Forking in simple theories and CM-triviality

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FORKING IN SIMPLE THEORIES AND CM-TRIVIALITY

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FORKING IN SIMPLE THEORIES
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Als meus pares i al meu germà,
per haver estat sempre al meu costat

A la Mar,
per estar-hi des que ens vem trobar

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Introduction

The development of stability, initiated by Saharon Shelah at the end of the sixties, required two crucial abstract notions: *forking independence*, and the related notion of *canonical base*. Forking independence generalizes the linear independence in vector spaces and the algebraic independence in algebraically closed fields. On the other hand, canonical bases generalize the field of definition of an algebraic variety. These key tools allowed model-theorists to deal abstractly with a range of mathematical examples. In parallel, Boris Zilber during the seventies studied $\aleph_1$-categorical structures. He introduced a variety of methods and technics which gave rise to the so-called *geometric stability theory*. The methods consisted in the analysis of the underlying pregeometry arising on strongly minimal sets. This analysis pushed Zilber to state his trichotomy: the geometry of a strongly minimal sets is essentially a vector space over a division ring, it interprets a field, or either it is degenerated.

In the eighties Ehud Hrushovski came into the scene and with him the geometric stability theory achieved its maturity. He generalized partial results of Zilber obtaining group existence theorems in stability. In addition, the interest in the geometry of forking independence reached a special rang when Hrushovski constructed a new strongly minimal set refuting Zilber's trichotomy. In particular, this was the beginning of *CM-triviality*.

Even though geometric stability was well-understood in the eighties, outside the stable realm there still were mathematical examples with a rudimentary notion of independence. In 1991, in a manuscript around pseudo-finite fields, Hrushovski developed the first notions of what would be *geometric simplicity theory* in a finite rank setting free of technical detail. But it was not until 1997, when Byunghan Kim and Anand Pillay developed the general theory of simplicity and proved the Independence Theorem for Lascar strong types. In addition, Kim showed that simple theories are those where forking independence is symmetric, something that Shelah was not able to prove when he introduced simple theories in 1980. After these seminal papers of Kim and Pillay, the independence theory of forking went through stability to simplicity, and the study of simple theories became an active area in model theory around the end of the millennium.

This non exhaustive brief resume on the history of simple theories brings us to the actual scene. The class of simple theories includes all stable theories as well as other important mathematical examples such as the random graph, pseudo-finite fields, and algebraically closed fields with a generic automorphism. The generalizations from stability to simplicity usually required the development of new methods. Typical notions of stability such as orthogonality, regularity, internality, analysability, generic types of groups, and Hrushovski's amalgamation were translated to simplicity. However, some translations were very technical: Canonical bases. While in stable theories canonical bases are sequences of imaginary elements which are given locally via definability of types, in simple theories they are defined as a single *hyperimaginary*, that is, as an equivalence class of an $\emptyset$-type-definable equivalence relation. Type-definability thus turns out to be essential for the understanding of the general theory. Of course, this is close to the absence of a local theory of forking in simple theories. This fact is related with a central question in simple theories which will be approach in this dissertation under geometric assumptions on forking independence: *elimination of hyperimaginaries*.

Problems around elimination of hyperimaginaries have spanned a lot of work in the initial years of simplicity. It was known that stable theories eliminate hyperimaginaries [58]. One of the most important results on simplicity is the elimination of hyperimaginaries in supersimple theories [10]. Furthermore, in [7] and [62] it is proved that simple low theories eliminate bounded hyperimaginaries – hyperimaginaries with a bounded orbit. The class of simple low theories will be introduced in **chapter 5**. Another interesting class of first-order theories is the class of *small theories*, that is, the class of first-order theories such that for any natural number $n < \omega$, $|S_n(\emptyset)| \leq \omega$. All small theories eliminate finitary hyperimaginaries [35]. That is, those hyperimaginaries that are classes of finite tuples modulo $\emptyset$-type-definable equivalence relations. Nevertheless, every theory with the strict order property has a hyperimaginary which is not eliminable [2]. Therefore, there are examples of small theories that do not eliminate hyperimaginaries; but the question is still open for small simple theories. In addition, there is an example of a theory without the strict order property which does not eliminate hyperimaginaries [18]. It is worth remarkable a theorem due to Lascar and Pillay showing that every bounded hyperimaginary can be replaced in favor of finitary bounded hyperimaginaries [43]. These results are all what we know around hyperimaginaries; in **chapter 4** we will make our contribution.

In the present PhD thesis we deal with topics around the geometry of forking and its applications to solve fundamental problems in simplicity. We present three distinct topics on simplicity whose common denominator is forking independence. The contributions done by Hrushovski outside pure model theory pointed out the relevance of the geometry of forking. He solved the Mordell-Lang conjecture in all characteristics and gave a new proof of the Manin-Mumford conjecture. It was noticed by Pillay and Ziegler that structural properties of forking play an essential role in both cases, the *Canonical Base Property*. One goal of this dissertation is to study possible ample hierarchies which code the complexity of forking, and to relate the first level of such hierarchies with a weak version of the Canonical Base Property; this is the core of **chapter 2**.

Before going on, we should introduce the reader to the ample world. During the eightieth decade *one-based theories*, i.e., those theories where two sets are independent over the intersection of their imaginary algebraic closures, were of general interest. Hrushovski's construction yielded another kind of stable theories which include all one-based theories: the CM-trivial ones. CM-triviality can be understood as the preservation of independence under projections. In fact, Hrushovski claimed that CM-triviality forbids the existence of certain point-line-plane configuration; similar to the one-based case, where there are no type-definable pseudoplanes. Pillay, and later Evans, generalized such notions providing a hierarchy which codes the complexity of forking. This is the *non n-ample hierarchy*, where the first level corresponds to one-basedness and the second level to CM-triviality. Pillay showed that any simple theory interpreting a field is n-ample for all $n < \omega$, and Evans obtained a top-level theory which does not interpret an infinite group.

In **chapters 4** and **5** we work inside the second level of the non ample hierarchy, and in **chapter 2** we will present two generalizations of the ample hierarchy which are relative to a given $\emptyset$-invariant family of partial types. An application of these generalizations is exhibited later in **chapter 3**.

This thesis is organized as follows. In **chapter 1** we give a quick account of simple theories. We start in section 1.1 introducing the basics of simplicity: forking independence, Morley sequences, Lascar strong types, *et cetera*. After that, we introduce hyperimaginaries and we present the convenient closure operators in this setting. In addition, we define complete hyperimaginary types and the equality of Lascar strong types for hyperimaginaries. Hyperimaginary forking is introduced and its main properties are collected in Theorem 1.1.19, including the Independence Theorem for hyperimaginary Lascar strong types. Amalgamation bases and canonical bases are defined. To finish the section, we define eliminability of hyperimaginaries and we pose a vexing question: are hyperimaginaries eliminable in all simple theories?

In section 1.2, the class of supersimple theories is introduced, as well as the two fundamental ranks for such theories: The Lascar rank and the D-rank. We characterize supersimplicity in terms of such ranks. Of course, we also mention the breathtaking theorem of Buechler, Pillay, and Wagner: elimination of hyperimaginaries in supersimple theories, Theorem 1.2.6.

In section 1.3 we recall the notions of regular type, internality, analysability, and foreignness.

Finally, in section 1.4, the ample hierarchy is in the spotlight. We define the concept of n-ample for partial types, and we relate this hierarchy with one-basedness and CM-triviality. We finish the section with a theorem of Wagner stating that non n-ampleness is preserved under analysis.

Chapter 2 is devoted to investigate new possible ample hierarchies. We will consider an $\emptyset$-invariant family of partial types Σ and we will introduce two versions of ampleness relative to Σ: weak Σ-ample and Σ-ample. For convenience we recall the notion of Σ-closure in section 2.1, and we collect the main properties. In addition, we introduce an interesting operator for the first level of the analysis. In section 2.2 we study a special kind of types: flat and ultraflat types, and we present the Theorem of Levels due to Wagner, Theorem 2.2.7. In section 2.3 we define what does it mean to be (weak) n-Σ-ample for an $\emptyset$-invariant family of types, and we do one of the main contributions of this chapter: to show that non (weak) n-Σ-ample is preserved under analysis.

Theorem 2.3.19. *Let Ψ be an $\emptyset$-invariant family of types. If Ψ is Φ-analysable and Φ is not (weak) n-Σ-ample, neither is Ψ.*

It must be said that this work was initiated by Frank O. Wagner, obtaining the mentioned result for non n-Σ-ample. In parallel, I was trying to understand Chatzidakis' Theorem [19, Proposition 1.16] [56] and it turns out that Chatzidakis' Theorem corresponds to a strong version of non 1-Σ-ampleness; this fact gives rise to the aforementioned concept of non weak 1-Σ-ample and so to non weak n-Σ-ampleness. With this ideology we can obtain a weak version of Chatzidakis' Theorem for any simple theory with enough regular types.

Corollary 2.4.2. *Suppose every type in T is non-orthogonal to a regular type. Then $\mathrm{tp}(\mathrm{Cb}(a/b)/a)$ is analysable in the family of all non one-based regular types, for all a, b.*

This result is stated in section 2.4. In addition, a general version of Chatzidakis' Theorem is given for any supersimple theory working modulo ω^α, Theorem 2.4.5. Recently, Wagner has given a proof of such a result using ultraimaginaries [70]. After that, we define the *Canonical Base Property* in section 2.5. To finish the chapter we give an application to groups in section 2.6. We shall extend a result of Kowalski and Pillay [41] to arbitrary strong Σ-based simple theories, Proposition 2.6.2.

Chapter 3 is based on a result around the stability of forking in supersimple CM-trivial theories due to Wagner [69]. Namely, Wagner showed that the relation $R(x, yz)$ defined as $x \mathop{\smile\!\!\!\!\!\!|}_z y$ is stable in any supersimple CM-trivial theory. Keeping ideas from his proof and using a result obtained in the previous chapter, we shall proof the following for $\mathcal{P}$ being the family of non one-based types:

Theorem 3.1.2. *In a simple theory, the relation $R(x; yz)$ defined by $x \mathop{\smile\!\!\!\!\!\!|}_{\mathrm{cl}_{\mathcal{P}}(z)} y$ is stable.*

This result is also included in [50]; we hope it can help to elucidate the stability of forking. Of course, this problem is related with the aforementioned vexing question of elimination of hyperimaginaries which is treated in the next chapter.

Chapter 4 is devoted to study the elimination of hyperimaginaries in simple CM-trivial theories. All the results exposed in this chapter are collected in [50]. They are the result of a question I addressed to Wagner in April 2010. An early version of [50] was written in June 2010, and the final version in March 2012. Elimination of hyperimaginaries problems are one of the most important open problems in simplicity, together with the related question of the stability of forking. The relevance of hyperimaginaries in simplicity lies in the fact that canonical bases are hyperimaginaries. Even though its model theory is well understood [26], the general theory of simplicity become more global and obscure. Stable theories eliminate hyperimaginaries since canonical bases are obtained via definability of types. The lack of this local approach in simplicity, it might be an obstacle to understand canonical bases and/or to develop the general theory. For instance, in this general setting the appropriates closure operators are not finitary anymore.

As we have pointed out, our main results need the additional assumption of CM-triviality. Even though, it is not completely satisfactory, we hope these ideas could shed some light to solve the general question. The chapter is split in two section. In section 4.1 we recall basic definitions and facts around hyperimaginaries. Following [43], we define a hyperimaginary to be *quasi-finitary* if it is bounded over a finite tuple. The relation between finitary and quasi-finitary hyperimaginaries is exhibited in:

Proposition 4.1.9. *If T eliminates finitary hyperimaginaries, then T eliminates quasi-finitary hyperimaginaries.*

Moreover, we present some consequences of elimination of hyperimaginaries in G-compact theories. For instance, we present a new proof of a fact due to Casanovas [12, Proposition 18.27]:

Proposition 4.1.12. *Assume that the ambient theory is G-compact. Then, the theory eliminates all bounded hyperimaginaries if and only if $a \equiv^{\mathrm{Ls}} b \Leftrightarrow a \equiv^{\mathrm{s}} b$ for all sequences a, b.*

We also relate elimination of finitary hyperimaginaries with the equality between Lascar strong types and strong types over parameter sets.

In section 4.2 we present our main contributions to the understanding of hyperimaginaries on simple CM-trivial theories. A description of canonical bases in simple CM-trivial theories as sequences of finitary hyperimaginaries, Proposition 4.2.2, allows us to obtain the main result:

Theorem 4.2.4. *In a simple CM-trivial theory, every hyperimaginary is interbounded with a sequence of finitary hyperimaginaries.*

Despite the fact that we cannot replace interbounded by interdefinable, by previous lemmata exposed in section 4.1 we obtain the following corollary:

Corollary 4.2.5. *A simple CM-trivial theory eliminates hyperimaginaries whenever it eliminates finitary ones.*

Using the elimination of finitary hyperimaginaries in small theories due to Kim [35], it turns out the following result:

Corollary 4.2.6. *A small simple CM-trivial theory eliminates hyperimaginaries.*

The exposed results form the core of **chapter 4**.

In **chapter 5** we investigate the class of simple low theories. Roughly speaking, simple low theories are those simple theories where dividing for a formula is type-definable. Main examples of simple low theories are stable theories and supersimple theories of finite D-rank. Our main contribution is to show that countable ω-categorical simple CM-trivial theories are low. Again we work with the additional assumption of CM-triviality. At the time of writing, all currently known countable ω-categorical simple theories are CM-trivial. To start, we will briefly recall basic aspects and results of simple low theories in section 5.1. In section 5.2 we recall the notions of pre-weight and weight for complete types, and we introduce the notion of bounded finite weight. Namely, there is a uniform finite bound on the weight of each n-types. As a result we give a new example of simple low theories.

Proposition 5.2.5. *Every simple theory of bounded finite weight is low.*

In section 5.3 we investigate some lemmata around the bounded closure in countable ω-categorical theories. In addition, some corollaries are obtained relating simple theories with finite coding and supersimplicity, Corollary 5.3.6.

Section 5.4 is focused to answer a question of Casanovas and Wagner posed in [17]: Is every countable ω-categorical simple theory low? We answer the question affirmatively under the assumption of CM-triviality.

Theorem 5.5.2. *A countable ω-categorical simple CM-trivial theory is low.*

To conclude the chapter, in section 5.5 we relate lowness with strong stable forking; such a notion was introduced by Kim and Pillay in [40]. A theory is said to have strong stable forking if for every complete type which forks over some parameter set, forking is witnessed by a stable formula. Thus, the forking independence has a stable-like behavior. Our result is to prove that all countable ω-categorical simple theories with strong stable forking are low, Theorem 5.5.2.

Notation and Conventions

Our notation is standard and follows [12]. We will work in a complete first-order theory with infinite models whose monster model will be denoted by $\mathfrak{C}$, and its corresponding imaginary monster model by $\mathfrak{C}^{eq}$.

We think $\mathfrak{C}$ as a proper class, so a small subset of the monster model have a cardinal size. Parameter sets are small subsets of the monster model and they are denoted by A, B, C; however, when working with Σ-closures A, B, C might be proper subclasses of the monster model. Tuples of imaginary elements are denoted by a, b, c; the notation do not distinguish between finite and infinite (but small) tuples. As usual, the union of two sets A and B is denoted by AB, and the concatenation of two tuples a and b by ab. We write $a \in A$ to express that all elements in the tuple a belong to A.

The automorphism group of the monster model fixing pointwise a set A is denoted by $\mathrm{Aut}(\mathfrak{C}/A)$. As every automorphism of the monster model extends uniquely to an automorphism of the imaginary monster model, we shall identify $\mathrm{Aut}(\mathfrak{C})$ with $\mathrm{Aut}(\mathfrak{C}^{eq})$. Given two tuples a, b of the same length and a set of parameters A, we write $a \equiv_A b$ whenever a and b have the same type over A (i.e., there exists some $f \in \mathrm{Aut}(\mathfrak{C}/A)$ such that $f(a) = b$). We shall write $a \equiv^s_A b$ if in addition a and b lie in the same class modulo all A-definable finite equivalence relations; equivalently, if a and b have the same type over $\mathrm{acl}^{eq}(A)$. For arbitrary sets A, B, C we shall write $A \equiv_C B$ whenever for implicit enumerations a and b of A and B, respectively, we have $a \equiv_C b$.

These conventions will be followed through the text but some exceptions might occur. In a such case, we will explicitly say it. For instance, in **chapter 2** tuples and sets will be tuples and sets of hyperimaginaries, respectively.

Chapter 1

Preliminaries

This first chapter is devoted to present the necessary concepts and facts to understand the rest of the work. The reader should not expect anything original; so, readers familiarized with general simplicity theory might want to skip this part, except possibly sections 1.3 and 1.4 which are less well-known. Of course, we refer the reader to [39, 67, 12] for a detailed exposition of the general theory. The chapter is organized as follows: we start with section 1.1 giving a brief introduction to simple theories and recalling basic notions which play an essential role in the development of simplicity, e.g., forking independence, hyperimaginaries, and canonical bases. Some well-known examples of simple and stable theories are discussed. In section 1.2 we will be focused on supersimple theories and ranks in this context. We will put some emphasis on orthogonal aspects inside such theories. In section 1.3, we recall some basic properties around analysability, internality, and foreignness. In addition, we discuss results on analysability of types in theories with enough regular types. Finally, the ample hierarchy is presented in section 1.4 as well as some results on n-ample types.

1.1 Simplicity and hyperimaginaries

We shall start with Shelah's definition of the tree property

Definition 1.1.1. A formula $\varphi(x, y) \in L$ has the *tree property* if there is a natural number $k \geq 2$ and some tree $(a_\eta : \eta \in \omega^{<\omega})$ such that

1. for each $f \in \omega^\omega$, $\{\varphi(x, a_{f|n}) : n < \omega\}$ is consistent, and
2. for each $\eta \in \omega^{<\omega}$, $\{\varphi(x, a_{\eta^\frown n}) : n < \omega\}$ is k-inconsistent.

A theory is *simple* if no formula has the tree property.

We shall recall the core of simplicity: dividing and forking.

Definition 1.1.2.

- A formula $\varphi(x, a)$ *divides* over a set A (with respect to $k < \omega$) if there is an A-indiscernible sequence $(a_i : i < \omega)$ such that $\{\varphi(x, a_i) : i < \omega\}$ is inconsistent (k-inconsistent, respectively). A partial type $\pi(x)$ divides over A if there is some formula $\varphi(x, a)$ which divides over A and $\pi(x) \vdash \varphi(x, a)$.
- A formula $\varphi(x, a)$ *forks* over A if there are $\psi_i(x, b_i)$ for $i < n$ such that $\varphi(x, a) \vdash \bigvee_{i<n} \psi_i(x, b_i)$ and for each $i < n$ $\psi_i(x, b_i)$ divides over A. We say that a partial type $\pi(x)$ forks over A if there is some formula $\varphi(x, a)$ which forks over A and $\pi(x) \vdash \varphi(x, a)$.

In an arbitrary first-order theory forking and dividing might not coincide, even over the empty set.

Example. Let L consist of a ternary relation $R(x, y, z)$ and let M be the circle S^1, where $R(x, y, z)$ holds if and only if "y lies on the shorter arc between x and z including the end-points". Let a, b, c be three equidistant points on the circle; so,

$$M \models \forall x(R(a, x, b) \vee R(b, x, c) \vee R(c, x, a)).$$

It is routine to check that the formulas $R(a, x, b)$, $R(b, x, c)$ and $R(c, x, a)$ divides (with respect to 2) over $\emptyset$. Therefore, the formula $x = x$ forks over $\emptyset$ but it does not divide over $\emptyset$.

In [34], Kim showed that forking and dividing coincide in simple theories. Moreover, Chernikov and Kaplan have shown that forking and dividing coincide over models for a wider class of first-order theories: NTP_2 theories [20]. Therefore, simplicity cannot be determined by the equality between forking and dividing.

Using the combinatorial notion of forking, we define the *forking independence* $\mathop{\smile\!\!\!\!\!\!\;|}$ as a ternary relation among small sets of the monster model such that

$$A \mathop{\smile\!\!\!\!\!\!\;|}_{C} B \Leftrightarrow \text{ for any enumeration } a \text{ of } A, \operatorname{tp}(a/BC) \text{ does not fork over } C.$$

Kim and Pillay showed that $\mathop{\smile\!\!\!\!\!\!\;|}$ satisfies a list of axioms and start the study of abstract independence relations. Many model-theorist have contributed to the study of such abstract relations; the main references are [38, 1, 12]. For our purposes it is not necessary to discuss abstractly the notion of independence, it will be enough to collect all properties of the non-forking independence.

Theorem 1.1.3. *In a simple theory, the non-forking independence* $\mathop{\smile\!\!\!\!\!\!\;|}$ *satisfies:*

1. *Invariance under* $\operatorname{Aut}(\mathfrak{C})$*: if* $A \mathop{\smile\!\!\!\!\!\!\;|}_C B$*, then* $f(A) \mathop{\smile\!\!\!\!\!\!\;|}_{f(C)} f(B)$*.*
2. *Finite character:* $A_0 \mathop{\smile\!\!\!\!\!\!\;|}_C B_0$ *for all finite* $A_0 \subseteq A$ *and* $B_0 \subseteq B$ *iff* $A \mathop{\smile\!\!\!\!\!\!\;|}_C B$*.*
3. *Symmetry: if* $A \mathop{\smile\!\!\!\!\!\!\;|}_C B$*, then* $B \mathop{\smile\!\!\!\!\!\!\;|}_C A$*.*
4. *Transitivity:* $A \mathop{\smile\!\!\!\!\!\!\;|}_C BD$ *iff* $A \mathop{\smile\!\!\!\!\!\!\;|}_C B$ *and* $A \mathop{\smile\!\!\!\!\!\!\;|}_{CB} D$*.*
5. *Extension: if* $a \mathop{\smile\!\!\!\!\!\!\;|}_C B$ *and* $D \supseteq B$*, then there is some* $a' \equiv_{CB} a$ *such that* $a' \mathop{\smile\!\!\!\!\!\!\;|}_C D$*.*
6. *Local character: for any finite tuple* a *and for any set* B*, there is some* $C \subseteq B$ *with* $|C| \leq |T|$ *such that* $a \mathop{\smile\!\!\!\!\!\!\;|}_C B$*.*
7. *Strictness: if* $A \mathop{\smile\!\!\!\!\!\!\;|}_C A$*, then* $A \subseteq \operatorname{acl}(C)$*.*
8. *Independence Theorem over models: if* $a_1 \equiv_M a_2$*,* $a_i \mathop{\smile\!\!\!\!\!\!\;|}_M b_i$ *for* $i = 1, 2$ *and* $b_1 \mathop{\smile\!\!\!\!\!\!\;|}_M b_2$*, then there is some* $a \equiv_{Mb_i} a_i$ *for* $i = 1, 2$ *such that* $a \mathop{\smile\!\!\!\!\!\!\;|}_M b_1 b_2$*.*

Shelah proved that simplicity is characterized by the local character of forking independence [63], and Kim showed that the forking independence is symmetric in simple theories [35]. In fact, he even proved that simplicity is equivalent to the symmetry of forking independence [36]. Furthermore, simplicity can also be characterized by the existence of an independence relation [38]:

Theorem 1.1.4. *A theory is simple if and only if there is a ternary relation among small sets of the monster model satisfying all properties collected in Theorem 1.1.3. In addition, if such a relation exists then it is the forking independence.*

It turns out that simplicity seems to be the good framework to develop a theory for independence. Next we discuss some well-known examples of first-order theories.

Example. We offer some examples of simple theories:

1. The *random graph*. Let R be a binary irreflexive symmetric relation, and consider the first-order theory axiomatized by: for each $n < \omega$, for all distinct $x_1 \ldots, x_n, y_1, \ldots, y_n$ there is some z such that $R(z, x_i)$ and $\neg R(z, y_i)$ for all $i < n$. By a back-and-forth argument, this theory is complete, ω-categorical, and admits elimination of quantifiers. One can check that the relation $A \mathop{\smile\!\!\!\!\!\!\;|}_C B \Leftrightarrow A \cap B \subseteq C$ satisfies all properties collected in Theorem 1.1.3. Hence, this theory is simple and $\mathop{\smile\!\!\!\!\!\!\;|}$ is the forking independence by Theorem 1.1.4. In fact, the only formulas that divide (fork) are the algebraic ones.

2. Let V be an *infinite vector space over a finite field with a non degenerate antisymmetric bilinear form* $[\ ,\ ]$. Then $\mathrm{Th}(V, [\ ,\ ])$ is ω-categorical and admits quantifier elimination. This can be seen by a back-and-forth argument. Furthermore, this theory is simple since the only formulas which divide are the algebraic ones.

Example. The theory of *the generic triangle-free countable graph* is not simple as forking independence does not satisfy the Independence Theorem over models.

Definition 1.1.5. A sequence $(a_i : i < \kappa)$ is independent over A (or A-independent) if $a_i \mathop{\smile\!\!\!\!\!\!\;|}_A (a_j : j < i)$ for all $i < \kappa$. In addition, we say that a sequence is Morley in $\mathrm{tp}(a/A)$ if it is A-independent and A-indiscernible in $\mathrm{tp}(a/A)$.

Remark 1.1.6. A sequence $(a_i : i < \kappa)$ is A-independent if and only if $a_i \mathop{\smile\!\!\!\!\!\!\;|}_A (a_j : j \neq i)$ for all $i < \kappa$.

Using the extension property one can obtain arbitrarily large independent sequences over any small set of parameters. To obtain indiscernible sequences on a given type one can apply the Erdős-Rado Theorem as follows.

Lemma 1.1.7. *If $\kappa \geq |T|+|A|$, $\lambda = \beth_{(2^\kappa)^+}$, and $(a_i : i < \lambda)$ is a sequence of tuples a_i of the same length $\leq \kappa$, then there is an A-indiscernible sequence $(b_i : i < \omega)$ such that for each $n < \omega$ there are $i_0 < \dots i_n < \lambda$ such that $b_0 \dots b_n \equiv_A a_{i_0} \dots a_{i_n}$.*

Morley sequences are one of the main tools in simplicity; for instance, dividing (and so, forking) is determined by Morley sequences. This is exemplified in the following result of Kim.

Theorem 1.1.8. *In a simple theory, a partial type $\pi(x, a)$ divides over A iff for some (any) Morley sequence $(a_i : i < \omega)$ in $\mathrm{tp}(a/A)$, $\bigcup_{i<\omega} \pi(x, a_i)$ is inconsistent.*

Another notion which plays a crucial role in the development of simplicity is due to Lascar.

Definition 1.1.9. Let A be set. We say that two tuples a and b have the same *Lascar strong type* over A if a and b lie in the same equivalence class in the least A-invariant equivalence relation. We write $\mathrm{Lstp}(a/A) = \mathrm{Lstp}(b/A)$ or $a \equiv^{\mathrm{Ls}}_A b$ for this.

In fact, Lascar introduced the Lascar strong types $\mathrm{Lstp}(a/A)$ as the orbit of the group of strong automorphism $\mathrm{Autf}(\mathfrak{C}/A)$, which is generated by all groups $\mathrm{Aut}(\mathfrak{C}/M)$ with $M \supseteq A$. Another approach to the Lascar equivalence relation can be given in terms of *thick* formulas. A formula $\theta(x, y) \in L(A)$ is said to be *thick over A* if it is symmetric and there is no infinite sequence of $(a_i : i < \omega)$ such that $\models \neg\theta(a_i, a_j)$ for $i < j$. Then

$$a \equiv^{\mathrm{Ls}}_A b \Leftrightarrow \; ab \models \bigvee_{n<\omega} \mathrm{nc}^n_A(x, y),$$

where $\mathrm{nc}_A(x, y)$ is the set of all thick formulas over A, and $\mathrm{nc}^n_A(x, y)$ is the n-times composition of nc_A as taking $\mathrm{nc}^0_A(x, y)$ the equation $x = y$. This characterization of the Lascar equivalence relation is presented in [15]. The reader can find a more detailed exposition around thick formulas and the Lascar equivalence relation in [12, Chapter 9].

A natural question is: when is the binary relation $\equiv^{\mathrm{Ls}}_A$ an A-type-definable equivalence relation?

Definition 1.1.10. A theory is *G-compact* if for any set A, $\equiv^{\mathrm{Ls}}_A$ is an A-type-definable equivalence relation for every possible length of tuples.

Kim showed that simple theories are G-compact [35], and Ziegler found the first example of a non G-compact theory [15]. Usually, the least A-type-definable equivalence relation is called the Kim-Pillay equivalence relation over A, denoted by $\equiv^{\mathrm{KP}}_A$, see [15, 46]. Therefore, in a simple theory

$$\equiv^{\mathrm{Ls}}_A = \equiv^{\mathrm{KP}}_A .$$

Proposition 1.1.11. *Let A be a set.*

1. *In a G-compact theory, the equivalence relation $\equiv^{\mathrm{Ls}}_A$ is defined by $\mathrm{nc}^n_A(x,y)$ for some $n < \omega$.*
2. *In a simple theory, the equivalence relation $\equiv^{\mathrm{Ls}}_A$ is defined by $\mathrm{nc}^2_A(x,y)$.*

As a corollary we have the following result. Despite the fact that it seems to be folklore, we cannot find a reference in the literature (for G-compact theories); for convenience we give a proof.

Lemma 1.1.12. *Assume the ambient theory is G-compact. Then, for all sequences a, b and for any set A:*

$$a \equiv^{\mathrm{Ls}}_A b \Leftrightarrow a \equiv^{\mathrm{Ls}}_{A_0} b \textit{ for all finite } A_0 \subseteq A.$$

Proof. It is enough to prove right to left direction. Let n be a natural number given by Proposition 1.1.11 such that $\equiv^{\mathrm{Ls}}_A$ is type-defined by $\mathrm{nc}^n_A(x,y)$. By compactness, it is enough to prove that $\models \theta^n(a,b)$ holds for every $\theta(x,y) \in \mathrm{nc}_A(x,y)$, where θ^n is the n-times composition of θ. Assume $a \equiv^{\mathrm{Ls}}_{A_0} b$ for all finite subset $A_0 \subseteq A$ and let $\theta(x,y) \in L(A)$ be a thick formula. In fact, the formula $\theta(x,y)$ has parameters over some finite subset A_0 of A, whence $\theta(x,y) \in \mathrm{nc}_{A_0}(x,y)$. Again by Proposition 1.1.11, there is some n_0 such that the relation $\equiv^{\mathrm{Ls}}_{A_0}$ is type-defined by $\mathrm{nc}^{n_0}_{A_0}(x,y)$. As $\equiv^{\mathrm{Ls}}_A$ implies $\equiv^{\mathrm{Ls}}_{A_0}$, $n_0 \leq n$. As $\theta(x,y) \in \mathrm{nc}_{A_0}(x,y)$ and $\mathrm{nc}^{n_0}_{A_0}(a,b)$ holds, $\models \theta^{n_0}(a,b)$ and hence, $\models \theta^n(a,b)$ since $n_0 \leq n$. $\square$

In simple theories, Lascar strong types are the analog of strong types in stable theories – most examples of simple theories are stable. In fact, both coincide in stable theories. On the other hand, Lascar strong types are relevant in simplicity as we have the Independence Theorem for them.

Theorem 1.1.13. *Assume the ambient theory is simple. If* $a_1 \equiv^{\mathrm{Ls}}_A a_2$, $a_i \mathop{\smile\!\!\!\!\!\!|}_A b_i$ *for* $i = 1, 2$ *and* $b_1 \mathop{\smile\!\!\!\!\!\!|}_A b_2$*, then there is some* $a \equiv^{\mathrm{Ls}}_{Ab_i} a_i$ *for* $i = 1, 2$ *such that* $a \mathop{\smile\!\!\!\!\!\!|}_A b_1 b_2$*.*

Next we shall comment some aspects of stable theories.

Example. The theory of an infinite set without any structure, algebraically/differentially/separably closed fields of a given characteristic, the theory of a finite generated free group, and the theory of an infinite vector space over a division ring are all stable, whence simple.

We also describe a combinatorial example whose theory is stable.

Example. The *free pseudoplane*. The language consist of two unary predicates P and Q for "points" and "lines" as well as a binary relation I for "incidence". The axioms say:

1. Everything is a point or a line, and nothing is both.
2. I is a symmetric relation between lines and points.
3. For any x there are infinitely many y such that $I(x, y)$.
4. There are no loops: for $n \geq 2$ there is no sequence $(x_i : i \leq n)$ of distinct elements such that $I(x_i, x_{i+1})$ holds for $i < n$ and $I(x_n, x_0)$ also holds.

This theory is complete and it admits elimination of quantifiers after adding for each $n < \omega$ the relation $d_n(x, y)$ interpreted as: there is an I-path between x and y of length n. The forking independence can be characterized as follows: $A \mathop{\smile\!\!\!\!\!\!|}_C B$ if for all $a \in A$ and $b \in B$, if there is a path between a and b then it passes through $\mathrm{acl}(C)$. It is well-known that this theory is ω-stable of Morley rank ω.

In a stable theory every complete type over a model (even over an imaginary algebraically closed set) is *stationary*, i.e., it has a unique global non-forking extension. In fact, a simple theory is stable if every type over a model is stationary. In addition, stability is equivalent to the fact that every φ-type is definable. Therefore, given a stationary type $p(x) \in S(A)$ one define the *canonical base* of p, $\mathrm{Cb}(p)$, as the definable closure of the canonical parameters of the φ-definitions of the unique global non-forking extension $\mathfrak{p} \in S(\mathfrak{C})$ of p. That is,

$$\mathrm{Cb}(p) = \mathrm{dcl}(\mathrm{Cb}_\varphi(\mathfrak{p}) : \varphi \in L),$$

where $\mathrm{Cb}_\varphi(\mathfrak{p})$ is the canonical parameter of some definition of $\mathfrak{p} \upharpoonright \varphi$. It follows that canonical bases in stable theories are sequences of imaginaries, and that any two stationary types with the same global non-forking extension have the same canonical base. The main properties of a canonical base of a global type $\mathfrak{p}$ are: $\mathfrak{p} \upharpoonright \mathrm{Cb}(\mathfrak{p})$ is stationary, and $\mathfrak{p}$ does not fork over $\mathrm{Cb}(\mathfrak{p})$. Moreover, $\mathrm{Cb}(\mathfrak{p})$ is fixed under $\mathrm{Aut}(\mathfrak{C})$ if and only if $\mathfrak{p}$ is $\mathrm{Aut}(\mathfrak{C})$-invariant.

Example. In the free pseudoplane, any type over an algebraically closed set is stationary. As for the canonical base of a type of a finite tuple a over a model M, for each $a' \in a$ either there is no I-path from a' to M or there is some $b \in M$ such that the distance between a' and b is minimal. Then the canonical base $\mathrm{Cb}(a/M)$ is (interalgebraic with) the set of all such $b \in M$.

In simple theories the picture might be completely different. For instance, there is no reason for stationarity of types.

Example. In the theory of the random graph, any type over a model has an unbounded number of non-forking extensions. Namely, any non-algebraic extension is a non-forking extension, and there are $2^{|T|+|A|}$ complete non-algebraic types over any set of parameters A.

In order to adapt the nice stable setting to simple theories, hyperimaginaries must be taken into account. The model-theoretic treatment of hyperimaginaries was done by Hart, Kim, and Pillay [26]. A more detailed approach to hyperimaginaries and forking for hyperimaginaries can be found in [12].

Definition 1.1.14. A *(finitary) hyperimaginary* is the equivalence class of a (finite) tuple modulo an $\emptyset$-type-definable equivalence relation.

It is clear that imaginaries are hyperimaginaries, and that $\mathrm{Aut}(\mathfrak{C})$ acts over the class of all hyperimaginaries. In contrast with imaginaries, hyperimaginaries are treated as external elements of the structure. In the imaginary framework equality defines a clopen set, whereas equality between hyperimaginaries of the same sort is given by a closed set. In fact, if we add a sort $\mathfrak{C}_E$ and a definable map $\pi_E : x \mapsto x_E$ for an $\emptyset$-type-definable equivalence relation $E(x,y)$, then two possible notions of *equality* between hyperimaginaries a_E and b_E come up. Namely, the one given by "$E(a,b)$ holds" and another one given by "$\pi_E(a) = \pi_E(b)$". These two candidates of equality differ in any saturated elementary extension of the monster model. For instance, if $E(x,y)$ is a bounded $\emptyset$-type-definable equivalence relation, in any enough saturated model there will be a bounded number of E-classes but by compactness there will be just a finite number of distinct projections. Therefore, we do not execute the corresponding hyperimaginary construction, that is, for an $\emptyset$-type-definable equivalence relation $E(x,y)$ we do not add a new sort $\mathfrak{C}_E$ and a new function π_E mapping a tuple of the right length with its corresponding E-equivalence class.

Note that a hyperimaginary is the equivalence class of a tuple of arbitrary length. Nevertheless, a hyperimaginary is interdefinable with a sequence of countable hyperimaginaries, and every sequence of hyperimaginaries is interdefinable with a single hyperimaginary. A justification of this can be found in [12, Chapter 15].

Definition 1.1.15. Let A be a set, possibly of hyperimaginaries.

- The *definable closure* of A, $\mathrm{dcl}(A)$, is the class of all hyperimaginaries which are fixed under $\mathrm{Aut}(\mathfrak{C}/A)$.
- The *bounded closure* of A, $\mathrm{bdd}(A)$, is the class of all hyperimaginaries which have a bounded (i.e., small) orbit under $\mathrm{Aut}(\mathfrak{C}/A)$.

Both are closure operator; however, they are not finitaries. As every hyperimaginary can be understood in terms of countable hyperimaginaries, and there is just a bounded number of $\emptyset$-type-definable equivalence relations on countable tuples, $\mathrm{dcl}(A)$ and $\mathrm{bdd}(A)$ can be seen as small sets of hyperimaginaries. Hence, $\mathrm{dcl}(A)$ and $\mathrm{bdd}(A)$ are interdefinable with a single hyperimaginary by remarks above.

The bounded closure operator allows us to understand KP-types and so Lascar strong types in G-compact theories: for any set A,

$$\equiv_A^{\mathrm{KP}} = \equiv_{\mathrm{bdd}(A)} .$$

In order to define types for hyperimaginaries, for a formula $\varphi(x,y) \in L$ and for two $\emptyset$-type-definable equivalence relations E and F, let $\Phi_\varphi(x,y)$ be the partial type given by

$$\exists uv(E(u,x) \wedge F(v,y) \wedge \varphi(u,v)).$$

Then the *type of a hyperimaginary a_E over a hyperimaginary b_F* is defined as the partial type:

$$\mathrm{tp}(a_E/b_F) = \bigcup_{\varphi \in \mathfrak{F}} \Phi_\varphi(x,y),$$

where $\mathfrak{F}$ is the set of all formulas $\varphi(x,y) \in L$ such that $\models \varphi(a', b')$ for some $a'Ea$ and $b'Fb$. Despite the fact that it is a partial type, it is complete in the following sense: $\mathrm{tp}(a_E/b_F) = \mathrm{tp}(c_E/b_F)$ if and only if there is some $f \in \mathrm{Aut}(\mathfrak{C}/b_F)$ such that $f(a_E) = c_E$ (i.e., $E(f(a),c)$ holds). Therefore, it is convenient to write $a_E \equiv_{b_F} c_E$ whenever a_E and c_E have the same type over b_F.

In addition, one can extend the Lascar equivalence to hyperimaginaries using the group characterization. For a hyperimaginary h, the Lascar group $\mathrm{Autf}(\mathfrak{C}/h)$ is generated by all the subgroups $\mathrm{Aut}(\mathfrak{C}/M)$ such that $h \in \mathrm{dcl}(M)$.

Definition 1.1.16. Two hyperimaginaries a and b have the same *Lascar strong type over a hyperimaginary h* if a and b have the same orbit under $\mathrm{Autf}(\mathfrak{C}/h)$. We write $a \equiv^{\mathrm{Ls}}_h b$ or $\mathrm{Lstp}(a/h) = \mathrm{Lstp}(b/h)$ for this.

Note that $a \equiv^{\mathrm{Ls}}_h b$ if and only if there are hyperimaginaries $a_0, \dots, a_{n+1}$ and models $M_0, \dots, M_n$ such that $h \in \mathrm{dcl}(M_i)$ for all $i \leq n$ and

$$a = a_0 \equiv_{M_0} a_1 \equiv_{M_1} \dots \equiv_{M_n} a_{n+1} = b.$$

In particular, types over models are clearly Lascar strong types. In a simple theory the Lascar equivalence between hyperimaginaries is characterized in terms of the bounded closure as expected:

$$\equiv^{\mathrm{Ls}}_h = \equiv_{\mathrm{bdd}(h)}.$$

A justification of these facts around the Lascar group and hyperimaginaries can be found in [12, Chapter 16]. Now we shall adapt dividing and forking to hyperimaginaries.

Definition 1.1.17. Let h be a hyperimaginary.

- A formula $\varphi(x,a)$ *divides* over h (with respect to $k<\omega$) if there is an h-indiscernible sequence $(a_i : i<\omega)$ such that $\{\varphi(x,a_i) : i<\omega\}$ is inconsistent (k-inconsistent, respectively). A partial type $\pi(x)$ divides over h if there is some formula $\varphi(x,a)$ which divides over A and $\pi(x)\vdash \varphi(x,a)$.

- A formula $\varphi(x,a)$ *forks* over h if there are $\psi_i(x,b_i)$ for $i<n$ such that $\varphi(x,a)\vdash \bigvee_{i<n}\psi_i(x,b_i)$ and for each $i<n$ $\psi_i(x,b_i)$ divides over h. We say that a partial type $\pi(x)$ forks over h if there is some formula $\varphi(x,a)$ which forks over h and $\pi(x)\vdash\varphi(x,a)$.

A complete hyperimaginary type $\mathrm{tp}(a/bc)$ *divides/forks* over h if it does as a partial type.

Forking and dividing coincide in the context of simple theories [26], and as expected, we define for hyperimaginaries a,b,c the independence relation as follows: $a \mathop{\smile\!\!\!\!\!\mid}_c b$ if and only if $\mathrm{tp}(a/bc)$ does not fork over c. Obviously, it coincides with the imaginary forking independence for imaginaries a,b,c. Next, we exhibit some properties satisfied by forking independence in this hyper-framework.

Lemma 1.1.18. *Let T be a simple theory. For hyperimaginaries a,b,c the following are equivalent:*

1. $a \mathop{\smile\!\!\!\!\!\mid}_c b$.
2. *There are representatives $\hat{a}$ of a and $\hat{b}$ of b such that $\hat{a} \mathop{\smile\!\!\!\!\!\mid}_c \hat{b}$.*
3. *$a \mathop{\smile\!\!\!\!\!\mid}_{\hat{c}} b$ for some (any) representative $\hat{c}$ of c such that $\hat{c} \mathop{\smile\!\!\!\!\!\mid}_c ab$.*

Theorem 1.1.19. *Let T be a simple theory and let a,b,c,d be hyperimaginaries. Then the forking independence for hyperimaginaries satisfies the following properties:*

1. *Finite character: for any sequence of hyperimaginaries $(a_i : i\in I)$ we have, $(a_i : i\in I) \mathop{\smile\!\!\!\!\!\mid}_c b$ if and only if $(a_i : i\in I_0) \mathop{\smile\!\!\!\!\!\mid}_c b$ for all finite $I_0\subseteq I$.*
2. *Symmetry: if $a \mathop{\smile\!\!\!\!\!\mid}_c b$, then $b \mathop{\smile\!\!\!\!\!\mid}_c a$.*

3. *Transitivity:* $a \mathop{\smile\!\!\!\!\!|}_c b$ *and* $a \mathop{\smile\!\!\!\!\!|}_{cb} d$ *if and only if* $a \mathop{\smile\!\!\!\!\!|}_c bd$.
4. *Local character: for all* a *and for any sequence of hyperimaginaries* $(b_i : i \in I)$ *there is some* $J \subseteq I$ *with* $|J| \leq |T|$ *such that* $a \mathop{\smile\!\!\!\!\!|}_{(b_i:i\in J)} (b_i : i \in I)$.
5. *Extension: if* $a \mathop{\smile\!\!\!\!\!|}_c b$ *and* $b \in \mathrm{dcl}(d)$*, then there exists some* $a' \equiv_{cb} a$ *such that* $a' \mathop{\smile\!\!\!\!\!|}_c d$.
6. *Strictness: if* $a \mathop{\smile\!\!\!\!\!|}_c a$*, then* $a \in \mathrm{bdd}(c)$.
7. *The Independence Theorem for Lascar strong types: if* $a_1 \equiv^{\mathrm{Ls}}_c a_2$*,* $a_i \mathop{\smile\!\!\!\!\!|}_c b_i$ *for* $i = 1, 2$ *and* $b_1 \mathop{\smile\!\!\!\!\!|}_c b_2$*, then there is some* $a \equiv^{\mathrm{Ls}}_{cb_i} a_i$ *for* $i = 1, 2$ *such that* $a \mathop{\smile\!\!\!\!\!|}_c b_1 b_2$.

Definition 1.1.20. We say that a type $p(x, a)$ (with a a hyperimaginary) is an *amalgamation base* if the Independence Theorem for hyperimaginaries holds over p.

Lascar strong types are amalgamation bases. In fact, a type $\mathrm{tp}(a/A)$ is an amalgamation base if and only if $\mathrm{tp}(a/A) \vdash \mathrm{Lstp}(a/A)$. For instance, stationary types are amalgamation bases: if $a' \equiv_A a$ and $\mathrm{tp}(a/A)$ is stationary, then $a' \equiv^{\mathrm{Ls}}_A a$ since $a \mathop{\smile\!\!\!\!\!|}_A \mathrm{bdd}(A)$ and $a' \mathop{\smile\!\!\!\!\!|}_A \mathrm{bdd}(A)$.

Definition 1.1.21. For an amalgamation base $p(x)$ we define its *amalgamation class* $\mathcal{P}_p$ as the class of all global types $\mathfrak{p}(x)$ such that for some $n < \omega$ there are global types $(\mathfrak{p}_i(x) : i \leq n)$ such that $\mathfrak{p}_0(x)$ is a global non-forking extension of $p(x)$, $\mathfrak{p}(x) = \mathfrak{p}_n(x)$ and for all $i < n$ $\mathfrak{p}_i(x)$ and $\mathfrak{p}_{i+1}(x)$ are global non-forking extensions of a common amalgamation base.

For a stationary type $p(x)$, its amalgamation class has a unique element: its unique global non-forking extension.

Definition 1.1.22. Let a be a hyperimaginary and let $p(x) \in S(a)$ be an amalgamation base. The canonical base of $p(x)$, $\mathrm{Cb}(p)$, is the smallest hyperimaginary $e \in \mathrm{dcl}(a)$ such that $p \upharpoonright e$ is an amalgamation base and p does not fork over e. For a Lascar strong type $\mathrm{Lstp}(b/A)$ we shall write $\mathrm{Cb}(b/A)$ instead of $\mathrm{Cb}(\mathrm{Lstp}(b/A))$.

Theorem 1.1.23. *Let* T *be a simple theory and let* $p(x) \in S(a)$ *be an amalgamation base, where* a *is a hyperimaginary. Then the canonical base of* p *exists and for every* $f \in \mathrm{Aut}(\mathfrak{C})$*:*

$$f(\mathrm{Cb}(p)) = \mathrm{Cb}(p) \Leftrightarrow f(\mathcal{P}_p) = \mathcal{P}_p.$$

Lemma 1.1.24. *In a simple theory, if* $a = (a_i : i \in I)$*, then*

$$\mathrm{Cb}(a/A) = \mathrm{dcl}(\bigcup_{I_0 \subseteq I \text{ finite}} \mathrm{Cb}(a_i : i \in I_0/A)).$$

Again, Morley sequences have something to say for canonical bases.

Proposition 1.1.25. *Let* T *be a simple theory and let* $(a_i : i < \omega)$ *be a Morley sequence in* $\mathrm{Lstp}(a/A)$*. Then* $\mathrm{Cb}(a/A) \subseteq \mathrm{dcl}(a_i : i < \omega)$*.*

Remark 1.1.26. In fact, if $(a_i : i < \omega)$ is a Morley sequence in $\mathrm{Lstp}(a/A)$, then

$$\mathrm{bdd}(\mathrm{Cb}(a/A)) = \mathrm{bdd}(a_i : i < \omega) \cap \mathrm{bdd}(A).$$

Next lemma is useful in order to deal with canonical bases in forking calculus.

Lemma 1.1.27. *Let* T *be a simple theory. For hyperimaginaries* a, b, c *the following are equivalent:*

1. $a \mathop{\smile\!\!\!\!\!\!|}_c b$.
2. $\mathrm{Cb}(a/bc) = \mathrm{Cb}(a/c)$.
3. $\mathrm{Cb}(a/bc) \subseteq \mathrm{bdd}(c)$.

In a stable theory, canonical bases coming from definitions of φ-types and canonical bases defined as in Definition 1.1.22 coincide. In fact, one of the main enigmas in pure simplicity theory is the following question: Are canonical bases sequences of imaginaries? This brings us to a central topic on simplicity: elimination of hyperimaginaries.

Definition 1.1.28. A hyperimaginary is *eliminable* if it is interdefinable with a sequence of imaginaries. We say that a theory *eliminates hyperimaginaries* if all hyperimaginaries are eliminable.

In **chapter 4** we approach elimination problems assuming that non forking is well-behaved, that is, via geometric assumptions on forking independence.

1.2 Supersimple theories

Definition 1.2.1. A theory is *supersimple* if for every complete finitary imaginary type $\mathrm{tp}(a/A)$ there is some finite $B \subseteq A$ such that $\mathrm{tp}(a/A)$ does not fork over B.

Example. The theory of the random graph, the theory of an algebraically closed field with an automorphism, the theory of any model-completion of pseudo-finite fields, and the theory of an infinite vector space over a finite field with a non-degenerated bilinear form are supersimple and unstable. The free pseudoplane is stable and supersimple; in fact, it is ω-stable.

In a supersimple theory forking independence has an associated ordinal-valued rank: the Lascar rank. One can define the Lascar rank for hyperimaginary types as follows:

Definition 1.2.2. The Lascar rank SU is the least function from the collection of all complete hyperimaginaries types to the class of ordinals or ∞ such that for every ordinal α:

$$\mathrm{SU}(\mathrm{tp}(a/b)) \geq \alpha + 1 \Leftrightarrow \text{there is some } c \not\perp_b a \text{ such that } \mathrm{SU}(\mathrm{tp}(a/bc)) \geq \alpha.$$

We write $\mathrm{SU}(a/b)$ for the Lascar rank of $\mathrm{tp}(a/b)$.

It turns out that SU is a foundation rank of finitary complete types over hyperimaginaries with the relation of being a forking extension. The properties satisfied by the Lascar rank can be summarized in:

Remark 1.2.3. The SU-rank has the following list of properties:

1. For an imaginary tuple a and an imaginary set A, there is an ordinal α such that: if $\mathrm{SU}(a/A) \geq \alpha$, then $\mathrm{SU}(a/A) = \infty$.
2. $\mathrm{SU}(a/b) \geq \mathrm{SU}(a/bc)$.
3. $\mathrm{SU}(a/c) \leq \mathrm{SU}(ab/c)$.
4. If $a \in \mathrm{bdd}(bA)$, then $\mathrm{SU}(a/A) \leq \mathrm{SU}(b/A)$.
5. If $\mathrm{SU}(a/b) < \infty$, then: $a \perp_b c \Leftrightarrow \mathrm{SU}(a/b) = \mathrm{SU}(a/bc)$.

Example.

1. The theory of the random graph have SU-rank 1. That is, every imaginary 1-type has Lascar rank ≤ 1. The reason for this is that forking extensions are algebraic (i.e., bounded).

2. The theory of the free pseudoplane has Lascar rank ω. In this theory, a 1-type $\mathrm{tp}(a/A)$ has Lascar rank n if there exists a path of length n between a and some element of A. In addition, the types $\mathrm{tp}(a/A)$ of Lascar rank ω are those where a and A are not connected.

Proposition 1.2.4. *The following are equivalent:*

1. *The ambient theory is supersimple.*

2. *Every finitary imaginary type* $\mathrm{tp}(a/A)$ *has ordinal* SU*-rank.*

3. *If a is a finite tuple and A an imaginary set, then there is no infinite sequence of imaginary sets* $(A_i : i < \omega)$ *such that* $A_0 = A$ *and for all* $i < \omega$, $A_i \subseteq A_{i+1}$ *and* $a \not\perp_{A_i} A_{i+1}$.

Example. Consider a simple theory with an infinite sequence of $\emptyset$-definable equivalence relations $E_0(x, y) \vdash E_1(x, y) \vdash \ldots$ such that every E_i-class splits into infinitely many E_{i+1}-classes. It follows that $a \not\perp_{a_{E_i}} a_{E_{i+1}}$ for all $i < \omega$ and hence the theory is not supersimple by proposition above.

Recall that every ordinal α can be decomposed in its Cantor normal form $\sum_{i=1}^{l} \omega^{\alpha_i} \cdot n_i$, where $\alpha_i > \alpha_{i+1}$ for $1 \leq i < l$; it is unique if we require all the $n_i \neq 0$. If $\beta = \sum_{i=1}^{l} \omega^{\alpha_i} \cdot m_i$, then the commutative ordinal sum $\alpha \oplus \beta$ is defined as $\sum_{i=1}^{l} \omega^{\alpha_i} \cdot (n_i + m_i)$. Obviously, for finite ordinal α and β, $\alpha + \beta = \alpha \oplus \beta$.

Theorem 1.2.5 (Lascar Inequalities)**.** *Assume* $\mathrm{SU}(ab/A) < \infty$.

1. $\mathrm{SU}(a/Ab) + \mathrm{SU}(b/A) \leq \mathrm{SU}(ab/A) \leq \mathrm{SU}(a/Ab) \oplus \mathrm{SU}(b/A)$.

2. *If* $a \perp_A b$*, then* $\mathrm{SU}(ab/A) = \mathrm{SU}(a/A) \oplus \mathrm{SU}(b/A)$.

For types of finite Lascar rank we have a stronger version:

$$\mathrm{SU}(ab/A) = \mathrm{SU}(a/Ab) + \mathrm{SU}(b/A).$$

One of the astonishing results for supersimple theories is due to Buechler, Pillay, and Wagner [10].

Theorem 1.2.6. *Supersimple theories eliminate hyperimaginaries.*

This result is a variation of Buechler's proof for equality between Lascar strong types and strong types in simple low theories, see [7]. The authors solve the problem via an analysability argument using the Lascar rank and the D-rank.

Definition 1.2.7. The D-rank is the least function from the set of formulas to the class of ordinals or ∞ defined as follows:

$$\mathrm{D}(\varphi) \geq \alpha + 1 \Leftrightarrow \text{there is some formula } \psi \text{ dividing over the domain of } \varphi \text{ such that } \psi \vdash \varphi \text{ and } \mathrm{D}(\psi) \geq \alpha.$$

For complete types we define the D-rank as $\mathrm{D}(p) = \min\{\mathrm{D}(\varphi) : \varphi \in p\}$.

D-rank is suitable for partial types and SU-rank for complete types. Inductively, one can check that $\mathrm{SU}(p) \leq \mathrm{D}(p)$ for complete types.

Proposition 1.2.8. *A theory is supersimple if and only if* $\mathrm{D}(p) < \infty$ *for every complete finitary imaginary type* p.

To finish this section we introduce the concept of orthogonality.

Definition 1.2.9. Two types $p(x) \in S(A)$ and $q(y) \in S(B)$ are *orthogonal* if for every realization $a \models p, b \models q$ and for any $C \mathop{\smile\!\!\!\!\!\!|}_A a$ and $C \mathop{\smile\!\!\!\!\!\!|}_B b$ we have $a \mathop{\smile\!\!\!\!\!\!|}_{ABC} b$.

Proposition 1.2.10. *If* $\mathrm{SU}(a/A) = \beta + \omega^\alpha \cdot n < \infty$ *with* $\beta \geq \omega^{\alpha+1}$ *and* $n > 0$, *then* $\mathrm{tp}(a/A)$ *is non-orthogonal to some type of Lascar rank* ω^α.

1.3 Analysability

In this section we assume that the ambient theory is simple. Tuples and sets are tuples and sets of hyperimaginaries. By a partial type π over A we mean an imaginary partial type whose set of realizations is A-invariant. In addition, π will implicitly have certain sort E, i.e., if $a \models \pi$, then $b \models \pi$ for all bEa.

Let Σ be an $\emptyset$-invariant family of partial types. We will say that a tuple b realizes types in Σ based on a set B if b realizes types of Σ with parameters over B.

Recall first the definitions of internality, analysability and foreignness.

Definition 1.3.1. Let π be a partial type over A. Then π is

- (*almost*) Σ-*internal* if for every realization a of π there are $B \mathop{\smile\!\!\!\!\!\mid}_A a$ and a tuple $\bar{b}$ of realizations of types in Σ based on B, such that $a \in \mathrm{dcl}(B\bar{b})$ (or $a \in \mathrm{bdd}(B\bar{b})$, respectively).
- Σ-*analysable* if for any realization a of π there are $(a_i : i < \alpha) \in \mathrm{dcl}(Aa)$ such that $\mathrm{tp}(a_i/A, a_j : j < i)$ is Σ-internal for all $i < \alpha$, and $a \in \mathrm{bdd}(A, a_i : i < \alpha)$.

A type $\mathrm{tp}(a/A)$ is *foreign* to Σ if $a \mathop{\smile\!\!\!\!\!\mid}_{AB} \bar{b}$ for all $B \mathop{\smile\!\!\!\!\!\mid}_A a$ and $\bar{b}$ realizing types in Σ over B.

Foreignness is preserved under non-forking extensions and non-forking restrictions. Also, observe that foreignness is stronger than orthogonality as we allow any possible extension, even the forking ones.

Lemma 1.3.2. *The type* $\mathrm{tp}(a/A)$ *is foreign to* Σ *if and only if it is foreign to the family of all* Σ*-analysable types.*

Next lemma exhibits useful properties of internality and analysability. A justification to this can be found in [67, Section 3.4] and [12, Chapter 19].

Lemma 1.3.3. *Let* Σ' *be an* $\emptyset$*-invariant family of partial types.*

1. *If* $\mathrm{tp}(a/A)$ *is (almost)* Σ*-internal and* $A \subseteq B$*, so is* $\mathrm{tp}(b/B)$ *where* $b \in \mathrm{dcl}(aB)$ *(or* $a \in \mathrm{bdd}(B\bar{b})$*, respectively).*
2. *If* $\mathrm{tp}(a/AB)$ *is (almost)* Σ*-internal and* $a \mathop{\smile\!\!\!\!\!\mid}_A B$*, then so is* $\mathrm{tp}(a/A)$*.*
3. *If* $\mathrm{tp}(a_i/A)$ *is (almost)* Σ*-internal for all* $i \in I$*, so is* $\mathrm{tp}(a_i : i \in I/A)$*.*
4. *If* $\mathrm{tp}(a/A)$ *is (almost)* Σ*-internal and every type of* Σ *is (almost)* Σ'*-internal, then* $\mathrm{tp}(a/A)$ *is (almost)* Σ*-internal.*

5. *If* $\mathrm{tp}(a/A)$ *is (almost)* Σ*-internal, then so is* $\mathrm{tp}(\mathrm{Cb}(a/B)/A)$ *for any* $B \supseteq A$.

The same is true for analysability.

Lemma 1.3.4. *If* $\mathrm{tp}(c/b)$ *is (almost)* Σ*-internal and* $a \mathop{\smile\!\!\!\!\!\!\mid}\, b$*, then* $\mathrm{tp}(\mathrm{Cb}(bc/a))$ *is (almost)* Σ*-internal. The same is true if we write analysable instead of internal.*

Proof. For any Morley sequence $(b_i c_i : i < \omega)$ in $\mathrm{tp}(bc/a)$ observe that $a \mathop{\smile\!\!\!\!\!\!\mid}\, (b_i : i < \omega)$ and $\mathrm{Cb}(bc/a) \in \mathrm{dcl}(b_i c_i : i < \omega)$, whence $\mathrm{tp}(\mathrm{Cb}(bc/a))$ is internal in the family $\{\mathrm{tp}(c_i/b_i) : i < \omega\}$. As $\mathrm{tp}(c/b)$ is (almost) Σ-internal, so is each $\mathrm{tp}(c_i/b_i)$ for $i < \omega$ and hence, $\mathrm{tp}(\mathrm{Cb}(bc/a))$ is (almost) Σ-internal. □

To finish this collection of results on internality, analysability, and foreignness, we present how to obtain internal types.

Lemma 1.3.5. *If* $\mathrm{tp}(a/A)$ *is not foreign to* Σ*, then there is some* $a_0 \in \mathrm{dcl}(Aa) \setminus \mathrm{bdd}(A)$ *such that* $\mathrm{tp}(a_0/A)$ *is* Σ*-internal.*

Using this lemma one can characterize analysability as follows:

Theorem 1.3.6. *A type* $\mathrm{tp}(a/A)$ *is* Σ*-analysable if and only if for every* $B \supseteq A$ *we have* $a \in \mathrm{bdd}(B)$ *or* $\mathrm{tp}(a/B)$ *is not* Σ*-foreign.*

Definition 1.3.7. A type is *regular* if it is not bounded and it is foreign to all its forking extensions.

By Lascar equations it is easy to see that a type of Lascar rank ω^α for some ordinal α is regular. Hence, in a supersimple theory there are many regular types. However, not all regular types have a monomial Lascar rank.

Example. Consider the theory of an equivalence relation with an infinite number of equivalence classes all whose classes are infinite. If we consider the type of an element a which is not equivalent to any element of A, $\mathrm{tp}(a/A)$ has Lascar rank 2 and it is regular.

In theories with enough regular types one can show that any type is analysable in the family of regular types. This can be done applying Theorem 1.3.6.

Proposition 1.3.8. *Assume that every type is non-orthogonal to a regular type. Then every type is analysable by a family of regular types.*

In fact, for ordinal Lascar ranked types we have a better result combining Proposition 1.2.10 and Theorem 1.3.6; the finite length of the analysis follows from the fact that there is no infinite descending chain of ordinals.

Proposition 1.3.9. *If* $\mathrm{SU}(a/A) = \omega^{\alpha_1}n_1 + \cdots + \omega^{\alpha_l}n_l < \infty$ *with* $\alpha_i > \alpha_{i+1}$ *for* $1 \leq i < l$*, then* $\mathrm{tp}(a/A)$ *is analysable (in a finite number of steps) by the family of types of Lascar rank* ω^{α_i} *for* $1 \leq i \leq l$*.*

Therefore supersimple theories are determined (in some sense) by its regular types; for instance, see Corollary 1.4.8. To finish this section we present a well-known example where analysability and internality differ.

Example. Consider the theory of the free pseudoplane. Let $\mathrm{tp}(a/A)$ be a type of SU-rank n and let $(a_i : i \leq n)$ be a path between a and A with $a_n = a$ and $a_0 \in A$. Observe that $\mathrm{tp}(a_i/A, a_j : j < i)$ has SU-rank 1 and $(a_i : i \leq n) \subseteq \mathrm{dcl}(Aa)$, whence the sequence $(a_i : i \leq n)$ is an analysis of $\mathrm{tp}(a/A)$ in the family of Lascar rank 1 types. However, it is not internal in the family of Lascar rank one types. In fact, for an element a, $\mathrm{tp}(a/A)$ is internal to the family of Lascar rank 1 types if and only if $\mathrm{SU}(a/A) = 1$, whence, analysability and internality differ.

1.4 The ordinary ample hierarchy

Assume the theory is simple. As before, tuples and sets will be tuples and sets of hyperimaginaries. In order to finish this preliminary chapter we recall the definition of n-ample for an arbitrary natural number $n < \omega$. We briefly present basic definitions and its correspondence with one-basedness and CM-triviality. Moreover, we make emphasis in the local version of n-ample, i.e., the definition for a single partial type.

The ample hierarchy was introduced by Pillay [55] and redefined by Nübling and Evans [22].

Definition 1.4.1. A theory is n-ample for $n \geq 1$ if there are $a_0, \ldots, a_n$ such that over some parameters A we have

1. $a_n \not\perp_A a_0$;
2. $a_{i+1} \perp_{Aa_i} a_0 \ldots a_{i-1}$ for $1 \leq i < n$;

3. $\mathrm{bdd}(Aa_0 \ldots a_{i-1}a_i) \cap \mathrm{bdd}(Aa_0 \ldots a_{i-1}a_{i+1}) = \mathrm{bdd}(Aa_0 \ldots a_{i-1})$ for $0 \leq i < n$.

Remark 1.4.2. Pillay only requires $a_n \mathop{\smile\!\!\!\!\!\mid}_{Aa_i} a_0 \ldots a_{i-1}$ for $1 \leq i < n$ in item (2). We follow the variant proposed by Evans and Nübling [22, 45] which seems more natural and which implies

$$a_n \ldots a_{i+1} \mathop{\smile\!\!\!\!\!\mid}_{Aa_i} a_0 \ldots a_{i-1}.$$

In particular, in this version we always have $a_i \notin \mathrm{bdd}(Aa_j : j \neq i)$ for all $i \leq n$, whereas in Pillay's original definition this is not the case.

Pillay observed that $(n+1)$-ample implies n-ample; therefore, he introduced a hierarchy which codes the complexity of forking in simple theories: the *non ample hierarchy*. Moreover, Pillay showed that any simple theory which interprets a field must be n-ample for all n [55], and Evans proved the existence of a top-level theory which does not interpret an infinite group [22]. In addition, Pillay defined n-ampleness for a partial type.

Definition 1.4.3. We say that a partial type $\pi(x)$ over A is *n-ample* if there are $a_0, \ldots, a_n$ satisfying all conditions of Definition 2.3.1 over some set of parameters including A such that a_n is a tuple of realizations of $\pi(x)$.

It is immediate from the definition that one can define n-ampleness for an arbitrary family of partial types Φ, just recalling that a_n is a tuple of realizations of types of Φ.

For $n = 1$ and $n = 2$ there are alternative definitions of non-n-ampleness, which of course, were the motivation of the whole hierarchy:

Definition 1.4.4. Let $p(x) \in S(A)$ be a complete type.

1. It is *one-based* if $\mathrm{Cb}(a/B) \subseteq \mathrm{bdd}(aA)$ for any tuple a of realizations of p and any $B \supseteq A$.

2. It is *CM-trivial* if $\mathrm{Cb}(a/AB) \subseteq \mathrm{bdd}(A, \mathrm{Cb}(a/AC))$ for any tuple a of realizations of p and any $B \subseteq C$ satisfying $\mathrm{bdd}(ABa) \cap \mathrm{bdd}(AC) = \mathrm{bdd}(AB)$.

If the ambient theory eliminates hyperimaginaries one may consider the algebraic closure instead of the bounded closure.

Natural examples of one-based theories are the random graph and the theory of an infinite vector space over finite field. CM-triviality is due to Hrushovski [29], who obtained a strongly minimal set which does not interpret an infinite group and whose geometry is not locally modular neither trivial; refuting a conjecture of Zil'ber.

Example. The free pseudoplane is CM-trivial but it is not one-based. To prove that it is not one-based consider two elements a and b such that $\models I(a, b)$. Then $a \notin \operatorname{acl}(b)$ as $\operatorname{tp}(a/b)$ is axiomatized by the formula $I(x, b)$ plus "x is a line" or "x is a point". Moreover, $\operatorname{Cb}(b/a) = a$ and therefore, $\operatorname{Cb}(b/a) \not\subseteq \operatorname{acl}(b)$.

To see that CM-triviality holds, it is enough to consider models $M \subseteq N$ and a (real) tuple a such that $\operatorname{acl}(aM) \cap N = M$, see [52]. If $\operatorname{Cb}(a/M)$ is not contained in $\operatorname{acl}(\operatorname{Cb}(a/N))$, then there is some $d \in N \setminus M$ which belongs to a path between some element of a and some element of $\operatorname{Cb}(a/M)$. But this implies that $d \in \operatorname{dcl}(aM)$, which is impossible.

It is not hard to see that one-basedness and CM-triviality correspond to the first and second level, respectively, of the hierarchy.

Lemma 1.4.5. *Let $p(x) \in S(A)$, then*

1. *p is one-based if and only if p is not 1-ample.*
2. *p is CM-trivial if and only if p is not 2-ample.*

One-based types have good properties in terms of analysis.

Theorem 1.4.6. *A complete type analysable by a family of one-based types is one-based.*

A partial result of this was obtained by Hrushovski [30]; Chatzidakis proved it for supersimple theories [19] and finally, Wagner generalized the result for an arbitrary simple theory [68]. In an earlier version of [49], Wagner proved the same for non n-ample types:

Theorem 1.4.7. *A family of partial types analysable by a family of non-n-ample types is non-n-ample.*

In fact, Wagner worked in a more general version, n-Σ-ample, which will be treated in section 2.3. In that chapter, this last theorem will be proved in a more general version.

Combining this result with Proposition 1.3.8 one obtains the following:

Corollary 1.4.8. *Assume that every type is non-orthogonal to some regular type. Then the theory is non-n-ample if all its regular types are non-n-ample.*

Chapter 2

Ample hierarchies

The (ordinary) ample hierarchy codes the degree of complexity of forking independence in simple theories. Nevertheless, there are simple theories which fall outside this hierarchy [55, 22]. The main goal of this chapter is to introduce new kinds of ample hierarchies which code the complexity of forking independence, and to relate these possible hierarchies with other geometric properties such as the *Canonical Base Property*.

The Canonical Base Property states that the canonical base of a type of finite Lascar rank is internal to the family of non one-based types of Lascar rank 1. It was introduced by Pillay and Ziegler [59]. They showed that the Canonical Base Property holds in difference fields and differential fields both of characteristic 0. Chatzidakis extended their result to difference fields of an arbitrary characteristic [19]. In addition, she even obtained a weak version of the Canonical Base Property: The canonical base of a type of finite Lascar rank is analysable in the family of non one-based types of Lascar rank 1. This property is studied in [19, 56]. Moosa and Pillay proved that the Canonical Base Property also holds for compact complex spaces [44].

It is worth mentioning that the Canonical Base Property plays an essential role in the proof of Mordell-Lang for function fields [59]. Hence, this property is not just interesting in itself, but also for the applications. Apparently, there is a probable example due to Hrushovski of a stable theory of finite rank where the Canonical Base Property fails. Nevertheless, it seems feasible that other similar structural properties will hold for simple theories of finite rank. Our goal is to relate the Canonical Base Property and the weak version studied by Chatzidakis and Pillay with relative versions of ampleness.

Wagner proved that non n-ample is preserved under analysis, see Theorem 1.4.7. In fact, we have pointed out that he worked in a more general context, non n-Σ-ample, which will be introduced in section 2.3. For $n = 1$, this corresponds to Σ-basedness [68]. Wagner worked with the analysable closure operator with respect to an $\emptyset$-invariant family of partial types: The Σ-closure cl_Σ. These considerations due to Wagner yielded the existence of the Σ-ample hierarchy; however, this generalization implies to work with cl_Σ-closed sets, which is quite restrictive for the applications. Even though, a priori it seems that one has to resign himself and to deal with types over cl_Σ-closed sets, the weak version of the Canonical Base Property studied by Chatzidakis and Pillay sheds some light to the discussion. This allows one to relate the Σ-ample hierarchy with more geometric properties and the weak Σ-ample hierarchy comes up.

Fixed an $\emptyset$-invariant family Σ of partial types we introduce two possible hierarchies relative to Σ: the Σ-ample hierarchy and the weak Σ-ample hierarchy. The ordinary ample hierarchy corresponds to the family Σ of bounded types. This consideration allows us to understand simple theories in terms of specific types. For instance, Corollary 2.4.2 tells us that supersimple theories belong to the first level of the weak ample hierarchy relative to the family of non one-based regular types. In fact, for Σ being the family of non one-based regular types, the first level of the weak Σ-ample hierarchy corresponds to the weak version of the Canonical Base Property studied by Chatzidakis and Pillay, i.e., to the Σ-analysability of $\mathrm{tp}(\mathrm{Cb}(a/b)/a)$. Replacing analysability by internality one obtains the Canonical Base Property.

To begin with, we shall recall the notions of Σ-closure and introduce an operator for the first level of the analysis. Mainly, all results are known and can be found in [67, Chapter 3.4] or [68], expect possibly Lemma 2.1.9 and the results concerned with the first level operator. Even though the reader might see these operators as unnecessary and artificial, this language permits to deal easily with highly nontrivial concepts such as analysability, internality, and foreignness in the forking calculus.

The theory of levels was introduced by Buechler in his study of Vaught's conjecture [8], and was developed by Prerna Juhlin [32] in her PhD dissertation. Namely, both authors decomposed analysable types level-by-level; Juhlin used this approach to study the Canonical Base Property. In section 2.2 we shall study the first level of the analysis, and those types which are internal whenever they are analysable: flat types. In addition, we study the stronger related notion of ultraflat type. We offer some known examples of ultraflat types: Lascar rank one types, generic types of fields, and generic types of simple definable groups. In addition, we prove that a finite Lascar rank type internal to a family of Lascar rank one types is flat; this is Corollary 2.2.14. To finish the section we discuss a general version of a theorem of Buechler on domination-equivalence between levels, Theorem 2.2.7. Mainly, this result is due to Wagner, and I contribute with the relative domination-equivalence version.

As we have pointed out, in section 2.3 we shall introduce two ample hierarchies relative to an $\emptyset$-invariant family Σ of partial types: the (weak) n-Σ-ample hierarchy. The main result in this section is Theorem 2.4.1, where we prove that non (weak) n-Σ-ampleness is preserved under analysis. The non n-Σ-ample is due to Wagner. The main idea of Wagner is to use an appropriate theory of levels (i.e., Theorem 2.2.7) to reduce the analysable case to the internal case. In fact, this idea comes from [55], where Pillay showed that a supersimple theory of finite Lascar rank is CM-trivial if all its regular types are. The weak case follows as an adaptation of Wagner's ideas modulo some previous extra-work. As all proofs in both cases are similar, we treat the two cases in parallel. In fact, we conjectured that the first and the second level of both hierarchies coincide; however, we cannot prove it.

It is worth remarkable that connections with other geometric properties come from the non weak Σ-ample hierarchy, and in particular its first level. In section 2.4 we study the analysability of canonical bases. As we have pointed out, simple theories with enough regular types are not weak 1-Σ^{nob}-ample (or equivalently, strongly Σ^{nob}-based), where Σ^{nob} is the family of non-one-based regular types. That is, for all tuples a and b we have, $\mathrm{tp}(\mathrm{Cb}(a/b)/a)$ is Σ^{nob}-analysable. This corresponds to Corollary 2.4.2 and it generalizes the weak version of the Canonical Base Property studied by Chatzidakis [19] and Pillay [56]. Nevertheless, Chatzidakis obtained a better result for the finite Lascar rank case; namely, for all tuples a and b with

$\mathrm{SU}(b) < \omega$, $\mathrm{tp}(\mathrm{Cb}(a/b)/\mathrm{bdd}(a) \cap \mathrm{bdd}(b))$ is analysable in the family of non-one-based types of Lascar rank one [19, Theorem 1.18]. Here we present this result as a corollary of Theorem 2.4.5 which indeed is a reinterpretation of Chatzidakis' theorem for ordinal-valued Lascar rank types. Recently, Wagner has improved this theorem. He obtains a general version of Chatzidakis' theorem for simple theories working with quasi-finitary ultraimaginaries [70, Corollary 5.7].

Finally, in section 2.5 the desired (but not reached) property is exposed: the Canonical Base Property. This property corresponds to Chatzidakis' theorem but asking for almost internality instead of analysability. There, we discuss such a property and we expose some (failed) approximation similar to the weak ample hierarchy but working with the cl^1_Σ instead of cl_Σ. In addition, some applications to groups due to Pillay and Kowalski are described in Section 2.6. Finally, a result on groups due to Wagner is stated.

This chapter was done while I was in the Institut Camille Jordan-CNRS during the spring of 2011. It is part of [49]: *Ample thought*, preprint 2011 (submitted).

2.1 Σ-closure

The bounded closure operator might not be appropriate in a proper simple theory. One might want to work with a more general notion of closure operator with respect to a given $\emptyset$-invariant family of partial types, say Σ. Next, we shall introduce a more general closure operator, which in fact generalizes the p-closure operator introduced by Hrushovski [27].

We will assume the ambient theory is simple. Tuples and sets are tuples and sets of hyperimaginaries, respectively. We allow parameter sets to be proper classes, i.e., we deal with $\mathrm{tp}(a/A)$ where A might be a proper subclass of the monster model. Nevertheless, by local character of non-forking and point (2) of Lemma 1.3.3, notions such as internality, analysability, and foreigness apply to this case. Namely, if A has the size of the monster model, $\mathrm{tp}(a/A)$ is Σ-analysable if and only if there is some small subset A_0 of A such that $a \mathop{\smile\!\!\!\!\!\!\mid}_{A_0} A$ and $\mathrm{tp}(a/A_0)$ is Σ-analysable. The same happens if we write internal or foreign instead of analysable.

Definition 2.1.1. The Σ-closure $\mathrm{cl}_\Sigma(A)$ of a set A is the collection of all hyperimaginaries a such that $\mathrm{tp}(a/A)$ is Σ-analysable.

Even though Σ might be a proper class, we think of Σ as small: When a type is Σ-analysable it is analysable in a subfamily of Σ which has a small size. Observe that we just need a small set of types of the family Σ to show that an arbitrary type is internal to Σ.

Only when Σ is the family of all bounded types, the Σ-closure coincides with the bounded closure. In any other case, the Σ-closure of a set has the size of the monster model.

Lemma 2.1.2. $\mathrm{cl}_\Sigma(\cdot)$ *is a closure operator. That is,*

1. $A \subseteq \mathrm{cl}_\Sigma(A)$.
2. *If* $A \subseteq B$, *then* $\mathrm{cl}_\Sigma(A) \subseteq \mathrm{cl}_\Sigma(B)$.
3. $\mathrm{cl}_\Sigma(\mathrm{cl}_\Sigma(A)) \subseteq \mathrm{cl}_\Sigma(A)$.

In addition, we always have $\mathrm{bdd}(A) \subseteq \mathrm{cl}_\Sigma(A)$.

Proof. As $\mathrm{tp}(\mathrm{bdd}(A)/A)$ is bounded, it is Σ-analysable, whence we obtain the "in addition" clause and (1). (2) follows from the fact: if $\mathrm{tp}(a/A)$ is Σ-analysable, then so is $\mathrm{tp}(a/B)$. To prove (3), assume $B \subseteq \mathrm{cl}_\Sigma(\mathrm{cl}_\Sigma(A))$; so, $\mathrm{tp}(B/\mathrm{cl}_\Sigma(A))$ is Σ-analysable. By the local character of forking, there is some small set $A_0 \subseteq \mathrm{cl}_\Sigma(A)$ such that $B \perp_{A_0} \mathrm{cl}_\Sigma(A)$. Thus, $\mathrm{tp}(A_0/A)$ and $\mathrm{tp}(B/AA_0)$ are Σ-analysable, and so is $\mathrm{tp}(B/A)$. That is, $B \subseteq \mathrm{cl}_\Sigma(A)$. □

Useful examples for Σ are: the family of all types of SU-rank $< \omega^\alpha$ for some ordinal α, the family of all types of valued SU-rank in a proper simple theory, or the family of p-simple types of p-weight 0 for some regular type p. The latter gives rise to the p-closure operator introduced by Hrushovski.

Next lemmata are proved in [67, Section 3.5] and [68]; we give proofs for the sake of completeness.

Lemma 2.1.3. *The following are equivalent:*

1. $\mathrm{tp}(a/A)$ *is foreign to* Σ.
2. $a \perp_A \mathrm{cl}_\Sigma(A)$.

3. $a \mathop{\smile\!\!\!\!\!\!\,|}_A \mathrm{dcl}(aA) \cap \mathrm{cl}_\Sigma(A)$.

4. $\mathrm{dcl}(aA) \cap \mathrm{cl}_\Sigma(A) \subseteq \mathrm{bdd}(A)$.

Proof. $(2) \Rightarrow (3) \Rightarrow (4)$ are obvious. To prove that $(4) \Rightarrow (1)$ assume $\mathrm{tp}(a/A)$ is not Σ-foreign. Then by Lemma 1.3.5 we obtain some $b \in \mathrm{dcl}(aA) \setminus \mathrm{bdd}(A)$ such that $\mathrm{tp}(b/A)$ is Σ-internal; in particular, $b \in \mathrm{cl}_\Sigma(A)$. But this implies that (4) does not hold; hence, (4) implies (1). It remains to check that (1) implies (2). If $\mathrm{tp}(a/A)$ is Σ-foreign, then it is foreign to the family of all Σ-analysable types by Lemma 1.3.2. Now, let B an arbitrary subset of $\mathrm{cl}_\Sigma(A)$; thus, $\mathrm{tp}(B/A)$ is Σ-analysable and hence, $a \mathop{\smile\!\!\!\!\!\!\,|}_A B$ by foreignness. As B was arbitrary, we obtain the result. □

Lemma 2.1.4. *Suppose $A \mathop{\smile\!\!\!\!\!\!\,|}_B C$. Then $\mathrm{cl}_\Sigma(A) \mathop{\smile\!\!\!\!\!\!\,|}_{\mathrm{cl}_\Sigma(B)} \mathrm{cl}_\Sigma(C)$. More precisely, for any $A_0 \subseteq \mathrm{cl}_\Sigma(A)$ we have $A_0 \mathop{\smile\!\!\!\!\!\!\,|}_{B_0} \mathrm{cl}_\Sigma(C)$, where $B_0 = \mathrm{dcl}(A_0B) \cap \mathrm{cl}_\Sigma(B)$. In particular, $\mathrm{cl}_\Sigma(AB) \cap \mathrm{cl}_\Sigma(BC) = \mathrm{cl}_\Sigma(B)$.*

Proof. Assume $A \mathop{\smile\!\!\!\!\!\!\,|}_B C$ and let $B' = \mathrm{dcl}(BC) \cap \mathrm{cl}_\Sigma(B)$. Then $A \mathop{\smile\!\!\!\!\!\!\,|}_B C$ implies that $A \mathop{\smile\!\!\!\!\!\!\,|}_{B'} C$ as $B' \subseteq \mathrm{dcl}(BC)$. Moreover, by Lemma 2.1.3 $\mathrm{tp}(C/B')$ is Σ-foreign, and so is $\mathrm{tp}(C/B'A)$. Thus, $C \mathop{\smile\!\!\!\!\!\!\,|}_{B'A} \mathrm{cl}_\Sigma(AB')$ again by Lemma 2.1.3 and so, $C \mathop{\smile\!\!\!\!\!\!\,|}_{B'} \mathrm{cl}_\Sigma(AB')$ by transitivity.

Now let $A_0 \subseteq \mathrm{cl}_\Sigma(A)$; so, $A_0 \mathop{\smile\!\!\!\!\!\!\,|}_{B'} C$. Put $B_0 = \mathrm{dcl}(A_0B) \cap \mathrm{cl}_\Sigma(B)$, then $A_0 \mathop{\smile\!\!\!\!\!\!\,|}_{B_0} \mathrm{cl}_\Sigma(B_0)$ again by Lemma 2.1.3. Note that $B' \subseteq \mathrm{cl}_\Sigma(B') = \mathrm{cl}_\Sigma(B) = \mathrm{cl}_\Sigma(B_0)$ and so, $A_0 \mathop{\smile\!\!\!\!\!\!\,|}_{B_0} B'$. On the other hand, as $C \mathop{\smile\!\!\!\!\!\!\,|}_{B'} \mathrm{cl}_\Sigma(AB')$ we have, $C \mathop{\smile\!\!\!\!\!\!\,|}_{B'} B_0$ and $C \mathop{\smile\!\!\!\!\!\!\,|}_{B'B_0} A_0$. Then by transitivity $A_0 \mathop{\smile\!\!\!\!\!\!\,|}_{B_0} C$ and hence, $\mathrm{tp}(A_0/B_0C)$ is foreign to Σ. But this means by Lemma 2.1.3 that $A_0 \mathop{\smile\!\!\!\!\!\!\,|}_{B_0C} \mathrm{cl}_\Sigma(B_0C)$ and so, $A_0 \mathop{\smile\!\!\!\!\!\!\,|}_{B_0} \mathrm{cl}_\Sigma(C)$ by transitivity and monotonicity. □

The following lemma tells us that the intersection $\mathrm{cl}_\Sigma(A) \cap \mathrm{cl}_\Sigma(B)$ is generated by a small set.

Lemma 2.1.5. *If $C = \mathrm{Cb}(AB/\mathrm{cl}_\Sigma(A) \cap \mathrm{cl}_\Sigma(B))$, then $\mathrm{cl}_\Sigma(A) \cap \mathrm{cl}_\Sigma(B) = \mathrm{cl}_\Sigma(C)$.*

Proof. We have $AB \mathop{\smile\!\!\!\!\!\!\,|}_C \mathrm{cl}_\Sigma(A) \cap \mathrm{cl}_\Sigma(B)$ and hence, $\mathrm{cl}_\Sigma(A) \cap \mathrm{cl}_\Sigma(B) = \mathrm{cl}_\Sigma(C)$ by Lemma 2.1.4. □

Lemma 2.1.6. $\mathrm{Cb}(a/\mathrm{cl}_\Sigma(A)) \subseteq \mathrm{bdd}(a\mathrm{Cb}(a/A)) \cap \mathrm{cl}_\Sigma(\mathrm{Cb}(a/A))$.

Proof. By definition of the canonical base, $a \mathop{\smile\!\!\!\!\!\!|}_{\mathrm{Cb}(a/A)} A$. Now applying Lemma 2.1.4 we obtain

$$a \mathop{\smile\!\!\!\!\!\!|}_{\mathrm{dcl}(a\mathrm{Cb}(a/A))\cap \mathrm{cl}_\Sigma(\mathrm{Cb}(a/A))} \mathrm{cl}_\Sigma(A).$$

As $\mathrm{dcl}(a\mathrm{Cb}(a/A)) \cap \mathrm{cl}_\Sigma(\mathrm{Cb}(a/A))$ is clearly contained in $\mathrm{cl}_\Sigma(A)$, we obtain the result. □

Question 2.1.7. Is $\mathrm{Cb}(a/A) \subseteq \mathrm{cl}_\Sigma(\mathrm{Cb}(a/\mathrm{cl}_\Sigma(A))$?

Lemma 2.1.8. *If* $b \in \mathrm{cl}_\Sigma(a)$ *and* $a \mathop{\smile\!\!\!\!\!\!|}_b A$, *then*

$$\mathrm{cl}_\Sigma(\mathrm{Cb}(a/\mathrm{cl}_\Sigma(A))) = \mathrm{cl}_\Sigma(\mathrm{Cb}(b/\mathrm{cl}_\Sigma(A))).$$

Proof. Let $X = \mathrm{Cb}(b/\mathrm{cl}_\Sigma(A))$ and $Y = \mathrm{Cb}(a/\mathrm{cl}_\Sigma(A))$. As $b \mathop{\smile\!\!\!\!\!\!|}_X \mathrm{cl}_\Sigma(A)$, $\mathrm{cl}_\Sigma(b) \mathop{\smile\!\!\!\!\!\!|}_{\mathrm{cl}_\Sigma(X)} \mathrm{cl}_\Sigma(A)$ by Lemma 2.1.4; again by Lemma 2.1.4 we also have that $a \mathop{\smile\!\!\!\!\!\!|}_{\mathrm{cl}_\Sigma(b)} \mathrm{cl}_\Sigma(A)$, whence $a \mathop{\smile\!\!\!\!\!\!|}_{\mathrm{cl}_\Sigma(X)} \mathrm{cl}_\Sigma(A)$ by transitivity and hence, $Y \subseteq \mathrm{cl}_\Sigma(X)$. To prove the other inclusion, note that by definition of canonical base and Lemma 2.1.4 we obtain

$$\mathrm{cl}_\Sigma(a) \mathop{\smile\!\!\!\!\!\!|}_{\mathrm{cl}_\Sigma(Y)} \mathrm{cl}_\Sigma(A),$$

whence $b \mathop{\smile\!\!\!\!\!\!|}_{\mathrm{cl}_\Sigma(Y)} \mathrm{cl}_\Sigma(A)$ and so, $X \subseteq \mathrm{cl}_\Sigma(Y)$. □

Point (2) of next result is new.

Lemma 2.1.9. *Suppose* $C \subseteq A \cap B \cap D$ *and* $AB \mathop{\smile\!\!\!\!\!\!|}_C D$.

1. *If* $\mathrm{cl}_\Sigma(A) \cap \mathrm{cl}_\Sigma(B) = \mathrm{cl}_\Sigma(C)$, *then* $\mathrm{cl}_\Sigma(AD) \cap \mathrm{cl}_\Sigma(BD) = \mathrm{cl}_\Sigma(D)$.
2. *If* $\mathrm{bdd}(A) \cap \mathrm{cl}_\Sigma(B) = \mathrm{bdd}(C)$, *then* $\mathrm{bdd}(AD) \cap \mathrm{cl}_\Sigma(BD) = \mathrm{bdd}(D)$.

Proof. (1) is [67, Lemma 3.5.6], which in turn adapts [52, Fact 2.4]. In fact, (1) is similar to (2).

To prove (2), assume $D \mathop{\smile\!\!\!\!\!\!|}_C AB$ and $\mathrm{bdd}(A) \cap \mathrm{cl}_\Sigma(B) = \mathrm{bdd}(C)$. Then observe that $AD \mathop{\smile\!\!\!\!\!\!|}_A AB$ and $BD \mathop{\smile\!\!\!\!\!\!|}_B AB$. The latter implies by Lemma 2.1.4 that

$$\mathrm{cl}_\Sigma(BD) \mathop{\smile\!\!\!\!\!\!|}_{\mathrm{cl}_\Sigma(B)\cap \mathrm{dcl}(AB)} AB.$$

Put $X = \mathrm{bdd}(AD) \cap \mathrm{cl}_\Sigma(BD)$; then

$$X \mathop{\smile\!\!\!\!\!\!|}_{A} AB \quad \text{and} \quad X \mathop{\smile\!\!\!\!\!\!|}_{\mathrm{cl}_\Sigma(B)\cap \mathrm{dcl}(AB)} AB$$

and so,

$$\mathrm{Cb}(X/AB) \subseteq \mathrm{bdd}(A) \cap \mathrm{cl}_\Sigma(B) = \mathrm{bdd}(C).$$

Thus $X \mathop{\smile\!\!\!\!\!\!|}_{C} AB$ and by transitivity $X \mathop{\smile\!\!\!\!\!\!|}_{D} ABD$ since $D \subseteq X$. Finally, as $X \subseteq \mathrm{bdd}(AD)$ we obtain $X \subseteq \mathrm{bdd}(D)$. Hence, the result. □

It turns out from previous lemmas that the Σ-closure has good properties with respect to the forking calculus. However, for some considerations such as the Canonical Base Property, one should like to work with the first level of the Σ-closure rather than with the full closure operator.

Definition 2.1.10. The first level of the Σ-closure of A is given by

$$\mathrm{cl}^1_\Sigma(A) = \{b : \mathrm{tp}(b/A) \text{ is almost } \Sigma\text{-internal}\}.$$

Unfortunately, cl^1_Σ might not be a closure operator.

Lemma 2.1.11. *Suppose $A \mathop{\smile\!\!\!\!\!\!|}_{B} C$ with $B \subseteq A \cap C$. Then*

$$\mathrm{cl}^1_\Sigma(A) \mathop{\smile\!\!\!\!\!\!|}_{\mathrm{cl}^1_\Sigma(B)} C.$$

More precisely, $\mathrm{cl}^1_\Sigma(A) \mathop{\smile\!\!\!\!\!\!|}_{\mathrm{cl}^1_\Sigma(B)\cap \mathrm{bdd}(C)} C$.

Proof. Consider $a \in \mathrm{cl}^1_\Sigma(A)$ and put $c = \mathrm{Cb}(Aa/C)$. Then $\mathrm{tp}(c/B)$ is internal to the family of $\emptyset$-conjugates of $\mathrm{tp}(a/A)$ by Lemma 1.3.4, whence it is almost Σ-internal and so, $c \in \mathrm{bdd}(C) \cap \mathrm{cl}^1_\Sigma(B)$. Hence, as a was arbitrary, we obtain the result. □

Lemma 2.1.12. *Suppose $C \subseteq A \cap B \cap D$ and $AB \mathop{\smile\!\!\!\!\!\!|}_{C} D$. If $\mathrm{bdd}(A) \cap \mathrm{cl}^1_\Sigma(B) = \mathrm{bdd}(C)$, then $\mathrm{bdd}(AD) \cap \mathrm{cl}^1_\Sigma(BD) = \mathrm{bdd}(D)$.*

Proof. This is similar to point (2) of Lemma 2.1.9; one has to apply Lemma 2.1.11 instead of Lemma 2.1.4. □

2.2 Levels

Definition 2.2.1. The *first* Σ*-level of* a *over* A is given by

$$\ell_1^\Sigma(a/A) = \{b \in \mathrm{bdd}(aA) : \mathrm{tp}(b/A) \text{ is almost } \Sigma\text{-internal}\}.$$

Inductively, $\ell_{\alpha+1}^\Sigma(a/A) = \ell_1^\Sigma(a/\ell_\alpha^\Sigma(a/A))$, and $\ell_\lambda^\Sigma(a/A) = \bigcup_{\alpha<\lambda} \ell_\alpha^\Sigma(a/A)$ for limit ordinals λ.

Note that $\ell_1^\Sigma(a/A)$ is boundedly closed and it is the maximal subset of $\mathrm{bdd}(aA)$ which is almost internal to Σ over A. In fact, $\ell_1^\Sigma(a/A) = \mathrm{bdd}(aA) \cap \mathrm{cl}_\Sigma^1(A)$. This motivates us to introduce the following definition:

Definition 2.2.2. We shall write $\ell_\infty^\Sigma(a/A)$ for the set of all hyperimaginaries $b \in \mathrm{bdd}(aA)$ such that $\mathrm{tp}(b/A)$ is Σ-analysable, that is,

$$\ell_\infty^\Sigma(a/A) = \mathrm{bdd}(aA) \cap \mathrm{cl}_\Sigma(A).$$

Remark 2.2.3. Clearly, $\mathrm{tp}(a/A)$ is Σ-analysable if and only if $\ell_\infty^\Sigma(a/A) = \mathrm{bdd}(aA)$ if and only if $\ell_\alpha^\Sigma(a/A) = \mathrm{bdd}(aA)$ for some ordinal α, and the minimal such α is the minimal length of a Σ-analysis of a over A.

Example. Consider the theory of the free pseudoplane and let Σ be the family of all types of Lascar rank 1. If a type $\mathrm{tp}(a/A)$ has finite Lascar rank, then the first level $\ell_1^\Sigma(a/A)$ is the algebraic closure of A together with the first element of the (unique) path between a and A.

Lemma 2.2.4. *If* $a \mathop{\smile\!\!\!\!\!\!|}\; b$*, then* $\ell_\alpha^\Sigma(ab) = \mathrm{bdd}(\ell_\alpha^\Sigma(a), \ell_\alpha^\Sigma(b))$ *for any* α*.*

Proof. Let $c = \ell_\alpha^\Sigma(ab)$. Firstly, we claim that may assume $c \not\mathop{\smile\!\!\!\!\!\!|}_a b$ as otherwise $c \subseteq \ell_\alpha^\Sigma(a)$ and so $\ell_\alpha^\Sigma(ab) = \ell_\alpha^\Sigma(a)$, as desired.

Let $a_0 = \mathrm{Cb}(bc/a)$ and assume $c \not\mathop{\smile\!\!\!\!\!\!|}_a b$; so, $\mathrm{tp}(a_0)$ is internal to the family of $\mathrm{bdd}(\emptyset)$-conjugates of $\mathrm{tp}(c/b)$ by Lemma 1.3.4. As $\mathrm{tp}(c/b)$ is Σ-analysable, so is $\mathrm{tp}(a_0)$.

Claim. *In fact,* $\mathrm{tp}(a_0)$ *is* Σ*-analysable in* α *steps.*

Proof. By the proof of Lemma 1.3.4, there is some Morley sequence $(b_i c_i : i < \omega)$ in $\mathrm{tp}(bc/a)$ such that $a_0 \mathop{\smile\!\!\!\!\!\!|}\; (b_i : i < \omega)$ and $a_0 \subseteq \mathrm{dcl}(b_i c_i : i < \omega)$. As $\mathrm{tp}(c/b)$ is Σ-analysable in α steps, so is each $\mathrm{tp}(c_i/b_i)$ for $i < \omega$ and hence,

$\mathrm{tp}(c_i : i < \omega / b_i : i < \omega)$ is Σ-analysable in α steps. Put $\bar{c} = (c_i : i < \omega)$, $\bar{b} = (b_i : i < \omega)$, and let $(\bar{c}_i : i < \alpha)$ be an analysis of $\mathrm{tp}(\bar{c}/\bar{b})$. Take $d_i = \mathrm{Cb}(\bar{b}, \bar{c}_j : j \leq i / a_0)$; note that each $d_i \in \mathrm{bdd}(a_0)$, and every $\mathrm{tp}(d_i / d_j : j < i)$ is internal to the family of $\emptyset$-conjugates of $\mathrm{tp}(\bar{c}_i / \bar{b}, \bar{c}_j : j < i)$ and so, it is Σ-internal. Finally, we have that

$$\begin{aligned} a_0 \subseteq \mathrm{bdd}(\mathrm{Cb}(\bar{b}, \bar{c}_i : i < \omega / a_0)) &= \mathrm{bdd}(\bigcup_{i<\omega} \mathrm{Cb}(\bar{b}, \bar{c}_j : j \leq i / a_0)) \\ &= \mathrm{bdd}(d_i : i < \omega). \end{aligned}$$

Hence, $(d_i : i < \alpha)$ is a Σ-analysis of $\mathrm{tp}(\mathrm{bdd}(a_0))$. To obtain an analysis of $\mathrm{tp}(a_0)$ it is enough to consider for each $i < \alpha$, the set d_i' of a-conjugates of d_i. As $\mathrm{tp}(d_i / d_j : j < i)$ is Σ-internal, so is each of its a-conjugates and hence, $\mathrm{tp}(d_i' / d_j' : j < i)$. By definition $d_i' \subseteq \mathrm{dcl}(a_0)$ for all $i < \alpha$ and clearly $a_0 \subseteq \mathrm{bdd}(d_i : i < \alpha) \subseteq \mathrm{bdd}(d_i' : i < \alpha)$, hence $(d_i' : i < \alpha)$ is a Σ-analysis of $\mathrm{tp}(a_0)$. □

Thus,

$$a_0 \subseteq \mathrm{bdd}(c) \cap \mathrm{bdd}(a) = \ell_\alpha^\Sigma(a).$$

As $bc \mathop{\smile\!\!\!\!\!\!\mid}_{a_0} a$, $bc \mathop{\smile\!\!\!\!\!\!\mid}_{\ell_\alpha^\Sigma(a)} a$ and so,

$$c \mathop{\smile\!\!\!\!\!\!\mid}_{\ell_\alpha^\Sigma(a) b} a.$$

Hence, $c \subseteq \mathrm{bdd}(\ell_\alpha^\Sigma(a), b)$. Similarly, we may assume $c \not\mathop{\smile\!\!\!\!\!\!\mid}_b a$ and hence, $c \subseteq \mathrm{bdd}(\ell_\alpha^\Sigma(b), a)$. This shows that

$$\ell_\alpha^\Sigma(ab) \subseteq \mathrm{bdd}(\ell_\alpha^\Sigma(a), b) \cap \mathrm{bdd}(\ell_\alpha^\Sigma(b), a).$$

On the other hand, as $a \mathop{\smile\!\!\!\!\!\!\mid} b$ we have

$$a \mathop{\smile\!\!\!\!\!\!\mid}_{\ell_\alpha^\Sigma(a) \ell_\alpha^\Sigma(b)} b;$$

thus, $\mathrm{bdd}(\ell_\alpha^\Sigma(a), b) \cap \mathrm{bdd}(\ell_\alpha^\Sigma(b), a) = \mathrm{bdd}(\ell_\alpha^\Sigma(a), \ell_\alpha^\Sigma(b))$. Hence, the result. □

Lemma 2.2.5. *If $\mathrm{tp}(a/A)$ is $\Sigma \cup \Sigma'$-internal, then there is some $b \in \mathrm{bdd}(Aa)$ such that $\mathrm{tp}(b/A)$ is Σ-internal and $\mathrm{tp}(a/Ab)$ is Σ'-internal.*

Proof. Let $B \mathop{\smile\!\!\!\!\!\!|}_A a$ and let $\bar{b}$ and $\bar{b}'$ be tuples of realizations of types in Σ and Σ' with parameters over B, respectively, such that $a \in \mathrm{dcl}(B, \bar{b}\bar{b}')$. Let $b = \mathrm{Cb}(B\bar{b}/Aa)$; thus, $\mathrm{tp}(b/A)$ is Σ-internal and $a \mathop{\smile\!\!\!\!\!\!|}_{Ab} B\bar{b}$. As $\bar{b}'$ is also a tuple of realizations of types in Σ' with parameters over $B\bar{b}$, the latter implies that $\mathrm{tp}(a/Ab)$ is Σ'-internal. $\square$

We shall see that the first level governs domination-equivalence.

Definition 2.2.6. An element a Σ-*dominates* an element b over A, denoted $a \trianglerighteq_A^\Sigma b$, if for all c such that $\mathrm{tp}(c/A)$ is Σ-analysable, $a \mathop{\smile\!\!\!\!\!\!|}_A c$ implies $b \mathop{\smile\!\!\!\!\!\!|}_A c$. Two elements a and b are Σ-*domination-equivalent* over A, denoted $a \sqsupseteq\!\sqsubseteq_A^\Sigma b$, if $a \trianglerighteq_A^\Sigma b$ and $b \trianglerighteq_A^\Sigma a$. If Σ is the set of all types, it is omitted.

The following generalizes a theorem of Buechler [8, Proposition 3.1] in the finite Lascar rank setting. This result is mainly due to Wagner. We include a proof for the sake of completeness.

Theorem 2.2.7. *Let Σ' be an $\emptyset$-invariant family of partial types.*

1. *a and $\ell_1^\Sigma(a/A)$ are Σ-domination-equivalent over A.*

2. *If $\mathrm{tp}(a/A)$ is Σ-analysable, then a and $\ell_1^\Sigma(a/A)$ are domination-equivalent over A.*

3. *If $\mathrm{tp}(a/A)$ is $\Sigma \cup \Sigma'$-analysable and foreign to Σ', then a and $\ell_1^\Sigma(a/A)$ are domination-equivalent over A.*

Proof. Since $\ell_1^\Sigma(a/A) \in \mathrm{bdd}(Aa)$, clearly a dominates (and Σ-dominates) $\ell_1^\Sigma(a/A)$ over A.

For the converse, suppose $\mathrm{tp}(b/A)$ is Σ-analysable and $b \not\mathop{\smile\!\!\!\!\!\!|}_A a$. Consider a sequence $(b_i : i < \alpha)$ in $\mathrm{dcl}(Ab)$ such that $\mathrm{tp}(b_i/A, b_j : j < i)$ is Σ-internal for all $i < \alpha$ and $b \in \mathrm{bdd}(A, b_i : i < \alpha)$. Since $a \not\mathop{\smile\!\!\!\!\!\!|}_A b$ there is a minimal $i < \alpha$ such that $a \not\mathop{\smile\!\!\!\!\!\!|}_{A,(b_j:j<i)} b_i$. Put $a' = \mathrm{Cb}(b_j : j \leq i/Aa)$. Then $a' \in \mathrm{bdd}(Aa)$ and $\mathrm{tp}(a'/A)$ is internal to the family of $\mathrm{bdd}(\emptyset)$-conjugates of $\mathrm{tp}(b_i/A, b_j : j < i)$ by Lemma 1.3.4. Thus, $\mathrm{tp}(a'/A)$ is Σ-internal and so, $a' \subseteq \ell_1^\Sigma(a/A)$. Clearly $a' \not\mathop{\smile\!\!\!\!\!\!|}_A (b_j : j \leq i)$, whence $a' \not\mathop{\smile\!\!\!\!\!\!|}_A b$ and finally $\ell_1^\Sigma(a/A) \not\mathop{\smile\!\!\!\!\!\!|}_A b$. This shows (1).

If $\mathrm{tp}(a/A)$ is Σ-analysable and $b \not\perp_A a$, we first consider $b' = \mathrm{Cb}(a/Ab)$. Then $\mathrm{tp}(b'/A)$ is Σ-analysable, $b' \in \mathrm{bdd}(Ab)$ and $a \not\perp_A b'$. Hence we obtain $\ell_1^\Sigma(a/A) \not\perp_A b'$, whence $\ell_1^\Sigma(a/A) \not\perp_A b$. This shows (2).

For (3), suppose $b \not\perp_A a$. We may assume that $b = \mathrm{Cb}(a/Ab)$, so $\mathrm{tp}(b/A)$ is $(\Sigma \cup \Sigma')$-analysable. Let $(b_i : i < \alpha)$ be a $(\Sigma \cup \Sigma')$-analysis of b over A; we can choose it such that for all $i < \alpha$ the type $\mathrm{tp}(b_i/A, b_j : j < i)$ is internal to either Σ or to Σ' by Lemma 2.2.5. Consider a minimal $i < \alpha$ such that $(b_j : j \leq i) \not\perp_A a$. Then $(b_j : j < i) \perp_A a$, so $\mathrm{tp}(a/A, b_j : j < i)$ is foreign to Σ'. Hence $\mathrm{tp}(b_i/A, b_j : j < i)$ must be Σ-internal as otherwise we would have $a \perp_{A(b_j:j<i)} b_i$ and so, $a \perp_A (b_j : j \leq i)$ by transitivity, a contradiction. As before, if $a' = \mathrm{Cb}(b_j : j \leq i/Aa)$, then $\mathrm{tp}(a'/A)$ is Σ-internal. Thus $a' \in \ell_1^\Sigma(a/A)$; since $(b_j : j \leq i) \not\perp_A a'$ we get $b \not\perp_A \ell_1^\Sigma(a/A)$. □

Remark 2.2.8. Observe that in point (3) of theorem above, if $\mathrm{tp}(a)$ is $\Sigma \cup \Sigma'$-analysable and we take $\hat{a} = \ell_\infty^{\Sigma'}(a)$, then the assumptions are satisfied by $\mathrm{tp}(a/\hat{a})$ and hence, $a \sqsubseteq_{\hat{a}} \ell_1^\Sigma(a/\hat{a})$. In particular, the reader will find an application of this in Theorem 2.4.5.

Definition 2.2.9.

- A type $\mathrm{tp}(a/A)$ is Σ-*flat* if $\ell_1^\Sigma(a/A) = \ell_2^\Sigma(a/A)$. It is *flat* if it is Σ-flat for all Σ. T is *flat* if all its types are.
- A type p is *ultraflat* if it is almost internal to any type it is non-orthogonal to.

Note that both notions are preserved under non-forking extensions and non-forking restrictions.

Remark 2.2.10. If $\mathrm{tp}(a/A)$ is Σ-flat, then $\ell_\alpha^\Sigma(a/A) = \ell_1^\Sigma(a/A)$ for all $\alpha > 0$. Moreover, ultraflat implies flat.

Proof. The first part easily follows from the definition of the α-level. For the second clause, assume $\mathrm{tp}(a/A)$ is ultraflat and it is not foreign to Σ, then $a \perp_A B$ and $a \not\perp_B b$ for some tuple of realizations of types in Σ over B. In particular, $\mathrm{tp}(a/A)$ is non orthogonal to $\mathrm{tp}(b/B)$ and so, $\mathrm{tp}(a/A)$ is $\mathrm{tp}(b/B)$-internal. Hence, $\mathrm{tp}(a/A)$ is Σ-internal since $\mathrm{tp}(b/B)$ is Σ-internal. □

Example. If there is no boundedly closed set between $\mathrm{bdd}(A)$ and $\mathrm{bdd}(aA)$, then $\mathrm{tp}(a/A)$ is ultraflat. In particular, this applies to types of Lascar rank one.

We give more examples of ultraflat types. Recall that a first-order theory is *small* if for any real finite set A and for any natural number n, $|S_n(A)| \leq \omega$.

Lemma 2.2.11. *Assume the ambient theory is small and let a and A be finite. Then the lattice of boundedly closed sets of $\mathrm{bdd}(Aa)$ is scattered, that is, no dense linear order can be embedded in the lattice of boundedly closed subsets of $\mathrm{bdd}(Aa)$.*

Proof. This is [67, Lemma 6.1.16] but working with the family of bounded types. In that case, the cl_P is just the bounded closure. □

It turns out from lemma above that in a small simple theory there will be many ultraflat types inside $\mathrm{bdd}(Aa)$ for finite a and A. Other examples of ultraflat types can be found in algebraic examples.

Recall that an *hyperdefinable group* is given by a hyperdefinable set G and a hyperdefinable binary function $*$ such that $(G, *)$ forms a group. More precisely, G is given by a type-definable equivalence relation $E(x, y)$, a partial type $\pi(x)$ and a type-definable relation $\mu(x, y, z)$ which are invariant under E-classes, such that $G = \pi(\mathfrak{C})/E$ and $* = \mu/E$.

In this context, for a set of parameters A we put $S_G(A)$ to denote the space of all types which are invariant under E-classes in G over A extending $\pi(x)$.

A type $p \in S_G(A)$ is said to be *generic* if for all b_E realizing a type in $S_G(A)$ and a_E realizing p with $a_E \mathop{\smile\!\!\!\!\!\mid}_A b_E$, we have $b_E a_E \mathop{\smile\!\!\!\!\!\mid} A, b_E$.

Theorem 2.2.12. *Suppose a generic type of a hyperdefinable group G is not foreign to some partial type π. Then there is a normal hyperdefinable normal subgroup N of unbounded index in G such that G/N is π-internal*

Proof. This is [67, Theorem 4.6.4]. □

Example. Generic types of fields and generic types of hyperdefinable simple groups are ultraflat. We follow [61, Corollary 2.27]. Let F be a field and let F^+ be its additive group. If a generic type of F^+ over A is not orthogonal to some type q, then there is some hyperdefinable subgroup N of unbounded index in F^+ such that F^+/N is q-internal by Theorem 2.2.12. Then if we consider the subgroup $\bar{N} = \bigcap_{a \in F \setminus \{0\}} aN$ of F^+, it is again hyperdefinable and in addition we have, $F^+/\bar{N}$ is q-internal. But this implies that F^+ is q-internal since $\bar{N}$ is an ideal and so, $\bar{N} = \{0\}$. In the case of a hyperdefinable simple group, if a generic type of G is non-orthogonal to a type q, then there is a normal hyperdefinable subgroup N of unbounded index in G such that G/N is q-internal, and so is G since N must be trivial.

Next we shall prove that any type internal to a family of Lascar rank one types is also flat.

Lemma 2.2.13. *It* $\mathrm{tp}(a/A)$ *is flat (ultraflat), then so is* $\mathrm{tp}(a_0/A)$ *for any* $a_0 \in \mathrm{bdd}(Aa)$.

Proof. The flat case is clear since $\ell^{\Sigma}_{\alpha}(a_0/A) = \ell^{\Sigma}_{\alpha}(a/A) \cap \mathrm{bdd}(Aa_0)$ for any $\alpha > 0$. Assume now $\mathrm{tp}(a/A)$ is ultraflat. Let $B \mathop{\smile\!\!\!\!\!\mid}_A a_0$ and $b \not\mathop{\smile\!\!\!\!\!\mid}_B a_0$ and let $\mathcal{P}$ be the family of $\mathrm{bdd}(\emptyset)$-conjugates of $\mathrm{tp}(b/B)$. We may assume $Bb \mathop{\smile\!\!\!\!\!\mid}_{Aa_0} a$. Thus $a \mathop{\smile\!\!\!\!\!\mid}_A B$ and $a \not\mathop{\smile\!\!\!\!\!\mid}_B b$, so $\mathrm{tp}(a/A)$ is almost $\mathcal{P}$-internal by ultraflatness, and so is $\mathrm{tp}(a_0/A)$. □

Corollary 2.2.14. *If* $\mathrm{tp}(a/A)$ *is an imaginary type which is almost internal to a family of Lascar rank one imaginaries types, then it is flat.*

Proof. Assume there is some $B \mathop{\smile\!\!\!\!\!\mid}_A a$ and some tuple $\bar{b}$ of realizations of types of Lascar rank one over B such that $a \subseteq \mathrm{bdd}(B, \bar{b})$. By assumption, $a \subseteq \mathrm{acl}(B, \bar{b})$ where B and $\bar{b}$ live in the imaginary universe; so, we may assume $\bar{b}$ is a finite tuple, whence $\mathrm{tp}(\bar{b}/B)$ is flat by Lemma 2.2.4. Thus, $\mathrm{tp}(a/B)$ is flat by Lemma 2.2.13 and so is $\mathrm{tp}(a/A)$. □

Observe that in the free pseudoplane a type of finite Lascar rank greater than one is analysable in the family of types of Lascar rank one, but it is not flat. Therefore, last result is the best we can obtain in terms of internality *versus* analysability.

2.3 Hierarchies

Let Φ and Σ be $\emptyset$-invariant families of partial types.

Definition 2.3.1. Φ is *n-Σ-ample* if there are tuples $a_0, \ldots, a_n$, with a_n a tuple of realizations of partial types in Φ over some parameters A, such that

1. $a_n \not\perp_{\mathrm{cl}_\Sigma(A)} a_0$;
2. $a_{i+1} \perp_{\mathrm{cl}_\Sigma(Aa_i)} a_0 \ldots a_{i-1}$ for $1 \leq i < n$;
3. $\mathrm{cl}_\Sigma(Aa_0 \ldots a_{i-1}a_i) \cap \mathrm{cl}_\Sigma(Aa_0 \ldots a_{i-1}a_{i+1}) = \mathrm{cl}_\Sigma(Aa_0 \ldots a_{i-1})$ for $0 \leq i < n$.

Remark 2.3.2. We follow the variant proposed by Evans and Nübling [22] which seems more natural and which implies

$$a_n \ldots a_{i+1} \underset{\mathrm{cl}_\Sigma(Aa_i)}{\perp} a_0 \ldots a_{i-1}.$$

Remark 2.3.3. In Definition 2.3.1 one may require $a_0, \ldots, a_{n-1}$ to lie in the definable closure of Φ^{heq}, replacing a_i by $a'_i = \mathrm{Cb}(a'_{i+1}/\mathrm{cl}_\Sigma(a_iA))$ for $i < n$, where $a'_n = a_n$.

Proof. This is similar to [55, Remark 3.7]. □

Remark 2.3.4. If every type in Σ' is Σ-analysable and $a_0, a_1, \ldots, a_n$ witness n-Σ-ampleness over A, the same $a'_0, a'_1, \ldots, a'_n$ as in Remark 2.3.3 witness n-Σ'-ampleness over $A' = \mathrm{cl}_\Sigma(A) \cap \mathrm{bdd}(Aa_0 \ldots a_n)$. In particular n-Σ-ample implies n-ample.

Remark 2.3.5. If $a_0, \ldots, a_n$ witness n-Σ-ampleness over A, then $a_i, \ldots, a_n$ witness $(n-i)$-Σ-ampleness over $Aa_0 \ldots a_{i-1}$. Thus n-Σ-ample implies i-Σ-ample for all $i \leq n$.

Proof. This is similar to [55, Lemma 3.2 and Corollary 3.3]. □

Remark 2.3.6. It is clear from the definition that even though Φ might be a complete type p, if p is not n-Σ-ample, neither is any extension of p, not only the non-forking ones.

For $n = 1, 2$ there are alternative definitions of non-n-Σ-ampleness: Σ-basedness and Σ-CM-triviality.

Definition 2.3.7.

- Φ is Σ-*based* if $\mathrm{Cb}(a/\mathrm{cl}_\Sigma(B)) \subseteq \mathrm{cl}_\Sigma(aA)$ for any tuple a of realizations of partial types in Φ over some parameters A and any $B \supseteq A$.
- Φ is Σ-*CM-trivial* if $\mathrm{Cb}(a/\mathrm{cl}_\Sigma(AB)) \subseteq \mathrm{cl}_\Sigma(A, \mathrm{Cb}(a/\mathrm{cl}_\Sigma(AC))$ for any tuple a of realizations of partial types in Φ over some parameters A and any $B \subseteq C$ such that $\mathrm{cl}_\Sigma(aAB) \cap \mathrm{cl}_\Sigma(AC) = \mathrm{cl}_\Sigma(AB)$.

Lemma 2.3.8.

1. *Φ is Σ-based if and only if Φ is not 1-Σ-ample.*
2. *Φ is Σ-CM-trivial if and only if Φ is not 2-Σ-ample.*

Proof. (1). Suppose Φ is Σ-based and consider a_0, a_1, A with $\mathrm{cl}_\Sigma(Aa_0) \cap \mathrm{cl}_\Sigma(Aa_1) = \mathrm{cl}_\Sigma(A)$. Put $a = a_1$ and $B = Aa_0$. By Σ-basedness

$$\mathrm{Cb}(a/\mathrm{cl}_\Sigma(B)) \subseteq \mathrm{cl}_\Sigma(Aa) \cap \mathrm{cl}_\Sigma(B) = \mathrm{cl}_\Sigma(A).$$

Hence $a \mathop{\smile\!\!\!\!\!\mid}_{\mathrm{cl}_\Sigma(A)} \mathrm{cl}_\Sigma(B)$, whence $a_1 \mathop{\smile\!\!\!\!\!\mid}_{\mathrm{cl}_\Sigma(A)} a_0$, so Φ is not 1-Σ-ample.

Conversely, if Φ is not Σ-based, let a, A, B be a counterexample. Put $a_0 = \mathrm{Cb}(a_1/\mathrm{cl}_\Sigma(B))$ and $a_1 = a$. Then $a_0 \notin \mathrm{cl}_\Sigma(Aa_1)$. Now take

$$A' = \mathrm{Cb}(Aa_0a_1/\mathrm{cl}_\Sigma(Aa_0) \cap \mathrm{cl}_\Sigma(Aa_1)).$$

Then by Lemma 2.1.5 we obtain $\mathrm{cl}_\Sigma(A'a_0) \cap \mathrm{cl}_\Sigma(A'a_1) = \mathrm{cl}_\Sigma(A')$.

Suppose $a_1 \mathop{\smile\!\!\!\!\!\mid}_{\mathrm{cl}_\Sigma(A')} a_0$. Then since $\mathrm{cl}_\Sigma(A') \subseteq \mathrm{cl}_\Sigma(Aa_0) \subseteq \mathrm{cl}_\Sigma(B)$ and $a_0 = \mathrm{Cb}(a_1/\mathrm{cl}_\Sigma(B))$ we have $a_1 \mathop{\smile\!\!\!\!\!\mid}_{a_0\mathrm{cl}_\Sigma(A')} \mathrm{cl}_\Sigma(B)$ and so, $a_1 \mathop{\smile\!\!\!\!\!\mid}_{\mathrm{cl}_\Sigma(A')} \mathrm{cl}_\Sigma(B)$ by transitivity. This implies

$$a_0 \subseteq \mathrm{cl}_\Sigma(A') \subseteq \mathrm{cl}_\Sigma(Aa_1),$$

a contradiction. Hence a_0, a_1, A' witness 1-Σ-ampleness of Φ.

(2). Suppose Φ is Σ-CM-trivial and consider a_0, a_1, a_2, A with

$$\mathrm{cl}_\Sigma(Aa_0) \cap \mathrm{cl}_\Sigma(Aa_1) = \mathrm{cl}_\Sigma(A),$$
$$\mathrm{cl}_\Sigma(Aa_0a_1) \cap \mathrm{cl}_\Sigma(Aa_0a_2) = \mathrm{cl}_\Sigma(Aa_0), \quad \text{and}$$
$$a_2 \mathop{\smile\!\!\!\!\!\mid}_{\mathrm{cl}_\Sigma(Aa_1)} a_0.$$

Put $a = a_2$, $B = a_0$ and $C = a_0a_1$. Then

$$a_2 \mathop{\downarrow}_{\mathrm{cl}_\Sigma(Aa_1)} \mathrm{cl}_\Sigma(Aa_0a_1),$$

so $\mathrm{Cb}(a/\mathrm{cl}_\Sigma(AC)) \subseteq \mathrm{cl}_\Sigma(Aa_1)$. Moreover $\mathrm{cl}_\Sigma(ABa) \cap \mathrm{cl}_\Sigma(AC) = \mathrm{cl}_\Sigma(AB)$, whence by Σ-CM-triviality

$$\begin{aligned} \mathrm{Cb}(a/\mathrm{cl}_\Sigma(AB)) &\subseteq \mathrm{cl}_\Sigma(A, \mathrm{Cb}(a/AC)) \cap \mathrm{cl}_\Sigma(AB) \\ &\subseteq \mathrm{cl}_\Sigma(Aa_1) \cap \mathrm{cl}_\Sigma(Aa_0) = \mathrm{cl}_\Sigma(A). \end{aligned}$$

Hence $a_2 \downarrow_{\mathrm{cl}_\Sigma(A)} \mathrm{cl}_\Sigma(B)$, whence $a_2 \downarrow_{\mathrm{cl}_\Sigma(A)} a_0$ and so, Φ is not 2-Σ-ample.

Conversely, if Φ is not Σ-CM-trivial, let a, A, B, C be a counterexample. Put

$$a_0 = \mathrm{Cb}(a/\mathrm{cl}_\Sigma(AB)), \quad a_1 = \mathrm{Cb}(a/\mathrm{cl}_\Sigma(AC)), \quad a_2 = a,$$
$$A' = \mathrm{Cb}(Aa_0a_1/\mathrm{cl}_\Sigma(Aa_0) \cap \mathrm{cl}_\Sigma(Aa_1)).$$

Then $a_2 \downarrow_{\mathrm{cl}_\Sigma(A'a_1)} a_0$ and $a_0 \notin \mathrm{cl}_\Sigma(Aa_1)$; by Lemma 2.1.5

$$\mathrm{cl}_\Sigma(A'a_0) \cap \mathrm{cl}_\Sigma(A'a_1) = \mathrm{cl}_\Sigma(A').$$

Moreover, $a_2 \downarrow_{a_0} \mathrm{cl}_\Sigma(AB)$ implies

$$\mathrm{cl}_\Sigma(A'a_0a_2) \mathop{\downarrow}_{\mathrm{cl}_\Sigma(A'a_0)} \mathrm{cl}_\Sigma(AB).$$

Thus

$$\begin{aligned} \mathrm{cl}_\Sigma(A'a_0a_2) \cap \mathrm{cl}_\Sigma(A'a_0a_1) &\subseteq \mathrm{cl}_\Sigma(ABa) \cap \mathrm{cl}_\Sigma(AC) \cap \mathrm{cl}_\Sigma(A'a_0a_2) \\ &\subseteq \mathrm{cl}_\Sigma(AB) \cap \mathrm{cl}_\Sigma(A'a_0a_2) = \mathrm{cl}_\Sigma(A'a_0). \end{aligned}$$

Suppose $a_2 \downarrow_{\mathrm{cl}_\Sigma(A')} a_0$. As above $a_2 \downarrow_{\mathrm{cl}_\Sigma(A')} \mathrm{cl}_\Sigma(B)$ and so

$$a_0 \subseteq \mathrm{cl}_\Sigma(A') \subseteq \mathrm{cl}_\Sigma(Aa_1),$$

a contradiction. Hence a_0, a_1, a_2, A' witness 2-Σ-ampleness of Φ. □

In our definition of Σ-ampleness, we only consider the type of a_n over a Σ-closed set, namely $\mathrm{cl}_\Sigma(A)$. This seems natural since the idea of Σ-closure is to work *modulo* Σ. However, sometimes one needs a stronger notion which takes care of all types. Let us first look at $n = 1$ and $n = 2$.

Definition 2.3.9.

- Φ is *strongly* Σ*-based* if $\mathrm{Cb}(a/B) \subseteq \mathrm{cl}_\Sigma(aA)$ for any tuple a of realizations of partial types in Φ over some A and any $B \supseteq A$.
- Φ is *strongly* Σ*-CM-trivial* if $\mathrm{Cb}(a/AB) \subseteq \mathrm{cl}_\Sigma(A, \mathrm{Cb}(a/\mathrm{cl}_\Sigma(AC))$ for any tuple a of realizations of partial types in Φ over some A and any $B \subseteq C$ with $\mathrm{cl}_\Sigma(aAB) \cap \mathrm{cl}_\Sigma(AC) = \mathrm{cl}_\Sigma(AB)$.

In Question 2.1.7 we asked if $\mathrm{Cb}(a/B) \subseteq \mathrm{cl}_\Sigma(\mathrm{Cb}(a/\mathrm{cl}_\Sigma(B)))$. If this were true, strong and ordinary Σ-basedness and Σ-CM-triviality would obviously coincide. Since we do not known wether it is true, we weaken our definition of ampleness.

Definition 2.3.10. Φ is *weakly* n*-*Σ*-ample* if there are tuples $a_0, \ldots, a_n$, where a_n is a tuple of realizations of partial types in Φ over A, with

1. $a_n \not\perp_A a_0$.
2. $a_{i+1} \perp_{\mathrm{cl}_\Sigma(Aa_i)} a_0 \ldots a_{i-1}$ for $1 \leq i < n$.
3. $\mathrm{bdd}(Aa_0) \cap \mathrm{cl}_\Sigma(Aa_1) = \mathrm{bdd}(A)$.
4. $\mathrm{cl}_\Sigma(Aa_0 \ldots a_{i-1}a_i) \cap \mathrm{cl}_\Sigma(Aa_0 \ldots a_{i-1}a_{i+1}) = \mathrm{cl}_\Sigma(Aa_0 \ldots a_{i-1})$ for $1 \leq i < n$.

Note that (3) implies that $\mathrm{tp}(a_0/A)$ is foreign to Σ by Lemma 2.1.3. If Σ is the family of bounded partial types, then (weak) n-Σ-ampleness just equals n-ampleness.

Remark 2.3.11. A n-Σ-ample family of types is weakly n-Σ-ample. If Σ' is Σ-analysable, then a weakly n-Σ-ample family is weakly n-Σ'-ample, and in particular n-ample.

Proof. As in Remark 2.3.3, if $a_0, \ldots, a_n$ witness n-Σ-ampleness over some set A, put $a'_n = a_n$ and $a'_i = \mathrm{Cb}(a'_{i+1}/\mathrm{cl}_\Sigma(Aa_i))$. Then $a'_0, \ldots, a'_n$ witness weak n-Σ-ampleness over $A' = \mathrm{bdd}(Aa'_0) \cap \mathrm{cl}_\Sigma(Aa'_1)$. Similarly, if $a_0, \ldots, a_n$ witness weak n-Σ-ampleness over A, then $a'_0, \ldots, a'_n$ witness weak n-Σ'-ampleness over $A' = \mathrm{bdd}(Aa'_0) \cap \Sigma'\mathrm{cl}(Aa'_1)$, as condition (4) follows from

$$a'_i \underset{Aa'_0 \ldots a'_{i-1}}{\perp} \mathrm{cl}_\Sigma(Aa_0 \ldots a_{i-1})$$

and Lemma 2.1.4 applied with respect to Σ'-closure. □

Lemma 2.3.12.

1. *Φ is strongly Σ-based if and only if Φ is not weakly 1-Σ-ample.*
2. *Φ is strongly Σ-CM-trivial if and only if Φ is not weakly 2-Σ-ample.*

Proof. This is similar to the proof of Lemma 2.3.8, so we shall be concise.

(1). Suppose Φ is strongly Σ-based and consider a_0, a_1, A with

$$\mathrm{bdd}(Aa_0) \cap \mathrm{cl}_\Sigma(Aa_1) = \mathrm{bdd}(A).$$

Put $a = a_1$ and $B = Aa_0$. By strong Σ-basedness

$$\mathrm{Cb}(a/B) \subseteq \mathrm{cl}_\Sigma(Aa) \cap \mathrm{bdd}(B) = \mathrm{bdd}(A),$$

whence $a_1 \mathop{\smile\!\!\!\!\!\mid}_A a_0$, so Φ is not weakly 1-Σ-ample.

Conversely, if Φ is not strongly Σ-based, let a, A, B be a counterexample. Put $a_0 = \mathrm{Cb}(a_1/B)$ and $a_1 = a$. Then $a_0 \notin \mathrm{cl}_\Sigma(Aa_1)$. Now take $A' = \mathrm{bdd}(Aa_0) \cap \mathrm{cl}_\Sigma(Aa_1)$. Clearly $A' = \mathrm{bdd}(A'a_0) \cap \mathrm{cl}_\Sigma(A'a_1)$. Suppose $a_1 \mathop{\smile\!\!\!\!\!\mid}_{A'} a_0$. Since $a_0 = \mathrm{Cb}(a_1/B)$ implies $a_1 \mathop{\smile\!\!\!\!\!\mid}_{a_0} A'$, we obtain

$$a_0 \subseteq \mathrm{bdd}(A') \subseteq \mathrm{cl}_\Sigma(Aa_1),$$

a contradiction. Hence a_0, a_1, A' witness weak 1-Σ-ampleness of Φ.

Suppose Φ is strongly Σ-CM-trivial and consider a_0, a_1, a_2, A with

$$\mathrm{bdd}(Aa_0) \cap \mathrm{cl}_\Sigma(Aa_1) = \mathrm{bdd}(A),$$
$$\mathrm{cl}_\Sigma(Aa_0a_1) \cap \mathrm{cl}_\Sigma(Aa_0a_2) = \mathrm{cl}_\Sigma(Aa_0), \quad \text{and}$$
$$a_2 \mathop{\smile\!\!\!\!\!\mid}_{\mathrm{cl}_\Sigma(Aa_1)} a_0.$$

Put $a = a_2$, $B = a_0$ and $C = a_0a_1$. Then $\mathrm{Cb}(a/\mathrm{cl}_\Sigma(AC)) \subseteq \mathrm{cl}_\Sigma(Aa_1)$ by Lemma 2.1.4. Moreover

$$\mathrm{cl}_\Sigma(ABa) \cap \mathrm{cl}_\Sigma(AC) = \mathrm{cl}_\Sigma(AB),$$

whence by strong Σ-CM-triviality

$$\begin{aligned}\mathrm{Cb}(a/AB) &\subseteq \mathrm{cl}_\Sigma(A, \mathrm{Cb}(a/AC)) \cap \mathrm{bdd}(AB)\\ &\subseteq \mathrm{cl}_\Sigma(Aa_1) \cap \mathrm{bdd}(Aa_0) = \mathrm{bdd}(A).\end{aligned}$$

Hence $a_2 \mathop{\smile\!\!\!\!\!\!\mid}_A a_0$, so Φ is not 2-Σ-ample.

Conversely, if Φ is not strongly Σ-CM-trivial, let a, A, B, C be a counterexample. Put

$$a_0 = \mathrm{Cb}(a/AB), \quad a_1 = \mathrm{Cb}(a/\mathrm{cl}_\Sigma(AC)), \quad a_2 = a,$$

$$A' = \mathrm{bdd}(Aa_0) \cap \mathrm{cl}_\Sigma(Aa_1).$$

Then $a_2 \mathop{\smile\!\!\!\!\!\!\mid}_{\mathrm{cl}_\Sigma(A'a_1)} a_0$ and $a_0 \notin \mathrm{cl}_\Sigma(Aa_1)$; moreover

$$\mathrm{bdd}(A'a_0) \cap \mathrm{cl}_\Sigma(A'a_1) = \mathrm{bdd}(A').$$

Now $a_2 \mathop{\smile\!\!\!\!\!\!\mid}_{a_0} AB$ implies

$$\mathrm{cl}_\Sigma(A'a_0a_2) \mathop{\smile\!\!\!\!\!\!\mid}_{\mathrm{cl}_\Sigma(A'a_0)} \mathrm{cl}_\Sigma(AB)$$

by Lemma 2.1.4, whence as before

$$\mathrm{cl}_\Sigma(A'a_0a_2) \cap \mathrm{cl}_\Sigma(A'a_0a_1) = \mathrm{cl}_\Sigma(A'a_0).$$

Suppose $a_2 \mathop{\smile\!\!\!\!\!\!\mid}_{A'} a_0$. Then $a_0 \subseteq \mathrm{bdd}(A') \subseteq \mathrm{cl}_\Sigma(Aa_1)$, a contradiction. Hence a_0, a_1, a_2, A' witness weak 2-Σ-ampleness of Φ. $\square$

Lemma 2.3.13. *If Φ is not (weakly) n-Σ-ample, neither is the family of $\emptyset$-conjugates of $\mathrm{tp}(a/A)$ for any $a \in \mathrm{cl}_\Sigma(\bar{a}A)$, where $\bar{a}$ is a tuple of realizations of partial types in Φ over A.*

Proof. Suppose the family of $\emptyset$-conjugates of $\mathrm{tp}(a/A)$ is n-Σ-ample, as witnessed by $a_0, \ldots, a_n$ over some parameters B. There is a tuple $\bar{a}$ of realizations of partial types in Φ over some $\emptyset$-conjugates of A inside B such that $a_n \in \mathrm{cl}_\Sigma(\bar{a}B)$; we may choose it such that

$$\bar{a} \mathop{\smile\!\!\!\!\!\!\mid}_{a_nB} a_0 \ldots a_{n-1}.$$

Then $\bar{a} \mathop{\smile\!\!\!\!\!\!\mid}_{a_{n-1}a_nB} a_0 \ldots a_{n-2}$, and hence

$$\bar{a} \mathop{\smile\!\!\!\!\!\!\mid}_{\mathrm{cl}_\Sigma(a_{n-1}a_nB)} a_0 \ldots a_{n-2}.$$

As $a_n \mathop{\smile\!\!\!\!\!\!\!\mid}_{\mathrm{cl}_\Sigma(a_{n-1}B)} a_0 \dots a_{n-2}$ implies

$$\mathrm{cl}_\Sigma(a_{n-1}a_nB) \mathop{\smile\!\!\!\!\!\!\!\mid}_{\mathrm{cl}_\Sigma(a_{n-1}B)} a_0 \dots a_{n-2}$$

by Lemma 2.1.4, we get

$$\bar{a} \mathop{\smile\!\!\!\!\!\!\!\mid}_{\mathrm{cl}_\Sigma(a_{n-1}B)} a_0 \dots a_{n-2}.$$

We also have $\bar{a} \mathop{\smile\!\!\!\!\!\!\!\mid}_{a_0 \dots a_{n-2}a_nB} a_{n-1}$, whence

$$\mathrm{cl}_\Sigma(a_0 \dots a_{n-2}\bar{a}B) \mathop{\smile\!\!\!\!\!\!\!\mid}_{\mathrm{cl}_\Sigma(a_0 \dots a_{n-2}a_nB)} \mathrm{cl}_\Sigma(a_0 \dots a_{n-2}a_{n-1}B);$$

since Σ-closure is boundedly closed,

$$\begin{aligned}\mathrm{cl}_\Sigma(a_0 \dots a_{n-2}\bar{a}B) \cap \mathrm{cl}_\Sigma(a_0 \dots a_{n-2}a_{n-1}B)& \\ &\subseteq \mathrm{cl}_\Sigma(a_0 \dots a_{n-2}a_nB) \cap \mathrm{cl}_\Sigma(a_0 \dots a_{n-2}a_{n-1}B) \\ &= \mathrm{cl}_\Sigma(a_0 \dots a_{n-2}B).\end{aligned}$$

Finally, $\bar{a} \mathop{\smile\!\!\!\!\!\!\!\mid}_{\mathrm{cl}_\Sigma(B)} a_0$ would imply $\mathrm{cl}_\Sigma(\bar{a}B) \mathop{\smile\!\!\!\!\!\!\!\mid}_{\mathrm{cl}_\Sigma(B)} a_0$ by Lemma 2.1.4, and hence $a_n \mathop{\smile\!\!\!\!\!\!\!\mid}_{\mathrm{cl}_\Sigma(B)} a_0$, a contradiction. Thus $\bar{a} \not\mathop{\smile\!\!\!\!\!\!\!\mid}_{\mathrm{cl}_\Sigma(B)} a_0$, and $a_0, \dots, a_{n-1}, \bar{a}$ witness n-Σ-ampleness of Φ over B, a contradiction.

Now suppose $a_0, \dots, a_n$ witness weak n-Σ-ampleness over B, and choose $\bar{a}$ as before. Then (2) and (4) from the definition follow as above. Suppose $\bar{a} \mathop{\smile\!\!\!\!\!\!\!\mid}_B a_0$. Since $\mathrm{tp}(a_0/B)$ is foreign to Σ, so is $\mathrm{tp}(a_0/B\bar{a})$. But then $a_0 \mathop{\smile\!\!\!\!\!\!\!\mid}_{B\bar{a}} \mathrm{cl}_\Sigma(B\bar{a})$ by Lemma 2.1.3, whence $a_0 \mathop{\smile\!\!\!\!\!\!\!\mid}_B a_n$, a contradiction. Thus $\bar{a} \not\mathop{\smile\!\!\!\!\!\!\!\mid}_B a_0$ and we have (1).

Finally (3) is trivial unless $n = 1$. In that case $\bar{a} \mathop{\smile\!\!\!\!\!\!\!\mid}_{Ba_1} a_0$ implies

$$\mathrm{cl}_\Sigma(B\bar{a}) \mathop{\smile\!\!\!\!\!\!\!\mid}_{\mathrm{cl}_\Sigma(Ba_1)} \mathrm{bdd}(Ba_0),$$

whence

$$\mathrm{bdd}(Ba_0) \cap \mathrm{cl}_\Sigma(B\bar{a}) \subseteq \mathrm{bdd}(Ba_0) \cap \mathrm{cl}_\Sigma(Ba_1) = \mathrm{bdd}(B).$$

So $a_0, \dots, a_{n-1}, \bar{a}$ witness weak n-Σ-ampleness of Φ over B, again a contradiction. □

Lemma 2.3.14. *Suppose $B \mathop{\smile\!\!\!\!\!\!|}_A a_0 \dots a_n$. If $a_0, \dots, a_n$ witness (weak) n-Σ-ampleness over A, they witness (weak) n-Σ-ampleness over B.*

Proof. Clearly $B \mathop{\smile\!\!\!\!\!\!|}_{a_0 \dots a_{i-1} A} a_0 \dots a_{i+1} A$, so Lemma 2.1.9 yields

$$\mathrm{cl}_\Sigma(Ba_0 \dots a_{i-1} a_i) \cap \mathrm{cl}_\Sigma(Ba_0 \dots a_{i-1} a_{i+1}) = \mathrm{cl}_\Sigma(Ba_0 \dots a_{i-1})$$

for $i < n$ in the ordinary and for $1 \le i < n$ in the weak case. In the latter, since $\mathrm{bdd}(Aa_0) \cap \mathrm{cl}_\Sigma(Aa_1) = \mathrm{bdd}(A)$ by assumption, Lemma 2.1.9 also gives

$$\mathrm{bdd}(Ba_0) \cap \mathrm{cl}_\Sigma(Ba_1) = \mathrm{bdd}(B).$$

Next, $a_{i+1} \mathop{\smile\!\!\!\!\!\!|}_{Aa_0 \dots a_i} B$, whence $a_{i+1} \mathop{\smile\!\!\!\!\!\!|}_{\mathrm{cl}_\Sigma(Aa_0 \dots a_i)} \mathrm{cl}_\Sigma(Ba_i)$ by Lemma 2.1.4. Now $a_{i+1} \mathop{\smile\!\!\!\!\!\!|}_{\mathrm{cl}_\Sigma(Aa_i)} a_0 \dots a_{i-1}$ implies $a_{i+1} \mathop{\smile\!\!\!\!\!\!|}_{\mathrm{cl}_\Sigma(Aa_i)} \mathrm{cl}_\Sigma(Aa_0 \dots a_i)$, whence

$$a_{i+1} \mathop{\smile\!\!\!\!\!\!|}_{\mathrm{cl}_\Sigma(Ba_i)} a_0 \dots a_{i-1}$$

for $1 \le i < n$ by transitivity.

Finally, $a_n \mathop{\smile\!\!\!\!\!\!|}_{\mathrm{cl}_\Sigma(A)} \mathrm{cl}_\Sigma(B)$ by Lemma 2.1.4, so $a_n \mathop{\smile\!\!\!\!\!\!|}_{\mathrm{cl}_\Sigma(B)} a_0$ would imply $a_n \mathop{\smile\!\!\!\!\!\!|}_{\mathrm{cl}_\Sigma(A)} a_0$, a contradiction. Hence $a_n \not\mathop{\smile\!\!\!\!\!\!|}_{\mathrm{cl}_\Sigma(B)} a_0$. In the weak case, $a_n \mathop{\smile\!\!\!\!\!\!|}_A B$ and $a_n \not\mathop{\smile\!\!\!\!\!\!|}_A a_0$ yield directly $a_n \not\mathop{\smile\!\!\!\!\!\!|}_B a_0$. □

Lemma 2.3.15. *Let Ψ be an $\emptyset$-invariant family of types. If Φ and Ψ are not (weakly) n-Σ-ample, neither is $\Phi \cup \Psi$.*

Proof. Suppose $\Phi \cup \Psi$ is weakly n-Σ-ample, as witnessed by $a_0, \dots, a_n = bc$ over some parameters A, where b and c are tuples of realizations of partial types in Φ and Ψ, respectively. Observe that $a_0, \dots, a_{n-1}, c$ satisfy the conditions $(2) - (4)$ of the weak n-Σ-ample definition. As Ψ is not weakly n-Σ-ample, we must have $c \mathop{\smile\!\!\!\!\!\!|}_A a_0$. Put $a_0' = \mathrm{Cb}(bc/a_0 A)$. Then $\mathrm{tp}(a_0'/A)$ is internal to $\mathrm{tp}(b/A)$ by Lemma 1.3.4. Put

$$a_n' = \mathrm{Cb}(a_0'/a_n A).$$

Then $\mathrm{tp}(a_n'/A)$ is $\mathrm{tp}(a_0'/A)$-internal and hence $\mathrm{tp}(b/A)$-internal. Note that $a_n \not\mathop{\smile\!\!\!\!\!\!|}_A a_0$ implies $a_n \not\mathop{\smile\!\!\!\!\!\!|}_A a_0'$, whence

$$a_n' \not\mathop{\smile\!\!\!\!\!\!|}_A a_0' \quad \text{and} \quad a_n' \not\mathop{\smile\!\!\!\!\!\!|}_A a_0.$$

Moreover $a_n' \in \operatorname{bdd}(Aa_n)$, so $a_0, \ldots, a_{n-1}, a_n'$ witness weak n-Σ-ampleness over A.

As $\operatorname{tp}(a_n'/A)$ is $\operatorname{tp}(b/A)$-internal, there are $B \mathop{\smile\!\!\!\!\!\mid}_A a_n'$ and a tuple $\bar{b}$ of realizations of $\operatorname{tp}(b/A)$ with $a_n' \in \operatorname{dcl}(B\bar{b})$. We may assume

$$B \mathop{\smile\!\!\!\!\!\mid}_{Aa_n'} a_0 \ldots a_{n-1},$$

whence $B \mathop{\smile\!\!\!\!\!\mid}_A a_0 \ldots a_{n-1}a_n'$. Hence, we obtain that $a_0, \ldots, a_{n-1}, a_n'$ witness weak n-Σ-ampleness over B by Lemma 2.3.14. As $a_n' \in \operatorname{dcl}(B\bar{b})$, this contradicts non weak n-Σ-ampleness of Φ by Lemma 2.3.13.

The proof in the ordinary case is analogous, replacing A by $\operatorname{cl}_\Sigma(A)$. □

Corollary 2.3.16. *For $i < \alpha$ let Φ_i be an $\emptyset$-invariant family of partial types. If Φ_i is not (weakly) n-Σ-ample for all $i < \alpha$, neither is $\bigcup_{i<\alpha} \Phi_i$.*

Proof. This just follows from the finite character of forking and Lemma 2.3.15. Assume $a_0, \ldots, a_n$ witness weak n-Σ-ampleness of $\bigcup_{i<\alpha} \Phi_i$ over a set A, where $a_n = (b_i : i < \alpha)$ and each b_i is a tuple of realizations of Φ_i. Then by the finite character of forking there is some finite subtuple $a_n' = (b_{i_0}, \ldots, b_{i_l})$ of a_n such that $a_n' \not\mathop{\smile\!\!\!\!\!\mid}_A a_0$ and so, $a_0, \ldots, a_n'$ witness weak n-Σ-ampleness of $\Phi_{i_0} \cup \ldots \cup \Phi_{i_l}$. But this is impossible since any finite union of Φ_i's is not weakly n-Σ-ample by Lemma 2.3.15.

The proof in the ordinary case is analogous, replacing A by $\operatorname{cl}_\Sigma(A)$. □

Lemma 2.3.17. *If the family of $\emptyset$-conjugates of $\operatorname{tp}(a/A)$ is not (weakly) n-Σ-ample and $a \mathop{\smile\!\!\!\!\!\mid} A$, then the family of $\emptyset$-conjugates of $\operatorname{tp}(a)$ is not (weakly) n-Σ-ample.*

Proof. Suppose $\operatorname{tp}(a)$ is (weakly) n-Σ-ample, as witnessed by $a_0, \ldots, a_n$ over some parameters B, where $a_n = (b_i : i < k)$ is a tuple of realizations of $\operatorname{tp}(a)$. For each $i < k$ choose $B_i \mathop{\smile\!\!\!\!\!\mid}_{b_i} (B, a_0 \ldots a_n, B_j : j < i)$ with $B_i b_i \equiv Aa$. Then $B_i \mathop{\smile\!\!\!\!\!\mid} b_i$, whence $(B_i : i < k) \mathop{\smile\!\!\!\!\!\mid} Ba_0 \ldots a_n$. Then $a_0, \ldots, a_n$ witness (weak) n-Σ-ampleness over $(B, B_i : i < k)$ by Lemma 2.3.14, a contradiction, since $\operatorname{tp}(b_i/B_i)$ is an $\emptyset$-conjugate of $\operatorname{tp}(a/A)$ for all $i < k$. □

Corollary 2.3.18. *Let Ψ be an $\emptyset$-invariant family of types. If Ψ is Φ-internal and Φ is not (weakly) n-Σ-ample, neither is Ψ.*

Proof. Immediate from Lemmas 2.3.13 and 2.3.17. □

Theorem 2.3.19. *Let Ψ be an $\emptyset$-invariant family of types. If Ψ is Φ-analysable and Φ is not (weakly) n-Σ-ample, neither is Ψ.*

Proof. Suppose Ψ is (weakly) n-Σ-ample exemplified by $a_0, \ldots, a_n$ over some parameters A, where a_n is a tuple of realizations of Ψ.

For the n-Σ-ample case, put $a'_n = \ell_1^\Phi(a_n/\mathrm{cl}_\Sigma(A) \cap \mathrm{bdd}(Aa_n))$. Then a_n and a'_n are domination-equivalent over $\mathrm{cl}_\Sigma(A) \cap \mathrm{bdd}(Aa_n)$ by Theorem 2.2.7; moreover a_n and hence a'_n are independent of $\mathrm{cl}_\Sigma(A)$ over $\mathrm{cl}_\Sigma(A) \cap \mathrm{bdd}(Aa_n)$ by Lemma 2.1.3, so a_n and a'_n are domination-equivalent over $\mathrm{cl}_\Sigma(A)$. Thus $a_0, \ldots, a_{n-1}, a'_n$ witness non-Σ-ampleness over A, in contradiction with Corollary 2.3.18.

For the weak case we put $a'_n = \ell_1^\Phi(a_n/A)$. So a_n and a'_n are domination-equivalent over A, whence $a'_n \not\perp_A a_0$. Thus $a_0, \ldots, a_{n-1}, a'_n$ witness weak non-Σ-ampleness over A, contradicting again Corollary 2.3.18. □

2.4 Analysability of Canonical Bases

As an immediate corollary to Theorem 2.3.19, we obtain the following:

Theorem 2.4.1. *Suppose every type in T is non-orthogonal to a regular type, and let Σ be the family of all n-ample regular types. Then T is non weakly n-Σ-ample.*

Proof. Clearly, as all n-ample types belong to Σ, every n-ample type is not weakly n-Σ-ample. In addition, a non n-ample type is not weakly Σ-ample by Remark 2.3.11. So all regular types are not weakly n-Σ-ample. But every type is analysable in regular types by the non-orthogonality hypothesis and Theorem 1.3.8. □

Corollary 2.4.2. *Suppose every type in T is non-orthogonal to a regular type. Then $\mathrm{tp}(\mathrm{Cb}(a/b)/a)$ is analysable in the family of all non one-based regular types, for all a, b.*

Proof. This is just Theorem 2.4.1 for $n = 1$. □

Remark 2.4.3. In fact the proof of Theorem 2.4.1 shows more. In any simple theory, if Σ is the family of all n-ample types, then T is non weakly n-Σ-

ample. In particular, $\mathrm{tp}(\mathrm{Cb}(a/b)/a)$ is analysable in the family of all non one-based types, for all a, b.

Note that a forking extension of a non one-based regular types may be one-based. For instance, in the free pseudoplane any type of Lascar rank ω is non one-based, but all its forking extensions are one-based; note that in such a theory, any type of finite Lascar rank is one-based.

Corollary 2.4.2 is due to Chatzidakis for types of finite SU-rank in simple theories [19, Proposition 1.14]. In fact, she even obtains $\mathrm{tp}(\mathrm{Cb}(a/b)/\mathrm{bdd}(a)\cap \mathrm{bdd}(b))$ to be analysable in the family of non one-based types of rank 1.

In infinite rank, one has to work modulo types of smaller rank. Let Σ_α be the collection of all partial types of SU-rank $< \omega^\alpha$. To easer notation, we will denote the closure operator associated to Σ_α by cl_α.

Proposition 2.4.4. *If* $\mathrm{tp}(a/A)$ *is analysable in a family of types of* SU*-rank* $< \omega^\alpha$*, then* $\mathrm{SU}(a/A) < \omega^\alpha$*. Hence,*

$$\mathrm{cl}_\alpha(A) = \{b : \mathrm{SU}(b/A) < \omega^\alpha\}.$$

Proof. Right to left inclusion is trivial. For the other, suppose, towards a contradicition, that $\mathrm{SU}(a/A) \geq \omega^\alpha$ but $\mathrm{tp}(a/A)$ is Σ_α-analysable. Then consider $A' \supseteq A$ such that $\mathrm{SU}(a/A') = \omega^\alpha$ and let $a' = \ell_1^{\Sigma_\alpha}(a/A')$. As $\mathrm{tp}(a/A')$ is Σ_α-analysable, $a' \sqsupseteq_{A'} a$ by Theorem 2.2.7(2). Let B be a set extending A' such that $B \mathop{\smile\!\!\!\!\!\mid}_{A'} a'$, and let $\bar b$ be a tuple of realizations of types of SU-rank $< \omega^\alpha$ be such that $a' \subseteq \mathrm{bdd}(B\bar b)$. As $a' \mathop{\smile\!\!\!\!\!\mid}_{A'} B$, we have $a \mathop{\smile\!\!\!\!\!\mid}_{A'} B$ and hence, $a' \sqsupseteq_B a$. Now, since $a' \not\subseteq \mathrm{bdd}(A')$, there is a finite subtuple $\bar b_0$ of $\bar b$ with $a' \not\mathop{\smile\!\!\!\!\!\mid}_B \bar b_0$; thus, $a \not\mathop{\smile\!\!\!\!\!\mid}_B \bar b_0$. In addition, note that $\mathrm{SU}(\bar b_0/B) < \omega^\alpha$ and $\mathrm{SU}(a/B\bar b_0) < \mathrm{SU}(a/B) = \omega^\alpha$; but then the Lascar inequalities imply that

$$\mathrm{SU}(a/A') = \mathrm{SU}(a/B) \leq \mathrm{SU}(a\bar b_0/B) \leq \mathrm{SU}(a/B\bar b_0) \oplus \mathrm{SU}(\bar b_0/B) < \omega^\alpha,$$

a contradiction. □

Now we state and prove Chatzidakis' Theorem for types of arbitrary ordinal SU-rank. Let $\mathcal{P}_\alpha$ be the family of non Σ_α-based types of SU-rank ω^α.

Theorem 2.4.5. *Let* $b = \mathrm{bdd}(\mathrm{Cb}(a/\mathrm{cl}_\alpha(b)))$ *be such that* $\mathrm{SU}(b) < \omega^{\alpha+1}$ *for some ordinal* α *and let* $A = \mathrm{cl}_\alpha(b) \cap \mathrm{cl}_\alpha(a)$*. Then* $\mathrm{tp}(b/A)$ *is* $(\Sigma_\alpha \cup \mathcal{P}_\alpha)$*-analysable.*

Proof. Firstly, if $a \in \mathrm{cl}_\alpha(b)$ then $a = b \in A$. Similarly, if $b \in \mathrm{cl}_\alpha(a)$ then $b \in A$; in both cases $\mathrm{tp}(b/A)$ is trivially $(\Sigma_\alpha \cup \mathcal{P}_\alpha)$-analysable. Hence we may assume $a \notin \mathrm{cl}_\alpha(b)$ and $b \notin \mathrm{cl}_\alpha(a)$.

Suppose towards a contradiction that the result is false and consider a counterexample a, b with $\mathrm{SU}(b)$ minimal modulo ω^α and then $\mathrm{SU}(b/\mathrm{cl}_\alpha(a))$ being maximal modulo ω^α. Note that this implies

$$\omega^\alpha \leq \mathrm{SU}(b/a) \leq \mathrm{SU}(b/A) \leq \mathrm{SU}(b) < \omega^{\alpha+1}.$$

Clearly (after adding parameters) we may assume $A = \mathrm{cl}_\alpha(\emptyset)$; this is possible by Lemma 2.1.5. Then for any c the type $\mathrm{tp}(c)$ is $(\Sigma_\alpha \cup \mathcal{P}_\alpha)$-analysable if and only if $\mathrm{tp}(c/A)$ is.

Claim. *We may assume $a = \mathrm{Cb}(b/\mathrm{cl}_\alpha(a))$.*

Proof. Put $\tilde{a} = \mathrm{Cb}(b/\mathrm{cl}_\alpha(a))$ and $\tilde{b} = \mathrm{Cb}(\tilde{a}/\mathrm{cl}_\alpha(b))$. Then $\tilde{a} \in \mathrm{cl}_\alpha(a)$ and $a \mathop{\smile\!\!\!\!\!\!\,|}_{\tilde{a}} b$. Hence $\mathrm{cl}_\alpha(b) = \mathrm{cl}_\alpha(\tilde{b})$ by Lemma 2.1.8, and $\mathrm{tp}(\tilde{b})$ is not $(\Sigma_\alpha \cup \mathcal{P}_\alpha)$-analysable either. Thus the pair $\tilde{a}, \tilde{b}$ also forms a counterexample. Moreover, $\mathrm{SU}(b)$ equals $\mathrm{SU}(\tilde{b})$ modulo ω^α and $\mathrm{SU}(b/\mathrm{cl}_\alpha(a)) = \mathrm{SU}(b/\mathrm{cl}_\alpha(\tilde{a}))$ equals $\mathrm{SU}(\tilde{b}/\mathrm{cl}_\alpha(\tilde{a}))$ modulo ω^α. $\square$

Since a is definable over a finite part of a Morley sequence in $\mathrm{Lstp}(b/a)$ by supersimplicity of $\mathrm{tp}(b)$, we see that $\mathrm{SU}(a) < \omega^{\alpha+1}$. On the other hand, $a \notin \mathrm{cl}_\alpha(b)$ implies $\mathrm{SU}(a/b) \geq \omega^\alpha$.

Let $\hat{a} \subseteq \mathrm{bdd}(a)$ and $\hat{b} \subseteq \mathrm{bdd}(b)$ be maximal $(\Sigma_\alpha \cup \mathcal{P}_\alpha)$-analysable. That is, $\hat{a} = \ell_\infty^{\Sigma_\alpha \cup \mathcal{P}_\alpha}(a)$ and $b = \ell_\infty^{\Sigma_\alpha \cup \mathcal{P}_\alpha}(b)$. Then $a \notin \mathrm{cl}_\alpha(\hat{a})$ and $b \notin \mathrm{cl}_\alpha(\hat{b})$, and $\mathrm{tp}(a/\hat{a})$ and $\mathrm{tp}(b/\hat{b})$ are foreign to $\Sigma_\alpha \cup \mathcal{P}_\alpha$. Since $\mathrm{Cb}(\hat{a}/b)$ and $\mathrm{Cb}(\hat{b}/a)$ are $(\Sigma_\alpha \cup \mathcal{P}_\alpha)$-analysable, we obtain

$$a \mathop{\smile\!\!\!\!\!\!\,|}_{\hat{a}} \hat{b} \quad \text{and} \quad b \mathop{\smile\!\!\!\!\!\!\,|}_{\hat{b}} \hat{a}.$$

Claim. $\mathrm{tp}(b/\hat{b})$ *and* $\mathrm{tp}(a/\hat{a})$ *are both Σ_α-based.*

Proof. Let Φ be the family of Σ_α-based types of SU-rank ω^α. Then $\mathrm{tp}(a/\hat{a})$ is $(\Sigma_\alpha \cup \mathcal{P}_\alpha \cup \Phi)$-analysable, but foreign to $\Sigma_\alpha \cup \mathcal{P}_\alpha$. Put $a_0 = \ell_1^\Phi(a/\hat{a})$ and $b_0 = \ell_1^\Phi(b/\hat{b})$. Then $a \sqsupseteq_{\hat{a}} a_0$ and $b \sqsupseteq_{\hat{b}} b_0$ by Theorem 2.2.7(3); as $a \mathop{\smile\!\!\!\!\!\!\,|}_{\hat{a}} \hat{b}$ and $b \mathop{\smile\!\!\!\!\!\!\,|}_{\hat{b}} \hat{a}$ we even have $a \sqsupseteq_{\hat{a}\hat{b}} a_0$ and $b \sqsupseteq_{\hat{a}\hat{b}} b_0$. Since $a \not\mathop{\smile\!\!\!\!\!\!\,|}_{\hat{a}\hat{b}} b$ we obtain

$a_0 \not\perp_{\hat{a}\hat{b}} b_0$. Moreover, $\mathrm{tp}(a_0/\hat{a})$ and $\mathrm{tp}(b_0/\hat{b})$ are Σ_α-based by Theorem 2.3.19 (or Theorem [68, Theorem 11]).

On the other hand, as $a_0 \not\perp_{\hat{b}} b_0$, we see that $b' = \mathrm{Cb}(a_0/\mathrm{cl}_\alpha(b_0))$ is not contained in $\hat{b}$ and hence is not $(\Sigma_\alpha \cup \mathcal{P}_\alpha)$-analysable. So a_0, b' is another counterexample; by minimality of SU-rank b and b' have the same SU-rank modulo ω^α, whence $b \in \mathrm{cl}_\alpha(b_0)$. Hence $\mathrm{tp}(b/\hat{b})$ is Σ_α-based, as is $\mathrm{tp}(a/\hat{a})$ since $a = \mathrm{Cb}(b/a)$ and $a \perp_{\hat{a}} \hat{b}$. □

Claim. $\mathrm{cl}_\alpha(a, \hat{b}) = \mathrm{cl}_\alpha(b, \hat{a}) = \mathrm{cl}_\alpha(a, b)$.

Proof. As $\mathrm{tp}(a/\hat{a})$ is Σ_α-based, we have

$$a \mathop{\perp}_{\mathrm{cl}_\alpha(a)\cap\mathrm{cl}_\alpha(\hat{a}b)} \hat{a}b,$$

whence

$$\mathrm{cl}_\alpha(a) \mathop{\perp}_{\mathrm{cl}_\alpha(a)\cap\mathrm{cl}_\alpha(\hat{a}b)} b$$

by Lemma 2.1.4. Thus $a = \mathrm{Cb}(b/\mathrm{cl}_\alpha(a)) \in \mathrm{cl}_\alpha(\hat{a}b)$. Similarly $b \in \mathrm{cl}_\alpha(\hat{b}a)$. □

Let now $(b)^\frown(b_j : j < \omega)$ be a Morley sequence in $\mathrm{tp}(b/a)$ and let $\hat{b}_j$ represent the part of b_j corresponding to $\hat{b}$. Then $(\hat{b}_j : j < \omega)$ is a Morley sequence in $\mathrm{tp}(\hat{b}/\hat{a})$ since $a \perp_{\hat{a}} \hat{b}$. As $\mathrm{SU}(\hat{b}) < \infty$ there is some minimal $m < \omega$ such that $\hat{a} = \mathrm{Cb}(\hat{b}/\hat{a}) \in \mathrm{cl}_\alpha(\hat{b}, \hat{b}_j : j < m)$. Then $m > 0$, as otherwise $\mathrm{cl}_\alpha(b) = \mathrm{cl}_\alpha(\hat{a}, b) \ni a$, which is impossible. Moreover, $a \in \mathrm{cl}_\alpha(\hat{a}, b_j)$ for all $j < m$ by invariance and hence, $a \in \mathrm{cl}_\alpha(\hat{b}, b_j : j < m)$.

Put $b' = \mathrm{Cb}(b_j : j < m/\mathrm{cl}_\alpha(b))$. Then $(b_j : j < m) \perp_{b'\hat{b}} \mathrm{cl}_\alpha(b)$, so by Lemma 2.1.4

$$\mathrm{cl}_\alpha(\hat{b}, b_j : j < m) \mathop{\perp}_{\mathrm{cl}_\alpha(b',\hat{b})} \mathrm{cl}_\alpha(b).$$

Then $a \perp_{\mathrm{cl}_\alpha(b',\hat{b})} \mathrm{cl}_\alpha(b)$, so $b = \mathrm{Cb}(a/\mathrm{cl}_\alpha(b)) \in \mathrm{cl}_\alpha(b', \hat{b})$. As $b \notin \mathrm{cl}_\alpha(\hat{b})$ we obtain $b' \notin \mathrm{cl}_\alpha(\hat{b})$.

Claim. $\mathrm{tp}(b'/\mathrm{cl}_\alpha(b') \cap \mathrm{cl}_\alpha(b_j : j < m))$ *is not* $(\Sigma_\alpha \cup \mathcal{P}_\alpha)$*-analysable.*

Proof. Note first that $(b_j : j < m) \perp_a b$ implies

$$\mathrm{cl}_\alpha(b_j : j < m) \mathop{\perp}_{\mathrm{cl}_\alpha(a)} \mathrm{cl}_\alpha(b)$$

by Lemma 2.1.4, whence

$$\mathrm{cl}_\alpha(b') \cap \mathrm{cl}_\alpha(b_j : j < m) \subseteq \mathrm{cl}_\alpha(b) \cap \mathrm{cl}_\alpha(a) = \mathrm{cl}_\alpha(\emptyset).$$

As $b \in \mathrm{cl}_\alpha(b', \hat{b})$ and $\mathrm{tp}(b/\hat{b})$ is not $(\Sigma_\alpha \cup \mathcal{P}_\alpha)$-analysable, neither is $\mathrm{tp}(b'/\hat{b})$, nor *a fortiori* $\mathrm{tp}(b'/\mathrm{cl}_\alpha(\emptyset))$. □

As $b' = \mathrm{Cb}(b_j : j < m/\mathrm{cl}_\alpha(b'))$, the pair $(b_j : j < m), b'$ forms another counterexample. By minimality $\mathrm{SU}(b)$ equals $\mathrm{SU}(b')$ modulo ω^α, which implies $\mathrm{cl}_\alpha(b) = \mathrm{cl}_\alpha(b')$.

As $\mathrm{tp}(b_j/\hat{b}_j)$ is foreign to $\Sigma_\alpha \cup \mathcal{P}_\alpha$ and $\mathrm{tp}(\hat{b})$ is $(\Sigma_\alpha \cup \mathcal{P}_\alpha)$-analysable, we obtain $\hat{b} \perp_{(\hat{b}_j : j<m)} (b_j : j < m)$ and hence by Lemma 2.1.4

$$\hat{b} \underset{\mathrm{cl}_\alpha(\hat{b}_j : j<m)}{\perp} \mathrm{cl}_\alpha(b_j : j < m).$$

On the other hand, as $\hat{a} \in \mathrm{cl}_\alpha(\hat{b}, \hat{b}_j : j < m)$ but $\hat{a} \notin \mathrm{cl}_\alpha(\hat{b}_j : j < m)$ by minimality of m, we get

$$\mathrm{SU}(\hat{b}/\mathrm{cl}_\alpha(\hat{b}_j : j < m)) >_\alpha \mathrm{SU}(\hat{b}/\hat{a}, \mathrm{cl}_\alpha(\hat{b}_j : j < m)),$$

where the subindex α indicates modulo ω^α.

Moreover, as $\hat{b} \perp_{\hat{a}} a$ we get $\hat{b} \perp_{\mathrm{cl}_\alpha(\hat{a})} \mathrm{cl}_\alpha(a)$, i.e.,

$$\mathrm{SU}(\hat{b}/\mathrm{cl}_\alpha(\hat{a})) = \mathrm{SU}(\hat{b}/\mathrm{cl}_\alpha(a)).$$

Since $\mathrm{cl}_\alpha(b) = \mathrm{cl}_\alpha(b')$ and $b \in \mathrm{cl}_\alpha(a\hat{b})$ we obtain

$$\begin{aligned}
\mathrm{SU}(b'/\mathrm{cl}_\alpha(b_j : j < m)) &=_\alpha \mathrm{SU}(b/\mathrm{cl}_\alpha(b_j : j < m)) \\
&\geq_\alpha \mathrm{SU}(\hat{b}/\mathrm{cl}_\alpha(b_j : j < m)) =_\alpha \mathrm{SU}(\hat{b}/\mathrm{cl}_\alpha(\hat{b}_j : j < m)) \\
&>_\alpha \mathrm{SU}(\hat{b}/\hat{a}, \mathrm{cl}_\alpha(\hat{b}_j : j < m)) =_\alpha \mathrm{SU}(\hat{b}/\mathrm{cl}_\alpha(\hat{a})) \\
&=_\alpha \mathrm{SU}(\hat{b}/\mathrm{cl}_\alpha(a)) =_\alpha \mathrm{SU}(b/\mathrm{cl}_\alpha(a)),
\end{aligned}$$

contradicting the maximality of $\mathrm{SU}(b/\mathrm{cl}_\alpha(a))$ modulo ω^α. This finishes the proof. □

As a corollary we obtain Chatzidakis' Theorem for the finite SU-rank case:

Corollary 2.4.6. *Let* $b = \mathrm{bdd}(\mathrm{Cb}(a/b))$ *be of finite* SU*-rank. Then* $\mathrm{tp}(b/\mathrm{bdd}(b) \cap \mathrm{bdd}(a))$ *is analysable in the family of all non one-based types of* SU*-rank* 1.

2.5 Around the Canonical Base Property

Let Σ^{nob} be the family of non one-based regular types. In fact, when we deal with types of finite Lascar rank, it is enough to consider the family of non one-based Lascar rank one types.

Definition 2.5.1. A theory has the *Canonical Base Property (CBP)* if for all tuples a, b $\mathrm{tp}(\mathrm{Cb}(a/b)/a)$ is almost Σ^{nob}-internal.

Remark 2.5.2. Chatzidakis [19] has proved that the Canonical Base Property implies that $\mathrm{tp}(\mathrm{Cb}(a/b)/\mathrm{bdd}(a) \cap \mathrm{bdd}(b))$ is almost Σ^{nob}-internal for $\mathrm{SU}(b) < \omega$. In addition, she has shown that the CBP coincide with the *Uniform Canonical Base Property* introduced by Moosa and Pillay [44].

Example. Types of finite Lascar rank have the CBP in:

- Differential closed fields in characteristic zero [59].
- Generic difference fields [19, 59].
- Compact complex spaces [44].

It had been conjectured that all simple theories of finite rank have the CBP, nevertheless there is a probable counter-example due to Hrushovski. In fact, Hrushovski has recently rewritten another version of a such counter-example.

Example. If a theory does not have finite Lascar rank then the CBP might fail. For this consider in the theory of the free pseudoplane two elements a and b such that $\mathrm{SU}(a/b) = 2$. Then $\mathrm{SU}(\mathrm{Cb}(a/b)/a) = 2$ and so, $\mathrm{tp}(\mathrm{Cb}(a/b)/a)$ is analysable in the family of non one-based Lascar rank 1 types but it is not almost internal to that family.

It follows from Theorem 2.4.2 that the failure of the CBP in a supersimple theory is given by the non Σ^{nob}-flatness of $\mathrm{tp}(\mathrm{Cb}(a/b)/a)$.

One may try to define strong Σ-basedness working with cl^1_Σ instead of cl_Σ. However, we cannot adapt the previous results as we have used the fact that cl_Σ is a closure operator. On the other hand, we define *a priori* the equivalent version of the CBP in ample terms.

Definition 2.5.3. A family of types Φ is weak n-Σ-ample with respect to cl^1_Σ if there are $a_0, \ldots a_n$, where a_n is a tuple of realizations of Φ, such that over some set of parameters A:

1. $a_n \not\perp_A a_0$.
2. $a_{i+1} \perp_A a_0 \ldots a_i$ for $1 \leq i < n$.
3. $\mathrm{bdd}(Aa_0) \cap \mathrm{cl}^1_\Sigma(Aa_1) = A$.
4. $\mathrm{cl}^1_\Sigma(Aa_0 \ldots a_{i-1}a_i) \cap \mathrm{cl}^1_\Sigma(Aa_0 \ldots a_{i-1}a_{i+1}) = \mathrm{cl}^1_\Sigma(Aa_0 \ldots a_{i-1})$ for $1 \leq i < n$.

Observe that weak n-Σ-ample implies weak n-Σ-ample with respect to cl^1_Σ, and the latter implies n-ample.

Remark 2.5.4. The equivalence between the non weak 1-Σ^{nob}-ampleness with respect to $\mathrm{cl}^1_{\Sigma^{nob}}$ and strongly Σ^{nob}-basedness with respect to $\mathrm{cl}^1_{\Sigma^{nob}}$ (i.e., the CBP) breaks down.

An inspection of the proofs given in section 2.3 around the preservation of non (weak) n-Σ-ampleness under internality/analysability yields the following result. The next lemma is just an adaption of lemmas 2.3.13, 2.3.14, 2.3.17, and Corollary 2.3.16, 2.3.18; we will omit the proof.

Lemma 2.5.5.

1. *If Φ is non weak n-Σ-ample with respect to cl^1_Σ, neither is $\mathrm{tp}(b/B)$ for $B \supseteq A$ and $b \in \mathrm{bdd}(\bar{a}B)$, where $\bar{a}$ is a tuple of realizations of types in Φ over A.*
2. *Assume $B \perp_A a_0 \ldots a_n$. If $a_0, \ldots, a_n$ witness the non weak n-Σ-ample with respect to cl^1_Σ of Φ over A, then so do over B.*
3. *Let Φ_i be an $\emptyset$-invariant family of partial types for all $i < \alpha$. If Φ_i is non weak n-Σ-ample with respect to cl^1_Σ for every $i < \alpha$, neither is $\bigcup_{i<\alpha} \Phi_i$.*
4. *If $a \perp A$ and $\mathrm{tp}(a/A)$ is non weak n-Σ-ample with respect to cl^1_Σ, neither is $\mathrm{tp}(a)$.*
5. *Let Ψ be an $\emptyset$-invariant family of partial types. If Ψ is Φ-internal and Φ is non weak n-Σ-ample with respect to cl^1_Σ, neither is Ψ.*

Furthermore, the fact that in item (3) we require $\mathrm{bdd}(Aa_0) \cap \mathrm{cl}^1_\Sigma(Aa_1) = A$ allows us to apply the Theorem of levels.

Theorem 2.5.6. *Let Ψ be an $\emptyset$-invariant family of types. If Ψ is Φ-analysable and Φ is not weakly n-Σ-ample with respect to cl^1_Σ, neither is Ψ.*

2.6 Applications to groups

Kowalski and Pillay [41, Section 4] have given some consequences of Σ-basedness in the context of groups. In fact, they work in a theory with the CBP, but they remark that their results hold, with *Σ-analysable* instead of *almost Σ-internal*, in all stable strongly Σ-based theories.

Recall that a type-definable group is said to be *connected* if H is the smallest type-definable subgroup of H of bounded index.

Theorem 2.6.1. *Let G be a type-definable strongly Σ-based group in a stable theory.*

1. *If $H \leq G$ is connected with canonical parameter c, then $\mathrm{tp}(c)$ is Σ-analysable.*
2. *$G/Z(G)$ is Σ-analysable.*

Proof. See [41]. □

An inspection of their proof shows that mere simplicity of the ambient theory is sufficient, replacing centers by approximate centers and connectivity by local connectivity. Recall that the *approximate center* of a group G is

$$\tilde{Z}(G) = \{g \in G : [G : C_G(g)] < \infty\}.$$

A subgroup $H \leq G$ is *locally connected* if for all group-theoretic or model-theoretic conjugates H^σ of H, if H and H^σ are commensurate, then $H = H^\sigma$. Recall that two hyperdefinable groups H and H' are commensurable if $H \cap H'$ has bounded index in both H and H'. Locally connected subgroups and their cosets have canonical parameters; every subgroup is commensurable with a unique minimal locally connected subgroup, its *locally connected component*. For more details about the approximate notions, the reader is invited to consult [67, Definition 4.4.9 and Proposition 4.4.10].

Proposition 2.6.2. *Let G be a hyperdefinable strongly Σ-based group in a simple theory.*

1. *If $H \leq G$ is locally connected with canonical parameter c, then $\mathrm{tp}(c)$ is Σ-analysable.*
2. *$G/\tilde{Z}(G)$ is Σ-analysable.*

Proof. (1). Take $h \in H$ generic over c and $g \in G$ generic over c, h. Let d be the canonical parameter of gH. Then $\mathrm{tp}(gh/g, c)$ is the generic type of gH, so d is interbounded with $\mathrm{Cb}(gh/g, c)$, see [67, Lemma 4.5.19]. By Σ-basedness, $\mathrm{tp}(d/gh)$ is Σ-analysable. But $c \in \mathrm{dcl}(d)$, so $\mathrm{tp}(c/gh)$ is Σ-analysable, as is $\mathrm{tp}(c)$ since $c \perp gh$.

(2). For generic $g \in G$ put $H_g = \{(x, x^g) \in G \times G : x \in G\}$, and let H_g^{lc} be the locally connected component of H_g. Then $g\tilde{Z}(G)$ is interbounded with the canonical parameter of H_g^{lc}, so $\mathrm{tp}(g\tilde{Z}(G))$ is Σ-analysable, as is $G/\tilde{Z}(G)$. □

Finally we finish this chapter with a result due to Wagner. He show how to obtain an almost internal quotient group without assuming the CBP. Nevertheless, one have to quotient by a nilpotent subgroup.

Theorem 2.6.3. *Let G be an $\emptyset$-hyperdefinable Σ-based group in a simple theory. If G is supersimple or type-definable, there is a normal nilpotent $\emptyset$-hyperdefinable subgroup N such that G/N is almost Σ-internal. In particular, a supersimple or type-definable group G in a simple theory has a normal nilpotent hyperdefinable subgroup N such that G/N is almost Σ^{nob}-internal.*

Chapter 3

Stable forking independence

One of the main open problems in simplicity is the stable forking conjecture: when forking is witnessed by a stable formula? This is the case in all current known examples of simple theories, and of course in any stable theory by an obvious reason. Peretz showed that this is the case for simple theories of Lascar rank 2 [51]. Moreover, Wagner has used Peretz's ideas to show that forking independence is a stable relation for supersimple CM-trivial theories [69]. An $\emptyset$-invariant relation $R(x, y)$ is said to be *stable* if there is no infinite indiscernible sequence $(a_i b_i : i < \omega)$ such that $R(a_i, b_j)$ holds if and only if $i < j$. The mentioned result of Wagner is the following:

Theorem 3.0.1. *In a supersimple CM-trivial theory the relation $R(x; yz)$ defined by $x \mathop{\smile\!\!\!\!\!|}_z y$ is stable.*

In fact, Wagner's result also is valid for any simple CM-trivial theory with enough regular types by Theorem 2.4.1. Following Wagner's ideas, we will obtain a similar result for any simple theory but working over a $\mathrm{cl}_{\mathcal{P}}$-closed set where $\mathcal{P}$ is the family of non one-based types. Even though these results are not completely satisfactory, they represent a first step to attack and solve (being optimistic) the stable forking conjecture.

3.1 Forking independence and strong Σ-basedness

Let Σ be an $\emptyset$-family of partial types.

Wagner in addition proves in a quick way an almost immediate corollary. For convenience we offer a proof.

Corollary 3.1.1. *In a supersimple CM-trivial theory the relation $R(x; yz)$ defined by $x \mathop{\smile\!\!\!\!\!\!|}_{\mathrm{cl}_\Sigma(z)} y$ is stable.*

Proof. Suppose there is an infinite indiscernible sequence $(a_i : i \in \mathbb{Q})$ and tuples b and c such that $a_i \mathop{\smile\!\!\!\!\!\!|}_{\mathrm{cl}_\Sigma(c)} b$ if and only if $i > 0$. Put $b' = \mathrm{bdd}(bc) \cap \mathrm{cl}_\Sigma(c)$; so, $b \mathop{\smile\!\!\!\!\!\!|}_{b'} \mathrm{cl}_\Sigma(b')$, whence $b \mathop{\smile\!\!\!\!\!\!|}_{b'} \mathrm{cl}_\Sigma(c)$ since $\mathrm{cl}_\Sigma(b') = \mathrm{cl}_\Sigma(c)$. Therefore $a_i \mathop{\smile\!\!\!\!\!\!|}_{b'} b$ for $i > 0$. Furthermore, observe that $a_i \not\mathop{\smile\!\!\!\!\!\!|}_{b'} b$ for $i < 0$ as $\mathrm{cl}_\Sigma(b') = \mathrm{cl}_\Sigma(c)$. By Theorem 3.0.1 we obtain the desired contradiction. □

In order to drop the assumption on CM-triviality we will consider the family $\mathcal{P}$ of non one-based types types.

Theorem 3.1.2. *In a simple theory, the relation $R(x; yz)$ given by $x \mathop{\smile\!\!\!\!\!\!|}_{\mathrm{cl}_\mathcal{P}(z)} y$ is stable.*

Proof. Suppose not. Then by a compactness argument and Erdős-Rado we obtain an indiscernible sequence $I = (a_i : i \in \mathbb{Q})$ and tuples b and c such that

- $I^+ = (a_i : i > 0)$ is indiscernible over $I^- bc$,
- $I^- = (a_i : i < 0)$ is indiscernible over $I^+ bc$, and
- $a_i \mathop{\smile\!\!\!\!\!\!|}_{\mathrm{cl}_\mathcal{P}(c)} b$ if and only if $i > 0$.

We consider limit types with respect to the cut at 0. Let $p = \lim(I/I)$, $p^+ = \lim(I^+/Ibc)$ and $p^- = \lim(I^-/Ibc)$. By indiscernibility everything is well-defined and all these types are complete. By finite satisfiability p is an amalgamation base. Note that p^+ and p^- are extension of p. In fact, by finite satisfiability both are non forking extensions since p^+ and p^- do not fork over I^+ and I^-, respectively; whence, p^+,p^-, and p belong to the same amalgamation class. Let

$$A = \mathrm{Cb}(p) = \mathrm{Cb}(p^+) = \mathrm{Cb}(p^-) \in \mathrm{bdd}(I^+) \cap \mathrm{bdd}(I^-).$$

As p^+ does not fork over I^+ and p^- does not fork over I^- by finite satisfiability, we obtain

$$a_i \mathop{\smile\!\!\!\!\!\!|}_{A} I^+bc \text{ for all } i < 0, \quad \text{and} \quad a_i \mathop{\smile\!\!\!\!\!\!|}_{A} I^-bc \text{ for all } i > 0.$$

We consider first $e = \mathrm{cl}_{\mathcal{P}}(a_1) \cap \mathrm{bdd}(A)$. By Remark 2.4.3 $a_1 \mathop{\smile\!\!\!\!\!\!|}_{e} A$; moreover, since I remains indiscernible over $\mathrm{bdd}(A)$ we have $a_{-1} \equiv_{\mathrm{bdd}(A)} a_1$, whence $e = \mathrm{cl}_{\mathcal{P}}(a_{-1}) \cap \mathrm{bdd}(A)$; in particular, $a_{-1} \mathop{\smile\!\!\!\!\!\!|}_{e} A$ again by Remark 2.4.3. On the other hand, since $e \in \mathrm{bdd}(A)$ and $a_i \mathop{\smile\!\!\!\!\!\!|}_{A} b$ for $i \in \mathbb{Q}$ we obtain,

$$a_1 \mathop{\smile\!\!\!\!\!\!|}_{e} bc \quad \text{and} \quad a_{-1} \mathop{\smile\!\!\!\!\!\!|}_{e} bc.$$

Now put $c' = \mathrm{dcl}(bc) \cap \mathrm{cl}_{\mathcal{P}}(c)$ and note that $\mathrm{cl}_{\mathcal{P}}(c) = \mathrm{cl}_{\mathcal{P}}(c')$. Then $b \mathop{\smile\!\!\!\!\!\!|}_{c'} \mathrm{cl}_{\mathcal{P}}(c')$ by Lemma 2.1.3. As $a_1 \mathop{\smile\!\!\!\!\!\!|}_{c} b$ we obtain $\mathrm{cl}_{\mathcal{P}}(a_1) \mathop{\smile\!\!\!\!\!\!|}_{\mathrm{cl}_{\mathcal{P}}(c)} b$ again by Lemma 2.1.3, whence $\mathrm{cl}_{\mathcal{P}}(a_1) \mathop{\smile\!\!\!\!\!\!|}_{c'} b$ by transitivity and so, $\mathrm{cl}_{\mathcal{P}}(a_1) \mathop{\smile\!\!\!\!\!\!|}_{c'} bc$ since $c \subseteq c'$. Hence, $e \mathop{\smile\!\!\!\!\!\!|}_{c'} bc$ since $e \in \mathrm{cl}_{\mathcal{P}}(a_1)$. But then as $c' \in \mathrm{dcl}(bc)$ and $a_{-1} \mathop{\smile\!\!\!\!\!\!|}_{e} bc$ we obtain that $a_{-1} \mathop{\smile\!\!\!\!\!\!|}_{c'} bc$ by transitivity and hence, $a_{-1} \mathop{\smile\!\!\!\!\!\!|}_{\mathrm{cl}_{\mathcal{P}}(c)} b$ since $\mathrm{cl}_{\mathcal{P}}(c) = \mathrm{cl}_{\mathcal{P}}(c')$, a contradiction. □

Remark 3.1.3. If the theory is supersimple we can take $\mathcal{P}$ the family of all non one-based regular types.

This result generalizes the fact that in one-based theories the relation $R(x; yz)$ given by $x \mathop{\smile\!\!\!\!\!\!|}_{z} y$ is stable. In order to generalize Wagner's result, Theorem 3.0.1 one should solve the following question.

Question 3.1.4. Is the result true for $\mathcal{P}$ being the family of 2-ample types?

Chapter 4

Hyperimaginaries in simple CM-trivial theories

The model-theoretic treatment of hyperimaginaries was required to obtain canonical bases in simple theories and so, to develop the general theory of independence. However, we are really far from the completely understanding of such elements in proper simple theories. Nevertheless, we have satisfactory answers for supersimple theories and small theories. Any small theory eliminates all finitary hyperimaginaries [35], and all hyperimaginaries are eliminable in any supersimple theory [10]. Even more, canonical bases are described by certain *nice* imaginaries in supersimple theories [40]. However, this description is not completely satisfactory since such imaginaries are not definitions of stable formulas. This is related with another big problem treated in **chapter 3**: the stability of forking independence.

In this short chapter we will discuss elimination problems under the extra assumption of CM-triviality, see Section 1.4. Mainly, under this assumption forking independence is well-behaved and this allows us to understand canonical bases. This is reflected in Proposition 4.2.2, where we show that canonical bases are interbounded with finitary canonical bases. Then it turns out that every hyperimaginary in a simple CM-trivial theory is interbounded with a sequence of finitary hyperimaginaries. This is Theorem 4.2.4. Even more, in Corollary 4.2.5 we reduce elimination of hyperimaginaries to elimination of finitary ones. As an immediate corollary we obtain elimination of hyperimaginaries in any small simple CM-trivial theory using the mentioned theorem of Byunghan Kim [35], Corollary 4.2.6.

The results presented in this chapter come from discussions with Frank O. Wagner, while I was doing a five-days stay at the Institut Camile Jordan. All these results are collected in [50].

4.1 Elimination problems

The goal of this section is to recall some basic notions and to expose some useful results around hyperimaginaries. In addition, we present different kinds of elimination of hyperimaginaries problems as well as some lemmata on eliminability under the assumption of G-compactness. Even though some of these result are probably folklore, we cannot find in the literature.

Definition 4.1.1. A hyperimaginary h is *bounded* if it is has a bounded orbit under $\mathrm{Aut}(\mathfrak{C})$; it is *finitary* if it is definable over some finite tuple; and it is *quasi-finitary* if it bounded over some finite tuple.

The following remark tells us that this definition of finitary hyperimaginary coincide with Definition 1.1.14.

Remark 4.1.2. A hyperimaginary $h \in \mathrm{dcl}(a)$ if and only if h is interdefinable with a_E for some $\emptyset$-type-definable equivalence relation E.

Proof. See [12, Proposition 15.6]. □

Recall that a relation F is type-definable over a hyperimaginary e, if it is type-definable and e-invariant. Note that this is equivalent to saying that F is type-definable over a representative of e.

Definition 4.1.3. Let e be a hyperimaginary. An e-hyperimaginary is an equivalence class b_F of an arbitrary tuple b modulo a type-definable over e equivalence relation F. If b has finite length, we say that b_F is a finitary e-hyperimaginary.

First we shall relate elimination of quasi-finitary hyperimaginaries with the eliminability of finitary ones.

Lemma 4.1.4. *Let e be a finitary hyperimaginary. If T eliminates finitary hyperimaginaries, then every finitary e-hyperimaginary is interdefinable over e with a sequence of imaginaries.*

Proof. Let a be a finite tuple and $E(x,y)$ be an $\emptyset$-equivalence relation such that $e = a_E$. Let h a finitary hyperimaginary over e. So there is a finite tuple b and a type-definable equivalence relation $F_a(u,v) = F(u,v;a)$ over a with $h = b_{F_a}$. For any $a' \equiv a$, $F_{a'}(u,v)$ is an equivalence relation. Moreover, $F_a(u,v)$ is e-invariant and so, it only depends on the conjugates of a which belong to the E-class of a, that is, if $a'Ea$ and $a' \equiv a$, then $F_{a'}(u,v) \equiv F_a(u,v)$ (as partial types).

Let $p(z) = \mathrm{tp}(a)$ and type-define an equivalence relation by

$$xy\bar{E}uv \quad \Leftrightarrow \quad (p(x) \wedge p(u) \wedge E(x,u) \wedge F(y,v;x)) \vee xy = uv.$$

It is easy to see that h is interdefinable with $(ab)_{\bar{E}}$ over e. Moreover, $(ab)_{\bar{E}}$ is clearly finitary, and hence eliminable in T. So h is interdefinable over e with a sequence of imaginaries. □

Next lemma exhibits a criterium to eliminate hyperimaginaries; it is a kind of sandwich lemma.

Lemma 4.1.5. *Let h be a hyperimaginary and let a be a sequence of imaginaries such that $a \in \mathrm{bdd}(h)$ and $h \in \mathrm{dcl}(a)$. Then, h is eliminable.*

Proof. This result first appear in [43, Proof of Proposition 2.2]. For the reader may be easier to consult [12, Lemma 18.6] or [67, Lemma 3.6.3]. □

Lemma 4.1.6. *Let h, e be hyperimaginaries with $h \in \mathrm{bdd}(e)$. Then the set of e-conjugates of h is interdefinable with a hyperimaginary h'.*

Proof. For the proof one should see [10, Lemma 2.18]. □

Another important result on hyperimaginaries is due to Lascar and Pillay [43]; roughly speaking, they described bounded hyperimaginaries as sequences of finitary (bounded) hyperimaginaries. Their proof uses the description of a compact Hausdorff group as an inverse limit of compact Lie groups – Peter-Weyl's Theorem.

Theorem 4.1.7. *A bounded hyperimaginary is interdefinable with a sequence of finitary hyperimaginaries.*

Proof. This corresponds to [43, Theorem 4.15]. □

In fact, an alternative treatment of this theorem can be found in [16], where an approach to the Lascar-Pillay Theorem is initiated in model theoretic terms.

Now we need the following lemma:

Lemma 4.1.8. *Let e, h be hyperimaginaries such that $e \in \mathrm{bdd}(h)$ and let A be the set of all h-conjugates of e. Then there is a hyperimaginary c such that $\mathrm{Aut}(\mathfrak{C}/c) = \{f \in \mathrm{Aut}(\mathfrak{C}) : f(A) = A\}$.*

Proof. We refer the reader to [12, Proposition 15.28]. □

Proposition 4.1.9. *If T eliminates finitary hyperimaginaries, then T eliminates quasi-finitary hyperimaginaries.*

Proof. Let h be a quasi-finitary hyperimaginary and let a be a finite tuple of imaginaries such that $h \in \mathrm{bdd}(a)$. By Neumann's Lemma, consider $a' \equiv_h a$ with $\mathrm{acl}^{eq}(a) \cap \mathrm{acl}^{eq}(a') = \mathrm{acl}^{eq}(h)$. By Lemma 4.1.8, let h' be the hyperimaginary corresponding to the set of aa'-conjugates of h. Then h' is aa'-invariant, and hence finitary. It is thus interdefinable with a sequence e of imaginaries. In particular, note that e might be an infinite tuple but it is finitary as a hyperimaginary: $e \in \mathrm{dcl}(h') \subseteq \mathrm{dcl}(aa')$, indeed.

On the other hand, $h \in \mathrm{bdd}(a) \cap \mathrm{bdd}(a')$, as are all its aa'-conjugates. Thus $h' \in \mathrm{bdd}(a) \cap \mathrm{bdd}(a')$ and so, $e \subseteq \mathrm{acl}^{eq}(a) \cap \mathrm{acl}^{eq}(a') = \mathrm{acl}^{eq}(h)$. Moreover, observe that an automorphism $f \in \mathrm{Aut}(\mathfrak{C}/h')$ permutes the orbit of h under $\mathrm{Aut}(\mathfrak{C}/aa')$ and so, f must send h to one of its aa'-conjugates. As there is just a bounded number aa'-conjugates of h, $h \in \mathrm{bdd}(h') = \mathrm{bdd}(e)$. Now, by Theorem 4.1.7 applied to $T(e)$, there is a sequence h'' of finitary hyperimaginaries interdefinable with h over e. Since e is finitary, by Lemma 4.1.4 and elimination of finitary hyperimaginaries, h'' is interdefinable over e with a sequence e' of imaginaries. So $h \in \mathrm{dcl}(ee')$ and $e' \in \mathrm{dcl}^{eq}(eh)$. Moreover, $ee' \subseteq \mathrm{acl}^{eq}(h)$ since $e \in \mathrm{acl}^{eq}(h)$. Hence h is eliminable by Lemma 4.1.5. □

Next results will need the assumption of G-compactness except Lemma 4.1.11, where the hypothesis implies that the theory is G-compact.

Remark 4.1.10. The following are equivalent for a G-compact theory:

1. $a \equiv_A^{\mathrm{Ls}} b$ iff $a \equiv_A^{\mathrm{s}} b$ for all sequences a, b.
2. $\mathrm{Aut}(\mathfrak{C}/\mathrm{bdd}(A)) = \mathrm{Aut}(\mathfrak{C}/\mathrm{acl}^{eq}(A))$.
3. $\mathrm{bdd}(A) = \mathrm{dcl}(\mathrm{acl}^{eq}(A))$.

Proof. This standard observation is left to the reader. □

Lemma 4.1.11. *Assume that $a \equiv_A^{\mathrm{Ls}} b \Leftrightarrow a \equiv_A^{\mathrm{s}} b$ for all sequences a, b and for any set A. Let now h be a hyperimaginary and let e be a sequence of imaginaries such that h and e are interbounded. Then h is eliminable.*

Proof. It follows from Remark 4.1.10 that $\mathrm{bdd}(e) = \mathrm{dcl}(\mathrm{acl}^{eq}(e))$. Fix an enumeration $\bar{e}$ of $\mathrm{acl}^{eq}(e)$ and observe that $h \in \mathrm{dcl}(\bar{e})$ and $\bar{e} \in \mathrm{bdd}(h)$. Then apply Lemma 4.1.5 to eliminate h. □

Next result corresponds to [12, Proposition 18.27]. As far as we know it is the only reference in the literature; however, we offer a distinct proof.

Proposition 4.1.12. *Assume that the ambient theory is G-compact. Then, the theory eliminates all bounded hyperimaginaries if and only if $a \equiv^{\mathrm{Ls}} b \Leftrightarrow a \equiv^{\mathrm{s}} b$ for all sequences a, b.*

Proof. If the theory eliminates all bounded hyperimaginaries, then it is easy to see that $\mathrm{bdd}(\emptyset) = \mathrm{dcl}(\mathrm{acl}^{eq}(\emptyset))$; thus, we obtain $\mathrm{Lstp} = \mathrm{stp}$ over the empty set by Remark 4.1.10. For the other direction, consider $h \in \mathrm{bdd}(\emptyset)$ and let $\bar{a}$ be an enumeration of $\mathrm{acl}^{eq}(\emptyset)$. It is clear that h and $\bar{a}$ are interbounded. Hence, by Lemma 4.1.11, h is eliminable. □

Finally, we show that elimination of finitary hyperimaginaries implies equality between Lascar strong types and strong types over any set of parameters.

Lemma 4.1.13. *Suppose the theory is G-compact and assume further that it eliminates finitary hyperimaginaries. Then $a \equiv_A^{\mathrm{Ls}} b$ iff $a \equiv_A^{\mathrm{s}} b$ for all sequences a, b and for any set A.*

Proof. Since T is G-compact, it is enough to check the condition for finite A. But then $T(A)$ eliminates finitary hyperimaginaries by Lemma 4.1.4, and hence all bounded hyperimaginaries by Theorem 4.1.7. Now applying Proposition 4.1.12 we obtain that $a \equiv^{\mathrm{Ls}} b$ iff $a \equiv^{\mathrm{s}} b$ in $T(A)$. □

4.2 Hyperimaginaries and CM-triviality

In this section we assume the ambient theory to be simple. Our aim is to prove elimination of hyperimaginaries for small simple CM-trivial theories.

Lemma 4.2.1. *Let* $h \in \mathrm{bdd}(b)$*, then* $\mathrm{Cb}(a/h) \subseteq \mathrm{dcl}(ab) \cap \mathrm{bdd}(h)$*. In particular, the canonical base of a finite tuple over a quasi-finitary hyperimaginary is a finitary hyperimaginary. Furthermore, for any* $h \in \mathrm{dcl}(a)$ *we have* $\mathrm{Cb}(a/h) = \mathrm{dcl}(a) \cap \mathrm{bdd}(h)$.

Proof. Recall that the equality of Lascar strong types over b is invariant under $\mathrm{Aut}(\mathfrak{C}/b)$. Since $h \in \mathrm{bdd}(b)$, the relation $\equiv^{\mathrm{Ls}}_b$ refines $\equiv^{\mathrm{Ls}}_h$ and so, the class of a modulo the latter is fixed under $\mathrm{Aut}(\mathfrak{C}/ab)$. Hence, $\mathrm{Cb}(a/h) \in \mathrm{dcl}(ab) \cap \mathrm{bdd}(h)$. Thus, the "in particular" clause follows taking a and b finite tuples.

Finally, if $h \in \mathrm{dcl}(a)$ we have $\mathrm{Cb}(a/h) \in \mathrm{dcl}(a) \cap \mathrm{bdd}(h)$. The other inclusion is essentially [10, Remark 3.8]; we offer a proof for convenience. Since $a \mathop{\smile\!\!\!\!\!\!\,|}_{\mathrm{Cb}(a/h)} h$ we have $\mathrm{dcl}(a) \cap \mathrm{bdd}(h) \subseteq \mathrm{bdd}(\mathrm{Cb}(a/h))$. Using that $\mathrm{tp}(a/\mathrm{Cb}(a/h))$ is an amalgamation base, we see that $\mathrm{dcl}(a) \cap \mathrm{bdd}(h)$ is definable over $\mathrm{Cb}(a/h)$. Set $e = \mathrm{Cb}(a/h)$ and consider $f \in \mathrm{Aut}(\mathfrak{C}/e)$; thus, $a \equiv_e f(a)$ and so, $a \equiv^{\mathrm{Ls}}_e f(a)$ as $\mathrm{tp}(a/e)$ is an amalgamation base. This implies that there is some $g \in \mathrm{Autf}(\mathfrak{C}/e)$ such that $g(a) = f(a)$, whence $f^{-1}(g(a)) = a$ and so, $f^{-1} \circ g$ fixes $\mathrm{dcl}(a) \cap \mathrm{bdd}(h)$. As $\mathrm{dcl}(a) \cap \mathrm{bdd}(h)$ is bounded over e, g fixes $\mathrm{dcl}(a) \cap \mathrm{bdd}(h)$ and hence, f^{-1} and so f fixes $\mathrm{dcl}(a) \cap \mathrm{bdd}(h)$, as desired. □

Proposition 4.2.2. *Assume the theory is simple CM-trivial. If a is a finite tuple, then*

$$\mathrm{bdd}(\mathrm{Cb}(a/B)) = \mathrm{bdd}(\mathrm{Cb}(a/b) : b \in X)$$

where X *is the set of all finitary* $b \in \mathrm{bdd}(\mathrm{Cb}(a/B))$.

Proof. As $\mathrm{Cb}(a/b) \subseteq \mathrm{bdd}(b)$ it is clear that $\mathrm{Cb}(a/b) \subseteq \mathrm{bdd}(\mathrm{Cb}(a/B))$ for all $b \in X$; so, it is enough to check the other inclusion. For this, let $\hat{b}$ be a representative of $\mathrm{Cb}(a/b)$ for every $b \in X$ such that

$$(\hat{b} : b \in X) \underset{(\mathrm{Cb}(a/b):b \in X)}{\perp\!\!\!\perp} aB,$$

whence $(\hat{b} : b \in X) \perp\!\!\!\perp_B a$. Now, if $a \not\perp\!\!\!\perp_{(\hat{b}:b\in X)} B$ then there is a finite tuple $b' \in B \cup \{\hat{b} : b \in X\}$ and a formula $\varphi(x, b') \in \mathrm{tp}(a/B, \hat{b} : b \in X)$ which divides over $(\hat{b} : b \in X)$. Put $\bar{b} = \mathrm{bdd}(ab') \cap \mathrm{bdd}(B, \hat{b} : b \in X)$. It turns out that $\bar{b}$ is a quasi-finitary hyperimaginary that satisfies

$$\mathrm{bdd}(a\bar{b}) \cap \mathrm{bdd}(B, \hat{b} : b \in X) = \mathrm{bdd}(\bar{b}).$$

Then by CM-triviality,

$$\mathrm{Cb}(a/\bar{b}) \subseteq \mathrm{bdd}(\mathrm{Cb}(a/B, \hat{b} : b \in X)) = \mathrm{bdd}(\mathrm{Cb}(a/B)).$$

Therefore, $\mathrm{Cb}(a/\bar{b})$ is finitary by Lemma 4.2.1 and so it belongs to X. Note that $b' \in \mathrm{dcl}(\bar{b})$; but $a \perp\!\!\!\perp_{\mathrm{Cb}(a/\bar{b})} \bar{b}$, so $\varphi(x, b')$ cannot divide over $\mathrm{Cb}(a/\bar{b})$, and even less over $(\hat{b} : b \in X)$ as this contains $\widehat{\mathrm{Cb}(a/\bar{b})}$. Thus, $a \perp\!\!\!\perp_{(\hat{b}:b\in X)} B$ and so $a \perp\!\!\!\perp_{(\mathrm{Cb}(a/b):b\in X)} B$ by transitivity. Thus,

$$\mathrm{Cb}(a/B) \subseteq \mathrm{bdd}(\mathrm{Cb}(a/b) : b \in X)$$

and hence, the result. □

Remark 4.2.3. The same proof will work without assuming CM-triviality if for every finite tuple $b \in B$ there is some quasi-finitary hyperimaginary $\bar{b} \in \mathrm{bdd}(B)$ with $b \in \mathrm{dcl}(\bar{b})$ such that $\mathrm{Cb}(a/\bar{b}) \subseteq \mathrm{bdd}(\mathrm{Cb}(a/B))$.

We can now state (and prove) the main result.

Theorem 4.2.4. *In a simple CM-trivial theory, every hyperimaginary is interbounded with a sequence of finitary hyperimaginaries.*

Proof. By Lemma 4.2.1 every hyperimaginary is interbounded with a canonical base. Since $\mathrm{Cb}(A/B)$ is interdefinable with $(\mathrm{Cb}(a/B) : a \in A$ finite), see [12, Lemma 18.8], it is enough to show that canonical bases of types of finite tuples are interbounded with a sequence of finitary hyperimaginaries. But this is precisely Proposition 4.2.2. □

As corollaries we obtain:

Corollary 4.2.5. *A simple CM-trivial theory eliminates hyperimaginaries whenever it eliminates finitary ones.*

Proof. By Theorem 4.2.4 every hyperimaginary is interbounded with a sequence of imaginaries. Since the ambient theory is simple, it is G-compact, whence $\mathrm{Lstp} = \mathrm{stp}$ over any set of parameters by Lemma 4.2.1. Therefore every hyperimaginary is eliminable by Lemma 4.1.11. □

Corollary 4.2.6. *A small simple CM-trivial theory eliminates hyperimaginaries.*

Proof. A small simple theory eliminates all the finitary hyperimaginaries by [35, Theorem 24]. Now apply Corollary 4.2.5. □

Chapter 5

ω-categorical simple theories and lowness

Buechler [7] and Shami [62] introduced the class of *simple low* theories using D-ranks. In [6], the authors observed that lowness is closely related to the non finite cover property. They defined a formula $\varphi(x, y) \in L$ to be *low* if there is some $k < \omega$ such that for every indiscernible sequence $(a_i : i < \omega)$ the following happens: if $\{\varphi(x, a_i) : i < \omega\}$ is inconsistent, then it is k-inconsistent. In a simple theory, it turns out that Buechler's and Shami's definition agree with this wider version. Roughly speaking, lowness implies that dividing for a formula is type-definable, see Lemma 5.1.5. As a consequence, Buechler and Shami showed independently that Lascar strong types and strong types coincide over any set of parameters for such theories, see [7, Corollary 3.11] and/or [62, Conclusion 6]. That is,

Theorem 5.0.1. *In a simple low theory for all tuples a, b and any set A we have:*

$$a \equiv^{\mathrm{Ls}}_A b \Leftrightarrow a \equiv^{\mathrm{s}}_A b.$$

Main examples of simple low theories are stable theories and supersimple theories of finite D-rank. In addition, we shall observe that any simple theory with bounded finite weight is low. Nevertheless, there is an example of a supersimple nonlow theory due to Casanovas and Kim [14]. Casanovas and Wagner asked in [17] if all countable ω-categorical simple theories are low. *A priori* there is no reason to believe in a positive answer. During the realization of this PhD, I have tried to obtain an ω-categorical simple non-low theory but I did not succeed. The first idea was to modify Casanovas' example [11] in order to give an axiomatization for a possible candidate. However, the condition of ω-categoricity restricts the choice of the language, fact that complicates the situation drastically. A natural alternative comes to mind: Hrushovski's construction.

At the time of writing, all currently known ω-categorical simple theories are CM-trivial, in particular those obtained with a predimension in a relational language via Hrushovski's construction [21, 67]. The aim of this chapter is to show that all countable ω-categorical simple CM-trivial theories are low. Therefore, this result should be seen as an obstacle to obtain the desired counterexample.

Another approach to our question is via stability of forking. Kim and Pillay introduced a strong version of the stable forking conjecture [40]. They proved that one-based theories with elimination of hyperimaginaries have strong stable forking, and so do the examples of simple nonlow theories due to Casanovas [11] and Casanovas and Kim [14] since both are one-based. However, in the ω-categorical context the picture is drastically different. We will show that ω-categorical simple theories with strong stable forking are low.

This chapter is based on the paper *On ω-categorical simple theories*, The Journal of Symbolic Logic (accepted).

5.1 Introduction to simple low theories

For our future purposes is convenient to analyse lowness in terms of the possible length of a dividing chain; so, we shall start recalling this notion.

Definition 5.1.1. Let α be an ordinal. A formula $\varphi(x; y) \in L$ divides α times if there is a sequence $(a_i : i < \alpha)$ in the monster model such that $\{\varphi(x, a_i) : i < \alpha\}$ is consistent and $\varphi(x, a_i)$ divides over $\{a_j : j < i\}$ for all $i < \alpha$. A such sequence $(a_i : i < \alpha)$ is called a dividing chain of length α.

The ideology behind this definition is exhibited in the following lemma, [11, Lemma 2.1]. It explains how to obtain a tree from a dividing chain, and *vice versa.*

Lemma 5.1.2. *Let α be an ordinal and let $(\varphi_i(x, y_i) : i < \alpha)$ be a sequence of formulas. Then the following are equivalent:*

1. *There is a sequence $(a_i : i < \alpha)$ such that the set $\{\varphi_i(x, a_i) : i < \alpha\}$ is consistent, and for all $i < \alpha$ $\varphi_i(x, a_i)$ divides over $\{a_j : j < i\}$ with respect to $k_i < \omega$.*

2. *There is some tree $(a_s : s \in \omega^{<\alpha})$ such that for every $\eta \in \omega^{<\alpha}$ the branch $\{\varphi_i(x, a_{\eta \restriction i+1}) : i < \alpha\}$ is consistent, and for all $i < \alpha$ and for all $s \in \omega^i$ the set $\{\varphi_i(x, a_{s \frown n}) : n < \omega\}$ is k_i-inconsistent.*

As an immediate corollary we obtain [11, Remark 2.2].

Remark 5.1.3. A theory is simple if and only if no formula divides ω_1 times; equivalently, if no formula divides ω times with respect to some fixed natural number $k < \omega$.

It turns out from remark above that the complexity of (local) forking might be characterized in terms of the length of possible dividing chains.

Definition 5.1.4. A formula $\varphi(x; y) \in L$ is *short* if it does not divide infinitely many times; and it is *low* if there is some $n < \omega$ such that it does not divide n times. We say a theory is short (low) if all formulas are short (low).

It follows from definition that low theories are short, and short theories are simple. Moreover, supersimple theories are short: if there is some tuple $b \models \bigwedge_{i<\omega} \varphi(x, a_i)$ and $\varphi(x, a_i)$ divides over $\{a_j : j < i\}$, then $b \not\perp_{(a_j : j<i)} a_i$ for all $i < \omega$, contradicting supersimplicity. An example of a simple non-short theory can be found in [11]. In fact, in Casanovas' example there is a formula which divides α times for any countable ordinal α.

Lemma 5.1.5. *The following are equivalent for a simple theory:*

1. *$\varphi(x, y) \in L$ is low.*

2. *There is a $k < \omega$ such that for every indiscernible sequence $(a_i : i < \omega)$ we have: if $\{\varphi(x, a_i) : i < \omega\}$ is inconsistent, then it is k-inconsistent.*

3. *The relation* $\{(y,z) \in \mathfrak{C} \times \mathfrak{C} : \varphi(x,y)$ *divides over* $z\}$ *is* $\emptyset$*-type-definable (*z *may be of infinite length).*

Proof. This is [6, Lemma 2.3]. □

Remark 5.1.6.

1. A stable formula is low.

2. A formula of finite D-rank is low.

Proof. For (1) we refer the reader to [6, Remark 2.2]. (2) follows from the fact that a formula $\varphi(x,a)$ has D-rank n if and only if there is a sequence $(\psi_i(x,a_i) : i \leq n)$ such that $\varphi(x,a) = \psi_0(x,a_0)$ and for all $i \leq n$ $\psi_i(x,a_i)$ divides over $\{a_j : j < i\}$. So, a formula of D-rank n divides at most $n+1$ times. □

In point 2 of remark above, the finiteness of the D-rank is essential. For instance, there is an example of a supersimple nonlow theory due to Casanovas and Kim [14]. To finish this introductory section we exhibit the relation between lowness and shortness in the ω-categorical simple context.

Proposition 5.1.7. *An* ω*-categorical simple short theory is low. More precisely, if* $\varphi(x,y) \in L$ *is a nonlow formula, then there is an infinite indiscernible sequence* $(a_i : i < \omega)$ *witnessing that* φ *is nonshort.*

Proof. This is [17, Proposition 19]. □

In fact, an inspection of their proof yields a stronger version which will be essential to achieve our goal.

Lemma 5.1.8. *Let* T *be a countable* ω*-categorical simple theory. If* $\varphi(x,y) \in L$ *is a nonlow formula, then there are a sequence* $(a_i : i < \omega)$ *and a tuple* c *realizing the set* $\{\varphi(x,a_i) : i < \omega\}$ *such that the sequence* $(a_i : i < \omega)$ *is indiscernible over* c*, and that for all* $i < \omega$ $\varphi(x,a_i)$ *divides over* $\{a_j : j < i\}$*.*

Proof. We offer a proof for convenience. Since the theory is ω-categorical and $\varphi(x,y)$ is nonlow, $\varphi(x,y)$ is nonshort by Proposition 5.1.7. Let $(a_i : i < \omega)$ be a sequence exemplifying that $\varphi(x,y)$ divides ω times. In particular, there is some $c \models \bigwedge_{i<\omega} \varphi(x, a_i)$. By ω-categoricity we may assume that $a_0 \equiv_c a_i$ for all $i < \omega$. Then, by ω-categoricity and Ramsey's Theorem, there is an infinite subsequence 2-indiscernible over c. Iterating this process we infer that for all $n \geq 1$ there is an infinite subsequence n-indiscernible over c. By compactness, the limit type q of these subsequences exists and if $(b_i : i < \omega) \models q$, then it is an indiscernible sequence over c. Moreover, for every $k < \omega$ there is a sequence $(n_i : i \leq k)$ such that $(b_i : i \leq k) \equiv_c (a_{n_i} : i \leq k)$. Thus, since $\varphi(x, a_{n_k})$ divides over $\{a_{n_i} : i < k\}$, so does $\varphi(x, b_k)$ over $\{b_i : i < k\}$. Moreover, for every $k < \omega$ we obtain $b_k \equiv_c a_{n_k}$ and hence, $\models \varphi(c, b_k)$ for all $k < \omega$. □

5.2 Low weight

In this section we will show that simple theories with bounded finite weight are also low. We shall recall the definitions of pre-weight and weight:

Definition 5.2.1.

- The pre-weight of a type $\mathrm{tp}(a/A)$, $\mathrm{pwt}(a/A)$, is the supremum of the set of all cardinals κ for which there is an independent over A sequence $(a_i : i < \kappa)$ such that $a \not\perp_A a_i$ for all $i < \kappa$.
- The weight of a type $\mathrm{tp}(a/A)$, denoted by $\mathrm{w}(a/A)$, is the supremum of the set of all pre-weights of the non-forking extensions of $\mathrm{tp}(a/A)$.

In a simple theory, by the local character of non-forking independence, every type has $< \infty$ (pre-)weight.

Definition 5.2.2. We say that a simple theory has *bounded (or uniform) finite weight* if there is some natural number n such that the weight of any type in one variable (over any set of parameters) is bounded by n.

Remark 5.2.3. If a simple theory has bounded finite weight then for any finite tuple of variable x there is some natural number $n_{|x|}$ such that the weight of any type on x (over any set of parameters) is bounded by $n_{|x|}$. See [53, Lemma 1.4.4.1]

Remark 5.2.4. In a supersimple theory, for any type $p(x)$ we always have $\mathrm{w}(p) \leq \mathrm{SU}(p)$; for instance, see [67, Theorem 5.2.5]. On the other hand, as we always have $\mathrm{SU}(p) \leq \mathrm{D}(x = x)$, every simple theory of finite D-rank has bounded finite weight.

There are examples of simple non-supersimple theories all whose types in one variable have weight ≤ 1.

Example. This example corresponds to [47, Observation 3.4]. Consider the theory of infinite equivalence relations $\{E_n(x, y) : n < \omega\}$ with infinitely many classes such that E_{n+1} refines E_n into infinitely many equivalence classes. It turns out that all its 1-types are regular and so have weight one. Of course, we can obtain a simple unstable example taking the disjoint union of this example with, say, the random graph.

Proposition 5.2.5. *Every simple theory of bounded finite weight is low.*

Proof. By Lemma 5.1.5 it is enough to show that dividing is $\emptyset$-type-definable. Let $\varphi(x, y) \in L$ be a formula with $|x| = n$. In fact, by [17, Proposition 18] we may assume $n = 1$. By assumption there is some $k < \omega$ such that every complete type on x has weight less than k. Firstly, we will check that for any tuple a and any set A, $\varphi(x, a)$ divides over A if and only if it divides over A with respect to k. For this, it is enough to check the k-inconsistency of $\{\varphi(x, a_i) : i < \omega\}$ for some Morley sequence $(a_i : i < \omega)$ in $\mathrm{tp}(a/A)$. So, consider a Morley sequence $(a_i : i < \omega)$ in $\mathrm{tp}(a/A)$; in particular, $\varphi(x, a_i)$ divides over A for all $i < \omega$. Thus, for any b such that $\models \varphi(b, a_i)$ with $i < \omega$ we have, $b \not\downarrow_A a_i$. As the a_i's are A-independent, the set $\{\varphi(x, a_i) : i < \omega\}$ must be k-inconsistent as otherwise we would obtain a type on x over A whose weight would be $\geq k$, a contradiction. Finally, it is clear that "$\varphi(x, y)$ divides over z with respect to $k_{|x|}$" is $\emptyset$-type-definable on y, z. $\square$

Remark 5.2.6. Observe that the size of A is irrelevant in the proof, the formula $\varphi(x, a)$ always divides over any set A with respect to $k_{|x|}$. If we require that for any possible cardinal λ, the weight of all types in one variable over any set of size λ is uniformly bounded by a finite number, then the same proof work. In that case, the natural number witnessing division will depend on the length of x and on the size of the parameter set A.

As an immediate corollary we get the following result:

Corollary 5.2.7. *In a simple theory of bounded finite weight, Lascar strong types and strong types coincide over any set of parameters.*

Proof. By Proposition 5.2.5 such a theory is low, then apply Theorem 5.0.1. □

To conclude this section we give a result on the pre-weight of types over finite sets in an ω-categorical simple theory.

Proposition 5.2.8. *Assume the ambient theory is ω-categorical and simple. For a finite tuple a and a finite set A, the type $\mathrm{tp}(a/A)$ has finite pre-weight.*

Proof. Assume not, then for every $n < \omega$ there is an A-independent sequence $(b_i : i < n)$ with $a \not\downarrow_A b_i$ for all $i < n$. For a cardinal κ big enough we consider a set of formulas $\Sigma(x_i : i < \kappa)$ expressing:

$$x_i \downarrow_A (x_j : j < i) \text{ and } a \not\downarrow_A x_i \text{ for all } i < \kappa.$$

The first part is type-definable over A by ω-categoricity since $(x_i : i < \kappa)$ is A-independent if and only if every finite subsequence is. In addition, $a \not\downarrow_A x_i$ is Aa-definable again by ω-categoricity. A compactness argument yields that $\Sigma(x_i : i < \kappa)$ is consistent and therefore, a realization of Σ witnesses that $\mathrm{tp}(a/A)$ has pre-weight at least κ. As the argument works for any κ, this contradicts simplicity. □

Remark 5.2.9. By Fact 5.1.8 and Lemma 5.2.8, if a formula is nonlow in an ω-categorical simple theory, then it is nonshort, and there is no $\emptyset$-independent sequence witnessing this.

5.3 A Lemma on the bounded closure operator

This section is devoted to study the bounded closure in ω-categorical simple theories. The results presented here are easy but we have not seen them in the literature. The imaginary version of the next lemma was suggested to us by David Evans. However, we present it in a hyperimaginary version. This generalization to hyperimaginaries is not difficult but will require the following lemma.

Lemma 5.3.1. *Any A-bounded hyperimaginary is an equivalence class of a bounded type-definable over A equivalence relation. More precisely, if a_E is an A-bounded hyperimaginary and $p(x) = \mathrm{tp}(a)$, then $\mathrm{dcl}(a_E) = \mathrm{dcl}(a_F)$, where $F(x, y)$ is given by $\exists z(p(z) \wedge E(x, z) \wedge E(z, y)) \vee x \equiv_A^{\mathrm{KP}} y$.*

Proof. This is [12, Proposition 15.27]. □

Lemma 5.3.2. *Assume T is a countable ω-categorical simple theory, let a be a finite tuple, and let A be an arbitrary set (possibly of hyperimaginaries). Then, there is some $e \in \mathfrak{C}^{eq}$ such that*

$$\mathrm{bdd}(e) = \mathrm{bdd}(a) \cap \mathrm{bdd}(A).$$

Proof. Let h be a hyperimaginary such that $\mathrm{dcl}(h) = \mathrm{bdd}(a) \cap \mathrm{bdd}(A)$. By simplicity we choose some $b \equiv_h a$ with $b \mathop{\smile\!\!\!\!\!\!\mid}_h a$ and so $\mathrm{bdd}(a) \cap \mathrm{bdd}(b) = \mathrm{bdd}(h)$. Note that $\mathrm{bdd}(a) \cap \mathrm{bdd}(b) = \mathrm{bdd}(a) \cap \mathrm{bdd}(A)$ and define the following relation

$$xyEuv \Leftrightarrow \mathrm{bdd}(x) \cap \mathrm{bdd}(y) = \mathrm{bdd}(u) \cap \mathrm{bdd}(v).$$

It is obvious that E is an $\emptyset$-invariant equivalence relation and so, it is $\emptyset$-definable by ω-categoricity. Let now $e = (ab)_E$ and notice that $e \in \mathrm{bdd}(a) \cap \mathrm{bdd}(b)$. So it remains to check that $\mathrm{bdd}(a) \cap \mathrm{bdd}(b) \subseteq \mathrm{bdd}(e)$. For this we consider the orbit of h under $\mathrm{Aut}(\mathfrak{C}/e)$, denoted by $\mathcal{O}_e(h)$, and we check that $\mathcal{O}_e(h)$ is small. Since $\mathrm{Aut}(\mathfrak{C}/e)$ fixes $\mathrm{bdd}(a) \cap \mathrm{bdd}(b)$ setwise and $h \in \mathrm{bdd}(a) \cap \mathrm{bdd}(b)$, $\mathcal{O}_e(h) \subseteq \mathrm{bdd}(a) \cap \mathrm{bdd}(b)$. Then, each e-conjugate of h is a-bounded and hence, each e-conjugate of h is an equivalence class of a bounded a-type-definable equivalence relation by Lemma 5.3.1. Since there is just a bounded number of such equivalence relations, the orbit of h under $\mathrm{Aut}(\mathfrak{C}/e)$ must be small. □

Corollary 5.3.3. *Let T be a countable ω-categorical simple theory. If a quasi-finitary hyperimaginary h is bounded over some other hyperimaginary h', then there is some $e \in \mathrm{acl}^{eq}(h')$ such that $h \in \mathrm{bdd}(e)$.*

Proof. Assume $h \in \mathrm{bdd}(a)$ for some finite tuple a and let h' be a hyperimaginary such that $h \in \mathrm{bdd}(h')$. By Lemma 5.3.2 there is some imaginary $e \in \mathfrak{C}^{eq}$ such that $\mathrm{bdd}(a) \cap \mathrm{bdd}(h') = \mathrm{bdd}(e)$. Hence, $e \in \mathrm{acl}^{eq}(h')$ and $h \in \mathrm{bdd}(e)$. □

Lemma above has immediate consequences in simple theories: supersimple theories and theories which admits finite coding coincide in the ω-categorical simple framework. We recall the definition of finite coding.

Definition 5.3.4. A simple theory *admits finite coding* if the canonical base of any finitary type is a quasi-finitary hyperimaginary.

In particular, one-based theories and supersimple theories admit finite coding. For readers interested in this topic we recommend [67, Chapter 6.1.3].

Lemma 5.3.5. *Let T be an ω-categorical simple theory. If $\mathrm{Cb}(a/A) \subseteq \mathrm{bdd}(B)$ for some finite set B, then $\mathrm{Cb}(a/A)$ is interbounded with an imaginary.*

Proof. By Lemma 5.3.2 as $\mathrm{bdd}(\mathrm{Cb}(a/A)) = \mathrm{bdd}(\mathrm{Cb}(a/A)) \cap \mathrm{bdd}(B)$. □

Corollary 5.3.6. *An ω-categorical simple theory which admits finite coding is supersimple.*

Proof. Let a be a finite tuple and let A be an arbitrary subset of the monster model. By assumption and Lemma 5.3.5, there is some e imaginary such that $\mathrm{bdd}(\mathrm{Cb}(a/A)) = \mathrm{bdd}(e)$, whence $e \in \mathrm{acl}^{eq}(A)$ and hence, $e \in \mathrm{acl}^{eq}(A_0)$ for some finite subset $A_0 \subseteq A$. On the other hand, $a \mathop{\smile\!\!\!\!\!\!|}_e A$ and so, $a \mathop{\smile\!\!\!\!\!\!|}_{A_0} A$. □

5.4 An approach via CM-triviality

In this section we shall state (and prove) that ω-categorical simple CM-trivial theories are low. At the time of writing, all known examples of ω-categorical simple theories are CM-trivial; in particular, those obtained via a Hrushovski construction with a standard predimension function. For background on such Hrushovski's construction, the reader may consult [21, 67].

In Section 1.4 we have introduced the definition of CM-triviality. For convenience, we shall recall its definition here.

Definition 5.4.1. A simple theory is CM-trivial if for all tuples a and for all sets $A \subseteq B$ such that $\mathrm{bdd}(aA) \cap \mathrm{bdd}(B) = \mathrm{bdd}(A)$ we have $\mathrm{Cb}(a/A) \subseteq \mathrm{bdd}(\mathrm{Cb}(a/B))$.

In the definition of CM-triviality we have to deal with the bounded closure operator since canonical bases are hyperimaginaries. By Corollary 4.2.6, in our context, each hyperimaginary is interdefinable with a sequence of imaginaries and so, we may replace the bounded closure bdd in favour of the imaginary algebraic closure.

Theorem 5.4.2. *A countable ω-categorical simple CM-trivial theory is low.*

Proof. As we have remarked above, we may assume that canonical bases are sequences of imaginaries and so, we may work only with imaginary elements.

Suppose, towards a contradiction, that there is a nonlow formula, say $\varphi(x, y) \in L$. Then by Lemma 5.1.8 there are some c and some c-indiscernible sequence $(a_i : i < \omega)$ such that for every $i < \omega$, $\varphi(x, a_i)$ divides over $\{a_j : j < i\}$ and $c \models \varphi(x, a_i)$. Now we prolong the sequence to a c-indiscernible sequence $(a_i : i \leq \omega)$. Since $\mathrm{tp}(a_\omega/a_i : i < \omega, c)$ is finitely satisfiable in $\{a_i : i < \omega\}$ we have,

$$a_\omega \underset{(a_i:i<\omega)}{\downarrow} c,$$

that is, $\mathrm{Cb}(a_\omega/a_i : i < \omega, c) = \mathrm{Cb}(a_\omega/a_i : i < \omega)$.

Let now $A = \mathrm{acl}(a_\omega c) \cap \mathrm{acl}(a_i : i < \omega, c)$. It follows that $c \in A = \mathrm{acl}(A)$ and that $\mathrm{acl}(A) = \mathrm{acl}(a_\omega A) \cap \mathrm{acl}(a_i : i < \omega, c)$. Now, by CM-triviality we obtain

$$\mathrm{Cb}(a_\omega/A) \subseteq \mathrm{acl}(\mathrm{Cb}(a_\omega/a_i : i < \omega, c)) = \mathrm{acl}(\mathrm{Cb}(a_\omega/a_i : i < \omega)),$$

whence $\mathrm{Cb}(a_\omega/A) \subseteq \mathrm{acl}(a_i : i < \omega)$. Also, observe that $\mathrm{Cb}(a_\omega/A) \subseteq \mathrm{acl}(A) \subseteq \mathrm{acl}(a_\omega c)$ and so, by Lemma 5.3.5 it is interalgebraic with a single imaginary element, say $e \in \mathfrak{C}^{eq}$. Thus, $a_\omega \downarrow_e A$ and hence, $a_\omega \downarrow_e c$. On the other hand, since e is a single imaginary, there exists some $n < \omega$ such that $e \in \mathrm{acl}(a_i : i < n)$. But by c-indiscernibility observe that $\models \varphi(c, a_\omega)$ and that $\varphi(x, a_\omega)$ divides over $\mathrm{acl}(a_i : i < n)$, and so does over e; a contradiction. Hence, we obtain the result. □

Remark 5.4.3. It is contained in the proof that ω-categorical simple CM-trivial theories satisfy the following property: if $a \downarrow_A b$ for a and b finite, then there is some finite $A_0 \subseteq \mathrm{acl}(A)$ such that $a \downarrow_{A_0} b$; namely, $A_0 =$

$\mathrm{acl}(ab) \cap \mathrm{acl}(A)$. In fact, this property appears isolated in [21] where it is called $(P7)$.

Question 5.4.4. The same proof would work without assuming CM-triviality if for all finite tuples a, b and for every set B with $b \in \mathrm{bdd}(B)$, there is some $\hat{b} \in \mathrm{bdd}(B)$ such that $b \in \mathrm{bdd}(\hat{b})$ and $\mathrm{Cb}(a/\hat{b}) \in \mathrm{bdd}(\mathrm{Cb}(a/B))$, where $\hat{b}$ might be a quasi-finitary hyperimaginary. Is this true in general?

This question was already stated in Section 4.2 where we observed that every theory satisfying this property would eliminate all hyperimaginaries if it eliminates finitary ones.

5.5 Forking stability and lowness

To finish this chapter we observe that ω-categorical simple theories whose forking independence is witnessed by a stable formula in a strong sense are low.

Definition 5.5.1. A simple theory has *strong stable forking* if whenever a type $\mathrm{tp}(a/B)$ forks over A, then there is a stable formula $\phi(x, y) \in L$ such that $\phi(x, b) \in \mathrm{tp}(a/B)$ forks over A. Observe that A might not be a subset of B; if we add the requirement $A \subseteq B$ this becomes the notion of *stable forking*.

Clearly, strong stable forking implies stable forking. Even though it is not known the existence of a simple theory without stable forking, any completion of the theory of pseudo-finite fields does not have strong stable forking, see [40].

Proposition 5.5.2. *A countable ω-categorical simple theory with strong stable forking is low.*

Proof. Assume the ambient theory has strong stable forking, but suppose, towards a contradiction, that there is a nonlow formula $\varphi(x, y) \in L$. By Lemma 5.1.8 (or Proposition 5.1.7, indeed) there is a dividing chain $(a_i : i < \omega)$ witnessing that $\varphi(x, y)$ is nonshort. Let b be a realization of the set $\{\varphi(x, a_i) : i < \omega\}$. Observe that $\mathrm{tp}(b/a_i)$ divides over $\{a_j : j < i\}$ for all $i < \omega$, so for each i there is a stable formula $\psi_i(x, y) \in L$ such that $\psi_i(x, a_i) \in \mathrm{tp}(b/a_i)$ divides over $\{a_j : j < i\}$. By ω-categoricity, there are just a finite number of formulas (up to equivalence) on x, y; thus, we may assume that all ψ_i's are equivalent to some $\psi(x, y) \in L$. Hence, $\psi(x, y)$ is a stable formula which divides ω times and so it is not low, a contradiction. Hence, we get the result. $\square$

Remark 5.5.3. In fact, for the proof above to work it suffices to assume strong stable forking over finite sets.

Resum en català

Introducció

El desenvolupament de l'estabilitat, iniciat per Saharon Shelah a finals dels anys seixanta, va requerir dues nocions crucials: la *independència del forking* i el concepte relacionat de *base canònica*. La independència del *forking* generalitza la independència lineal en els espais vectorials i la independència algebraica en els cossos algebraicament tancats. D'altra banda, les bases canòniques generalitzen el concepte de cos de definició d'una varietat algebraica. Aquestes eines han permès als model-teòrics tractar una sèrie d'objectes matemàtics de manera totalment abstracta. En paral·lel, Boris Zilber durant els anys setanta va estudiar les estructures $\aleph_1$-categòriques. Zilber va introduir una gran varietat de tècniques que van donar lloc a l'anomenada *teoria geomètrica de l'estabilitat*. Aquests mètodes consistien en analitzar les pregeometries obtingudes en considerar conjunts fortament minimals. Aquest anàlisis va empènyer a Zilber a establir la seva tricotomia: la geometria d'un conjunt fortament minimal és o bé degenerada, o bé interpreta un cos, o bé és bàsicament un espai vectorial sobre un anell de divisió.

A la dècada dels anys vuitanta, Ehud Hrushovski apareix a l'escena model-teòrica i amb ell, la teoria geomètrica de l'estabilitat assoleix la seva maduresa. Hrushovski generalitzà resultats parcialment obtinguts per Zilber, provant teoremes sobre l'existència de grups en estabilitat. A més, l'interès per a la geometria del *forking* assoleix un rang d'importància elevat en el moment en que Hrushovski obté un nou conjunt fortament minimal refutant la tricotomia de Zilber. En particular, aquest va ésser l'inici de la *CM-trivialitat*.

Malgrat que la teoria geomètrica de l'estabilitat era entesa durant els vuitanta, fora del paradís estable encara hi havia exemples matemàtics amb una noció primària d'independència. El 1991, en un manuscrit sobre cossos pseudo-finits, Hrushovski va desenvolupar en un context de rang finit, lliure de tecnicismes, les primeres nocions del que acabaria essent amb el pas dels anys la *teoria geomètrica de la simplicitat*. Però no va ésser fins al 1997, quan Byunghan Kim i Anand Pillay van desenvolupar la teoria general de la simplicitat i van demostrar el Teorema de Independència per a tipus forts de Lascar. A més, Kim va demostrar que les teories simples són aquelles teories de primer orde on la independència del *forking* és simètrica, fet que no va poder demostrar Shelah quan va introduir les teories simples al 1980. Després del treball desenvolupat per Kim i Pillay, la teoria de la independència del *forking* es va traslladar del context estable al simple; l'estudi de les teories simples va esdevenir una àrea molt activa de la teoria de models a finals del mil·leni.

Aquest escuet resum sobre la història de les teories simples ens porta al moment actual. La classe de les teories simples inclou totes les teories estables així com altres exemples matemàtics importants com el *random graph*, els cossos pseudo-finits i els cossos algebraicament tancats amb un automorfisme genèric. El pas de les teories estables a les teories simples va requerir el desenvolupament de nous mètodes. Nocions típiques d'estabilitat com ortogonalitat, regularitat, internalitat, analitzabilitat, tipus genèric de grups i les construccions de Hrushovski van ésser traslladades a la simplicitat. Malauradament, algunes traduccions foren molt tècniques com la existència de bases canòniques. Mentre que en les teories estables les bases canòniques són seqüències d'imaginaris obtinguts localment via la definibilitat de tipus, en teories simples aquestes són definides com un únic *hiperimaginari*, i.e., com una classe d'equivalència d'una relació d'equivalència tipus-definible sobre $\emptyset$. Així doncs, la tipus-definibilitat esdevé essencial per entendre la teoria general. Evidentment, aquest fet està estretament relacionat amb l'absència d'una teoria local del *forking* en teories simples. Una qüestió relacionada i que és abordada en aquesta tesi sota la hipòtesis de CM-trivialitat és la eliminació d'hiperimaginaris. És a dir, és tot hiperimaginari interdefinible amb una seqüència d'imaginaris?

L'eliminació d'hiperimaginaris va desencadenar grans esforços durant els inicis de la simplicitat. Era sabut que les teories estables eliminen els hiperimaginaris [58]. Un dels resultats més importants en teories simples és l'eliminació d'hiperimaginaris en teories supersimples [10]. A [7] i [62] es demostra que tota teoria simple baixa elimina els hiperimaginaris acotats – aquells hiperimaginaris que tenen una òrbita acotada per l'acció del grup d'automorfismes del model monstre. La classe de les teories simples baixes és introduïda al **capítol 5**. Una altra classe de teories de primer ordre que ha estat objecte d'estudi és la classe de les teories *baixes*. Recordem que una teoria baixa és aquella on per a qualsevol nombre natural $n < \omega$, $|S_n(\emptyset)| \leq \omega$. Kim va demostrar que qualsevol teoria petita, no necessàriament simple, elimina els hiperimaginaris finitaris [35]. És a dir, tota teoria petita elimina aquells hiperimaginaris que són classes d'equivalència d'una tupla finita mòdul una relació d'equivalència $\emptyset$-tipus-definible. No obstant, qualsevol teoria de primer ordre amb la *propietat de l'ordre estricte* té un hipermaginari que no pot ésser eliminat [2]. En particular, hi ha teories petites que no eliminen els hiperimaginaris; la pregunta encara és oberta per a teories petites simples. També es coneix una teoria sense la propietat de l'ordre estricte que no elimina els hiperimaginaris [17]. Finalment, un resultat relacionat que ens permet entendre millor la naturalesa dels hiperimaginaris acotats va ser obtingut per Lascar i Pillay: tot hiperimaginari acotat pot ésser reemplaçat a favor d'una seqüència d'hiperimaginaris finitaris acotats [43]. Aquests resultats són tot el que es coneix sobre hiperimaginaris; al **capítol 4** aportarem el nostre granet de sorra.

En aquesta tesi doctoral es tracten temes relacionats amb la geometria del *forking* i les seves aplicacions a problemes fonamentals de la simplicitat. Presentem tres tòpics diferents de les teories simples on el denominador comú és la independència del *forking*. Les contribucions dutes a terme per Hrushovski, més enllà de la teoria de models pura, han posat de manifest la rellevància de la geometria del *forking*. Hrushovski ha resolt la conjectura de Mordell-Lang en qualsevol característica i ha donat una altra prova a la conjectura de Manin-Munford. Pillay i Ziegler [59] van observar que una propietat estructural del *forking* juga un paper clau en la resolució de les dues conjectures: la *propietat de la base canònica*. Un dels objectius d'aquesta tesi és estudiar possibles jerarquies amples que codifiquin la complexitat del *forking* i relacionar el primer nivell d'aquestes jerarquies amb la propietat de la base canònica. Aquesta és la motivació principal del **capítol 2**.

Anem a endinsar-nos en la jerarquia ample. Durant la dècada dels vuitanta les teories *monobasades*, i.e., aquelles teories on dos conjunts són independents sobre la intersecció de les seves clausures algebraiques, eren d'interès general. Amb l'aparició de les construccions de Hrushovski, van sorgir un nou tipus de teories estables que generalitzaven les monobasades: les teories estables CM-trivials. La CM-trivialitat pot ésser entesa com la preservació de la independència sota projeccions. Hrushovski afirmà que la hipòtesi de CM-trivialitat prohibeix l'existència d'una certa configuració punt-línia-pla, similar al cas monobasat on no existeixen pseudoplans tipus-definibles. Pillay va generalitzar aquestes nocions obtenint una jerarquia que codifica la complexitat del *forking*. Aquesta s'anomena la *jerarquia no n-ample*, on el primer nivell correspon a monobasat i el segon a CM-trivialitat. Pillay demostrà que qualsevol teoria simple que interpreti un cos és n-ample per a tot $n < \omega$. Uns anys més tard, David Evans va adoptar una definició més natural de n-ample i va obtenir una teoria que és n-ample per a tot n però no interpreta un grup infinit.

En els **capítols 4** i **5** treballem dins del segon nivell de la jerarquia, mentre que en el **capítol 2** donem dues generalitzacions de la jerarquia relativa a una família $\emptyset$-invariant de tipus parcials. Una aplicació d'aquestes generalitzacions és donada al **capítol 3**.

Contingut i resultats

Toto seguit exposarem com s'organitza aquesta tesi i els resultats obtinguts. En el **capítol 1** es dóna una ràpida introducció a les teories simples. Comencem amb la secció 1.1 introduint les nocions bàsiques de la simplicitat: la independència del *forking*, les seqüències de Morley, els tipus forts de Lascar, *et cetera*. Després, introduïm els hiperimaginaris i presentem els corresponents operadors de clausura en aquest context. A més, definim que són els tipus complets per a hiperimaginaris i la igualtat de tipus forts de Lascar. El *forking* per a hiperimaginaris és introduït i les seves principals propietats són exposades al Teorema 1.1.19, incloent-hi el Teorema d'Independència per tipus forts de Lascar per a hiperimaginaris. Les nocions de base d'amalgamació i de base canònica també són definides. Per finalitzar la secció, definim el concepte d'elimininació d'hiperimaginaris i ens preguntem: tota teoria simple elimina els hiperimaginaris?

A la secció 1.2, la classe de les teories supersimples és introduïda, així com els dos rangs fonamentals per aquestes teories: el rang de Lascar i el D-rang. Tot seguit, caracteritzem la supersimplicitat en funció d'aquests dos rangs. També esmentem l'importantíssim teorema de Buechler, Pillay i Wagner: les teories supersimples eliminen els hiperimaginaris, Teorema 1.2.6.

A la secció 1.3 recordem les nocions de tipus regular, d'intern, d'analitzable i de aliè.

Finalment, a la secció 1.4, la jerarquia ample és l'objecte d'estudi i es defineix el concepte d'n-ample per tipus parcials, i relacionem aquestes teories amb les teories monobasades i les teories CM-trivials. Finalitzem la secció amb un teorema de Wagner: ésser no n-ample és preservat sota analitzabilitat.

El **capítol 2** està dedicat a investigar noves possibles jerarquies amples. Es considera una família $\emptyset$-invariant de tipus parcials Σ i s'introdueixen dues versions d'ample relatives a Σ: dèbil Σ-ample i Σ-ample. També es recorda la noció de Σ-clausura a la secció 2.1 i es recullen les seves principals propietats. A més, es presenta algun lema nou i s'introdueix un operador corresponent al primer nivell de l'anàlisi; malauradament, aquest operador no és un operador de clausura. A la secció 2.2 s'estudia una categoria especial de tipus: tipus plans i tipus ultraplans. També es presenta el Teorema dels Nivells de Wagner, Teorema 2.2.7. A la secció 2.3 es defineix el concepte de (dèbilment) n-Σ-ample per a una família $\emptyset$-invariant de tipus i es dóna una de les principals contribucions del capítol: ésser no (dèbilment) Σ-ample es preserva sota analitzabilitat.

Teorema 2.3.19. *Sigui Ψ una família $\emptyset$-invariant de tipus. Si Ψ és Φ-analitzable i Φ no és (dèbilment) n-Σ-ample, tampoc ho és Ψ.*

Cal remarcar que aquest treball va ésser iniciat per Frank O. Wagner, obtenint el resultat esmentat per a no Σ-ample. En paral·lel, jo estava intentant entendre el Teorema de Chatzidakis [19, Proposició 1.16] i [56]. El Teorema de Chatzidakis correspon a una versió forta de no 1-Σ-ample. Amb aquest fet en ment, podem obtenir una versió dèbil de la jerarquia n-Σ-ample i adaptar els resultats de Wagner.

Corollari 2.4.2. *Suposem que tot tipus de la teoria és no ortogonal a un tipus* *regular. Aleshores,* $\mathrm{tp}(\mathrm{Cb}(a/b)/a)$ *és analitzable en la família de tipus regulars no monobasats, per a tot* a*,*b*.*

Aquest resultat apareix a la secció 2.4. A més, també presentem una versió general del Teorema de Chatzidakis per a qualsevol teoria supersimple treballant mòdul ω^{α}, Teorema 2.4.5. Recentment, Wagner ha donat una prova d'aquest resultat utilitzant ultraimaginaris [70]. Tot seguit, a la secció 2.5 definim la *Propietat de la Base Canònica*. Per finalitzar el capítol s'ofereix una aplicació a grups a la secció 2.6. En aquesta secció, es generalitza un resultat de Kowalski i Pillay [41] a qualsevol teoria simple fortament Σ-basada, Proposició 2.6.2.

El **capítol 3** està basat en un resultat obtingut per Wagner sobre la estabilitat del *forking* en teories supersimples CM-trivials. A saber, Wagner demostrà que la relació $R(x;yz)$ definida per $x \mathop{\smile\!\!\!\!\!\!\mid}_z y$ és estable en qualsevol teoria supersimple CM-trivial. Utilitzant les seves idees i un resultat obtingut en el capítol anterior, nosaltres demostrem el següent resultat respecte a la família $\mathcal{P}$ de tipus no monobasats:

Teorema 3.1.2. *En una teoria simple, la relació* $R(x;yz)$ *definida per* $x \mathop{\smile\!\!\!\!\!\!\mid}_{\mathrm{cl}_{\mathcal{P}}(z)} y$ *és estable.*

Aquest resultat està inclòs a [50]. El fet que treballem respecte a la família $\mathcal{P}$ fa que aquest resultat no sigui del tot satisfactori; no obstant, esperem que aporti idees noves per treballar al voltant de la estabilitat del *forking*. Aquest problema està fortament lligat amb la ja esmentada qüestió de l'eliminació dels hiperimaginaris en les teories simples; problema que és tractat en el següent capítol.

El **capítol 4** està dedicat a estudiar l'eliminació dels hiperimaginaris en les teories simples CM-trivials. Els resultats que s'exposen estan recollits a [50]. Aquests, són el resultat d'una pregunta que vaig formular a Frank O. Wagner a l'Abril del 2010. La primera versió de [50] va ésser escrita al Juny del 2010, i la versió final al Març del 2012. Els problemes sobre l'eliminació dels hiperimaginaris és un dels topics més importants en simplicitat, juntament amb la qüestió sobre l'estabilitat del *forking*. La importància dels hiperimaginaris recau en el fet que les bases canòniques – eina imprescindible de la simplicitat – són obtingudes com un hiperimaginari. Tot i que la teoria de models dels hiperimaginaris està perfectament estudida [26], el desenvolupament de la teoria general de la simplicitat esdevé complex i més tècnic. En comparació al cas estable, la falta d'una aproximació local és un obstacle a l'hora de treballar en teories simples. Per exemple, en aquest context més general els operadors de clausura apropiats ja no són finitaris.

Tal i com hem remarcat anteriorment, els nostre resultat principal requereix la hipòtesi de CM-trivialitat. Per tant, aquest resultat no és completament satisfactori; no obstant, esperem que pugui aportar idees i esperança per poder resoldre el problema definitivament. El capítol es divideix en dues seccions. A la secció 4.1 recordem definicions bàsiques i fets sobre hiperimaginaris. Seguint a [43], diem que un hiperimaginari és *quasi-finitari* si és acotat sobre una tuple finita. La relació entre hiperimaginaris finitaris i hiperimaginaris quasi-finitaris és donada a:

Proposició 4.1.9. *Si T elimina els hiperimaginaris finitaris, aleshores T elimina els hiperimaginaris quasi-finitaris.*

En aquesta secció també presentem conseqüències de l'eliminació dels hiperimaginaris en teories G-compactes. Per exemple, oferim un prova nova d'un resultat de Casanovas [12, Proposition 18.27]:

Proposició 4.1.12. *Suposem que la teoria ambient és G-compacta. Aleshores, la teoria elimina els hiperimaginaris acotats, si i només si, $a \equiv^{\mathrm{Ls}} b \Leftrightarrow a \equiv^{\mathrm{s}} b$ per a qualssevol seqüències a i b.*

A la secció 4.2 presentem les principals contribucions per entendre la naturalesa dels hiperimaginaris en les teories simples CM-trivials. Veiem que tota base canònica es pot entendre com una seqüència d'hiperimaginaris finitaris, Proposició 4.2.2, aquest fet ens permet obtenir el següent resultat:

Teorema 4.2.4. *En una teoria simple CM-trivial, tot hiperimaginari és interacotat amb una seqüència d'hiperimaginaris finitaris.*

Tot i que no podem reemplaçar interacotat per interdefinible, per lemes obtinguts i exposats en la secció 4.1, obtenim el següent corol·lari:

Corol·lari 4.2.5. *Tota teoria simple CM-trivial elimina els hiperimaginaris si elimina els hiperimaginaris finitaris.*

Utilitzant un teorema de Kim [35] sobre la eliminació dels hiperimaginaris finitaris en les teories petites, podem obtenir el següent resultat:

Corol·lari 4.2.6. *Tota teoria petita simple CM-trivial elimina els hiperimaginaris.*

Els resultats que acabem de presentar són l'ànima del **capítol 4**.

En el **capítol 5** investiguem la classe de les teories simples baixes. Les teories simples baixes són aquelles teories simples on dividir per una fórmula és tipus-definible. Els principals exemples de teories simples baixes són les teories estables i les teories supersimples amb D-rang finit. La nostra major contribució és demostrar que les teories numerables ω-categòriques simples CM-trivials són baixes. Altre cop treballem amb l'hipòtesi afegida de CM-trivialitat. En el moment d'escriure, tots els exemples coneguts de teories numerables ω-categòriques simples són CM-trivials.

Breument, recordem els aspectes bàsics i resultats de les teories simples baixes a la secció 5.1. A la secció 5.2 recordem les nocions de *pre-pes* i *pes* per a tipus complets. També, introduïm el concepte de *pes finit acotat*: existència d'una cota uniforme finita dels pesos dels 1-tipus. Com a resultat donem un nou exemple de teories simples baixes.

Proposition 5.2.5. *Tota teoria simple amb pes finit acotat és baixa.*

A la secció 5.3 s'investiga un lema sobre l'operador de clausura acotada en les teories numerables ω-categòriques. A més, s'exposen alguns corol·laris que ens permeten relacionar les teories supersimples amb les teories simples amb codificació finita, Corol·lari 5.3.6.

La secció 5.4 està enfocada a contestar una pregunta de Casanovas i Wagner [17]: tota teoria numerable ω-categòrica simple és baixa? Sota la hipòtesi de CM-trivialitat, podem resoldre la qüestió afirmativament.

Theorem 5.5.2. *Tota teoria numerable ω-categòrica simple CM-trivial és baixa.*

Per finalitzar el capítol, a la secció 5.5 relacionem el fet d'ésser una teoria simple baixa amb el fet que el *forking* sigui fortament estable. La darrera noció fou introduïda per Kim i Pillay a [40]. Una teoria es diu que té *forking* fortament estable si per a qualsevol tipus complet que bifurca sobre un conjunt de paràmetres, la bifurcació és exemplificada mitjançant una fórmula estable. Per tant, en aquestes teories la independència del *forking* té un comportament similar a les teories estables. El nostre resultat demostra que tota teoria numerable ω-categòrica simple amb *forking* fortament estable és baixa, Teorema 5.5.2.

Per acabar volem remarcar que aquesta tesi doctoral ha donat lloc als següents articles d'investigació:

[1] Daniel Palacín and Frank O. Wagner. *Hyperimaginaries and stable independence in simple CM-trivial theories*. En Proceedings (ed: Chatzidakis et al.), Olerón 2011. Apareixerà al número especial de The Notre Dame Journal of Formal Logic.

[2] Daniel Palacín. *On ω-categorical simple theories*. The Journal of Symbolic Logic (acceptat).

[3] Daniel Palacín and Frank O. Wagner. *Ample thoughts*. The Journal of Symbolic Logic (enviat).

Possibles direccions

Per finalitzar aquest escrit exposem breument una possible direcció futura d'investigació. En primer lloc, la jerarquia dèbilment n-Σ-ample descrita al **capítol 2** ens ha permès entendre el Teorema de Chatzidakis, una versió dèbil de la propietat de la base canònica, situant-lo en el primer nivell de la jerarquia. Una pregunta oberta és saber si tota teoria simple de rang finit satisfà la propietat de la base canònica. Malauradament, Hrushovski ha afirmat que hi ha un contraexemple. D'altra banda i tenint en ment el probable contraexemple de Hrushovski, sembla factible definir altres variant de la jerarquia dèbil per obtenir una corresponent jerarquia per a la propietat de la base canònica i així poder classificar les teories simples de rang finit. És d'esperar que debilitacions de la propietat de la base canònica tinguin un paper rellevant en les aplicacions de la teoria de models tal i com l'ha tingut la propietat de la base canònica.

En segon lloc, en aquesta tesi doctoral s'han resolt diferents problemes sota la hipòtesi de CM-trivialitat. Una direcció natural a seguir és intentar millorar aquests resultats eliminant CM-trivialitat. Per exemple, una pas natural seria demostrar que tota teoria simple petita elimina els hiperimaginaris. Per solucionar el problema es podria demostrar que en tota teoria simple qualsevol hiperimaginari és interacotat amb una seqüència d'imaginaris; aquest fet generalitzaria el Corol·lari 4.2.4.

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