Computer Science. The final authenticated version is available online at:
http://dx.doi.org/10.1007/978-3-319-45587-7_4

Reducing the Clique and Chromatic Number via Edge Contractions and Vertex Deletions

Daniël Paulusma[1], Christophe Picouleau[2(✉)], and Bernard Ries[3]

[1] Durham University, Durham, UK
daniel.paulusma@durham.ac.uk
[2] CNAM, Laboratoire CEDRIC, Paris, France
christophe.picouleau@cnam.fr
[3] University of Fribourg, Fribourg, Switzerland
bernard.ries@unifr.ch

Abstract. We consider the following problem: can a certain graph parameter of some given graph G be reduced by at least d, for some integer d, via at most k graph operations from some specified set S, for some given integer k? As graph parameters we take the chromatic number and the clique number. We let the set S consist of either an edge contraction or a vertex deletion. As all these problems are NP-complete for general graphs even if d is fixed, we restrict the input graph G to some special graph class. We continue a line of research that considers these problems for subclasses of perfect graphs, but our main results are full classifications, from a computational complexity point of view, for graph classes characterized by forbidding a single induced connected subgraph H.

1 Introduction

When considering a graph modification problem, we usually fix a graph class $\mathcal{G}$ and then, given a graph G, a set S of one or more graph operations and an integer k, we ask whether G can be transformed into a graph $G' \in \mathcal{G}$ using at most k operations from S. Now, instead of fixing a particular graph class, one may be interested in *fixing a certain graph parameter* π. In this setting we ask, given a graph G, a set S of one or more graph operations and an integer k, whether G can be transformed into a graph G' by using at most k operations from S such that $\pi(G') \leq \pi(G) - d$, for some *threshold* $d \geq 0$. Such problems are called *blocker problems*, as the set of vertices or edges involved can be seen as "blocking" some desirable graph property (such as being colorable with only a few colors). Identifying the part of the graph responsible for a significant decrease of the graph parameter under consideration gives crucial information on the graph.

Blocker problems have been given much attention over the last years [1–4,6,7,13,15,16]. Graph parameters considered were the chromatic number, the independence number, the clique number, the matching number and the

D. Paulusma—Author supported by EPSRC (EP/
K025090/1).

vertex cover number. So far, the set S always consisted of a single graph operation, which was a vertex deletion, edge deletion, edge contraction, or an edge addition. Here, we consider the chromatic number and the clique number. We keep the restriction on the size of S and let S consist of an edge contraction or a vertex deletion. Thus, we continue the research initiated by Bentz et al. [4] and Diner et al. [7]. In the latter paper, classes of perfect graphs are considered. Here, we also consider classes of perfect graphs, but in our main results we restrict the input to graphs that are defined by a single forbidden induced subgraph H, that is, to so-called *H-free graphs*.

Definitions. The *contraction* of an edge uv of a graph G removes the vertices u and v from G, and replaces them by a new vertex made adjacent to precisely those vertices that were adjacent to u or v in G (neither self-loops nor multiple edges are introduced). Then G can be *k-contracted* into a graph H if G can be modified into H by a sequence of at most k edge contractions. For a subset $V' \subseteq V$, let $G - V'$ be the graph obtained from G after deleting the vertices of V'. Let $\chi(G)$ and $\omega(G)$ denote the chromatic number and the clique number of G. We now define our two blocker problems formally, where $\pi \in \{\chi, \omega\}$ is the (fixed) graph parameter:

Contraction Blocker(π)
Input: A graph G and two integers $d, k \geq 0$.
Question: Can G be k-contracted into a graph G' such that $\pi(G') \leq \pi(G) - d$?

Deletion Blocker(π)
Input: A graph $G = (V, E)$ and two integers $d, k \geq 0$.
Question: Is there a set $V' \subseteq V$, with $|V'| \leq k$, such that $\pi(G - V') \leq \pi(G) - d$?

If we remove d from the input and fix it instead, we call the resulting problems d-Contraction Blocker(π) and d-Deletion Blocker(π), respectively.

Relations to known problems. In Sect. 3, we will pinpoint a close relationship between the blocker problem and the problem of deciding whether the graph parameter under consideration (chromatic number or clique number) is bounded by some constant (in order to prove a number of hardness results). We also observe that blocker problems generalize graph transversal problems. To explain the latter type of problems, for a family of graphs $\mathcal{H}$, the $\mathcal{H}$-transversal problem is that of finding a set $V' \subseteq V$ in a graph $G = (V, E)$ of size $|V'| \leq k$ for some integer k, such that $G - V'$ contains no induced subgraph isomorphic to a graph in $\mathcal{H}$. By letting, for instance, $\mathcal{H}$ be the family of all complete graphs on at least two vertices, we find that $\mathcal{H}$-transversal is equivalent to Deletion Blocker(ω) restricted to instances $(G, d = \omega(G) - 1, k)$.

Our Results. In Sect. 2, we introduce some more terminology and give a number of known results used to prove our results. In Sect. 3, we show how the computational hardness of the decision problems for χ, ω relates to the computational hardness of the blocker variants. There, we also give a number of additional results on subclasses of perfect graphs. We need these results for our proofs. However, these results may be of independent interest, as they continue similar work on perfect graphs in [7]. In Sect. 4 we present our results for CONTRACTION BLOCKER(π) and d-CONTRACTION BLOCKER(π) for H-free graphs, where $\pi \in \{\chi, \omega\}$. Amongst others we prove complete dichotomies for all connected graphs H. In Sect. 5 we perform the same study for DELETION BLOCKER(π) and d-DELETION BLOCKER(π), where $\pi \in \{\chi, \omega\}$ to obtain complete dichotomies for all connected graphs H. We conclude our paper in Sect. 6.

2 Preliminaries

All graphs considered are finite, undirected and without self-loops or multiple edges. The *complement* of G is the graph $\overline{G} = (V, \overline{E})$ with vertex set V and an edge between two vertices u and v if and only if $uv \notin E$. For a subset $S \subseteq V$, we let $G[S]$ denote the subgraph of G *induced* by S, which has vertex set S and edge set $\{uv \in E \mid u, v \in S\}$. We write $H \subseteq_i G$ if a graph H is an induced subgraph of G. For a vertex $v \in V$, we write $G - v = G[V \setminus \{v\}]$. Recall that for a subset $V' \subseteq V$ we write $G - V' = G[V \setminus V']$. When we contract an edge uv, we may also say that a vertex u is *contracted onto* v, and we use v to denote the new vertex resulting from the edge contraction.

Let $G = (V_G, E_G)$ and $H = (V_H, E_H)$ be two vertex-disjoint graphs. The *disjoint union* $G + H$ has vertex set $V_G \cup V_H$ and edge set $E_G \cup E_H$. The disjoint union of k copies of G is denoted by kG. Let $\{H_1, \ldots, H_p\}$ be a set of graphs. We say that G is $(H_1, \ldots, H_p)$-*free* if G has no induced subgraph isomorphic to a graph in $\{H_1, \ldots, H_p\}$. If $p = 1$ we may write H_1-free instead of (H_1)-free. A subset $C \subseteq V$ is called a *clique* of G if any two vertices in C are adjacent to each other. The *clique number* $\omega(G)$ is the number of vertices in a maximum clique of G. The CLIQUE problem tests if a graph contains a clique of size at least k for some given integer $k \geq 0$. For a positive integer k, a k-*coloring* of G is a mapping $c : V \to \{1, 2, \ldots, k\}$ such that $c(u) \neq c(v)$ whenever $uv \in E$. The *chromatic number* $\chi(G)$ is the smallest integer k for which G has a k-coloring. The COLORING problem tests if a graph has a k-coloring for some given integer k. If k is fixed, that is, not part of the input, then we write k-COLORING instead.

A graph $G = (V, E)$ is a *split graph* if G has a *split partition*, which is a partition of its vertex set into a clique K and an independent set I. A graph is *cobipartite* if it is the complement of a *bipartite* (2-colorable) graph. A graph is *chordal* if it has no induced cycles on more than three vertices. A graph is *perfect* if the chromatic number of every induced subgraph equals the size of a largest clique in that subgraph. Let C_n, P_n and K_n denote the n-vertex cycle, path and clique, respectively. Let $K_{n,m}$ denote the complete bipartite graph with partition classes of size m and n, respectively. The *cobanner*, *bull* and *butterfly* are displayed in Fig. 1. We finish this section by stating some known results.

Lemma 1 ([14]). CLIQUE *is* NP-*complete for the following classes:* (C_5, P_5)-*free graphs,* $K_{1,3}$-*free graphs, cobanner-free graphs and* $(bull, P_5)$-*free graphs.*

Lemma 2 ([10]). *Let H be a graph. For the class of H-free graphs,* COLORING *is polynomial-time solvable if H is an induced subgraph of P_4 or of $P_1 + P_3$ and* NP-*complete otherwise.*

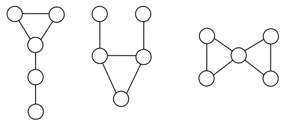

Fig. 1. (a) Cobanner. (b) Bull. (c) Butterfly.

Lemma 3 ([11]). 3-Coloring *is* NP*-complete for the class of* K_3*-free graphs.*

Lemma 4 ([7]). 1-Contraction Blocker(ω) *is* NP*-complete for graphs with clique number* 3*.*

Lemma 5 ([7]). *For* $\pi \in \{\chi, \omega\}$*, both problems* Contraction Blocker(π) *and* Deletion Blocker(π) *can be solved in polynomial time for* P_4*-free graphs, but are* NP*-compete on split graphs.*

3 Hardness Conditions and Results for Perfect Graphs

In this section we give some results that we need for the proofs of our main results in later sections. In the proof of Lemma 4 [7] it is readily seen that the graph obtained in the reduction as input graph for 1-Contraction Blocker(ω) is in fact $(K_4, \overline{2P_1 + P_2}$, butterfly)-free. This gives us the following result.

Lemma 6 ([7]). 1-Contraction Blocker(ω) *is* NP*-complete for the class of* $(K_4, \overline{2P_1 + P_2}, butterfly)$*-free graphs.*

Let $\mathcal{G}$ be a graph class closed under adding a vertex-disjoint copy of the same graph or of a complete graph. We call such a graph class *clique-proof.* The following result establishes a close relation between Coloring (resp. Clique) and 1-Contraction Blocker(χ) (resp. 1-Contraction Blocker(ω)).

Theorem 1. *Let* $\mathcal{G}$ *be a clique-proof graph class. Then the following two statements hold:*

(i) if Coloring *is* NP*-complete for* $\mathcal{G}$*, then so is* 1-Contraction Blocker(χ)*.*
(ii) if Clique *is* NP*-complete for* $\mathcal{G}$*, then so is* 1-Contraction Blocker(ω)*.*

Proof. We only give the proof for Coloring and 1-Contraction Blocker(χ), as the proof for Clique and 1-Contraction Blocker(ω) can be obtained by the same arguments. Let $\mathcal{G}$ be a graph class that is clique-proof. From a given graph $G \in \mathcal{G}$ and integer $\ell \geq 1$ we construct the graph $G' = 2G + K_{\ell+1}$. Note that $G' \in \mathcal{G}$ by definition and that $\chi(G') = \max\{\chi(G), \ell + 1\}$. We claim that G is ℓ-colorable if and only if G' can be 1-contracted into a graph G^* with $\chi(G^*) \leq \chi(G') - 1$. First suppose that G is ℓ-colorable.

Then, in G', we contract an edge of $K_{\ell+1}$ in order to obtain a graph G^* that is ℓ-colorable. Conversely, suppose that G' can be 1-contracted into a graph G^* with $\chi(G^*) \leq \chi(G') - 1$. As contracting an edge in a copy of G does not lower the chromatic number, the contracted edge must be in $K_{\ell+1}$. Then, as $\chi(G^*) \leq \chi(G') - 1$, this implies that $\chi(G') = \ell + 1$ and $\chi(G^*) = \ell$. Hence, as $\chi(G^*) = \max\{\chi(G), \ell\}$, we conclude that $\chi(G) \leq \ell$. $\square$

Our next result is on cobipartite graphs (we omit its proof).

Theorem 2. *For $\pi \in \{\chi, \omega\}$,* Contraction Blocker(π) *is* NP*-complete for cobipartite graphs.*

As cobipartite graphs are $3P_1$-free, we immediately obtain the following.

Corollary 1. *For $\pi \in \{\chi, \omega\}$,* Contraction Blocker(π) *is* NP*-complete for $3P_1$-free graphs.*

We will continue with some further results on subclasses of perfect graphs. We need a known lemma.

Lemma 7 ([7]). *Let $G = (V, E)$ be a C_4-free graph and let $v_1v_2 \in E$. Let $G|v_1v_2$ be the graph obtained after the contraction of v_1v_2 and let v_{12} be the new vertex replacing v_1 and v_2. Then every maximal clique K in $G|v_1v_2$ containing v_{12} corresponds to a maximal clique K' in G and vice versa, such that*

(a) either $|K| = |K'|$ and $K \setminus \{v_{12}\} = K' \setminus \{v_1\}$;
(b) or $|K| = |K'|$ and $K \setminus \{v_{12}\} = K' \setminus \{v_2\}$;
(c) or $|K| = |K'| - 1$ and $K \setminus \{v_{12}\} = K' \setminus \{v_1, v_2\}$.

Moreover, every maximal clique in $G|v_1v_2$ not containing v_{12} is a maximal clique in G and vice versa.

Theorem 3. *For $\pi \in \{\chi, \omega\}$,* 1-Contraction Blocker(π) *is* NP*-complete for chordal graphs.*

Proof. Since chordal graphs are perfect and closed under taking edge contractions, we may assume without loss of generality that $\pi = \omega$. Let $G = (V, E)$ be a graph that together with an integer k forms an instance of Vertex Cover, which is the problem of deciding whether a graph G has a *vertex cover* of size at most k, that is, a subset S of vertices of size at most k such that each edge is incident with at least one vertex of S. Vertex Cover is a well-known NP-complete problem (see [9]).

From G we construct a chordal graph G' as follows. We introduce a new vertex y not in G. We represent each edge e of G by a clique K_e in G' of size $|V|$ so that $K_e \cap K_f = \emptyset$ whenever $e \neq f$. We represent each vertex v of G by a vertex in G' that we also denote by v. Then we let the vertex set of G' be $V \cup \bigcup_{e \in E} K_e \cup \{y\}$. We add an edge between every vertex in K_e and a vertex $v \in V$ if and only if v is incident with e in G. In G' we let the vertices of V form a clique. Finally, we add all edges between y and any vertex in $V \cup \bigcup_{e \in E} K_e$.

Note that the resulting graph G' is indeed chordal. Also note that $\omega(G') = |V|+3$ (every maximum clique consists of y, the vertices of a clique K_e and their two neighbours in V).

We claim that G has a vertex cover of size at most k if and only if G' can be k-contracted to a graph H with $\omega(H) \leq \omega(G') - 1$. First suppose that G has a vertex cover U of size at most k. For each vertex $v \in U$, we contract the corresponding vertex v in G' to y. As $|U| \leq k$, this means that we k-contracted G' into a graph H. Since U is a vertex cover, we obtain $\omega(H) \leq |V|+2 = \omega(G')-1$.

Now suppose that G' can be k-contracted to a graph H with $\omega(H) \leq \omega(G')-1$. Let S be a corresponding sequence of edge contractions (so $|S| \leq k$ holds). By Lemma 7 and the fact that chordal graphs are closed under taking edge contractions, we find that no contraction in S results in a new maximum clique. Hence, as we need to reduce the size of each maximum clique $K_{uv} \cup \{u, v, y\}$ by at least 1, we may assume without loss of generality that each contraction in S concerns an edge with both its end-vertices in $V \cup \{y\}$. We construct a set U as follows. If S contains the contraction of an edge uy we select u. If S contains the contraction of an edge uv, we select one of u, v arbitrarily. Because each maximum clique $K_{uv} \cup \{u, v, y\}$ must be reduced, we find that $U \subseteq V$ is a vertex cover. By construction, $|U| \leq k$. This completes the proof. □

Similar arguments as in the above proof can be readily used to show the following.

Theorem 4. *For $\pi \in \{\chi, \omega\}$,* 1-Deletion Blocker(π) *is* NP*-complete for chordal graphs.*

We will finish this section with a result on C_4-free perfect graphs.

Theorem 5. *For $\pi \in \{\chi, \omega\}$,* 1-Contraction Blocker(π) *is* NP*-complete for the class of C_4-free perfect graphs.*

Proof. Let $\pi = \omega$, or equivalently, $\pi = \chi$. We adapt the construction used in the proof of Lemma 4 by doing as follows for each edge e of the graph G in this proof. First we subdivide e. This gives us two new edges e_1 and e_2. We introduce two new non-adjacent vertices u_e and v_e and make them adjacent to both end-vertices of e_1. Denote the resulting graph by G^*. Notice that we do not create any induced C_4 this way. Hence G^* is C_4-free. The vertices of the original graph together with the subdivision vertices form a bipartite graph on top of which we placed a number of triangles. Hence, G^* contains no induced hole of odd size and no induced antihole of odd size, where a *hole* is an induced cycle on at least five vertices and an *antihole* is the complement of a hole. Then, by the Strong Perfect Graph Theorem [5], G^* is perfect as well.

We increase the allowed number of edge contractions accordingly and observe that, because of the presence of the vertices u_e and v_e for each edge e, we are always forced to contract the edge e_1, which gives us back the original construction extended with a number of pendant edges (which do not play a role). Note that we have left the class of C_4-free perfect graphs after contracting away the triangles, but this is allowed. □

4 Contraction Blocker in H-Free Graphs

In this section, we will consider both problems CONTRACTION BLOCKER(π) and d-CONTRACTION BLOCKER(π) for $\pi \in \{\omega, \chi\}$ and present our classification results for H-free graphs. We start with $\pi = \chi$ and H being a connected graph. In this case, we obtain a complete dichotomy for both problems CONTRACTION BLOCKER(χ) and d-CONTRACTION BLOCKER(χ) concerning their computational complexity.[1]

Theorem 6. *Let H be a connected graph. If H is an induced subgraph of P_4 then* CONTRACTION BLOCKER(χ) *is polynomial-time solvable for H-free graphs. Otherwise even* 1-CONTRACTION BLOCKER(χ) *is* NP*-hard for H-free graphs.*

Proof. Let H be a connected graph. If H is an induced subgraph of P_4, then we use Lemma 5. Now suppose that H is not an induced subgraph of P_4. Then COLORING is NP-complete for H-free graphs by Lemma 2. If H is not a clique, then the class of H-free graphs is clique-proof. Hence, we can use Theorem 1. So suppose H is a clique. It suffices to show NP-completeness for $H = K_3$. We reduce from 3-COLORING restricted to K_3-free graphs. This problem is NP-complete by Lemma 3. Let G be a K_3-free graph representing an instance of 3-COLORING. We obtain an instance of 1-CONTRACTION BLOCKER(χ) as follows. Take two copies of G and the 4-chromatic Grötzsch graph F (see [17], p. 184). Call the resulting graph G', i.e. $G' = 2G + F$. We claim that G is 3-colorable if and only if it is possible to contract precisely one edge of G' so that the new graph G^* has chromatic number $\chi(G') - 1$. We prove this claim via similar arguments as used in the proof of Theorem 1. ☐

For the case when H is a general graph (not necessarily connected), we obtain a complete dichotomy for CONTRACTION BLOCKER(χ).

Theorem 7. *Let H be a graph. If H is an induced subgraph of P_4 then* CONTRACTION BLOCKER(χ) *is polynomial-time solvable for H-free graphs, otherwise it is* NP*-hard for H-free graphs.*

Proof. If H is connected then we use Theorem 6. Suppose H is disconnected. If H contains a component that is not an induced subgraph of P_4 then we use Theorem 6 again. Assume that each connected component of H is an induced subgraph of P_4. If $2P_2 \subseteq_i H$ or $3P_1 \subseteq_i H$ then we use Lemma 5 and the fact that split graphs are $(2P_2, C_4, C_5)$-free (see [8]) or Corollary 1, respectively. Hence, $H \in \{2P_1, P_2 + P_1\}$, so $H \subseteq_i P_4$ and we can use again Theorem 6. ☐

Completing the classification of the computational complexity of d-CONTRACTION BLOCKER(χ) for general graphs H (not necessarily connected) is still open.

We now consider the case $\pi = \omega$. Also in this case we obtain a complete dichotomy when H is connected.

[1] We can modify the gadgets for proving NP-completeness for the case $d = 1$ in a straightforward way to obtain NP-completeness for every constant $d \geq 2$. A similar remark holds for other theorems. Details will be given in the journal version.

Theorem 8. *Let H be a connected graph. If H is an induced subgraph of P_4 or of $\overline{P_1+P_3}$ then* Contraction Blocker(ω) *is polynomial-time solvable for H-free graphs. Otherwise* 1-Contraction Blocker(ω) *is* NP*-hard for H-free graphs.*

Proof. Let H be a connected graph. If H contains an induced C_4, use Theorem 5. If H has an induced K_4, $\overline{2P_1+P_2}$ or butterfly, use Lemma 6. If H contains an induced $K_{1,3}$, C_5, P_5, bull or cobanner, use Lemma 1 with Theorem 1. So we may assume that H is $(C_4, C_5, P_5, K_{1,3}, K_4, \overline{2P_1+P_2}$, bull, butterfly, cobanner)-free.

We claim that H is an induced subgraph of P_4 or of $\overline{P_1+P_3}$. For contradiction, assume that $H \not\subseteq_i P_4$ and $H \not\subseteq_i \overline{P_1+P_3}$. First suppose that H contains no cycle. Then, as H is connected, H is a tree. Because H is $K_{1,3}$-free, H is a path. Our assumption that H is not an induced subgraph of P_4 or of $\overline{P_1+P_3}$ implies that H contains an induced P_5, which is not possible as H is P_5-free.

Now suppose that H contains a cycle C. Then C must have exactly three vertices, because H is (C_4, C_5, P_5)-free. As H is not an induced subgraph of $\overline{P_1+P_3}$, we find that H contains at least one vertex x not on C. As H is connected, we may assume that x has a neighbour on C. Because H is $(\overline{2P_1+P_2}, K_4)$-free, x has exactly one neighbour on C. Let v be this neighbour. Hence, H contains an induced $\overline{P_1+P_3}$ (consisting of x, v and the other two vertices of C). As H is not an induced subgraph of $\overline{P_1+P_3}$ and H is connected, it follows that H contains a vertex $y \notin V(C) \cup \{x\}$ that is adjacent to a vertex on C or to x.

First suppose y is adjacent to a vertex of C. Then, as H is $(\overline{2P_1+P_2}, K_4)$-free, y has exactly one neighbour u in C. If $u = v$ then H either contains an induced claw (if x and y are non-adjacent) or an induced butterfly (if x and y are adjacent). Since, by our assumption, this is not possible, it follows that $u \neq v$. Then, because H is bull-free, we deduce that x and y are adjacent. However, then the vertices, u, v, x, y form an induced C_4, which is not possible as H is C_4-free. We conclude that y is not adjacent to a vertex of C, so y must be adjacent to x only. But then H contains an induced cobanner, a contradiction. Hence, H is an induced subgraph of P_4 or of $\overline{P_1+P_3}$ as we claimed.

If H is an induced subgraph of P_4 then we use Lemma 5. If H is an induced subgraph of $\overline{P_1+P_3}$, then we know from [12] that either G is K_3-free or G is complete multipartite. In the first case one must contract all the edges of an H-free graph in order to decrease its clique number. Hence Contraction Blocker(ω) is polynomial-time solvable for K_3-free graphs. In the second case H is P_4-free, so we can use Lemma 5 again. □

For general graphs H, we have one open case for Contraction Blocker(ω) (while for d-Contraction Blocker(ω) there are many more open cases).

Theorem 9. *Let $H \neq K_3 + P_1$ be a graph. If H is an induced subgraph of P_4 or of $\overline{P_1+P_3}$ then* Contraction Blocker(ω) *is polynomial-time solvable for H-free graphs, otherwise it is* NP*-hard for H-free graphs.*

Proof. If H is connected, use Theorem 8. Suppose H is disconnected. If H contains a component that is not an induced subgraph of P_4 or $\overline{P_1+P_3}$ then we use

Theorem 8 again. Assume that each component of H is an induced subgraph of P_4 or $\overline{P_1 + P_3}$. If $2P_2 \subseteq_i H$ or $3P_1 \subseteq_i H$ then we use Lemma 5 or Corollary 1, respectively. Hence, $H \in \{2P_1, P_2 + P_1, K_3 + P_1\}$. In the first two cases $H \subseteq_i P_4$ and thus we can use Theorem 8, whereas we excluded the last case. $\square$

5 Deletion Blocker in H-Free Graphs

We adapt the proof of Theorem 1 to present relations between Coloring and 1-Deletion Blocker(χ) and between Clique and 1-Deletion Blocker(ω).

Theorem 10. *Let $\mathcal{G}$ be a clique-proof graph class. Then the following two statements hold:*

(i) if Coloring *is* NP*-complete for $\mathcal{G}$, then so is* 1-Deletion Blocker(χ).
(ii) if Clique *is* NP*-complete for $\mathcal{G}$, then so is* 1-Deletion Blocker(ω).

We notice a relation between 1-Deletion Blocker(ω) and Vertex Cover.

Lemma 8. *Let G be a triangle-free graph containing at least one edge and let $k \geq 1$ be an integer. Then (G, k) is a yes-instance for* 1-Deletion Blocker(ω) *if and only if (G, k) is a yes-instance for* Vertex Cover.

Proof. Let $G = (V, E)$ be a triangle-free graph with $|E| \geq 1$. Thus, $\omega(G) = 2$. Let $k \geq 1$ be an integer. First suppose that (G, k) is a yes-instance for Vertex Cover and let V' be a solution, i.e. for every edge $e \in E$, there exists a vertex $v \in V'$ such that v is an endvertex of e. It follows that by deleting all vertices in V', we obtain a graph G' containing no edges and hence $\omega(G') \leq 1$. We conclude that (G, k) is a yes-instance for 1-Deletion Blocker(ω). Conversely, suppose that (G, k) is a yes-instance for 1-Deletion Blocker(ω) and let V' be a solution, i.e. the graph obtained form G by deleting the vertices in V' satisfies $\omega(G') \leq 1$. But this implies that G' contains no edges and thus V' is a vertex cover of size at most k. So (G, k) is a yes-instance for Vertex Cover. $\square$

Corollary 2. 1-Deletion Blocker(ω) *is* NP*-complete for the class of (C_3, C_4)-free graphs.*

Proof. This follows immediately from Lemma 8 and the fact that Vertex Cover is NP-complete for (C_3, C_4)-free graphs (see [14]). $\square$

We are now ready to prove the first main result of this section.

Theorem 11. *Let H be a connected graph. If H is an induced subgraph of P_4, then* Deletion Blocker(ω) *is polynomial-time solvable on H-free graphs. Otherwise* 1-Deletion Blocker(ω) *is* NP*-hard for H-free graphs.*

Proof. If H contains a cycle C_r, $r \in \{3, 4\}$, we use Corollary 2. If H contains a cycle C_r, ≥ 5, we use Lemma 1 combined with Theorem 10. Hence, we may assume now that H is a tree. If H contains an induced $K_{3,1}$, we use Lemma 1 combined with Theorem 10. Thus, H is a path. If this path has length at most 4, we use Lemma 5. Otherwise, we use Lemma 1 combined with Theorem 10. This completes the proof. $\square$

If H is disconnected, finding such a dichotomy is open. In particular, the cases when $H \in \{2P_2, 3P_1\}$ are unknown. Moreover, in contrast to the Contraction Blocker(ω) problem, Deletion Blocker(ω) is polynomial-time solvable on cobipartite graphs [6], which form a subclass of $3P_1$-free graphs. We now focus on $\pi = \chi$. The proof of Theorem 6 can easily be adapted to get the following.

Theorem 12. *Let H be a connected graph. If H is an induced subgraph of P_4, then* Deletion Blocker(χ) *is polynomial-time solvable on H-free graphs. Otherwise,* 1-Deletion Blocker(χ) *is* NP*-hard for the class of H-free graphs.*

If H is disconnected, it seems much harder to get a dichotomy even when d is part of the input. In contrast to the case of ω, we can prove that Deletion Blocker(χ) is polynomial-time solvable for $3P_1$-free graphs.

Theorem 13. Deletion Blocker(χ) *can be solved in polynomial time for the class of $3P_1$-free graphs.*

Proof. Let $G = (V, E)$ be a $3P_1$-free graph with $|V| = n$ and let $k \geq 1$ be an integer. Consider an instance (G, k, d) of Deletion Blocker(χ). We proceed as follows. First consider an optimal coloring of G, which can be obtained in polynomial time [10]. Since G is $3P_1$-free, the size of each color class is at most 2. Also the number of color classes of size 1 is the same for every optimal coloring of G. Let ℓ be this number. Hence, there are $\frac{n-\ell}{2}$ color classes of size 2 and $\chi(G) = \ell + \frac{n-\ell}{2}$. Now (G, k, d) is a yes-instance if and only if we can obtain a graph G' from G by deleting at most k vertices such that $\chi(G') \leq \chi(G) - d = \ell + \frac{n-\ell}{2} - d$. Since G' is also $3P_1$-free, the color classes in any optimal coloring of G' have size at most 2 and thus, G' contains at most $2(\ell + \frac{n-\ell}{2} - d) = n + \ell - 2d$ vertices. In other words, we need to delete at least $2d - \ell$ vertices from G in order to get such a graph G'. So (G, k, d) is clearly a no-instance if $k < 2d - \ell$. Next we will show that if $k \geq 2d - \ell$, then (G, k, d) is a yes-instance and this will complete the proof. If $d \leq \ell$, we delete d vertices representing color classes of size 1. If $d > \ell$, we delete the ℓ vertices representing the color classes of size 1 and $2(d-\ell)$ vertices of $d - \ell$ color classes of size 2. This way, we clearly obtain a graph G' whose chromatic number is exactly $\chi(G) - d$. $\square$

6 Conclusions

We considered the problems (d-)Contraction Blocker(π) and (d-)Deletion Blocker(π), where $\pi \in \{\chi, \omega\}$. We mainly focused on H-free graphs and analyzed the computational complexity of these problems. We obtained a complete dichotomy for both problems and both when d is fixed and when d is part of the input, if H is a connected graph. If H is an arbitrary graph that is not necessarily connected, further research is needed: What is the complexity of the problems d-Contraction Blocker(χ) and d-Contraction Blocker(ω) for H-free graphs when H is disconnected? What is the complexity of Contraction Blocker(ω) for $(K_3 + P_1)$-free graphs? What are the complexities of

Table 1. Results for subclasses of perfect graphs closed under edge contraction (apart from the classes of bipartite and perfect graphs), where NP-c stands for NP-complete and P for polynomial-time solvable; results marked with a * correspond to results of this paper; the unmarked results for perfect graphs follow directly from other results.

	Contraction Blocker(π)		Deletion Blocker(π)	
Class	$\pi = \alpha$	$\pi = \omega(= \chi)$	$\pi = \alpha$	$\pi = \omega(= \chi)$
Bipartite	?	P (trivial)	P [6]	P*
Cobipartite	$d = 1$: NP-c [7]	NP-c*; d fixed: P [7]	P*	P [6]
Chordal	?	$d = 1$: NP-c*	?	$d = 1$: NP-c*
Interval	?	P [7]	?	P [7]
Split	NP-c; d fixed: P [7]	NP-c; d fixed: P [7]	NP-c; d fixed: P [6]	NP-c; d fixed: P [6]
Cograph	P [7]	P [7]	P [7]	P [7]
C_4-free Perfect	?	$d = 1$: NP-c*	?	?
Perfect	$d = 1$: NP-c	$d = 1$: NP-c	NP-c; d fixed: ?	$d = 1$: NP-c

(d-)Deletion Blocker(χ) and (d-)Deletion Blocker(ω) for H-free graphs when H is disconnected? In particular, what is the complexity of d-Deletion Blocker(ω) for $2P_2$-free graphs and $3P_1$-free graphs?

Besides considering the parameters χ and ω, we may of course choose any other graph parameter π, such as $\pi = \alpha$, where α is the independence number (the size of a largest independent set in a graph). Note that d-Deletion Blocker(ω) in a graph G is equivalent to d-Deletion Blocker(α) in its complement $\overline{G}$. Studying the complexity of d-Contraction Blocker(α) and d-Deletion Blocker(α) for H-free graphs is left as future research.

In addition to our results on H-free graphs, we also obtained some new results for subclasses of perfect graphs. We used these as auxiliary results for our classifications but also in order to continue a line of research started in [7]. Table 1 gives an overview of the known results and the new results of this paper for such classes of graphs. Notice that $\chi = \omega$ holds by definition of a perfect graph. In the table we also added results for Contraction Blocker(α) and Deletion Blocker(α), since these problems have been studied in [6,7] and since some of our new results immediately imply corresponding results for the case $\pi = \alpha$. In particular, the polynomial-time solvability of d-Deletion Blocker(ω) for bipartite graphs (and therefore d-Deletion Blocker(α) in cobipartite graphs) follows from Corollary 2 and the fact that Vertex Cover is polynomial-time solvable in bipartite graphs. The proof that shows that Contraction Blocker(ω) is polynomial-time solvable for interval graphs can easily be adapted to show that Deletion Blocker(ω) is polynomial-time solvable for interval graphs.

As can be seen from Table 1 there are several open cases (marked by "?"). Some of these open cases form challenging open problems related to interval and chordal graphs, namely what is the complexity of Contraction Blocker(α) and d-Contraction Blocker(α) for interval graphs and for chordal graphs? What are the complexities of the problems Deletion Blocker(α) and d-Deletion Blocker(α) for interval graphs and for chordal graphs?

References

1. Bazgan, C., Bentz, C., Picouleau, C., Ries, B.: Blockers for the stability number and the chromatic number. Graphs Comb. **31**, 73–90 (2015)
2. Bazgan, C., Toubaline, S., Tuza, Z.: Complexity of most vital nodes for independent set in graphs related to tree structures. In: Iliopoulos, C.S., Smyth, W.F. (eds.) IWOCA 2010. LNCS, vol. 6460, pp. 154–166. Springer, Heidelberg (2011)
3. Bazgan, C., Toubaline, S., Tuza, Z.: The most vital nodes with respect to independent set and vertex cover. Discrete Appl. Math. **159**, 1933–1946 (2011)
4. Bentz, C., Costa, M.-C., de Werra, D., Picouleau, C., Ries, B.: Weighted Transversals and blockers for some optimization problems in graphs. In: Progress in Combinatorial Optimization. ISTE-WILEY (2012)
5. Chudnovsky, M., Robertson, N., Seymour, P.D., Thomas, R.: The strong perfect graph theorem. Ann. Math. **164**, 51–229 (2006)
6. Costa, M.-C., de Werra, D., Picouleau, C.: Minimum d-blockers and d-transversals in graphs. J. Comb. Optim. **22**, 857–872 (2011)
7. Diner, Ö.Y., Paulusma, D., Picouleau, C., Ries, B.: Contraction blockers for graphs with forbidden induced paths. In: Paschos, V.T., Widmayer, P. (eds.) CIAC 2015. LNCS, vol. 9079, pp. 194–207. Springer, Heidelberg (2015)
8. Földes, S., Hammer, P.L.: Split graphs. Congressus Numerantium **19**, 311–315 (1977). 8th South-Eastern Conference on Combinatorics, Graph Theory and Computing
9. Garey, M.R., Johnson, D.S.: Computers and Intractability: A Guide to the Theory of NP-Completeness. Freeman, New York (1979)
10. Král', D., Kratochvíl, J., Tuza, Z., Woeginger, G.J.: Complexity of coloring graphs without forbidden induced subgraphs. In: Brandstädt, A., Le, V.B. (eds.) WG 2001. LNCS, vol. 2204, pp. 254–262. Springer, Heidelberg (2001)
11. Maffray, F., Preissmann, M.: On the NP-completeness of the k-colorability problem for triangle-free graphs. Discrete Math. **162**, 313–317 (1996)
12. Olariu, S.: Paw-free graphs. Inf. Process. Lett. **28**, 53–54 (1988)
13. Pajouh, F.M., Boginski, V., Pasiliao, E.L.: Minimum vertex blocker clique problem. Networks **64**, 48–64 (2014)
14. Poljak, S.: A note on the stable sets and coloring of graphs. Comment. Math. Univ. Carolin. **15**, 307–309 (1974)
15. Ries, B., Bentz, C., Picouleau, C., de Werra, D., Costa, M.-C., Zenklusen, R.: Blockers and transversals in some subclasses of bipartite graphs: when caterpillars are dancing on a grid. Discrete Math. **310**, 132–146 (2010)
16. Toubaline, S.: Détermination des éléments les plus vitaux pour des problèmes de graphes. Ph.D. Thesis, Université Paris-Dauphine (2010)
17. West, D.B.: Introduction to Graph Theory. Prentice-Hall, Upper Saddle River (1996)