, \omega\rangle = \langle k, U^\top\omega\rangle = \frac{1}{\lambda}\langle k, \omega\rangle.$$

It follows that if $k \in \mathcal{A}$, then also $Uk \in \mathcal{A}$. We say k is primitive if $k \in \mathcal{A}$ and $U^{-1}k$ is not in $\mathcal{A}$. We see that k is *primitive* if and only if

$$\frac{1}{2|\lambda|} < |\langle k, \omega\rangle| < \frac{1}{2}. \tag{7.3}$$

Admissible vectors can be presented in form

$$k^0(j) = (-\mathrm{rint}\,(j_1\Omega + j_2\Omega^2), j_1, j_2),$$

where $j = (j_1, j_2) \neq (0,0)$ and $\mathrm{rint}\,(a)$ is the closest integer to a. We say that two-dimensional integer vector j is primitive if $k^0(j)$ is primitive. Let $\mathcal{P}$ be the set of the primitive integer vectors j.

For each $j \in \mathcal{P}$, we define the *resonant sequence*

$$s(j,n) = U^n k^0(j), \quad \forall n = 0, 1, 2 \ldots$$

We prove the following result:

Proposition 7.2. *For any primitive j, one has*

(a) $|s(j,n)|^2 = \lambda^n[(c_2^j)^2 + (c_3^j)^2](K_1 + K_2\cos[2n\varphi - 2\psi_j - \theta]) + O(\lambda^{-n/2})$

(b) $\gamma_{s(j,n)} = |\langle s(j,n), \omega\rangle||s(j,n)|^2 = A_j(K_1 + K_2\cos[2n\varphi + 2\psi_j - \theta]) + O(\lambda^{-3n/2})$, *where*

- ◇ $A_j = |\langle k^0(j), \omega\rangle|\,[(c_2^j)^2 + (c_3^j)^2]$
- ◇ $K_1 = \frac{1}{2}(|w_1|^2 + |w_2|^2)$;
- ◇ $K_2 = \frac{1}{2}\sqrt{(|w_1|^2 - |w_2|^2)^2 + 4(\langle w_1, w_2\rangle)^2}$;
- ◇ $c_2^j = \frac{\langle z_1,w_1\rangle\langle z_1,k^0(j)\rangle + \langle z_1,w_2\rangle\langle z_2,k^0(j)\rangle}{(\langle z_1,w_1\rangle)^2 + (\langle z_1,w_2\rangle)^2}$, $c_3^j = \frac{\langle z_1,w_2\rangle\langle z_1,k^0(j)\rangle - \langle z_1,w_1\rangle\langle z_2,k^0(j)\rangle}{(\langle z_1,w_1\rangle)^2 + (\langle z_1,w_2\rangle)^2}$;

◇ $\varphi = \arg(\lambda_2^T)$;

◇ $\cos\psi_j = \frac{c_2^j}{\sqrt{(c_2^j)^2+(c_3^j)^2}}, \quad \sin\psi_j = \frac{c_3^j}{\sqrt{(c_2^j)^2+(c_3^j)^2}}$;

◇ $\cos\theta = \frac{|w_1|^2-|w_2|^2}{\sqrt{(|w_1|^2-|w_2|^2)^2+4(\langle w_1,w_2\rangle)^2}}, \qquad \sin\theta = \frac{2\langle w_1,w_2\rangle}{\sqrt{(|w_1|^2-|w_2|^2)^2+4(\langle w_1,w_2\rangle)^2}}$;

(c) As $n \to \infty$, $\gamma_{s(j,\cdot)}$ oscillates between two values

$$\gamma_j^- = |\langle k^0(j),\omega\rangle|[(c_2^j)^2+(c_3^j)^2](K_1-K_2), \quad \gamma_j^+ = |\langle k^0(j),\omega\rangle|[(c_2^j)^2+(c_3^j)^2](K_1+K_2); \tag{7.4}$$

(d) a lower bound for γ_j^- is

$$\gamma_j^- \geq \frac{1}{|\lambda||u_2|^2}\left[|j| - \frac{|u_1|}{2|\langle u_1,\omega\rangle|}\right]^2 (K_1 - K_2) \tag{7.5}$$

Proof. We present the vector $k^0(j)$ as a linear combination of u_1, w_1 and w_2

$$k^0(j) = c_1^j u_1 + c_2^j w_1 + c_3^j w_2,$$

where taking a scalar product with ω, z_1 and z_2 one can obtain the values of the coefficients c_i^j, $i = 1,2,3$. Thus, one finds

$$c_1^j = \frac{\langle k^0(j),\omega\rangle}{\langle u_1,\omega\rangle}, \tag{7.6}$$

as well as c_j^2 and c_j^3 are the solution of

$$c_2^j\langle z_1,w_1\rangle + c_3^j\langle z_1,w_2\rangle = \langle z_1,k^0(j)\rangle, \quad c_2^j\langle z_2,w_1\rangle + c_3^j\langle z_2,w_2\rangle = \langle z_2,k^0(j)\rangle,$$

given in (b). Using these formulae, we find

$$\begin{aligned} s(j,n) = & \; U^n k^0(j) = c_1^j U^n u_1 + c_2^j U^n w_1 + c_3^j U^n w_2 \\ = & \; c_1^j \lambda^{-n} u_1 + c_2^j r^{-n}[w_1\cos(n\varphi) + w_2\sin(n\varphi)] + c_3^j r^{-n}[w_2\cos(n\varphi) - w_1\sin(n\varphi)] \\ = & \; c_1^j \lambda^{-n} u_1 + r^{-n}[c_2^j\cos(n\varphi) - c_3^j\sin(n\varphi)]w_1 + r^{-n}[c_3^j\cos(n\varphi) + c_2^j\sin(n\varphi)]w_2 \\ = & \; c_1^j \lambda^{-n} u_1 + r^{-n}\sqrt{(c_2^j)^2 + (c_3^j)^2}\cos(n\varphi + \psi_j)w_1 \\ + & \; r^{-n}\sqrt{(c_2^j)^2 + (c_3^j)^2}\sin(n\varphi + \psi_j)w_2, \end{aligned}$$

where ψ_j is defined in (b).

We calculate further

$$|\langle s(j,n),\omega\rangle| = \frac{|\langle k^0(j),\omega\rangle|}{\lambda^n}. \tag{7.7}$$

It remains to find $|s(j,n)|^2$. We obtain

$$\begin{aligned} |s(j,n)|^2 = {} & \langle s(j,n), s(j,n)\rangle = r^{-2n}[(c_2^j)^2 + (c_3^j)^2]\,\{|w_1|^2 \cos^2[n\varphi + \psi_j] \\ + {} & |w_2|^2 \sin^2[n\varphi + \psi_j] + 2\langle w_1, w_2\rangle \sin[n\varphi + \psi_j] \cos[n\varphi + \psi_j]\} + \mathcal{O}(\lambda^{-n/2}) \\ = {} & \tfrac{1}{2}\lambda^n[(c_2^j)^2 + (c_3^j)^2]\,\{(|w_1|^2 + |w_2|^2) + (|w_1|^2 - |w_2|^2)\cos[2n\varphi + 2\psi_j] \\ + {} & 2\langle w_1, w_2\rangle \sin[2n\varphi + 2\psi_j]\} + \mathcal{O}(\lambda^{-n/2}) \\ = {} & \lambda^n[(c_2^j)^2 + (c_3^j)^2](K_1 + K_2\cos[2n\varphi + 2\psi_j - \theta]) + \mathcal{O}(\lambda^{-n/2}) \end{aligned}$$

where θ is defined in (b).

Therefore,

$$\gamma_{s(j,n)} = A_j(K_1 + K_2\cos[2n\varphi + 2\psi_j - \theta]) + O(\lambda^{-3n/2}).$$

Using (7.3), we obtain asymptotical bounds for $n \to \infty$

$$\begin{aligned} \limsup_{n\to\infty} \gamma_{s(j,n)} & \le \tfrac{1}{2}[(c_2^j)^2 + (c_3^j)^2](K_1 + K_2) \\ \liminf_{n\to\infty} \gamma_{s(j,n)} & \ge \tfrac{1}{\lambda}[(c_2^j)^2 + (c_3^j)^2](K_1 - K_2) \end{aligned}$$

We see that the sequence $\gamma_{s(j,n)}$ oscillates between two values γ_j^- and γ_j^+, defined in (7.4) Also we have that

$$\begin{aligned} |k^0(j) - c_1^j u_1|^2 &= |c_2^j w_1 + c_3^j w_2|^2 \le (c_2^j)^2|w_1|^2 + (c_3^j)^2|w_2|^2 \\ &\le [(c_2^j)^2 + (c_3^j)^2]|w_1|^2 + [(c_2^j)^2 + (c_3^j)^2]|w_2|^2 = [(c_2^j)^2 + (c_3^j)^2](|w_1|^2 + |w_2|^2) \\ &= [(c_2^j)^2 + (c_3^j)^2]|u_2|^2, \end{aligned}$$

then due to (7.6) and (7.3), one gets

$$(c_2^j)^2 + (c_3^j)^2 \ge \frac{|k^0(j) - c_j^1 u_1|^2}{|u_2|^2} \ge \frac{1}{|u_2|^2}\left[|k^0(j)| - \frac{|u_1|}{2|\langle u_1, \omega\rangle|}\right]^2 \ge \frac{1}{|u_2|^2}\left[|j| - \frac{|u_1|}{2|\langle u_1, \omega\rangle|}\right]^2$$

Applying this inequality and (7.3) again, one obtains the lower bound (7.5) □

The sequence of integer vectors $s(j,n)$ which gives the minimal lower γ_j^- and upper γ_j^+ bounds are called *primary resonances* and we denote them by $s_0(n) = s(j_0, n)$. Integer vectors are called *secondary resonances* if they belong to any of the remaining sequences $s(j,n)$, $j \ne j_0$. This can be defined if γ_j^- and γ_j^+ are simultaneously minimal (as happens in the golden cubic case). Such primary resonances can easily be detected thanks to Proposition 7.2(d): although $\gamma_j^\pm$ are not increasing in general, we have $\lim\limits_{|j|\to\infty} \gamma_j^\pm = \infty$, and then one has to check only a finite number of primitive vectors j in order to find the minimal γ_j^- and γ_j^+ and, hence, the primary resonances.

7.1.1 The cubic golden number

Assume that in (7.1) Ω is the real root of equation $x^3 + x = 1$ ($\Omega \approx 0.6823$). In this case the matrix T can be written in the form

$$T = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

and its eigenvalues (the roots of $x^3 - x^2 = 1$) and the associated eigenvectors are

$$\begin{aligned} \lambda_1 &= \lambda = \frac{1}{\Omega} \approx 1.4656, & \omega &= (1, \Omega, \Omega^2)^\top \\ \lambda_2 &= -\frac{\Omega^2}{2} + i\Omega L, & v_2 &= \left(1, -\frac{\Omega}{2} - iL, -1 - \frac{\Omega^2}{2} + i\Omega L\right)^\top \\ \lambda_3 &= -\frac{\Omega^2}{2} - i\Omega L, & v_3 &= \left(1, -\frac{\Omega}{2} + iL, -1 - \frac{\Omega^2}{2} - i\Omega L\right)^\top, \end{aligned}$$

where $|\lambda_1| > 1$, $|\lambda_2| = |\lambda_3| < 1$ and

$$L = \sqrt{\frac{3\Omega^2 + 4}{4}} = \frac{6\Omega^2 + 9\Omega + 4}{2\sqrt{31}} \approx 1.1615.$$

The matrix $U = (T^{-1})^\top$ is

$$U = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

The eigenvalues (the roots of $x^3 + x = 1$) and the eigenvectors of matrix U are

$$\begin{aligned} \lambda_1^U &= \Omega \approx 0.6823, & u_1 &= (1, \Omega^2, \Omega)^\top \\ \lambda_2^U &= -\frac{\Omega}{2} - iL, & u_2 &= \left(1, -1 - \frac{\Omega^2}{2} + i\Omega L, -\frac{\Omega}{2} - iL\right)^\top \\ \lambda_3^U &= -\frac{\Omega}{2} + iL, & u_3 &= \left(1, -1 - \frac{\Omega^2}{2} - i\Omega L, -\frac{\Omega}{2} + iL\right)^\top \end{aligned}$$

with $|\lambda_1^U| < 1$, $|\lambda_2^U| = |\lambda_3^U| \approx 1.2106 > 1$.

Therefore, following the notation of this section, we have

$$\begin{aligned} z_1 &= \left(1, -\frac{\Omega}{2}, -1 - \frac{\Omega^2}{2}\right)^\top, & z_2 &= (0, -L, \Omega L)^\top \\ w_1 &= \left(1, -1 - \frac{\Omega^2}{2}, -\frac{\Omega}{2}\right)^\top, & w_2 &= (0, \Omega L, -L)^\top. \end{aligned}$$

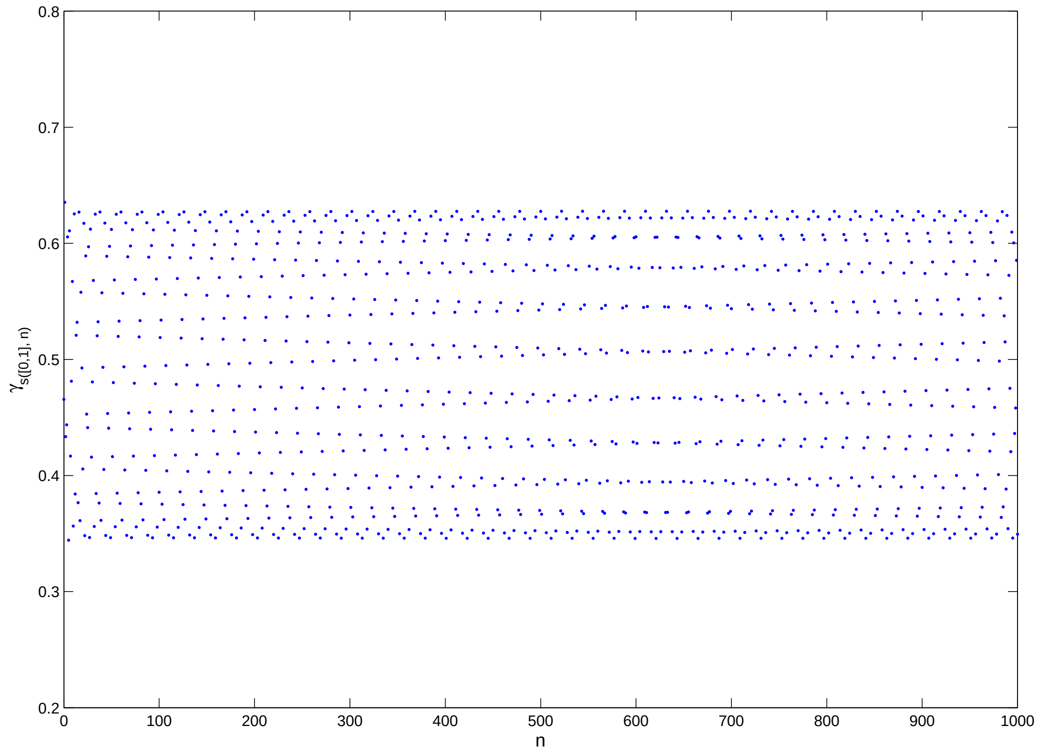


Figure 7.1: The values of $\gamma_{s_0(n)}$ oscillate between two quantities.

It is easy to check the following data

$$
\begin{array}{ll}
\langle z_1, w_1\rangle = -\langle z_2, w_2\rangle = \dfrac{3}{2} + \dfrac{\Omega}{2}, & \langle z_2, w_1\rangle = \langle z_2, w_2\rangle = L, \\
\langle w_1, \omega\rangle = \langle w_2, \omega\rangle = 0, & \langle z_1, u_1\rangle = \langle z_2, u_1\rangle = 0 \\
\langle \omega, u_1\rangle = 1 + 2\Omega^3 = 3 - 2\Omega, & \\
|w_1|^2 = \Omega^2 + \dfrac{\Omega}{4} + 2 & |w_2|^2 = \Omega^2 + \dfrac{3\Omega}{4} + 1 \\
\langle w_1, w_2\rangle = -\frac{L}{2}, & \\
K_1 = \Omega^2 + \dfrac{\Omega}{2} + \dfrac{3}{2}, & K_2 = \dfrac{1}{2}\sqrt{\Omega^2 - \Omega + 2}.
\end{array}
$$

Numerical experiments verify the statements of Proposition 7.2, namely, that for each primitive j the values of $\gamma_{s(j,n)}$ oscillate between two limit bounds γ_j^- and γ_j^+. For example, you can see in Figure 7.1 the graph $\gamma_{s(j,n)}$ depending on integer numbers n for the primitive vector $j = [0, 1]$, the corresponding limit bounds are $\gamma_{[0,1]}^- \approx 0.3459$ and $\gamma_{[0,1]}^+ \approx 0.6276$.

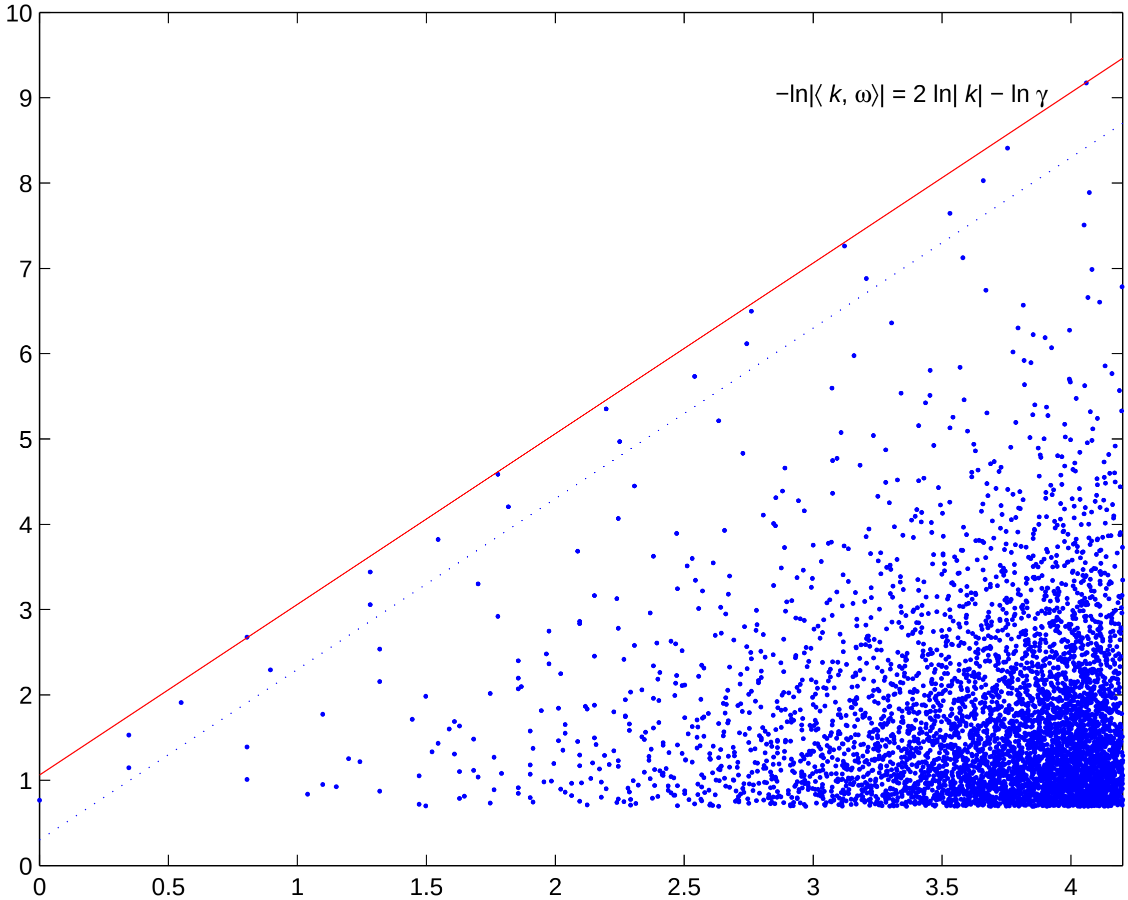


Figure 7.2: Points $(\ln|k|, -\ln|\langle k, \omega\rangle|)$.

Taking logarithm of the both hands of the Diophantine condition (7.2), we can write it as

$$-\ln|\langle k, \omega\rangle| \leq 2\ln|k| - \ln\gamma.$$

If we draw all the points with coordinates $(\ln|k|, -\ln|\langle k, \omega\rangle|)$ (see the Figure 7.2), we can see the sequence of the points laying closer to the straight line $-\ln|\langle k, \omega\rangle| = 2\ln|k| - \ln\gamma$. Such points correspond to integer vectors with minimal lower and upper bounds which are the primary resonances $s_0(\cdot)$.

In Table 7.3 we write down the values of the bounds γ_j^- and γ_j^+ for the resonant sequences induced by different primitive $k_0(j)$. In turns, we compute also the quantities c_2^j, c_3^j, $(c_2^j)^2 + (c_3^j)^2$, $A_j = |\langle k^0(j), \omega\rangle|[(c_2^j)^2 + (c_3^j)^2]$ and $B_j = \frac{A_j}{A_{[0,1]}}$ in Tables 7.1 and 7.1.1. The smallest ones correspond the first primitive vector $k_0([0,1]) = [0,0,1]$ (primary resonances). We can see a good separation of the primary resonances from the secondary ones.

To complete the study we give the values of the remaining quantities of Proposi-

$k_0(j)$	c_2^j	c_3^j	$(c_2^j)^2+(c_3^j)^2$
[0, 0, 1]	$-\frac{1}{31}(6-2\Omega+9\Omega^2)$	$-\frac{2}{31}L(2+9\Omega)$	$\frac{4}{31}(1+3\Omega+\Omega^2)\approx 0.4532$
[-1, 2, 0]	$-\frac{2}{31}(13+6\Omega+4\Omega^2)$	$\frac{8}{31}L(3-2\Omega)$	$\frac{4}{31}(7+4\Omega+8\Omega^2)\approx 1.7360$
[-2, 1, 2]	$-\frac{1}{31}(52-7\Omega+16\Omega^2)$	$-\frac{2}{31}L(7+16\Omega)$	$\frac{12}{31}(9+2\Omega+5\Omega^2)\approx 4.9132$
[-1, 0, 3]	$-\frac{1}{31}(36-12\Omega+23\Omega^2)$	$-\frac{2}{31}L(12+23\Omega)$	$\frac{4}{31}(24+19\Omega+18\Omega^2)\approx 5.8509$
[2, -3, 1]	$\frac{1}{31}(42+17\Omega+\Omega^2)$	$-\frac{2}{31}L(17-\Omega)$	$\frac{4}{31}(23+11\Omega+10\Omega^2)\approx 4.5369$
[-3, 3, 1]	$-\frac{1}{31}(72+7\Omega+15\Omega^2)$	$\frac{2}{31}L(7-15\Omega)$	$\frac{4}{31}(40+12\Omega+19\Omega^2)\approx 7.3592$
[0, -2, 2]	$-\frac{1}{31}(4-22\Omega+6\Omega^2$	$-\frac{4}{31}L(11+3\Omega)$	$\frac{16}{31}(5+\Omega+\Omega^2)\approx 3.8399$
$\lvert j\rvert_1>5$			≥ 2.1977

Table 7.1: Computing data: quantities c_2^j, c_3^j, $(c_2^j)^2+(c_3^j)^2$

$k_0(j)$	$\lvert\langle k^0(j),\omega\rangle\rvert$	A_j	B_j
[0, 0, 1]	$\Omega^2\approx 0.4656$	$4(3-2\Omega)/31\approx 0.2110$	1
[-1, 2, 0]	$-1+2\Omega\approx 0.3647$	$12(3-2\Omega)/31\approx 0.6330$	3
[-2, 1, 2]	$2-\Omega-2\Omega^2\approx 0.3866$	$36(3-2\Omega)/31\approx 1.8991$	9
[-1, 0, 3]	$-1+3\Omega^2\approx 0.3967$	$44(3-2\Omega)/31\approx 2.3211$	11
[2, -3, 1]	$2-3\Omega+\Omega^2\approx 0.4186$	$36(3-2\Omega)/31\approx 1.8991$	9
[-3, 3, 1]	$3-3\Omega-\Omega^2\approx 0.4745$	$68(3-2\Omega)/31\approx 3.5872$	17
[0, -2, 2]	$2\Omega-2\Omega^2\approx 0.4335$	$32(3-2\Omega)/31\approx 1.6881$	8
$\lvert j\rvert_1>5$		≥ 0.7498	≥ 3.55

Table 7.2: Computation data: $|\langle k^0(j),\omega\rangle|$, A_j, B_j

tion 7.2

$$\begin{array}{ll}
\varphi=\arg(-\frac{1}{2}\Omega^2+i\Omega L)=-\arctan\frac{2L}{\Omega}+\pi\approx 1.8565, & \frac{\pi}{2}<\varphi<\pi\\
\theta=-\arctan\frac{2L}{2-\Omega}\approx -1.0548, & -\frac{\pi}{2}<\theta<0\\
\psi_{[0,1]}=\arctan\frac{2L(2+9\Omega)}{6-2\Omega+9\Omega^2}-\pi\approx -2.0074, & -\pi<\psi_{[0,1]}<-\frac{\pi}{2}
\end{array}$$

Note that φ can be approximated by

$$\varphi\approx\frac{13\pi}{22}.$$

7.2 Asymptotic estimates

To justify that the Poincaré-Melnikov approach predicts correctly the size of splitting of separatrices, we need to provide asymptotic estimates for the dominant harmonics of the Melnikov potential L together with bounds of the sum of the remaining terms of L. These estimates have to be large enough such that the corresponding harmonics of the splitting $\mathcal{L}$ overcome the error of the Poincaré-Melnikov method. We work with functions L and $\mathcal{L}$, since its nondegenerate critical points correspond to simple zeros of

$k_0(j)$	$\gamma_j^{\pm}$	γ_j^-	γ_j^+
[0, 0, 1]	$\frac{2}{31}\left(5+\Omega+4\Omega^2 \pm (3-2\Omega)\sqrt{2-\Omega+\Omega^2}\right)$	0.3459	0.6276
[-1, 2, 0]	$\frac{6}{31}\left(5+\Omega+4\Omega^2 \pm (3-2\Omega)\sqrt{2-\Omega+\Omega^2}\right)$	1.0376	1.8829
[-2, 1, 2]	$\frac{18}{31}\left(5+\Omega+4\Omega^2 \pm (3-2\Omega)\sqrt{2-\Omega+\Omega^2}\right)$	3.1127	5.6488
[-1, 0, 3]	$\frac{22}{31}\left(5+\Omega+4\Omega^2 \pm (3-2\Omega)\sqrt{2-\Omega+\Omega^2}\right)$	3.8044	6.904
[2, -3, 1]	$\frac{18}{31}\left(5+\Omega+4\Omega^2 \pm (3-2\Omega)\sqrt{2-\Omega+\Omega^2}\right)$	3.1127	5.6488
[-3, 3, 1]	$\frac{34}{31}\left(5+\Omega+4\Omega^2 \pm (3-2\Omega)\sqrt{2-\Omega+\Omega^2}\right)$	5.8796	10.6699
[0, -2, 2]	$\frac{16}{31}\left(5+\Omega+4\Omega^2 \pm (3-2\Omega)\sqrt{2-\Omega+\Omega^2}\right)$	2.7669	5.0211
$\lvert j\rvert_1 > 5$		≥ 1.2289	≥ 2.2302

Table 7.3: Computation and numerical data: $\gamma_j^{\pm}$

M and $\mathcal{M}$, respectively, and give rise to transverse homoclinic orbits to the whiskered torus.

In order to find the dominant harmonics of the Melnikov potential L we proceed in the same way as for quadratic frequencies in Chapter 6. Substituting functions f and h from (5.6–5.7) into (5.12), we get the Fourier series of the Melnikov potential (in the section $s = const$):

$$L(\theta) = \sum_k L_k \cos(\langle k, \theta\rangle - \sigma_k), \tag{7.8}$$

where

$$L_k = \frac{2\pi |\langle k, \hat\omega_\varepsilon\rangle| e^{-\rho|k|}}{b \sinh |\frac{\pi}{2}\langle k, \hat\omega_\varepsilon\rangle|} \quad \text{and} \quad \hat\omega_\varepsilon = \frac{\tilde\omega_\varepsilon}{b} = \frac{b'\omega}{b\sqrt{\varepsilon}}.$$

We can present the coefficients L_k in the exponential form as

$$L_k = \alpha_k e^{-\beta_k},$$

where

$$\alpha_k = \frac{4\pi b' |\langle k, \omega\rangle|}{b^2\sqrt{\varepsilon}(1 - e^{-\{\pi|\langle k,\omega\rangle| \frac{b'}{b\sqrt{\varepsilon}}\}})}, \quad \beta_k = \rho|k| + \frac{\pi}{2}|\langle k, \omega\rangle| \frac{b'}{b\sqrt{\varepsilon}}. \tag{7.9}$$

Therefore, the largest L_k corresponds to the smallest β_k, and this depends strongly on the quantity $\gamma_k = |\langle k, \omega\rangle||k|^2$ studied in Section 7.1, p. 156.

We find the smallest β_k in a similar way as in Chapter 6. We write the functions β_k of (7.9) in the form

$$\beta_k(\varepsilon) = \varrho|k| + \frac{\pi}{2}\frac{\gamma_k}{|k|^2}\frac{b'}{b\sqrt{\varepsilon}} = \frac{C_\mu g_k(\varepsilon)}{\varepsilon^{1/6}}, \quad \text{where } g_k(\varepsilon) = \gamma_k^{1/3}\left(\frac{\varepsilon^{1/6}}{\varepsilon_k^{1/6}} + \frac{\varepsilon_k^{1/3}}{2\varepsilon^{1/3}}\right), k \in \mathbb{Z}^3\backslash\{0\} \tag{7.10}$$

where

$$C_\mu = \rho^{2/3}\left(\frac{\pi b'}{b}\right)^{1/3}, \quad \varepsilon_k^{1/2} = D_\mu \frac{\gamma_k}{|k|^3}, \quad D_\mu = \frac{\pi b'}{b\rho}.$$

Here we denote C_μ and D_μ to depend implicitly on μ, since b and b' are μ-close to 1. Indeed, C_μ and D_μ are μ-close to the constants

$$C_0 = \sqrt[3]{\rho^2\pi} \text{ and } D_0 = \pi/\rho. \tag{7.11}$$

For all k we have

$$\beta_k \geq \frac{3C_\mu\gamma_k^{1/3}}{2\varepsilon^{1/6}}. \tag{7.12}$$

which suggests that the size of β_k is related to the sequence $s(j,\cdot)$ to which k belongs, due to the fact that γ_k oscillates between two values (see Proposition 7.2). If γ_k is big, the corresponding L_k will not be dominant. We prove below that due to the fact that the separation between the primary and secondary resonances is big enough for the golden cubic number, at least 4 dominant harmonics of L are given by the primary vectors $k \in s(j_0,n) = s_0(\cdot)$.

We consider the functions g_k which contain the main information on the size of β_k. First, we consider the primary functions

$$g_n(\varepsilon) := \gamma_{s_0(n)}^{1/3}\left(\frac{\varepsilon^{1/6}}{\varepsilon_n^{1/6}} + \frac{\varepsilon_n^{1/3}}{2\varepsilon^{1/3}}\right), \quad \varepsilon_n^{1/2} := D_0\frac{\gamma_{s_0(n)}}{|s_0(n)|^3}, \tag{7.13}$$

where according to Proposition 7.2, we use the following approximations:

$$\gamma_{s_0(n)} \simeq A+B\cos(2\varphi n+\alpha), \quad |s_0(n)|^2 \simeq \lambda^n\frac{\gamma_{s_0(n)}}{|\langle k^0(j_0),\omega\rangle|} = \lambda^n\Omega^{-2}(A+B\cos(2\varphi n+\alpha)), \tag{7.14}$$

where we denote $A = A_jK_1 \approx 0.4867$, $B = A_jK_2 \approx 0.1408$, $\alpha = 2\psi_{j_0} - \theta$ (all the constants are defined in Proposition 7.2). Then, we obtain

$$\varepsilon_n = \frac{D_0^2\Omega^6}{\lambda^{3n}(A+B\cos(2\varphi n+\alpha))}. \tag{7.15}$$

We can see the functions g_n depends on the value of $\cos(2\varphi n+\alpha)$. Consider together with $g_n(\varepsilon)$ also two auxiliary functions $g_n^-(\varepsilon)$ and $g_n^+(\varepsilon)$ with the values $\cos(\ldots) = -1$ and $\cos(\ldots) = 1$, respectively:

$$g_n^\pm(\varepsilon) = \left(\gamma_n^\pm\right)^{1/3}\left[\frac{\varepsilon^{1/6}}{(\varepsilon_n^\pm)^{1/6}} + \frac{(\varepsilon_n^\pm)^{1/3}}{2\varepsilon^{1/3}}\right], \quad \gamma_n^\pm = A \pm B, \ \varepsilon_n^\pm = \frac{D^2\Omega^6}{\lambda^{3n}(A\pm B)}.$$

We prove the following proposition

Proposition 7.3 (Properties of g_n). *(a) The function $g_n(\varepsilon)$ has the minimum $3\gamma_{s_0(n)}^{1/3}/2$ at $\varepsilon = \varepsilon_n$. The points ε_n satisfy the formula*

$$\varepsilon_n^{1/2} = |\lambda|^{-n}\frac{1}{|s_0(n)|}\varepsilon_0^{1/2} \tag{7.16}$$

and they are decreasing, i.e. $\varepsilon_{n+1} \leq \varepsilon_n$ for all $n \geq 0$.

(b) *The intersection point of the functions $g_m(\varepsilon)$ and $g_n(\varepsilon)$, $m \neq n$, is*

$$\varepsilon_{m,n}^{1/2} = \frac{1}{2}\frac{\varepsilon_m^{1/2}|s_0(m)| - \varepsilon_n^{1/2}|s_0(n)|}{|s_0(n)| - |s_0(m)|} = \frac{1}{2}D_0\Omega^2\frac{\lambda^{-m} - \lambda^{-n}}{|s_0(n)| - |s_0(m)|}.$$

Assume $m > n$, then one has

- (b.1) *The intersection point $\varepsilon_{m,n}$ exists if $|s_0(m)| > |s_0(n)|$; moreover, $\varepsilon_{m,n} < \varepsilon_n$ for $m \geq n+3$ and $\varepsilon_{m,n} > \varepsilon_m$ for $m \geq n+2$.*
- (b.2) *The intersection point doesn't exist if $|s_0(m)| \leq |s_0(n)|$ that may take place if $m - n = 1$.*

(c) *The functions $g_n^-(\varepsilon)$ and $g_n^+(\varepsilon)$ have minimums at ε_n^- and ε_n^+, respectively. The points $\varepsilon_n^\pm$ are decreasing. The functions $g_n^\pm(\varepsilon)$ and $g_m^\pm(\varepsilon)$, $m \neq n$, intersect at*

$$(\varepsilon_{m,n}^\pm)^{1/2} = \frac{1}{2}(\varepsilon_m^\pm)^{1/6}(\varepsilon_n^\pm)^{1/6}\left[(\varepsilon_m^\pm)^{1/6} + (\varepsilon_n^\pm)^{1/6}\right].$$

(d) *The function $g_n(\varepsilon)$ is between $g_n^-(\varepsilon)$ and $g_n^+(\varepsilon)$.*

Proof. The first statement of part (a) is simple. For the remaining statement of (a), we get by definition of $\gamma_{s_0(n)}$ (and using also (7.7)):

$$|s_0(n)|^2 = \frac{|\lambda|^n\gamma_{s_0(n)}}{|\langle k^0([0,1]),\omega\rangle|}$$

Therefore, we have

$$\frac{\varepsilon_{n+1}^{1/2}}{\varepsilon_n^{1/2}} = \frac{D\gamma_{s_0(n+1)}}{|s_0(n+1)|^2} \times \frac{|s_0(n)|^2}{D\gamma_{s_0(n)}} \times \frac{|s_0(n)|}{|s_0(n+1)|} = \frac{1}{|\lambda|}\frac{|s_0(n)|}{|s_0(n+1)|}$$

From the recurrent formula

$$\varepsilon_{n+1}^{1/2} = \frac{1}{|\lambda|}\frac{|s_0(n)|}{|s_0(n+1)|}\varepsilon_n^{1/2}$$

we get

$$\varepsilon_n^{1/2} = |\lambda|^{-n}\frac{|s_0(0)|}{|s_0(n)|}\varepsilon_0^{1/2} = |\lambda|^{-n}\frac{1}{|s_0(n)|}\varepsilon_0^{1/2}.$$

Here we use the fact that, in the golden cubic case, $s_0(0) = k^0(j_0) = [0,0,1]^\top$ and $|s_0(0)| = 1$.
Using (7.15), we obtain

$$\frac{\varepsilon_{n+1}}{\varepsilon_n} \leq \frac{1}{|\lambda|^3}\frac{A + B\cos(2\varphi n + \alpha)}{A + B\cos(2\varphi n + 2\varphi + \alpha)} \leq \frac{1}{|\lambda|^3}\frac{A+B}{A-B} \approx 0.5764 < 1$$

Hence, the points are decreasing. Part (c) is proved in a similar way.
(b) Equating $g_n(\varepsilon)$ and $g_m(\varepsilon)$, we obtain the intersection point $\varepsilon_{m,n}$. Then if $m < n$, we write the intersection point as

$$\varepsilon_{m,n}^{1/2} = \frac{1}{2}\lambda^{-m} D\Omega^2 \frac{\lambda^{m-n} - 1}{|s_0(m)| - |s_0(n)|}$$

The last expression makes sense if the fraction on the right hand is positive that is true if $|s_0(m)| > |s_0(n)|$, since $\lambda > 1$, and, hence, the point $\varepsilon_{m,n}$ exists. Otherwise, the function $g_m(\varepsilon)$ and $g_n(\varepsilon)$ don't intersect. The inequality $|s_0(m)| \leq |s_0(n)|$ is equivalent to $\lambda^m(A + B\cos(2\varphi m + \alpha)) \leq \lambda^n(A + B\cos(2\varphi n + \alpha))$ that is true when

$$\lambda^{m-n} \leq \frac{A + B\cos(2\varphi n + \alpha)}{A + B\cos(2\varphi m + \alpha)} \tag{7.17}$$

The right hand of the inequality (7.17) can be bounded from above by $(A+B)/(A-B)$. In the case of the cubic golden number, we have $\lambda \leq \frac{A+B}{A-B} \leq \lambda^2$, since $\lambda \approx 1.4656$, $(A + B)/(A - B) \approx 1.8147$, $\lambda^2 \approx 2.1480$. Thus, the inequality (7.17) is satisfied only for $m = n + 1$.

In the case $\varepsilon_{m,n}$ exists, $|s_0(m)| > |s_0(n)|$ for $m > n$, this follows that

$$\lambda^{m-n} > \frac{A + B\cos(2\varphi n + \alpha)}{A + B\cos(2\varphi m + \alpha)} \tag{7.18}$$

we get (using 7.14)

$$\frac{\varepsilon_{m,n}^{1/2}}{\varepsilon_n^{1/2}} = \frac{1}{2}\frac{1 - \lambda^{-(m-n)}}{\frac{|s_0(m)|}{|s_0(n)|} - 1} < \frac{1}{2}\frac{1 - \lambda^{-(m-n)}}{\lambda^{\frac{m-n}{2}}\sqrt{\frac{A-B}{A+B}} - 1}.$$

This quotient is smaller than 1 if

$$\lambda^{-(m-n)} + 2\lambda^{\frac{m-n}{2}}\sqrt{\frac{A - B}{A + B}} - 3 > 0.$$

The last inequality holds for $m-n > 2$, excluding the values $m-n = 1$ and $m-n = 2$. Considering separately these 2 cases, we get for $m - n = 1$ that

$$\frac{\varepsilon_{n+1,n}^{1/2}}{\varepsilon_n^{1/2}} = \frac{1}{2}\frac{1 - \lambda^{-1}}{\frac{|s_0(n+1)|}{|s_0(n)|} - 1} < \frac{1}{2}\frac{1 - \lambda^{-1}}{\lambda^{1/2}\sqrt{\frac{A+B\cos(2\varphi(n+1)+\alpha)}{A+B\cos(2\varphi n+\alpha)}} - 1}$$

is smaller than 1 when

$$\frac{A + B\cos(2\varphi(n + 1) + \alpha)}{A + B\cos(2\varphi n + \alpha)} > \frac{(3 - \lambda^{-1})^2}{4\lambda} \approx 0.9163.$$

Also we can conclude from (7.18) that for the existing point $\varepsilon_{n+1,n}$ the following is fulfilled:

$$\frac{A + B\cos(2\varphi(n+1)+\alpha)}{A + B\cos(2\varphi n+\alpha)} > \lambda^{-1} \approx 0.6823.$$

Thus, there can exist intersection points $\varepsilon_{n+1,n}$ that are on the right of ε_n if

$$\lambda^{-1} < \frac{A + B\cos(2\varphi(n+1)+\alpha)}{A + B\cos(2\varphi n+\alpha)} < \frac{(3-\lambda^{-1})^2}{4\lambda}.$$

On the other hand, to have $\varepsilon_{n+2,n} < \varepsilon_n$, the inequality

$$\frac{A + B\cos(2\varphi(n+2)+\alpha)}{A + B\cos(2\varphi n+\alpha)} > \frac{(3-\lambda^{-2})^2}{4\lambda^2} \approx 0.7476$$

should be true. But it is not the case for some n satisfying

$$\lambda^{-2} < \frac{A + B\cos(2\varphi(n+2)+\alpha)}{A + B\cos(2\varphi n+\alpha)} < \frac{(3-\lambda^{-2})^2}{4\lambda^2}.$$

In a similar way one can prove that $\varepsilon_{m,n} > \varepsilon_m$ for $m-n \le 2$ and that there possible points $\varepsilon_{n+1,n} < \varepsilon_n$ if

$$\frac{A + B\cos(2\varphi(n+1)+\alpha)}{A + B\cos(2\varphi n+\alpha)} > \frac{4}{\lambda(3-\lambda)^2} \approx 1.1592.$$

To prove (d) we equate $g_n^-(\varepsilon)$ and $g_n^+(\varepsilon)$ and get no intersection points. □

Remark 7.2. It is important to know the location of the intersection point $\varepsilon_{m,n}$, if exists, with respect to ε_m and ε_n $(m > n)$, since $\varepsilon_m < \varepsilon_{m,n} < \varepsilon_n$ implies that the right branch of $g_m(\varepsilon)$ intersect the left branch of $g_n(\varepsilon)$ the fact that we will use later in Lemma 7.1.

For secondary vectors k we have

$$g_k = \gamma_k^{1/3}\left(\frac{\varepsilon^{1/6}}{\varepsilon_k^{1/3}} + \frac{\varepsilon_k^{1/3}}{2\varepsilon^1/3}\right) \ge \frac{3}{2}\gamma_k^{1/3} \ge \frac{3}{2}(\gamma_{[2,0]}^-)^{1/3} \approx 1.5186, \quad k \ne s_0(\cdot). \tag{7.19}$$

For each ε we find vectors $S_i = S_i(\varepsilon)$, $i = 1,2,3,4,5$ giving the 5 smallest values of $g_k(\varepsilon)$, $k \in \mathbb{Z}^3\backslash\{0\}$. This means that

$$g_{S_1}(\varepsilon) \le g_{S_2}(\varepsilon) \le g_{S_3}(\varepsilon) \le g_{S_4}(\varepsilon) \le g_{S_5}(\varepsilon) \le g_k(\varepsilon), k \ne S_1,\dots,S_5.$$

Then we define the function

$$h_1(\varepsilon) = \min_{k\in\mathbb{Z}^3\backslash\{0\}} g_k(\varepsilon) = g_{S_1}(\varepsilon). \tag{7.20}$$

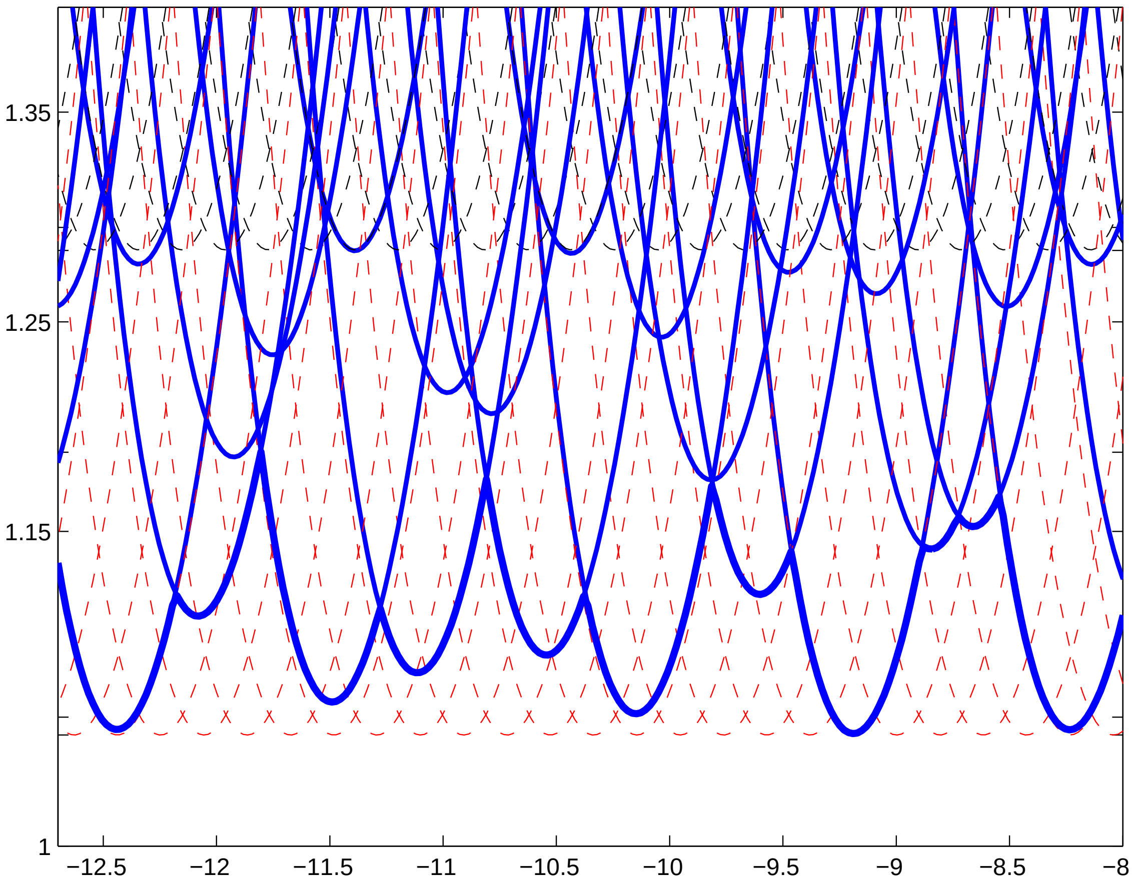


Figure 7.3: Graphs of the functions $g_n(\varepsilon)$ (solid), $g_n^{\pm}(\varepsilon)$ (dashed), and $h_1(\varepsilon)$ (thick solid) using a logarithmic scale for ε

See Figure 7.3 for an illustration of the function $h_1(\varepsilon)$. In a similar way we define the functions $h_1^{\pm}(\varepsilon)$ for the functions $g_n^{\pm}(\varepsilon)$ (dashed lines in Figure 7.3). The functions $h_1^{\pm}(\varepsilon)$ are continuous and $3\ln\lambda$-periodic in $\ln\varepsilon$. One can write the following expression for them

$$h_1^-(\varepsilon) = \min_n g_n^-(\varepsilon), \text{ and } h_1^+(\varepsilon) = \min_n g_n^+(\varepsilon). \tag{7.21}$$

One can show that

$$\begin{aligned} C_1^- &= \min h_1^-(\varepsilon) = \tfrac{3}{2}(A-B)^{1/3} \approx 1.0529, \\ C_2^- &= \max h_1^-(\varepsilon) = \frac{(A-B)^{1/3}\left(\lambda+\sqrt{\lambda}+1\right)}{\left(2\lambda\left(\sqrt{\lambda}+1\right)\right)^{2/3}} \approx 1.0625, \\ C_1^+ &= \min h_1^+(\varepsilon) = \tfrac{3}{2}(A+B)^{1/3} \approx 1.2843, \\ C_2^+ &= \max h_1^+(\varepsilon) = \frac{(A+B)^{1/3}\left(\lambda+\sqrt{\lambda}+1\right)}{\left(2\lambda\left(\sqrt{\lambda}+1\right)\right)^{2/3}} \approx 1.2960. \end{aligned} \tag{7.22}$$

We prove the following result (note that we have introduced the notation of '$\sim$' and '$\preceq$' just before Theorem 5.1 in Chapter 5):

Theorem 7.1 ((Maximal) splitting distance)**.** *For the Hamiltonian system (5.1-5.7) with* $n = 3$*, assume that* $0 < \varepsilon \preceq 1$ *and* $\mu = \varepsilon^p$*,* $p > p^*$ *with* $p^* = 2$ *if* $\nu = 1$ *and* $p^* = 3$ *if* $\nu = 0$*, then the following asymptotic estimate holds*

$$\max_{\theta \in \mathbb{T}^3} |\mathcal{M}(\theta)| \sim \frac{\mu}{\sqrt[3]{\varepsilon}} \exp\left\{ -\frac{C_0 h_1(\varepsilon)}{\varepsilon^{1/6}} \right\}$$

where C_0 *is the constant given in (7.11) and the function* $h_1(\varepsilon)$*, defined in (7.20), satisfies the following bounds:*

- *"Constant bound":* $0 < C_1^- \leq h_1(\varepsilon) \leq C_2^+$ *with constants* C_1^- *and* C_2^+*, defined in (7.22);*
- *"Periodic bound":* $0 < h_1^-(\varepsilon) \leq h_1(\varepsilon) \leq h_1^+(\varepsilon)$*, where* $h^-(\varepsilon)$*,* $h^+(\varepsilon)$ *are the* $3\ln\lambda$*-periodic functions in* $\ln\varepsilon$ *defined in (7.21);* $\min h_1^- = C_1^-, \max h_1^- = C_2^-, \min h_1^+ = C_1^+, \max h_1^+ = C_2^+$*, the constants* C_2^-, C_1^+ *are defined in (7.22).*

See Figure 7.4 for an illustration of the bounds of the function $h_1(\varepsilon)$ from Theorem 7.1. Also looking at the form of $h_1(\varepsilon)$ in Figure 7.4, it seems plausible to make the following

Conjecture 7.1. *The function* $h_1(\varepsilon)$ *is quasi periodic in* $\ln\varepsilon$ *with periods* $T_1 = 3\ln\lambda \approx 1.1467$ *and* $T_2 = 3\pi\ln\lambda/\varphi \approx 1.9406$ *for all* ε*.*

A heuristic argument is presented now. The first period T_1 is the period of the functions $h_1^+(\varepsilon)$ and $h_1^-(\varepsilon)$, while T_2 is related to the presence of cos(...) in the formulae for $\gamma_{s_0(n)}$ and ε_n in (7.14) and (7.15). Therefore, for $\varepsilon \approx \varepsilon_n$, we get applying logarithm to (7.15)

$$\ln\varepsilon \approx 2\ln(D_0\Omega^3) - 3n\ln\lambda.$$

We solve the equation for n:

$$n \approx \frac{2\ln(D_0\Omega^3) - \ln\varepsilon}{3\ln\lambda}.$$

Thus,

$$\cos(2\varphi n + \alpha) \approx \cos\left(-\frac{2\varphi\ln\varepsilon}{3\ln\lambda} + \widetilde{\alpha} \right),$$

and, hence, $T_2 \approx 2\pi \Big/ \frac{2\varphi}{3\ln\lambda} = 3\pi\ln\lambda/\varphi$.

The difficulty presented in numerical checking is due to the fact that $h_1(\varepsilon)$ is not differentiable at some points.

We also define the functions h_i, $i = 2, 3, 4, 5$ in a similar way as $h_1(\varepsilon)$:

$$\begin{aligned} h_2(\varepsilon) &= \min_{k \neq S_1} g_k = g_{S_2}(\varepsilon); & h_3(\varepsilon) &= \min_{k \neq S_1, S_2} g_k = g_{S_3}(\varepsilon); \\ h_4(\varepsilon) &= \min_{k \neq S_1, S_2, S_3} g_k = g_{S_4}(\varepsilon); & h_5(\varepsilon) &= \min_{k \neq S_1, S_2, S_3, S_4} g_k = g_{S_5}(\varepsilon); \end{aligned} \tag{7.23}$$

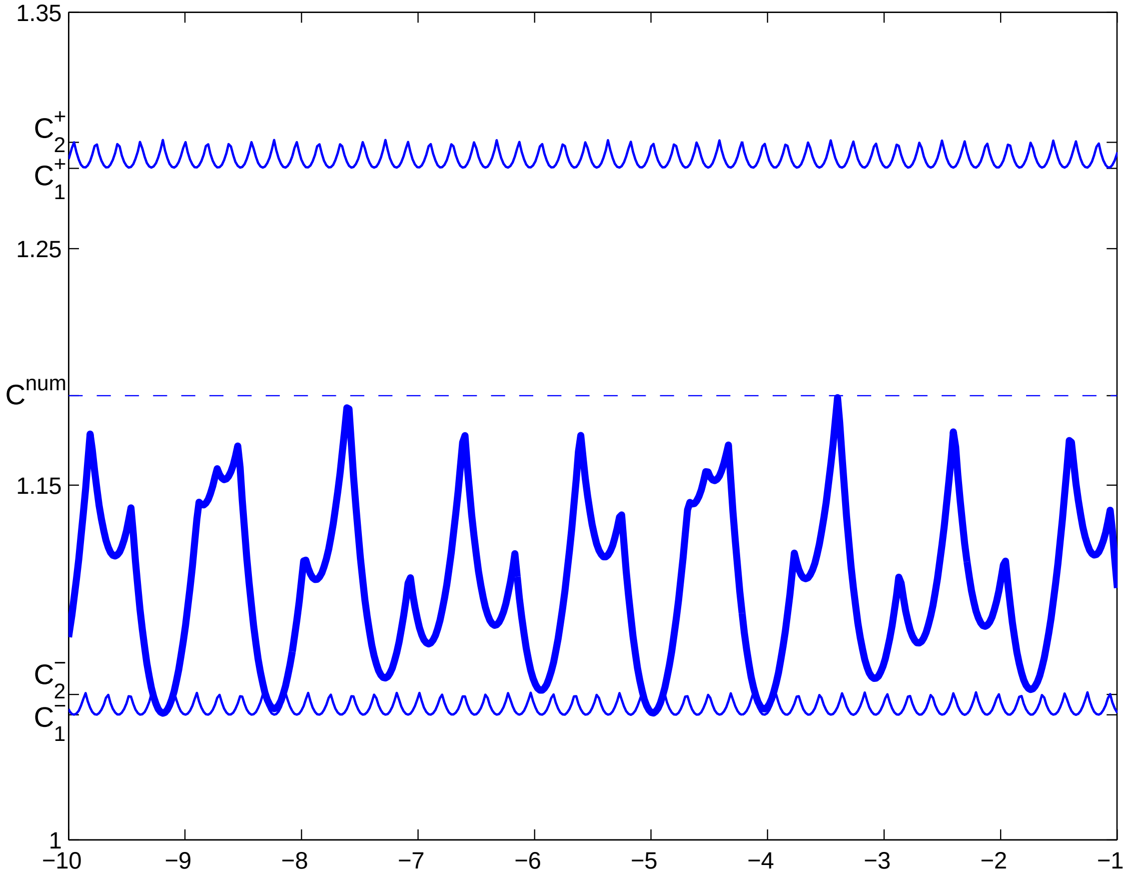


Figure 7.4: Graphs of the functions $h_1(\varepsilon)$ (thick solid) and $h_1^{\pm}(\varepsilon)$ (solid) using a logarithmic scale for ε. Notice that the upper bound C_2^+ is not sharp, but a numerical sharp upper bound C^{num} can be obtained

Numerical experiments suggest that if we consider only primary functions $g_n(\varepsilon)$ in (7.20) and (7.23), the functions $h_i(\varepsilon)$ have following bounds:

$$h_1(\varepsilon) \le 1.1888 \le h_2(\varepsilon) \le 1.3037 \le h_3(\varepsilon) \le 1.3255 \le h_4(\varepsilon) \le 1.4234 \le h_5(\varepsilon) \le 1.5459$$

Due to (7.19), the vectors S_i, $i = 1, \ldots, 4$ are given by primary resonances, instead S_5 may be a secondary resonance for some intervals of ε.

We also define the following 2 discrete sets:

$$\mathcal{E}_1 = \{\varepsilon : h_3(\varepsilon) = h_4(\varepsilon)\}, \quad \mathcal{E}_2 = \{\varepsilon : h_4(\varepsilon) = h_5(\varepsilon)\}. \tag{7.24}$$

In the next theorem we find the simple zeros of the splitting function $\mathcal{M}(\theta)$ which give rise to transverse homoclinic orbits associated to the whiskered torus considered. These zeros are determined essentially by 3 dominant harmonics if these dominant harmonics are given by independent vectors of indexes S_1, S_2, S_3, and by 4 dominant harmonics of indexes S_1, S_2, S_3, S_4 if $\det(S_1, S_2, S_3) = 0$, $\det(S_1, S_2, S_4) \neq 0$. Fortunately, in intervals of ε where 4 dominant harmonics are required, the 5th dominant harmonic is given by a primary vector S_5 as it seems from the numerical experiments. We also give an estimate for the minimal eigenvalue (in modulus) of the splitting matrix $\partial_\theta \mathcal{M}$ at each zero. This estimate provides a measure for the transversality of the homoclinic orbits. Note that the notations of '$\sim$' and '$\preceq$' has been introduced just before Theorem 5.1 in Chapter 5.

Theorem 7.2 (Transversality). *For the Hamiltonian system (5.1-5.7) with $n = 3$, assume that $\varepsilon \preceq 1$ and $\mu = \varepsilon^p$, $p > p^*$ with $p^* = 2$ if $\nu = 1$ and $p^* = 3$ if $\nu = 0$, then one has:*

- *If the indexes of the first 3 dominant harmonics satisfy:* $\det(S_1, S_2, S_3) \neq 0$, *the Melnikov function $\mathcal{M}(\theta)$ has exactly 8 zeros θ_*, all simple, for all ε except of some small neighbourhood of the discrete set $\mathcal{E}_1$;*

  The minimal eigenvalue of $\partial_\theta \mathcal{M}(\theta_*)$ satisfies

  $$m_* \sim \mu \varepsilon^{1/2} \exp\left\{-\frac{C_0 h_3(\varepsilon)}{\varepsilon^{1/6}}\right\}.$$

- *If* $\det(S_1, S_2, S_3) = 0$, $\det(S_1, S_2, S_4) \neq 0$, *the Melnikov function $\mathcal{M}(\theta)$ has exactly 8 zeros θ_*, all simple, for all ε except for some small neighbourhood of the discrete set $\mathcal{E}_2$;*

  The minimal eigenvalue of $\partial_\theta \mathcal{M}(\theta_*)$ satisfies

  $$m_* \sim \mu \varepsilon^{1/2} \exp\left\{-\frac{C_0 h_4(\varepsilon)}{\varepsilon^{1/6}}\right\}.$$

The rest of the chapter is devoted to the proof of the main results: Theorem 7.1 and Theorem 7.2. We get estimates for the dominant harmonics of L (Lemma 7.1, p. 174) and of $\mathcal{L}$ (Lemma 7.2, p. 178) as well as for the sums of the remaining terms in each case. Then we find the critical points of $\mathcal{L}$ and give estimates for the minimal eigenvalue of $D^2\mathcal{L}$ in both cases $\det(S_1, S_2, S_3) \neq 0$ and $\det(S_1, S_2, S_3) = 0$, $\det(S_1, S_2, S_4) \neq 0$. Theorems 7.1 and 7.2 follow from these lemmas, taking into account that $\mathcal{M}(\theta) = \partial_\theta \mathcal{L}(\theta)$.

7.3 Dominant harmonics of the Melnikov potential

In order to estimate the coefficients L_k of the Melnikov potential we use the following approximations obtained from (7.14) and (7.15):

$$|s_0(n)|^2 \sim \lambda^n, \quad \varepsilon_n \sim \lambda^{-3n}$$

Therefore, if we fix on the interval of ε where $s_0(n)$ is the dominant vector, i.e. $\varepsilon \approx \varepsilon_n$, we find

$$|s_0(n)|^2 \sim \varepsilon^{-1/3}.$$

Substituting (7.16) into (7.13) and using the definition of $\gamma_{s_0(n)}$ and (7.7), we present the functions g_n in form

$$g_n(\varepsilon) = \Omega^{2/3}\varepsilon_0^{-1/6}|s_0(n)|\varepsilon^{1/6} + \frac{1}{2}\varepsilon_1^{1/3}\lambda^{-n}\varepsilon^{-1/3},$$

where the first and the second summands correspond to the right and left branches of $g_n(\varepsilon)$, respectively. Due to part (b.1) of Proposition 7.3, we ensure that at ε (fixed near to ε_n) the value of g_{n+l_1}, $l_1 \geq 2$, is estimated by its right branch, while $g_{n-l_2}(\varepsilon)$, $l_2 \leq 1$, is evaluated by its left branch, i.e.

$$\begin{array}{l} \text{for } l_1 > 1 : g_{n+l_1}(\varepsilon) \sim |s_0(n+l_1)|\varepsilon^{1/6} \sim \lambda^{l_1/2}|s_0(n)|\varepsilon^{1/6} \sim \lambda^{l_1/2} \\ \text{for } l_2 > 2 : g_{n-l_2}(\varepsilon) \sim \lambda^{l_2-n}\varepsilon^{-1/3}. \end{array} \tag{7.25}$$

Note that the dominant vectors S_i, $i = 1, \ldots, 5$ get estimates

$$|S_i|^2 \sim \varepsilon^{-1/3}. \tag{7.26}$$

We want to evaluate the sum (7.8) for all $k \in \mathcal{Z}$. Our idea is to find the dominant terms (that are among the primary resonances and correspond to minimal values of $g_n(\varepsilon)$ for a fixed ε) and then to provide estimates for the sum of others terms. To this end, we divide the sum (7.8) into three ones: the first sum is for the dominant terms S_i, the second one is for primary k excluding S_i and the third sum is for secondary vectors k. We prove the lemma

Lemma 7.1. *Assuming $\varepsilon \preceq 1$ and $\mu \preceq \delta^{q_2}\varepsilon^{1/6}$, one has*

1. $L_{S_i} \sim \frac{1}{\varepsilon^{1/6}} \exp\left\{-\frac{Ch_i(\varepsilon)}{\varepsilon^{1/6}}\right\}, \quad i = 1, \ldots, 5;$
2. $\sum\limits_{k \neq S_1, \ldots, S_l, k = s_0(\cdot)} |k|^m L_k \sim \frac{1}{\varepsilon^{m/6}} L_{S_{l+1}}$, $0 \leq l \leq 4$, $m \geq 0$;
3. $\sum\limits_{k \neq s_0(\cdot)} |k|^m L_k \ll \frac{1}{\varepsilon^{m/6}} L_{S_{l+1}}$, $m \geq 0$;

Proof. The largest coefficients L_k, defined in 7.9, are given by the smallest exponents β_k. Due to (7.10), the main information on the size of $\beta_k(\varepsilon)$ is contained in the functions $g_k(\varepsilon)$ whose minimal values are given by the dominant vectors S_i, $i = 1, \ldots, 5$. By the definitions of the functions $h_i(\varepsilon)$, $i = 1, \ldots, 5$, we get for S_i

$$\beta_{S_i} \simeq \frac{C_0 h_i(\varepsilon)}{\varepsilon^{1/6}}.$$

In fact, if we look at the true exponents β_{S_i}, the constant C_μ (that depends on b and b') is $\mu\delta^{-q_2}$-close to C_0 (see the corresponding estimates for b and b' in Theorem 5.1, p. 118, in Chapter 5). Also there is an error when we substitute the approximations (7.14)-(7.15) into (7.13) and define the functions $h_i(\varepsilon)$. According to Proposition 7.2, this error is $\mathcal{O}(\lambda^{-3n/2})$ for $\gamma_{s_0(n)}$ and $\mathcal{O}(\lambda^{-n/2})$ for $|s_0(n)|^2$. Having $\lambda^{-3n/2} \sim \sqrt{\varepsilon}$, we get for the exponent

$$\beta_{S_i} = \frac{C_0 h_i(\varepsilon) + \mathcal{O}(\sqrt{\varepsilon}, \mu\delta^{-q_2})}{\varepsilon^{1/6}}.$$

Since $\sqrt{\varepsilon} \preceq \varepsilon^{1/6}$ and $\mu\delta^{-q_2} \preceq \varepsilon^{1/6}$ under the smallness condition given in the lemma, we can neglect the error.

We show that the factor α_k in (7.9) does not influence the dominance in L_k. Indeed, if we assume that $[1 - \exp\{\ldots\}] \sim 1$ in the denominator of α_k, then we deduce from (7.9) that $\alpha_k \preceq \beta_k$ and, hence, $|\ln \alpha_k| \ll \beta_k$. If $[1 - \exp\{...\}]$ is too small (that happens when $|k|^2 \succeq \gamma_k/\sqrt{\varepsilon}$), we obtain $\alpha_k \preceq 1$, since $\frac{x}{1-\exp(-Cx)} \sim C$ for small x. Concerning the dominant vectors, one can prove that for S_i that $[1 - \exp\{...\}] \sim 1$ and, reminding (7.26) and the fact that $\gamma^-_{[0,1]} \leq \gamma_{S_i} \leq \gamma^+_{[0,1]}$, we get

$$\alpha_{S_i} \sim \frac{\gamma_{S_i}}{|S_i|^2\sqrt{\varepsilon}} \sim \frac{1}{\varepsilon^{1/6}}.$$

This estimate, together with the estimate for β_{S_i}, implies part (a).

Let us prove (b) for $l = 1$. Then to bound the sum of L_k for the primary vectors $k \in \{s_0(\cdot)\}$, excluding the dominant vector S_1, say $S_1 = s_0(n)$, we fix $\varepsilon \approx \varepsilon_{s_0(n)}$ and we divide the vectors k into the ones previous to S_1 and the ones after S_1. For $s_0(n + l_1)$, $l_1 > 1$ (vectors after S_1), we use (7.25) and get

$$\beta_{s_0(n+l_1)} = \frac{C_0 g_{n+l_1}(\varepsilon)}{\varepsilon^{1/6}} \sim \frac{\lambda^{l_1/2}}{\varepsilon^{1/6}}$$

Hence

$$\begin{aligned}\Sigma_1 &= \sum_{l_1>1} |s_0(n+l_1)|^m L_{s_0(n+l_1)} \sim \sum_{l_1>1} \lambda^{ml_1/2}|s_0(n)|^m \alpha_{s_0(n+l_1)} \exp\{-C_0\lambda^{l_1/2}\varepsilon^{-1/6}\} \\ &\sim \sum_{l_1>1} \varepsilon^{-(m+1)/6} \frac{\lambda^{(m-2)l_1/2}}{1-\exp\{-C\lambda^{-l_1}\varepsilon^{-1/6}\}} \exp\left\{-\frac{C_0\lambda^{l_1/2}}{\varepsilon^{1/6}}\right\} = \sum_{l_1>1} a_{l_1},\end{aligned}$$

where

$$\begin{aligned}\frac{a_{l_1+1}}{a_{l_1}} &= \lambda^{m/2-1} \frac{1-\exp\{-C\lambda^{-l_1}\varepsilon^{-1/6}\}}{1-\exp\{-C\lambda^{-l_1}\varepsilon^{-1/6}/\lambda\}} \exp\left\{-\frac{C_0\lambda^{l_1/2}(\sqrt{\lambda}-1)}{\varepsilon^{1/6}}\right\} \\ &\leq \lambda^{m/2} \exp\left\{-\frac{C_0\sqrt{\lambda}(\sqrt{\lambda}-1)}{\varepsilon^{1/6}}\right\} = r.\end{aligned}$$

We have bounded the quotient, since $\frac{1-\exp(-x)}{1-\exp(-x/\lambda)} \leq \lambda$ for sufficiently small $x > 0$, and $\lambda^{l_1/2} \geq \lambda^{1/2}$ for $l_1 \geq 1$. Thus, the sum is bounded by a geometric series that is estimated essentially by the first term a_2 if $0 < r < 1$

$$\sum_{l_1>1} a_{l_1} = a_2(1+\frac{a_3}{a_2}+\frac{a_4}{a_2}+\ldots) = a_2(1+\frac{a_3}{a_2}+\frac{a_4}{a_3}\cdot\frac{a_3}{a_2}+\ldots) \leq a_2(1+r+r^2+\ldots) = a_2\frac{1}{1-r}$$

Taking $\varepsilon^{1/6} < 2C_0\sqrt{\lambda}(\sqrt{\lambda}-1)/(m\ln\lambda)$ (to have $r < 1$), Σ_1 is bounded by a convergent geometric series and is determined by the first term (i.e. $l_1 = 2$)

$$\Sigma_1 \sim |s_0(n+2)|^m L_{s_0(n+2)}.$$

For $s_0(n-l_2)$ with $l_2 > 2$ (vectors previous to S_1) we get by (7.25)

$$\beta_{s_0(n-l_2)} \sim \frac{\lambda^{l_2-n}}{\sqrt{\varepsilon}}.$$

Then, the sum is

$$\Sigma_2 = \sum_{l_2=3}^{n} |s_0(n-l_2)|^m L_{s_0(n-l_2)} \sim \sum_{l_2=3}^{n} \lambda^{-(m/2-1)l_2}\varepsilon^{-(m+1)/6}\frac{\exp\{-C_0\lambda^{l_2-n}\varepsilon^{-1/2}\}}{1-\exp\{-C\lambda^{l_2}\varepsilon^{-1/6}\}} = \sum_{l_2=3}^{n} b_{l_2}$$

can also be bounded by a geometric series, since

$$\frac{b_{l_2+1}}{b_{l_2}} \leq \lambda^{1-m/2}\exp\left\{-\frac{C_0\lambda(\lambda-1)}{\lambda^n\varepsilon^{1/2}}\right\} = r.$$

In this case $r < 1$ for any $\varepsilon > 0$ if $m \geq 2$ and one has to take $\varepsilon^{1/2} < C_0\lambda^{1-n}(\lambda-1)/((1-m/2)\ln\lambda)$ if $m = 0$ and $m = 1$. Thus, sum is bounded by a convergent geometric series and, thus, is estimated essentially by the first term b_3:

$$\sum_{l_2=3}^{n} b_{l_2} \leq \sum_{l_2=3}^{\infty} b_{l_2} = \frac{b_3}{1-r}.$$

Hence, we obtain the estimate

$$\Sigma_2 \sim |s_0(n-3)|^m L_{s_0(n-3)}.$$

Therefore, we conclude that the sum of the coefficients $|s_0(\cdot)|L_{s_0(\cdot)}$ is estimated by a finite number of terms (including also those that have not entered in Σ_1 and Σ_2):

$$\sum_{i\neq n} |s_0(i)|^m L_{s_0(i)} \sim \Sigma_1+\Sigma_2+|s_0(n+1)|^m L_{s_0(n+1)}+|s_0(n-1)|^m L_{s_0(n-1)}+|s_0(n-2)|^m L_{s_0(n-2)}.$$

The other dominant terms corresponding to the vectors S_i are among those that are presented in the last estimate. Thus, excluding a necessary number of dominant harmonics, one obtains the expected estimate of part (b) in the case $l \neq 1$.

To bound the sum of secondary resonances ($k \notin \{s_0(\cdot)\}$) in part (c), we divide all such vectors into the cases $|k|^2 \leq \varepsilon^{-1/6}$ and $|k|^2 \geq \varepsilon^{-1/6}$. In the first case, we have a finite number of vectors (bounded by a ball $|k_1|^2 + |k_2|^2 + |k_3|^2 \leq \varepsilon^{-1/6}$), this number can be estimated by $O(\varepsilon^{-1})$. Recall that for all $k \notin \{s_0(\cdot)\}$, we have the bound $\gamma_k \geq \gamma^-_{[2,0]} \approx 1.0376$ and, hence, by (7.12) one gets

$$\beta_k \geq \frac{3C_0\gamma^{1/3}_{[2,0]}}{2\varepsilon^{1/6}} > \beta_{S_l}, \quad 1 \leq l \leq 5.$$

Also we conclude that in this case $[1-\exp\{\dots\}]$ in the denominator of α_k in (7.9) and, hence,

$$\alpha_k \sim \frac{\gamma_k}{|k|^2\sqrt{\varepsilon}} = \frac{|\langle k, \omega\rangle|}{\sqrt{\varepsilon}} \leq \frac{|k||\omega|}{\sqrt{\varepsilon}} \preceq \varepsilon^{-7/12}.$$

The sum for these vectors is

$$\begin{aligned}\sum_{\substack{k\neq s_0(\cdot)\\ |k|^2\preceq\varepsilon^{-1/6}}} |k|^m L_k \;&\preceq\; \varepsilon^{-m/12}\varepsilon^{-7/12}\exp\left\{-\frac{3C_0\gamma^{1/3}_{[2,0]}}{2\varepsilon^{1/6}}\right\} \preceq \varepsilon^{-m/12-19/12}\exp\left\{-\frac{3C_0\gamma^{1/3}_{[2,0]}}{2\varepsilon^{1/6}}\right\}\\ &\preceq\; |S_{l+1}|^m L_{S_{l+1}}.\end{aligned}$$

In the case $k \notin \{s_0(\cdot)\}$ and $|k|^2 \succeq \varepsilon^{-1/6}$ we can say that for large enough k the functions β_k in (7.9) behave as $\beta_k \sim \rho|k|$. Concerning α_k, there are vectors satisfying $|k|^2 \succeq \gamma_k/\sqrt{\varepsilon}$, then in this case $\alpha_k \preceq 1$ and the sum is estimated as

$$\begin{aligned}\sum_{|k|^2\succeq\gamma_k/\sqrt{\varepsilon}} |k|^m L_k \;&\sim\; \sum_{|k|^2\succeq\gamma_k/\sqrt{\varepsilon}} |k|^m \exp\{-\rho|k|\} \preceq \sum_{K^2\succeq\varepsilon^{-1/6}} K^m \exp(\rho K)\\ &\sim\; \varepsilon^{-m/12}\exp\{-\rho\varepsilon^{-1/12}\} \preceq |S_{l+1}|^m L_{S_{l+1}}\end{aligned}$$

Here we have bounded the sum by the sum of a geometric series that is assessed by its first term. Otherwise, for $\varepsilon^{-1/6} \preceq |k^2| \preceq \gamma_k/\sqrt{\varepsilon}$, we get

$$\alpha_k \sim \frac{\gamma_k}{|k|^2\sqrt{\varepsilon}} = \frac{|\langle k, \omega\rangle|}{\sqrt{\varepsilon}} \leq \frac{|k||\omega|}{\sqrt{\varepsilon}} \preceq |k|\varepsilon^{-1/2},$$

and we proceed analogously to estimate the sum

$$\begin{aligned}\sum_{\varepsilon^{-1/6}\preceq|k|^2\preceq\gamma_k/\sqrt{\varepsilon}} |k|^m L_k \;&\preceq\; \varepsilon^{-1/6} \preceq \sum_{\varepsilon^{-1/2}\preceq|k|^2\preceq\gamma_k/\sqrt{\varepsilon}} |k|^{m+1}e^{-\rho|k|} = \varepsilon^{-1/2}\sum_{K^2\geq\varepsilon^{-1/6}} K^m e^{-\rho K}\\ &\sim\; \varepsilon^{-1/2-m/12}\exp\{-\rho\varepsilon^{-1/12}\} \preceq |S_{l+1}|^m L_{S_{l+1}}\end{aligned}$$

Finally, to complete the proof of Lemma, we consider the non-admissible vectors $k \notin \mathcal{A}$, for which $|\langle k, \omega\rangle| > 1/2$ and, hence, $\gamma_k \geq |k|^2/2$. We find in (7.9) that

$$\alpha_k \preceq \frac{|k|}{\sqrt{\varepsilon}(1-\exp(-C/\sqrt{\varepsilon}))} \text{ and } \beta_k \geq \rho|k| + C/\sqrt{\varepsilon}$$

and we again bound the sum by a geometric series

$$\begin{aligned}\sum_{k\notin\mathcal{A}}|k|^m L_k &\preceq \varepsilon^{-1/2}\exp(-C/\sqrt{\varepsilon})\sum_{k\notin\mathcal{A}}|k|^{m+1}\exp(-\rho|k|)\preceq\sum_{K\geq 1}K^{m+1}\exp(-\rho K)\\ &\preceq \varepsilon^{-1/2}\exp(-\rho-C/\sqrt{\varepsilon}).\end{aligned}$$

The obtained upper bound is smaller than the one from part (b). □

7.4 Dominant harmonics of the splitting potential

The estimates obtained for in Lemma 7.1 can be used as a first approximation for the dominant harmonics of the splitting potential $\mathcal{L}(\theta)$. We prove that assuming $\mu=\varepsilon^p$, for a suitable $p>0$, the dominant harmonics don't change essentially in $\mathcal{L}$ if we add the error term of order $\mathcal{O}(\mu^2)$.

Lemma 7.2. *Assuming $\varepsilon\preceq 1$, $\mu=\varepsilon^p$ with $p>p^*$, with $p^*=2$ if $\nu=1$ and $p^*=3$ if $\nu=0$, one has:*

(a) $\mathcal{L}_{S_i}\sim\frac{\mu}{\varepsilon^{1/6}}\exp\{-\frac{C_0h_i(\varepsilon)}{\varepsilon^{1/6}}\}$,
$|\tau_{S_i}-\sigma_{S_i}-s^{(0)}\langle S_i,\hat\omega_\varepsilon\rangle|\preceq\frac{\mu}{\varepsilon^{p^*}}$, $i=1,\dots,5$;

(b) $\displaystyle\sum_{k\neq S_1,\dots,S_l}|k|^m\mathcal{L}_k\preceq\frac{1}{\varepsilon^{m/6}}\mathcal{L}_{S_{l+1}},\quad 0\leq l\leq 4, m\geq 0.$

Proof. The proof is analogous to the quadratic case (Lemma 6.2 in Chapter 6), we adapt it to the cubic case having only the difference in some exponents.

Taking into account that $\mathcal{L}$ is $\hat\omega_\varepsilon$-quasiperiodic, we can consider the following Fourier expansion:

$$\mathcal{L}(s,\theta)=\sum_{k\in\mathbb{Z}^2}\mathcal{L}_k^* e^{i\langle k,\theta-\hat\omega_\varepsilon s\rangle}=\sum_{k\in\mathcal{Z}}\mathcal{L}_k\cos(\langle k,\theta-\hat\omega_\varepsilon s\rangle-\tau_k),$$

where $\mathcal{L}_k,\tau_k$ are real, $\mathcal{L}_k\geq 0$ and $\mathcal{Z}$ is defined in (5.8). For every $k\in\mathcal{Z}$, the exponential and the trigonometric forms are related by $\mathcal{L}_k^*=\frac{1}{2}\mathcal{L}_k e^{-i\tau_k}$. Then the corresponding Fourier coefficients of the splitting function $\mathcal{M}(s,\theta)\partial_\theta\mathcal{L}$ and the Melnikov function $M(s,\theta)$ are (in the exponential form) $\mathcal{M}_k^*=ik\mathcal{L}_k^*$ and $M_k^*=ikL_k^*$, respectively.

By Theorem 5.1, the splitting function $\mathcal{M}(s,\theta)$ can be defined on a complex domain: $|\mathrm{Im}\ s|<\frac{\pi}{2}-\delta$, $|\mathrm{Im}\ \theta|<\rho$, where δ is a small reduction (to be chosen), and we have the upper bound (5.14) for the error term. Due to $\hat\omega_\varepsilon$-quasiperiodicity of $\mathcal{R}$, we apply a standard result (Lemma 11 of [DGS04]) to it to get bounds for its Fourier coefficients:

$$|\mathcal{R}_k^*|\preceq\left(\frac{\mu^2}{\delta^{q_3}}+\frac{\mu^2}{\delta^{q_4}\sqrt{\varepsilon}}\right)e^{-\tilde\beta_k(\varepsilon)},\quad \tilde\beta_k(\varepsilon)=(\rho-\delta)|k|+\frac{(\pi/2-\delta)b'\gamma_k}{b|k|\sqrt{\varepsilon}}.$$

From (5.13) we deduce that the Fourier coefficients of the error term satisfy $\mathcal{R}_k^* = ik(\mathcal{L}_k^* - \mu L_k^* e^{-is^{(0)}\langle k,\hat{\omega}_\varepsilon\rangle})$, $k \neq 0$. Taking modulus and argument, we get

$$|\mathcal{L}_k - \mu L_k| \preceq \frac{|\mathcal{R}_k^*|}{|k|}, \quad |\tau_k - \sigma_k - s^{(0)}\langle k, \hat{\omega}_\varepsilon\rangle| \preceq \frac{|\mathcal{R}_k^*|}{|k|\mu L_k}.$$

We present the function $\tilde{\beta}_k$ as in (7.10), but with $\rho - \delta$ and $\pi/2 - \delta$ instead of ρ and $\pi/2$

$$\tilde{\beta}_k(\varepsilon) = \frac{C_{\mu,\delta}\tilde{g}_k(\varepsilon)}{\varepsilon^{1/6}}, \text{ where } \tilde{g}_k = \gamma_k^{1/3}\left[\frac{\varepsilon^{1/6}}{\tilde{\varepsilon}_k^{1/6}} + \frac{\tilde{\varepsilon}_k^{1/3}}{2\varepsilon^{1/3}}\right], \quad \tilde{\varepsilon}_k^{1/4} = \frac{D_{\mu,\delta}\sqrt{\tilde{\gamma}_k}}{|k|},$$

now with

$$C_{\mu,\delta} = (\rho - \delta)^{2/3}\left(\frac{2(\pi/2 - \delta)b'}{b}\right)^{1/3} = C_0 + \mathcal{O}(\mu\delta^{-q_2}, \delta),$$
$$D_{\mu,\delta} = \frac{2(\pi/2 - \delta)b'}{b(\rho - \delta)} = D_0 + \mathcal{O}(\mu\delta^{-q_2}, \delta).$$

In fact, we consider $\tilde{\beta}_k$ as a perturbation of β_k. Indeed, proceeding as in the proof of Lemma 7.1, we get for the most dominant terms:

$$\tilde{\beta}_{S_i} = \frac{C_0 h_i(\varepsilon) + \mathcal{O}(\sqrt{\varepsilon}, \mu\delta^{-q_2}, \delta)}{\varepsilon^{1/6}}.$$

We can neglect the perturbation term if $\mu\delta^{-q_2} \preceq \varepsilon^{1/6}$, $\delta \preceq \varepsilon^{1/6}$. So we choose $\delta = \varepsilon^{1/6}$. The smallness conditions on μ in Theorem 5.1 become $\mu \preceq \varepsilon^{q_1/6}$ (the condition containing the exponent q_2 can be ignored, since $q_1 \geq q_2 + 4$). Then using (7.26), we conclude

$$|\mathcal{L}_{S_i} - \mu L_{S_i}| \preceq \frac{|\mathcal{R}_{S_i}^*|}{|S_i|} \preceq \frac{\mu^2}{\varepsilon^{(q_3-1)/6}} \exp\left\{-\frac{C_0 h_i(\varepsilon)}{\varepsilon^{1/6}}\right\}.$$

We have ignored the term containing q_4, since $q_3 \geq q_4 + 3$.

Therefore, the term $|\mu L_{S_i}| \sim \frac{\mu}{\varepsilon^{1/6}} \exp\{-\frac{C_0 h_i(\varepsilon)}{\varepsilon^{1/6}}\}$ (estimated in Lemma 7.1) dominates if

$$\frac{\mu}{\varepsilon^{1/6}} \succeq \frac{\mu^2}{\varepsilon^{(q_3-1)/6}}.$$

If one takes $\mu = \varepsilon^p$, the last condition is fulfilled at $p > (q_3 - 2)/6$. We get $p^* = \max\{(q_3 - 2)/6, q_1/6\} = (q_3 - 2)/6$, since $q_3 - 2 \geq q_1$. In fact, for $\nu = 0$ we have $q_3 = 20$ and hence $p^* = 3$, and for $\nu = 1$ we have $q_3 = 14$ and $p^* = 2$. The remaining statement of (a) is proved similarly.

To prove (c), we bound the sum of $|k|^l \mathcal{L}_k$, $k \neq S_i, S'$, $i = 1, \ldots, m$, by geometric series analogously as it was done in the proof of Lemma 7.1. □

7.5 Critical points of the splitting potential

In this section we study the critical points of the splitting potential

$$\mathcal{L}(\theta) = \sum_{i=1}^{\infty} \mathcal{L}_{S_i} \cos(\langle S_i, \theta\rangle - \sigma_{S_i}),$$

where

$$\mathcal{L}_{S_1} \geq \mathcal{L}_{S_2} \geq \mathcal{L}_{S_3} \geq \dots . \tag{7.27}$$

The goal is (i) to show the nondegenericity of the critical points and (ii) to give estimates for the minimal eigenvalue of $D^2\mathcal{L}(\theta)$. To these ends, we need at least 3 dominant harmonics of $\mathcal{L}(\theta)$. We consider two different cases: 1. when the only 3 dominant harmonics $\mathcal{L}_{S_1}, \mathcal{L}_{S_2}\mathcal{L}_{S_3}$ are sufficient (it the case when the corresponding vectors S_1, S_2, S_3 are linearly independent, i.e. $\Delta = \det(S_1, S_2, S_3) \neq 0$) and 2. when we need 4 dominant harmonics $\mathcal{L}_{S_1}, \mathcal{L}_{S_2}\mathcal{L}_{S_3}$ and $\mathcal{L}_{S_4}$ (the case $\Delta = \det(S_1, S_2, S_3) = 0$ and $\Delta_1 = \det(S_1, S_2, S_4) \neq 0$). Therefore, we consider, first, the approximations given by 3 or 4 dominant harmonics:

$$\mathcal{L}^{(3)}(\theta) = \sum_{i=1}^{3} \mathcal{L}_{S_i} \cos(\langle S_i, \theta\rangle - \sigma_{S_i}) \text{ and } \mathcal{L}^{(4)}(\theta) = \sum_{i=1}^{4} \mathcal{L}_{S_i} \cos(\langle S_i, \theta\rangle - \sigma_{S_i}),$$

according to the case studied, and complete (i)-(ii). Afterwards, we apply Lemma B.3 (the Fixed Point Theorem from Appendix B) to validate these critical points in the whole function $\mathcal{L}(\theta)$.

Remark 7.3. Numerical explorations show that Δ takes values $1, -1, 0$, and if $\Delta = 0$, Δ_1 is equal to 1 or -1.

7.5.1 The case $\Delta = \det(S_1, S_2, S_3) \neq 0$.

In the case of nonzero determinant $\det(S_1, S_2, S_3)$ we consider the approximation $\mathcal{L}^{(3)}$ and carry out the linear change

$$\psi_i = \langle S_i, \theta\rangle - \sigma_{S_i}, \quad i = 1, 2, 3, \tag{7.28}$$

that can be written as

$$\psi = \mathcal{A}_n\theta - b_n, \text{ where } \mathcal{A}_n = (S_1; S_2; S_3)^\top, b_n = (\sigma_{S_1}, \sigma_{S_2}, \sigma_{S_3})^\top$$

and turns $\mathcal{L}^{(3)}(\theta)$ into

$$K^{(3)}(\psi) = \mathcal{L}_{S_1} \cos\psi_1 + \mathcal{L}_{S_2} \cos\psi_2 + \mathcal{L}_{S_3} \cos\psi_3.$$

It easy to see that $K^{(3)}$ has 8 nondegenerate critical points $\psi^{[3]}_{(1),0} = (0, 0, 0)$, $\psi^{[3]}_{(2),0} = (0, 0, \pi)$, $\psi^{[3]}_{(3),0} = (0, \pi, 0)$, $\psi^{[3]}_{(4),0} = (0, \pi, \pi)$, $\psi^{[3]}_{(5),0} = (\pi, 0, 0)$, $\psi^{[3]}_{(6),0} = (\pi, \pi, 0)$, $\psi^{[3]}_{(7),0} =$

$(\pi, 0, \pi)$, $\psi^{[3]}_{(8),0} = (\pi, \pi, \pi)$. Note that $\det \mathcal{A}_n = \Delta \neq 0$ and, moreover, numerically $|\Delta| = 1$ (if $\Delta \neq 0$, see Remark 7.3). Therefore, applying the inverse change of (7.28) and the fact that $|\det \mathcal{A}| = 1$, we obtain 8 nondegenerate critical points of $\mathcal{L}^{(3)}(\theta)$:

$$\theta^{[3]}_{(j),0} = \mathcal{A}_n^{-1}(\psi^{[3]}_{(j),0} + b_n), \quad j = 1, \dots, 8. \tag{7.29}$$

To find the eigenvalues of $D^2\mathcal{L}^{(3)}(\theta^{[3]}_{(j),0})$ and give estimates for the minimal in modulus of them, we note that the eigenvalues of the second derivative $D^2K^{(3)}(\psi)$ of $K^{(3)}(\psi)$ at the critical points $\psi^{[3]}_{(j),0}$, $j = 1, \dots, 8$, are $\lambda_1 = \pm\mathcal{L}_{S_1}$, $\lambda_2 = \pm\mathcal{L}_{S_2}$, $\lambda_3 = \pm\mathcal{L}_{S_3}$.

On the other hand, the second derivative of $\mathcal{L}^{(3)}$ can be presented in terms of $D^2K^{(3)}$ as

$$D^2\mathcal{L}^{(3)}(\theta^{[3]}_{(j),0}) = \mathcal{A}_n^\top D^2K^{(3)}(\psi^{[3]}_{(j),0})\mathcal{A}_n = \pm\mathcal{L}_{S_1} S_1 \cdot S_1^\top \pm \mathcal{L}_{S_2} S_2 \cdot S_2^\top \pm \mathcal{L}_{S_3} S_3 \cdot S_3^\top.$$

By assumption (7.27), we write the eigenvalues of $D^2K^{(3)}$ in form $\lambda_1 = \pm\mathcal{L}_{S_1} = A$, $\lambda_2 = \pm\mathcal{L}_{S_2} = A\delta_1$, $\lambda_3 = \pm\mathcal{L}_{S_3} = A\delta_1\delta_2$ with δ_1 and δ_2 small. Note that A, δ_1, δ_2 can be both positive and negative (depending on the sign of the corresponding eigenvalue).

Assume μ is an eigenvalue and v the associated eigenvector of $D^2\mathcal{L}^{(3)}$. Then the following is true:

$$D^2\mathcal{L}^{(3)}v = \mathcal{A}_n^\top D^2K^{(3)}\mathcal{A}_n v = \mu v.$$

Multiplying on the left by $\mathcal{A}_n$ and denoting $w = \mathcal{A}_n v$ and $B = \mathcal{A}_n\mathcal{A}_n^\top$, we get $BD^2K^{(3)}w = \mu w$. This means that μ is also an eigenvalue of $BD^2K^{(3)}$ with a new associated eigenvector w. Hence, instead of finding the eigenvalues of $D^2\mathcal{L}^{(3)} = \mathcal{A}_n^\top D^2K^{(3)}\mathcal{A}_n$ directly, we look for the eigenvalues of the matrix $BD^2K^{(3)}$, since these matrices have the same eigenvalues (but different eigenvectors). Note that B is a symmetric matrix with components $b_{ij} = S_i^\top S_j$ and the eigenvalues of $D^2K^{(3)}$, as said above, are A, $A\delta_1$ and $A\delta_1\delta_2$. We use the following lemma to find the eigenvalues of the product of these matrices:

Lemma 7.3. *Assume δ_1, δ_2 are small enough. Then the eigenvalues of the matrix BK, where B is a symmetric 3×3-matrix with positive components $b_{i,j}$ and $K = diag(A_1, A_2\delta_1, A_3\delta_1\delta_2)$ is a diagonal matrix with some constants A_1, A_2, A_3, are:*

$$\begin{aligned}
\mu_1^{[3]} &= A_1 b_{11} + A_2\tfrac{b_{12}^2}{b_{11}}\delta_1 + A_3\tfrac{b_{13}^2}{b_{11}}\delta_1\delta_2 + \mathcal{O}(\delta_1^2) + \delta_1\delta_2\mathcal{O}(\delta_1, \delta_2);\\
\mu_2^{[3]} &= A_2\tfrac{b_{11}b_{22} - b_{12}^2}{b_{11}}\delta_1 + A_3\left(b_{33} - \tfrac{b_{13}^2}{b_{11}}\tfrac{\det(B)}{b_{22}b_{11} - b_{12}^2}\right)\delta_1\delta_2 + \mathcal{O}(\delta_1^2) + \delta_1\delta_2\mathcal{O}(\delta_1, \delta_2);\\
\mu_3^{[3]} &= \tfrac{\det(B)}{b_{22}b_{11} - b_{12}^2}A_3\delta_1\delta_2 + \mathcal{O}(\delta_1^2) + \delta_1\delta_2\mathcal{O}(\delta_1, \delta_2).
\end{aligned}$$

Proof. We write the product

$$BK = \begin{pmatrix} A_1 b_{11} & A_2 b_{12}\delta_1 & A_3 b_{13}\delta_1\delta_2 \\ A_1 b_{12} & A_2 b_{22}\delta_1 & A_3 b_{23}\delta_1\delta_2 \\ A_1 b_{13} & A_2 b_{23}\delta_1 & A_3 b_{33}\delta_1\delta_2 \end{pmatrix}$$

1. In the easiest case $\delta_1 = 0$, the matrix has eigenvalues $\mu_1^{[1]} = A_1 b_{11}$ and $\mu_2^{[1]} = \mu_3^{[1]} = 0$.

2. Assume that $\delta_2 = 0$, but $\delta_1 \neq 0$. In this case we perturb eigenvalues obtained in the case 1 as

$$\mu_1^{[2]} = Ab_{11} + c_1^{[2]}\delta_1 + \mathcal{O}(\delta_1^2), \mu_2^{[2]} = c_2^{[2]}\delta_1 + \mathcal{O}(\delta_1^2), \mu_3^{[2]} = c_3^{[2]}\delta_1 + \mathcal{O}(\delta_1^2)$$

and try to find coefficients c_i such that $\mu_i^{[2]}$ are eigenvalues of

$$\begin{pmatrix} A_1 b_{11} & A_2 b_{12}\delta_1 & 0 \\ A_1 b_{21} & A_2 b_{22}\delta_1 & 0 \\ A_1 b_{31} & A_2 b_{32}\delta_1 & 0 \end{pmatrix}$$

We get equations for $\mu_i^{[2]}$

$$\mu_1^{[2]} + \mu_2^{[2]} = A_1 b_{11} + A_2 b_{22}\delta_1; \quad \mu_1^{[2]}\mu_2^{[2]} = A_1 A_2 (b_{11}b_{22} - b_{12}^2)\delta_1; \quad \mu_3^{[2]} = 0.$$

From the last equation we obtain $c_3^{[2]} = 0$, while for the other coefficients we have

$$c_1^{[2]}\delta_1 + c_2^{[2]}\delta_2 = A_2 b_{22}\delta_1; \quad (A_1 b_{11} + c_1^{[2]}\delta_1)c_2^{[2]}\delta_1 = A_1 A_2 (b_{11}b_{22} - b_{12}^2)\delta_1.$$

Neglecting terms of order $\mathcal{O}(\delta_1^2)$, we conclude

$$c_1^{[2]} + c_2^{[2]} = b_{22}A_2; \quad b_{11}c_2^{[2]} = A_2(b_{11}b_{22} - b_{12}^2)$$

From this, it follows

$$c_1^{[2]} = A_2\frac{b_{12}^2}{b_{11}}; \quad c_2^{[2]} = A_2\frac{b_{11}b_{22} - b_{12}^2}{b_{11}}$$

Thus, in the case $\delta_2 = 0$ and $\delta_1 \neq 0$, the eigenvalues are

$$\mu_1^{[2]} = A_1 b_{11} + A_2\frac{b_{12}^2}{b_{11}}\delta_1 + \mathcal{O}(\delta_1^2); \quad \mu_2^{[2]} = A_2\frac{b_{11}b_{22} - b_{12}^2}{b_{11}}\delta_1 + \mathcal{O}(\delta_1^2); \quad \mu_3^{[2]} = \mathcal{O}(\delta_1^2).$$

3. In order to find the eigenvalues in the whole case $\delta_1, \delta_2 \neq 0$ we perturb $\mu_i^{[2]}$ as follows

$$\begin{aligned} \mu_1^{[3]} &= Ab_{11} + A\tfrac{b_{12}^2}{b_{11}}\delta_1 + c_1^{[3]}\delta_1\delta_2 + \mathcal{O}(\delta_1^2) + \delta_1\delta_2\mathcal{O}(\delta_1, \delta_2); \\ \mu_2^{[3]} &= c_2^{[2]}\delta_1 + c_2^{[3]}\delta_1\delta_2 + \mathcal{O}(\delta_1^2) + \delta_1\delta_2\mathcal{O}(\delta_1, \delta_2); \\ \mu_3^{(3)} &= c_3^{[3]}\delta_1\delta_2 + \mathcal{O}(\delta_1^2) + \delta_1\delta_2\mathcal{O}(\delta_1, \delta_2). \end{aligned}$$

We have equations

$$\begin{array}{l} \mu_1^{[3]} + \mu_1^{[3]} + \mu_1^{[3]} = T; \\ \mu_1^{[3]}\mu_2^{[3]} + \mu_1^{[3]}\mu_3^{[3]} + \mu_2^{[3]}\mu_3^{[3]} = \frac{1}{2}[T^2 - T_1]; \\ \mu_1^{[3]}\mu_2^{[3]}\mu_3^{[3]} = D; \end{array}$$

where

$$\begin{array}{l} T = tr(BK) = A_1 b_{11} + A_2 b_{22}\alpha + A_3 b_{33}\beta, \\ T_1 = tr[(BK)^2] = A_1^2 b_{11}^2 + 2A_1A_2 b_{12}^2\alpha + 2A_1A_3 b_{13}^2\beta + O_2(\alpha,\beta), \\ D = \det(BK) = \det(B) A_1A_2A_3\alpha\beta. \end{array}$$

Substituting $\mu^{[3]}$ into the last equation, we find

$$\begin{aligned} \mu_3^{[3]} &= \frac{\det(B)A_1A_2A_3\delta_1^2\delta_2}{\mu_1^{[3]}\mu_2^{[3]}} \\ &= \frac{\det(B)A_1A_2A_3^3\delta_1^2\delta_2}{(A_1b_{11} + A_2\frac{b_{12}^2}{b_{11}}\delta_1 + c_1^{[3]}\delta_1\delta_2 + \ldots)(c_2^{[2]}\delta_1 + c_2^{[3]}\delta_1\delta_2 + \ldots)} \\ &= \frac{\det(B)A_1A_2A_3\delta_1^2\delta_2}{A_1b_{11}c_2^{[2]}\delta_1}\left(1 - \frac{A_2b_{12}^2}{A_1b_{11}^2}\delta_1 - \frac{c_1^{[3]}}{A_1b_{11}}\delta_1\delta_2 + \ldots\right)\left(1 - \frac{c_2^{[3]}}{c_2^{[2]}}\delta_2 + \ldots\right) \\ &= \frac{\det(B)A_3\delta_1\delta_2}{b_{11}b_{22} - b_{12}^2}\left(1 + \mathcal{O}(\delta_1,\delta_2)\right). \end{aligned}$$

Other equations becomes (neglecting the higher order terms)

$$c_1^{[3]} + c_2^{[3]} = A_3b_{33} - \frac{A_3\det(B)}{b_{11}b_{22} - b_{12}^2}; \quad A_1b_{11}c_2^{[3]} + A_1A_3b_{11}\frac{\det(B)}{b_{11}b_{22} - b_{12}^2} = A_1A_3(b_{11}b_{33} - b_{13}^2)$$

Thus, we get

$$c_1^{[3]} = A_3\frac{b_{13}^2}{b_{11}}; \quad c_2^{[3]} = A_3\left(b_{33} - \frac{b_{13}^2}{b_{11}}\frac{\det(B)}{b_{22}b_{11} - b_{12}^2}\right); \quad c_3^{[3]} = \frac{A_3\det(B)}{b_{22}b_{11} - b_{12}^2},$$

that gives the expected values for the eigenvalues $\mu_i^{[3]}$.

□

In our case, we have $b_{ij} = S_i^\top S_j$, $i,j = 1,2,3$, and $A_1 = A_2 = A_2 = A$. Then according to the Lemma 7.3, then the minimum eigenvalue is

$$m_{(j),0}^{[3]} = \mu_3^{[3]} \sim \frac{\det(B)}{b_{11}b_{22} - b_{12}^2}A\delta_1\delta_2 = \frac{\det(B)}{b_{11}b_{22} - b_{12}^2}\mathcal{L}_{S_3}.$$

Note that the denominator $b_{11}b_{22} - b_{12}^2 = |S_1|^2|S_2|^2 - (\langle S_1, S_2\rangle)^2 = |S_1|^2|S_2|^2\sin\phi$, where ϕ is the angle between S_1 and S_2, is far from zero. It could be equal to 0, if the vector S_1 and S_2 were collinear, but it is not the case, since $\Delta = \det(S_1, S_2, S_3) \neq 0$. Moreover, due to (7.26), we can estimate

$$b_{11}b_{22} - b_{12}^2 \sim \varepsilon^{-2/3}.$$

Therefore, taking into account also the numerical result that $\det B = 1$, we get the estimate for the minimum eigenvalue

$$m^{[3]}_{(j),0} \sim \varepsilon^{2/3}\mathcal{L}_{S_3}. \tag{7.30}$$

Thus, we have proven the following lemma:

Lemma 7.4 (Critical points of $\mathcal{L}^3(\theta)$)**.** *The function $\mathcal{L}^{(3)}$ has exactly 8 nondegenerate critical points $\theta^{[3]}_{(j),0}$, $j = 1, \ldots, 8$, given by (7.29), and the minimal eigenvalue of $D^2\mathcal{L}(\theta^{[3]}_{(j),0})$ satisfies the estimate (7.30).*

To prove the persistence of the critical points $\theta^{[3]}_{(j),0}$ with some small perturbations in the whole splitting potential $\mathcal{L}(\theta)$, we proceed as in the quadratic case in Chapter 6 (Section 6.2.5, p. 139). We consider the function

$$\mathcal{L}(\theta) = \mathcal{L}^{(3)}(\theta) + \mathcal{L}_{S_4}F(\theta),$$

where the term $\mathcal{L}_{S_4}F(\theta)$ stands for the remainder and, according to Lemma 7.2, function F satisfies:

$$\left|\partial_{\psi_j}F\right| \sim \frac{1}{\varepsilon^{1/6}}, \quad \left|\partial^2_{\psi_i\psi_j}F\right| \sim \frac{1}{\varepsilon^{1/3}}, \quad i,j = 1,2,3.$$

By the linear change (7.28), $\mathcal{L}(\theta)$ is transformed to

$$K(\psi) = K^{(3)}(\psi) + \mathcal{L}_{S_4}G(\psi),$$

where function G satisfies the bounds

$$\left|\partial_{\psi_j}G\right| \sim \varepsilon^{-1/2}, \quad \left|\partial^2_{\psi_i\psi_j}G\right| \sim \varepsilon^{-1}, \quad i,j = 1,2,3. \tag{7.31}$$

The critical points of $K(\psi)$ are the solutions of

$$\sin\psi_1 = \frac{\mathcal{L}_{S_4}}{\mathcal{L}_{S_1}}\partial_{\psi_1}G(\psi),\ \sin\psi_2 = \frac{\mathcal{L}_{S_4}}{\mathcal{L}_{S_2}}\partial_{\psi_2}G(\psi),\ \sin\psi_3 = \frac{\mathcal{L}_{S_4}}{\mathcal{L}_{S_3}}\partial_{\psi_3}G(\psi).$$

We denote

$$\eta = \max\left\{\frac{\mathcal{L}_{S_4}}{\mathcal{L}_{S_1}}, \frac{\mathcal{L}_{S_4}}{\mathcal{L}_{S_2}}, \frac{\mathcal{L}_{S_4}}{\mathcal{L}_{S_3}}\right\} = \frac{\mathcal{L}_{S_4}}{\mathcal{L}_{S_3}}$$

Then by Lemma B.3 (the Fixed Point Theorem, see Appendix B) the equations have nondegenerate solutions if

$$\left(\frac{\eta}{\varepsilon^{1/6}}\right)^2 + \left(\frac{3\eta}{\varepsilon^{1/3}}\right)^2 < 1$$

is satisfied. The inequality is true for $\eta \preceq \varepsilon$.

Under this condition, by Lemma B.3 there are 8 simple solutions:

$$\psi^{[3]}_{(j)} = \psi^{[3]}_{(j,0)} + \mathcal{O}(\eta), j = 1, \ldots, 8.$$

Executing the inverse change of (7.28), we get the critical points of $\mathcal{L}$:

$$\theta^{[3]}_{(j)} = \mathcal{A}_n^{-1}(\psi^{[3]}_{(j)} + b), \quad j = 1, \ldots, 8. \tag{7.32}$$

Remark 7.4. By the inverse change, we obtain 8 critical points, since we suppose that $|\mathcal{A}| = 1$, the fact confirmed numerically.

Regarding the eigenvalues of $D^2\mathcal{L}$ at each critical point, we have $D^2\mathcal{L}(\theta^{[3]}_{(j)}) = \mathcal{A}_n^\top D^2K(\psi^{[3]}_{(j)})\mathcal{A}_n$, where

$$D^2K(\psi^{[3]}_{(j)}) = D^2K^{(3)}(\psi^{[3]}_{(j)}) + \mathcal{O}(\eta) = D^2K^{(3)}(\psi^{[3]}_{(j,0)}) + \mathcal{O}(\eta),$$

and, therefore, the eigenvalues of $D^2\mathcal{L}$ are a perturbation of order $\mathcal{O}(\eta)$ of the eigenvalues of $\mathcal{L}^{(3)}$. Hence, the minimal (in modulus) eigenvalue satisfies

$$m^{[3]}_{(j)} \sim \varepsilon^{2/3}\mathcal{L}_{S_3}. \tag{7.33}$$

7.5.2 The case $\Delta = 0$, but $\Delta_1 = \det(S_1, S_2, S_4) \neq 0$.

Due to a peculiarity of the matrix U, the following equality takes place:

$$s(j, n+3) = s(j, n) - s(j, n+1) \text{ for any } j \in \mathcal{P} \tag{7.34}$$

and, therefore, it can occur that the dominant vectors S_1, S_2, S_3 satisfy this equality, i.e. $\Delta = 0$. We assume that $\Delta_1 = \det(S_1, S_2, S_4) \neq 0$.

Remark 7.5. Numerical explorations show that both Δ and Δ_1 never vanish simultaneously, i.e. if $\Delta = 0$, then $\Delta_1 \neq 0$ (in fact, Δ_1 takes values -1 and 1).

In this case, we consider the approximation $\mathcal{L}^{(4)}(\theta)$ by 4 dominant harmonics. We carry out the change

$$\psi_1 = \langle S_1, \theta\rangle - \sigma_{S_1}, \quad \psi_2 = \langle S_2, \theta\rangle - \sigma_{S_2}, \quad \psi_3 = \langle S_4, \theta\rangle - \sigma_{S_4} \tag{7.35}$$

or $\psi = \mathcal{A}'_n\theta - b'_n$ with $\mathcal{A}'_n = (S_1; S_2; S_4)^\top, b'_n = (\sigma_{S_1}, \sigma_{S_2}, \sigma_{S_4})^\top$. We suppose, due to (7.34), $S_3 = \tau S_1 + \sigma S_2$, $\sigma, \tau = \pm 1$. Then, under the change (7.35), $\mathcal{L}^{(4)}(\theta)$ turns into

$$K^{(4)}(\psi) = \mathcal{L}_{S_1}\cos\psi_1 + \mathcal{L}_{S_2}\cos\psi_2 + \mathcal{L}_{S_3}\cos(\tau\psi_1 + \delta\psi_2 - \triangle\sigma) + \mathcal{L}_{S_4}\cos\psi_3,$$

where $\triangle\sigma = \sigma_3 - \tau\sigma_1 - \delta\sigma_2$.

The case of 4 dominant harmonics is more difficult than the previous one, since the second derivative $D^2K^{(4)}$ is not diagonal anymore (but it is diagonalizable) and a special study is required to find the critical points of $\mathcal{K}^{(\triangle)}$. Thus, we divide this problem in some parts where we find critical points of $K^{(4)}$, reduce $D^2K^{(4)}$ to the diagonal form (and, thus, we find eigenvalues of $D^2K^{(4)}$), and, finally, find estimates for the eigenvalues of $D^2\mathcal{L}^{(4)}$ (using the Lemma 7.3 through the eigenvalues of matrix $Bdiag(D^2K^{(4)})$ with a new B).

Critical points of $K^{(4)}(\psi)$

The critical points $\psi \in \mathbb{T}^3$ of $K^{(4)}(\psi)$ are the zeros of its gradient:

$$\begin{pmatrix} \mathcal{L}_{S_1} \sin\psi_1 + \tau\mathcal{L}_{S_3} \sin(\tau\psi_1 + \sigma\psi_2 - \triangle\sigma) \\ \mathcal{L}_{S_2} \sin\psi_2 + \sigma\mathcal{L}_{S_3} \sin(\tau\psi_1 + \sigma\psi_2 - \triangle\sigma) \\ \mathcal{L}_{S_4} \sin\psi_3 \end{pmatrix}.$$

For the last component of ψ we have $\sin\psi_3 = 0$ and simply find $\psi_3^{(1)} = 0$ and $\psi_3^{(2)} = \pi$. To find the other components we work with equations

$$\begin{aligned} \mathcal{L}_{S_1} \sin\psi_1 + \tau\mathcal{L}_{S_3} \sin(\tau\psi_1 + \sigma\psi_2 - \triangle\sigma) = 0 \\ \mathcal{L}_{S_2} \sin\psi_2 + \sigma\mathcal{L}_{S_3} \sin(\tau\psi_1 + \sigma\psi_2 - \triangle\sigma) = 0 \end{aligned}$$

As in the case of 3 dominant harmonics, we denote again: $\mathcal{L}_{S_1} = A$, $\mathcal{L}_{S_2} = A\delta_1$, $\mathcal{L}_{S_3} = A\delta_1\delta_2$ and, hence, get

$$\begin{aligned} \sin\psi_1 &= -\delta_1\delta_2\tau \sin(\psi_1 - \psi_2 - \triangle\sigma), \\ \sin\psi_2 &= -\delta_2\sigma \sin(\psi_1 - \psi_2 - \triangle\sigma). \end{aligned} \tag{7.36}$$

By Lemma B.2, these equations have 4 simple solutions if the following holds:

$$3(\delta_1\delta_2)^2 < 1, \quad 3\delta_2^2 < 1.$$

The second inequality implies the first one. Choosing $\tilde{\eta} = \max\{\delta_1\delta_2, \delta_2\} = \delta_2$. Thus, if $\tilde{\eta} < 1/\sqrt{3}$ (that is $\mathcal{L}_{S_3} < \mathcal{L}_{S_2}/\sqrt{3}$), function $K^{(4)}$ has 8 nondegenerate critical points:

$$\begin{aligned} &\psi^{[4]}_{(1),\tilde{\eta}} = (\mathcal{O}(\tilde{\eta}), \mathcal{O}(\tilde{\eta}), 0), && \psi^{[4]}_{(2),\tilde{\eta}} = (\pi + \mathcal{O}(\tilde{\eta}), \mathcal{O}(\tilde{\eta}), 0), \\ &\psi^{[4]}_{(3),\tilde{\eta}} = (\mathcal{O}(\tilde{\eta}), \pi + \mathcal{O}(\tilde{\eta}), 0), && \psi^{[4]}_{(4),\tilde{\eta}} = (\pi + \mathcal{O}(\tilde{\eta}), \pi + \mathcal{O}(\tilde{\eta}), 0), \\ &\psi^{[4]}_{(5),\tilde{\eta}} = (\mathcal{O}(\tilde{\eta}), \mathcal{O}(\tilde{\eta}), \pi), && \psi^{[4]}_{(6),\tilde{\eta}} = (\pi + \mathcal{O}(\tilde{\eta}), \mathcal{O}(\tilde{\eta}), \pi), \\ &\psi^{[4]}_{(7),\tilde{\eta}} = (\mathcal{O}(\tilde{\eta}), \pi + \mathcal{O}(\tilde{\eta}), \pi), && \psi^{[4]}_{(8),\tilde{\eta}} = (\pi + \mathcal{O}(\tilde{\eta}), \pi + \mathcal{O}(\tilde{\eta}), \pi). \end{aligned}$$

Thus, by the inverse change of (7.35), we get 8 critical points of $\mathcal{L}^{[4]}$

$$\theta^{[4]}_{(j),\tilde{\eta}} = (\mathcal{A}'_n)^{-1}(\psi^{[4]}_{(j),\tilde{\eta}} + b'_n), \quad j = 1, \ldots, 8. \tag{7.37}$$

Diagonal form of $D^2K^{(4)}$.

The matrix $D^2K^{(4)}$ in sot diagonal, but at critical points it has form

$$D^2K^{(4)}(\psi^{[4]}_{(j),\tilde{\eta}}) = \begin{pmatrix} a & c & 0 \\ c & b & 0 \\ 0 & 0 & d \end{pmatrix}$$

being

$$\begin{aligned} a &= -\left(\mathcal{L}_{S_1}\cos\psi_1 + \mathcal{L}_{S_3}\cos(\tau\psi_1+\sigma\psi_2-\triangle\sigma)\right)\big|_{\psi^{[4]}_{(j),\tilde\eta}}, \\ b &= -\left(\mathcal{L}_{S_2}\cos\psi_2 + \mathcal{L}_{S_3}\cos(\tau\psi_1-\sigma\psi_2-\triangle\sigma)\right)\big|_{\psi^{[4]}_{(j),\tilde\eta}}, \\ c &= -\tau\sigma\mathcal{L}_{S_3}\cos(\tau\psi_1+\sigma\psi_2-\triangle\sigma)\big|_{\psi^{[4]}_{(j),\tilde\eta}} \\ d &= \pm\mathcal{L}_{S_4} \end{aligned}$$

We mention that $D^2K^{(4)}$ is a symmetric real matrix and, hence, can be reduced to a diagonal form, say, D, by an orthogonal matrix $Q^{-1}=Q^\top$, i.e.

$$D^2K^{(4)} = Q^{-1}DQ = Q^\top DQ$$

Thus,

$$D^2\mathcal{L}^{(4)} = (\mathcal{A}'_n)^\top D^2K^{(4)}\mathcal{A}'_n = (\mathcal{A}'_n)^\top Q^\top DQ\mathcal{A}'_n = W^\top DW, \text{ with } W = Q\mathcal{A}'_n \tag{7.38}$$

The eigenvalues of $D^2K^{(4)}$ (and the diagonal elements of D) are

$$\lambda_{1,2} = \frac{a+b\pm\sqrt{(a+b)^2-4ab+4c^2}}{2} \text{ and } \lambda_3 = d.$$

For example, at the point $\psi^{[4]}_{(1),\tilde\eta}$, we have (we also denote here $\mathcal{L}_{S_4} = A\delta_1\delta_2\delta_3$, where $\delta_3 = \mathcal{L}_{S_4}/\mathcal{L}_{S_3} < 1$):

$$\begin{aligned} &a(\psi^{[4]}_{(1),\tilde\eta}) = -A(1+\delta_1\delta_2\cos\triangle\sigma + \mathcal{O}(\tilde\eta^2)), \quad b(\psi^{[4]}_{(1),\tilde\eta}) = -A\delta_1(1+\tilde\eta\cos\triangle\sigma+\mathcal{O}(\tilde\eta^2)), \\ &c(\psi^{[4]}_{(1),\tilde\eta}) = -\tau\sigma A\delta_1\delta_2\cos\triangle\sigma + \mathcal{O}(\tilde\eta^2), \\ &d(\psi^{[4]}_{(1),\tilde\eta}) = -A\delta_1\delta_2\delta_3, \end{aligned}$$

and the eigenvalues are

$$\begin{aligned} \lambda_1(\psi^{[4]}_{(1),\tilde\eta}) &= -A - A\delta_1\delta_2\cos\triangle\sigma + \mathcal{O}(\tilde\eta^2); \\ \lambda_2(\psi^{[4]}_{(1),\tilde\eta}) &= -A\delta_1 - A\delta_1\delta_2\cos\triangle\sigma + \mathcal{O}(\tilde\eta^2); \\ \lambda_3(\psi^{[4]}_{(1),\tilde\eta}) &= -A\delta_1\delta_2\delta_3. \end{aligned}$$

We write the diagonal matrix

$$D = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} = \begin{pmatrix} A' & 0 & 0 \\ 0 & A''\delta_1 & 0 \\ 0 & 0 & A'''\delta_1\delta_2 \end{pmatrix},$$

where $A' = -A - A\delta_1\delta_2\cos\triangle\sigma + \mathcal{O}(\tilde\eta^2)$, $A'' = -A - A\delta_2\cos\triangle\sigma + \mathcal{O}(\tilde\eta)$, $A''' = -A\delta_3$.

As in the case of 3 dominant harmonics, the matrix $D^2\mathcal{L}^{(4)}$ has the same eigenvalues as the matrix BD, where $B = WW^\top$ with W from (7.38). In fact, $B = A'_n(A'_n)^\top$. Then according to Lemma 7.3, the minimum (in modulus) eigenvalue is

$$\mu^{[4]}_{(1),\tilde\eta} = \frac{|A'''|\det(B)\delta_1\delta_2}{b_{11}b_{22}-b_{12}^2} + \ldots \sim \frac{A\det(B)\delta_1\delta_2\delta_3}{b_{11}b_{22}-b_{12}^2} \sim \varepsilon^{2/3}\mathcal{L}_{S_4},$$

where $b_{11} = |S_1|^2$, $b_{22} = |S_2|^2$, $b_{12} = \langle S_1, S_2\rangle$ are elements of B. Since $\Delta_1 = \det(S_1, S_2, S_4) \neq 0$, the denominator $b_{11}b_{22} - b_{12}^2$ does not vanish, and, analogously to the case of $\Delta \neq 0$, one give the estimate $b_{11}b_{22} - b_{12}^2 \sim \varepsilon^{-2/3}$.

Thus, we have proven the following lemma:

Lemma 7.5 (Critical points of $\mathcal{L}^{(4)}(\theta)$)**.** *Assume $\tilde{\eta} = \mathcal{L}_{S_3}/\mathcal{L}_{S_2} \preceq \varepsilon$. Then function $\mathcal{L}^{(4)}$ has exactly 8 nondegenerate critical points $\theta^{[4]}_{(j),\tilde{\eta}}$, $j = 1, \dots, 8$, given by (7.37) , and the minimal eigenvalue of $D^2\mathcal{L}^{[4]}(\theta^{[4]}_{(j),\tilde{\eta}})$ satisfies*

$$\mu^{[4]}_{(j),\tilde{\eta}} \sim \varepsilon^{2/3}\mathcal{L}_{S_4}.$$

Critical points of $\mathcal{L}(\theta)$ in the case $\Delta = 0, \Delta_1 \neq 0$.

Concerning the critical points in the whole splitting potential, in the case of the 4 dominant harmonics, we consider

$$\mathcal{L}(\theta) = \mathcal{L}^{(4)}(\theta) + \mathcal{L}_{S_5}\widetilde{F}(\theta),$$

where we denote by $\mathcal{L}_{S_5}\widetilde{F}(\theta)$ the sum of the non-dominant terms which satisfies by Lemma 7.2:

$$\left|\partial_{\psi_j} F\right| \sim \frac{1}{\varepsilon^{1/6}}, \quad \left|\partial^2_{\psi_i\psi_j} F\right| \sim \frac{1}{\varepsilon^{1/3}}, \quad i,j = 1,2,3.$$

After the change (7.35), the splitting potential becomes

$$K(\psi) = K^{(4)}(\psi) + \mathcal{L}_{S_5}\widetilde{G}(\psi),$$

whose critical points are the solutions of

$$\begin{aligned} \sin\psi_1 &= -\tau\delta_1\delta_2\sin(\tau\psi_1 + \sigma\psi_2 - \triangle\sigma) + \tfrac{\mathcal{L}_{S_5}}{\mathcal{L}_{S_1}}\partial_{\psi_1}\widetilde{G}(\psi), \\ \sin\psi_2 &= -\sigma\delta_2\sin(\tau\psi_1 + \sigma\psi_2 - \triangle\sigma) + \tfrac{\mathcal{L}_{S_5}}{\mathcal{L}_{S_2}}\partial_{\psi_2}\widetilde{G}(\psi), \\ \sin\psi_3 &= \tfrac{\mathcal{L}_{S_5}}{\mathcal{L}_{S_4}}\partial_{\psi_3}\widetilde{G}(\psi). \end{aligned} \tag{7.39}$$

Note that function $\widetilde{G}$ also satisfies the bounds (7.31). We denote

$$\bar{\eta} = \max\left\{\frac{\mathcal{L}_{S_5}}{\mathcal{L}_{S_1}}, \frac{\mathcal{L}_{S_5}}{\mathcal{L}_{S_2}}, \frac{\mathcal{L}_{S_5}}{\mathcal{L}_{S_4}}\right\} = \frac{\mathcal{L}_{S_5}}{\mathcal{L}_{S_4}}$$

Then in order to have nondegenerate solutions, it is required by Lemma B.3 of the Appendix B that the following inequalities hold:

$$\begin{aligned} &\left(\delta_1\delta_2 + \bar{\eta}\varepsilon^{-1/2}\right)^2 + \left(\delta_1\delta_2 + 3\bar{\eta}\varepsilon^{-1}\right)^2 < 1, \\ &\left(\delta_2 + \bar{\eta}\varepsilon^{-1/2}\right)^2 + \left(\delta_2 + 3\bar{\eta}\varepsilon^{-1}\right)^2 < 1, \\ &\left(\bar{\eta}\varepsilon^{-1/2}\right)^2 + \left(3\bar{\eta}\varepsilon^{-1}\right)^2 < 1. \end{aligned}$$

We choose $\overline{\overline{\eta}} = \max\{\bar{\eta}, \delta_1\delta_2, \delta_2\} = \max\{\bar{\eta}, \tilde{\eta}\}$ and it turns out the inequalities are true for $\overline{\overline{\eta}} \preceq \varepsilon$.

Under this condition, the equations (7.39) have 8 solutions $\psi^{[4]}_{(j)} = \psi^{[4]}_{(j,0)} + \mathcal{O}(\overline{\overline{\eta}}), j = 1, \ldots, 8$. Apply the inverse change of (7.35), we get the critical points of $\mathcal{L}$:

$$\theta^{[4]}_{(j)} = (\mathcal{A}'_n)^{-1}(\psi^{[4]}_{(j)} + b'_n), \quad j = 1, \ldots, 8. \tag{7.40}$$

Remark 7.6. By the inverse change, we obtain 8 critical points, since we suppose that $|\mathcal{A}'_n| = 1$, the fact confirmed numerically.

One can show that the eigenvalues of $D^2\mathcal{L}$ are a perturbation of order $\mathcal{O}(\overline{\overline{\eta}})$ of the eigenvalues of $\mathcal{L}^{(4)}$ (for the same reason as in the case of the 3 dominant harmonics). Hence, the minimal (in modulus) eigenvalue satisfies

$$m^{[4]}_{(j)} \sim \varepsilon^{2/3}\mathcal{L}_{S_4}. \tag{7.41}$$

In the end, we summarize the results of this section in the following lemma:

Lemma 7.6 (Critical points of $\mathcal{L}(\theta)$)**.** *Assume that $\varepsilon \preceq 1$. Then one has:*

- *In the case that the indexes of the 3 dominant harmonics satisfy:* $\det(S_1, S_2, S_3) \neq 0$*, if $\eta = \mathcal{L}_{S_4}/\mathcal{L}_{S_3} \preceq \varepsilon$, then the splitting potential $\mathcal{L}(\theta)$ has 8 critical points $\theta^{[3]}_{(j)}$, $j = 1, \ldots, 8$, defined in (7.32), all nondegenerate, and the minimal eigenvalue (in modulus) $m^{[3]}_{(j)}$ of $D^2\mathcal{L}(\theta^{[3]}_{(j)})$ satisfies (7.33).*
- *In the case* $\det(S_1, S_2, S_3) = 0, \det(S_1, S_2, S_4) \neq 0$*, if* $\overline{\overline{\eta}} = \max\{\mathcal{L}_{S_5}/\mathcal{L}_{S_4}, \mathcal{L}_{S_3}/\mathcal{L}_{S_2}\} \preceq$ *ε, then the splitting potential $\mathcal{L}(\theta)$ has 8 critical points $\theta^{[4]}_{(j)}$, $j = 1, \ldots, 8$, defined in (7.40), all nondegenerate, and the minimal eigenvalue (in modulus) $m^{[4]}_{(j)}$ of $D^2\mathcal{L}(\theta^{[3]}_{(j)})$ satisfies (7.41).*

Remark 7.7. In each case, the conditions $\eta \preceq \varepsilon$ and $\overline{\overline{\eta}} =\preceq \varepsilon$ exclude from consideration some neighborhoods of the points ε where the last dominant harmonics, $\mathcal{L}_{S_3}$ and $\mathcal{L}_{S_4}$, respectively, coincides with the next harmonic, $\mathcal{L}_{S_4}$ and $\mathcal{L}_{S_5}$, respectively. These points belong to the discrete sets $\mathcal{E}_1$ and $\mathcal{E}_2$, introduced in (7.24). The study of these points still remains an open problem.

7.6 Proof of Theorems 7.1 and 7.2

Theorem 7.1, p. 171, follows from the estimate for the most dominant harmonic $\mathcal{L}_{S_1}$ given in Lemma Lemma 7.2, p. 178, and the fact that $\mathcal{M}(\theta) = \partial_\theta\mathcal{L}(\theta)$. We consider the approximation $\mathcal{L}^{(3)}$ by 3 dominant harmonics and deduce the following estimates:

$$|\partial_\theta\mathcal{L}^{(3)}| \sim |S_1|\mathcal{L}_{S_1} \sim \frac{1}{\varepsilon^{1/6}}\mathcal{L}_{S_1}, \quad |\partial\mathcal{L} - \partial_\theta\mathcal{L}^{(3)}| \sim \frac{1}{\varepsilon^{1/6}}\mathcal{L}_{S_4}.$$

Since $h_4(\varepsilon) > h_1(\varepsilon)$, we get the estimate

$$|\mathcal{M}| = |\partial_\theta \mathcal{L}| \sim \frac{\mu}{\sqrt[3]{\varepsilon}} \exp\left\{-\frac{C_0 h_1(\varepsilon)}{\varepsilon^{1/6}}\right\}.$$

Theorem 7.2, p. 173, is a consequence of Lemma 7.6 and that the nondegenerate critical points of $\mathcal{L}(\theta)$ correspond to simple zeros of $\mathcal{M}(\theta)$. Applying the estimate for $\mathcal{L}_{S_3}$ or $\mathcal{L}_{S_4}$ given in Lemma 7.2, one can obtain the expected estimate for the minimal (in modulus) eigenvalue of the splitting matrix $\partial_\theta \mathcal{M} = D^2\mathcal{L}$.

Appendix B

The fixed point theorems

To find zeros of a function $f(x) = 0$, $x \in \mathbb{R}$, we rewrite the equation in the form $x = g(x)$ and use the iteration method to define the sequence $x_{n+1} = g(x_n)$, $n = 0, 1, 2, \ldots$, and if this sequence converges, it converges to the solution of the equation. First, we state the well-known theorems for the cases when the dimension of the variable x is 1 or n, then we adapt them to solve specific equations of type $\sin X = F(X)$ ($X \in \mathbb{T}^m$, $m = 1, 2$ or 3). Note that the last equations appear often throughout Part II of this thesis in finding the nondegenerate critical points of the splitting potential $\mathcal{L}$.

Theorem B.1 (The 1-dimensional fixed point theorem)**.** *Let $I = [a, b]$ and $g : I \to \mathbb{R}$ such that*

(i) $g(I) \subset I$

(ii) *g is a Lipschitz function with a Lipschitz constant* $0 < L < 1$, *i.e.* $|g(x_1) - g(x_2)| \leq L|x_1 - x_2|, \forall x_1, x_2 \in I$

Then for any starting point $x_0 \in I$ the sequence $x_{n+1} = g(x_n), n = 0, 1, 2, \ldots$ converges to the unique solution $s \in I$ of $x = g(x)$ and

$$|x_n - s| \leq \frac{L^n}{1 - L}|x_1 - x_0|$$

Remark B.1. By the Mean Value Theorem, $|g\prime(x)| \leq L < 1$ implies (ii)

In the case of dimension 2, 3 or higher, we act in the analogous way, we rearrange the equation $F(z) = 0, z \in \mathbb{R}^n, F \in \mathbb{R}^n$ in the form $z = G(z)$ and find the sequence $z_{n+1} = G(z_n)$ with some starting point $z_0 \in \mathbb{R}^n$.

Theorem B.2 (The n-dimensional fixed point theorem)**.** *Let $I = \{z \in \mathbb{R}^n : a_i \leq z \leq b_i\}$ and $G : I \to \mathbb{R}^n$ such that*

(i) $G(I) \subset I$;

(ii) There exists a constant $0 < L < 1$ *such that* $\|G(z_1) - G(z_2)\| \leq L\|z_1 - z_2\|$ *for all* $z_1, z_2 \in I$*;*

Then

(a) the equation $z = G(z)$ *has a unique solution* $s \in I$*;*

(b) for any $z_0 \in I$ *the sequence* $z_{n+1} = G(z_n)$*,* $n = 0, 1, 2, \ldots$ *converges to* s*;*

(c) for $n = 1, 2, \ldots$ *the following inequality holds*

$$\|z_n - s\| \leq \frac{L^n}{1-L}\|z_1 - z_0\|.$$

Remark B.2. By the Mean Value theorem ($G : \mathbb{R}^n \to \mathbb{R}^m$ differentiable, $z, \bar{z} \in \mathbb{R}^n$, then $\exists y : y_i = z_i + t_i(z_i - \bar{z}_i), t_i \in [0,1]$ st $G(z) - G(\bar{z}) = DG(y)(z - \bar{z})$)

$$\|G(z) - G(\bar{z})\| = \|DG(y)(z - \bar{z})\| \leq \|DG(y)\|\|z - \bar{z}\|,$$

where we use the 1-norm $\|DG\| = \|DG\|_1 = \max\limits_{1\leq i\leq m} \sum\limits_{j=1}^{n} \left|\frac{\partial G_i}{\partial z_j}\right|$. Therefore, $L = \sup\limits_{y} \|DG(y)\|$ and the condition $0 < L < 1$ is equivalent to

$$\begin{array}{lcl} \left|\dfrac{\partial G_1}{\partial z_1}\right| + \left|\dfrac{\partial G_1}{\partial z_2}\right| + \ldots + \left|\dfrac{\partial G_1}{\partial z_n}\right| & < & 1 \\ \vdots & & \\ \left|\dfrac{\partial G_m}{\partial z_1}\right| + \left|\dfrac{\partial G_m}{\partial z_2}\right| + \ldots + \left|\dfrac{\partial G_m}{\partial z_n}\right| & < & 1 \end{array} \tag{B.1}$$

Now we apply these theorems to equations of type $\sin X = F(X), X \in \mathbb{T}^m, m = 1, 2, 3$. Note that we deal with equations of type $f(x) = \mathcal{O}(\eta)$ throughout Part II. In the one-dimensional case, one gets:

Lemma B.1. *If* $f : \mathbb{T} \to \mathbb{R}$ *is differentiable and satisfies*

$$(f')^2 + f^2 < 1, \tag{B.2}$$

then the equation

$$\sin x = f(x) \tag{B.3}$$

has exactly two solutions x^* *and* x^{**} *which are simple.*

If $f(x) = \mathcal{O}(\eta)$ *for any* $x \in \mathbb{T}$ *with* η *sufficiently small, then the solutions of (B.3) are* $x^* = \mathcal{O}(\eta)$ *and* $x^{**} = \pi + \mathcal{O}(\eta)$*.*

Proof. It is clear from (B.2) that $|f(x)| < 1$ for all $x \in \mathbb{T}$, then the equation (B.3) can be rewritten as

$$x = \arcsin f(x) = g(x), x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \quad \text{and} \quad x = \pi - g(x), x \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right].$$

In the first case, for $x \in [-\pi/2, \pi/2]$, the values of function $g(x)$ lie in $[-\pi/2, \pi/2]$ (the first condition of Theorem B.1 is fulfilled) and, moreover, by (B.2)

$$|g'(x)| = \left|\frac{f'}{\sqrt{1-f^2}}\right| < 1$$

is satisfied (the second condition of Theorem B.1 is fulfilled too), then by the Fixed Point Theorem B.1 there exists a unique solution x^* that is the limit of the sequence $x_n = g(x_{n-1}), n = 1, 2, \ldots$ and $|x_n - x^*| \leq \frac{L^n}{1-L}|x_1 - x_0|$ with

$$x_n = \arcsin(\arcsin(\ldots(\arcsin f(x_0)))).$$

In the second case, for $x \in [\pi/2, 3\pi/2]$, we shift $y = x - \pi$ and get the equation $y = \tilde{g}(y)$, where $y \in [-\pi/2, \pi/2]$, $\tilde{g}(y) = g(y + \pi)$ satisfies $\tilde{g}(y) \in [-\pi/2, \pi/2]$ and $|\tilde{g}'(y)| < 1$, then there exist a unique solution y^*, thus one gets $x^{**} = y^* + \pi$.

The solutions are simple, since if x^* (or x^{**}) is a degenerate zero of $F(x) = \sin x - f(x)$, i.e. it satisfies $F(x^*) = 0$ and $F'(x^*) = 0$, then the following is true

$$\sin x^* = f(x^*), \quad \cos x^* = f'(x^*)$$

and one gets

$$1 = \sin^2 x^* + \cos^2 x^* = (f(x^*))^2 + (f'(x^*))^2,$$

a contradiction to (B.2). □

In the two-dimensional case, manipulating with equations

$$\sin x = f(x, y), \quad \sin y = g(x, y) \tag{B.4}$$

we get 4 systems

$$\begin{array}{lllll}
(a) & x = \arcsin f(x,y), & y = \arcsin g(x,y), & x \in [-\frac{\pi}{2}, \frac{\pi}{2}], & y \in [-\frac{\pi}{2}, \frac{\pi}{2}]; \\
(b) & x = \pi - \arcsin f(x,y), & y = \arcsin g(x,y), & x \in [\frac{\pi}{2}, \frac{3\pi}{2}], & y \in [-\frac{\pi}{2}, \frac{\pi}{2}]; \\
(c) & x = \arcsin f(x,y), & y = \pi - \arcsin g(x,y), & x \in [-\frac{\pi}{2}, \frac{\pi}{2}], & y \in [\frac{\pi}{2}, \frac{3\pi}{2}]; \\
(d) & x = \pi - \arcsin f(x,y), & y = \pi - \arcsin g(x,y), & x \in [\frac{\pi}{2}, \frac{3\pi}{2}], & y \in [\frac{\pi}{2}, \frac{3\pi}{2}].
\end{array}$$

Consider, for example, the system (a). Here, in the notation of Theorem B.2, $I = [-\frac{\pi}{2}, \frac{\pi}{2}] \times [-\frac{\pi}{2}, \frac{\pi}{2}]$, $z = (x, y)$ and $G = (\arcsin f(x, y), \arcsin g(x, y))$. It is clear that $G(I) \subset I$. Also the conditions (B.1) become

$$\begin{aligned}
\left|\frac{\partial(\arcsin f)}{\partial x}\right| + \left|\frac{\partial(\arcsin f)}{\partial y}\right| < 1 \quad &\leftrightarrow \quad f^2 + (|f'_x| + |f'_y|)^2 < 1 \\
\left|\frac{\partial(\arcsin g)}{\partial x}\right| + \left|\frac{\partial(\arcsin g)}{\partial y}\right| < 1 \quad &\leftrightarrow \quad g^2 + (|g'_x| + |g'_y|)^2 < 1.
\end{aligned} \tag{B.5}$$

If these conditions are satisfied, then by the 2-dimensional Fixed Point Theorem B.2 (with $n = 2$), the system (a) has a unique solution $z^* = (x^*, y^*) \in I$ to which the sequence $z_n = G(z_{n-1})$ converges for any initial point $z_0 = (x_0, y_0) \in I$. The solution z^* is simple: if we consider the vector function

$$F = (\sin x - f(x,y), \sin y - g(x,y))^\top ,$$

the determinant of its Hessian matrix does not vanish at z^* under conditions (B.5).

In the same way, we can prove (shifting variables, if necessary) that each system $(b)-(d)$ has a unique simple solution, and, therefore, the equations (B.4) have 4 simple solutions. Thus, the following lemma is true.

Lemma B.2. *If $f, g : \mathbb{T}^2 \to \mathbb{R}$ are differentiable and satisfy*

$$f^2 + (|f'_x| + |f'_y|)^2 < 1, \quad g^2 + (|g'_x| + |g'_y|)^2 < 1$$

the equations (B.4) have 4 solutions which are simple.

Moreover, if $f(z) = \mathcal{O}(\eta), g(z) = \mathcal{O}(\eta)$ for any point $z = (x, y) \in \mathbb{T}^2$, then the solutions are small perturbations $z^{(j)} = z_*^{(j)} + \mathcal{O}(\eta)$, $j = 1, 2, 3, 4$ of

$$z_*^{(1)} = (0,0), \quad z_*^{(2)} = (\pi,0), \quad z_*^{(3)} = (0,\pi), \quad z_*^{(4)} = (\pi,\pi).$$

We can proceed analogously in the three-dimensional case and prove that

Lemma B.3. *If $f, g, h : \mathbb{T}^3 \to \mathbb{R}$ are differentiable and satisfy*

$$f^2+(|f'_x|+|f'_y|+|f'_z|)^2 < 1, \quad g^2+(|g'_x|+|g'_y|+|g'_z|)^2 < 1, \quad h^2+(|h'_x|+|h'_y|+|h'_z|)^2 < 1$$

the equations

$$\sin x = f(x,y,z), \quad \sin y = g(x,y,z), \quad \sin z = h(x,y,z) \tag{B.6}$$

have 8 simple solutions. Moreover, if $f(u) = \mathcal{O}(\eta), g(u) = \mathcal{O}(\eta), h(u) = \mathcal{O}(\eta)$ for any point $u = (x, y, z) \in \mathbb{T}^3$, the solutions are $u^{(j)} = u_*^{(j)} + \mathcal{O}(\eta)$, $(j = 1, \ldots, 8)$, where

$$\begin{aligned} &u_*^{(1)} = (0,0,0), \quad u_*^{(2)} = (\pi,0,0), \quad u_*^{(3)} = (0,\pi,0), \quad u_*^{(4)} = (\pi,\pi,0), \\ &u_*^{(5)} = (0,0,\pi), \quad u_*^{(6)} = (\pi,0,\pi), \quad u_*^{(7)} = (0,\pi,\pi), \quad u_*^{(8)} = (\pi,\pi,\pi). \end{aligned}$$

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