Relation Modules and Identities for Presentations of Inverse Monoids

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Abstract

We investigate the Squier complexes of presentations of groups and inverse monoids using the theory semiregular, regular, and pseudoregular groupoids. Our main interest is the class of regular groupoids, and the new class of pseudoregular groupoids.

Our study of group presentations uses monoidal, regular groupoids. These are equivalent to crossed modules, and we recover the free crossed module usually associated to a group presentation, and a free presentation of the relation module with kernel the fundamental group of the Squier complex, the module of identities among the relations.

We carry out a similar study of inverse monoid presentations using pseudoregular groupoids. The relation module is defined via an intermediate construction – the derivation module of a homomorphism, – and a key ingredient is the factorisation of the presentation map from a free inverse monoid as the composition of an idempotent pure map and an idempotent separating map. We can then use the properties of idempotent separating maps, and properties of the derivation module as a left adjoint, to derive a free presentation of the relation module. The construction of its kernel – the module of identities – uses further key facts about pseudoregular groupoids.

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Contents

Chapter 1

Introduction

The relationships between presentations of semigroups, monoids, and groups, and systems of rewriting rules has led to a network of ideas that has drawn together concepts from group and semigroup theory, low-dimensional topology, and theoretical computer science. In [37], Squier addressed the question of whether a finitely presented monoid with solvable word problem is necessarily presented by a finite, complete, string rewriting system. He proved that a monoid presented by a finite, complete, string rewriting system must satisfy the homological finiteness condition FP_3: indeed, an earlier result of Anick [2] implies that such a monoid satisfies the stronger condition FP_∞. These ideas are concisely surveyed by Cohen [9], and more extensively by Otto and Kobayashi in [31]. Since examples are known of finitely presented monoids with solvable word problem that do not satisfy FP_3, Squier's work shows that such monoids need not be presented by finite, complete, string rewriting systems.

Squier, Otto and Kobayashi [38] go on to study finite, complete, string rewriting systems for monoids and proved that the existence of such a system presenting a monoid M implies a homotopical property – *finite derivation type* – defined for a graph that encodes the rewriting system. Moreover, they show that having finite derivation type does not depend on the particular rewriting system used to present

M, and so is a property of M itself and hence is a necessary condition that M should be presented by a finite, complete string rewriting system.

Finite derivation type is naturally thought of as a property of a 2–complex – the *Squier complex* – associated to a monoid presentation, and obtained by adjoining certain 2–cells to the graph of [38]. This point of view was introduced independently by Pride [33] and Kilibarda [20], and then extensively developed by Guba and Sapir in terms of both string-rewriting systems [16] and more geometrically, in terms of directed 2-complexes [17]. The theory developed by Kilibarda and then by Guba and Sapir focusses on the properties of *diagram groups*, which are fundamental groups of the Squier complex. Pride's results in [33] focus upon the low-dimensional homology of the Squier complex.

In [13] Gilbert looked at the structure of the fundamental groupoid of the Squier complex associated to a monoid presentation $[X : R]$. The local groups of this groupoid are precisely the diagram groups. He showed that the fundamental groupoid is a monoid in the category of groupoids, and that the universal group of this monoid then gives the low-dimensional homotopy invariants for the identities among the relations of the *group* presentation $\langle X : R \rangle$ (see [8]) and also explained Pride's corresponding theory of diagram groups for monoid presentations of groups [34] in these terms.

In this thesis, we take the work of Pride [33, 34] and Gilbert [13] as the starting point for an investigation of the Squier complexes of presentations of groups and of inverse monoids, and the algebraic structures that arise from these presentations, in general these structures depend on the chosen presentation and will not be invariants of the group or inverse monoid being presented. After preliminary material in chapters 1 and 2, we study the classes of semiregular, regular, and pseudoregular groupoids in chapter 3. These are groupoids with extra structure that describe the fundamental groupoids of our version of the Squier complexes obtained from monoid, group, and inverse semigroup presentations. Semiregular and regular groupoids were introduced in [13], and we review the basic properties of semiregular and regular groupoids in sections 3.1 and 3.2. Our main interest is in the class of regular groupoids and in the new class of pseudoregular groupoids introduced in section 3.3. The classes are nested, since regular implies pseudoregular, and pseudoregular implies semiregular, but new features arise for the study of a pseudoregular groupoid. All three classes are related to the class of monoidal groupoids, and we note here that semiregular groupoids are also called *whiskered groupoids*, see [4].

In Chapter 4 we use regular and pseudoregular groupoids to classify congruences on groups and inverse monoids. For groups the classification by normal subgroups is well-known, and our classification in Proposition 4.2.1 merely restates this in different terms. For inverse monoids, the classification of congruences by congruence pairs, due to Petrich [32], is also well-known, but its restatement in terms of pseudoregular groupoids is more involved.

In chapter 5 we carry out a study of group presentations using regular groupoids. To each group presentation $\mathcal{P} = \langle X : R \rangle$ we associate the fundamental groupoid $\Pi(\mathrm{Sq}(\mathcal{P}), F(X))$ of a version $\mathrm{Sq}(\mathcal{P})$ of the Squier complex of $\mathcal{P}$ that has vertex set the free group $F(X)$ with basis X. Our complex $\mathrm{Sq}(\mathcal{P})$ may be considered as a subcomplex of the variant of the Squier complex used by Pride in [34] (and there denoted $\mathcal{D}(\mathcal{P})^*$) to study group presentations, and called the *Pride complex* in [13] (and there denoted $K^+(\mathcal{P})$). The groupoid $\Pi(\mathrm{Sq}(\mathcal{P}), F(X))$ is a regular, monoidal groupoid, and so is equivalent to a *crossed module* of groups. Crossed modules are introduced in chapter 2, and their role in understanding group presentations is outlined in section 2.1.3, and fully elaborated in [8] and [36]. We show that the crossed module equivalent to $\Pi(\mathrm{Sq}(\mathcal{P}), F(X))$ is isomorphic to the free crossed module usually associated to the presentation $\mathcal{P}$, and so in Proposition 5.2.6 we obtain a free presentation (5.10) of the relation module of $\mathcal{P}$ with kernel the fundamental group $\pi_1(\mathrm{Sq}(\mathcal{P}, 1))$, which is now identified with the module of identities among the relations of $\mathcal{P}$, as in [8].

In chapter 6 we begin a similar study of inverse monoid presentations using pseudoregular groupoids. The first task is to define the relation module of an inverse monoid presentation $\mathcal{P} = [X : R]$ of an inverse monoid S. For this we use the approach of Gilbert in [14], and construct from any homomorphism $\phi : T \to S$ of inverse monoids, an S–module $\mathcal{D}_\phi$ called the derivation module of ϕ. The presentation $\mathcal{P}$ gives rise to a presentation map $\theta : \mathrm{FIM}(X) \to S$ from the free inverse monoid $\mathrm{FIM}(X)$ on X, and in [14] the relation module is defined as the kernel of a canonical map $\mathcal{D}_\theta \to \mathbb{Z}S$ from the derivation module, to the module $\mathbb{Z}S$ defined by Loganathan [24]. However, we improve upon the approach of [14] by using the factorisation of the presentation map θ as a composition $\mathrm{FIM}(X) \xrightarrow{\tau} \mathcal{T}(X) \xrightarrow{\psi} S$ of an idempotent pure map τ and an idempotent separating map ψ. It turns out that $\mathcal{D}_\theta$ is isomorphic to $\mathcal{D}_\psi$, and we can then use the favourable properties of idempotent separating maps, and properties of the construction $\mathcal{D}$ as a left adjoint functor, to get a better understanding of the relation module. We also show, again following [14], how the relation module can be obtained from the Schützenberger graphs of S.

In chapter 7 we obtain a presentation of the relation module of the presentation $\mathcal{P} = [X : R]$ derived from a Squier complex $\mathrm{Sq}(\mathcal{P})$ with vertex set $\mathcal{T}(X)$. We show that its fundamental groupoid is pseudoregular and monoidal, and that there is an associated free crossed module – now a crossed module of groupoids, using the interpretation of $\mathcal{T}(X)$ as an inductive groupoid with vertex set $E(\mathcal{T})$ (the set of idempotents of $\mathcal{T}$, which is isomorphic to $E(S)$). From this free crossed module we obtain a free presentation of the relation module with kernel a collection of subgroups of the fundamental groups of the fundamental groupoid $\Pi(\mathrm{Sq}(\mathcal{P}), \mathcal{T}(X))$. The construction of this kernel – the module of identities of $\mathcal{P}$ – uses key facets of the theory of pseudoregular groupoids from chapter 3.

Finally, in chapter 8, we look at our construction in action in a number of illustrative examples.

Throughout this thesis we have included proofs of known results for completeness and to make the thesis more self-contained, the proofs given are close copies of the originals unless otherwise stated.

1.1 Regular and Inverse Semigroups

A *semigroup* is defined as a non-empty set S on which an associative binary operation is defined. If a semigroup has an identity element $1 \in S$, then S is called a *monoid.* If a semigroup S has the property that, for all $x, y \in S$ $xy = yx$, we shall say that S is *commutative.*

An element x of a semigroup S is called *regular,* if there exists $y \in S$ such that $x = xyx$ and $y = yxy$. A semigroup S is called *regular* if all its elements are regular. A semigroup S is called an *inverse* semigroup if for each $s \in S$ there is a unique element $s^{-1} \in S$ such that

$$ss^{-1}s = s \quad and \quad s^{-1}ss^{-1} = s^{-1}.$$

If there exists an identity element $1 \in S$ then S is called an inverse monoid. An element e of a semigroup S is an *idempotent*, if $e^2 = e$. The set of idempotents of an inverse semigroup S is a commutative inverse subsemigroup of S. It is a *semilattice*, by which we mean a commutative semigroup in which every element is an idempotent. We shall denote the set of idempotents of an inverse semigroup S by $E(S)$. Inverse semigroups may also be characterized as the regular semigroups in which the idempotents commute. For this and other background information on inverse semigroups, we refer to Lawson's book [22].

1.1.1 Remark. A meet semilattice is a partially ordered set P in which any two elements $a, b \in P$ have a greatest lower bound $a \wedge b \in P$

If S is a commutative semigroup with $S = E(S)$, i.e. a semilattice, then defining a partial order on S by

$$a \leqslant b \Leftrightarrow ab = a$$

gives S the structure of a meet semilattice.

Conversely, a meet semilattice P becomes a commutative semigroup with $P = E(P)$, i.e. a semilattice, if we define the binary operation on P to be the greatest lower bound.

A non-empty subset T of S is called a *subsemigroup* of S, if it is closed with respect to multiplication, that is , if $xy \in T$ for all $x, y \in T$. Furthermore, if T is closed under taking inverses, then it is called an *inverse subsemigroup* of S. An inverse subsemigroup T of S is said to be *full* if $E(T) = E(S)$, and is *normal* if it is full, and if in addition, $s^{-1}Ts \subseteq T$ for all $s \in S$.

A map $\phi : S \to T$, where S and T are semigroups, is called a *morphism* (or *homomorphism*) if

$$(xy)\phi = (x\phi)\,(y\phi) \quad \text{for all} \quad x, y \in S.$$

If S and T are monoids, with identity elements $1_S, 1_T$ respectively, then ϕ will be called a *monoid morphism* if it has the additional property $1_S\,\phi = 1_T$. Because of the uniqueness of inverses, a homomorphism $\phi : S \to T$ between inverse semigroups automatically preserves the inverse: for all $s \in S$ we have $s^{-1}\phi = (s\phi)^{-1}$.

1.1.1 Definition. The *kernel* of a homomorphism of inverse semigroups $\phi : S \to T$ is the inverse image of $E(T)$: that is,

$$\ker \phi = \{s \in S : s\phi \in E(T)\}\,.$$

We now record some useful results drawn from of [22, Section 1.4].

1.1.2 Proposition. *Let S be an inverse semigroup.*

(1) For any $s \in S$, both $s^{-1}s$ and ss^{-1} are idempotents and $s(s^{-1}s) = s$ and $(ss^{-1})s = s$.

(2) $(s^{-1})^{-1} = s$ *for every* $s \in S$.

(3) For any idempotent e *in* S *and any* $s \in S$, *the element* $s^{-1}es$ *is an idempotent.*

(4) If e *is an idempotent in* S, *then* $e^{-1} = e$.

(5) $(s_1 \cdots s_n)^{-1} = {s_n}^{-1} \cdots {s_1}^{-1}$ *for all* $s_1, \ldots, s_n \in S$ *where* $n \geqslant 2$.

1.1.3 Lemma. *Let* S *be an inverse semigroup.*

(1) For every idempotent e *and element* s *there exists an idempotent* f *such that* $e\,s = s\,f$.

(2) For every idempotent e *and element* s *there exists an idempotent* f *such that* $s\,e = f\,s$.

1.1.4 Proposition. *Groups are precisely the inverse semigroups with exactly one idempotent*

1.1.1 The Natural Partial Order

An inverse semigroup S comes equipped with a *natural partial order* defined by:

$$s \leqslant t \quad \text{if and only if} \quad s = te \quad \textit{for some} \quad e \in E(S).$$

We record the following properties of the natural partial order from [22, Lemma 1.4.6].

1.1.5 Lemma. *Let* S *be an inverse semigroup with semilattice* E *of idempotents, and let* $s, t \in S$. *The following statements are equivalent:*

(1) $s \leqslant t$;

(2) $s = ft$ *for some idempotent* $f \in E$;

(3) $s^{-1} \leqslant t^{-1}$;

(4) $s = ss^{-1}t$;

(5) $s = ts^{-1}s$;

(6) $ss^{-1} = ts^{-1}$;

(7) $s = s\,t^{-1}\,s$;

(8) $ss^{-1} = st^{-1}$;

(9) $s^{-1}s = t^{-1}s$;

(10) $s^{-1}s = s^{-1}t$;

The following result presents a number of further properties of the natural partial order

1.1.6 Proposition. *[22, Proposition 1.4.7] Let S be an inverse semigroup.*

(1) The relation $\leqslant$ is a partial order on S.

(2) For idempotents $e, f \in S$, we have that $e \leqslant f$ if and only if $e = ef = fe$.

(3) If $s \leqslant t$ and $u \leqslant v$ then $su \leqslant tv$.

(4) If $s \leqslant t$ then $s^{-1}s \leqslant t^{-1}t$ and $ss^{-1} \leqslant tt^{-1}$.

1.1.2 Definition. An inverse semigroup is *E–unitary* if, whenever e is an idempotent and $e \leqslant s$ then s is an idempotent.

1.2 Congruences and Green's Relations

A *congruence* on a semigroup S is an equivalence relation ρ on S such that, if $(a, b) \in \rho$ and $(c, d) \in \rho$ then $(ac, bd) \in \rho$. An equivalence relation is called a *left congruence* if $(a, b) \in \rho$ implies that $(ca, cb) \in \rho$ for any $c \in S$. *Right congruences* are defined dually. An equivalence relation is a congruence if it is a left and right congruence.

We recall the definitions of Green's relations $\mathcal{L}, \mathcal{R}, \mathcal{H}$ and $\mathcal{D}$ on an inverse semigroup. We shall subsequently relate these to the interpretation of an inverse semigroup as an inductive groupoid in section 1.4.2.

1.2.1 Definition. We define Green's $\mathcal{L}$ and $\mathcal{R}$ relations as follows:

$$(s,t) \in \mathcal{L} \Longleftrightarrow s^{-1}s = t^{-1}t \quad \text{and} \quad (s,t) \in \mathcal{R} \Longleftrightarrow ss^{-1} = tt^{-1}\,.$$

Both $\mathcal{L}$ and $\mathcal{R}$ are equivalence relations: indeed $\mathcal{L}$ is a right congruence and $\mathcal{R}$ is a left congruence. We can consider the equivalence classes of these relations, $\mathcal{L}$-classes and $\mathcal{R}$-classes respectively, such that for $s \in S$ the $\mathcal{L}$-class of s is $L_s = \{t \in S : t^{-1}t = s^{-1}s\}$ and the $\mathcal{R}$-class of s is $R_s = \{t \in T : tt^{-1} = ss^{-1}\}$.

The equivalence relation $\mathcal{H}$ is defined by $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$, and the equivalence relation $\mathcal{D}$ is defined by $\mathcal{D} = \mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}$. The $\mathcal{D}-$relation on an inverse semigroup can be characterized explicitly by:

$$s\,\mathcal{D}\,t \iff \exists z \text{ such that } s^{-1}s = zz^{-1} \text{ and } z^{-1}z = tt^{-1}.$$

The set of equivalence classes S/ρ of a congruence on S is a semigroup in a natural way: if $[a]$ denotes the equivalence class of $a \in S$, then we define $[a][b] = [ab]$. Any semigroup homomorphism determines a congruence on its domain: given $\phi : S \to T$ we define

$$a\,\rho\,b \qquad \iff \qquad a\phi = b\phi\,.$$

Conversely, the canonical map $S \to S/\rho$ is a semigroup morphism. We then have the following useful result and its corollary, (see [19, Lemma 2.4.3]).

1.2.1 Lallement's Lemma. *If ρ is a congruence on a regular semigroup S, and for $a \in S$, we have $a\,\rho\,a^2$, then $a\,\rho\,e$ for some $e \in E(S)$.*

Proof. Let $v \in S$ satisfy $a^2va^2 = a^2$ and $va^2v = v$. Set $e = ava$. Then $e^2 = ava^2va = ava = e$, so $e \in E(S)$, and $a\,\rho\,a^2 = a^2va^2\,\rho\,ava = e$. □

1.2.2 Corollary. *If ρ is a congruence on an inverse semigroup, then S/ρ is an inverse semigroup.*

1.2.2 Definition.

(a) A congruence ρ on an inverse semigroup S is said to be *idempotent pure* if $a \in S$ and $a \,\rho\, e$ for some $e \in E(S)$ imply that $a \in E(S)$.

(b) A congruence ρ on an inverse semigroup S is said to be *idempotent separating* if $e, f \in E(S)$ and $e \,\rho\, f$ imply that $e = f$.

1.2.3 Remark. Any inverse semigroup homomorphism $\phi : S \to T$ induces a congruence χ_ϕ on S by

$$a \,\chi_\phi\, b \iff a\phi = b\phi\,.$$

We say that ϕ is idempotent pure (respectively, idempotent separating) if χ_ϕ has this property. To avoid excessive notation we have chosen in later chapters to use ϕ interchangeably as both the homomorphism and the congruence.

We conclude this introductory account of congruences on inverse semigroups with an important example.

1.2.4 Theorem. *[22, Theorem 2.4.1] The relation σ defined on the inverse semigroup S by*

$$s \,\sigma\, t \text{ if and only if there exists } u \in S \text{ such that } u \leqslant s \text{ and } u \leqslant t$$

is a congruence on S, and S/σ is a group. Moreover, if ρ is any congruence on S such that S/ρ is a group, then $\sigma \subseteq \rho$.

The congruence σ is the *minimum group congruence*. We shall denote the quotient S/σ by $\widehat{S}$.

1.2.5 Theorem. *[22, Theorem 2.4.6] Let S be an inverse semigroup. Then the minimum group congruence is idempotent pure if and only if S is E–unitary.*

1.3 Clifford Semigroups

Clifford semigroups constitute a class of inverse semigroups that will be of importance in the description of relation modules in section 6.2.4.

Let $(E, \leqslant)$ be a meet semilattice, and let $\{G_e : e \in E\}$ be a family of disjoint groups indexed by the elements of E, the identity of G_e being denoted by 1_e. For each pair e, f of elements of E where $e \geqslant f$ let $\phi_{e,f} : G_e \to G_f$ be a group homomorphism, such that the following two axioms hold:

- $\phi_{e,e}$ is the identity homomorphism on G_e,
- if $e \geqslant f \geqslant g$ then $\phi_{e,f}\, \phi_{f,g} = \phi_{e,g}$.

We call such a family

$$(G_e, \phi_{e,f}) = (\{G_e : e \in E\}, \{\phi_{e,f} : e, f \in E,\ f \leqslant e\})$$

a presheaf of groups over E.

Let $(G_e, \phi_{e,f})$ be a presheaf of groups. Let $S = S(G_e, \phi_{e,f})$ be the union of the G_e equipped with the product $\otimes$ on S defined by:

$$x \otimes y = (x\phi_{e,ef})(y\phi_{f,ef}),$$

where $x \in G_e$ and $y \in G_f$. Then $(S, \otimes)$ is an inverse semigroup, called a *Clifford* semigroup.

1.3.1 Proposition. *If a homomorphism $\phi : S \to T$ of inverse semigroups is idempotent separating then its kernel is a Clifford semigroup over $E(S)$.*

Proof. Suppose that ϕ is idempotent separating and that $a \in \ker \phi$. Then $a\phi = x \in E(T)$ and so

$$(aa^{-1})\phi = (a\phi)(a\phi)^{-1} = x = (a\phi)^{-1}(a\phi) = (a^{-1}a)\phi.$$

Hence $aa^{-1} = a^{-1}a$.

For $e \in E(S)$ we set

$$K_e = \{a \in \ker\phi : aa^{-1} = e = a^{-1}a\}\,.$$

The it is easy to see that K_e is a group with identity e. Now if $a \in K_e$ and $f \in E(S)$ then $af \in \ker\phi$, and so

$$(af)(af)^{-1} = afa^{-1} = (af)^{-1}(af) = fa^{-1}a = faa^{-1}\,.$$

Hence

$$af = af(a^{-1}a) = (afa^{-1})a = (faa^{-1})a = fa$$

and so K_e centralises $E(S)$. If $e \geqslant f$ we may then define $\kappa_{e,f} : K_e \to K_f$ by $a \mapsto fa \quad (= af)$, and in S, for $a \in K_e$ and $b \in K_f$,

$$(a\kappa_{e,ef})(b\kappa_{f,ef}) = (aef)(efb) = (ae)(fb) = (aa^{-1}a)(bb^{-1}b) = ab\,.$$

□

1.4 Groupoids and Inverse Semigroups

1.4.1 Groupoids

We shall give two definitions of a groupoid, which are equivalent. The first definition presents a groupoid as an algebraic structure similar to a group, but in which the binary operation is no longer always defined. This idea goes back to the introduction of the groupoid concept by Brandt [3].

1.4.1 Definition. A *groupoid* is a set G with a unary operation, $^{-1} : G \to G$, and a partial function, $\circ : G \times G \to G$, such that the following axioms hold, for $a, b, c \in G$:

- **Associativity:** if both $a \circ b$ and $b \circ c$ are defined then $(a \circ b) \circ c$ and $a \circ (b \circ c)$ are defined and equal.
- **Inverses:** $a \circ a^{-1}$ and $a^{-1} \circ a$ are always defined.
- **Identities:** if $a \circ b$ is defined then $a \circ b \circ b^{-1} = a$ and $a^{-1} \circ a \circ b = b$.

The second definition is based in category theory, and presents a groupoid as a special kind of category.

1.4.2 Definition. A *category* $\mathcal{C}$ consists of:

- a collection of objects $\mathcal{C}^0 0$,
- for any two objects $A, B \in \mathcal{C}^0 0$ a set of arrows from A to B. We will denote the collection of all arrows as $\mathcal{C}^1$, and the set of arrows from A to B as $\mathcal{C}(A, B)$. Two such arrows, say from A to B and from B to C can be composed to obtain an arrow from A to C. If $f \in \mathcal{C}(A, B)$ we say that f has *domain* A and *range* B and write $f\mathbf{d} = A$ and $f\mathbf{r} = B$.
- for each object $A \in \mathcal{C}^0$ we have an identity arrow, $1_A \in \mathcal{C}(A, A)$.

These satisfy the following:

- **Associativity:** for each $f \in \mathcal{C}(A, B)$, $g \in \mathcal{C}(B, C)$ and $h \in \mathcal{C}(C, D)$ we have

$$(f \circ g) \circ h = f \circ (g \circ h)$$

- **Identity:** for each $f \in \mathcal{C}(A, B)$ we have:

$$f \circ 1_B = f = 1_A \circ f$$

1.4.3 Definition. A *small category* is a category where the collections of objects is a set.

1.4.4 Definition. A *groupoid* is a small category in which all of the arrows are invertible. So given $f \in \mathcal{C}(A, B)$ there exists a (unique) $f^{-1} \in \mathcal{C}(B, A)$ such that $f \circ f^{-1} = 1_A$ and $f^{-1} \circ f = 1_B$.

A morphism of groupoids is then just a functor between such categories.

Given a topological space we can consider a groupoid associated with it, its fundamental groupoid. We will use these fundamental groupoids in later sections.

1.4.5 Definition. Let X be a topological space, then $\Pi(X)$, the *fundamental groupoid* of X, is the groupoid with objects X, and with arrows from x to y the homotopy classes, $[\alpha]$, of continuous maps $\alpha : [0, 1] \to X$ whose endpoints map to x and y. Composition is then concatenation of representative maps.

1.4.2 Inverse Semigroups and Inductive Groupoids

We consider a groupoid as an algebraic structure following [22, Chapter 4]: the elements are the morphisms, and composition is an associative partial binary operation. The set of identities in G is denoted G^0, and an element $g \in G$ has domain $g\mathbf{d} = gg^{-1}$ and range $g\mathbf{r} = g^{-1}g$. (Note that this reverses the conventions of [22, Chapter 4]). For each $x \in G^0$ the *star* at x is the set $\text{star}_x(G) = \{g \in G : g\mathbf{d} = x\}$, and the set $G(x) = \{g \in G : g\mathbf{d} = x = g\mathbf{r}\}$ is a subgroup of G, called the *local subgroup* at x.

A groupoid G is *connected* if, for any $x, y \in G^0$, there exists at least one $g \in G$ with $g\mathbf{d} = x$ and $g\mathbf{r} = y$, and G is *unicursal* if there exists at most one $g \in G$ with $g\mathbf{d} = x$ and $g\mathbf{r} = y$. A connected, unicursal groupoid therefore has exactly one edge from x to y for any $x, y \in G^0$. Such a groupoid is isomorphic to the *simplicial* groupoid on the set G^0, where for any set X the simplical groupoid $\Delta(X)$ has set of arrows $X \times X$: the arrows of the form (x, x) are the identities, (x, y) has inverse (y, x), and the composition of arrows is given by the rule $(x, y)(y, z) = (x, z)$.

In a connected groupoid G, all local subgroups are isomorphic, and for any such local subgroup L there is an isomorphism $G \cong G_0 \times L \times G_0$, where the latter set carries the groupoid composition $(x, k, y)\,(y, l, z) = (x, kl, z)$.

An *ordered groupoid* $(G, \leqslant)$ is a groupoid G with a partial order $\leqslant$ satisfying the following axioms:

(OG1) for all $g, h \in G$, if $g \leqslant h$ then $g^{-1} \leqslant h^{-1}$,

(OG2) if $g_1 \leqslant g_2\,, h_1 \leqslant h_2$ and if the compositions g_1h_1 and g_2h_2 are defined, then $g_1h_1 \leqslant g_2h_2$,

(OG3) if $g \in G$ and x is an identity of G with $x \leqslant g\mathbf{d}$, there exists a unique element $(x|g)$, called the *restriction* of g to x, such that $(x|g)\mathbf{d} = x$ and $(x|g) \leqslant g$,

As a consequence of (OG3) we also have:

(OG3*) if $g \in G$ and y is an identity of G with $y \leqslant g\mathbf{r}$, there exists a unique element $(g|y)$, called the *corestriction* of g to y, such that $(g|y)\mathbf{r} = y$ and $(g|y) \leqslant g$,

since the corestriction of g to y may be defined as $(y|g^{-1})^{-1}$.

Let G be an ordered groupoid and let $a, b \in G$. If $a\mathbf{r}$ and $b\mathbf{d}$ have a greatest lower bound $\ell \in G^0$, then we may define the *pseudoproduct* of a and b in G as:

$$a \otimes b = (a|\ell)\,(\ell|b),$$

where the right-hand side is now a composition defined in G. As Lawson shows in [22, Lemma 4.1.6], this is a partially defined associative operation on G.

If G^0 is a meet semilattice (see Remark 1.1.1) then G is called an *inductive* groupoid. The pseudoproduct is then everywhere defined and $(G, \otimes)$ is an inverse semigroup. On the other hand, given an inverse semigroup S with semilattice of idempotents $E(S)$, then S is a poset under the natural partial order, and the restriction of its multiplication to the partial composition

$$a \cdot b = ab \in S \;\text{ defined when }\; a^{-1}a = bb^{-1}$$

gives S the structure of an inductive groupoid, which we denote by $\vec{S}$, with $\mathrm{obj}(\vec{S}) = E(S)$. These constructions give rise to an isomorphism between the categories of inverse semigroups and inductive groupoids: this is the *Ehresmann-Schein-Nambooripad Theorem* [22, Theorem 4.1.8].

It is sometimes useful to adopt a less formal version of this correspondence, and to think of an element s of an inverse semigroup S as an arrow joining the idempotent ss^{-1} to the idempotent $s^{-1}s$.

1.4.3 Strict Monoidal Groupoids

We can form products of small categories and of groupoids by taking the set-theoretic product of the sets of objects and arrows, and performing all operations componentwise. Indeed, the category of small categories and the category of groupoids are each examples of *complete* categories, in that they admit all (category-theoretic) *limits*: see, for example, [18, Proposition 17].

Let $\mathbb{1}$ denote the trivial groupoid containing just one object and its associated identity arrow. A groupoid G (considered as a small category) is *monoidal* if there exist functors $\mu : G \times G \to G$ and $\eta : \mathbb{1} \to G$ such that the following diagrams commute:

$$\begin{array}{ccc} G \times G \times G & \xrightarrow{\mu \times \mathbf{1}} & G \times G \\ {\scriptstyle \mathbf{1} \times \mu}\downarrow & & \downarrow{\scriptstyle \mu} \\ G \times G & \xrightarrow[\mu]{} & G \end{array}$$

and

$$\begin{array}{ccccc} \mathbb{1} \times G & \xrightarrow{\eta \times \mathbf{1}} & G & \xleftarrow{\mathbf{1} \times \eta} & G \times \mathbb{1} \\ & \searrow_{\pi_2} & {\scriptstyle \mu}\downarrow & \swarrow_{\pi_1} & \\ & & G & & \end{array}$$

where $\mathbf{1}$ is the identity functor on G, and π_1, π_2 are the projection maps.

We note that $\eta : \mathbb{1} \to G$ simply selects a distinguished object $e \in G^0$ (and its associated identity arrow $\mathbf{1}_e$).

On restriction to objects, the diagrams above describe a monoid structure on G^0, with identity e. We write this as juxtaposition: so for $x, y \in G^0$, $(x, y)\mu = xy$. The functor μ defines an associative everywhere-defined composition on the arrows of G. For $f, g \in G^1$ we write $(f, g)\mu = f \otimes g$. Functorialty of μ then implies that if $f \in G(u, v)$ and $g \in G(x, y)$ then $f \otimes g \in G(ux, vy)$. Furthermore, for all $f, g, h, k \in G^1$ we have

$$(f \otimes g) \circ (h \otimes k) = (f \circ h) \otimes (g \circ k)\,. \qquad (1.1)$$

Equation (1.1) is known as the *interchange law.* The *Eckmann-Hilton argument* [11] shows that if G is a group then the operations $\circ$ (the group multiplication) and $\otimes$ coincide and are commutative.

A monoidal groupoid is a monoid object in the category of groupoids, using the standard product structure. We shall construct examples in later chapters.

1.5 Free Monoids and Free Inverse Monoids

Let A be a set, and let A^* be the set of all finite strings of elements of A. The empty string, denoted by ε, is an element of A^*. The length of a string $w \in A^*$ is denoted by $|w|$. Then A^* is a monoid under the operation of concatenation of strings, with identity element ε, and is called the *free monoid* on A. Note that the strings of length 1 are the elements of A, so that $A \subset A^*$. The freeness property possessed by A^* is the extension of functions to monoid homomorphisms: given any function $f : A \to M$ from A to a monoid M, there exists a unique monoid homomorphism $f^* : A^* \to M$ such that, for all $a \in A$, $af^* = af$.

The *free inverse monoid* $\mathrm{FIM}(X)$ on a set X satisfies the following freeness property: given any function $f : X \to M$ from X to an inverse monoid M, there exists a unique inverse monoid homomorphism $f^{\#} : \mathrm{FIM}(X) \to M$ such that, for all $x \in X$, $xf^{\#} = xf$. The elements of FIM(X) were described by Munn [28] using what are now called *Munn trees*: see [22, Section 6.4].

Let $F(X)$ be the free group on X, we will think of this as the collection of reduced words on the alphabet $X \cup X^{-1}$. The Cayley graph Γ of $F(X)$, with respect to the free generating set X, is a tree with vertex set $V(\Gamma) = F(X)$, and an element of FIM(X) is a pair (P, u) where P is a finite connected subtree of Γ with $1 \in V(P)$, and $u \in V(P)$. The pair (P, u) is a *Munn tree*. The multiplication in FIM(X) is then given by:

$$(P, u)(Q, v) = (P \cup uQ, uv).$$

We note that a generator $x \in X$ is then represented by the pair (e_x, x) where e_x is the directed edge in Γ from 1 to x. The natural partial order on FIM(X) is given by

$$(P, u) \leqslant (Q, v) \iff P \supseteq Q \text{ and } u = v\,.$$

We may carry out the same construction in any Cayley graph $\mathrm{Cay}(G, X)$ of a group G with generating set X. A *pattern* in $\mathrm{Cay}(G, X)$ is a connected subgraph that contains the vertex $1 \in G$, and a *pointed pattern* is a pair (P, g) where P is a pattern and $g \in G$ is a distinguished vertex of P. Pointed patterns in $\mathrm{Cay}(G, X)$ may then be multiplied by the rule

$$(P, g)(Q, h) = (P \cup gQ, gh)$$

and in this way we obtain an inverse monoid $\mathscr{M}(G, X)$, the *Margolis-Meakin* or *graph expansion* of (G, X). Margolis and Meakin [26, Theorem 2.2] show that any X–generated E–unitary inverse monoid M with maximum group image G is an idempotent pure image of $\mathscr{M}(G, X)$.

1.6 Modules for Inverse Semigroups

Modules for inverse semigroups were first defined by Lausch [21].

1.6.1 Definition. Let S be an inverse semigroup with semilattice of idempotents $E(S)$. Consider a Clifford semigroup $\mathcal{A} = (A_e, \alpha_{e,f})$ (see section 1.3), in which each A_e is an additively written abelian group with identity 0_e. The disjoint union $A = \bigsqcup_{e \in E(S)} A_e$ is a commutative inverse semigroup under the operation

$$a \oplus b = a\alpha_{e,ef} + b\alpha_{f,ef}$$

for $a \in A_e$ and $b \in A_f$. Then $\mathcal{A}$ is an S*–module* [21, section 2] if there exists a map $A \times S \to A$, written $(a, s) \mapsto a \lhd s$, such that

(i) $(a \oplus b) \lhd s = a \lhd s \oplus b \lhd s$ for all $a, b \in A$ and $s \in S$,

(ii) $a \lhd st = (a \lhd s) \lhd t$ for all $a \in A$ and $s, t \in S$,

(iii) $a \lhd e = a \oplus 0_e$ for all $a \in A$ and $e \in E(S)$,

(iv) $0_e \lhd s = 0_{s^{-1}es}$ for all $e \in E(S)$ and $s \in S$.

Loganathan [24] then showed that Lausch's S–modules could also be described as modules for a left-cancellative category $\mathfrak{L}(S)$ associated to S – that is, as functors from $\mathfrak{L}(S)$ to the category of abelian groups. The category $\mathfrak{L}(S)$ is defined as follows.

1.6.2 Definition. For an inverse semigroup S, we construct the category $\mathfrak{L}(S)$ with set of objects $E(S)$, and set of arrows

$$\{(e, s) : e \in E(S), s \in S, e \geqslant ss^{-1}\}.$$

We define $(e, s)\mathbf{d} = e$, $(e, s)\mathbf{r} = s^{-1}s$, and (e, e) to be the identity arrow at e. Composition is defined by $(e, s)(f, t) = (e, st)$ whenever $s^{-1}s = f$.

1.6.1 Lemma. *The category $\mathfrak{L}(S)$ is left cancellative, and a homomorphism of inverse semigroups $\phi : S \to T$ induces a functor $F_\phi : \mathfrak{L}(S) \to \mathfrak{L}(T)$ which maps $(e, s) \mapsto (e\phi, s\phi)$.*

Proof. Suppose that $(e, s)(f, t) = (e, s)(g, u)$ in $\mathfrak{L}(S)$. Then $(e, st) = (e, su)$ and $f = s^{-1}s = g$. So $f = g$ and

$$t = ft = s^{-1}st = s^{-1}su = gu = u\,.$$

The existence of F_ϕ is clear. □

Loganathan defines an *S–module* to be a functor $\mathcal{A}$ from $\mathfrak{L}(S)$ to the category of abelian groups, and shows in [24, Lemma 2.6] that this defines a category of S–modules isomorphic to that defined by Lausch in [21], (see Definition 1.6.1). We sketch Loganathan's constructions of the connections of the two notions of modules in the next result.

1.6.2 Proposition. *[24, Lemma 2.6] Let S be an inverse semigroup and let $\mathcal{A}$ be an S–module in the sense of Lausch. Then $\mathcal{A}$ determines a functor $\mathcal{A}_{\mathfrak{L}}$ from $\mathfrak{L}(S)$ to abelian groups, and so gives a module in the sense of Loganathan. Conversely, any functor $\mathcal{B}$ from $\mathfrak{L}(S)$ to abelian groups determines a Lausch S–module $\mathcal{B}_L$, and these constructions are inverse to one another.*

Proof. For a Lausch S–module $\mathcal{A}$, the functor $\mathcal{A}_{\mathfrak{L}}$ carries $e \in E(S)$ to the abelian group A_e, and an arrow (e, s) to the map $A_e \to A_{s^{-1}s}$ given by $a \mapsto a \lhd s$. Part (i) of Definition 1.6.1 ensures that this map is a homomorphism of abelian groups. If $a \in A_e$, then

$$0_{s^{-1}es} = 0_e \lhd s = (a - a) \lhd s = (a \lhd s) \oplus (-a \lhd s) = (a \lhd s) - (a \lhd s)$$

and so $a \lhd s \in A_{s^{-1}es}$. Hence, if $e \geqslant ss^{-1}$, then $a \lhd s \in A_{s^{-1}s}$.

Conversely, a functor $\mathcal{B}$ from $\mathfrak{L}(S)$ to abelian groups determines a Clifford semigroup $\mathcal{B}_L = (B_e, \beta_{e,f})$ with $B_e = e\mathcal{B}$ and $\beta_{e,f} = (e,f)\mathcal{B}$. We then define, for $a \in B_e$ and $s \in S$,

$$a \lhd s = a((e, es)\mathcal{B}) \,.$$

We check that this action satisfies the four conditions given in Definition 1.6.1.

For (i) we have, for $a \in B_e$ and $b \in B_f$, using the functorial properties of $\mathcal{B}$,

$$\begin{aligned}
(a \oplus b) \lhd s &= ((a\beta_{e,ef}) + (b\beta_{f,ef})) \lhd s \\
&= [a(e,ef)\mathcal{B} + b(f,ef)\mathcal{B}](ef, efs)\mathcal{B} \\
&= a(e,ef)\mathcal{B}(ef,efs)\mathcal{B} + b(f,ef)\mathcal{B}(ef,efs)\mathcal{B} \\
&= a(e,efs)\mathcal{B} + b(f,efs)\mathcal{B} \\
&= a((e,es)(s^{-1}es, s^{-1}efs))\mathcal{B} + b((f,fs)(s^{-1}fs, s^{-1}efs))\mathcal{B} \\
&= (a(e,es)\mathcal{B})\beta_{s^{-1}es, s^{-1}efs} + (b(f,fs)\mathcal{B})\beta_{s^{-1}fs, s^{-1}efs} \\
&= (a \lhd s) \oplus (b \lhd s) \,.
\end{aligned}$$

For (ii) we have, for $a \in B_e$ and $s, t \in S$,

$$\begin{aligned}
a \lhd st = a(e, est)\mathcal{B} &= a((e,es)(s^{-1}es, s^{-1}est))\mathcal{B} \\
&= (a((e,es)\mathcal{B})(s^{-1}es, s^{-1}est))\mathcal{B} = (a \lhd s) \lhd t \,.
\end{aligned}$$

For (iii) we have, for $a \in B_e$ and $f \in E(S)$,

$$a \lhd f = a(e,ef)\mathcal{B} = a\beta_{e,ef} = a\beta_{e,ef} + 0_{ef} = a\beta_{e,ef} + 0_f\beta_{f,ef} = a \oplus 0_f \,.$$

Finally, for (iv) we have, for $e \in E(S)$ and $s \in S$,

$$0_e \lhd s = 0_e(e, es)\mathcal{B} = 0_{s^{-1}es} \,.$$

Hence $\mathcal{B}_L$ is a Lausch S–module.

It is clear that the constructions given above are inverse to one another. □

Remark. The correspondence in Proposition 1.6.2 may be readily extended to a correspondence between the obvious notions of S–module morphism, to establish the full strength of [24, Lemma 2.6], that the two categories of S–modules in the sense of Lausch and of Loganathan, are isomorphic.

1.6.3 Example. Any abelian group A extends to the *constant* or *homogeneous* S–module $\underline{A}$, in which $\underline{A}_e = A$ and all the maps $(e,s)\underline{A}$ are identities. In particular we can take $A = \mathbb{Z}$ to obtain the constant S–module $\underline{\mathbb{Z}}$.

A further construction of Loganathan [24] is the $\mathfrak{L}(S)$–module $\mathbb{Z}S$, defined as follows. For each idempotent $e \in E(S)$, let L_e be the $\mathcal{L}$–class of e, that is $L_e = \{s \in S : s^{-1}s = e\}$, and let $\mathbb{Z}S_e$ be the free abelian group with basis L_e. Now if $a \in L_e$ and $(e,s) \in \mathfrak{L}(S)$ we define $a \lhd (e,s) = as$. Since $e = a^{-1}a \geqslant ss^{-1}$, it follows that $(as)^{-1}(as) = s^{-1}a^{-1}as = s^{-1}s$, so that $as \in L_{s^{-1}s}$ and the mapping $a \mapsto as$ induces a homomorphism $\mathbb{Z}S_e \to \mathbb{Z}S_{s^{-1}s}$. The *augmentation* map $\varepsilon_S : \mathbb{Z}S \to \underline{\mathbb{Z}}$ is the $\mathfrak{L}(S)$–map defined on the basis L_e of $\mathbb{Z}S_e$ by $s \mapsto 1 \in \underline{\mathbb{Z}}_e$. It is clear that this is an $\mathfrak{L}(S)$–map, and its kernel is the *augmentation module* IS of S.

1.6.4 Lemma. *For each $e \in E(S)$, the abelian group IS_e is freely generated by the elements $s - e$ with $e \neq s \in L_e$.*

Proof. Let $x = \sum_{i\in I} n_i s_i \in IS_e$, so that each $s_i \in L_e$. Since $x\varepsilon_S = 0$, we have $\sum_{i\in I} n_i = 0$ and hence $\sum_{i\in I} n_i e = 0$. Then

$$x = \sum_{i\in I} n_i s_i - \sum_{i\in I} n_i e = 0 = \sum_{i\in I} n_i(s_i - e) = 0.$$

Hence the elements $s-e$ with $s \in L_e$ generate IS_e, and since $\mathbb{Z}S_e$ is freely generated by L_e, it is clear that the elements $s - e$ with $e \neq s \in L_e$ are a basis. □

We remark that essentially the same definitions of $\mathbb{Z}G$ and its augmentation module were given for a groupoid G by Brown and Higgins [6].

Loganathan's description of an S–module also allows a clear definition of freeness (see [25, section 2]). We begin with an $E(S)$–set $\mathcal{X}$, that is a family of disjoint sets X_e indexed by $E(S)$. (This is equivalent to a functor from the trivial category with object set $E(S)$ to the category of sets.) An $\mathfrak{L}(S)$–module $\mathcal{F}$ is then *free on* $\mathcal{X}$ if, for each $e \in E(S)$, we have a map $i_e : X_e \to F_e$ such that, for every $\mathfrak{L}(S)$–module $\mathcal{A}$ and for each family of maps $j_e : X_e \to A_e$, there exists a unique $\mathfrak{L}(S)$–map $\phi : \mathcal{F} \to \mathcal{A}$ such that $i_e\phi_e = j_e$ for all $e \in E(S)$. To construct $\mathcal{F}$ on basis $\mathcal{X}$, we set the group F_e to be the free abelian group on the basis

$$\{(x, (f, s)) : x \in X_f, (f, s) \in \mathfrak{L}(S), s^{-1}s = e\}\,.$$

The action $F_e \to F_k$ of an arrow (e, t) with $t^{-1}t = k$ is given by $(x, (f, s)) \lhd (e, t) = (x, (f, st))$. Loganathan notes [24, Remark 4.2] that if S is an inverse monoid then $\mathbb{Z}S$ is a free $\mathfrak{L}(S)$–module. A basis in this case is the $E(S)$–set $\mathcal{X}$ with X_1 equal to a singleton set, and with $X_e = \emptyset$ for $e \neq 1$.

Finally in this section, we give the details of one further example which will be of interest later. Let $\phi : S \to T$ be an idempotent separating surjective homomorphism of inverse semigroups, so that, by Proposition 1.3.1, $K = \ker\phi$ is a Clifford semigroup $(K_e, \kappa_{e,f})$. Abelianise each K_e and let $\overline{\kappa}_{e,f}$ be the induced map $K_e^{ab} \to K_f^{ab}$. Set $\mathcal{K} = (K_e^{ab}, \overline{\kappa}_{e,f})$. Then:

1.6.5 Proposition. *$\mathcal{K}$ is a T–module, with the T– action defined by*

$$\overline{k} \lhd t = \overline{s^{-1}ks} \quad \textit{where} \quad s\phi = t\,.$$

Proof. We first check that the action is well-defined. Suppose that $a\phi = b\phi$, with

$aa^{-1} = x = bb^{-1}$ and $a^{-1}a = y = b^{-1}b$. Let $k \in K_e$. Then

$$\begin{aligned} a^{-1}ka &= a^{-1}aa^{-1}kaa^{-1}a = a^{-1}bb^{-1}kbb^{-1}a \\ &= (a^{-1}b)(b^{-1}kb)(b^{-1}a) \\ &= (a^{-1}b)\kappa_{y,ye}(b^{-1}kb)(b^{-1}a)\kappa_{y,ye}\,. \end{aligned}$$

The last line here is a conjugation in the group K_{ye}^{ab}, and so on abelianisation we find that

$$\overline{a^{-1}ka} = \overline{b^{-1}kb} \in K_{ye}^{ab}\,.$$

It is then easy to check that the axioms listed in Definition 1.6.1 all hold, so that $\mathcal{K}$ is a T–module. $\square$

1.7 Presentations

We shall consider presentations of groups and of inverse monoids, and we establish our conventions here.

1.7.1 Definition. A group *presentation* $\mathcal{P} = \langle X : \mathcal{R} \rangle$ of a group G, consists of a set of *generators* X, and a set of *relators* $\mathcal{R} \subseteq (X \cup X^{-1})^* \times (X \cup X^{-1})^*$. We write $A = X \cup X^{-1}$. Our formalism allows for relators that are not freely reduced: by allowing a multiset $\mathcal{R} \subseteq A^* \times A^*$ we can also allow for repeated relators. We let $\rho : A^* \to F(X)$ be the canonical map, and define $\widehat{\rho} : \mathcal{R} \to F(X)$ by $(\ell, r)\widehat{\rho} = (\ell^{-1}r)\rho$. We let R be the image of $\widehat{\rho}$ in $F(X)$, and define $N = \langle\langle R \rangle\rangle$ to be the normal closure of R in F, so that a typical element of N has the form

$$u_1^{-1}(r_1\widehat{\rho})^{\varepsilon_1}u_1 \cdots u_k^{-1}(r_k\widehat{\rho})^{\varepsilon_k}u_k\,,$$

where, for $1 \leqslant j \leqslant k$, we have $u_j \in F$, $r_j \in \mathcal{R}$, and $\varepsilon_j = \pm 1$. Then G is the quotient group $F(X)/N$, and we have a canonical *presentation map* $\theta : F(X) \to G$.

We note that $N = \ker\theta$ and denote the abelianisation of N by N^{ab}, these groups will be important in later chapters.

In most cases, however, we shall regard $\mathcal{R}$ as a subset of $F(X) \times F(X)$, with $(\ell, r) \in \mathcal{R}$ then corresponding to the relation $\ell = r$, which is to hold in the group G.

1.7.2 Definition. An inverse monoid *presentation* $\mathcal{Q} = [X : \mathcal{R}]$ of an inverse monoid S, consists of a set of *generators* X, and a set of *relations* $\mathcal{R} \subseteq A^* \times A^*$, where again we write $A = X \cup X^{-1}$. The canonical image of $\mathcal{R}$ in $\mathrm{FIM}(X) \times \mathrm{FIM}(X)$ generates a congruence θ on $\mathrm{FIM}(X)$, and S is the quotient $\mathrm{FIM}(X)/\theta$. We also use θ to denote the *presentation map* $\mathrm{FIM}(X) \to S$.

Two inverse monoid presentations $\mathcal{Q} = [X : \mathcal{R}]$ and $\mathcal{P} = [X : \mathcal{T}]$ with the same set of generators are said to be *equivalent* if $\mathcal{R}$ and $\mathcal{T}$ generate the same congruence on $\mathrm{FIM}(X)$.

The fact that we use a congruence to obtain the presented inverse monoid S, where as for a group presentation we use a quotient group by a normal subgroup, reflects the structural difference between congruences on groups and on inverse semigroups. We shall return to this in chapter 4.

The (left) *Schützenberger* graph $\mathrm{Sch}^{\mathcal{L}}(S, X)$ of an inverse monoid S generated by a set X is a directed graph with vertex set S. There is a directed edge, labelled by $(x, s) \in X \times S$, from s to xs whenever $xs \,\mathcal{L}\, s$, as in Green's $\mathcal{L}$ relation, Definition 1.2.1, or equivalently whenever $x^{-1}x \geqslant ss^{-1}$, which we recall from Definition 1.2.1 of Green's $\mathcal{L}$ relation.

1.7.1 Lemma. *Suppose that* $s \in S$ *and that* $s = x_{i_1}^{\varepsilon_1} \cdots x_{i_k}^{\varepsilon_k}$ *with* $x_{i_j} \in X$ *and* $\varepsilon_j = \pm 1$. *Then there exists a path in* $\mathrm{Sch}^{\mathcal{L}}(S, X)$ *from* $s^{-1}s$ *to* s *of the form*

$$(x_{i_k}, u_k)^{\varepsilon_k} \cdots (x_{i_1}, u_1)^{\varepsilon_1}$$

where $u_j \,\mathcal{L}\, s$.

Proof. Let v_j $(1 \leqslant j \leqslant k)$ be the suffix $x_{i_{k-j+1}}^{\varepsilon_{k-j+1}} \cdots x_{i_k}^{\varepsilon_k}$ of the given expression for s of length j : so $v_k = s$ and we set $v_0 = 1$. Let p_j be the associated prefix $x_{i_1}^{\varepsilon_1} \cdots x_{i_{k-j}}^{\varepsilon_{k-j}}$, so that $s = p_j v_j$, and set $u_j = v_j s^{-1} s$. Then for all j,

$$u_j^{-1} u_j = s^{-1} s v_j^{-1} v_j = s^{-1} p_j v_j v_j^{-1} v_j = s^{-1} p_j v_j = s^{-1} s \,.$$

Let $0 \leqslant m \leqslant k-1$ and suppose that $\varepsilon_{k-m} = 1$. Then,

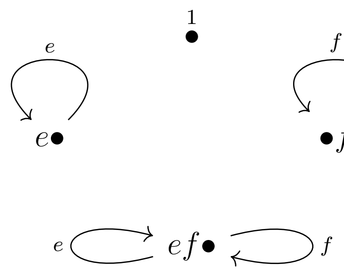
$$x_{k-m} u_m = x_{k-m} v_m s^{-1} s = v_{m+1} s^{-1} s = u_{m+1}$$

and there is an edge labelled (x_{k-m}, u_m) connecting u_m and u_{m+1} in $\mathrm{Sch}^{\mathcal{L}}(S, X)$. Similarly, if $\varepsilon_{k-m} = -1$, there is an edge labelled (x_{k-m}, u_{m+1}) connecting u_{m+1} and u_m. □

The connected components of $\mathrm{Sch}^{\mathcal{L}}(S, X)$ are therefore in one-to-one correspondence with the $\mathcal{L}$–classes of S, and so each connected component contains a unique idempotent vertex. The component containing $e \in E(S)$ will be denoted $\mathrm{Sch}^{\mathcal{L}}(S, X, e)$. There exists a dual version, the right Schützenberger graph $\mathrm{Sch}^{\mathcal{R}}(S, X)$ whose connected components correspond to the $\mathcal{R}$–classes of S, and this right version is the one more usually discussed, see [26, 40].

1.7.1 Examples of Schützenberger Graphs

1.7.2 Example. Let M be the semilattice $\{1, e, f, ef\}$, generated as an inverse monoid by $X = \{e, f\}$. Each element of M is its own $\mathcal{L}$–class, and the Schützenberger graph is



1.7.3 Example. The *bicyclic monoid* B is the inverse monoid presented by $[x : xx^{-1} = 1]$. An element of $b \in B$ has a unique expression as $b = x^{-p}x^q$ with $p, q \geqslant 0$, and such an element is an idempotent if and only if $p = q$. Clearly $bb^{-1} = x^{-p}x^p$ and $b^{-1}b = x^{-q}x^q$: hence the $\mathcal{L}$–class of $b = x^{-p}x^q$, with $q \geqslant 0$, is

$$L_{x^{-p}x^q} = \{x^{-k}x^q : k \geqslant 0\}\,.$$

Left multiplication by x does not change the $\mathcal{L}$–class, and so the Schützenberger graph $\mathrm{Sch}(B, x, x^{-q}x^q)$ is the semi-infinite path

$$x^q \xleftarrow{\;x\;} x^{-1}x^q \xleftarrow{\;x\;} x^{-2}x^q \xleftarrow{\;x\;} \dots \xleftarrow{\;x\;} x^{-k}x^q \xleftarrow{\;x\;} \dots$$

1.7.4 Example. Given an inverse monoid M with presentation $[Y : R]$, we add a zero to M to obtain M^0. For M^0 we take the generating set $X = Y \cup \{z\}$ (with $z \notin Y$), and we have a presentation $\mathcal{Q}$ of M^0 given by

$$\mathcal{Q} = [Y, z : R, z^2 = z, yz = z = zy \;(y \in Y)]\,.$$

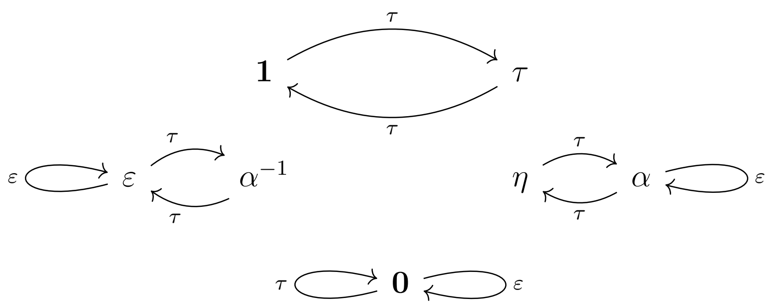
The element $0 \in M^0$ is now an $\mathcal{L}$–class, and in the Schützenberger graph there is a loop at 0 labelled for each element of Y.

1.7.5 Example. The symmetric inverse monoid $\mathcal{I}_2$ on the set $\{1, 2\}$ has seven elements, represented in the usual matrix notation with $*$ indicating 'undefined', as

$$\mathbf{1} = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \tau = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \varepsilon = \begin{pmatrix} 1 & 2 \\ 1 & * \end{pmatrix}, \alpha = \begin{pmatrix} 1 & 2 \\ 2 & * \end{pmatrix},$$

$$\alpha^{-1} = \begin{pmatrix} 1 & 2 \\ * & 1 \end{pmatrix}, \eta = \begin{pmatrix} 1 & 2 \\ * & 2 \end{pmatrix}, \mathbf{0} = \begin{pmatrix} 1 & 2 \\ * & * \end{pmatrix}$$

The $\mathcal{I}_2$ is generated by τ and ε, and the Schützenberger graph $\mathrm{Sch}^{\mathcal{L}}(\mathcal{I}_2, \{\tau, \varepsilon\})$ is:



1.7.2 Schützenberger Automata

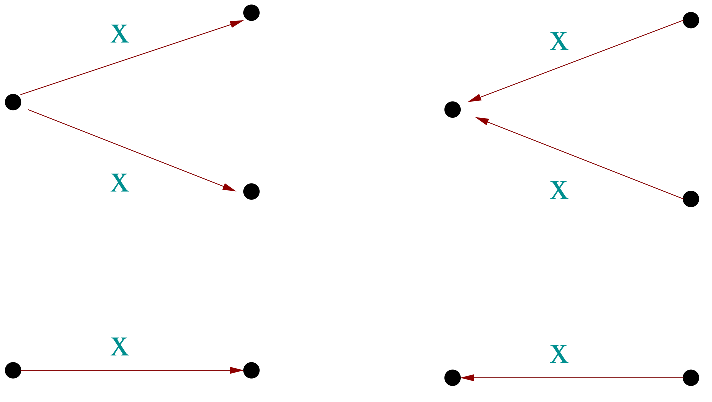
For each $s \in S$, the connected component $\mathrm{Sch}^{\mathcal{L}}(S, X, s)$ of $\mathrm{Sch}^{\mathcal{L}}(S, X)$ that contains s may be thought of as a deterministic automaton – the *Schützenberger automaton* with input alphabet A, with $s^{-1}s$ as its start state and s as the only accept state. This automaton accepts a language $\mathcal{L}(S, X, s)$. The connections between inverse monoid presentations and Schützenberger automata are due to Stephen [40] and are summarised in the next result.

1.7.6 Theorem. *[40, Theorem 3.1] Let S be an inverse monoid generated by X, and let $A = X \cup X^{-1}$. Let $\theta : A^* \to S$ be the canonical map. Then for any $u \in A^*$, the language $\mathcal{L}(S, X, w\theta)$ accepted by $\mathrm{Sch}^{\mathcal{L}}(S, X, u\theta)$ is*

$$\mathcal{L}(S, X, u\theta) = \{w \in A^* : w\theta \geqslant u\theta\}\,.$$

Theorem 1.7.6 points to a method for solving the word problem for S, since for $u, v \in A^*$ we have $u\theta = v\theta$ if and only if $u \in \mathcal{L}(S, X, v\theta)$ and $v \in \mathcal{L}(S, X, u\theta)$. Of course, the automata $\mathcal{L}(S, X, w\theta)$ needs to be constructible, and Stephen [40, section 5] gives an iterative construction from a presentation $\mathcal{Q} = [X, \mathcal{R}]$ of S. The starting point is the *linear graph* of w: if $w = x_{i_1}^{\varepsilon_1} \cdots x_{i_k}^{\varepsilon_k}$ its linear graph is the path from $(w^{-1}w)\theta$ to $w\theta$ constructed in the proof of Lemma 1.7.1. The linear graph is then modified by a sequence of *expansions* and *folds*. If (ℓ, r) is a relation in $\mathcal{R}$ and

ℓ labels a path in some automaton, then an expansion adjoins a new path (with the same endpoints) labelled by r. A fold (as defined by Stallings [39]) collapses edges with the same label and same initial vertex:



We refer to [40] for further details.

Chapter 2

Crossed Modules and Group Presentations

2.1 Crossed Modules

Here we introduce precrossed modules and crossed modules (of groups). These will be the algebraic models of group presentations that we shall use in our formulation of the relation module and the module of identities for a group presentation. For a more detailed account of these topics , we refer to [8].

A *precrossed module* (of groups) is a group homomorphism $\partial : T \to \Gamma$ together with an action of Γ on T (written $(t, g) \mapsto t^g$) such that ∂ is Γ–equivariant, that is, for all $t \in T$ and $g \in \Gamma$ we have

$$(t^g)\partial = g^{-1}(t\partial)g \, . \tag{2.1}$$

Here Γ is considered to act on itself by conjugation. A *crossed module* is a precrossed module that additionally satisfies the rule, that for all $t, u \in T$, we have:

$$t^{u\partial} = u^{-1}tu \, . \tag{2.2}$$

2.1.1 Example. Examples of crossed modules include the following:

- any Γ–module M with the trivial map $M \xrightarrow{0} \Gamma$,
- the inclusion of any normal subgroup $N \hookrightarrow \Gamma$,
- the map $T \to \operatorname{Aut} T$ that associates to $t \in T$ the inner automorphism of T defined by $a \mapsto t^{-1}at$,
- any surjection $T \to \Gamma$ with central kernel, where Γ acts on T by lifting and conjugation,
- the boundary map $\pi_2(X, Y) \to \pi_1(Y)$ from the second relative homotopy group of a pair of spaces (X, Y) with $Y \subseteq X$ [41, Section IV.1].

This last example motivated the introduction of the crossed module idea by J.H.C. Whitehead [42]. The main concern here will be the special case in which X is a 2–complex and Y its 1–skeleton: a theorem of Whitehead then asserts that $\pi_2(X, Y) \to \pi_1(Y)$ is a *free* crossed module (in a sense that we shall clarify in section 2.1.1) and $\pi_1(Y)$ is of course a free group.

Let $\partial : T \to \Gamma$ be a crossed module, and let N be the image of ∂. The following properties are easy consequences of (2.1) and (2.2).

- N is normal in Γ, and so if we set $G = \Gamma/N$ we get the short exact sequence of groups:
$$1 \to N \to \Gamma \to G \to 1\,.$$
- $\ker \partial \subseteq Z(T)$, the center of T, so $\ker \partial$ is abelian.
- $\ker \partial$ is invariant under the Γ–action on T, and so is a Γ–module.
- N acts trivially on $Z(T)$ and thus on $\ker \partial$, hence $\ker \partial$ inherits an action of G to become a G–module.
- the abelianisation T^{ab} of T inherits the structure of a G-module.

Morphisms between precrossed Γ–modules and between crossed Γ-modules are defined in the same way, and so we give a combined definition.

2.1.1 Definition. Let (A, ∂) and (A', ∂') be (pre)crossed Γ-modules. A *morphism of (pre)crossed* Γ*-modules* $\phi : (A, \partial) \to (A', \partial')$ is a group homomorphism $\phi : A \to A'$ such that for $a \in A$, and $u \in \Gamma$, $(a^u)\phi = (a\phi)^u$ and $\phi\partial' = \partial$, i.e.

$$\begin{array}{ccc} A & \xrightarrow{\partial} & \Gamma \\ {\scriptstyle \phi}\downarrow & \nearrow_{\partial'} & \\ A' & & \end{array}$$

commutes.

We can also have (pre)crossed module morphisms between (pre)crossed modules associated with different groups. If we have a (pre)crossed Γ_1-module (A_1, ∂_1) and a (pre)crossed Γ_2-module (A_2, ∂_2), a (pre)crossed module morphism is now a pair $(\phi, \psi) : (A_1, \partial_1) \to (A_2, \partial_2)$ consisting of a group homomorphism $\phi : A_1 \to A_2$ and a group homomorphism $\psi : \Gamma_1 \to \Gamma_2$ such that

$$\begin{array}{ccc} A_1 & \xrightarrow{\partial_1} & \Gamma_1 \\ {\scriptstyle \phi}\downarrow & & \downarrow{\scriptstyle \psi} \\ A_2 & \xrightarrow[\partial_2]{} & \Gamma_2 \end{array}$$

commutes and $(a^u)\phi = (a\phi)^{(u\psi)}$, for $a \in A$ and $u \in \Gamma$.

2.1.1 Free (Pre)Crossed Modules

2.1.2 Definition. Let (A, ∂) be a (pre)crossed Γ-module, let R be a set, and let $v : R \to A$ be a function. We say (A, ∂) is a *free (pre)crossed* Γ*-module with basis* v if for any (pre)crossed Γ-module (A', ∂') and function $v' : R \to A'$ such that $\partial' v' = \partial v$, that is, such that

$$\begin{array}{ccc} R & \xrightarrow{v} & A \\ {\scriptstyle v'}\downarrow & & \downarrow{\scriptstyle \partial} \\ A' & \xrightarrow[\partial']{} & \Gamma \end{array}$$

commutes, then there exists a unique morphism of (pre)crossed modules $\phi : (A, \partial) \to (A', \partial')$ such that $v\phi = v'$, that is,

$$\begin{array}{ccc} R & \xrightarrow{v} & A \\ {\scriptstyle v'}\downarrow & \phi & \downarrow{\scriptstyle \partial} \\ A' & \xrightarrow[\partial']{} & \Gamma \end{array}$$

commutes.

We may also choose to emphasise $w = v\partial : R \to \Gamma$ by saying that a free (pre)crossed module (A, ∂) with basis v is a *free (pre)crossed module on w.*

2.1.2 Proposition. *Let Γ be a group, R a set, and $w : R \to \Gamma$ a function. Then a free precrossed Γ-module on w exists and is unique up to isomorphism.*

Proof. Let F be the free group on the basis $R \times \Gamma$. Then Γ acts on F by right multiplication of basis elements: for $r \in R$ and $u, v \in \Gamma$ we have $(r, u)^v = (r, uv)$. We map $(r, u) \mapsto u^{-1}(rw)u$ and this induces a group homomorphism $\delta : F \to \Gamma$. It is easy to check that this is a free precrossed Γ–module on w. Uniqueness up to isomorphism follows from the usual universal argument. □

2.1.2 From Precrossed to Crossed Modules

We now discuss a procedure to obtain, from any precrossed Γ-module, a universal crossed Γ–module quotient. This procedure is discussed in detail in [8, Section 2], whose account we follow. From a precrossed module (X, δ) we shall obtain a crossed Γ–module (A, ∂) with $A = X/P$ for some normal subgroup P. The definition of P will ensure that (2.2) holds in A.

2.1.3 Definition. Let (X, δ) be a precrossed Γ-module. A *Peiffer element* of X is an element of the form

$$\langle x, y \rangle = x^{-1}y^{-1}xy^{x\delta}$$

for some $x, y \in X$. We call the subgroup of X generated by all the Peiffer elements the *Peiffer group of* X, denoted P.

2.1.3 Proposition. *[8, Proposition 2] The Peiffer group* P *of a precrossed* Γ*–module* (X, δ) *is normal in* X*, is invariant under the* Γ*–action, and is contained in* $\ker \delta$*.*

Proof. Let $x, y, z \in X$, then

$$\begin{aligned} z^{-1}\langle x, y \rangle z &= z^{-1}x^{-1}y^{-1}xy^{x\delta}z \\ &= z^{-1}x^{-1}y^{-1}x\,(zy^{(xz)\delta}(y^{(xz)\delta})^{-1}z^{-1})\,y^{x\delta}z \\ &= (xz)^{-1}y^{-1}(xz)y^{(xz)\delta}\left(z^{-1}(y^{x\delta})^{-1})z(y^{x\delta})^{z\delta}\right)^{-1} \\ &= \langle xz, y \rangle \langle z, y^{x\delta} \rangle^{-1} \end{aligned}$$

So a conjugate of a Peiffer element is a product of Peiffer elements, and P is normal in X. Now for $u \in \Gamma$ and $x, y \in X$,

$$\begin{aligned} \langle x^u, y^u \rangle &= (x^u)^{-1}(y^u)^{-1}x^u(y^u)^{(x^u)\delta} \\ &= (x^u)^{-1}(y^u)^{-1}x^u(y^u)^{u^{-1}(x\delta)u} \\ &= (x^u)^{-1}(y^u)^{-1}x^u(y^{x\delta})^u \\ &= \langle x, y \rangle^u \end{aligned}$$

So a Peiffer element acted on by an element of Γ is another Peiffer element, and we see that P is invariant under the Γ–action. Lastly we have

$$\begin{aligned}\langle x, y\rangle\delta = (x^{-1}y^{-1}xy^{x\delta})\delta &= (x^{-1})\delta(y^{-1})\delta x\delta(y^{x\delta})\delta \\ &= (x\delta)^{-1}(y\delta)^{-1}(x\delta)(x\delta)^{-1}(y\delta)(x\delta) \\ &= 1\end{aligned}$$

and so $P \subseteq \ker\delta$. □

2.1.4 Proposition. *[8, Corollary to Proposition 2] Let (X, δ) be a precrossed Γ-module. Then there exists a crossed Γ-module (A, ∂) and a precrossed Γ-module morphism $\phi : (X, \delta) \to (A, \partial)$, such that ϕ is universal for morphisms from (X, δ) to crossed Γ-modules.*

Proof. Let P be the Peiffer group of (X, δ) and set $A = X/P$, with $\phi : X \to A$ the quotient map. Moreover, since $P \subseteq \ker\delta$, there is an induced homomorphism $\partial : A \to \Gamma$ with $\delta = \phi\partial$. Furthermore, there is an action of Γ on A defined by $(xP)^u = (x^u)P$: this is well-defined since if $xP = yP$, then

$$\begin{aligned}x^u(y^u)^{-1} &= x^u(y^{-1})^u \\ &= (xy^{-1})^u.\end{aligned}$$

Since $xP = yP$ we know $xy^{-1} \in P$, and by Proposition 2.1.3, P is invariant under the Γ–action on X. Thus $(xy^{-1})^u \in P$, and our action is well-defined.

Now we must show that (A, ∂) is in fact a crossed Γ–module, and so we verify the two conditions (2.1) and (2.2) for the Γ–action on A. For (2.1):

$$\begin{aligned}(xP)^u\partial &= (x\phi)^u\partial \\ &= (x^u)\phi\partial \\ &= (x^u)\delta \\ &= u^{-1}(x\delta)u \\ &= u^{-1}(x\phi\partial)u \\ &= u^{-1}(xP)\partial u\,.\end{aligned}$$

Now the subgroup P is precisely defined to ensure that (2.2) holds:

$$(x^{-1}yx)P = (y^{x\delta})P = (yP)^{x\delta} = (yP)^{(xP)\partial}\,.$$

Finally, given any morphism τ from (X,δ) to a crossed Γ–module (T,d), we have $P \subseteq \ker\tau$ and so τ induces a group homomorphism $A \to T$ which gives a crossed Γ–module morphism $(A,\partial) \to (T,d)$. □

2.1.5 Corollary. *Let Γ be a group, R a set, and $w : R \to \Gamma$ a function. Then a free crossed Γ-module on w exists and is unique up to isomorphism.*

Proof. We form the quotient F/P of the free precrossed Γ–module F from Proposition 2.1.2. □

2.1.6 Proposition. *[8, Proposition 7] Let (C,∂) be the free crossed Γ-module with basis $v' : \mathcal{R} \to C$, and set $Q = \operatorname{coker}\partial = \Gamma/C\partial$. Then C^{ab} is a free Q-module on the image of the composition $\overline{v} : \mathcal{R} \xrightarrow{v'} C \to C^{ab}$.*

Proof. Let $p : \Gamma \to Q$ be the quotient map and let M be any Q–module. Then $C\partial \times M$, with the projection map π_1 onto $C\partial \subseteq \Gamma$, is a crossed Γ–module, with action given by conjugation on $\mathcal{C}\partial$ and on M via p. To verify this we check (2.1) and (2.2). For $(c\partial, m) \in C\partial \times M$ and $g \in \Gamma$, we have

$$((c\partial, m)^g)\pi_1 = (g^{-1}(c\partial)g, m^g p)\pi_1 = g^{-1}(c\partial)g = g^{-1}(c\partial, m)\pi_1 g\,.$$

and for $(c\partial, m), (d\partial, n) \in C\partial \times M$ we have

$$\begin{aligned}(c\partial, m)^{(d\partial,n)\pi_1} &= (c\partial, m)^{d\partial} = ((d^{-1}cd)\partial, m^{d\partial p})\\ &= ((d^{-1}cd)\partial, m) \quad (\text{since } d\partial p = 1_Q)\\ &= (d\partial, n)^{-1}(c\partial, m)(d\partial, n).\end{aligned}$$

Therefore, both (2.1) and (2.2) hold, and $\pi_1 : C\partial \times M \to \Gamma$ is a crossed module.

For any function $v'' : \mathcal{R} \to M$, define $v^* : \mathcal{R} \to C\partial \times M$ by $r \mapsto (rv'\partial, rv'')$. Then the diagram

$$\begin{array}{ccc} \mathcal{R} & \xrightarrow{v'} & C \\ {\scriptstyle v^*}\downarrow & & \downarrow{\scriptstyle \partial} \\ \mathcal{C}\partial \times M & \longrightarrow & \Gamma \end{array}$$

commutes and so by freeness of C , we get $\phi : C \to \Gamma \times M$ such that $v'\phi = v^*$:

$$\begin{array}{ccc} \mathcal{R} & \xrightarrow{v'} & C \\ {\scriptstyle v^*}\downarrow & \overset{\phi}{\swarrow} & \downarrow{\scriptstyle \partial} \\ \mathcal{C}\partial \times M & \longrightarrow & \Gamma \end{array}$$

Composing with projection to M then gives us a a morphism of groups $C \to M$ which factors through the abelianisation C^{ab}, and $C^{ab} \to M$ is then the required Q–morphism. □

2.1.7 Remark. Proof given in [8] uses $\Gamma \times M$ but it seemed to us that we need to restrict to $C\partial \times M$ as shown above.

2.1.3 Crossed Modules from Group Presentations

Let $\mathcal{P} = \langle X : \mathcal{R} \rangle$ be a presentation of a group G. Recalling our conventions from section 1.7, we write $A = X \cup X^{-1}$ and let $\rho : A^* \to F(X)$ be the canonical map, and define $\widehat{\rho} : \mathcal{R} \to F(X)$ by $(\ell, r)\widehat{\rho} = (\ell^{-1}r)\rho$. We let R be the image of $\widehat{\rho}$ in $F(X)$, and define $N = \langle\langle R \rangle\rangle$ to be the normal closure of R in F.

We now let $(C(\mathcal{P}), \partial)$ be the free crossed $F(X)$–module on the function $\widehat{\rho} : \mathcal{R} \to F(X)$. An element of $C = C(\mathcal{P})$ is represented by a product

$$(r_1, w_1)^{\varepsilon_1} \cdots (r_k, w_k)^{\varepsilon_k}$$

where $r_j \in \mathcal{R}, w_j \in F(X)$ and $\varepsilon_k = \pm 1$. A typical Peiffer element (trivial in C) has the form

$$(r, u)^{-1}(s, v)^{-1}(r, u)(s, vu^{-1}(r\widehat{\rho})u)\,.$$

For $(r, w) \in C$ we have $\partial : (r, w) \mapsto w^{-1}(r\widehat{\rho})w$, and the image of ∂ is N. We denote $\ker \partial$ by $\pi = \pi(\mathcal{P})$ (and we will explain this choice of notation in section 2.1.4). We therefore have short exact sequences of groups

$$1 \to N \to F(X) \to G \to 1 \tag{2.3}$$

and

$$0 \to \pi(\mathcal{P}) \to C(\mathcal{P}) \to F(X) \to 1\,, \tag{2.4}$$

with π central in C and a G–module.

2.1.8 Proposition. *[8, Corollary to Proposition 7] The free crossed module C is isomorphic as a group to $\pi \times N$. Its abelianisation C^{ab} is a free G–module, and the induced map $\pi \to C^{ab}$ is injective, so that we have a short exact sequence of G–modules.*

$$0 \to \pi \to C^{ab} \to N^{ab} \to 0 \tag{2.5}$$

Proof. Since F is free, (2.4) splits, and since π is central in C we have $C \cong \pi \times F$. It follows that $[C, C] \cong \{0\} \times [F, F]$ and so $\pi \to C^{ab}$ is injective. C^{ab} is free by Proposition 2.1.6. □

In the sequence (2.5), the G–module N^{ab} is the *relation module* of the presentation $\mathcal{P}$, and the G–module π is the *module of identities*. The sequence (2.5) then gives a free presentation of the relation module.

2.1.4 The Presentation Complex

We conclude this chapter with a brief review of the classical approach to the relation module and the module of identities, based on the topology of a 2–complex $K(\mathcal{P})$ associated to a group presentation $\mathcal{P}$. This material is standard, and we follow the account in [8, Section 5]. The first stage is to replace the derivation of the sequence (2.5) from crossed modules, by a description of the sequence using some basic homological algebra.

Given a group presentation $\mathcal{P} = \langle X : \mathcal{R} \rangle$ of a group G, we can construct an associated chain complex of free G-modules. Consider the free G-module $P_2 = \oplus_{\rho \in \mathcal{R}} \mathbf{Z} G e^2_\rho$ with basis e^2_ρ. Then by Proposition 2.1.6, P_2 is isomorphic to C^{ab}, where (C, ∂) is the free crossed module derived from $\mathcal{P}$, and (2.5) becomes:

$$0 \to \pi \to P_2 \to N^{ab} \to 0\,. \tag{2.6}$$

There are two further short exact sequences related to (2.6). Let $P_0 = \mathbb{Z}G$, and regard $\mathbb{Z}$ as a G-module with trivial G-action. The augmentation map $P_0 \to \mathbb{Z}$ has kernel the augmentation ideal IG of G and we get the short exact sequence:

$$0 \to IG \to P_0 \to \mathbb{Z} \to 0 \tag{2.7}$$

Now define P_1 to be the free module $P_1 = \oplus_X \mathbb{Z}G$, with basis e^1_x. Then we have a short exact sequence:

$$0 \to N^{ab} \to P_1 \to IG \to 0 \tag{2.8}$$

with the map $P_1 \to IG$ given by

$$e^1_x \to 1 - xN$$

with $x \in X$, and the map $N^{ab} \to P_1$ given by

$$uN' \to \sum_X \tau\left(\frac{\partial u}{\partial x}\right) e^1_x$$

with $\frac{\partial}{\partial x} : \mathbb{Z}F \to \mathbb{Z}F$ the Fox derivative (see [8, Section 4]), and $\tau : \mathbb{Z}F \to \mathbb{Z}G$ is induced by the natural epimorphism $F \to G$. We can put (2.6), (2.7) and

$$0 \to \pi \to P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \to \mathbb{Z} \to 0 \tag{2.9}$$

with $\pi = \ker \partial_2$.

Now given a presentation $\mathcal{P} = \langle X; \mathcal{R} \rangle$ of a group G, we can form a 2–complex $K = K(\mathcal{P})$ with;

- a single vertex, $*$;
- a 1-cell, e^1_x, for each $x \in X$;
- a 2-cell, e^2_ρ, for each $\rho \in \mathcal{R}$, attached by the image of the relator $r = \widehat{\rho}$ in F

We call $K = K(\mathcal{P})$ the *presentation complex* of $\mathcal{P}$. We can now identify the chain complex (2.9) with the cellular chain complex of the universal cover, $\tilde{K}$ of K. The complex $\tilde{K}$ has

- vertex set, $\tilde{K}^0 = G$;
- the 1-cells of $\tilde{K}$ are bijective with $X \times G$, so we can label them $e^1_{(x,g)}$, with

$(x, g) \in X \times G$, where $e^1_{(x,g)}$ joins g to $g(x\theta)$, with $\theta : F \to G$ the presentation map,

- the 2-cells of $\tilde{K}$, are bijective with $\mathcal{R} \times G$, so we label them $e^2_{(\rho,g)}$, with $(\rho, g) \in \mathcal{R} \times G$,

2.1.9 Proposition. *[8, Proposition 9] The cellular chain complex of $\tilde{K}$ is G-isomorphic to the chain complex $P_2 \to P_1 \to P_0$ associated to the presentation $\mathcal{P}$.*

2.1.10 Corollary. *Let $\mathcal{P} = \langle X; \mathcal{R} \rangle$ be a presentation of a group G, and let $K = K(\mathcal{P})$ be its presentation complex. Then the module of identities π for $\mathcal{P}$ is naturally isomorphic to the second homology group $H_2(\tilde{K})$ of the universal cover $\tilde{K}$ of K, and hence also to $\pi_2(K)$, the second homotopy group of K.*

Proof. We see from (2.6) that $\pi = \ker \partial_2$ and so by Proposition 2.1.9 we have $\pi = H_2(\tilde{K})$. Then the Hurewicz theorem (see, for example, [41, Theorem 2.5.2]) gives $H_2(\tilde{K}) \cong \pi_2(K)$. □

Chapter 3

Regular and Semiregular Groupoids

3.1 Semiregular Groupoids

We now introduce some additional structure on a groupoid. This idea originates in work of Brown and Gilbert [5], and was further developed by Gilbert in [13] and by Brown in [4]. Brown uses the terminology *whiskered* groupoid for what Gilbert had called a *semiregular* groupoid. We shall use the semiregular terminology, and will also discuss in detail the two special cases of *regular* groupoids and *pseudoregular* groupoids.

3.1.1 Definition. Let $\mathcal{G}$ be a groupoid, with object set $\mathcal{G}^0$, morphism set $\mathcal{G}^1$ and domain and range maps $\mathbf{d}, \mathbf{r} : \mathcal{G}^1 \to \mathcal{G}^0$ as in Definition 1.4.2. Then $\mathcal{G}$ is *semiregular* if

- $\mathcal{G}^0$ is a monoid, with identity $\mathbf{1} \in \mathcal{G}^0$, and

- there are left and right actions of $\mathcal{G}^0$ on $\mathcal{G}^1$, denoted $x \triangleright \alpha$, $\alpha \triangleleft x$, which for all $x, y \in \mathcal{G}^0$ and $\alpha, \beta \in \mathcal{G}^1$ satisfy:

1. $(xy) \triangleright \alpha = x \triangleright (y \triangleright \alpha)$; $\alpha \triangleleft (xy) = (\alpha \triangleleft x) \triangleleft y$; $(x \triangleright \alpha) \triangleleft y = x \triangleright (\alpha \triangleleft y)$,
2. $\mathbf{1} \triangleright \alpha = \alpha = \alpha \triangleleft \mathbf{1}$,
3. $(x \triangleright \alpha)\mathbf{d} = x(\alpha\mathbf{d})$; $(\alpha \triangleleft x)\mathbf{d} = (\alpha\mathbf{d})x$; $(x \triangleright \alpha)\mathbf{r} = x(\alpha\mathbf{r})$; $(\alpha \triangleleft x)\mathbf{r} = (\alpha\mathbf{r})x$,
4. $x \triangleright (\alpha \circ \beta) = (x \triangleright \alpha) \circ (x \triangleright \beta)$; $(\alpha \circ \beta) \triangleleft x = (\alpha \triangleleft x) \circ (\beta \triangleleft x)$, whenever $\alpha \circ \beta$ is defined, and
5. $x \triangleright 1_y = 1_{xy} = 1_x \triangleleft y$.

3.1.1 Proposition. *[13, Proposition 1.1] Let $\mathcal{G}$ be a semiregular groupoid. Then there are two everywhere defined binary operations on $\mathcal{G}^1$ given by:*

$$\alpha * \beta = (\alpha \triangleleft \beta\mathbf{d}) \circ (\alpha\mathbf{r} \triangleright \beta)$$

$$\alpha \circledast \beta = (\alpha\mathbf{d} \triangleright \beta) \circ (\alpha \triangleleft \beta\mathbf{r})\,.$$

Each of the binary operations $*$ and $\circledast$ make $\mathcal{G}^1$ into a monoid, with identity $1_{\mathbf{1}}$.

Proof. We have $*$ associative:

$$\begin{aligned}(\alpha * \beta) * \gamma &= \big((\alpha \triangleleft \beta\mathbf{d}) \circ (\alpha\mathbf{r} \triangleright \beta)\big) * \gamma \\ &= \big(\alpha \triangleleft (\beta\mathbf{d})(\gamma\mathbf{d})\big) \circ \big(\alpha\mathbf{r} \triangleright \beta \triangleleft \gamma\mathbf{d}\big) \circ \big((\alpha\mathbf{r})(\beta\mathbf{r}) \triangleright \gamma\big) \\ &= \alpha * \big((\beta \triangleleft \gamma\mathbf{d}) \circ (\beta\mathbf{r} \triangleright \gamma)\big) \\ &= \alpha * (\beta * \gamma)\end{aligned}$$

and we can see that $1_{\mathbf{1}}$ is the identity under $*$:

$$\begin{aligned}1_{\mathbf{1}} * \alpha &= (1_{\mathbf{1}} \triangleleft \alpha\mathbf{d}) \circ \alpha \\ &= 1_{\alpha\mathbf{d}} \circ \alpha \\ &= \alpha\end{aligned}$$

$$\begin{aligned} \alpha * 1_{\mathbf{1}} &= \alpha \circ (\alpha \mathbf{r}) \rhd 1_{\mathbf{1}}) \\ &= \alpha \circ 1_{\alpha \mathbf{r}} \\ &= \alpha \end{aligned}$$

Similar calculations can be carried out for $\circledast$. $\square$

3.1.2 Proposition. *[13, Proposition 1.2] The binary operation $*$ and the monoid structure on $\mathcal{G}^0$ make the semiregular groupoid $\mathcal{G}$ into a strict monoidal groupoid, as in section 1.4.3, if and only if the operations $*$ and $\circledast$ on $\mathcal{G}^1$ coincide.*

Proof. The requirement that the monoid structure on $\mathcal{G}^0$ and $*$ on $\mathcal{G}^1$ make $\mathcal{G}$ into a monoidal groupoid, so that $\mathcal{G} \times \mathcal{G} \to \mathcal{G}$ is a functor, is equivalent to the interchange law:

$$(\alpha * \beta) \circ (\gamma * \delta) = (\alpha \circ \gamma) * (\beta \circ \delta) \tag{3.1}$$

For the equation (3.1) to be defined we will have $\alpha \mathbf{r} = \gamma \mathbf{d}$, $\beta \mathbf{r} = \delta \mathbf{d}$, so we can rewrite the left hand side of (3.1) as:

$$\begin{aligned} (\alpha * \beta) \circ (\gamma * \delta) &= (\alpha \lhd \beta \mathbf{d}) \circ (\alpha \mathbf{r} \rhd \beta) \circ (\gamma \lhd \delta \mathbf{d}) \circ (\gamma \mathbf{r} \rhd \delta) \\ &= (\alpha \lhd \beta \mathbf{d}) \circ (\gamma \mathbf{d} \rhd \beta) \circ (\gamma \lhd \beta \mathbf{r}) \circ (\gamma \mathbf{r} \rhd \delta) \\ &= (\alpha \lhd \beta \mathbf{d}) \circ (\gamma \circledast \beta) \circ (\gamma \mathbf{r} \rhd \delta) \end{aligned} \tag{3.2}$$

Similarly

$$(\alpha \circ \gamma) * (\beta \circ \delta) = (\alpha \lhd \beta \mathbf{d}) \circ (\gamma * \beta) \circ (\gamma \mathbf{r} \rhd \delta) \tag{3.3}$$

So it is clear that the interchange law holds if $* = \circledast$.

Conversely if (3.2) and (3.3) are always equal, we can set $\alpha = 1_{\gamma \mathbf{d}}$ and $\delta = 1_{\beta \mathbf{r}}$ to see that $* = \circledast$. $\square$

3.2 Regular Groupoids

3.2.1 Definition. A semiregular groupoid $\mathcal{G}$ is a *regular* groupoid if $\mathcal{G}^0$ is a group.

3.2.1 Proposition. *[13, Proposition 1.3(i)] Let $\mathcal{G}$ be a regular groupoid. Then the two binary operations $*$ and $\circledast$ given in Proposition 3.1.1 makes $\mathcal{G}^1$ into a group, with identity $1_{\mathbf{1}}$.*

Proof. The inverse of α with respect to $*$ is

$$\alpha^{-*} = \alpha\mathbf{r}^{-1} \triangleright \alpha^{-\circ} \triangleleft \alpha\mathbf{d}^{-1}$$

and $\circledast$ is

$$\alpha^{-\circledast} = \alpha\mathbf{d}^{-1} \triangleright \alpha^{-\circ} \triangleleft \alpha\mathbf{r}^{-1}$$

where $^{-\circ}$ is the inverse of α with respect to the groupoid operation, and $^{-1}$ is the inverse in the group $\mathcal{G}^0$. □

We remark here that the formula for α^{-*} is mis-stated in [13].

Recall from section 1.4.2 that the star at $\mathbf{1}$ of $\mathcal{G}$ is defined by $\{\alpha \in \mathcal{G} : \alpha\mathbf{d} = \mathbf{1}\}$. We will denote this by $\mathrm{star}_{\mathbf{1}}(\mathcal{G})$, or just $\mathrm{star}_{\mathbf{1}}$.

3.2.2 Lemma. $\mathrm{star}_{\mathbf{1}}$ *is a subgroup of* $(\mathcal{G}^1, *)$.

Proof. $\mathrm{star}_{\mathbf{1}}$ is closed under $*$, since $(\alpha * \beta)\mathbf{d} = (\alpha\mathbf{d})(\beta\mathbf{d})$, and α^{-*} for $\alpha \in \mathrm{star}_{\mathbf{1}}$ is equal to $\alpha\mathbf{r}^{-1} \triangleright \alpha^{-\circ}$, which is also clearly in $\mathrm{star}_{\mathbf{1}}$. □

3.2.3 Proposition. *[13, Proposition 1.3(i)] $(\mathcal{G}^1, *)$ admits a group action of $\mathcal{G}^0$ by automorphisms defined by $\alpha^w = w^{-1} \triangleright \alpha \triangleleft w$, where w^{-1} is the inverse in the group $\mathcal{G}^0$. This action makes $\mathbf{r} : \mathrm{star}_{\mathbf{1}}(\mathcal{G}) \to \mathcal{G}^0$ into a precrossed module.*

Proof. We have $(\mathrm{star}_{\mathbf{1}}(\mathcal{G}), *)$ a subgroup by Lemma 3.2.2, and it is clear that it is invariant under the action of $\mathcal{G}^0$. Then for $\mathrm{star}_1 \xrightarrow{\mathbf{r}} \mathcal{G}^1$ to be a precrossed module we need $\mathbf{r}$ to be a group homomorphism which satisfies (2.1):

$$(\alpha * \beta)\mathbf{r} = (\alpha \circ (\alpha\mathbf{r} \triangleright \beta))\mathbf{r} = (\alpha\mathbf{r})(\beta\mathbf{r})$$

$$(\alpha^w)\mathbf{r} = (w^{-1} \triangleright \alpha \triangleleft w)\mathbf{r} = w^{-1}(\alpha\mathbf{r})w$$

by the semiregularity conditions given in Definition 3.1.1. □

3.2.4 Proposition. *[13, Proposition 1.3(ii)] The categories of precrossed modules and regular groupoids are equivalent.*

Proof. One half of the equivalence is given in Propositions 3.2.1, and 3.2.3. The other half is the construction of a category from a precrossed module $\delta : H \to F$. The set of objects is F, and the set of morphisms is the semidirect product $F \ltimes H$, with $(u,h)\mathbf{d} = u$ and $(u,h)\mathbf{r} = u(h\delta)$, composition of morphisms is defined by $(u_1,h_1) \bullet (u_2,h_2) = (u_1, h_1h_2)$ whenever $u_2 = u_1(h_1\delta)$. The left and right actions of F on $F \rtimes H$ are given by the operation in the semidirect product:

$$v \triangleright (u,h) = (v,1)(u,h) = (vu,h) \text{ and } (u,h) \triangleleft w = (u,h)(w,1) = (uw, h^w)\,.$$

□

3.2.5 Proposition. *[13, Proposition 1.3(iii)] A regular groupoid is equivalent as in Proposition 3.2.4 to a crossed module if and only if $*$ and $\circledast$ coincide.*

Proof.

$$\begin{aligned}(u,h) * (v,g) &= ((u,h) \triangleleft v) \bullet (u(h\delta) \triangleright (v,g)) \\ &= (uv, h^v) \bullet (u(h\delta)v, g) \\ &= (uv, h^v g)\end{aligned}$$

$$\begin{aligned}(u,h) \circledast (v,h) &= (u \rhd (v,g)) \bullet ((u,h) \lhd v(g\delta)) \\ &= (uv,g) \bullet (uv(g\delta), h^{v(g\delta)}) \\ &= (uv, gh^{v(g\delta)})\end{aligned}$$

These are equal if and only if $h^{(g\delta)} = g^{-1}hg$ for all $h, g \in H$, that is if and only if $\delta : H \to F$ is a crossed module. □

Let $\pi_{\mathbf{1}} = \{\alpha \in \mathcal{G} : \alpha\mathbf{d} = \mathbf{1} = \alpha\mathbf{r}\}$, the local group at $\mathbf{1}$. For a regular groupoid $\mathcal{G}$, $\pi_{\mathbf{1}}$ is the kernel of its associated precrossed module.

3.2.6 Proposition. *In a regular groupoid $\mathcal{G}$ for which $*$ equals $\circledast$, the group $\pi_{\mathbf{1}}$ is a $\mathcal{G}^0$-module.*

Proof. For $\alpha, \beta \in \pi_{\mathbf{1}}$,

$$\alpha * \beta = (\alpha \lhd \beta\mathbf{d}) \circ (\alpha\mathbf{r} \rhd \beta) = (\alpha \lhd \mathbf{1}) \circ (\mathbf{1} \rhd \beta) = \alpha \circ \beta\,.$$

Since $* = \circledast$ we have

$$\begin{aligned}\alpha * \beta &= \alpha \circledast \beta \\ &= (\alpha\mathbf{d} \rhd \beta) \circ (\alpha \lhd \beta\mathbf{r}) \\ &= (\mathbf{1} \rhd \beta) \circ (\alpha \lhd \mathbf{1}) \\ &= \beta \circ \alpha \\ &= \beta * \alpha\,.\end{aligned}$$

So $(\pi_{\mathbf{1}}, *)$ is abelian.

Now consider the action of $w \in \mathcal{G}^0$ on $\alpha \in \pi_{\mathbf{1}}$. Since $(\alpha^w)\mathbf{r} = w^{-1}(\alpha\mathbf{r})w = w^{-1}\mathbf{1}w = w^{-1}w = \mathbf{1}$, then $\alpha^w \in \pi_{\mathbf{1}}$, and so $\pi_{\mathbf{1}}$ is a $\mathcal{G}^0$–module. □

3.3 Pseudoregular Groupoids

In considering presentations of inverse monoids in subsequent chapters, we shall want to consider semiregular groupoids in which the vertex set is an inverse monoid.

3.3.1 Definition. A semiregular groupoid $\mathcal{G}$ is a *pseudoregular* groupoid if $\mathcal{G}^0$ is an inverse monoid.

The name *pseudoregular* is chosen to reflect the close structural connection between inverse monoids and pseudogroups, which are inverse semigroups of partial homeomorphisms of topological spaces (see [22, section 1.1]).

3.3.1 Proposition. *In a pseudoregular groupoid $\mathcal{G}$, the operations, $*$ and $\circledast$ given in Proposition 3.1.1 each make $\mathcal{G}^1$ into a monoid, but $\mathcal{G}^1$ is not necessarily an inverse monoid.*

3.3.2 Example. We give an example of a pseudoregular groupoid $\mathcal{G}$ in which $(\mathcal{G}^1, *)$ is not inverse. Let $\partial : T \to G$ be a crossed module of groups. Add a zero 0 to G to form G^0 and let $0 \in G^0$ act on T as the trivial endomorphism $t \mapsto 1_T$. The semidirect product $G^0 \ltimes T$ is then the disjoint union

$$G^0 \ltimes T = G \ltimes T \sqcup \{(0,t) : t \in T\}.$$

The group semidirect product $G \ltimes T$ is, by Proposition 3.2.5 equivalent to a regular groupoid with vertex set G and arrow set $G \ltimes T$, with $(g,t)\mathbf{d} = g$ and $(g,t)\mathbf{r} = g(t\partial)$. Extending this structure to the additional arrows in $\{0\} \times T$ we have $(0,t)\mathbf{d} = 0 = (0,t)\mathbf{r}$ and composition $(0,t) \circ (0,u) = (0,tu)$ and so the local group at 0 is a copy of T. The left and right actions of $0 \in G^0$ are obtained from left and right multiplication by $(0,1)$ in $G^0 \ltimes T$:

$$\begin{aligned}
0 \rhd (g,t) &= (0,1)(g,t) = (0,t),\\
(g,t) \lhd 0 &= (g,t)(0,1) = (0,1),\\
0 \rhd (0,t) &= (0,1)(0,t) = (0,t),\\
(0,t) \lhd 0 &= (0,t)(0,1) = (0,1).
\end{aligned}$$

Then $\partial : T \to G^0$ is a *crossed monoid* (originally *monoïde croisé*) in the sense of Lavendhomme and Roisin [23, Example 1.3C]. The structure just described on $G^0 \ltimes T$ is a pseudoregular groupoid $\mathcal{G}$, with vertex set G^0. The $*$–operation on $\mathcal{G}$ recovers the semidirect product:

$$(g,t) * (h,u) = (gh, t^h u) \in G^0 \ltimes T$$

and the operations $*$ and $\circledast$ coincide, but the semidirect product is not inverse. This follows from the results of [29], but can also be seen directly, as follows.

For any $t \in T$, the element $(0,t)$ is an idempotent in $(G^0 \ltimes T, *)$:

$$(0,t) * (0,t) = (0, t^0 t) = (0, 1_T t) = (0,t)\,.$$

But for distinct $s, t \in T$ we have

$$(0,s) * (0,t) = (0, s^0 t) = (0,t) \text{ and } (0,t)(0,s) = (0,s)$$

and so the idempotents in $G^0 \ltimes T$ do not commute, thus $G^0 \ltimes T$ is not inverse. Since $G \ltimes T$ is a subgroup of $G^0 \ltimes T$ and the other elements are idempotents, $G^0 \ltimes T$ is regular (and indeed *orthodox*, since $E(G^0 \ltimes T)$ is a subsemigroup).

3.3.1 Inverse Monoids as Pseudoregular Groupoids

We saw in section 1.4.2 that an inverse monoid S can be considered as an inductive groupoid $\vec{S}$, with vertex set $E(S)$. Since we have natural left and right actions of $E(S)$ on S by left and right multiplication, and $E(S)$ is an inverse monoid, we may ask when is $\vec{S}$ pseudoregular?

Defining the action of $E(S)$ on S as proposed, by $e \triangleright s = es$ and $s \triangleleft e = se$, we consider the five semiregularity conditions in Definition 3.1.1, for $s, t \in S$ and $e, f \in E(S)$:

1. For $e, f \in E(S)$, and $s \in S$, associativity in S implies that

$$e \triangleright (f \triangleright s) = ef \triangleright s\,, (s \triangleleft e) \triangleleft f = s \triangleleft ef\,, \text{ and } (e \triangleright s) \triangleleft f = e \triangleright (s \triangleleft f)\,.$$

2. $1 \triangleright s = 1s = s = s1 = s \triangleleft 1$

3. We have $(e \triangleright s)\mathbf{d} = (es)\mathbf{d} = (es)(es)^{-1} = ess^{-1}$, and $e(s\mathbf{d}) = ess^{-1}$ as required. But for the range map we have $(e \triangleright s)\mathbf{r} = (es)\mathbf{r} = (es)^{-1}(es) = s^{-1}es$, and $e(s\mathbf{r}) = es^{-1}s$. Now

$$s^{-1}es = es^{-1}s \implies s^{-1}ess^{-1} = es^{-1}ss^{-1} \implies s^{-1}e = es^{-1} \implies s^{-1}es = es^{-1}s\,.$$

   We see that $(e \triangleright s)\mathbf{r} = e(s\mathbf{r})$ if and only if the idempotents of S are central in S. This condition is also necessary and sufficient for the equation $(s \triangleleft e)\mathbf{d} = (s\mathbf{d})e$ to hold, whilst $(s \triangleleft e)\mathbf{r} = (s\mathbf{r})e$ always holds.

4. Given $s, t \in S$, the composition $s \circ t = st$ is defined in $\vec{S}$ only when $s^{-1}s = tt^{-1}$. However, assuming that idempotenst are central in S, we have for all $s, t \in S$ and $e \in E(S)$ that $est = eest = (es)(et)$ and so certainly $e \triangleright (s \circ t) = (e \triangleright s) \circ (e \triangleright t)$. Similarly $(s \circ t) \triangleleft e = (s \triangleleft e) \circ (t \triangleleft e)$.

5. The identity arrow $1_e \in \vec{S}$ at $e \in E(S)$ is just e itself, and so $e \triangleright 1_f = ef =$

$1_{ef} = 1_e \lhd f$.

These considerations, together with [19, Theorem 4.2.1], establish:

3.3.3 Proposition. *The inductive groupoid $\vec{S}$ associated to an inverse monoid S is pseudoregular if and only if S is a Clifford semigroup.*

3.3.2 Stars in Pseudoregular Groupoids

In a pseudoregular groupoid, it is natural to consider

$$\mathrm{star}_e(\mathcal{G}) = \{\alpha \in \mathcal{G} : \alpha\mathbf{d} = e\}$$

for each idempotent $e \in E(\mathcal{G}^0)$. It is then easy to see the following:

3.3.4 Proposition. *In any pseudoregular groupoid $\mathcal{G}$, and for any $e \in E(\mathcal{G}^0)$, the operation $*$ makes $\mathrm{star}_e(\mathcal{G})$ into a semigroup.*

However, we do not always have a monoid with identity 1_e here, as the action of e is not necessarily trivial:

$$\begin{aligned} \alpha * 1_e &= (\alpha \lhd e) \circ (\alpha\mathbf{r} \rhd 1_e) \\ &= (\alpha \lhd e) \circ 1_{\mathbf{r}(\alpha)e} \\ &= \alpha \lhd e \end{aligned}$$

3.3.5 Example. Let E be the semilattice $\{1, e, f, 0\}$ with $ef = 0$ and consider the subgroupoid U of the simplicial groupoid $\Delta(E)$ (see section 1.4.2) defined by

$$U = \{(x, y) \in \Delta(E) : x \neq 1 \neq y\} \cup \{(1, 1)\}.$$

Right and left actions of E on the arrows of $\Delta(E)$ may be defined just by multiplication:

$$x \triangleright (y,z) = (xy, xz) \quad \text{and} \quad (x,y) \triangleleft z = (xz, yz)\,,$$

making U pseudoregular. The $*$–operation on edges is then given by

$$(u,v) * (x,y) = ((u,v) \triangleleft x)(v \triangleright (x,y)) = (ux, vx)(vx, vy) = (ux, vy)\,.$$

The star at 0 is

$$\mathrm{star}_0 = \{(0,e), (0,f), (0,0)\}\,,$$

which we relabel using only the second component as $\mathrm{star}_0 = \{e, f, 0\}$ and the $*$–operation is then just identical to multiplication in E. In particular, $(\mathrm{star}_0, *)$ is not a monoid.

The construction in Example 3.3.5 will be generalised in chapter 4 to study congruences.

We can, however, remedy the problem illustrated in Example 3.3.5 by passing to a subsemigroup that does admit 1_e as an identity. For $e \in E(\mathcal{G}^0)$ we define

$$\mathrm{star}_e^{\bowtie}(\mathcal{G}) = \{e \triangleright \alpha \triangleleft e : \alpha \in \mathrm{star}_e(\mathcal{G})\}\,.$$

3.3.6 Proposition. *In any pseudoregular groupoid $\mathcal{G}$, and for any $e \in E(\mathcal{G}^0)$, the operation $*$ makes $\mathrm{star}_e^{\bowtie}(\mathcal{G})$ into a monoid with identity 1_e. The range map $\mathbf{r} : \mathcal{G} \to \mathcal{G}^0$ restricts to a semigroup morphism $\mathbf{r}_e : \mathrm{star}_e^{\bowtie}(\mathcal{G}) \to \mathcal{G}^0$ whose image is a monoid U_e with identity e.*

We now define

$$\pi_e^{\bowtie}(\mathcal{G}) = \{\alpha^{\bowtie} \in \mathrm{star}_e^{\bowtie}(\mathcal{G}) : \mathbf{r}(\alpha^{\bowtie}) = e\}\,.$$

3.3.7 Proposition. *The binary operation $*$ and the groupoid composition $\circ$ coincide on $\pi_e^{\bowtie}(\mathcal{G})$ and under each operation $\pi_e^{\bowtie}(\mathcal{G})$ is a group. Furthermore if the binary* *operations $*$ and $\circledast$ are equal, then $\pi_e^{\bowtie}(\mathcal{G})$ is abelian.*

Proof. For $\alpha^{\bowtie}, \beta^{\bowtie} \in \pi_e^{\bowtie}(\mathcal{G})$ we have

$$\begin{aligned}\alpha^{\bowtie} * \beta^{\bowtie} &= (\alpha^{\bowtie} \lhd \mathbf{d}(\beta^{\bowtie})) \circ (\mathbf{r}(\alpha^{\bowtie}) \rhd \beta^{\bowtie}) \\ &= (\alpha^{\bowtie} \lhd e) \circ (e \rhd \beta^{\bowtie}) \\ &= \alpha^{\bowtie} \circ \beta^{\bowtie}\end{aligned}$$

Since $e \rhd \alpha^{-\circ} \lhd e = (e \rhd \alpha \lhd e)^{-\circ}$ it is clear that $\pi_e^{\bowtie}(\mathcal{G})$ is a subgroup of the local group $\pi_1(\mathcal{G}, e)$ at e in the groupoid $\mathcal{G}$.

If $*$ and $\circledast$ coincide, then:

$$\begin{aligned}\alpha^{\bowtie} \circ \beta^{\bowtie} &= \alpha^{\bowtie} * \beta^{\bowtie} \\ &= \alpha^{\bowtie} \circledast \beta^{\bowtie} \\ &= (\mathbf{d}(\alpha^{\bowtie}) \rhd \beta^{\bowtie}) \circ (\alpha^{\bowtie} \lhd \mathbf{r}(\beta^{\bowtie})) \\ &= (e \rhd \beta^{\bowtie}) \circ (\alpha^{\bowtie} \lhd e) \\ &= \beta^{\bowtie} \circ \alpha^{\bowtie}\end{aligned}$$

So $\pi_e^{\bowtie}$ is abelian. □

3.3.8 Example. In the setting of Example 3.3.5 we have

$$\mathrm{star}_0^{\bowtie} = \{0\}\,.$$

In Proposition 3.2.5 we saw that a regular groupoid $\mathcal{G}$ in which we have $*$ coinciding with $\circledast$, is equivalent to a crossed module, and so as a consequence of section 2.1, the group $\pi_{\mathbf{1}}(\mathcal{G})$ is a $\mathcal{G}^0$–module. We now want to show that, for a pseudoregular groupoid $\mathcal{G}$, the collection of abelian groups $\{\pi_e^{\bowtie}(\mathcal{G}) : e \in E(\mathcal{G}^0)\}$ is a $\mathcal{G}^0$–module.

3.3.9 Proposition. *Let $\mathcal{G}$ be a pseudoregular groupoid in which the operations $*$ and $\circledast$ are equal. Then the family of abelian groups $\pi_e^{\bowtie}(\mathcal{G})$, $e \in E(\mathcal{G}_0)$, is a $\mathcal{G}^0$-module.*

Proof. By Proposition 3.3.7, each $\pi_e^{\bowtie}(\mathcal{G})$ is an abelian group, and for $e \geqslant f$ we define,

$$
\begin{aligned}
\varphi_f^e : \pi_e^{\bowtie}(\mathcal{G}) &\to \pi_f^{\bowtie}(\mathcal{G}) \quad \text{by} \\
\alpha^{\bowtie} &\mapsto f \rhd \alpha^{\bowtie} \lhd f \\
&= f \rhd e \rhd \alpha \lhd e \lhd f \\
&= fe \rhd \alpha \lhd ef \\
&= f \rhd \alpha \lhd f \in \pi_f \\
&= f \rhd f \rhd \alpha \lhd f \lhd f \in \pi_f^{\bowtie}
\end{aligned}
$$

Now for $\alpha^{\bowtie}, \beta^{\bowtie} \in \pi_e^{\bowtie}$:

$$
\begin{aligned}
(\alpha^{\bowtie} * \beta^{\bowtie})\varphi_f^e &= f \rhd (\alpha^{\bowtie} * \beta^{\bowtie}) \lhd f \\
&= f \rhd (\alpha^{\bowtie} \circ \beta^{\bowtie}) \lhd f \\
&= (f \rhd \alpha^{\bowtie} \lhd f) \circ (f \rhd \beta^{\bowtie} \lhd f) \\
&= (f \rhd \alpha^{\bowtie} \lhd f) * (f \rhd \beta^{\bowtie} \lhd f) \\
&= \alpha^{\bowtie}\varphi_f^e * \beta^{\bowtie}\varphi_f^e
\end{aligned}
$$

and so each φ_f^e is a homomorphism. Furthermore

$$\alpha^{\bowtie}\varphi_f^e\varphi_g^f = g \rhd (f \rhd \alpha^{\bowtie} \lhd f) \lhd g = gf \rhd \alpha^{\bowtie} \lhd fg = g \rhd \alpha^{\bowtie} \lhd g = \alpha^{\bowtie}\varphi_g^e$$

and,

$$\alpha^{\bowtie}\varphi_e^e = e \rhd \alpha^{\bowtie} \lhd e = e \rhd e \rhd \alpha \lhd e \lhd e = e \rhd \alpha \lhd e = \alpha^{\bowtie}.$$

So these φ_f^e satisfy Definition 1.6.1.

The $\mathcal{G}^0$–action is now given by $\alpha \lhd s = s^{-1} \rhd \alpha \lhd s \in \pi_{s^{-1}es}^{\bowtie}$, and it is easy to check that the conditions for a Lausch $\mathcal{G}^0$–module are satisfied. $\square$

Chapter 4

Congruences and Groupoids

Supose that ρ is an equivalence relation on a set X. Then we may define a groupoid G_ρ whose set of arrows are the pairs $(x, y) \in X \times X$ with $x \; \rho \; y$. The identity arrows are those of the form (x, x), an arrow (x, y) has inverse (y, x), and the composition of arrows is given by the rule $(x, y)(y, z) = (x, z)$. The groupoid G_ρ is unicursal, and so embeds in the simplicial groupoid $\Delta(X)$ as a *wide* subgroupoid, that is, one containing all the identities. Conversely, given a wide subgroupoid H of $\Delta(X)$, we obtain an equivalence relation χ on X by defining $x \; \chi \; y$ if and only if there exists $h \in H$ with $h\mathbf{d} = x$ and $h\mathbf{r} = y$. Hence equivalence relations on a set X are in one-to-one correspondence with unicursal groupoids $\mathcal{G}$ with vertex set $\mathcal{G}^0 = X$. If X supports a semigroup structure, then we can consider congruences, rather than just equivalence relations on X.

4.1 Congruences on Semigroups

4.1.1 Proposition. *Unicursal semiregular groupoids $\mathcal{G}$ with vertex set $\mathcal{G}^0 = S$ are in one-to-one correspondence with congruences on the semigroup S.*

Proof. Suppose that $\mathcal{G}$ is unicursal and semiregular, and that $\mathcal{G}$ corresponds to the equivalence relation $\simeq$ on S. If $a \simeq b$ and $c \simeq d$, with $\alpha \in \mathcal{G}(a, b)$ and $\gamma \in \mathcal{G}(c, d)$, then $\alpha * \gamma \in \mathcal{G}(ac, bd)$ and so $ac \simeq bd$. Hence $\simeq$ is a congruence on S. Conversely, if $\simeq$ is known to be a congruence on S, and $a \simeq b$ corresponds to $\alpha \in \mathcal{G}(a, b)$, then for all $x, y \in S$ we may define $x \triangleright \alpha$ to be the unique arrow in $\mathcal{G}(xa, xb)$ correpsonding to $xa \simeq xb$, and $\alpha \triangleleft y$ to be the unique arrow in $\mathcal{G}(ay, by)$ corresponding to $ay \simeq by$. It is then easy to check that this defines a semiregular structure on $\mathcal{G}$. □

We note that in the proof of Proposition 4.1.1 we could have used the operation $\circledast$ instead of $*$:

4.1.2 Lemma. *In a unicursal semiregular groupoid $\mathcal{G}$, the binary operations $*$ and $\circledast$ coincide.*

Proof. If $a, b, c, d \in S$ and $\alpha \in \mathcal{G}(a, b), \beta \in \mathcal{G}(c, d)$ then both $\alpha * \beta$ and $\alpha \circledast \beta$ are arrows in $\mathcal{G}(ac, bd)$ and so must be equal, by the unicursal property of $\mathcal{G}$. □

4.2 Congruences on Groups

As an easy consequence of Proposition 4.1.1, we obtain the following well-known classification of congruences on a group.

4.2.1 Proposition.

(a) *Unicursal regular groupoids $\mathcal{G}$ with vertex set $\mathcal{G}^0 = G$ are in one-to-one correspondence with congruences on the group G.*

(b) *The congruences on a group G are in one-to-one correspondence with the normal subgroups of G.*

Proof. Part (a) is just the regular case of Proposition 4.1.1. For part (b), we note that, in a unicursal regular groupoid, the homomorphism $\mathbf{r} : \mathrm{star}_{\mathbf{1}}(\mathcal{G}) \to \mathcal{G}^0$ is injective and so identifies $\mathrm{star}_{\mathbf{1}}(\mathcal{G})$ with its image N, which is a normal subgroup of $\mathcal{G}^0$. Then there exists $\alpha \in \mathcal{G}(a, b)$ if and only if $a^{-1}b \in N$. □

Although Proposition 4.2.1 describes the standard correspondence between congruences on a group and its normal subgroups, we do not immediately recover the regular groupoid corresponding to a congruence from the crossed module determined by a normal subgroup, as we might expect from Proposition 3.2.5.

Let N be a normal subgroup of a group G considered as a crossed module $\iota : N \hookrightarrow G$. Then from Proposition 3.2.4 we can construct a regular groupoid $\Gamma = \Gamma(G, N)$ as follows: $\Gamma^0 = G$ and $\Gamma^1 = G \times N$, with $\mathbf{d}(g, n) = g$ and $\mathbf{r}(g, n) = gn$. The composition is given by $(g, x) \cdot (h, y) = (g, xy)$ when $h = gx$, and the actions by $h \triangleright (g, x) = (hg, x)$ and $(g, x) \triangleleft h = (gh, x^h) = (gh, h^{-1}xh)$. We let $\mathcal{G}(G, N)$ be the regular groupoid corresponding to the congruence determined by N.

4.2.2 Theorem. *The regular groupoids $\Gamma(G, N)$ and $\mathcal{G}(G, N)$ are isomorphic.*

Proof. We define the functor $\phi : \mathcal{G}(G, N) \to \Gamma(G, N)$ to be the identity on the common object set G, and to be given on arrows by $\phi : (g, h) \mapsto (g, g^{-1}h)$. Since (g, h) is an arrow in $\mathcal{G}(G, N)$ if and only if $g^{-1}h \in N$ this is well-defined, and ϕ is a functor, since

$$\phi((g, h) \circ (h, k)) = \phi(g, k) = (g, g^{-1}k)$$

whereas

$$\phi(g, h) \cdot \phi(h, k) = (g, g^{-1}h) \cdot (h, h^{-1}k) = (g, g^{-1}hh^{-1}k) = (g, g^{-1}k)\,.$$

The inverse of ϕ is clearly given by $(g, x) \to (g, gx)$. □

This is all seen to be an elaborate version of the following elementary (but not necessarily well-known fact).

4.2.3 Corollary. *Let N be a normal subgroup of the group G, and form the semidirect product $G \ltimes N$ with G acting by conjugation. Then $G \ltimes N$ is isomorphic to the direct product $G \times N$.*

Proof. The group $(\Gamma(G, N), *)$ is the semidirect product $G \ltimes N$, whereas $(\mathcal{G}(G, N), *)$ is the direct product $G \times N$. $\square$

4.3 Congruences on Inverse Monoids

The classification of congruences on inverse semigroups is more subtle than that for congruences on groups, and has been treated by different authors in different ways: see for example [15, 32, 35]. An approach to the classification using ordered goupoids has been given by AlYamani and Gilbert [1].

Let ρ be a congruence on an inverse semigroup S. Following [32], the *trace*, $\mathrm{tr}(\rho)$ of ρ is its restriction to $E(S)$, and the *kernel*, $\ker \rho$ is the set

$$\ker \rho = \{s \in S : s \,\rho\, e \text{ for some } e \in E(S)\}.$$

4.3.1 Definition. Recall from [32] that a congruence ρ on the semilattice of idempotents $E(S)$ of S is *normal* if, for all $s \in S$, $e \,\rho\, f$ implies that $s^{-1}es \,\rho\, s^{-1}fs$. Then [32, Definition 4.2] a *congruence pair* (K, ν) on S consists of a normal inverse subsemigroup K of S and a normal congruence ν on $E(S)$ such that

(4.1) if $e \in E(S)$ and $s \in S$ satisfy $se \in K$ and $s^{-1}s \,\nu\, e$ then $s \in K$,

(4.2) if $u \in K$ then $uu^{-1} \,\nu\, u^{-1}u$.

For any congruence ρ, its kernel and trace form a congruence pair. Conversely, given a congruence pair (K, ν) the relation $\rho_{(K,\nu)}$ defined by

$$s \,\rho_{(K,\nu)}\, t \iff s^{-1}t \in K \text{ and } ss^{-1} \,\nu\, tt^{-1} \tag{4.3}$$

is a congruence with kernel K and trace ν. Congruence pairs form the basis of the classification of congruences on inverse semigroups in [32, Theorem 4.4].

Any normal congruence ν on $E(S)$ determines a relation $\nu_{\min}$ on S as follows: for $a, b \in S$,

$$a\,\nu_{\min}\,b \iff \text{there exists } e \in E(S) \text{ with } ae = be \text{ and } a^{-1}a\;\nu\;e\;\nu\;b^{-1}b\,. \tag{4.4}$$

4.3.1 Lemma. *[35, Theorem 4.2] The relation $\nu_{\min}$ is a congruence on S and is the smallest congruence on S whose trace is equal to ν.*

Proof. It is clear that $\nu_{\min}$ is reflexive and symmetric. Suppose that $a, b, c \in S$ with $a\ \nu_{\min}\ b\ \nu_{\min}\ c$: so there exist $e, f \in E(S)$ with $ae = be, bf = cf$ and $a^{-1}a\;\nu\;e\;\nu\;b^{-1}b\;\nu\;f\;\nu\;c^{-1}c$. Then

$$a(ef) = (ae)f = (be)f = b(ef) = b(fe) = (bf)e = (cf)e = c(fe) = c(ef)$$

and, since ν is a congruenec on $E(S)$,

$$a^{-1}a = (a^{-1}a)(a^{-1}a)\;\nu\;ef\;\nu\;(c^{-1}c)(c^{-1}c) = c^{-1}c\,.$$

Hence $\nu_{\min}$ is an equivalence relation on S.

To show that $\nu_{\min}$ is a congruence on S, we shall show that it is both a left congruence and a right congruence.

Suppose $a, b, c \in S$ with $a\ \nu_{\min}\ b$. So there exists $e \in E(S)$ with $ae = be$, and therefore

$$ac(c^{-1}ec) = aec = bec = bc(c^{-1}ec)$$

with $(ac)^{-1}(ac) = c^{-1}a^{-1}ac\;\nu\;c^{-1}b^{-1}bc = (bc)^{-1}(bc)$, since ν is a normal congruence on $E(S)$. This shows that $\nu_{\min}$ is a right congruence.

Now set $f = e(a^{-1}c^{-1}ca)(b^{-1}c^{-1}cb) \in E(S)$. Since $ae = be$, we have $caf = cbf$.

Then

$$\begin{aligned}(ca)^{-1}(ca) = a^{-1}c^{-1}ca &= a^{-1}aa^{-1}c^{-1}caa^{-1}a \\ &\nu\ ea^{-1}c^{-1}cae = eb^{-1}c^{-1}cbe \\ &\nu\ b^{-1}bb^{-1}c^{-1}cbb^{-1}b = b^{-1}c^{-1}cb \\ &= (cb)^{-1}(cb)\,.\end{aligned}$$

and

$$(ca)^{-1}(ca) = (ca)^{-1}(ca)(ca)^{-1}(ca)\ \nu\ ea^{-1}c^{-1}cab^{-1}c^{-1}cb = f\,.$$

Therefore $\nu_{\min}$ is also a left congruence.

If ν' is a congruence on S with trace ν and $a\ \nu_{\min}\ b$ then $ae = be$ for some $e \in E(S)$ with $a^{-1}a\ \nu\ e\ \nu\ b^{-1}b$ and so $a^{-1}a\ \nu'\ e\ \nu'\ b^{-1}b$. Then

$$a = aa^{-1}a\ \nu'\ ae = be\ \nu'\ bb^{-1}b = b$$

and so $a\ \nu'\ b$. □

Definition. We recall from Definition 1.2.2:

(a) A congruence ρ on an inverse semigroup S is said to be *idempotent pure* if $a \in S$ and $a\ \rho\ e$ for some $e \in E(S)$ imply that $a \in E(S)$.

(b) A congruence ρ on an inverse semigroup S is said to be *idempotent separating* if $e, f \in E(S)$ and $e\ \rho\ f$ imply that $e = f$.

Since the trace of an idempotent separating congruence on S is the identity relation on $E(S)$, an idempotent separating congruence ρ is completely determined by its kernel K: from (4.3) we see that

$$s\ \rho\ t \iff s^{-1}t \in K \text{ and } ss^{-1} = tt^{-1}\,. \tag{4.5}$$

We can describe the classes of ρ as cosets of K in S, as follows:

4.3.2 Definition. Let S be an inverse semigroup, and K a normal subsemigroup of S. Then the *left trace coset* of K determined by $s \in S$ is:

$$\vec{s}K = \{sk : k \in K, s^{-1}s = kk^{-1}\}.$$

Similarly, the *right trace coset* of K determined by s is

$$K\vec{s} = \{ks : k \in K, k^{-1}k = ss^{-1}\}$$

4.3.2 Proposition. *Let S be an inverse semigroup and ρ an idempotent separating congruence on S with kernel K. Then the congruence classes of ρ are exactly the trace cosets of K.*

Proof. Suppose that $s \;\rho\; t$. Then by (4.5) we have $t = tt^{-1}t = ss^{-1}t$, and $s^{-1}t \in K$. Since $s^{-1}t(s^{-1}t)^{-1} = s^{-1}tt^{-1}s = s^{-1}s$ we see that $t \in \vec{s}K$.

Conversely, if $t \in \vec{s}K$ then $t = sk$ for some $k \in K$ with $s^{-1}s = kk^{-1}$. But then $s^{-1}t = s^{-1}sk \in K$ and $tt^{-1} = skk^{-1}s^{-1} = ss^{-1}ss^{-1} = ss^{-1}$, and so $s \;\rho\; t$. □

Restating Proposition 1.3.1 for congruences, we have:

4.3.3 Lemma. *The kernel K of an idempotent separating congruence ρ on an inverse semigroup S is a Clifford semigroup.*

We now turn to a factorization result for congruences that will be important for our discussion of relation modules in sections 6.2.3 and 6.2.4. It is closely related to Lemma 4.3.1. Our discussion is based on [26, page 265], to which we refer for further details. The result originates in [35, Theorem 4.2], as did Lemma 4.3.1.

4.3.4 Proposition. *Let ρ be a congruence on the inverse semigroup S. Let $\rho_{\min}$ be the smallest congruence on S with the same trace as ρ: this is equal to the congruence $\operatorname{tr}(\rho)_{\min}$ of Lemma 4.3.1 defined in (4.4).*

1. *For $a, b \in S$ we have*

$$a \; \rho_{\min} \; b \iff \text{there exists } c \in S \text{ with } a \geqslant c \leqslant b \text{ and } a \; \rho \; c \; \rho \; b. \tag{4.6}$$

2. *The canonical map $\psi : S/\rho_{\min} \to S/\rho$ is idempotent separating,*

3. *If S is E–unitary then the canonical map $\tau : S \to S/\rho_{\min}$ is idempotent pure.*

Proof. (1) We first show that the conditions (4.4) and (4.6) are equivalent. First assume that (4.4) holds and set $c = ae = be$. Then $a \geqslant c \leqslant b$ and, since $a^{-1}a \; \rho \; e \; \rho \; b^{-1}b$ we have

$$a = aa^{-1}a \; \rho \; ae = be \; \rho \; bb^{-1}b = b\,.$$

Now if (4.6) holds, take $e = c^{-1}c$. Since $a \geqslant c \leqslant b$ we have $ae = c = be$, and since $a \; \rho \; c$ we have $a^{-1}a \; \rho \; c^{-1}c = e$. Similarly $b^{-1}b \; \rho \; c^{-1}c$.

(2) Suppose that $a, b \in S$ with $a \; \rho_{\min} \; a^2$ and $b \; \rho_{\min} \; b^2$. By Lallement's Lemma 1.2.1, there exist $e, f \in S$ with $a \; \rho_{\min} \; e$ and $b \; \rho_{\min} \; f$. If now $e \; \rho \; f$ then $e \; \rho_{\min} \; f$, and so $a \; \rho_{\min} \; b$. Hence ψ is idempotent separating.

(3) Suppose that, for $s \in S$ and $x \in E(S)$, we have $s \; \rho_{\min} \; x$, then there exists $e \in E(S)$ with $se = xe$ and $xe \in E(S)$. Then if S is E–unitary, we have $s \in E(S)$ and τ is idempotent pure. □

4.3.1 Congruences and Pseudoregular Groupoids

Our aim is now to use pseudoregular groupoids to classify congruences. For consistency with our earlier definitions, we shall work with inverse monoids, but the extension to inverse semigroups is straightforward.

Let S be an inverse monoid. A congruence χ on S is then equivalent to a unicursal (and so monoidal) pseudoregular groupoid with vertex set S. We shall now take such a groupoid as our starting point, look at properties of the congruence that it represents, and find an equivalent algebraic description that can be considered as a generalisation of Proposition 4.2.1.

Let Γ be a unicursal pseudoregular groupoid, whose vertex set Γ^0 is an inverse monoid S. We let χ be the relation on S defined by

$$a \, \chi \, b \iff a \text{ and } b \text{ are in the same connected component of } \Gamma \, .$$

4.3.5 Lemma. *The relation χ is a congruence on S and its trace is a normal congruence on $E(S)$.*

Proof. It is clear that χ is an equivalence relation on S, whose set of equivalence classes may be identified with the set $\pi_0(\Gamma)$ of connected components of Γ. Moreover, if $a, b, c, d \in S$ and if $\eta \in \Gamma(a, b)$ and $\xi \in \Gamma(c, d)$, then

$$\eta * \xi = (\eta \lhd c)(b \rhd \xi) \in \Gamma(ac, bd)$$

and so χ is a congruence on S. Moreover, if $e, f \in E(S)$, and if $\eta \in \Gamma(e, f)$ and $s \in S$, then

$$s^{-1} \rhd \gamma \lhd s \in \Gamma_E(s^{-1}es, s^{-1}fs) \, ,$$

and so χ is a normal congruence on $E(S)$. □

By Lemma 4.4, the normal congruence χ on $E(S)$ determines the congruence $\chi_{\min}$ on S, with quotient inverse monoid $T = S/\chi_{\min}$: we let τ denote the quotient map $S \to T$.

We define

$$K(\Gamma) = \{ s \in S : \text{ there exists } e \in E(S) \text{ with } s \, \chi \, e \} \, .$$

4.3.6 Lemma. *$K(\Gamma)$ is a normal inverse submonoid of S, and is a disjoint union of inverse subsemigroups $K_e = \{ s \in S : s \, \chi \, e \}$.*

Proof. $K(\Gamma)$ is just the kernel of the congruence χ and so is known to be a normal inverse subsemigroup of S. However, it is worthwhile to show how the necessary properties are obtained from the structure of Γ.

It is clear that K is a subsemigroup of Γ, using the proof that χ is a congruence on S, given above. Moreover, if $a \in K$ then $a^{-1} \in K$, since if $\gamma \in \Gamma(a, e)$ with $e \in E(S)$ then

$$(a^{-1} \triangleright \gamma \triangleleft a^{-1})(a^{-1} \triangleright \gamma^{-1} \triangleleft ea^{-1})(e \triangleright \gamma^{-1} \triangleleft a^{-1})(eaa^{-1} \triangleright \gamma^{-1} \triangleleft e)(e \triangleright \gamma \triangleleft e)$$

is a path in $\Gamma(a^{-1}, e)$. Hence each K_e is an inverse subsemigroup of K, and their union K is an inverse subsemigroup.

It is obvious that $E(S) \subseteq K$, and if $s \in S$ and $\gamma \in \Gamma(a, e)$ then $s^{-1} \triangleright \gamma \triangleleft s \in \Gamma(s^{-1}as, s^{-1}es)$, and so K is normal. □

4.3.7 Lemma. *The image U of $K(\Gamma)$ under the quotient map $\tau : S \to T$ is a normal inverse subsemigroup of T that is a disjoint union of groups, indexed by $E(T)$.*

Proof. Let $a \in K(\Gamma)$ with $a \;\chi\; e$. Then $a^{-1} \;\chi\; e$ and so $aa^{-1} \;\chi\; e \;\chi\; a^{-1}a$. Since χ and $\chi_{\min}$ have the same trace, $aa^{-1} \;\chi_{\min}\; a^{-1}a$ and so $a\tau$ is an element of a subgroup of T, with identity $e\tau$. □

The pseudoregular groupoid Γ has now determined the following diagram of inverse monoids:

$$\begin{array}{ccccc} & & S & \hookleftarrow & E(S) \\ & & \downarrow{\scriptstyle\tau} & & \downarrow{\scriptstyle\tau} \\ U & \hookrightarrow & T & \hookleftarrow & E(T) \end{array} \tag{4.7}$$

in which τ is the quotient map induced by the congruence χ_{min}, and $U \hookrightarrow T$ is the inclusion of a disjoint union of groups indexed by $E(T)$.

4.3.8 Proposition. *The orginal congruence χ on S is recoverable from diagram*

(4.7) *as follows: for* $a, b \in S$,

$$a \chi b \iff (ab^{-1})\tau \in U \text{ and } (a^{-1}a)\tau = (b^{-1}b)\tau\,. \tag{4.8}$$

Proof. If $a \chi b$ then $ab^{-1} \chi bb^{-1}$ and so $ab^{-1} \in K(\Gamma)$ and so $(ab^{-1})\tau \in U$: furthermore, $a^{-1}a \chi b^{-1}b$.

For the converse, consider the relation χ defined by (4.8). Since U is full, χ is reflexive, and is clearly transitive. Moreover, if $a, b, c \in S$ and if $(ab^{-1})\tau, (bc^{-1})\tau$ are in U, and $(a^{-1}a)\tau = (b^{-1}b)\tau = (c^{-1}c)\tau$, then

$$(ac^{-1})\tau = (aa^{-1}ac^{-1})\tau = (ab^{-1}bc^{-1})\tau = (ab^{-1})\tau\,(bc^{-1})\tau \in U\,.$$

Hence χ is an equivalence relation. It is a right congruence since, if $a \chi b$ and $c \in S$ then $(ab^{-1})\tau \in U$

$$\begin{aligned}((ac)(bc)^{-1})\tau &= (acc^{-1}b^{-1})\tau = (acc^{-1}b^{-1}bb^{-1})\tau \\ &= (ab^{-1}bcc^{-1}b^{-1})\tau = (ab^{-1})\tau\,(bcc^{-1}b^{-1})\tau \in U\end{aligned}$$

since U is full in T, and

$$(ac)^{-1}(ac) = c^{-1}a^{-1}ac \,\chi\, c^{-1}b^{-1}bc = (bc)^{-1}(bc)$$

since χ is a normal congruence on $E(S)$.

To show that χ is a left congruence, again given $a, b, c \in S$ with $a \chi b$, we have $(ab^{-1})\tau = u$ from some $u \in U$ and

$$\begin{aligned}((ca)(cb)^{-1})\tau &= (cab^{-1}c)\tau = (c\tau)\,(ab^{-1})\tau\,(c\tau)^{-1} \\ &= (c\tau)u(c\tau)^{-1} \in U\end{aligned}$$

since U is normal in T. Set $e = c^{-1}c$: then $(ca)^{-1}(ca) = a^{-1}ea$ and $(cb)^{-1}(cb) =$

$b^{-1}eb$. Now

$$\begin{aligned}(a^{-1}ea)\tau &= (a^{-1}eaa^{-1}aa^{-1}ea)\tau \\ &= (a^{-1}eab^{-1}ba^{-1}ea)\tau \\ &= (a^{-1}e)\tau\,(ab^{-1})\tau\,(ba^{-1})\tau\,(ea)\tau \\ &= (a^{-1}e)\tau\,(ab^{-1})\tau\,((ab^{-1})\tau)^{-1}\,(ea)\tau \\ &= (a^{-1}e)\tau\,((ab^{-1})\tau)^{-1}\,(ab^{-1})\tau\,(ea)\tau\end{aligned}$$

since $(ab^{-1})\tau \in U$

$$\begin{aligned}&= (a^{-1}eba^{-1}ab^{-1}ea)\tau \\ &= (a^{-1})\tau\,((ab^{-1}e)\tau)^{-1}\,(ab^{-1}e)\tau\,(a\tau) \\ &= (a^{-1})\tau\,(ab^{-1}e)\tau\,((ab^{-1}e)\tau)^{-1}\,(a\tau)\end{aligned}$$

since $(ab^{-1}e)\tau \in U$

$$\begin{aligned}&= (a^{-1}ab^{-1}eba^{-1}a)\tau \\ &= (b^{-1}bb^{-1}ebb^{-1}b)\tau \\ &= (b^{-1}eb)\tau\,.\end{aligned}$$

$\square$

Chapter 5

The Squier Complex of a Group Presentation

In this chapter we study group presentations using regular groupoids. To a group presentation $\mathcal{P} = \langle X : R \rangle$ we associate the fundamental groupoid $\Pi(\mathrm{Sq}(\mathcal{P}), F(X))$ of a version of the Squier complex of $\mathcal{P}$, that has vertex set the free group $F(X)$ with basis X. Our complex $\mathrm{Sq}(\mathcal{P})$ may be considered as a subcomplex of the variant of the Squier complex used by Pride in [34] (and there denoted $\mathcal{D}(\mathcal{P})^*$) to study group presentations, and called the *Pride complex* in [13]. The groupoid $\Pi(\mathrm{Sq}(\mathcal{P}), F(X))$ is a regular, monoidal groupoid, and so is equivalent to a crossed module of groups.

5.1 The Squier Complex and its Fundamental Groupoid

Consider a group presentation $\mathcal{P} = \langle X : R \rangle$, where R is a set of relations whose elements we denote in the form $(l = r)$. For convenience we shall assume that for $(l = r) \in R$ we have $l, r \in F(X)$ and that if $(l = r) \in R$ then $(r = l) \notin R$.

5.1.1 Definition. For a group presentation $\mathcal{P}$, its *Squier complex* $\mathrm{Sq}(\mathcal{P})$ is the 2–complex defined as follows:

- the vertex set of Sq($\mathcal{P}$) is the free group $F(X)$ on X,
- the edge set of Sq($\mathcal{P}$) will consist of all 3-tuples $(p, l = r, q)$ with $p, q \in F(X)$ and $(l = r) \in R$. Such an edge will start at $\overline{plq}$ and end at $\overline{prq}$, where these are the reduced forms of plq and prq respectively, we will drop the bars for convenience and consider plq and prq as elements of the free group. Each edge corresponds to the application of a relation. An edge path corresponds to a succession of such applications.
- the 2-cells correspond to applications of pairs non-overlapping relations, and so a 2-cell is attached along every edge path of the form:

$$\begin{array}{ccc} plqp'l'q' & \xrightarrow{(p,l=r,qp'l'q')} & prqp'l'q' \\ {\scriptstyle (plqp',l'=r',q')}\downarrow & & \downarrow{\scriptstyle (prqp',l'=r',q')} \\ plqp'r'q' & \xrightarrow[(p,l=r,qp'r'q')]{} & prqp'r'q' \end{array}$$

  This makes the two edge paths

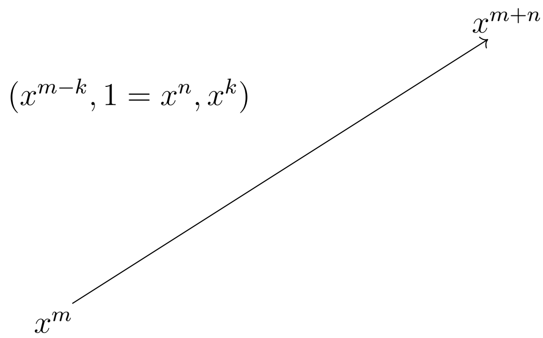
$$(p, l = r, qp'l'q')(prqp', l' = r', q') \quad \text{and} \quad (plqp', l' = r', q')(p, l = r, qp'r'q')$$

  homotopic in Sq($\mathcal{P}$).

5.1.1 Example. Let $\mathcal{P} = \langle x; 1 = x^n \rangle$ presenting the cyclic group of order n. Then we can build $Sq(\mathcal{P})$ as follows.

Vertex set $V(Sq(\mathcal{P})) = F(x)$, so we have a vertex x^m for each $m \in \mathbb{Z}$.

Edges correspond to factorisations, at x^m we have $x^m = x^{m-k} \cdot x^k$ so we have an edge for each $k \in \mathbb{Z}$:

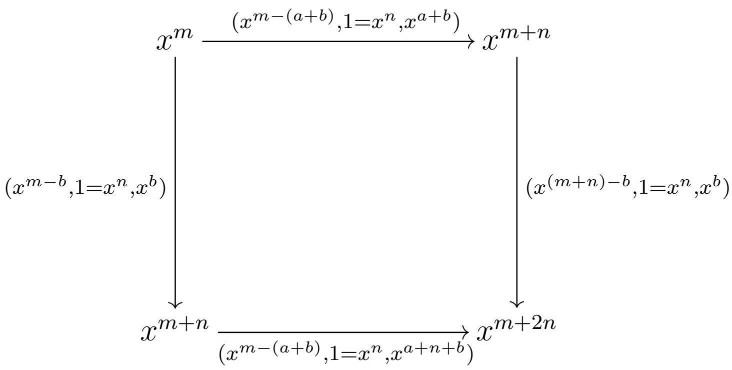


So for each pair $(m, k) \in \mathbb{Z} \times \mathbb{Z}$ we have an edge from x^m to x^{m+n}.

2-cells correspond to 3-way factorisations of the form,

$$x^m = x^{m-(a+b)} \cdot x^a \cdot x^b$$

for $a, b \in \mathbb{Z}$ and this factorisation gives us the 2-cell:



So we have a 2-cell for each triple $(m, a, b) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$.

5.1.2 Theorem. *The fundamental groupoid* $\Pi(\mathrm{Sq}(\mathcal{P}), F(X))$ *of the Squier complex* $\mathrm{Sq}(\mathcal{P})$ *of a group presentation* $\mathcal{P}$ *is a regular groupoid.*

Proof. The vertex set of $\Pi = \Pi(\mathrm{Sq}(\mathcal{P}), F(X))$ is the group $F(X)$. We need to define left and right actions of $F(X)$ on homotopy classes of paths in $\mathrm{Sq}(\mathcal{P})$. We first define such actions for single edges. Let $u, v \in F(X)$ and suppose that $(p, l = r, q)$ is an edge in $\mathrm{Sq}(\mathcal{P})$. We define

$$u \rhd (p, l = r, q) = (up, l = r, q) \tag{5.1}$$

$$(p, l = r, q) \lhd v = (p, l = r, qv)\,. \tag{5.2}$$

We now define the actions on a path $\alpha = \alpha_1 \circ \alpha_2 \circ \cdots \circ \alpha_n$ with α_i single edges in $\mathrm{Sq}(\mathcal{P})$ by:

$$u \rhd \alpha = (u \rhd \alpha_1) \circ (u \rhd \alpha_2) \circ \cdots \circ (u \rhd \alpha_n)$$

$$\alpha \lhd v = (\alpha_1 \lhd v) \circ (\alpha_2 \lhd v) \circ \cdots \circ (\alpha_n \lhd v)\,.$$

Since the left and right actions of $F(X)$ will transform the boundary of a 2–cell in $\mathrm{Sq}(\mathcal{P})$ into the boundary of another 2–cell, it follows that these actions are well-defined on homotopy classes of paths in Π, and it is clear that the conditions defining a semiregular groupoid given in Definition 3.1.1 then hold. Therefore Π is a regular groupoid. □

In what follows it will be convenient to work directly with edge paths in $\mathrm{Sq}(\mathcal{P})$, even though these are to be interpreted as representatives of homotopy classes in the fundamental groupoid $\Pi(\mathrm{Sq}(\mathcal{P}), F(X))$. In particular, we shall apply the operations $*$ and $\circledast$ directly to edge paths.

5.1.3 Theorem. *In the regular groupoid* $\Pi(\mathrm{Sq}(\mathcal{P}), F(X))$ *, the two everywhere defined operations* $*$ *and* $\circledast$*, as in Proposition 3.1.1, coincide.*

Proof. Recall that

$$\alpha * \beta = (\alpha \lhd \beta\mathbf{d}) \circ (\alpha\mathbf{r} \rhd \beta)$$

$$\alpha \circledast \beta = (\alpha\mathbf{d} \rhd \beta) \circ (\alpha \lhd \beta\mathbf{r})\,.$$

First consider single edge paths $\alpha = (p, l = r, q)$ and $\beta = (p', l' = r', q')$. Then

$$\alpha * \beta = (p, l = r, qp'l'q') \circ (prqp', l' = r', q')$$
$$\alpha \circledast \beta = (plqp', l' = r', q') \circ (p, l = r, qp'r'q') .$$

These paths comprise the boundary of a 2–cell in $\mathrm{Sq}(\mathcal{P})$ and are thus homotopic in $\Pi(\mathrm{Sq}(\mathcal{P}), F(X))$: hence $\alpha * \beta = \alpha \circledast \beta$.

Now consider edge paths $\alpha = \alpha_1 \circ \alpha_2 \circ \cdots \circ \alpha_k$ and $\beta = \beta_1 \circ \beta_2 \circ \cdots \circ \beta_m$ and with each α_i, β_j single edges. Inductively we may assume that if β is the single edge β_1 then

$$(\alpha_1 \circ \cdots \circ \alpha_{k-1}) * \beta_1 = (\alpha_1 \circ \cdots \circ \alpha_{k-1}) \circledast \beta_1 .$$

Then

$$\begin{aligned}
\alpha * \beta &= (\alpha \lhd \beta_1\mathbf{d}) \circ (\alpha_k\mathbf{r} \rhd \beta) \\
&= (\alpha_1 \lhd \beta_1\mathbf{d}) \circ (\alpha_2 \lhd \beta_1\mathbf{d}) \circ \cdots \circ (\alpha_k \lhd \beta_1\mathbf{d}) \circ (\alpha_k\mathbf{r} \rhd \beta_1) \\
&= (\alpha_1 \lhd \beta_1\mathbf{d}) \circ \cdots \circ (\alpha_{k-1} \lhd \beta_1\mathbf{d}) \circ (\alpha_k * \beta_1) \\
&= (\alpha_1 \lhd \beta_1\mathbf{d}) \circ \cdots \circ (\alpha_{k-1} \lhd \beta_1\mathbf{d}) \circ (\alpha_k \circledast \beta_1) \\
&= (\alpha_1 \lhd \beta_1\mathbf{d}) \circ \cdots \circ (\alpha_{k-1} \lhd \beta_1\mathbf{d}) \circ (\alpha_k\mathbf{d} \rhd \beta_1) \circ (\alpha_k \lhd \beta_1\mathbf{r}) \\
&= (\alpha_1 \lhd \beta_1\mathbf{d}) \circ \cdots \circ (\alpha_{k-1} \lhd \beta_1\mathbf{d}) \circ (\alpha_{k-1}\mathbf{r} \rhd \beta_1) \circ (\alpha_k \lhd \beta_1\mathbf{r}) \\
&= ((\alpha_1 \circ \cdots \circ \alpha_{k-1}) * \beta) \circ (\alpha_k \lhd \beta_1\mathbf{r}) \\
&= ((\alpha_1 \circ \cdots \circ \alpha_{k-1}) \circledast \beta) \circ (\alpha_k \lhd \beta_1\mathbf{r}) \\
&= (\alpha_1\mathbf{d} \rhd \beta_1) \circ (\alpha_1 \lhd \beta_1\mathbf{r}) \circ \cdots \circ (\alpha_{k-1} \lhd \beta_1\mathbf{r}) \circ (\alpha_k \lhd \beta_1\mathbf{r}) \\
&= \alpha \circledast \beta
\end{aligned}$$

So by induction on k, we have $\alpha * \beta = \alpha \circledast \beta$, whenever $m = 1$. We now assume inductively that, for any edge path α,

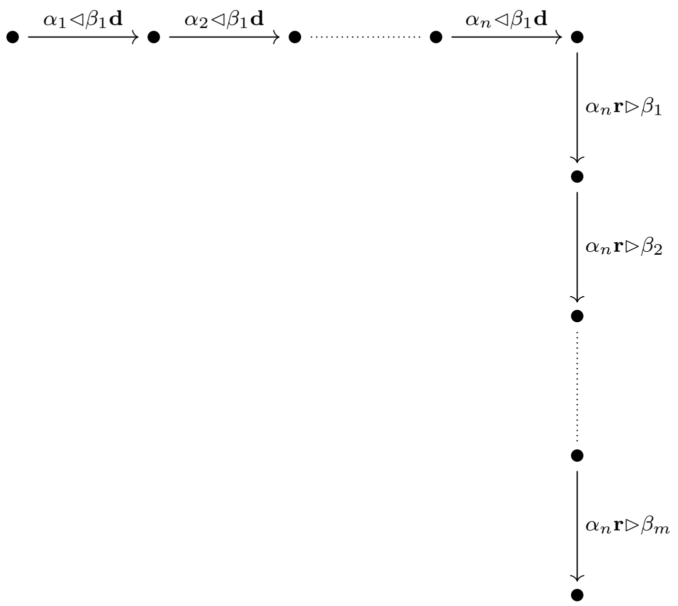
$$\alpha * (\beta_1 \circ \cdots \circ \beta_{m-1}) = \alpha \circledast (\beta_1 \circ \cdots \circ \beta_{m-1}) \,.$$

Then

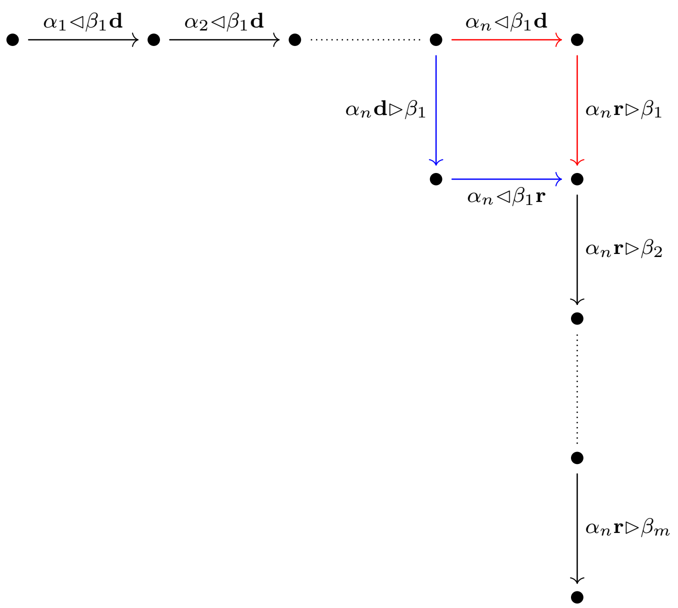
$$\begin{aligned}
\alpha * \beta &= (\alpha \lhd \beta_1 \mathbf{d}) \circ (\alpha_n \mathbf{r} \rhd \beta) \\
&= \big((\alpha_1 \circ \cdots \circ \alpha_n) \lhd \beta_1 \mathbf{d}\big) \circ \big(\alpha_n \mathbf{r} \rhd (\beta_1 \circ \cdots \circ \beta_{m-1})\big) \circ \big(\alpha_n \mathbf{r} \rhd \beta_j\big) \\
&= \big(\alpha * (\beta_1 \circ \cdots \circ \beta_{m-1})\big) \circ \big(\alpha_n \mathbf{r} \rhd \beta_m\big) \\
&= \big(\alpha \circledast (\beta_1 \circ \cdots \circ \beta_{m-1})\big) \circ \big(\alpha_n \mathbf{r} \rhd \beta_m\big) \\
&= \big(\alpha_1 \mathbf{d} \rhd (\beta_1 \circ \cdots \circ \beta_{m-1})\big) \circ \big(\alpha \lhd \beta_{m-1} \mathbf{r}\big) \circ \big(\alpha_n \mathbf{r} \rhd \beta_j\big) \\
&= \big(\alpha_1 \mathbf{d} \rhd (\beta_1 \circ \cdots \circ \beta_{m-1})\big) \circ \big(\alpha \lhd \beta_m \mathbf{d}\big) \circ \big(\alpha_n \mathbf{r} \rhd \beta_j\big) \\
&= \big(\alpha_1 \mathbf{d} \rhd (\beta_1 \circ \cdots \circ \beta_{m-1})\big) \circ \big(\alpha * \beta_m\big) \\
&= \big(\alpha_1 \mathbf{d} \rhd (\beta_1 \circ \cdots \circ \beta_{m-1})\big) \circ \big(\alpha \circledast \beta_m\big) \\
&= \big(\alpha_1 \mathbf{d} \rhd (\beta_1 \circ \cdots \circ \beta_{m-1})\big) \circ \big(\alpha_1 \mathbf{d} \rhd \beta_m\big) \circ \big(\alpha \lhd \beta_m \mathbf{r}\big) \\
&= \big(\alpha \mathbf{d} \rhd \beta\big) \circ \big(\alpha \rhd \beta \mathbf{r}\big) \\
&= \alpha \circledast \beta
\end{aligned}$$

Thus by induction we have that $\alpha * \beta = \alpha \circledast \beta$, for all edge paths α, β in $\mathrm{Sq}(\mathcal{P})$. $\square$

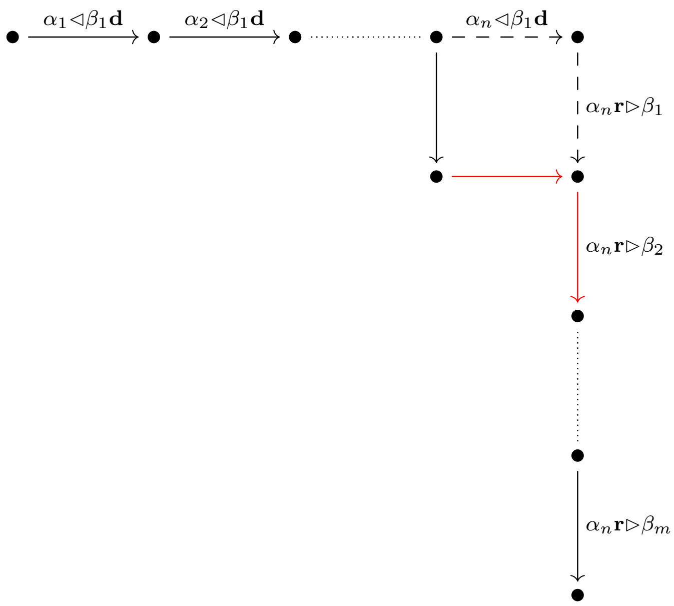
This result is perhaps easier to see graphically, we can represent $\alpha * \beta$ by:



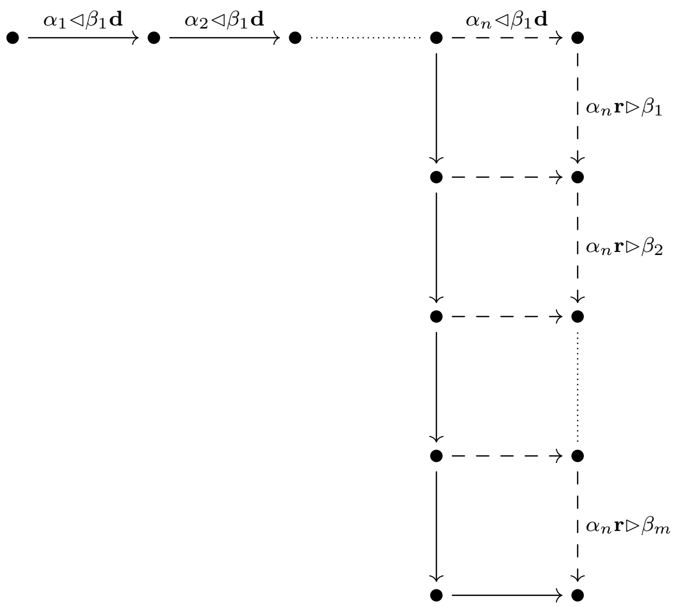
and recognise that $(\alpha_n \lhd \beta_1 \mathbf{d}) \circ (\alpha_n \mathbf{r} \rhd \beta_1)$ shown in red equals $\alpha_n * \beta_1 = \alpha_n \circledast \beta_1$ which equals $((a_n)\mathbf{d} \rhd \beta_1) \circ (\alpha_n \lhd \beta_1 \mathbf{r})$ shown in blue and thus we have two sides of a two-cell which we can complete:



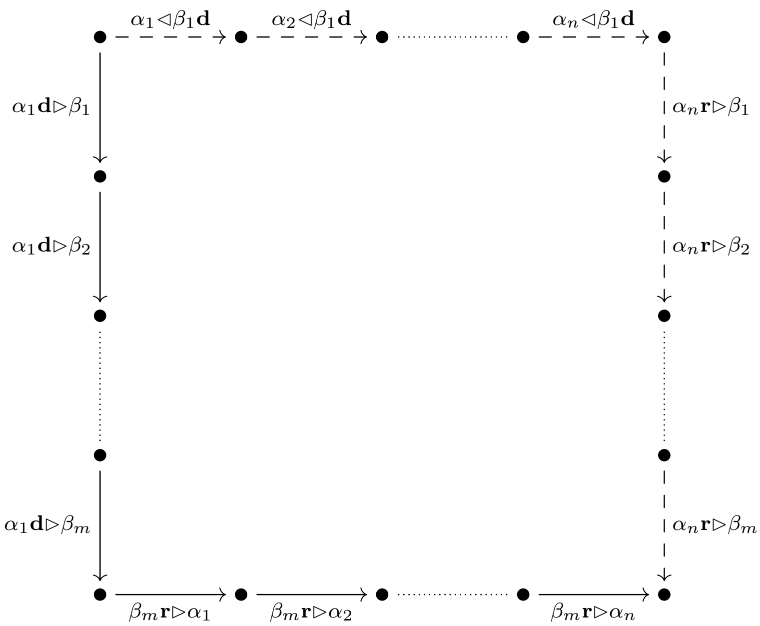
It is homotopically equivalent to follow the blue path and thus we get a new diagram, where we can see another pair making up 2 sides of a 2-cell:



We can continue to use the 2-cells in this way until we reach the end of the rightmost column:



We can then repeat this process across each of the remaining columns until we have:



and so $\alpha * \beta$ is homotopic to $\alpha \circledast \beta$ via the Squier 2-cells.

5.1.4 Corollary. *The set of edge paths* $\mathrm{star}_1(\Pi(\mathrm{Sq}(\mathcal{P}), F(X)))$ *at* $1 \in F(X)$ *in the fundamental groupoid of the Squier complex* $\mathrm{Sq}(\mathcal{P})$ *is a group under the binary operation* $*$*, and the restriction of the range map to* $\mathrm{star}_1(\Pi(\mathrm{Sq}(\mathcal{P}), F(X)))$ *is a crossed module*

$$\mathbf{r} : \mathrm{star}_1(\Pi(\mathrm{Sq}(\mathcal{P}), F(X))) \to F(X)\,.$$

Proof. This follows from Theorem 5.1.3 and Proposition 3.2.5. □

5.2 The Crossed Module of a Squier Complex

Our aim is now to show that the crossed module in Corollary 5.1.4 is isomorphic to the free crossed module $C \xrightarrow{\partial} F(X)$ derived from the presentation $\mathcal{P}$, as in Section 2.1.3. Furthering our blurring of the distinction between an edge path and its homotopy class in the fundamental groupoid, we shall abbreviate $\mathrm{star}_1(\Pi(\mathrm{Sq}(\mathcal{P}), F(X)))$ as $\mathrm{star}_1(\mathrm{Sq}(\mathcal{P}))$. We denote by S_1 the set of all edges $e \in \mathrm{Sq}(\mathcal{P})$ with $e\mathbf{d} = 1$, that is

$$S_1 = \{(p, l = r, q) : p, q \in F(X), (l = r) \in R, plq = 1\} \tag{5.3}$$
$$= \{(q^{-1}l^{-1}, l = r, q) : q \in F(X), (l = r) \in R\} \,. \tag{5.4}$$

We shall denote the edge $(q^{-1}l^{-1}, l = r, q)$ by $\lambda_{l=r,q}$.

Let $e = (p, l = r, q)$ be an edge of $\mathrm{Sq}(\mathcal{P})$ in the connected component of $1 \in F(X)$, and define

$$e\lambda = (e\mathbf{d})^{-1} \triangleright e = (q^{-1}l^{-1}, l = r, q) \in S_1 \,.$$

5.2.1 Proposition. *Let α be an edge path in $\mathrm{star}_1(\mathrm{Sq}(\mathcal{P}))$. Then α is equal to a $*$-product of single edges in S_1. Thus the group $(\mathrm{star}_1(\mathrm{Sq}(\mathcal{P})), *)$ is generated by S_1.*

Proof. The claim is trivial for edge paths α of length 1, so now suppose that

$$\alpha = \alpha_1 \circ \alpha_2 \circ \cdots \circ \alpha_n$$

for some $n > 1$, with each α_i a single edge. Set $\lambda_i = \alpha_i\lambda = (\alpha_i\mathbf{d})^{-1} \triangleright \alpha_i$. Then $\lambda_i \in S_1$, and $\alpha_1 = \lambda_1$. We now assume inductively that

$$\alpha_1 \circ \alpha_2 \circ \cdots \circ \alpha_{n-1} = \lambda_1 * \lambda_2 * \cdots * \lambda_{n-1} \,.$$

Then

$$\begin{aligned}
\alpha &= (\alpha_1 \circ \cdots \circ \alpha_{n-1}) \circ \alpha_n \\
&= (\alpha_1 \circ \cdots \circ \alpha_{n-1}) \circ (\alpha_n\mathbf{d} \triangleright \lambda_n) \\
&= (\alpha_1 \circ \cdots \circ \alpha_{n-1}) \circ (\alpha_{n-1}\mathbf{r} \triangleright \lambda_n) \\
&= (\alpha_1 \circ \cdots \circ \alpha_{n-1}) * \lambda_n \\
&= \lambda_1 * \lambda_2 * \cdots * \lambda_{n-1} * \lambda_n \,.
\end{aligned}$$

Therefore $\alpha = \lambda_1 * \cdots * \lambda_n$. $\Box$

5.2.1 Definition. We denote the product $\lambda_1 * \cdots * \lambda_n$ used to represent $\alpha \in \text{star}_1(\text{Sq}(\mathcal{P}))$ in Proposition 5.2.1 by $\alpha\lambda$.

5.2.2 Lemma. *Suppose that $\alpha \circ \beta \in \text{star}_1(\text{Sq}(\mathcal{P}))$. Then $(\alpha \circ \beta)\lambda = \alpha\lambda * \beta\lambda$.*

We now want to understand the effect of homotopy of edge paths in $\text{Sq}(\mathcal{P})$ on the $*$–products defined in Proposition 5.2.1. We first consider a 1–homotopy, that is, the insertion of deletion of a pair of inverse edges. Let $\xi = \rho \circ \sigma$ in $\text{Sq}(\mathcal{P})$, with $\rho \in \text{star}_1(\text{Sq}(\mathcal{P}))$. Then consider the homotopic path $\xi' = \rho \circ \alpha \circ \alpha^{-\circ} \circ \sigma$, with α a single edge. Then

$$\begin{aligned}
\xi'\lambda &= \rho\lambda * \alpha\lambda * (\alpha^{-\circ})\lambda * \sigma\lambda \\
&= \rho\lambda * [(\alpha\mathbf{d})^{-1} \rhd \alpha * (\alpha^{-\circ}\mathbf{d})^{-1} \rhd \alpha^{-\circ}] * \sigma\lambda \\
&= \rho\lambda * [(\alpha\mathbf{d})^{-1} \rhd \alpha * (\alpha\mathbf{r})^{-1} \rhd \alpha^{-\circ}] * \sigma\lambda \\
&= \rho\lambda * [(\alpha\mathbf{d})^{-1} \rhd \alpha \lhd 1) \circ (\alpha\mathbf{d})^{-1}\alpha\mathbf{r} \rhd ((\alpha\mathbf{r})^{-1} \rhd \alpha^{-\circ})] * \sigma\lambda \\
&= \rho\lambda * [(\alpha\mathbf{d})^{-1} \rhd \alpha) \circ (\alpha\mathbf{d})^{-1} \rhd \alpha^{-\circ})] * \sigma\lambda \\
&= \rho\lambda * \sigma\lambda \qquad\qquad = \xi\lambda
\end{aligned}$$

Therefore a 1–homotopy applied to an edge path α does not change the $*$–product.

Suppose that we have a 2–cell

$$\begin{array}{ccc}
plqtsu & \xrightarrow{(p,l=r,qtsu)} & prqtsu \\
{\scriptstyle (plqt,s=d,u)}\downarrow & & \downarrow{\scriptstyle (prqt,s=d,u)} \\
plqtdu & \xrightarrow[(p,l=r,qtdu)]{} & prqtdu
\end{array} \tag{5.5}$$

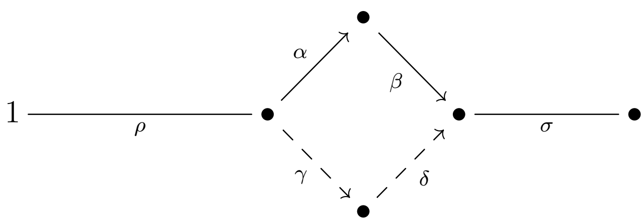
in the connected component of $1 \in F(X)$ in $\mathrm{Sq}(\mathcal{P})$, with

$$\alpha = (p, l = r, qtsu), \beta = (prqt, s = d, u), \gamma = (plqt, s = d, u), \delta = (p, l = r, qtdu)\,. \tag{5.6}$$

This 2–cell gives a homotopy between $\alpha \circ \beta$ and $\gamma \circ \delta$, or equivalently tells us that in $\Pi(\mathrm{Sq}(\mathcal{P}))$ we have

$$(p, l = r, q) * (t, s = d, u) = (p, l = r, q) \circledast (t, s = d, u)\,.$$

If this 2–cell is involved in a 2–homotopy between edge paths ξ and ξ', we may assume using 1–homotopies where necessary, that we have $\xi = \rho \circ \alpha \circ \beta \circ \sigma$ and $\xi' = \rho \circ \gamma \circ \delta \circ \sigma$, that is a configuration



Then, using $\simeq$ to denote homotopy of edge paths in $\mathrm{Sq}(\mathcal{P})$, we have

$$\begin{aligned} \xi\lambda &= \rho\lambda * \alpha\lambda * \beta\lambda * \sigma\lambda \\ &= \rho\lambda * (\alpha \circ \beta) * \sigma\lambda \\ &\simeq \rho\lambda * (\gamma \circ \delta) * \sigma\lambda \\ &= \rho\lambda * \gamma\lambda * \delta\lambda * \sigma\lambda \\ &= \xi'\lambda\,. \end{aligned} \tag{5.7}$$

The above considerations show that, for a given homotopy class in $\mathrm{star}_1(\mathrm{Sq}(\mathcal{P}))$, we may select a representative edge path ξ in the form of its $*$–product $\xi\lambda$ and that this product will be unique up to changes induced by the 2–cells in $\mathrm{Sq}(\mathcal{P})$, which may modify the product as in equations (5.7) above. We can be more precise.

5.2.3 Proposition. *Given $q \in F(X)$ and $(l = r) \in R$, we set*

$$\lambda_{l=r,q} = (q^{-1}l^{-1}, l = r, q) \in S_1 \,.$$

Then the following are a set of defining relations for the group $(\mathrm{star}_1(\mathrm{Sq}(\mathcal{P})), *)$ *on the generating set S_1:*

$$\lambda_{l=r,vsu} * \lambda_{s=d,u} = \lambda_{s=d,u} * \lambda_{l=r,vdu} \,, \tag{5.8}$$

where $(l = r), (s = d) \in R$ and $u, v \in F(X)$.

Proof. Since

$$\lambda_{l=r,vsu} * \lambda_{s=d,u} = (u^{-1}s^{-1}v^{-1}l^{-1}, l = r, vsu) \circ (u^{-1}s^{-1}v^{-1}l^{-1}rv, s = d, u)$$

and

$$\lambda_{s=d,u} * \lambda_{l=r,vdu} = (u^{-1}s^{-1}, s = d, u) \circ (u^{-1}s^{-1}v^{-1}l^{-1}, l = r, vdu) \,,$$

we see that the stated relations are true in $(\mathrm{star}_1(\mathrm{Sq}(\mathcal{P})), *)$ since they record the equality of the two paths around the sides of the 2–cell

$$\begin{array}{ccc}
1 & \xrightarrow{(u^{-1}s^{-1}v^{-1}l^{-1},l=r,vsu)} & u^{-1}s^{-1}v^{-1}l^{-1}rvsu \\
\Big\downarrow{\scriptstyle (u^{-1}s^{-1},s=d,u)} & & \Big\downarrow{\scriptstyle (u^{-1}s^{-1}v^{-1}l^{-1}rv,s=d,u)} \\
u^{-1}s^{-1}du & \xrightarrow[(u^{-1}s^{-1}v^{-1}l^{-1},l=r,vdu)]{} & u^{-1}s^{-1}v^{-1}l^{-1}rvdu
\end{array}$$

On the other hand, to accomplish the rewriting in (5.7), we need to identify the paths around the boundary of a general 2–cell as in (5.5) and, in the notation of

(5.6), use the relation

$$\alpha\lambda * \beta\lambda = \gamma\lambda * \delta\lambda\,.$$

Now

$$\alpha\lambda = (u^{-1}s^{-1}t^{-1}q^{-1}l^{-1}, l = r, qtsu) = (l^{-1}, l = r, 1)^{qtsu}\,,$$

$$\beta\lambda = (u^{-1}s^{-1}, s = d, u) = (s^{-1}, s = d, 1)^{u} = \gamma\lambda$$

and

$$\delta\lambda = (u^{-1}d^{-1}t^{-1}q^{-1}l^{-1}, l = r, qtdu) = (l^{-1}, l = r, 1)^{qtdu}\,.$$

If we set $v = qt$ then

$$\alpha\lambda = (u^{-1}s^{-1}v^{-1}l^{-1}, l = r, vsu) = \lambda_{l=r,vsu}\,,$$

$$\beta\lambda = \lambda_{s=d,u} = \gamma\lambda$$

and

$$\delta\lambda = (u^{-1}d^{-1}v^{-1}l^{-1}, l = r, vdu) = \lambda_{l=r,vdu}\,.$$

and the required relation is

$$\lambda_{l=r,vsu} * \lambda_{s=d,u} = \lambda_{s=d,u} * \lambda_{l=r,vdu}\,.$$

□

5.2.4 Example. Recall the presentation $\mathcal{P} = \langle x : 1 = x^n \rangle$ from Example 5.1.1 we can now use Proposition 5.2.3 to construct star_1. We have the generating set:

$$S_1 = \{(x^{-k}, 1 = x^n, x^k) : k \in \mathbb{Z}\}$$

where we can denote $(x^{-k}, 1 = x^n, x^k)$ by λ_k. The defining relations are

$$\lambda_{k+l} * \lambda_l = \lambda_l * \lambda_{k+n+l} \tag{5.9}$$

If we set $k = 0$ we can see that we have $\lambda_l = \lambda_{n+l}$ for all $l \in \mathbb{Z}$, so in star_1 only n of the generators remain distinct. Applying $\lambda_l = \lambda_{n+l}$ to λ_{k+l} we can then see that (5.9) becomes:

$$\lambda_{k+l} * \lambda_l = \lambda_l * \lambda_{k+l}$$

for all $k, l \in \mathbb{Z}$ and so all relations assert that the generators commute. Hence star_1 is free abelian on n generators.

5.2.5 Theorem. *The crossed $F(X)$-module $\mathrm{star}_1(\mathrm{Sq}(\mathcal{P})) \xrightarrow{\mathbf{r}} F(X)$ derived from the Squier complex $\mathrm{Sq}(\mathcal{P})$ of a group presentation $\mathcal{P} = \langle X : R \rangle$, is isomorphic to the free crossed $F(X)$-module $C \xrightarrow{\partial} F$ derived from $\mathcal{P}$, as in section 2.1.3.*

Proof. Recall from section 2.1.3 that the free crossed module $C \xrightarrow{\partial} F$ has basis function $v : R \to C$, $v : (l = r) \mapsto (l = r, 1)$. We define $\overline{v} : R \to \mathrm{star}_1(\mathrm{Sq}(\mathcal{P}))$ by $\overline{v} : (l = r) \mapsto (l^{-1}, l = r, 1)$. Then $v\partial = \overline{v}\mathbf{r}$, and thus by freeness of (C, ∂), we have a crossed module morphism $\phi : C \to \mathrm{star}_1(\mathrm{Sq}(\mathcal{P}))$, defined on generators by $(l = r, u) \mapsto (u^{-1}l^{-1}, l = r, u) = \lambda_{l=r,u}$. We note that this is a bijection from the group generating set of C to S_1, a generator for C is a pair $(l = r, u)$ with $(l = r) \in R$ and $u \in F(X)$, these are exactly the pairs used to form $\lambda_{l=r,u}$.

To obtain an inverse to ϕ, we therefore wish to map $\lambda_{l=r,u} \mapsto (l = r, u)$. This will be well-defined and a homomorphism if and only if the defining relations given in (5.8) in Proposition 5.2.3 are mapped to an equation that holds in the group C.

Now the left-hand side of (5.8) maps to

$$(l = r, vsu)(s = d, u)$$

and the right-hand side to

$$(s = d, u)(l = r, vdu)\,.$$

and in the crossed $F(X)$–module C we do indeed have

$$(s = d, u)^{-1}(l = r, vsu)(s = d, u) = (l = r, vsu(u^{-1}s^{-1}du)) = (l = r, vdu)\,.$$

□

The kernel of the map $\mathrm{star}_1(\mathrm{Sq}(\mathcal{P})) \xrightarrow{\mathbf{r}} F(X)$ is the local group at $1 \in F(X)$ of the groupoid $\Pi(\mathrm{Sq}(\mathcal{P}), F(X))$, that is the fundamental group $\pi_1(\mathrm{Sq}(\mathcal{P}), 1)$. Then from Proposition 2.1.8 we obtain:

5.2.6 Proposition. *Let $\mathcal{P} = \langle X : R \rangle$ be a presentation of a group G with presentation map $\theta : F(X) \to G$ and let $N = \ker\theta$, so that N^{ab} is the relation module of $\mathcal{P}$. Then we have a short exact sequence of G–modules:*

$$0 \to \pi_1(\mathrm{Sq}(\mathcal{P}), 1) \to \bigoplus_{r \in R} \mathbb{Z}G \to N^{ab} \to 0\,. \tag{5.10}$$

5.2.7 Example. Returning again to the presentation $\mathcal{P} = \langle x : 1 = x^n \rangle$ we have that the relation module for $\mathcal{P}$ is the abelianisation of the image of the crossed module homomorphism $\mathbf{r} : \mathrm{star}_1 \to F(x)$. The image of $\mathbf{r}$ is the vertex set of the component of the Squier complex containing 1, that is $\{x^{kn} : k \in \mathbb{Z}\}$. So we can see that $N^{ab} \cong \mathbb{Z}$. Then we can see that (5.10) becomes:

$$0 \to F^{ab}_{n-1} \to F^{ab}_{n} \to F^{ab}_{1} \to 0$$

where F_n^{ab} is the free abelian group on n generators.

The action of G on the relation module F_1^{ab} is induced by conjugation in $F(x)$ and so the action is trivial.

F_n^{ab} is generated as an abelian group by $\{\lambda_0, \dots, \lambda_{n-1}\}$, we can see that

$$x^{-1} \triangleright \lambda_k \triangleleft x = (x^{-k-1}, 1 = x^n, x^{k+1}) = \lambda_{k+1}$$

so $\lambda_l = x^{-l} \triangleright \lambda_0 \triangleleft x^l$, and so λ_0 generates F_n^{ab} as a free G-module.

For all k we have that $\mathbf{r} : \lambda_k \mapsto x^n$, so we can see that $\ker \mathbf{r}$ is generated by the set

$$\{\lambda_0^{-*} * \lambda_1, \lambda_1^{-*} * \lambda_2, \dots, \lambda_{n-2}^{-*} * \lambda_{n-1}\}$$

which has size $n-1$, thus is a basis for F_{n-1}^{ab} as a free abelian group. Write $\kappa_i = \lambda_{i-1}^{-*} * \lambda_i$ for $i \in [1, n-1]$. Then the G-action on F_{n-1}^{ab}, induced by $\lambda_k \mapsto \lambda_{k+1}$, is

$$x^{-1} \triangleright \kappa_i \triangleleft x = \kappa_{i+1} \qquad \text{if } 1 \leqslant i \leqslant n-2$$

$$x^{-1} \triangleright \kappa_{n-1} \triangleleft x = -(\kappa_1 + \kappa_2 + \dots + \kappa_{n-1})$$

Chapter 6

Derivation and Relation Modules

6.1 Derivation Module of a Group Homomorphism

We will begin by reviewing some results from group theory given by Crowell in [10].

6.1.1 Definition. Given an arbitrary group homomorphism $\phi : G \to H$ and a right H-module A, then a *ϕ-derivation* $\delta : G \to A$ is a mapping which satisfies:

$$(g_1 g_2)\delta = (g_1\delta) \triangleleft (g_2\phi) + (g_2\delta)\,.$$

If ϕ is the identity map $G \to G$ then we just refer to δ as a *derivation*.

6.1.2 Definition. The *derivation module* of a group homomorphism, $\phi : G \to H$ consists of an H-module D_ϕ and a ϕ-derivation $\delta : G \to D_\phi$, such that for any H-module A and ϕ-derivation $\delta' : G \to A$ there exists a unique H-morphism $\lambda : D_\phi \to A$, such that

$$\begin{array}{ccc} G & \xrightarrow{\delta} & D_\phi \\ & \searrow^{\delta'} & \downarrow \lambda \\ & & A \end{array}$$

commutes.

In [10] Crowell proves that for a homomorphism of groups $\phi : G \to H$, the derivation module exists and is unique up to isomorphism, and in fact is given by the tensor product

$$D_\phi = IG \otimes_G \mathbb{Z}H\,,$$

with $\delta : G \to D_\phi$ defined by $g \mapsto 1 \otimes (g-1)$. Where IG is the *augmentation ideal* of G, the kernel of the natural map $\varepsilon : \mathbb{Z}G \to \mathbb{Z}$ that maps $g \mapsto 1$ for all $g \in G$. Crowell goes on to show that for a given short exact sequence of groups:

$$1 \to K \xrightarrow{\psi} G \xrightarrow{\phi} H \to 1 \tag{6.1}$$

we can construct a short exact sequence of H-modules:

$$0 \to K^{ab} \xrightarrow{\psi_*} D_\phi \xrightarrow{\phi_*} IH \to 0 \tag{6.2}$$

where D_ϕ is the derivation module of ϕ, IH is the augmentation ideal of H, and K^{ab} is the abelianisation of K. Here, for $g \in G$ and $k \in K$ with image $\overline{k} \in K^{ab}$, we have

$$\phi_* : 1 \otimes (1-g) \mapsto 1 - g\phi$$

and

$$\psi_* : \overline{k} \mapsto 1 \otimes (1-k)\,.$$

Of particular interest is the case of a presentation map. Let $\mathcal{P} = \langle X; \mathcal{R} \rangle$ present the group G, with $\theta : F(X) \to G$ the presentation map. Then the exact sequences (6.1) and (6.2) become:

$$1 \to N \to F(X) \xrightarrow{\theta} G \to 1 \tag{6.3}$$

$$0 \to N^{ab} \to D_\theta \xrightarrow{\theta*} IG \to 0 \tag{6.4}$$

and we recognise N^{ab} from section 2.1.3 as the relation module for $\mathcal{P}$, which can now be defined as $\ker(D_\theta \to IG)$.

6.2 Derivation Module of an Inverse Monoid Homomorphism

This section begins with the construction of the derivation module for an inverse monoid morphism given in [14].

6.2.1 Definition. Let M, N be inverse monoids and let $\mathcal{A}$ be an $\mathfrak{L}(M)$-module. Suppose that we have an inverse monoid homomorphism $\phi : N \to M$. Then a *ϕ-derivation* $\eta : N \to \mathcal{A}$ is a function $N \to \bigsqcup_{e \in E(M)} A_e$ such that:

- if $a \in N$ then $a\eta \in A_{(a^{-1}a)\phi}$, and
- if $a, b \in N$ and $a^{-1}a \geqslant bb^{-1}$ then

$$(ab)\eta = a\eta \lhd ((a^{-1}a)\phi, b\phi) + b\eta \tag{6.5}$$

If $\phi : M \to M$ is the identity map, we just refer to η as a *derivation*.

6.2.1 Example. Let IM be the augmentation ideal of M. Define $\kappa : M \to IM$ by $m \mapsto m - m^{-1}m$. Then κ is a derivation:

- for $m \in M$ we have $m\kappa = m - (m^{-1}m)$, which is an element of $(IM)_{m^{-1}m}$,

- For $l, m \in M$ such that $l^{-1}l \geqslant mm^{-1}$ we have:

$$\begin{aligned} l\kappa \lhd (l^{-1}l, m) + m\kappa &= (l - l^{-1}l) \lhd (l^{-1}l, m) + (m - m^{-1}m) \\ &= lm - l^{-1}lm + m - m^{-1}m \\ &= lm - m + m - m^{-1}m \\ &= lm - m^{-1}m \\ &= lm - m^{-1}l^{-1}lm \\ &= (lm) - (lm)^{-1}(lm) \\ &= (lm)\kappa \end{aligned}$$

6.2.1 Constructing the Derivation Module

Let $\phi : N \to M$ be an inverse monoid homomorphism, and for each $e \in E(M)$ define

$$(N \between M)_e = \left\{(a, m) : a \in N, m \in M, (a^{-1}a)\phi \geqslant mm^{-1}, m^{-1}m = e\right\}.$$

Now define $D_{\phi,e}$ as the abelian group generated by $(N \between M)_e$ subject to all relations of the form

$$(ab, m) = (a, (b\phi)m) + (b, m) \quad \text{where} \quad a, b \in N, m \in M, \text{ and } a^{-1}a \geqslant bb^{-1}. \tag{6.6}$$

We denote the image of (a, m) in $D_{\phi,e}$ by $\langle a, m\rangle$.

The **derivation module** $\mathcal{D}_\phi$ of ϕ is then the $\mathfrak{L}(M)$–module with, for $e \in E(M)$, $(\mathcal{D}_\phi)_e$ defined to be $D_{\phi,e}$, and the action of (e, n) on a generator $\langle a, m\rangle$ given by

$$\langle a, m\rangle \lhd (e, n) = \langle a, mn\rangle,$$

Since $m^{-1}m = e \geqslant nn^{-1}$, we have $\langle a, mn\rangle \in D_{\phi,n^{-1}n}$, and $\mathcal{D}_\phi$ is an $\mathfrak{L}(M)$-module.

6.2.2 Proposition. *[14, Proposition 3.5] There exists a canonical ϕ-derivation δ :*

$N \to \mathcal{D}_\phi$ such that given any ϕ-derivation $\eta : N \to \mathcal{A}$, with $\mathcal{A}$ an $\mathfrak{L}(M)$-module, there is a unique $\mathfrak{L}(M)$-map $\xi : \mathcal{D}_\phi \to \mathcal{A}$ such that

$$\begin{array}{ccc} N & \xrightarrow{\delta} & \mathcal{D}_\phi \\ & {\scriptstyle\eta}\searrow & \downarrow{\scriptstyle\xi} \\ & & \mathcal{A} \end{array}$$

commutes.

Proof. Define the map $\delta : N \to \mathcal{D}_\phi$ by $a \mapsto \langle a, (a^{-1}a)\phi\rangle$. Then $a\delta \in D_{\phi,(a^{-1}a)\phi}$ and δ satisfies the derivation property (6.5) since, for $a, b \in N$, with $a^{-1}a \geqslant bb^{-1}$, we have

$$\begin{aligned} (ab)\delta &= \langle ab, (b^{-1}a^{-1}ab)\phi\rangle \\ &= \langle a, b\phi(b^{-1}a^{-1}ab)\phi\rangle + \langle b, (b^{-1}a^{-1}ab)\phi\rangle \\ &= \langle a, ((a^{-1}a)\phi)(b\phi)\rangle + \langle b, (b^{-1}b)\phi\rangle \\ &= \langle a, (a^{-1}a)\phi\rangle \lhd ((a^{-1}a)\phi, b\phi) + \langle b, (b^{-1}b)\phi\rangle \\ &= a\delta \lhd ((a^{-1}a)\phi, b\phi) + b\delta\,. \end{aligned}$$

Let $\mathcal{A}$ be an $\mathfrak{L}(M)$-module and suppose that an $\mathfrak{L}(M)$-map, $\xi : \mathcal{D}_\phi \to \mathcal{A}$ satisfies $\eta = \delta\xi$. Let $\langle a, m\rangle \in \mathcal{D}_\phi$: then $(a^{-1}a)\phi \geqslant mm^{-1}$, and

$$\begin{aligned} \langle a, m\rangle\xi &= \big(\langle a, (a^{-1}a)\phi\rangle \lhd ((a^{-1}a)\phi, m)\big)\xi \\ &= \langle a, (a^{-1}a)\phi\rangle\xi \lhd ((a^{-1}a)\phi, m) \\ &= a\delta\xi \lhd ((a^{-1}a)\phi, m) \\ &= a\eta \lhd ((a^{-1}a)\phi, m) \end{aligned}$$

So ξ is completely determined by η. Given a derivation $\eta : N \to \mathcal{A}$ we now define $\xi : \mathcal{D}_\phi \to \mathcal{A}$ by

$$\langle a, m\rangle = a\eta \lhd ((a^{-1}a)\phi, m)\,.$$

This is well-defined since for $a, b \in N$, $a^{-1}a \geqslant bb^{-1}$ we have:

$$\begin{aligned}
(\langle ab, m\rangle - \langle b, m\rangle)\xi &= ((ab)\eta \lhd ((b^{-1}b)\phi, m)) - (b\eta \lhd ((b^{-1}b)\phi, m)) \\
&= ((ab)\eta - b\eta) \lhd ((b^{-1}b)\phi, m) \\
&= (a\eta \lhd ((a^{-1}a)\phi, b\phi) + b\eta - b\eta) \lhd ((b^{-1}b)\phi, m) \\
&= a\eta \lhd ((a^{-1}a_\phi, b\phi)((b^{-1}b)\phi, m) \\
&= a\eta \lhd ((a^{-1}a)\phi, (b\phi)m) \\
&= \langle a, (b\phi)m\rangle\xi\,.
\end{aligned}$$

It is also an $\mathfrak{L}(M)$-map since:

$$\begin{aligned}
(\langle a, m\rangle \lhd (e, x))\xi &= \langle a, mx\rangle\xi \\
&= a\eta \lhd ((a^{-1}a)\phi, mx) \\
&= a\eta \lhd ((a^{-1}a)\phi, m) \lhd (e, x) \\
&= \langle a, m\rangle\xi \lhd (e, x)\,.
\end{aligned}$$

□

6.2.3 Example. In Example 6.2.1 we saw the derivation $\kappa : M \to IM$ mapping $m \mapsto m - m^{-1}m$. This induces an M–module map $\xi : \mathcal{D} \to IM$ that maps $\langle a, b\rangle \mapsto ab - b$.

We note some consequences of the relations (6.6) for later use.

6.2.4 Lemma. *[14, Lemmas 3.2 and 3.4]*

(a) If $(a, m) \in (N \between M)_e$ then $\langle aa^{-1}, m\rangle = 0$ in $D_{\phi,e}$.

(b) If $a, b \in N$, with $b \geqslant a$ and $m \in M$ with $(a^{-1}a)\phi \geqslant mm^{-1}$ then

$$\langle a, m \rangle = \langle b, m \rangle$$

in $D_{\phi,e}$.

(c) Suppose N is generated as an inverse semigroup by the subset X. Then $D_{\phi,e}$ is generated as an abelian group by the subset

$$(X \between M)_e = \{\langle x, m \rangle : x \in X,\, m \in M, (x^{-1}x)\phi \geqslant mm^{-1}, m^{-1}m = e\}$$

Proof. *(a)* We have:

$$\begin{aligned} 0 &= \langle a, m \rangle - \langle a, m \rangle \\ &= \langle aa^{-1}a, m \rangle - \langle a, m \rangle \\ &= \langle a, ((a^{-1}a)\phi)m \rangle + \langle a^{-1}a, m \rangle - \langle a, m \rangle \\ &= \langle a, m \rangle + \langle a^{-1}a, m \rangle - \langle a, m \rangle \\ &= \langle a^{-1}a, m \rangle . \end{aligned}$$

(b) Note that when $b \geqslant a$ we have $a = ba^{-1}a$ and $b^{-1}b \geqslant a^{-1}a$, and so:

$$\langle a, m \rangle = \langle ba^{-1}a, m \rangle = \langle b, ((a^{-1}a)\phi)m \rangle + \langle a^{-1}a, m \rangle = \langle b, m \rangle .$$

(c) Suppose $a \in N$, with $a = bu$ for some $b \in N$ and $u \in X \cup X^{-1}$, then

$$a = b(b^{-1}b)u \text{ and } b^{-1}b \geqslant b^{-1}buu^{-1} = (b^{-1}bu)(b^{-1}bu)^{-1}$$

Then the relations, (6.6), imply that if $(a, m) \in (N \between M)_e$, then

$$\langle a, m \rangle = \langle b, ((b^{-1}bu)\phi)m \rangle + \langle b^{-1}bu, m \rangle.$$

By *(b)* $\langle b^{-1}bu, m\rangle = \langle u, m\rangle$ and so

$$\langle a, m\rangle = \langle b, ((b^{-1}bu)\phi)m\rangle + \langle u, m\rangle$$

It follows by induction on the minimum length of a product of elements in $X \cup X^{-1}$ representing $a \in N$, that $D_{\phi,e}$ is generated as an abelian group by the subset $\{\langle x, m\rangle : x \in X \cup X^{-1}\}$. By part *(a)*, and relations (6.6), for any $(a, m) \in (N \between M)_e$

$$0 = \langle aa^{-1}, m\rangle = \langle a, (a^{-1}\phi)m\rangle + \langle a^{-1}, m\rangle$$

and thus $\langle x^{-1}, m\rangle = -\langle x, (x^{-1}\phi)m\rangle$. $\square$

6.2.5 Corollary. *Consider an inverse monoid, M, with presentation $\mathcal{P} = [X; \mathcal{R}]$, and associated presentation map $\theta : \mathrm{FIM}(X) \to M$. Then $D_{\theta,e}$ is generated as an abelian group by the subset*

$$\{\langle x, m\rangle : x \in X, (x^{-1}x)\theta \geqslant mm^{-1}, m^{-1}m = e\}.$$

and $\mathcal{D}_\theta$ is generated as an $\mathfrak{L}(M)$-module by

$$\{\langle x, (x^{-1}x)\theta\rangle : x \in X\}.$$

6.2.2 Examples of Derivation Modules

6.2.6 Example. Let $K = (K_e, \kappa_{e,f})$ be a Clifford monoid and let $\mathcal{K} = (K_e^{ab}, \overline{\kappa}_{e,f})$ be its abelianisation. We have an inverse monoid map $\varepsilon : K \to E(K)$ that maps each $k \in K_e$ to $e \in E(K)$, and $(\mathcal{D}_\varepsilon)_f$ is generated by elements $\langle a, f\rangle$ with $a \in K_e$ and $e \geqslant f$, subject to all relations of the form $\langle ab, x\rangle = \langle a, x\rangle + \langle b, x\rangle$, where $a \in K_e, b \in K_f$ and $e \geqslant f \geqslant x$. It is then easy to see that the map $\alpha : \langle a, f\rangle \mapsto \overline{a\kappa_{e,f}} \in K_f^{ab}$ is an abelian group homomorphism, with inverse given by $\overline{a} \mapsto \langle a, e\rangle$ for $a \in K_e$. Hence the Clifford monoids $\mathcal{K}$ and $\mathcal{D}_\varepsilon$ are isomorphic. The $E(K)$–module structure on $\mathcal{K}$ is determined by the $\overline{\kappa_{e,f}}$: in Loganathan's formalism (see section 1.6), $\overline{a} \lhd (e,f) = \overline{a\kappa_{e,f}}$. Hence

$$(\langle a,f\rangle \lhd (f,x))\alpha = \langle a,x\rangle\alpha = \overline{a\kappa_{e,x}} = \overline{a\kappa_{e,f}\kappa_{f,x}} = \langle a,f\rangle\alpha \lhd (f,x)\,,$$

and we have an isomorphism of $E(K)$–modules $\mathcal{D} \cong \mathcal{K}$.

6.2.7 Example. Consider the identity map $\mathbf{1} : M \to M$. For $e \in E(M)$, the abelian group $(\mathcal{D}_{\mathbf{1}})_e$ is generated by the elements $\langle a,b\rangle$ with $a^{-1}a \geqslant bb^{-1}$, subject to relations $\langle ab,c\rangle = \langle a,bc\rangle + \langle b,c\rangle$. It follows that we have an abelian group homomorphism $(\mathcal{D}_{\mathbf{1}})_e \to (IM)_e$ given by $\langle a,b\rangle \mapsto ab - b$. This has inverse given by $m - m^{-1}m \mapsto \langle m, m^{-1}m\rangle$ and we have an isomorphism of M–modules $\mathcal{D}_{\mathbf{1}} \cong IM$.

Now we have an analogue to Crowell's derivation module for an inverse monoid presentation so we can generalise the exact sequence 6.4 and replace it with

$$0 \to \mathcal{M}_\theta \to \mathcal{D}_\theta \xrightarrow{\theta_*} IM \to 0 \tag{6.7}$$

where $\mathcal{D}_\theta$ is the derivation module for the presentation map θ. The map θ_* is induced by the θ–derivation $\mathrm{FIM}(X) \to \mathbb{Z}M$ mapping $w \mapsto w\theta - (w^{-1}w)\theta$, which is the composite of θ with the derivation κ of Example 6.2.1. It follows that, for $\langle w,m\rangle \in \mathcal{D}_\theta$,

$$\theta_* : \langle w,m\rangle \mapsto (w\theta)m - m\,.$$

6.2.2 Definition. The M–module $\ker\theta_* = \mathcal{M}_\theta$ is the *relation module* of the presentation $\mathcal{P}$.

6.2.3 Factorising the Presentation Map

Our aim in this section is to obtain a more explicit description of the relation module $\mathcal{M}_\theta$, which will be analogous to the identification of the relation module of a group presentation map $\theta : F(X) \to G$ as the abelianisation of $\ker \theta$. Our first result is taken from [26], and characterizes the idempotent pure quotients of $\mathrm{FIM}(X)$.

6.2.8 Lemma. *[26, Lemma 1.6] Let $\mathcal{P} = [X; T]$ be a presentation of an inverse monoid M with associated presentation map $\theta : \mathrm{FIM}(X) \to M$. Then the following are equivalent:*

(a) $\mathcal{P}$ is equivalent to a presentation of the form $\mathcal{P}_1 = [X; T_1]$ where $T_1 = \{e_i = f_i : i \in I\}$ for some set I and idempotents e_i, f_i of $\mathrm{FIM}(X)$.

(b) the presentation map $\theta : \mathrm{FIM}(X) \to M$ is idempotent pure.

(c) Each Schützenberger graph, $\mathrm{Sch}^{\mathcal{L}}(X, T, e)$ is a tree.

Proof. *(a) ⇒ (b)*: The group presented by $\langle X; T_1 \rangle$ is clearly $F(X)$ and so we have a commutative triangle

$$\begin{array}{ccc} \mathrm{FIM}(X) & \xrightarrow{\theta} & M \\ & {\scriptstyle\sigma}\searrow \quad \swarrow{\scriptstyle\sigma_M} & \\ & F(X) & \end{array}$$

Suppose that $w \in \mathrm{FIM}(X)$ and that $w\theta \in E(M)$. Then $w\sigma = w\theta\sigma_M = 1_F$, but σ is idempotent pure, and so $w \in E(\mathrm{FIM}(X))$.

(b) ⇒ (c): Suppose that $a, b \in \mathrm{FIM}(X)$ and that $a\theta = b\theta$. Then $(aa^{-1})\theta = (ba^{-1})\theta \in E(M)$ and so $ba^{-1} \in E(\mathrm{FIM}(X))$. Therefore $(ba^{-1})\sigma = 1_F$ and so $(ba^{-1})\sigma = (b\sigma)(a^{-1}\sigma) = (b\sigma)(a\sigma)^{-1} = 1_F$ and $b\sigma = a\sigma$. So the congruence θ (and in fact any idempotent pure congruence) is contained in σ.

In particular $\sigma : \mathrm{FIM}(X) \to F(X)$ factors through M:

$$\begin{array}{ccc} \mathrm{FIM}(X) & \xrightarrow{\theta} & M \\ & {\scriptstyle\sigma}\searrow \quad \swarrow{\scriptstyle\sigma_M} & \\ & F(X) & \end{array}$$

and since σ is idempotent pure so is σ_M.

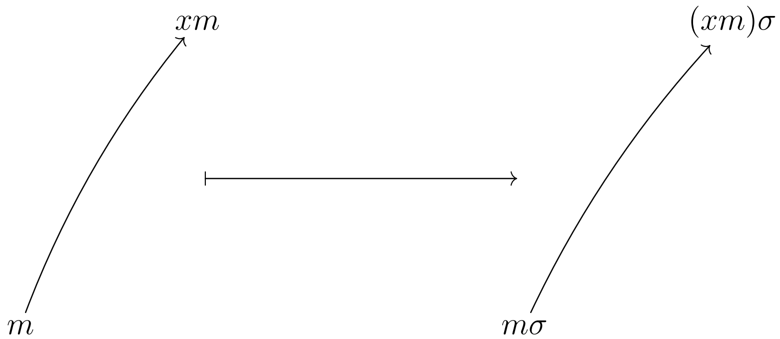
Now suppose that $m \mathcal{L} n$ in M, so that $m^{-1}m = n^{-1}n$. Then $m = mm^{-1}m = mn^{-1}n$, and if $m\sigma_M = n\sigma_M$ then in $F(X)$:

$$\begin{aligned} m\sigma_M &= (mn^{-1})\sigma_M(n\sigma_M) \\ &= (mn^{-1})\sigma_M(m\sigma_M) \end{aligned}$$

and so $(mn^{-1})\sigma_M = 1_F$. Then since σ_M is idempotent pure, $mn^{-1} \in E(M)$ and $m = (mn^{-1})n \leqslant n$, and by symmetry we have $m = n$. Hence σ_M is injective on each $\mathcal{L}$-class, and the mapping

$$\mathrm{Sch}^{\mathcal{L}}(M, X, e) \to \mathrm{Cay}(F(X), X)$$

defined on edges in $\mathrm{Sch}^{\mathcal{L}}(M, X)$ by



is an embedding, and so $\mathrm{Sch}^{\mathcal{L}}(M, X, e)$ is a tree.

(c) ⇒ (b): Suppose that $w \in \mathrm{FIM}(X)$ and that $w\theta \in E(M)$. Now follow the path labelled by w in the tree $\mathrm{Sch}^{\mathcal{L}}(M, X, w\theta)$, starting at $w\theta$. This ends at $(w\theta)(w\theta) = w\theta$ and therefore is a closed path in the tree. Thus the path reduces to the empty path, and so w is an idempotent in $\mathrm{FIM}(X)$. Thus θ is idempotent pure.

(b) ⇒ (a): Assume $\theta : \mathrm{FIM}(X) \to M$ is idempotent pure. Let $\theta_{\min}$ be the smallest congruence on $\mathrm{FIM}(X)$ that has the same trace as the congruence χ_θ induced by θ on $E(\mathrm{FIM}(X))$. Then by Lemma 4.3.4, since $\mathrm{FIM}(X)$ is E–unitary, $\theta_{\min}$ is

also idempotent pure, and so χ_θ and $\theta_{\min}$ have the same trace and kernel, and are therefore the same congruence on FIM(X). Now θ_{min} is clearly generated by the set

$$T_1 = \{(e, f) : e, f \in E(\mathrm{FIM}(X)) \text{ and } e\theta = f\theta\}.$$

□

6.2.3 Definition (Gilbert [14]). An inverse monoid M is *arboreal* if it satisfies the conditions of Lemma 6.2.8.

6.2.9 Corollary. *An arboreal inverse monoid M is E–unitary.*

Proof. As shown in the proof of Lemma 6.2.8, the map $\sigma_M : M \to F(X)$ is idempotemt pure, and so the result follows from Theorem 1.2.5. □

6.2.10 Corollary. *Let $\mathcal{P} = [X : R]$ be a presentation of an inverse monoid M, with presentation map $\theta : \mathrm{FIM}(X) \to M$. Then θ has a canonical factorization as*

$$\mathrm{FIM}(X) \xrightarrow{\tau} \mathcal{T}(X, M) \xrightarrow{\psi} M$$

where $\mathcal{T}(X, M)$ is arboreal and ψ is idempotent separating.

Proof. This follows from Proposition 4.3.4 and Lemma 6.2.8, since FIM(X) is E–unitary. □

Since we define τ to be θ_{min} as in Proposition 4.3.4 this factorisation is completely determined by the presentation.

6.2.4 The Relation Module

The factorization of a presentation map $\theta : \mathrm{FIM}(X) \to M$ given in Corollary 6.2.10 gives us an idempotent separating homomorphism $\psi : \mathcal{T}(X, M) \to M$, for which we can compute the derivation module $\mathcal{D}_\psi$. In our next result, we see that $\mathcal{D}_\psi$ is isomorphic to $\mathcal{D}_\theta$, which gave rise to the relation module $\mathcal{M}_\theta$ in (6.7).

6.2.11 Theorem. *For an inverse monoid presentation $\mathcal{P} = [X; R]$ presenting M, with presentation map θ, which factorises as $\tau\psi$ as in Corollary 6.2.10, the derivation modules $\mathcal{D}_\theta$ and $\mathcal{D}_\psi$ are isomorphic as M-modules.*

Proof. There is a map $\mathcal{D}_\theta \to \mathcal{D}_\psi$ induced by τ, which a maps generator $\langle w, m\rangle$ to $\langle w\tau, m\rangle$. This mapping is surjective, since τ is a quotient map and is therefore surjective.

We want to reverse this map, and so take $\langle w\tau, m\rangle \mapsto \langle w, m\rangle$. Observe that if we have $\langle w\tau, m\rangle \in \mathcal{D}_\psi$ then $((w\tau)^{-1}(w\tau))\psi \geqslant mm^{-1}$, and so

$$mm^{-1} \leqslant (w\tau)^{-1}\psi(w\tau)\psi = (w\tau\psi)^{-1}(w\tau\psi) = (w\theta)^{-1}(w\theta)\,.$$

Therefore $\langle w, m\rangle$ exists in $\mathcal{D}_\theta$. However given $\langle w\tau, m\rangle$ we do not have a canonical choice of w.

Recall that τ is the quotient map $\mathrm{FIM}(X) \to \mathrm{FIM}(X)/\theta_{\min}$, as in Proposition 4.3.4. Suppose that $w_1\tau = w_2\tau$. Then $w_1\,\theta_{min}\,w_2$ and so by part (1) of Proposition 4.3.4, there exists $z \in \mathrm{FIM}(X)$ with $w_1 \geqslant z \leqslant w_2$ and $w_1\tau = z\tau = w_2\tau$, If $e = z^{-1}z$ then $w_1e = z = w_2e$. Now if $\langle w_1, m\rangle$ exists in $\mathcal{D}_\theta$ then so do $\langle w_2, m\rangle$ and $\langle z, m\rangle$, and moreover $e\theta \geqslant mm^{-1}$. Therefore, using the defining relations (6.6) for $\mathcal{D}_\theta$, we have

$$\langle z, m\rangle = \langle w_1e, m\rangle = \langle w_1, (e\theta)m\rangle + \langle e, m\rangle = \langle w_1, m\rangle$$

and similarly $\langle z, m\rangle = \langle w_2, m\rangle$. So for $\langle w_1\tau, m\rangle \in \mathcal{D}_\psi$, and $w_2 \in \mathrm{FIM}(X)$ such that $w_1\tau = w_2\tau$, $\langle w_1, m\rangle = \langle w_2, m\rangle$ in $\mathcal{D}_\theta$, and so we have a prospective inverse map.

Finally we must check if this map is in fact well defined on $\mathcal{D}_\psi$, so let's consider a defining relation:

$$\langle (u\tau)(v\tau), m\rangle = \langle u\tau, ((v\tau)\psi)m\rangle + \langle v\tau, m\rangle \tag{6.8}$$

where $(u\tau)^{-1}(u\tau) \geqslant (v\tau)(v\tau)^{-1}$. Now set $\bar{v} = u^{-1}uv$, then $\bar{v}\tau = (u^{-1}uv)\tau = v\tau$, and thus $(u\tau)(\bar{v}\tau) = (u\tau)(v\tau)$. So we can write (6.8) as

$$\begin{aligned} \langle (u\tau)(\bar{v}\tau), m\rangle &= \langle u\tau, (\bar{v}\tau)\psi m\rangle + \langle \bar{v}\tau, m\rangle \\ &= \langle u\tau, (\bar{v}\theta)m\rangle + \langle \bar{v}\tau, m\rangle \end{aligned} \tag{6.9}$$

Then the left hand side of (6.9) lifts to $\langle u\bar{v}, m\rangle$ and the right hand side lifts to $\langle u, (\bar{v}\theta)m\rangle + \langle \bar{v}, m\rangle$. Since

$$\bar{v}\bar{v}^{-1} = u^{-1}uvv^{-}u^{-1}uu^{-1}uv^{-1} \leqslant u^{-1}u$$

this is a defining relation of $\mathcal{D}_\theta$, and therefore the inverse map is well-defined. □

We now return our focus to the analogue of Crowell's exact sequences, but now using $\mathcal{T}(X, M) \xrightarrow{\psi} M$ rather than the presentation map θ. We set

$$U = \ker\psi = \{w \in \mathcal{T}(X, M) : w\psi \in E(M)\}.$$

and consider the sequence

$$0 \to U \hookrightarrow \mathcal{T}(X, M) \xrightarrow{\psi} M \to 0\,. \tag{6.10}$$

of inverse monoid maps as our analogue of (6.3). By Lemma 4.3.3, we know that U is a Clifford semigroup, and so U is a union of groups U_e, indexed by the idempotents of M. Hence U has a natural abelianisation

$$\mathcal{U} = \bigcup_{e \in E(M)} U_e^{ab}\,.$$

6.2.12 Lemma. *$\mathcal{U}$ is an M–module.*

Proof. This is a special case of Proposition 1.6.5. □

We have a sequence of M–modules

$$\mathcal{U} \xrightarrow{\alpha} \mathcal{D}_\psi \to IM \to 0. \tag{6.11}$$

where, for $u \in U_e$ with image $\overline{u} \in U_e^{ab}$ we have $\overline{u}\alpha = \langle u, (u^{-1}u)\psi \rangle$. We wish to show that this sequence is exact at $\mathcal{D}_\psi$, and that α is injective: the resulting short exact sequence

$$0 \to \mathcal{U} \xrightarrow{\alpha} \mathcal{D}_\psi \to IM \to 0 \tag{6.12}$$

will be our analogue of (6.4).

To establish the exactness of (6.11) at $\mathcal{D}_\psi$ we use functorial properties of $\mathcal{D}$ established in [14]. We include the details here, for the reader's convenience.

For an inverse monoid M, the *slice category* $\mathbf{InvMon} \downarrow_M$ over M has as its objects all inverse monoid homomorphisms with codomain M, and as its morphisms the obvious commutative triangles. The derivation module construction defines, from an inverse monoid homomorphism $\phi : N \to M$, an M–module $\mathcal{D}_\phi$ and this gives us a functor from the slice category $\mathbf{InvMon} \downarrow_M$ to the category Mod_M of M–modules.

We claim that $\mathcal{D}$ has a left adjoint, and so is right exact. The adjoint construction, from Mod_M to $\mathbf{InvMon} \downarrow_M$, is a semidirect product $M \ltimes -$ equipped with the projection map to M. The details are as follows. Let $\mathcal{A}$ be an M–module, and define

$$M \ltimes \mathcal{A} = \{(m,a) : m \in M, a \in A_{m^{-1}m}\}.$$

For $(p,a),(q,b) \in M \ltimes \mathcal{A}$ we define

$$(p,a)(q,b) = (pq, a \lhd q \oplus b),$$

where we are using Lausch's conventions for describing an M–module, explained in Definition 1.6.1: this gives a more compact notation for $M \ltimes \mathcal{A}$ than the Loganathan conventions would do. $M \ltimes \mathcal{A}$ is the *semidirect product* of M and $\mathcal{A}$. Gilbert [14] states the following without giving the details of the proof:

6.2.13 Proposition. *[14, Proposition 2.2] $M \ltimes \mathcal{A}$ is an inverse monoid, with $E(M \ltimes \mathcal{A})$ isomorphic to $E(M)$.*

Proof. We have

$$(p,a)((q,b)(r,c)) = (p,a)(qr, b \lhd r \oplus c) = (pqr, a \lhd (qr) \oplus b \lhd r \oplus c)$$

and

$$((p,a)(q,b))(r,c) = (pq, a \lhd q \oplus b)(r,c) = (pqr, (a \lhd q \oplus b) \lhd r \oplus c)$$

and so associativity of the given multiplication follows from conditions (i) and (ii) of Definition 1.6.1. The element $(1_M, 0_{1_M})$ is an identity for $M \ltimes \mathcal{A}$. Now if $(p,a) \in M \ltimes \mathcal{A}$ with $p^{-1}p = f$ then $(p^{-1}, -a \lhd p^{-1}) \in M \ltimes \mathcal{A}$ and

$$\begin{aligned}(p,a)(p^{-1}, -a \lhd p^{-1})(p,a) &= (p,a)(f, -a \lhd f \oplus a)\\ &= (p,a)(f, -a \oplus 0_f \oplus a)\\ &= (p,a)(f, 0_f) = (p,a)\end{aligned}$$

so that $M \ltimes \mathcal{A}$ is regular.

Now suppose that (p,a) is an idempotent, so that $(p,a)^2 = (p^2, a \lhd p \oplus a) = (p,a)$. Thus $p \in E(M)$, and $a \lhd p \oplus a = a \oplus 0_p \oplus a = a + a$ in A_p, by part (iii) of Definition 1.6.1. But then $a + a = a$ and so $a = 0_p$. Hence

$$E(M \lhd \mathcal{A}) = \{(e, 0_e) : e \in E(M)\}$$

and it is now clear that the idempotents in $M \ltimes \mathcal{A}$ commute, and that $E(M \ltimes \mathcal{A})$ is isomorphic to E(M). □

It is obvious that the map $(p, a) \mapsto p$ is an inverse monoid homomorphism $M \ltimes \mathcal{A} \to M$, and so we may regard the semidirect product $M \ltimes \mathcal{A}$ as an object of $\mathbf{InvMon}{\downarrow_M}$.

6.2.14 Proposition. *[14, Proposition 3.7] The derivation module, regarded as a functor* $\mathbf{InvMon}{\downarrow_M} \to \mathrm{Mod}_M$ *given by* $(N \xrightarrow{\phi} M) \mapsto \mathcal{D}_\phi$*, is left adjoint to the semidirect product functor* $\mathrm{Mod}_M \to \mathbf{InvMon}{\downarrow_M}$ *given by* $\mathcal{A} \mapsto M \ltimes \mathcal{A}$*.*

Proof. We need to establish a natural isomorphism

$$\mathbf{InvMon}{\downarrow_M} (N, M \ltimes \mathcal{A}) \cong \mathrm{Mod}_M(\mathcal{D}_\phi, \mathcal{A})\,, \tag{6.13}$$

where $\phi : N \to M$.

A morphism $\nu : N \to M \ltimes \mathcal{A}$ in $\mathbf{InvMon}{\downarrow_M}$ is an inverse monoid homorphism of the form $n \mapsto (n\phi, n\delta)$ for some function $\delta : N \to \mathcal{A}$ such that $n\delta \in A_{(n^{-1}n)\phi}$. Then if $u, v \in N$ with $u^{-1}u \geqslant vv^{-1}$ we have

$$(u\nu)(v\nu) = (u\phi, u\delta)(v\phi, v\delta) = ((u\phi)(v\phi), (u\delta) \lhd v\phi \oplus v\delta\,.$$

However, both $(u\delta) \lhd v\phi$ and $v\delta$ are in $A_{(v^{-1}v)\phi}$: hence $\oplus$ here is just addition in $A_{(v^{-1}v)\phi}$, and so δ is a ϕ–derivation (cf Definition 6.2.1). By Proposition 6.2.2, δ induces an M–module map $\widehat{\nu} : \mathcal{D}_\phi \to \mathcal{A}$ given by $\langle u, m\rangle \mapsto u\delta \lhd m$.

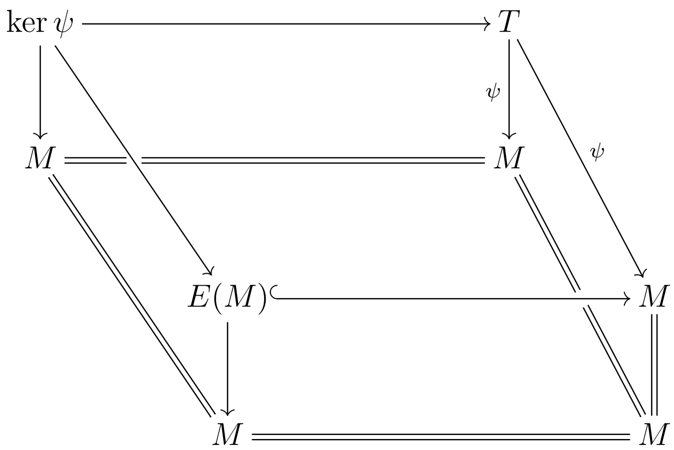
Conversely, an M–map $\gamma : \mathcal{D}_\phi \to \mathcal{A}$ induces a ϕ–derivation $\gamma' : N \to \mathcal{A}$ given by $n\gamma' = \langle n, (n^{-1}n)\phi\rangle\gamma$, and hence an inverse monoid map $\gamma^\dagger : N \to M \ltimes \mathcal{A}$ given by $n\gamma^\dagger = (n\phi, \langle n, (n^{-1}n)\phi\rangle\gamma)$. Clearly this gives a morphism $\gamma^\dagger : N \to M \ltimes \mathcal{A}$ in $\mathbf{InvMon}{\downarrow_M}$.

The correspondences $\nu \mapsto \widehat{\nu}$ and $\gamma \mapsto \gamma^\dagger$ are then mutually inverse natural bijections, as required to establish (6.13). □

6.2.15 Corollary. *The functor* $\mathcal{D} : \mathbf{InvMon}{\downarrow_M} \to \mathrm{Mod}_M$ *preserves colimits.*

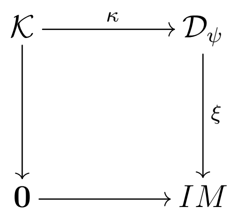
Proof. This is a standard fact from elementary category theory: see, for example, [18, Proposition 15]. □

6.2.16 Lemma. *Let M be the quotient of an inverse monoid T by an idempotent separating congruence, and let $\psi : T \to M$ be the natural map. Then the following diagram is a pushout in the slice category* $\mathbf{InvMon}{\downarrow_M}$*:*



Proof. Suppose we are given $\nu : V \to M$ and $\phi : T \to V$, $\varepsilon : E(M) \to V$ such that $\phi\nu = \psi$ and $\varepsilon\nu$ is equal to the inclusion $i : E(M) \hookrightarrow M$. We define $\mu : M \to V$ by $\sigma : t\psi \mapsto t\phi$. This is well-defined: if $a\psi = b\psi$ then, since ψ is idempotent separating, $aa^{-1} = bb^{-1}$ and moreover $(a^{-1}b)\psi \in E(M)$, and so $(a^{-1}b)\phi = (a^{-1}b)\psi\varepsilon \in E(M)$. Then $b\phi = (bb^{-1}b)\phi = (aa^{-1}b)\phi = a\phi(a^{-1}b)\phi \leqslant a\phi$. By symmetry, $b\phi = a\phi$. So μ is well-defined, and is a map in $\mathbf{InvMon}\downarrow_M$ since, for $m = t\psi \in M$, we have $m\mu\nu = t\phi\nu = t\psi = m$. □

6.2.17 Proposition. *Let $\psi : T \to M$ be an idempotent separating homomorphism of inverse monoids. Its kernel $K = \ker\psi$ is a Clifford semigroup $K = \sqcup_{e\in E(M)}K_e$, and we let $\mathcal{K}$ be the M–module $\mathcal{K} = \sqcup_{e\in E(M)}K_e^{ab}$. Then the diagram*



(6.14)

in the module category Mod_M *is a pushout, and so the sequence of M–modules*

$$\mathcal{K} \xrightarrow{\kappa} \mathcal{D}_\psi \xrightarrow{\xi} IM \longrightarrow \mathbf{0} \tag{6.15}$$

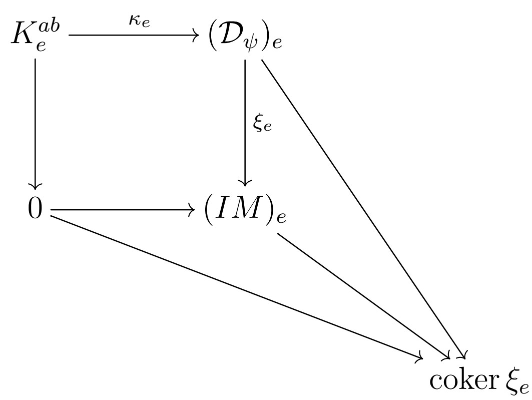
is exact.

Proof. That the square is a pushout follows from Lemma 6.2.16, Corollary 6.2.15, and the computations of derivation modules in section 6.2.2.

We need only verify exactness of (6.15) at $\mathcal{D}_\psi$, and this is equivalent to the exactness of the sequence

$$K_e^{ab} \longrightarrow (\mathcal{D}_\psi)_e \xrightarrow{\xi_e} IM_e \to 0$$

for each $e \in E(M)$. It is clear that $(\mathcal{D}_\psi)_e \to IM_e$ is surjective, so only the exactness at $(\mathcal{D}_\psi)_e$ needs to be checked. It is clear from the commutativity of the square (6.14) that the image of κ_e is contained in $\ker \xi_e$. The pushout property of IM then implies that there is a map $(IM)_e \to \operatorname{coker} \xi_e$ making the diagram



commute, and so $\ker \xi_e \subseteq \operatorname{im} \kappa_e$. □

6.2.18 Proposition. *The mapping α in sequence (6.11) splits as a homomorphism of abelian groups, and therefore*

$$0 \to \mathcal{U} \xrightarrow{\alpha} \mathcal{D}_\psi \to IM \to 0 \tag{6.16}$$

is a short exact sequence of M–modules.

Proof. We want to define a map $\beta : \langle x, m\rangle \mapsto \bar{u} \in \mathcal{U}$ such that $\alpha\beta = id_{\mathcal{U}}$. Since we have presentations for each $D_{\psi,e}$ as an abelian group, we shall construct β_e as an abelian group homomorphism. This will be sufficient to show that $\alpha_e : U_e^{ab} \to D_{\psi_e}$ is injective.

We consider the left trace cosets of U in $\mathcal{T}(X, M)$ and let $V \subseteq \mathcal{T}(X, M)$ be a transversal. By Proposition 4.3.2 the elements of M are in one-to-one correspondence with the trace cosets of U, and so we may define a map $\upsilon : M \to V$ by $m\upsilon = t$ if $m = t\psi$: that is, υ maps $m \in M$ to the transversal element that represents its coset. Therefore $\upsilon\psi = \mathrm{id}_M$ and υ splits ψ. Furthermore, since ψ is idempotent separating, each coset contains at most one idempotent, and so we may assume that V is chosen so that, for each $e \in E(\mathcal{T}(X, M))$, we have $(e\psi)\upsilon = e$.

Now suppose that $w \in \mathcal{T}(X, M)$ and $m \in M$ with $(w^{-1}w)\psi \geqslant mm^{-1}$. Then $\langle w, m\rangle \in \mathcal{D}_\psi$ and we define $\kappa : \mathcal{D}_\psi \to U$ by

$$\langle w, m\rangle\kappa = (((w\psi)m)\upsilon)^{-1}w(m\upsilon) \tag{6.17}$$

Clearly $\langle w, m\rangle\kappa \in \mathcal{T}(X, M)$, and applying ψ we find

$$\begin{aligned}
\langle w, m\rangle\kappa\psi &= \big((((w\psi)m)\upsilon)^{-1}w(m\upsilon)\big)\psi \\
&= (((w\psi)m)\upsilon\psi)^{-1}w\psi(m\upsilon\psi) \\
&= ((w\psi)m)^{-1}(w\psi)m \\
&= m^{-1}(w^{-1}w)\psi m \\
&= m^{-1}m
\end{aligned}$$

Hence $\langle w, m\rangle\kappa \in U_{m^{-1}m}$. We now compose κ with the abelianisation map on U and so define

$$\overline{\kappa} : \langle w, m\rangle \mapsto \overline{\langle w, m\rangle\kappa} \in U^{ab}_{m^{-1}m}\,.$$

We check that $\overline{\kappa}$ induces an abelian group homomorphism $\mathcal{D}_{\psi,m^{-1}m} \to U^{ab}_{m^{-1}m}$. Consider a defining relation of $\mathcal{D}_{\psi,m^{-1}m}$, where $a, b \in \mathcal{T}(X, M)$ with $a^{-1}a \geqslant bb^{-1}$ and $m \in M$ with $(b^{-1}b)\psi \geqslant mm^{-1}$:

$$\langle ab, m\rangle = \langle a, (b\psi)m\rangle + \langle b, m\rangle \,. \tag{6.18}$$

In what follows we shall for clarity suppress the bars above elements of $\mathcal{U}$, but it is crucial to remember that we are indeed mapping $\mathcal{D}_{\psi,m^{-1}m}$ into an abelian group using $\overline{\kappa}$. If we apply $\overline{\kappa}$ to the left hand side of (6.18) we obtain

$$\langle ab, m\rangle\overline{\kappa} = [(((ab)\psi)m)\upsilon]^{-1} \cdot ab \cdot m\upsilon \,. \tag{6.19}$$

Applying $\overline{\kappa}$ to the right-hand side, we obtain

$$[(a\psi \cdot b\psi \cdot m)\upsilon]^{-1} \cdot a \cdot ((b\psi)m)\upsilon \cdot ((b\psi)m)\upsilon^{-1} \cdot b \cdot m\upsilon \,.$$

Now

$$[((b\psi)m)\upsilon \cdot ((b\psi)m)\upsilon^{-1}]\psi = (b\psi)m \cdot ((b\psi)m)^{-1} = [b \cdot m\upsilon \cdot (m\upsilon)^{-1} \cdot b^{-1}]\psi$$

and since ψ is idempotent separating, we deduce that

$$((b\psi)m)\upsilon \cdot ((b\psi)m)\upsilon^{-1} = b \cdot m\upsilon \cdot (m\upsilon)^{-1} \cdot b^{-1} \,.$$

Hence, on applying $\overline{\kappa}$ to the right-hand side of (6.18), we obtain

$$\begin{aligned}[(a\psi \cdot b\psi \cdot m)\upsilon]^{-1} \cdot ab{\cdot}m\upsilon \cdot (m\upsilon)^{-1} \cdot b^{-1}b \cdot m\upsilon& \\ = [(a\psi \cdot b\psi \cdot m)\upsilon]^{-1} \cdot ab \cdot m\upsilon&\end{aligned}$$

and comparison with (6.19) shows that the relation (6.18) is preserved. Therefore $\overline{\kappa}$

does define an abelian group homomorphism $\mathcal{D}_{\psi,m^{-1}m} \to U^{ab}_{m^{-1}m}$.

It remains to check that $\overline{\kappa}$ splits $\alpha : u \to \langle u, (u^{-1}u)\psi\rangle$. Now

$$\begin{aligned}\overline{\kappa} : \langle u, (u^{-1}u)\psi\rangle &\mapsto [((u\psi)(u^{-1}u)\psi)\upsilon]^{-1} \cdot u \cdot (u^{-1}u)\psi\upsilon \\ &= ((u\psi)\upsilon)^{-1} \cdot u \cdot (u^{-1}u)\psi\upsilon\end{aligned}$$

Since $u\psi \in E(M)$ then $u\psi = (uu^{-1})\psi$, and we have assumed that $\psi\upsilon$ is the identity on $E(\mathcal{T}(X, M))$. Thus

$$\langle u, (u^{-1}u)\psi\rangle\overline{\kappa} = ((u\psi)\upsilon)^{-1} \cdot u \cdot (u^{-1}u)\psi\upsilon = (((uu^{-1})\psi)\upsilon)^{-1} \cdot u \cdot (u^{-1}u)\psi\upsilon = u\,.$$

Therefore $\overline{\kappa}$ splits α as an abelian group homomorphism, and each $\alpha_e : U^{ab}_e \to D_{\psi,e}$ is injective. □

6.2.19 Corollary. *The module $\mathcal{U}$ and the relation module $\mathcal{M}_\theta$ are isomorphic M–modules.*

Proof. By Theorem 6.2.11, the relation module $\mathcal{M}_\theta$ is isomorphic to $\mathcal{M}_\psi$, defined as the kernel of the canonical map $\mathcal{D}_\psi \to IM$ given in Example 6.2.3, and by Proposition 6.2.18 this is isomorphic to $\mathcal{U}$. □

6.2.20 Example. Let $S = \{1, e, f, ef\}$ be the semilattice presented as an inverse monoid by $\mathcal{P} = [e, f : e^2 = e, f^2 = f]$. Then $\text{FIM}(e, f)$ is partitioned as

$$\text{FIM}(e, f) = \{1\} \cup (\text{FIM}(e) \setminus \{1\}) \cup (\text{FIM}(f) \setminus \{1\}) \cup (\text{FIM}(e, f) \setminus (\text{FIM}(e) \cup \text{FIM}(f)))$$

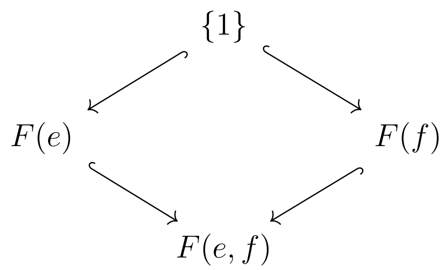
with the parts mapping to $1, e, f$ and ef respectively. We represent elements of $\text{FIM}(e, f)$ by Munn trees in the Cayley graph of the free group $F(e, f)$, as explained in section 1.5. We want to factorize $\theta : \text{FIM}(e, f) \to S$ into

$$\text{FIM}(e, f) \xrightarrow[\tau]{} \mathcal{T}(e, f) \xrightarrow[\psi]{} S, \qquad \theta = \tau\psi$$

Since S is a semilattice, $\mathcal{T} = \ker\psi = U$, and so $\mathcal{T}$ is a Clifford semigroup. The congruence on $\mathrm{FIM}(e, f)$ determined by τ is, by Proposition 4.3.4, the following: $(P, u)\ \tau\ (Q, v)$ if and only if there exists an idempotent $(R, 1) \in \mathrm{FIM}(e, f)$ such that $(P \cup uR, u) = (Q \cup vR, v)$ and $(u^{-1}P, 1)\theta = (v^{-1}Q, 1)\theta = (R, 1)\theta$. Now $(P \cup uR, u) = (Q \cup vR, v)$ implies that $u = v$: if we then define $R = u^{-1}(P \cup Q)$ it follows that $(P \cup uR, u) = (Q \cup uR, u)$. The condition $(u^{-1}P, 1)\theta = (u^{-1}Q, 1)\theta = (R, 1)\theta$ then holds if and only if each of $(u^{-1}P, 1), (u^{-1}Q, 1)$ and $(R, 1)$ is in the same part of the partition of $\mathrm{FIM}(e, f)$ given above. Therefore $(P, u)\ \tau\ (Q, v)$ if and only if one of the following four mutually exclusive conditions holds:

- $P = Q = \{1\}$ and $u = v = 1$, or
- P and Q involve only e–labelled edges in the Cayley graph $\mathrm{Cay}(F(e, f), \{e, f\})$, and $u = v \in F(e)$, or
- P and Q involve only f–labelled edges in the Cayley graph $\mathrm{Cay}(F(e, f), \{e, f\})$, and $u = v \in F(f)$, or
- P and Q each involves both e–labelled edges and f–labelled edges in $\mathrm{Cay}(F(e, f), \{e, f\})$, and $u = v \in F(e, f)$.

In each case, the τ–class is determined by the element $u \in \{1\} \sqcup F(e) \sqcup F(f) \sqcup F(e, f)$. The Clifford semigroup $\mathcal{T}$ is then the disjoint union of groups



Given τ and $\mathcal{T}$, we can easily define ψ by mapping $\{1\} \mapsto 1$, $F(e) \mapsto e$, $F(f) \mapsto f$ and $F(e, f) \mapsto ef$. This is clearly idempotent separating since each group contains exactly one idempotent, and $\tau\psi = \theta$ since τ respects the partition we used to characterise θ. Now we have ψ defined we can construct $\mathcal{D}_\psi$. For $y \in E(S)$ each $D_{\psi,y}$ is generated as an abelian group by:

$$X_y = \{(a,s) : a \in \mathcal{T}, s \in S, (a^{-1}a)\psi \geqslant ss^{-1}, s^{-1}s = y\}$$

subject to relations:

$$(ab, s) = (a, (b\psi)s) + (b, s)$$

with $a, b \in \mathcal{T}$, $s \in S$ and $a^{-1}a \geqslant bb^{-1}$. We can take each element of S in turn:

- $D_{\psi,1}$

  Generators:

$$\begin{aligned} X_1 &= \{(a,1) : a \in \mathcal{T}, (a^{-1}a)\psi \geqslant 1\} \\ &= \{(a,1) : a \in \{1\}\} \\ &= \{(1,1)\} \end{aligned}$$

  Relations:

$$\begin{aligned} (1 \cdot 1, 1) &= (1,1) + (1,1) \\ &= (1,1) \end{aligned}$$

  So $D_{\psi,1}$ is the trivial group.

- $D_{\psi,e}$

  Generators:

$$\begin{aligned} X_e &= \{(a,e) : a \in \mathcal{T}, (a^{-1}a)\psi \geqslant e\} \\ &= \{(a,e) : a \in F(e) \sqcup \{1\}\} \end{aligned}$$

Relations:

$$\begin{aligned}(ab, e) &= (a, (b\psi)e) + (b, e)\\ &= (a, e) + (b, e)\end{aligned}$$

So $D_{\psi,e}$ is the free abelian group, $F^{ab}(e)$.

- $D_{\psi,f}$

  This follows the same construction as $D_{\psi,e}$, thus $D_{\psi,f} = F^{ab}(f)$.

- $D_{\psi,ef}$

  Generators:

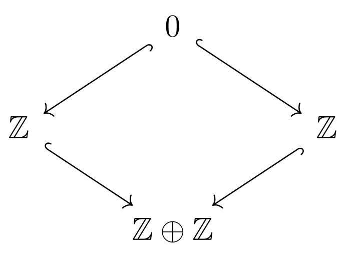
$$\begin{aligned}X_{ef} &= \{(a, ef) : a \in \mathcal{T}, (a^{-1}a)\psi \geqslant ef\}\\ &= \{(a, ef) : a \in \mathcal{T}\}\end{aligned}$$

  Relations:

$$\begin{aligned}(ab, ef) &= (a, (b\psi)ef) + (b, ef)\\ &= (a, ef) + (b, ef)\end{aligned}$$

  So $D_{\psi,ef}$ is the free abelian group $F^{ab}(e, f)$.

Therefore $\mathcal{D}_\psi =$



with the S–module structure given by inclusion maps.

We then recall from (6.16) that we have the exact sequence of S-modules:

$$0 \to \mathcal{U} \to \mathcal{D}_\psi \to IS \to 0$$

with $\mathcal{U} = \mathcal{M}_\theta$ the relation module for our presentation. We have already constructed $\mathcal{D}_\psi$, it remains to consider IS. From Lemma 1.6.4 we know that for each $y \in E(S)$, IS_y is freely generated by

$$\{s - y : y \neq s \in L_y\}$$

However since S a semilattice each of these generating sets is empty, and so IS is trivial, and we have $\mathcal{M}_\theta = \mathcal{D}_\psi$.

6.2.5 The Relation Module and the Schützenberger Graphs

As shown in [8, Corollary 5.1], the relation module of a group presentation $\langle X : R\rangle$ of a group G – that is, the abelianisation of the kernel of the quotient map $F(X) \to G$ – is isomorphic to the first homology group of the Cayley graph $\text{Cay}(G, X)$. Hence in settings in which the Cayley graph is well understood, the computation of the relation module is straightforward. Similar considerations apply to the relation module of an inverse monoid presentation $[X : R]$, if we replace the Cayley graph by the Schützenberger graph (see section 1.7).

Consider the Schützenberger graph $\text{Sch}^{\mathcal{L}}(M, X)$ for an inverse monoid M generated by X, and let $\theta : \text{FIM}(X) \to M$ be the presentation map. We recall that $\text{Sch}^{\mathcal{L}}(M, X)$ has a connected component $\text{Sch}^{\mathcal{L}}(M, X, e)$ for each idempotent $e \in E(M)$, and that the vertex set of $\text{Sch}^{\mathcal{L}}(M, X, e)$ is the $\mathcal{L}$–class of e. Hence the cellular chain group $C_0(\text{Sch}^{\mathcal{L}}(M, X, e))$ in dimension 0, which is the free abelian group on the vertex set of $\text{Sch}^{\mathcal{L}}(M, X, e)$, is the free abelian group on the $\mathcal{L}$–class L_e and so is isomorphic to $\mathbb{Z}M_e$. The cellular chain group $C_1(\text{Sch}^{\mathcal{L}}(M, X, e))$ in dimension 1 is the free abelian group on the edge set, and hence is the free abelian group on the set

$$\{(x, s) : x \in X, s \in S, (x^{-1}x)\theta \geqslant ss^{-1}, s^{-1}s = e\}$$

and the boundary map $\partial : C_1(\mathrm{Sch}^{\mathcal{L}}(M, X, e)) \to C_0(\mathrm{Sch}^{\mathcal{L}}(M, X, e))$ is then given by $(x, s) \mapsto (x\theta)s - s$. For the Schützenberger graph $\mathrm{Sch}^{\mathcal{L}}(M, X)$ we have

$$C_0(\mathrm{Sch}^{\mathcal{L}}(M, X)) = \bigoplus_{e \in E(M)} C_0(\mathrm{Sch}^{\mathcal{L}}(M, X, e))$$

and

$$C_1(\mathrm{Sch}^{\mathcal{L}}(M, X)) = \bigoplus_{e \in E(M)} C_1(\mathrm{Sch}^{\mathcal{L}}(M, X, e))\,.$$

Let $\mathfrak{L}(M)$ be Loganathan's left-cancellative category , see Definition 1.6.2. Suppose that $s^{-1}s = e$ and that $(e, t) \in \mathfrak{L}(M)$. Then by defining

$$s \lhd (e, t) = st \text{ and } (a, s) \lhd (e, t) = (a, st)$$

we get an M–module structure on each of $C_0(\mathrm{Sch}^{\mathcal{L}}(M, X))$ and $C_1(\mathrm{Sch}^{\mathcal{L}}(M, X))$, with that on $C_0(\mathrm{Sch}^{\mathcal{L}}(M, X))$ making $C_0(\mathrm{Sch}^{\mathcal{L}}(M, X))$ isomorphic to $\mathbb{Z}M$ as an M–module. The cellular chain complex

$$C_1(\mathrm{Sch}^{\mathcal{L}}(M, X)) \xrightarrow{\partial} C_0(\mathrm{Sch}^{\mathcal{L}}(M, X))$$

of $\mathrm{Sch}^{\mathcal{L}}(S, X)$ is then a complex of M–modules.

6.2.21 Theorem. *[14, Theorem 4.3] Suppose that M is an inverse monoid presented by $\mathcal{P} = [X : R]$ with presentation map $\theta : \mathrm{FIM}(X) \to M$. Then the derivation module $\mathcal{D}_\theta$ is isomorphic as a group to the cellular chain group $C_1(\mathrm{Sch}^{\mathcal{L}}(M, X))$ and the relation module $\mathcal{M}_\theta$ is isomorphic as a group to the first homology group of the Schützenberger graph* $\mathrm{Sch}^{\mathcal{L}}(M, X)$.

Proof. The sequence 6.7 gives us a commutative diagram of $\mathfrak{L}(M)$–maps

$$\begin{array}{ccccc} H_1(\mathrm{Sch}^{\mathcal{L}}(M,X)) & \hookrightarrow & C_1(\mathrm{Sch}^{\mathcal{L}}(M,X)) & \xrightarrow{\partial} & C_0(\mathrm{Sch}^{\mathcal{L}}(M,X)) \\ \downarrow{\scriptstyle\kappa} & & \downarrow{\scriptstyle\kappa} & & \downarrow{\scriptstyle\cong} \\ \mathcal{M}_\theta & \hookrightarrow & \mathcal{D}_\theta & \xrightarrow{\theta_*} & \mathbb{Z}M \end{array}$$

in which the middle map κ is the $\mathfrak{L}(M)$–map induced (as a homomorphism of abelian groups) by $(x,m) \mapsto \langle x,m \rangle$, the two left-hand horizontal maps are inclusions, and κ therefore restricts to a surjection $H_1(\mathrm{Sch}^{\mathcal{L}}(M,X)) \to \mathcal{M}_\theta$.

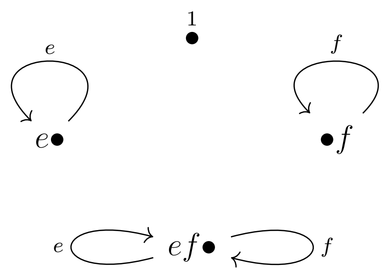
Now the function $X \to M \ltimes C_1(\mathrm{Sch}^{\mathcal{L}}(M,X))$ that maps $x \mapsto (x\theta, (x,(x^{-1}x)\theta)$ induces an inverse monoid homomorphism $\mathrm{FIM}(X) \to M \ltimes C_1(\mathrm{Sch}^{\mathcal{L}}(M,X))$ and by Proposition 6.2.14 there is then an $\mathfrak{L}(M)$–map $\mathcal{D}_\theta \to C_1(\mathrm{Sch}^{\mathcal{L}}(M,X))$ mapping $\langle x,(x^{-1}x)\theta \rangle \mapsto (x,(x^{-1}x)\theta)$ giving an inverse to κ. $\square$

6.2.22 Remark. Description of $\mathcal{D}_\theta$ and $\mathcal{M}_\theta$ in term of Schützenberger graphs makes clear the possible dependence of these modules on the choice of generating set.

6.2.6 Examples of Relation Modules

We return to the examples of Schützenberger graphs in Section 1.7.1.

6.2.23 Example. Let M be the semilattice $\{1,e,f,ef\}$, generated as an inverse monoid by $X = \{e,f\}$. The Schützenberger graph is



The relation module is therefore

$$\begin{array}{ccccc} & & 0 & & \\ & \swarrow & & \searrow & \\ \mathbb{Z} & & & & \mathbb{Z} \\ & \searrow & & \swarrow & \\ & & \mathbb{Z} \oplus \mathbb{Z} & & \end{array}$$

and all the structure maps are inclusions. This of course recovers the relation module described in Example 6.2.20.

6.2.24 Example. The *bicyclic monoid* B is the inverse monoid presented by $[x : xx^{-1} = 1]$. The Schützenberger graph $\mathrm{Sch}(B, x, x^{-q}x^q)$ is the semi-infinite path

$$x^q \xleftarrow{x} x^{-1}x^q \xleftarrow{x} x^{-2}x^q \xleftarrow{x} \dots \xleftarrow{x} x^{-k}x^q \xleftarrow{x} \dots$$

The relation module $\mathcal{M}$ is therefore trivial. This is no surprise: B is an arboreal inverse monoid, and this example illustrates Lemma 6.2.8.

6.2.25 Example. Given an inverse monoid M with presentation $[Y : R]$, we add a zero to M to obtain M^0. For M^0 we take the generating set $X = Y \cup \{z\}$ (with $z \notin Y$), and we have a presentation $\mathcal{Q}$ of M^0 given by

$$\mathcal{Q} = [Y, z : R, z^2 = z, yz = z = zy \; (y \in Y)] \, .$$

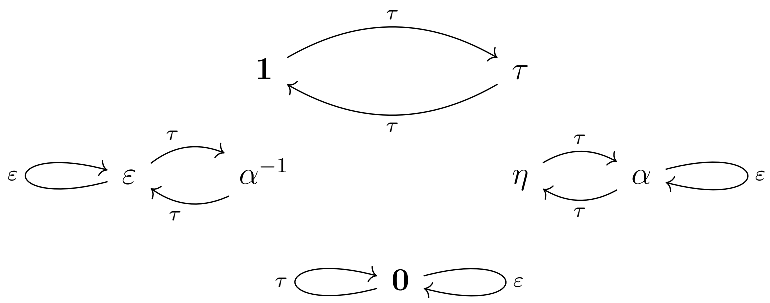
In the Schützenberger graph there is a loop at 0 labelled for each element of X. If $[Y : R]$ has relation module $\mathcal{M}$ then the relation module of $\mathcal{Q}$ can be thought of schematically as

$$\begin{array}{c} \mathcal{M} \\ \Big\downarrow \zeta \\ \mathbb{Z}^{|X|} \end{array}$$

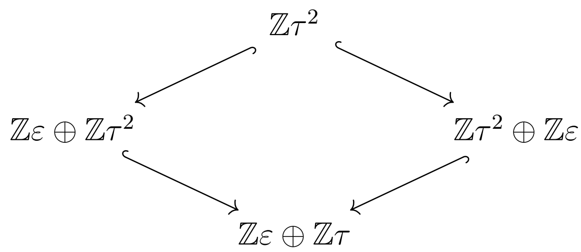
where the map ζ carries a circuit in $\mathrm{Sch}^{\mathcal{L}}(M, Y)$ labelled by a word $w \in (Y \sqcup Y^{-1})^*$

to the element of $\mathbb{Z}^{|Y|} \subset \mathbb{Z}^{|X|}$ determined by w.

6.2.26 Example. The symmetric inverse monoid $\mathcal{I}_2$ on the set $\{1, 2\}$ has Schützenberger graph $\mathrm{Sch}^{\mathcal{L}}(\mathcal{I}_2, \{\tau, \varepsilon\})$



The relation module is therefore



Chapter 7

Squier Complexes for Inverse Monoid Presentations

7.1 The Squier Complex

We now wish to extend the constructions described in Chapter 5 and show that we can obtain a presentation of a relation module $\mathcal{M}_\theta$, derived from an inverse monoid presentation $\mathcal{P} = [X : R]$ with presentation map $\theta : \mathrm{FIM}(X) \to M$, from a free crossed module that is in turn derived from a Squier complex $\mathrm{Sq}(\mathcal{P})$ associated to $\mathcal{P}$. However, in the more general setting of inverse monoid presentations, we will obtain a free crossed module of groupoids. This notion will be reviewed in section 7.2.

The fundamental groupoid of $\mathrm{Sq}(\mathcal{P})$ will be a pseudoregular groupoid, and in the construction of $\mathrm{Sq}(\mathcal{P})$ we are faced with the problem of selecting the correct vertex set. The obvious analogy with group presentations would lead us to use the free inverse monoid $\mathrm{FIM}(X)$. However, to establish a connection with our description of the relation module in chapter 6 as the kernel of the idempotent separating map $\mathcal{T}(X, M) \to M$, it is more appropriate to make a definition of $\mathrm{Sq}(\mathcal{P})$ with vertex set $\mathcal{T}(X, M)$.

7.1.1 Definition. Let $\mathcal{P} = [X : R]$ be an inverse monoid presentation, with presentation map $\theta : \mathrm{FIM}(X) \to M$ factorised, as in Corollary 6.2.10, as

$$\mathrm{FIM}(X) \xrightarrow{\tau} \mathcal{T}(X, M) \xrightarrow{\psi} M$$

with τ idempotent pure and ψ idempotent separating. Then $\mathrm{Sq}(\mathcal{P}, \mathcal{T}(X, M))$ is the 2–complex constructed as follows:

- The vertex set is $\mathcal{T}(X, M)$.
- The edge set consists of all 3-tuples $(p, l = r, q)$ with $p, q \in \mathcal{T}(X, M))$ and $(l = r) \in R$. Such an edge will start at $p(l\tau)q$ and end at $p(r\tau)q$, so each edge corresponds to the application of a relation. An edge path in $\mathrm{Sq}(\mathcal{P})$ therefore corresponds to a succession of such applications.
- The 2-cells correspond to applications of non-overlapping relations, and so a 2-cell is attached along every edge path of the form:

$$\begin{array}{ccc}
p(l\tau)qp'(l'\tau)q' & \xrightarrow{(p,\, l=r,\, qp'(l'\tau)q')} & p(r\tau)qp'(l'\tau)q' \\
\Big\downarrow {\scriptstyle (p(l\tau)qp',\, l'=r',\, q')} & & \Big\downarrow {\scriptstyle (p(r\tau)qp',\, l'=r',\, q')} \\
p(l\tau)qp'(r'\tau)q' & \xrightarrow[(p,\, l=r,\, qp'(r'\tau)q')]{} & p(r\tau)qp'(r'\tau)q'
\end{array}$$

  This attachment of 2–cells makes these two edge paths between $p(l\tau)qp'(l'\tau)q'$ and $p(r\tau)qp'(r'\tau)q'$ homotopic in $\mathrm{Sq}(\mathcal{P}, \mathcal{T}(X, M))$.

We can see that this is essentially the same definition as for groups, see Definition 5.1.1, however now we have the vertex set $\mathcal{T}(X, M)$ and this means we must map our relations using τ into $\mathcal{T}(X, M)$. For clarity from here we will suppress the use of τ, and just write plq and prq for the endpoints of $(p, l = r, q)$.

7.1.1 Proposition. *The fundamental groupoid* $\Pi(\mathrm{Sq}(\mathcal{P}, \mathcal{T}(X, M)))$ *is pseudoregular, and the binary operations* $*$ *and* $\circledast$ *coincide.*

Proof. This follows in exactly the same way as the proof of regularity given in Theorem 5.1.3. □

Let $e \in E(\mathcal{T}(X, M))$. From Proposition 3.3.6 we see that $\mathrm{star}_e^{\bowtie}$ in $\Pi(\mathrm{Sq}(\mathcal{P}, \mathcal{T}(X, M)))$ is a monoid. Moreover, the vertex set of $\mathrm{star}_e^{\bowtie}$ is equal to

$$\begin{aligned} U_e &= \{a \in \mathcal{T}(X, M) : a\psi = e\psi\} \\ &= \{a \in \mathcal{T}(X, M) : a\psi = e\} \end{aligned}$$

since we can identify $E(\mathcal{T}(X, M)$ and $E(M)$ as ψ is idempotent separating. We recognise this set as one of the groups that make up the kernel of ψ. (see Lemma 4.3.3). However, we shall now see that in fact $\mathrm{star}_e^{\bowtie}$ is a group, and U_e is a subgroup.

7.1.2 Lemma. *Let* $e \in E(\mathcal{T}(X, M))$*. In* $\Pi(\mathrm{Sq}(\mathcal{P}, \mathcal{T}(X, M)))$ *the set* $\mathrm{star}_e^{\bowtie}$*, with the operation* $*$*, is a group.*

Proof. By Proposition 3.3.6, the set of arrows $\mathrm{star}_e^{\bowtie}$ is a monoid under the binary operation $*$. As in the regular case, for $\alpha \in \mathrm{star}_e^{\bowtie}$ we define

$$\alpha^{-*} = (\alpha\mathbf{r})^{-1} \rhd \alpha^{-\circ} \lhd (\alpha\mathbf{d})^{-1},$$

where a superscript $^{-1}$ denotes the inverse in the inverse monoid $\mathcal{T}(X, M)$ and a superscript $^{-\circ}$ denotes the inverse in the groupoid $\mathrm{Sq}(\mathcal{P})$.

Now

$$\begin{aligned} \alpha * (\alpha)^{-*} &= \alpha * \big((\alpha\mathbf{r})^{-1} \rhd (\alpha)^{-\circ} \lhd (\alpha\mathbf{d})^{-1}\big) \\ &= \big(\alpha \lhd (\alpha\mathbf{r})^{-1}((\alpha\mathbf{d})^{-\circ})(\alpha\mathbf{d})^{-1}\big) \circ \big((\alpha\mathbf{r})(\alpha\mathbf{r})^{-1} \rhd (\alpha)^{-\circ} \lhd (\alpha\mathbf{d})^{-1}\big) \\ &= \big(\alpha \lhd (\alpha\mathbf{r})^{-1}(\alpha\mathbf{r})(\alpha\mathbf{d})^{-1}\big) \circ \big((\alpha\mathbf{r})(\alpha\mathbf{r})^{-1} \rhd (\alpha)^{-\circ} \lhd (\alpha\mathbf{d})^{-1}\big). \end{aligned}$$

Since $\alpha \in \mathrm{star}_e^{\bowtie}$ we have $\alpha\mathbf{d} = e$, and since $\alpha\mathbf{r} \in U_e$ and U_e is a subgroup of $\mathcal{T}(X, M)$ with identity e, then $(\alpha\mathbf{r})^{-1}(\alpha\mathbf{r}) = e = (\alpha\mathbf{r})(\alpha\mathbf{r})^{-1}$. So

$$\begin{aligned} \alpha * (\alpha)^{-*} &= (\alpha \lhd e) \circ (e \rhd (\alpha)^{-\circ} \lhd e) \\ &= \alpha \circ (\alpha)^{-\circ} \\ &= 1_e \end{aligned}$$

Similarly $(\alpha)^{-*} * \alpha = 1_e$, and therefore each $\mathrm{star}_e^{\bowtie}$ is a group. □

7.1.3 Lemma. *Suppose that $(ep, l = r, qe) \in \mathrm{star}_e^{\bowtie}$. Then $eplqe = e$, and therefore*

$$(ep)(lqe)(ep) = eep = ep$$
$$(lqe)(ep)(lqe) = lqee = lqe$$

and so $ep = (lqe)^{-1}$ in $\mathcal{T}(X, M)$. Therefore $(ep, l = r, qe) = (eq^{-1}l^{-1}, l = r, qe)$: moreover $e = eq^{-1}l^{-1}lqe$ and so $e \leqslant q^{-1}l^{-1}lq$.

7.1.4 Lemma. *A path $\alpha \in \mathrm{star}_e^{\bowtie}$ can be rewritten as a $*$-product of single edges in $\mathrm{star}_e^{\bowtie}$. Thus each $\mathrm{star}_e^{\bowtie}$ is generated by the subset $\mathcal{S}_e^{\bowtie}$ of single edges in $\mathrm{star}_e^{\bowtie}$, and these have the form*

$$\lambda^e_{l=r,q} = (eq^{-1}l^{-1}, l = r, qe)\,,$$

with $e \leqslant q^{-1}l^{-1}lq$.

Proof. The rewriting of a path α to a $*$–product $\alpha\lambda$ of edges in $\mathcal{S}_e^{\bowtie}$ is essentially that defined in Proposition 5.2.1. The details are as follows.

The vertex set of the component of $\mathrm{Sq}(\mathcal{P})$ that contains e is the group U_e (with identity e), and so for a path α in this component we define

$$\alpha\lambda = (\alpha\mathbf{d})^{-1} \rhd \alpha \lhd e$$

If $\alpha = (p, l = r, q)$ is a single edge, then

$$\alpha\lambda = (q^{-1}l^{-1}p^{-1}p, l = r, qe) = (eq^{-1}l^{-1}p^{-1}p, l = r, qe)$$

since $plq \in U_e$. Then by Lemma 7.1.3, we have $eq^{-1}l^{-1}p^{-1}p = (lqe)^{-1}$ and so

$$\alpha\lambda = (eq^{-1}l^{-1}, l = r, qe) = \lambda^e_{l=r,q}\,,$$

and $(\alpha\lambda)\mathbf{d} = eq^{-1}l^{-1}lqe = e$.

Now if $\alpha = \alpha_1 \circ \alpha_2$ then

$$\begin{aligned}\alpha\lambda &= (\alpha_1\mathbf{d})^{-1} \rhd (\alpha_1 \circ \alpha_2) \lhd e \\ &= ((\alpha_1\mathbf{d})^{-1} \rhd \alpha_1 \lhd e) \circ ((\alpha_1\mathbf{d})^{-1} \rhd \alpha_2 \lhd e) \\ &= \alpha_1\lambda \circ ((\alpha_1\mathbf{d})^{-1} \rhd \alpha_2 \lhd e)\,,\end{aligned}$$

and

$$\alpha_1\lambda * \alpha_2\lambda = (\alpha_1\lambda \lhd e) \circ ((\alpha_1\mathbf{d})^{-1}(\alpha_1\mathbf{r})e(\alpha_2\mathbf{d})^{-1} \rhd \alpha_2 \lhd e)\,.$$

But $\alpha_1\mathbf{r} = \alpha_2\mathbf{d} \in U_e$ and so $(\alpha_1\mathbf{d})^{-1}(\alpha_1\mathbf{r})e(\alpha_2\mathbf{d})^{-1} = (\alpha_1\mathbf{d})^{-1}$ and therefore $\alpha\lambda = \alpha_1\lambda * \alpha_2\lambda$. The lemma then follows easily by induction on the length of a path. □

Now suppose that the path α is a composition $\alpha = \alpha_1 \circ \alpha_2$ and that β is the path

$$\beta = \alpha_1 \circ \gamma \circ \gamma^{-\circ} \circ \alpha_2$$

for some path γ. Then

$$\beta\lambda = \alpha_1\lambda * \gamma\lambda * \gamma^{-\circ}\lambda * \alpha_2\lambda\,.$$

Now if $d = \gamma\mathbf{d}$ and $r = \gamma\mathbf{r}$ then

$$\begin{aligned}
\gamma\lambda * \gamma^{-\circ}\lambda &= (d^{-1} \triangleright \gamma \triangleleft e) * (r^{-1} \triangleright \gamma^{-\circ} \triangleleft e) \\
&= (d^{-1} \triangleright \gamma \triangleleft e)(d^{-1}r \triangleright r^{-1} \triangleright \gamma^{-\circ} \triangleleft e) \\
&= (d^{-1} \triangleright \gamma \triangleleft e)(d^{-1} \triangleright \gamma^{-\circ} \triangleleft e) \\
&= d^{-1} \triangleright (\gamma \circ \gamma^{-\circ}) \triangleleft e \\
&= e\,.
\end{aligned}$$

Hence if α and β are paths differing by a 1–homotopy in $\mathrm{Sq}(\mathcal{P})$ then $\alpha\lambda = \beta\lambda$ in the group $(\mathrm{star}_e^{\bowtie}, *)$.

Now consider a 2–cell in the component of $\mathrm{Sq}(\mathcal{P})$ containing e: Suppose that we have a 2–cell

$$\begin{array}{ccc}
plqtsu & \xrightarrow{(p,l=r,qtsu)} & prqtsu \\
{\scriptstyle (plqt,s=d,u)}\Big\downarrow & & \Big\downarrow{\scriptstyle (prqt,s=d,u)} \\
plqtdu & \xrightarrow[(p,l=r,qtdu)]{} & prqtdu
\end{array} \tag{7.1}$$

with

$$\alpha = (p, l = r, qtsu), \beta = (prqt, s = d, u), \gamma = (plqt, s = d, u), \delta = (p, l = r, qtdu)\,. \tag{7.2}$$

Then

$$\begin{aligned}\alpha\lambda &= (p, l = r, qtsu)\lambda \\ &= (u^{-1}s^{-1}t^{-1}q^{-1}l^{-1}p^{-1}p, l = r, qtsue) \\ &= (eu^{-1}s^{-1}t^{-1}q^{-1}l^{-1}p^{-1}p, l = r, qtsue) \quad (\text{since } plqtsu \in U_e) \\ &= (eu^{-1}s^{-1}t^{-1}q^{-1}l^{-1}, l = r, qtsue) \\ &= \lambda^e_{l=r,qtsu} \quad (\text{by Lemma 7.1.3}).\end{aligned}$$

Similarly $\beta\lambda = \lambda^e_{s=d,u} = \gamma\lambda$ and $\delta\lambda = \lambda^e_{l=r,qtdu}$. Hence if two paths differ by a 2–homotopy in $\mathrm{Sq}(\mathcal{P})$ their λ–rewrites in $\mathrm{star}_e^{\bowtie}$ are equal as a consequence of the relation

$$\lambda^e_{l=r,vsu} * \lambda^e_{s=d,u} = \lambda^e_{s=d,u} * \lambda^e_{l=r,vdu}$$

(where $v = qt$ above). These considerations show that:

7.1.5 Proposition. *Given $e \in E(\mathcal{T}(X, M)), q \in \mathcal{T}(X, M)$ and $(l = r) \in R$ such that $e \leqslant q^{-1}l^{-1}lq$, we set $\lambda^e_{l=r,q} = (eq^{-1}l^{-1}, l = r, qe)$. Then the following are a set of defining relations for the group $(\mathrm{star}_e^{\bowtie}, *)$ on the generating set $\mathcal{S}_e^{\bowtie}$:*

$$\lambda^e_{l=r,vsu} * \lambda^e_{s=d,u} = \lambda^e_{s=d,u} * \lambda^e_{l=r,vdu}\,.$$

7.2 Crossed Modules of Groupoids

We now present the rudiments of the theory of crossed modules of groupoids. For further information we refer to [7].

7.2.1 Definition. Let $\mathcal{G}$ be a groupoid with vertex set E, which we denote by $\mathcal{G} \rightrightarrows E$, where $\rightrightarrows$ represent the maps $\mathbf{d}$ and $\mathbf{r}$ defined in Definition 1.4.2. Then a *crossed $\mathcal{G}$-module of groupoids*

$$\mathcal{C} \xrightarrow{\partial} \mathcal{G} \rightrightarrows E$$

consists of:

1. a disjoint union of groups $\mathcal{C} = \bigsqcup_{e \in E} C_e$, indexed by E,
2. a homomorphism ∂ of groupoids,
3. an action of $\mathcal{G}$ on $\mathcal{C}$, denoted $(c, g) \mapsto c^g$, such that an edge g in $\mathcal{G}$ with $g\mathbf{d} = e$ and $g\mathbf{r} = f$, acts on $c \in C_e$ with $c^g \in C_f$.

The action of $\mathcal{G}$ on $\mathcal{C}$ satisfies

$$(c^g)\partial = g^{-1}(c\partial)g \tag{7.3}$$

whenever c^g is defined, and

$$c^{a\partial} = a^{-1}ca \tag{7.4}$$

for some $e \in E$, $a, c \in C_e$.

Just as for groups we have precrossed modules of groupoids which are as above but need not satisfy (7.4). We also have analogous definitions of a free precrossed and crossed module of groupoids, see [7, Section 7.3].

7.2.2 Definition. Consider a (pre)crossed module of groupoids

$$\mathcal{C} \xrightarrow{\partial} \mathcal{G} \rightrightarrows E$$

along with a set R and a function $\omega : R \to \mathcal{G}$ such that $\omega\mathbf{d} = \omega\mathbf{r}$. Then $\mathcal{C}$ is said to be the *free (pre)crossed $\mathcal{G}$-module with basis ω* if for any (pre)crossed $\mathcal{G}$-module

$$\mathcal{C}' \xrightarrow{\partial'} \mathcal{G} \rightrightarrows E$$

and function $\nu' : R \to \mathcal{C}'$ such that $\omega = \nu'\partial'$ there exists a unique morphism of (pre)crossed modules $\phi : \mathcal{C} \to \mathcal{C}'$ such that $\partial = \phi\partial'$.

We shall now discuss the construction of the free (pre)crossed module of groupoids again following [7, Section 7.3] and amplifying the details.

7.2.1 Proposition. *Given a groupoid $\mathcal{G}$, a set R and a function $\omega : R \to \mathcal{G}$ such that $\omega\mathbf{d} = \omega\mathbf{r}$, then a free precrossed $\mathcal{G}$-module with basis ω exists and is unique up to isomorphism.*

Proof. For each $e \in E$ we define $R_e = \{r \in R : (r\omega)\mathbf{r} = e = (r\omega)\mathbf{d}\}$ and

$$(R \between \mathcal{G})_e = \{(r, g) \in R \times \mathcal{G} : r \in R_{gg^{-1}}, g^{-1}g = e\}.$$

We define F_e to be the free group on $(R \between \mathcal{G})_e$, and $\mathcal{F} = \bigsqcup_{e \in E} F_e$. Then $\mathcal{F}$ along with the map $\delta : \mathcal{F} \to \mathcal{G}$, defined on generators by $(r, g) \mapsto g^{-1}(r\omega)g$, and an action of $\mathcal{G}$ on $\mathcal{F}$, defined on generators by $(r, g)^h = (r, gh)$ whenever $g^{-1}g = hh^{-1}$ is a free precrossed $\mathcal{G}$-module. Uniqueness follows from the usual universal argument. □

Then recall from Definition 2.1.3 that the *Peiffer elements* of F_e are the elements of the form $\langle x, y\rangle = x^{-1}y^{-1}xy^{x\delta}$, for $x, y \in F_e$. We observe that for $x, y \in F_e$, the action of $x\delta$ on y takes us from F_e to $F_{(x\delta)^{-1}(x\delta)}$: however $(x\delta)^{-1}(x\delta) = (x^{-1}x)\delta = e\delta = e$, and so $\langle x, y\rangle$ is defined in the group F_e. We let P_e denote the subgroup of F_e generated by the Peiffer elements of F_e.

7.2.3 Definition. For a precrossed module of groupoids $\mathcal{F} \xrightarrow{\delta} \mathcal{G} \rightrightarrows E$, the *Peiffer subgroupoid* P is the disjoint union of the subgroups P_e of F_e.

7.2.2 Proposition. *For P the Peiffer subgroupoid of a precrossed module of groupoids $\mathcal{F} \xrightarrow{\delta} \mathcal{G} \rightrightarrows E$:*

1. *Each P_e is a normal subgroup of F_e.*

2. *P is invariant under the $\mathcal{G}$ action.*

3. *P is contained in the kernel of δ.*

Proof. The proof of this proposition follows the proof of Proposition 2.1.3. However now we have to note carefully how the action shifts elements.

1. Let $x, y, z \in F_e$, then

$$\begin{aligned} z^{-1}\langle x,y\rangle z &= z^{-1}x^{-1}y^{-1}xy^{x\delta}z \\ &= z^{-1}x^{-1}y^{-1}x\,(zy^{(xz)\delta}(y^{(xz)\delta})^{-1}z^{-1})\,y^{x\delta}z \\ &= (xz)^{-1}y^{-1}(xz)y^{(xz)\delta}\left(z^{-1}(y^{x\delta})^{-1})z(y^{x\delta})^{z\delta}\right)^{-1} \\ &= \langle xz,y\rangle\langle z,y^{x\delta}\rangle^{-1} \end{aligned}$$

   Therefore a conjugate of a Peiffer element is a product of Peiffer elements, and P_e is normal in F_e.

2. Now consider $g \in \mathcal{G}$ with $gg^{-1} = e$ and a Peiffer element $\langle x, y\rangle$ of F_e. Then

$$\begin{aligned} \langle x^g, y^g\rangle &= (x^g)^{-1}(y^g)^{-1}x^g(y^g)^{(x^g)\delta} \\ &= (x^g)^{-1}(y^g)^{-1}x^g(y^g)^{g^{-1}(x\delta)g} \\ &= (x^g)^{-1}(y^g)^{-1}x^g(y^{x\delta})^g \\ &= \langle x,y\rangle^g \end{aligned}$$

   So a Peiffer element in F_e acted on by an appropriate element of $\mathcal{G}$ is another Peiffer element, now in $F_{g^{-1}g}$, and we see that P is invariant under the $\mathcal{G}$-action, and that acting on elements of P shifts us in the subgroups P_e just as it does around the subgroups F_e of $\mathcal{F}$.

3. Lastly, for $\langle x, y\rangle \in P_e$ we have,

$$\begin{aligned} \langle x,y\rangle\delta = (x^{-1}y^{-1}xy^{x\delta})\delta &= (x^{-1})\delta(y^{-1})\delta x\delta(y^{x\delta})\delta \\ &= (x\delta)^{-1}(y\delta)^{-1}(x\delta)(x\delta)^{-1}(y\delta)(x\delta) \\ &= e \in E(\mathcal{G}) \end{aligned}$$

and so $P \subseteq \ker \delta$.

□

7.2.3 Theorem. *Let $\mathcal{F} \xrightarrow{\delta} \mathcal{G} \rightrightarrows E$ be a precrossed $\mathcal{G}$-module of groupoids. Then there exists a crossed $\mathcal{G}$-module of groupoids, $\mathcal{A} \xrightarrow{\partial} \mathcal{G} \rightrightarrows E$ and a morphism of precrossed $\mathcal{G}$-modules $\phi : \mathcal{F} \to \mathcal{A}$ which is universal for morphisms from $(\mathcal{F}, \delta)$ to crossed $\mathcal{G}$-modules of groupoids.*

Proof. Let $P = \sqcup_{e \in E} P_e$ be the Peiffer subgroupoid of $(\mathcal{F}, \delta)$, and set $\mathcal{A} = \sqcup_{e \in E} A_e$, where $A_e = F_e / P_e$, regarded as the set of left cosets of P_e in F_e, with ϕ the quotient map. Then since each $P_e \subseteq \ker \delta$ we have an induced homomorphism $\partial : \mathcal{A} \to \mathcal{G}$.

We can define an action of $\mathcal{G}$ on $\mathcal{A}$ by

$$(xP_e)^g = (x^g)P_{g^{-1}g} \quad \text{whenever } gg^{-1} = e.$$

This is well-defined, since if $xP_e = yP_e$, and $g \in \mathcal{G}$ such that $gg^{-1} = e$, then

$$x^g (y^g)^{-1} = x^g (y^{-1})^g = (xy^{-1})^g.$$

and since $xy^{-1} \in P_e$, we have $(xy^{-1})^g \in P_{g^{-1}g}$. Therefore $x^g P_{g^{-1}g} = y^g P_{g^{-1}g}$.

It remains to check the conditions (7.3) and (7.4). For (7.3) we have

$$\begin{aligned}
(xP_e)^g \partial &= (x\phi)^g \partial \\
&= (x^g)\phi\partial \\
&= (x^g)\delta \\
&= g^{-1}(x\delta)g \\
&= g^{-1}(x\phi\partial)g \\
&= g^{-1}(xP_e\partial)g\,,
\end{aligned}$$

and for (7.4)

$$(x^{-1}yx)P_e = (y^{x\delta})P_e = (yP_e)^{x\delta} = yP_e^{(xP_e)\partial}.$$

Then given any morphism τ from $(\mathcal{F}, \delta)$ to a crossed $\mathcal{G}$-module of groupoids $(\mathcal{T}, d)$ we will have $P \in \ker \tau$, and thus there will be an induced crossed module morphism $(\mathcal{A}, \partial) \to (\mathcal{T}, d)$. □

Via the free precrossed module, we arrive at the free crossed module, as in [7, Proposition 7.3.7].

7.2.4 Corollary. *Given a groupoid $\mathcal{G}$, a set R and a function $\omega : R \to \mathcal{G}$ with $\omega\mathbf{d} = \omega\mathbf{r}$, then a free crossed $\mathcal{G}$-module with basis R exists, and is unique up to isomorphism.*

7.2.5 Example. Consider a crossed module $\mathcal{A} \xrightarrow{\partial} \mathcal{G} \rightrightarrows E$ in which ∂ is trivial: that is, ∂ maps each $a \in A_e$ to $e \in E$. We write $\partial = \varepsilon$. By (7.4) each A_e is then abelian, and $\mathcal{A}$ is a $\mathcal{G}$–module. The concept of a *free* $\mathcal{G}$–module then follows: given a set R and a function $\omega : R \to \mathcal{G}$ with $\omega\mathbf{d} = \omega\mathbf{r}$, a $\mathcal{G}$–module $\mathcal{A}$ is *free with basis* ω, if for any $\mathcal{G}$–module $\mathcal{B}$ and function $\nu : R \to \mathcal{B}$ such that $\nu\varepsilon = \omega$, there exists a unique morphism $\phi : \mathcal{A} \to \mathcal{B}$ of $\mathcal{G}$–modules.

7.2.1 Modules and Crossed Modules

Suppose that (as in Example 7.2.5) in a crossed $\mathcal{G}$-module $\mathcal{C} \xrightarrow{\partial} \mathcal{G} \rightrightarrows E$, the map ∂ has image E. Then, by (7.4), for $a, c \in C_e$ we have $c = a^{-1}ca$ and so C_e is abelian, and $\mathcal{C}$ is then a $\mathcal{G}$–module: that is, the association $e \mapsto C_e$ is the object part of a functor from $\mathcal{G}$ to the category of abelian groups, and an arrow $g \in \mathcal{G}$ is then associated to the action homomorphism $c \mapsto c^g$ defined for $c \in C_{g\mathbf{d}}$.

More generally, we have:

7.2.6 Proposition.

1. *Let* $\mathcal{C} \xrightarrow{\partial} \mathcal{G} \rightrightarrows E$ *be a crossed module of groupoids, and let* $\mathcal{Q}$ *be the quotient groupoid* $\mathcal{G}/\mathcal{C}\partial$*, with* $\phi : \mathcal{G} \to \mathcal{Q}$ *the natural map Then* $\mathcal{C}^{ab} = \bigsqcup_{e \in E} C_e^{ab}$ *is a* $\mathcal{Q}$*–module, where for* $c \in C_e$ *and* $q = g\phi$ *with* $gg^{-1} = e$ *we have*

$$\overline{c} \lhd q = \overline{c^g} \,.$$

2. *If* $\mathcal{C} \xrightarrow{\partial} \mathcal{G} \rightrightarrows E$ *is a free crossed* $\mathcal{G}$*-module of groupoids with basis* $\omega : R \to \mathcal{G}$ *then* $\mathcal{C}^{ab}$ *is a free* $\mathcal{Q}$*–module with basis the image of the induced map* $R \to \mathcal{C} \to \overline{\mathcal{C}}$*.*

Proof. The claimed $\mathcal{Q}$–action is well-defined, since if $q = g\phi = h\phi$ with $g, h \in \mathcal{G}$, then $h = (a\partial)g$ for some $a \in \mathcal{C}$, and then

$$\overline{c^h} = \overline{c^{(a\partial)g}} = \overline{(a^{-1}ca)^g} = \overline{c^g} \,,$$

just as for crossed modules of groups in section 2.1.

Now let $\mathcal{A}$ be an arbitrary $\mathcal{Q}$-module, and consider the disjoint union of groups $\Lambda = \sqcup_{e \in E} \Lambda_e$, where $\Lambda_e = C_e\partial \times A_e$. We let $\mathcal{G}$ act on Λ by conjugation on each C_e and via ϕ on A_e. Let $\pi_1 : \Lambda \to \mathcal{G}$ be the projection map: we claim that $\Lambda \xrightarrow{\pi_1} \mathcal{G} \rightrightarrows E$ is a crossed module of groupoids. For $(c\partial, a) \in \Lambda_e$ and $g \in \mathcal{G}$ with $g\mathbf{d} = e$ we have:

$$((c\partial, a)^g)\pi_1 = (g^{-1}(c\partial)g, a^{g\phi})\pi_1 = g^{-1}(c\partial)g = g^{-1}(c\partial, a)\pi_1 g \,,$$

and for $(c_1\partial, a_1), (c_2\partial, a_2) \in \Lambda_e$,

$$\begin{aligned}(c_1\partial, a_1)^{(c_2\partial, a_2)\pi_1} &= (c_1\partial, a_1)^{c_2\partial} \\ &= ((c_2\partial)^{-1}(c_1\partial)(c_2\partial), a_1^{(c_2\partial)\phi}) \\ &= \big((c_2\partial)^{-1}(c_1\partial)(c_2\partial), a_1\big)\end{aligned}$$

since $c_2\partial\phi = e$, and

$$= (c_2\partial, a_2)^{-1}(c_1\partial, a_1)(c_2\partial, a_2)\,,$$

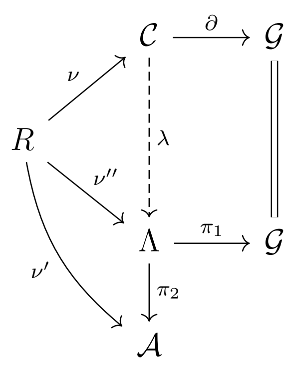
since A_e is abelian. So $\Lambda \xrightarrow{\pi_1} \mathcal{G} \rightrightarrows E$ is a crossed module of groupoids.

Now let $\nu' : R \to \mathcal{A}$ be given by a family of functions $\nu' : R_e \to A_e$, and then define

$$\nu'' = (\nu\partial, \nu') : \mathcal{R} \to \Lambda\,.$$

We note that $\nu''\pi_1 = \nu\partial$, and so by freeness of $\mathcal{C}$, there is an induced morphism $\lambda : \mathcal{C} \to \Lambda$ of crossed $\mathcal{G}$–modules, with $\nu\lambda = \nu''$. Composing λ with the second projection $\pi_2 : \Lambda \to \mathcal{A}$ gives a morphism $\mathcal{C} \to \mathcal{A}$ that factors through $\mathcal{C}^{ab} \to \mathcal{A}$, and is easily seen to be a map of $\mathcal{Q}$–modules.

The maps used in the proof are illustrated below.



□

Part (2) of Proposition 7.2.6 generalises Proposition 2.1.6.

7.2.2 A Crossed Module from an Inverse Monoid Presentation

Consider an inverse monoid presentation $\mathcal{P} = [X : \mathcal{R}]$ of M, along with its presentation map $\theta : \mathrm{FIM}(X) \to M$, and its factorisation given in Lemma 4.3.4

$$\mathrm{FIM}(X) \xrightarrow{\tau} \mathcal{T}(X, M) \xrightarrow{\psi} M \, .$$

Let $E = E(\mathcal{T}(X, M))$. We regard $\mathcal{T}(X, M)$ as an inductive groupoid with vertex set E, as discussed in section 1.4.2, and define

$$S^{\bowtie} = \bigsqcup_{e \in E} \mathrm{star}_e^{\bowtie} \, .$$

7.2.7 Proposition. *$S^{\bowtie} \xrightarrow{\mathbf{r}} \mathcal{T}(X, M) \rightrightarrows E$ is a crossed module of groupoids.*

Proof. By Lemma 7.1.2, each $\mathrm{star}_e^{\bowtie}$ is a group, and so $S^{\bowtie}$ is a disjoint union of groups indexed by E, and is a groupoid with vertex set E. Then $\mathbf{r}$ is a groupoid homomorphism, and is the identity on E.

An action of $\mathcal{T}(X, M)$ on $S^{\bowtie}$ is defined using the actions in the pseudoregular groupoid $\Pi(\mathrm{Sq}(\mathcal{P}, \mathcal{T}(X, M)))$ as follows: for $w \in \mathcal{T}(X, M)$ and $\alpha \in \mathrm{star}_{ww^{-1}}^{\bowtie}$ we define

$$\alpha^w = w^{-1} \rhd \alpha \lhd w \in \mathrm{star}_{w^{-1} \cdot \mathbf{d}(\alpha) \cdot w}^{\bowtie} = \mathrm{star}_{w^{-1}ww^{-1}w}^{\bowtie} = \mathrm{star}_{w^{-1}w}^{\bowtie} \, .$$

Then (7.3) holds, since

$$(\alpha)^w \mathbf{r} = (w^{-1} \rhd \alpha \lhd w)\mathbf{r} = w^{-1}(\alpha\mathbf{r})w \, ,$$

For (7.4), since the binary operations $*$ and $\circledast$ on $\Pi(\mathrm{Sq}(\mathcal{P}, \mathcal{T}(X, M)))$ coincide by

Proposition 7.1.1, then

$$\alpha * \beta = \alpha \circledast \beta = \beta \circ (\alpha \lhd \beta \mathbf{r})$$

and

$$\beta * (\alpha)^{\beta \mathbf{r}} = \beta * ((\beta \mathbf{r})^{-1} \rhd \alpha \lhd \beta \mathbf{r}) = \beta \circ (\alpha \lhd \beta \mathbf{r}) \,.$$

So $\alpha * \beta = \beta * \alpha^{\beta \mathbf{r}}$. Therefore $S^{\bowtie} \xrightarrow{\mathbf{r}} \mathcal{T}(X, M) \rightrightarrows E$ is a crossed module of groupoids. □

We now wish to define a free crossed $\mathcal{T}(X, M)$–module directly from an inverse monoid presentation $\mathcal{P} = [X : \mathcal{R}]$ of M. We recall that $\psi : \mathcal{T}(X, M) \to M$ is idempotent separating, and since if $(l = r) \in \mathcal{R}$ then $l\psi = r\psi$ we have $(l^{-1}r)\psi \in E(M)$. Hence $(l^{-1}r)\psi = e\psi$ for some (unique) $e \in E(\mathcal{T}(X, M))$ and

$$(l^{-1}r)\psi = (l^{-1}l)\psi = (r^{-1}r)\psi \quad \text{and} \quad (lr^{-1})\psi = (ll^{-1})\psi = (rr^{-1})\psi \,.$$

Hence $l^{-1}l = x = r^{-1}r$ and $ll^{-1} = rr^{-1}$: that is, l and r are $\mathcal{H}$–related in $\mathcal{T}$.

Now for $x \in E$ we define

$$R_x = \{(l = r, x) \in \mathcal{R} \times E : (l^{-1}r)\psi \geqslant x\psi\}$$

and consider the set $R = \bigsqcup_{x \in E} R_x$, along with the function $\omega : R \to \mathcal{T}(X, M)$ which maps $(l = r, x) \mapsto xl^{-1}rx$. (We note that, since $l^{-1}rx$ is in the $\mathcal{H}$–class at x, then $l^{-1}rx = xl^{-1}rx$: we write $xl^{-1}rx$ for its convenient symmetry.) Then

$$(l = r, x)\omega\mathbf{d} = xl^{-1}rxr^{-1}lx \text{ and } (l = r, x)\omega\mathbf{r} = xr^{-1}lxl^{-1}rx \,,$$

with

$$(l = r, x)\omega\mathbf{d}\psi = (xl^{-1}rxr^{-1}lx)\psi = x\psi = (xr^{-1}lxl^{-1}rx)\psi = (l = r, x)\omega\mathbf{r}\psi \,.$$

Since ψ is idempotent separating, we conclude that $(l = r, x)\omega\mathbf{d} = (l = r, x)\omega\mathbf{r}$. As in Proposition 7.2.1 and Theorem 7.2.3 we can construct the free precrossed and crossed $\mathcal{T}(X, M)$-modules with basis ω. To this end, we define F_e to be the free group with basis

$$(R \between \mathcal{T})_e = \{(l = r, u) \in \mathcal{R} \times \mathcal{T}(X, M) : (l = r, uu^{-1}) \in R_{uu^{-1}}, u^{-1}u = e\}\,. \quad (7.5)$$

and set $\mathcal{F} = \bigsqcup_{e \in E} F_e$. Then $\mathcal{F}$ along with $\delta : \mathcal{F} \to \mathcal{T}(X, M)$ mapping $(l = r, u) \mapsto u^{-1}l^{-1}ru$, and the action of $\mathcal{T}(X, M)$ on $\mathcal{F}$ given on generators by $(l = r, u)^v = (l = r, uv)$ (where $(l = r, u) \in (R \between \mathcal{T})_e$ and $vv^{-1} = e$) is the free precrossed $\mathcal{T}(X, M)$-module on ω. Then by Theorem 7.2.3 we can factor out the Peiffer subgroupoid to obtain $\mathcal{C} \xrightarrow{\partial} \mathcal{T}(X, M) \rightrightarrows E(M)$ as the free crossed $\mathcal{T}(X, M)$-module on ω.

7.2.8 Theorem. *The crossed $\mathcal{T}(X, M)$-module*

$$S^{\bowtie} \xrightarrow{\mathbf{r}} \mathcal{T}(X, M) \rightrightarrows E(M)$$

is isomorphic to the free crossed $\mathcal{T}(X, M)$–module

$$\mathcal{C} \xrightarrow{\partial} \mathcal{T}(X, M) \rightrightarrows E(M)\,.$$

Proof. Define $\nu : R \to S^{\bowtie}$ by $(l = r, x) \mapsto (xl^{-1}, l = r, x)$. We then have $(xl^{-1}, l = r, x)\mathbf{d} = xl^{-1}lx = x$, and so $(l = r, x)\nu \in \mathrm{star}_x^{\bowtie}$. Moreover,

$$(l = r, x)\nu\mathbf{r} = (xl^{-1}, l = r, x)\mathbf{r} = xl^{-1}rx = (l = r, x)\omega\,.$$

Therefore $\nu\mathbf{r} = \omega$ and by freeness of $\mathcal{C} \xrightarrow{\partial} \mathcal{T}(X, M) \rightrightarrows E(M)$ there exists a crossed module morphism $\eta : \mathcal{C} \to S^{\bowtie}$ mapping

$$(l = r, u) \mapsto (u^{-1}l^{-1}, l = r, u) \in \mathrm{star}^{\bowtie}(u^{-1}u)\,,$$

where $(l^{-1}r)\psi \geqslant (uu^{-1})\psi$ and $l^{-1}l = r^{-1}r \geqslant uu^{-1}$. We claim that η is an isomorphism, and we verify this by constructing its inverse.

Suppose that $(ep, l = r, qe) \in \mathrm{star}_e^{\bowtie}$. Then byLemma 7.1.3, we have $ep = (lqe)^{-1}$ in $\mathcal{T}(X, M)$ and therefore $(ep, l = r, qe) = (eq^{-1}l^{-1}, l = r, qe)$. We can define a mapping μ on the set of single edges $(ep, l = r, qe)$ in $\mathrm{star}_e^{\bowtie}$ into the basis $(R \between \mathcal{T})_e$ of the free group H_e defined in (7.5) by

$$\mu : (ep, l = r, qe) \mapsto (l = r, qe)\,.$$

and this is a bijection. We consider the effect of this map on a defining relation

$$\lambda^e_{l=r,vsu} * \lambda^e_{s=d,u} = \lambda^e_{s=d,u} * \lambda^e_{l=r,vdu}\,.$$

as given in Proposition 7.1.5. We have

$$\begin{aligned}
\lambda^e_{l=r,vsu} &\overset{\mu}{\mapsto} (l = r, vsue) \\
\lambda^e_{s=d,u} &\overset{\mu}{\mapsto} (s = d, ue) \\
\lambda^e_{l=r,vdu} &\overset{\mu}{\mapsto} (l = r, vdue)\,.
\end{aligned}$$

In the group $C_e = F_e/P_e$ we have

$$(s = d, ue)^{-1}(l = r, vsue)(s = d, ue) = (l = r, vsueu^{-1}s^{-1}due)\,.$$

Now $s\psi = d\psi$ and so

$$[(sue)(sue)^{-1}]\psi = [(due)(due)^{-1}]\psi\,.$$

Since ψ is idempotent separating, $(sue)(sue)^{-1} = (due)(due)^{-1}$ and therefore

$$vsueu^{-1}s^{-1}due = v(sue)(sue)^{-1}due = v(due)(due)^{-1}(due) = vdue\,.$$

So in C_e we have

$$(s = d, ue)^{-1}(l = r, vsue)(s = d, ue) = (l = r, vdue)$$

and μ induces a homomorphism $\mathrm{star}_e^{\bowtie} \to C_e$ that is the inverse of η. $\square$

By Proposition 7.2.6,

$$(S^{\bowtie})^{ab} = \bigsqcup_{e \in E} (\mathrm{star}_e^{\bowtie})^{ab}$$

is a free $\vec{M}$–module with basis function

$$(R \between \mathcal{T}) \to S^{\bowtie} \to (S^{\bowtie})^{ab}\,, \quad (l = r, u) \mapsto (u^{-1}l^{-1}, l = r, u)^{ab}\,.$$

Here the $\vec{M}$–module structure only takes account of the groupoid action of $\vec{M}$. However, we can say more.

7.2.9 Proposition. *$(S^{\bowtie})^{ab}$ is the free $\mathfrak{L}(M)$–module on the $E(M)$–set $\mathcal{Z}$ in which $Z_e = \{(l = r) \in R : (l^{-1}r)\psi = e\psi\}$.*

Proof. We need to extend the action of $\vec{M}$ to one of $\mathfrak{L}(M)$, and if $e, f \in E(M)$ with $e \geqslant f$, and if $\alpha \in \mathrm{star}_e^{\bowtie}$ with image $\overline{\alpha} \in (\mathrm{star}_e^{\bowtie})^{ab}$, then we define $\overline{\alpha} \lhd (e, f) = f \rhd \alpha \lhd f$, and so for $(e, m) \in \mathfrak{L}(M)$ with $m = w\psi$, we have

$$\overline{\alpha} \lhd (e, m) = \overline{w^{-1} \rhd \alpha \lhd w}\,.$$

We note that $(w^{-1} \rhd \alpha \lhd w)\mathbf{d}\psi = (w^{-1} \rhd \alpha\mathbf{d} \lhd w)\psi = m^{-1}em = m^{-1}m$, since $e \geqslant mm^{-1}$.

As a component of the free $\vec{M}$–module $(S^{\bowtie})^{ab}$, the group $(\mathrm{star}_e^{\bowtie})^{ab}$ is the free abelian group with basis

$$Y_e = \{(l = r, m) : (l^{-1}r)\psi \geqslant mm^{-1}, m^{-1}m = e\}\,.$$

and the $\mathfrak{L}(M)$–action on a basis element is $(l = r, m) \lhd (e, n) = (l = r, mn)$. Now Loganathan's free $\mathfrak{L}(M)$–module $\mathcal{F}$ with basis $\mathcal{Z}$ is, according to the description in section 1.6, constructed by taking F_e to be free abelian on the basis

$$B_e = \{(l = r, (f, s)) : (l^{-1}r)\psi = f\psi, f\psi \geqslant ss^{-1}, s^{-1}s = e\}\,.$$

Since ψ is idempotent separating, there is a unique idempotent $f \in E(\mathcal{T})$ such that $(l^{-1}r)\psi = f\psi$ and so f is determined by the relation $l = r$. Therefore Y_e and B_e are in one-to-one correspondence, and the bijection $(l = r, a) \mapsto (l = r, (f, a))$ induces an isomorphism between $(\mathrm{star}_e^{\bowtie})^{ab}$ and F_e as abelian groups. However, since the $\mathfrak{L}(M)$–action on a basis element $(l = r, (f, s)) \in B_e$ is given by $(l = r, (f, s)) \lhd (e, n) = (l = r, (f, sn)$, this isomorphism is an $\mathfrak{L}(M)$–map. $\square$

We recall that, in the fundamental groupoid $\Pi(\mathrm{Sq}(\mathcal{P}), \mathcal{T})$, an element is a homotopy class of sequence of edges $(p, l = r, q)$ (and their inverses) that forms a path in the Squier complex $\mathrm{Sq}(\mathcal{P}, \mathcal{T})$. Such a path starting at $e \in \mathcal{T}$ ends at an element u such that $u\psi = e\psi$, and since the congruence induced by ψ is generated by the relations $l = r$, any such u occurs at the end of a path. Hence the image of the restriction of the range map

$$\mathbf{r} : \mathrm{star}_e^{\bowtie} \to \mathcal{T}$$

is the subgroup U_e of $\mathcal{T}$, whose abelianisation U_e^{ab} is a component of the relation module for $\mathcal{P}$ by Corollary 6.2.19. We therefore have a crossed module of groupoids

$$S^{\bowtie} \longrightarrow U \rightrightarrows E(M)\,. \tag{7.6}$$

We wish to use this crossed module to understand more about the structure of the relation module $\mathcal{U} = \bigsqcup_{e \in E(M)} U_e^{ab}$. To this end, we shall show that (7.6) is a free crossed U–module. We shall generalise a result of Ellis and Porter [12], who show for crossed modules of groups that, given a free crossed G-module $\partial : C \to G$ on R

with $\upsilon : R \to C$ and with $N = \operatorname{im}\partial$, then $C \xrightarrow{\partial} N$ is a free crossed module on $R \times T$ with $\upsilon' : (r,t) \mapsto t^{-1}\upsilon(r)t$, where T is a transversal for N in G.

Once again we shall use trace cosets, as in Definition 4.3.2.

7.2.10 Proposition. *Let $\mathcal{C} \xrightarrow{\partial} \mathcal{G} \rightrightarrows E$ be a free crossed module of groupoids with basis $\omega : R \to \mathcal{G}$, and let N be the image of ∂. Let Z be a transversal to the trace cosets of N in $\mathcal{G}$. Then the crossed module $\mathcal{C} \xrightarrow{\partial} N \rightrightarrows E$ is free with basis $\overline{\omega} : (R \between Z) \to N$, where*

$$(R \between Z) = \bigsqcup_{e \in E} (R \between Z)_e = \bigsqcup_{e \in E} \{(r,z) \in R \times Z : r\omega = zz^{-1}\,, z^{-1}z = e\}$$

and $\overline{\omega} : (r,z) \mapsto z^{-1}(r\omega)z$.

Proof. Suppose that $\mathcal{K} \xrightarrow{\partial} N \rightrightarrows E$ is a crossed module and that we have a function $\nu : (R \between Z) \to \mathcal{K}$ such that $\overline{\omega} = \nu\partial$. Recalling the construction of $\mathcal{C}$ from Proposition 7.2.1 and Corollary 7.2.4, for each $e \in E$ we have a free group F_e with basis

$$(R \between \mathcal{G})_e = \{(r,g) : (r\omega)\mathbf{d} = gg^{-1} = (r\omega)\mathbf{r}\,, g^{-1}g = e\}\,.$$

For $g \in G$ we may write $g = z_g n_g$ for unique $n_g \in N$ and $z_g \in Z$. We now define a group homomorphism $\phi_e : F_e \to K_e$ by $\phi_e : (r,g) \mapsto ((r,z_g)\nu)^{n_g}$.

Two basis elements (r,g) and (s,h) of F_e yield a Peiffer element

$$\langle (r,g),(s,h)\rangle = (r,g)^{-1}(s,h)^{-1}(r,g)(s,hg^{-1}(r\omega)g) \in F_e\,. \tag{7.7}$$

Now $hg^{-1}(r\omega)g = z_h(n_h g^{-1}(r\omega)g)$ with $n_h g^{-1}(r\omega)g \in N$, and so

$$\begin{aligned}(s,hg^{-1}(r\omega)g)\phi_e &= ((s,z_h)\nu)^{n_h g^{-1}(r\omega)g}\\ &= (((s,z_h)\nu)^{n_h})^{g^{-1}(r\omega)g}\\ &= ((s,h)\phi_e)^{g^{-1}(r\omega)g}.\end{aligned}$$

Now

$$
\begin{aligned}
(r,g)\phi_e\partial &= ([(r,z_g)\nu]^{n_g})\partial \\
&= n_g^{-1}((r,z_g)\nu\partial)n_g \\
&= n_g^{-1}(r,z_g)\overline{\omega}n_g \\
&= n_g^{-1}z_g^{-1}(r\omega)z_g n_g \\
&= g^{-1}(r\omega)g\,.
\end{aligned}
$$

Hence

$$
\begin{aligned}
(s,hg^{-1}(r\omega)g)\phi_e &= ((s,h)\phi_e)^{g^{-1}(r\omega)g} \\
&= ((s,h)\phi_e)^{(r,g)\phi_e\partial} \\
&= (r,g)^{-1}\phi_e(s,h)\phi_e(r,g)\phi_e
\end{aligned}
$$

Comparing with (7.7) we see that each Peiffer element $\langle (r,g),(s,h)\rangle$ is in the kernel of ϕ_e , and since the Peiffer subgroup P_e is normally generated by such elements, we have $\mathcal{P}_e \subseteq \ker\phi_e$, and so we obtain an induced homomorphism $C_e \to K_e$. It is then easy to see that this gives the morphism of crossed N–modules required by the universal property of a free crossed module. $\Box$

7.2.3 A Presentation for the Relation Module

From an inverse monoid presentation $\mathcal{P} = [X : R]$ of an inverse monoid M we have now constructed a free crossed module $S^{\bowtie} \xrightarrow{\partial} U \rightrightarrows E$ as in (7.6), and for each $e \in E$ we have a crossed module of groups $\mathrm{star}_e^{\bowtie} \xrightarrow{\partial} U_e$. Since U_e is the vertex set of the component of $\mathrm{Sq}(\mathcal{P})$ containing e, the map $\partial : \mathrm{star}_e^{\bowtie} \to U_e$ is surjective. We define

$$
\pi_1^{\bowtie}(\mathrm{Sq}(\mathcal{P}),e) = \{\alpha \in \mathrm{star}_e^{\bowtie} : \alpha\mathbf{r} = e\}\,,
$$

and abbreviate this to $\pi_e^{\bowtie}$. By Propositions 3.3.7 and 7.1.1, $\pi_e^{\bowtie}$ is abelian and so we have a short exact sequence of groups

$$0 \to \pi_e^{\bowtie} \to \mathrm{star}_e^{\bowtie} \to U_e \to 1\,, \tag{7.8}$$

7.2.11 Lemma. *Each group U_e is free, and so the sequence splits and, as groups, $\mathrm{star}_e^{\bowtie}$ and $\pi_e^{\bowtie} \times U_e$ are isomorphic.*

Proof. The group U_e is a subgroup of $\mathcal{T}(X, M)$ and the maximum group image map $\sigma_M : \mathcal{T}(X, M) \to F(X)$ is idempotent pure. Its restriction $\sigma_m : U_e \to F(X)$ therefore has trivial kernel and so is injective. Hence U_e is isomorphic to a subgroup of a free group and is free. As noted in section 2.1, U_e acts trivially on $\pi_e^{\bowtie}$ and so the splitting of the sequence (7.8) induces an isomorphism $\mathrm{star}_e^{\bowtie} \cong \pi_e^{\bowtie} \times U_e$. □

7.2.12 Theorem. *Let $\mathcal{P}$ be an inverse monoid presentation of an inverse monoid M. There exists a short exact sequence of $\mathfrak{L}(M)$–modules*

$$0 \to \bigsqcup_{e \in E(M)} \pi_e^{\bowtie} \to (S^{\bowtie})^{ab} \xrightarrow{\overline{\partial}} \mathcal{U} \to 0 \tag{7.9}$$

in which $(S^{\bowtie})^{ab}$ is a free $\mathfrak{L}(M)$–module, $\overline{\partial}$ is induced by $S^{\bowtie} \xrightarrow{\partial} U$ and $\mathcal{U}$ is isomorphic to the relation module for $\mathcal{P}$.

Proof. The M–module structure on $\bigsqcup_{e \in E(M)} \pi_e^{\bowtie}$ is given by Proposition 3.3.9, that on $(S^{\bowtie})^{ab}$ by Proposition 7.2.9, and that on $\mathcal{U}$ by Lemma 6.2.12. Lemma 7.2.11 gives us, for each $e \in E(M)$, a short exact sequence of abelian groups

$$0 \to \pi_e^{\bowtie} \to \mathrm{star}_e^{\bowtie} \xrightarrow{\overline{\partial}} U_e^{ab} \to 0$$

and these assemble into the sequence (7.9). It remains to check that $\overline{\partial}$ is then a map of M–modules.

Let $\alpha \in \mathrm{star}_e^{\bowtie}$ with image $\overline{\alpha} \in (S^{\bowtie})^{ab}$, and let $m \in M$ with $mm^{-1} = e$. Then the action of m on $\overline{\alpha}$ is defined by lifting m to $\mathcal{T}(X, M)$ and acting on α in the crossed module $S^{\bowtie} \xrightarrow{\partial} \mathcal{T}$:

$$\overline{\alpha} \triangleleft m = \overline{t^{-1} \triangleright \alpha \triangleleft t}$$

where $t\psi = m$. Hence

$$(\overline{\alpha} \triangleleft m)\overline{\partial} = \overline{(t^{-1} \triangleright \alpha \triangleleft t)\mathbf{r}} = \overline{t^{-1}(\alpha\mathbf{r})t} \in U^{ab}_{m^{-1}m} .$$

But $\overline{\alpha}\,\overline{\partial} = \overline{\alpha\mathbf{r}}$ and $\overline{\alpha\mathbf{r}} \triangleleft m = \overline{t^{-1}(\alpha\mathbf{r})t}$. $\square$

Chapter 8

Further Examples

8.1 Free Semilattices

The inverse monoid presentation $\mathcal{P} = [x : x = x^2]$ presents the two-element semilattice $E = \{1, e\}$. The trace of the congruence associated to the presentation map $\theta : \text{FIM}(x) \to E$ has two classes, $\{1\}$ and $E(\text{FIM}(X)) \setminus \{1\}$ and it follows from the factorisation of the presentation map (see Corollary 6.2.10) and the definition of τ (see Proposition 4.3.4) that $\mathcal{T}(x, E)$ is the disjoint union of groups $\{1\} \sqcup F(x)$, with $\psi : \mathcal{T}(x, E) \to E$ mapping $1 \mapsto 1$ and $F(x) \to \{e\}$. The relation module is then equal to $\mathcal{T}(x, E)$ (with $F(x) \cong \mathbb{Z}$ of course).

Following the recipe (and using the notation) of section 7.2.2, we have $R_1 = \emptyset$ and $R_e = \{(x = x^2, e)\}$, with $\omega : (x = x^2, e) \mapsto x$. The free crossed $\mathcal{T}$–module $\mathcal{C} \xrightarrow{\partial} \mathcal{T}(x, E) \rightrightarrows E$ is the union of the trivial crossed module $\{1\} \to \{1\}$ with the crossed module of groups $\partial : C \to F(x)$ on the single-element basis R_e. As a group, C is isomorphic to $F(x)$ and is a crossed $F(x)$–module with trivial action (the conjugation action in the abelian group $F(x)$). This can be seen from the construction of free crossed modules described in section 2.1.1 but also follows from results in chapter 7. It is clear that 1 is an isolated vertex in the Squier complex

Sq($\mathcal{P}$), and by Lemma 7.1.4, the group $(\text{star}^{\bowtie}(e), *)$ is generated by the set

$$\mathcal{S}_e^{\bowtie} = \{(x^{-q-1}, x = x^2, x^q) : q \in \mathbb{Z}\}\,,$$

subject to the defining relations given in Proposition 7.1.5. Writing $\lambda_q = (x^{-q-1}, x = x^2, x^q)$, these relations are

$$\lambda_{v+u+1} * \lambda_u = \lambda_u * \lambda_{v+u+2} \tag{8.1}$$

where $u, v \in \mathbb{Z}$. Setting $v = -1$ we obtain $\lambda_u = \lambda_{u+1}$ for all $u \in \mathbb{Z}$. Hence, for all $p \in \mathbb{Z}$ we have $\lambda_p = \lambda_0$, and once these identifications are made, the relations (8.1) are trivial and so $\text{star}^{\bowtie}(e)$ is freely generated by λ_0. Hence, for each $q \in \mathbb{Z}$, there is a unique homotopy class of paths in $\text{star}^{\bowtie}(e)$ from e to x^q, and $\pi_1^{\bowtie}(e)$ is trivial. The short exact sequence (7.9) reduces to an isomorphism $(S^{\bowtie})^{ab} \to \mathcal{T}(x, E)$.

8.2 Polycyclic Monoids

The *polycyclic monoid* P_n, introduced by Nivat and Perrot in [30], is the inverse monoid with zero presented by

$$[a_1, \ldots, a_n : a_i a_i^{-1} = 1, a_j a_k^{-1} = 0\ (j \neq k)]$$

and so presented as an inverse monoid by the presentation

$$\mathcal{P}_n = [a_1, \ldots, a_n, z : a_i a_i^{-1} = 1, a_j a_k^{-1} = z\,(j \neq k), z^2 = z, a_i z = z = z a_i]\,.$$

We set $A = \{a_1, \ldots, a_n\}$ and $A^z = A \sqcup \{z\}$. Non-zero elements of P_n are uniquely represented in the form $u^{-1}v$ for $u, v \in A^*$ and multiplied as follows:

$$(p^{-1}q)(u^{-1}v) = \begin{cases} p^{-1}tv & \text{if } q = tu \text{ for some } t \in A^*\,, \\ p^{-1}t^{-1}v & \text{if } u = tq \text{ for some } t \in A^*\,, \\ 0 & \text{otherwise.} \end{cases}$$

The non-zero idempotents of P_n are the elements of the form $u^{-1}u$ for $u \in A^*$, and elements $p^{-1}q$ and $u^{-1}v$ are $\mathcal{L}$–related if and only if $q = v$. The natural partial order on the non-zero idempotents is induced by the suffix ordering on A^*: we have $q^{-1}q \leqslant u^{-1}u$ if and only if u is a suffix of q, if and only if $q = tu$ for some $t \in A^*$. Left multiplication by an element $a \in A$ preserves the $\mathcal{L}$-class of $p^{-1}q$ if and only if p^{-1} begins with a^{-1}, and in this case, if $p = ta$ then $a(p^{-1}q) = t^{-1}q$. Hence a typical a–labelled edge in the Schützenberger graph $\mathrm{Sch}^{\mathcal{L}}(P_n, A^z, q^{-1}q)$ starts at $a^{-1}t^{-1}q$ and ends at $t^{-1}q$. Identifying a vertex $p^{-1}q$ with the word $p \in A^*$, we see that $\mathrm{Sch}^{\mathcal{L}}(P_n, A^z, q^{-1}q)$ is isomorphic to the A–regular tree rooted at vertex q. Hence the relation module at every non-zero idempotent is zero, and at the idempotent 0 is free abelian of rank $n+1$, since $\mathrm{Sch}^{\mathcal{L}}(P_n, A^z, 0)$ consists of a loop at 0 for each element of A^z.

The $E(P_n)$–set $\mathcal{Z}$ of Proposition 7.2.9 is given by

$$\begin{aligned} &Z_1 = \{a_i a_i^{-1} = 1 : 1 \leqslant i \leqslant n\}\,, \\ &Z_{q^{-1}q} = \emptyset\,, \\ &Z_0 = \{a_j a_k^{-1} = z, z = z^2, a_i z = z, z a_i = z : 1 \leqslant i, j, k \leqslant n, j \neq k\}\,. \end{aligned}$$

In the free $\mathfrak{L}(P_n)$–module $\mathcal{F}$ on $\mathcal{Z}$, F_1 is free abelian on the basis

$$B_1 = \{(a_i a_i^{-1} = 1, (1, p^{-1})) : 1 \leqslant i \leqslant n, \quad p \in A^*\}$$

which we rewrite as the set $B_1 = \{b_1^{p^{-1}}, \ldots, b_n^{p^{-1}} : p \in A^*\}$; the group $F_{q^{-1}q}$ is free abelian on the basis

$$B_{q^{-1}q} = \{(a_i a_i^{-1} = 1, (1, p^{-1}q)) : 1 \leqslant i \leqslant n, \quad p \in A^*\}$$

which we rewrite as the set $B_{q^{-1}q} = \{b_1^{p^{-1}q}, \ldots, b_n^{p^{-1}q} : p \in A^*\}$; and the group F_0 is free abelian on the basis $B_0 \sqcup Y_0$, where $B_0 = \{b_1^0, \ldots, b_n^0\}$ and

$$Y_0 = \{(a_j a_k^{-1} = z, (0,0)) : j \neq k\} \cup \{(z = z^2, (0,0))\} \cup \{(a_i z = z, (0,0)) : 1 \leqslant i \leqslant n\} \\ \cup \{(z a_i = z, (0,0)) : 1 \leqslant i \leqslant n\}.$$

We rewrite Y_0 as

$$Y_0 = \{y_{jk} : j \neq k\} \cup \{y_z\} \cup \{x_i : 1 \leqslant i \leqslant n\} \cup \{y_i : 1 \leqslant i \leqslant n\}.$$

The action of $\mathfrak{L}(P_n)$ is then

$$b_i^{p^{-1}} \lhd (1, u^{-1}v) = b_i^{p^{-1}u^{-1}v} \in F_{v^{-1}v},$$
$$b_i^{p^{-1}q} \lhd (q^{-1}q, u^{-1}v) = b_i^{p^{-1}t^{-1}v} \in F_{v^{-1}v} \quad \text{where } u = tq,$$

and the basis elements in Y_0 are fixed by the action of $(0,0) \in \mathfrak{L}(P_n)$.

For $q \in A^*$, a basis element $b_j^{p^{-1}q} \in F_{q^{-1}q}$ corresponds in $(S^{\bowtie})^{ab}$, under the isomorphism of Proposition 7.2.9, to $(q^{-1}p, a_j a_j^{-1}, p^{-1}q)$ and this edge $(q^{-1}p, a_j a_j^{-1}, p^{-1}q)$ is a loop in $\mathrm{Sq}(\mathcal{P})$ at the vertex $q^{-1}q$. Hence the map $\mathbf{r} : \mathrm{star}^{\bowtie}(q^{-1}q) \to \mathcal{T}$ is zero, the relation module at $q^{-1}q$ is $\{0\}$ as already noted, and the module of identities $\pi_1(\mathrm{Sq}(\mathcal{P}), q^{-1}q)$ is isomorphic to $\mathrm{star}^{\bowtie}(q^{-1}q)$.

However, at the idempotent 0, the relation module is free abelian of rank $n+1$ and $\mathrm{star}^{\bowtie}(0)$ is free abelian of rank $n(n-1) + 3n + 1 = (n+1)^2$. Therefore, the module of identities at 0 is free abelian of rank $(n+1)^2 - (n+1) = n(n+1)$.

8.3 The Free Inverse Monoid Mc_2 on Two Commuting Generators

The free inverse semigroup on two commuting generators was introduced by McAlister and McFadden [27] as an example of a universal construction of inverse semigroups from semigroups. We consider here the *inverse monoid on two commuting generators*, which is presented by

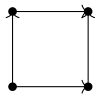
$$\mathcal{P} = [x, y : xy = yx] .$$

We follow [26] and denote this inverse monoid by Mc_2. McAlister and McFadden show [27, Proposition 2.5] that Mc_2 is E–unitary, and as explained in [26, Example 4.2], it follows that elements of Mc_2 can be represented by certain pointed patterns in the Cayley graph $\mathrm{Cay}(F(x,y)^{ab}, \{x,y\})$ of the free abelian group of rank two.

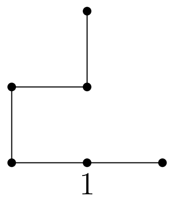
Specifically, let $A = \{x, x^{-1}, y, y^{-1}\}$ and, for $w \in A^*$, let P_w be the pattern spanned by w in $\mathrm{Cay}(F(x,y)^{ab}, \{x,y\})$. Let $\overline{w}$ be the image of w in $F(x,y)^{ab}$. We now complete P_w as follows. If $x^i y^j$ and $x^k y^l$ are vertices of P_w, we add all vertices of the form $x^r y^s$ with $i \leqslant r \leqslant k$ and $j \leqslant s \leqslant l$, and all edges between such vertices. The result is the *box completion* $P_w^{\square}$ of P_w, and we say that $P_w^{\square}$ is *complete*. The pointed pattern $(P_w^{\square}, \overline{w})$ is then a canonical representative of w, in the sense that words $u, v \in A^*$ represent the same element of Mc_2 if and only if $P_u^{\square} = P_v^{\square}$ and $\overline{u} = \overline{v}$.

8.3.1 Example. The box completion P of a pattern P is obtained by repeatedly completing either of the subgraphs

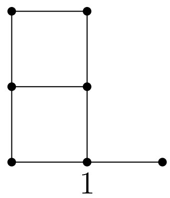
to the box



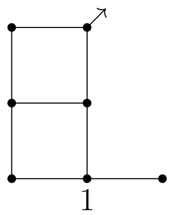
If $w = xx^{-2}yxy \in A^*$ then P_w is the pattern



and its box completion is the pattern



Since $\overline{w} = y^2$, the pointed pattern representing $w \in \mathrm{Mc}_2$ is



with the distinguished vertex y^2 indicated by the small arrow.

Multiplication in Mc_2 is obtained by the multiplication and box completion of pointed patterns: if P and Q are complete patterns then

$$(P, u) \circ (Q, v) = ((P \cup u \cdot Q)^{\square}, uv)\,.$$

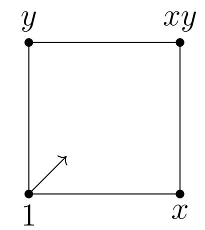
The idempotents in Mc_2 are the complete pointed patterns $(P, 1)$, and since the complete pointed pattern (Q, v) has $(Q, v)^{-1} = (v^{-1}Q, v^{-1})$, and $(Q, v)^{-1}(Q, v) = (v^{-1}Q, 1)$, we see that $(Q, v)\ \mathcal{L}\ (P, 1)$ if and only if $Q = vP$. Hence the $\mathcal{L}$–class of $(P, 1)$ is

$$L_{(P,1)} = \{(uP, u) : u^{-1} \in P\}\,.$$

In the Schützenberger graph $\mathrm{Sch}^{\mathcal{L}}(\mathrm{Mc}_2, \{x, y\}, (P, 1))$ there is an edge labelled x from (uP, u) to (xuP, xu) if and only if $u^{-1}, u^{-1}x^{-1} \in P$, that is, if and only if there is an edge labelled x from $u^{-1}x^{-1}$ to u^{-1} in P. Therefore the mapping $(uP, u) \mapsto u^{-1}$ induces an edge-reversing isomorphism of labelled graphs

$$\mathrm{Sch}^{\mathcal{L}}(\mathrm{Mc}_2, \{x, y\}, (P, 1)) \to P \, .$$

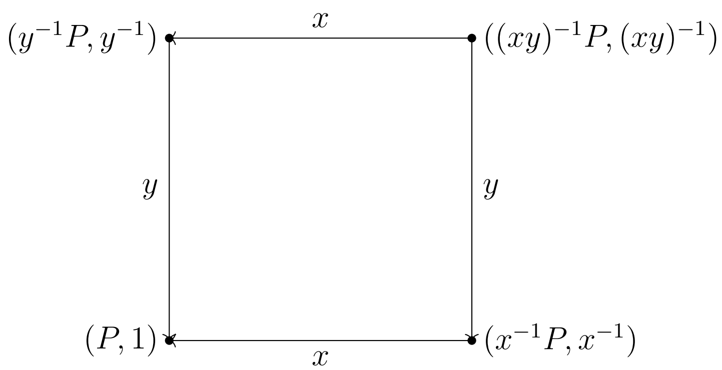
8.3.2 Example. Let $e = xyy^{-1}x^{-1}$, represented by the complete pattern $(P, 1)$:



The $\mathcal{L}$-class of e, and therefore the vertex set of $Sch^{\mathcal{L}}(\mathrm{Mc}_2, \{x, y\}, e)$ is

$$L_e = \{(P, 1), (x^{-1}P, x^{-1}), (y^{-1}P, y^{-1}), ((xy)^{-1}P, (xy)^{-1})\} \, .$$

The edges of $Sch^{\mathcal{L}}(\mathrm{Mc}_2, \{x, y\}, e)$ replicate those in P, and so $Sch^{\mathcal{L}}(\mathrm{Mc}_2, \{x, y\}, e)$ is:



Let K be any complete pattern in $\mathrm{Cay}(F(x, y)^{ab}, \{x, y\})$. The relation module $\mathcal{M}$ at the idempotent $(K, 1) \in \mathrm{Mc}_2$ is then the first homology $H_1(K)$ of the pattern K. Under the isomorphism $\mathrm{Sch}^{\mathcal{L}}(\mathrm{Mc}_2, \{x, y\}, (K, 1)) \to K$ the action of $(K, (Q, v)) \in \mathfrak{L}(\mathrm{Mc}_2)$ (where $K \subseteq Q$) on $\mathrm{Sch}^{\mathcal{L}}(\mathrm{Mc}_2, \{x, y\}, (K, 1))$ is given by the composition of left multiplication by v^{-1} and embedding the pattern $v^{-1}K$ into $v^{-1}Q$:

$$(K, (Q, v)) : K \to v^{-1}K \hookrightarrow v^{-1}Q\,.$$

The induced map on homology $H_1(K) \to H_1(v^{-1}Q)$ is therefore injective.

Let B be the complete pattern consisting of the square with boundary $y^{-1}x^{-1}yx$ (starting from 1) and let $\mathcal{Z}$ be the $E(\mathrm{Mc}_2)$–set with $Z_B = \{*\}$ a singleton, and $Z_Q =$

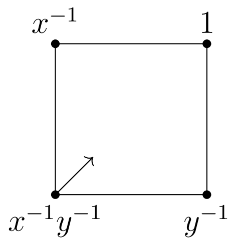


Figure 8.1: The pattern B

$\emptyset$ for $B \neq Q$. Then the free $\mathfrak{L}(\mathrm{Mc}_2)$–module $\mathcal{F}$ on $\mathcal{Z}$ (see section 1.6) has, for a complete pattern K in $\mathrm{Cay}(F(x,y)^{ab}, \{x,y\})$, the group F_K free abelian on the basis $\{(B, (wK, w)) : B \subseteq wK, w^{-1} \in K\}$, and since B is fixed, this basis is in one-to-one correspondence with $\{(wK, w) : w^{-1}B \subseteq K\}$, where $w^{-1}B$ is the complete pattern consisting of the square with boundary $y^{-1}x^{-1}yx$ starting from w^{-1}. Therefore F_K is free abelian on a basis that is in one-to-one correspondence with the unit squares in the pattern K. The boundaries of these squares then give a cycle basis for the homology $H_1(K)$, so that $H_1(K)$ and F_K are isomorphic as abelian groups.

The $\mathfrak{L}(\mathrm{Mc}_2)$–action of $(K, (Q, v))$ (with $K \subseteq Q$) on a basis element (wK, w) is then given by

$$(wK, w) \lhd (K, (Q, v)) = (wQ, wv)\,.$$

Now (wK, w) corresponds to the square $w^{-1}B$ in K with boundary $y^{-1}x^{-1}yx$ starting from w^{-1}, and (wQ, wv) corresponds to the square $v^{-1}w^{-1}B$ in $(wv)^{-1}wQ = v^{-1}Q$ with boundary $y^{-1}x^{-1}yx$ starting from $(wv)^{-1} = v^{-1}w^{-1}$. The action of $(K, (Q, v))$ on $H_1(K)$ first translates by v^{-1} on the left, then includes $H_1(v^{-1}K)$ into

$H_1(v^{-1}Q)$: in particular, it maps the basis element $w^{-1}B \subseteq K$ to $v^{-1}w^{-1}B \subseteq v^{-1}Q$. Hence the isomorphisms $F_K \to H_1(K)$ combine to give a $\mathfrak{L}(\mathrm{Mc}_2)$–module isomorphism $\mathcal{F} \to \mathcal{M}$, and the relation module of the presentation $[x, y : xy = yx]$ of Mc_2 is a free $\mathfrak{L}(\mathrm{Mc}_2)$–module.

Now we can consider using the factorised presentation map $\theta = \tau\psi$. First we have to construct the inverse monoid $\mathcal{T}(\{x, y\}, \mathrm{Mc}_2)$, where we have

$$\mathrm{FIM}(x, y) \xrightarrow{\tau} \mathcal{T}(\{x, y\}, \mathrm{Mc}_2) \xrightarrow{\psi} \mathrm{Mc}_2$$

Let $(P, u), (Q, v) \in \mathrm{FIM}(x, y)$ then by Lemma 4.3.1 the relation τ is characterised as follows:

$(P, u)\tau(Q, v) \Leftrightarrow$

1. $\exists(K, 1) \in E(\mathrm{FIM}(x, y))$ such that $(P, u)(K, 1) = (Q, v)(K, 1)$, and
2. $(P, u)^{-1}(P, u)\theta(K, 1)\theta(Q, v)^{-1}(Q, v)$

Firstly for $P \subseteq Cay(F(x, y), \{x, y\})$ we have to define what we mean by $P^{\square}$. The abelianisation $F(x, y) \to F^{ab}(x, y)$ induces a graph map

$$\alpha : Cay(F(x, y), \{x, y\}) \to Cay(F^{ab}(x, y), \{x, y\})$$

so, for a pattern $P \subseteq Cay(F(x, y), \{x, y\})$ we let $P^{\square}$ denote the box completion of $P\alpha$.

Proposition. For $(P, u), (Q, v) \in \mathrm{FIM}(x, y)$, $(P, u)\tau(Q, v) \Leftrightarrow P^{\square} = Q^{\square}$ and $u = v$.

Proof. Assume that $(P, u)\tau(Q, v)$, then since $(P, u)(K, 1) = (P \cup uK, u)$ and $(Q, v)(K, 1) = (Q \cup vK, v)$ we deduce that $u = v$ and that $P \cup uK = Q \cup uK$. Now $(P, u)^{-1}(P, u) = (u^{-1}P, 1)$ is θ-related to $(Q, u)^{-1}(Q, u) = (u^{-1}Q, 1)$ if and only if $(u^{-1}P, 1)$ and $(u^{-1}Q, 1)$ represent the same idempotent in Mc_2, and so $(u^{-1}P)^{\square} = (u^{-1}Q)^{\square}$. Hence $u^{-1}P^{\square} = u^{-1}Q^{\square}$, and $P^{\square} = Q^{\square}$.

Now assume that $P^\square = Q^\square$ and $u = v$. Let $K = u^{-1}(P \cup Q)$, then given that $u = v$ we have that $(P, u)(K, 1) = (Q, u)(K, 1)$. We have that $(P, u)^{-1}(P, u) = (u^{-1}P, 1)$ represents the idempotent $(\overline{u}^{-1}P^\square, 1) \in \mathrm{Mc}_2$. Since $P^\square = Q^\square$ this is the same as the idempotent represented by $(Q, u)^{-1}(Q, u)$ in Mc_2, so $(P, u)^{-1}(P, u)\theta(Q, v)^{-1}(Q, v)$. $(K, 1) = (u^{-1}(P \cup Q), 1)$ represents the idempotent $((\overline{u}^{-1}(P \cup Q))^\square, 1)$ in Mc_2.

$$\begin{aligned}(P \cup Q)^\square &= (P^\square \cup Q^\square)^\square \\ &= (P^\square \cup P^\square)^\square \\ &= (P^\square)^\square \\ &= P^\square\end{aligned}$$

Thus $(K, 1)$ represents the idempotent $(\overline{u}^{-1}P^\square, 1)$, and so

$$(P, u)^{-1}(P, u)\theta(K, 1)\theta(Q, v)^{-1}(Q, v)$$

$\square$

So $(P, u)\tau(Q, v) \Leftrightarrow P^\square = Q^\square$ and $u = v$. Thus $\mathcal{T}(\{x, y\}, \mathrm{Mc}_2) = \{(L, u) : L$ is a closed pattern in $Cay(F^{ab}(x, y), \{x, y\}), u \in F(x, y)$ readable in L from $1\}$
Recall that $(P, u)\theta(Q, v) \Leftrightarrow P^\square = Q^\square$ and $\overline{u} = \overline{v}$, with $\overline{w}$ the image of $w \in F(x, y)$ in $F(x, y)^{ab}$. So we can define

$$\begin{aligned}\psi : \mathcal{T}(\{x, y\}, \mathrm{Mc}_2) &\to \mathrm{Mc}_2 \text{ by} \\ (P^\square, u) &\mapsto (P^\square, \overline{u})\end{aligned}$$

Consider $\ker\psi$, for a complete pattern K, we have that $(K, u) \mapsto (K, 1)$, exactly when $u \in \ker\phi$ where $\phi : F(x, y) \to F^{ab}(x, y)$. So we have $\ker\psi = \{(L, u) : L$ a closed pattern in $Cay(F^{ab}(x,y),\{x,y\}), u \in F(x,y)$ readable in L as a circuit at $1\}$. For a given closed pattern K the elements $u \in F(x,y)$ readable as a circuit as 1 in K form a subgroup, C_K, of $F(x,y)$. It is clear that $1 \in C_K$ represented by the empty circuit, for $u, v \in C_K$ we can read u then v and we still have a closed circuit at 1, and given $u \in C_K$, let $u^{-1} \in C_K$ be u read backwards. Then C_K is isomorphic to the fundamental group of K. Then the relation module for $\mathcal{P}$ is the collection of abelianisations of the fundamental groups of closed pattern in $Cay(F^{ab}(x,y),\{x,y\})$ which is exactly the first homology groups of the closed patterns as we saw above.

8.4 The Bicyclic Monoid

Let $\mathcal{P} = [x, 1 = xx^{-1}]$, then $\mathcal{P}$ presents the bicyclic monoid, B. Firstly it is useful to determine the form of the elements of B. Elements of $\mathrm{FIM}(x)$ can be expressed in Schlieblich normal form, that gives us any word $w \in \mathrm{FIM}(x)$ as $x^{-p}x^{p}x^{q}x^{-q}x^{r}$ with $p, q \geqslant 0$ and $-p \leqslant r \leqslant q$. If we then consider the presentation map $\theta : \mathrm{FIM}(x) \to B$, we can see that this would map $x^{-p}x^{p}x^{q}x^{-q}x^{r} \mapsto x^{-p}x^{p}x^{r} = x^{-p}x^{p+r}$, and thus elements $b \in B$ are can all be expressed in the form $x^{-p}x^{t}$ for $p, t \geqslant 0$ just as in Example 1.7.3.

We know from Corollary 6.2.19 that the relation module of $\mathcal{P}$ is equal to the kernel of the mapping $\mathcal{D}_\psi \to IB$, where we can factorise the presentation map θ as in Lemma 4.3.4 as follows

$$\begin{array}{ccc} \mathrm{FIM}(x) & \xrightarrow{\tau} & \mathrm{FIM}(x)/\tau \\ & \searrow^{\theta} & \downarrow \psi \\ & & \mathcal{B} \end{array}$$

Consider the maps θ and τ as congruences on $\mathrm{FIM}(x)$, τ is the minimum congruence on $\mathrm{FIM}(x)$ with trace equal to the trace of θ, defined on $u, v \in \mathrm{FIM}(x)$ by:

$$u\tau b \Leftrightarrow \exists e \in E(\mathrm{FIM}(x)) : ue = ve \,\text{and}\, u^{-1}u\theta e\theta v^{-1}v$$

Let $u = x^{-a}x^{a}x^{b}x^{-b}x^{c}$, and $v = x^{-p}x^{p}x^{q}x^{-q}x^{r}$, then

$$
\begin{aligned}
(u^{-1}u)\theta &= (x^{-c}x^{b}x^{-b}x^{-a}x^{a}x^{-a}x^{a}x^{b}x^{-b}x^{c})\theta \\
&= (x^{-c}x^{-a}x^{a}x^{b}x^{-b}x^{c})\theta \\
&= x^{-c}x^{-a}x^{a}x^{c} \\
&\text{similarly} \\
(v^{-1}v)\theta &= x^{-r}x^{-p}x^{p}x^{r}
\end{aligned}
$$

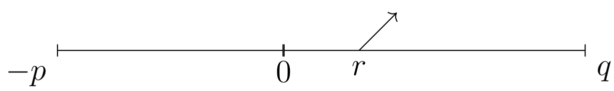
We have $u^{-1}u\theta v^{-1}v \Leftrightarrow a + c = p + r$, and since idempotents in FIM(x) are of the form $e = x^{-k}x^{k}x^{l}x^{-l}$, for $u^{-1}u\theta e\theta v^{-1}v$, we set $k = a + c = p + r$.

Moving onto the second condition , $ue = ve$, to investigate this condition it is convenient to consider the geometric representation if FIM(x), described in section 1.5.

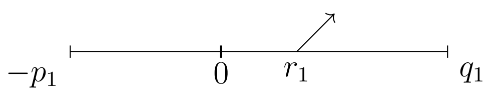
Geometric Representation of FIM(x)**:**

Consider elements of FIM(x) in Schleiblich normal form as above, we can represent this geometrically by considering the route along the number line it would traverse, with $\nearrow$ representing where this terminates.

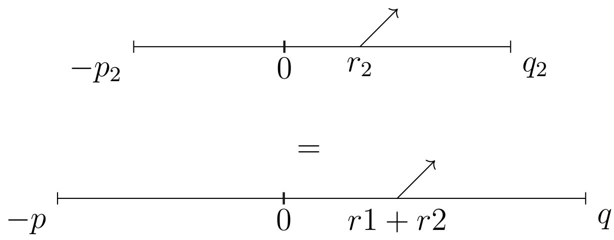
8.4.1 Example. $x^{-p}x^{p}x^{q}x^{-q}x^{r}$ becomes:



We can compose two of these pictures by "attaching" the start, or 0, of the second at the point ext point of the first, giving:



$\circ$



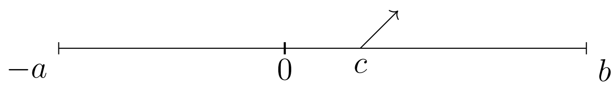
Where $-p = \min\{-p_1, -p_2 + r_1\}$ $q = \max\{q_1, q_2 + r_1\}$.

Recall $u = x^{-a}x^a x^b x^{-b} x^c$, and $v = x^{-p}x^p x^q x^{-q} x^r$, with $e = x^{-k}x^k x^l x^{-l}$, such that $a + c = p + r = k$, then

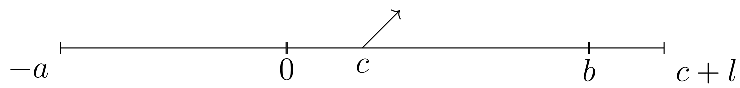
$$\begin{aligned}
ue &= x^{-a}x^a x^b x^{-b} x^c x^{-k} x^k x^l x^{-l} \\
&= x^{-a}x^a x^b x^{-b} x^c x^{-(a+c)} x^{a+c} x^l x^{-l} \\
&= x^{-a}x^a x^b x^{-b} x^c x^{-c} x^{-a} x^a x^c x^l x^{-l} \\
&= x^{-a}x^a x^b x^{-b} x^c x^l x^{-l} \\
&\text{similarly} \\
ve &= x^{-p}x^p x^q x^{-q} x^r x^l x^{-l}
\end{aligned}$$

Consider the geometric representations:

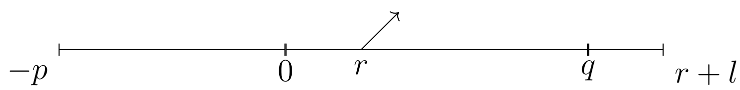
u becomes



and ue becomes



when $c + l > b$, otherwise $ue = u$. Similarly, in the case where $r + l \leqslant r$, $ve = v$, otherwise ve is represented by:



For $ue = ve$ it is clear we must have $a = p$ and $c = r$, then we can choose an l such that $l + c = l + r \geqslant \max(b, q)$. Then $u\tau v$.

Recall the definition of θ as a congruence, $u\theta v \Leftrightarrow u\theta = v\theta$ which is exactly when $a = p$ and $c = r$. Since given such a u and v, we can construct an idempotent $e = x^{-k}x^k x^l x^l$, by setting $k = a + c$, and $l = \max(b, q) - c$, so that $ue = ve$ and $u^{-1}u\theta e\theta v^{-1}v$. Therefore $\theta = \tau$ as a congruence, and $\mathrm{FIM}(x)/\tau = B$, and ψ is then the identity morphism on B.

From Theorem 6.2.11 we know $\mathcal{D}_\theta = \mathcal{D}_\psi$, and we know from Example 6.2.7 that the derivation module of the identity morphism is the augmentation ideal, IB. Therefore the sequence

$$0 \to \mathcal{M}_\theta \to \mathcal{D}_\theta \to IB \to 0$$

becomes

$$0 \to \mathcal{M}_\theta \to IB \to IB \to 0$$

and the relation module $\mathcal{M}_\theta = \ker(\mathcal{D}_\theta \to IB)$ is then trivial, as we saw in Example 6.2.24.

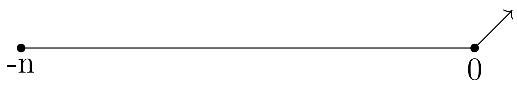
8.5 A Squier Complex Construction

Here we will construct the Squier complex for the presentation $\mathcal{P} = [x; 1 = xx^{-1}]$, presenting the bicyclic monoid as in section 8.4. In this section I will consider elements of B to be of the form $x^{-p}x^p x^r$ with $p \geqslant 0$ and $r \geqslant -p$, this is equivalent to $x^{-p}x^t$ where $t = p + r$, but separation here will be convenient.

We have above in section 8.4 that $\mathcal{T} = B$, so we can define $Sq(\mathcal{P}, \mathcal{T})$ to be the two complex defined as in Definition 7.1.1, with vertex set B. Let's consider the components of $Sq(\mathcal{P}, B)$ containing idempotents, $E(B) = \{x^{-n}x^n | n \geqslant 0\}$, we will construct the component of $Sq(\mathcal{P}, B)$ containing $x^{-n}x^n$, $Sq(\mathcal{P}, B)_{x^{-n}x^n}$.

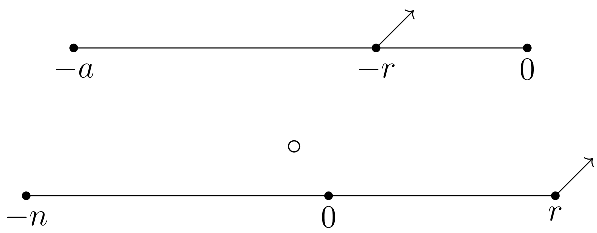
Since two elements $u, v \in B$ are in the same connected component of $Sq(\mathcal{P}, B)$ if and only if $u\psi = v\psi$ and ψ is the identity morphism each connected component contains exactly one vertex.

The edge set of $Sq(\mathcal{P}, B)_{x^{-n}x^n}$, consists of insertions and deletions of xx^{-1}, and thus correspond to factorisations of $x^{-n}x^n$. We can consider these factorisations geometrically. $x^{-n}x^n$ is represented geometrically by:



There are 4 types of factorisation, as follows:

Type 1:



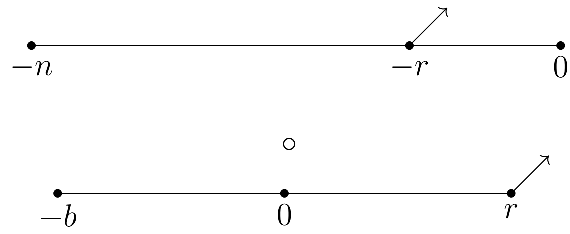
where $0 \leqslant r \leqslant a < n$. Converting to our standard form we get:

$$\begin{aligned}
x^{-a}x^a x^{-r} \cdot x^{-(n-r)}x^{n-r}x^r &= x^{-a}x^a x^{-r} \cdot x^r x^{-r} x^{-(n-r)} x^{n-r} x^r \\
&= x^{-a}x^a x^{-r} \cdot x^r x^{-(n-r)-r} x^{(n-r)+r} \\
&= x^{-a}x^a x^{-r} \cdot x^r x^{-n} x^n
\end{aligned}$$

for a equals	0	1	2	...	n-2	n-1
# choices for r	1	2	3	...	n-1	n

So the number of factorisations of type 1 is $\frac{1}{2}n(n+1)$.

<u>Type 2:</u>



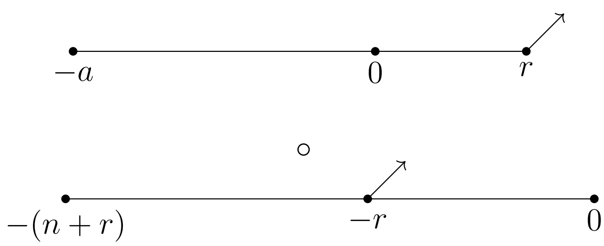
where $0 \leqslant r \leqslant n$ and $0 \leqslant b \leqslant n-r$. Converting to our standard form we get:

$$\begin{aligned} x^{-n}x^{n}x^{-r} \cdot x^{-b}x^{b}x^{r} &= x^{-n}x^{n}x^{-r} \cdot x^{r}x^{-r}x^{-b}x^{b}x^{r} \\ &= x^{-n}x^{n}x^{-r} \cdot x^{r}x^{-(b+r)}x^{b+r} \end{aligned}$$

for r equals	0	1	2	...	n-1	n
# choices for b	n+1	n	n-1	...	2	1

So the number of factorisations of type 2 is $\frac{1}{2}(n+1)(n+2)$.

<u>Type 3:</u>

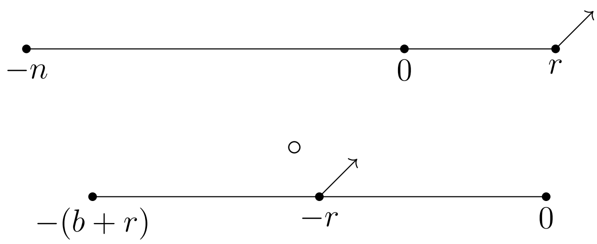


where $0 < r$ and $0 \leqslant a < n$. Converting to our standard form we get:

$$
\begin{aligned}
x^{-a}x^{a}x^{r}\cdot x^{-(n+r)}x^{n+r}x^{-r} &= x^{-a}x^{a}x^{r}\cdot x^{-r}x^{-n}x^{n}x^{r}x^{-r}\\
&= x^{-a}x^{a}x^{r}\cdot x^{-r}x^{-n}x^{n}
\end{aligned}
$$

In this case we now have r unbounded, we have n choices for a, and so we have n infinite families of type 3.

<u>Type 4:</u>

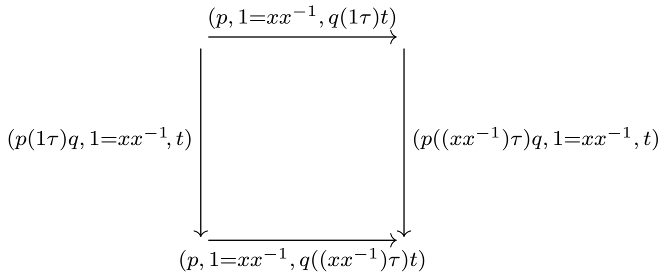


where $0 < r$ and $0 \leqslant b \leqslant n$. Converting to our standard form we get:

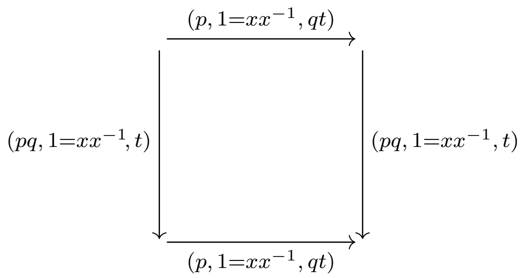
$$
\begin{aligned}
x^{-n}x^{n}x^{r}\cdot x^{-(b+r)}x^{(b+r)}x^{-r} &= x^{-n}x^{n}x^{r}\cdot x^{-r}x^{-b}x^{b}x^{r}x^{-r}\\
&= x^{-n}x^{n}x^{r}\cdot x^{-r}x^{-b}x^{b}
\end{aligned}
$$

Again we have r unbounded, we have $n+1$ choices for b, and so we have $n+1$ infinite families of type 4.

We can now consider how the 2-cells affect these edges. 2-cells correspond to three-way factorisations, $x^{-n}x^{n} = p\cdot q\cdot t$, as follows:



Then we recall that $1\tau = 1 = (xx^{-1})\tau$, and so this becomes:



Which gives us a commutativity relation: $(p,\, 1 = xx^{-1},\, qt) \circ (pq,\, 1 = xx^{-1},\, t) = (pq,\, 1 = xx^{-1},\, t) \circ (p,\, 1 = xx^{-1},\, qt)$.

It remains to check when we have these commutativity relations, and so which of the edges commute. Given two edges $(p, 1 = xx^{-1}, u)$ and $(v, 1 = xx^{-1}, t)$, if we can find $q \in \mathcal{T}$ such that $u = qt$ and $v = pq$, then we can use a 2-cell to show that these two edges commute.

We have four types of edges and so we have ten ways in which to form pairs:

1. **Type 1 & Type 1:**

   Take

   $$(x^{-a}x^{a}x^{-r}, 1 = xx^{-1}, x^{r}x^{-n}x^{n}) \qquad 0 \leqslant r \leqslant a < n$$

   and

   $$(x^{-b}x^{b}x^{-s}, 1 = xx^{-1}, x^{s}x^{-n}x^{n}) \qquad 0 \leqslant s \leqslant b < n$$

   and let $a \leqslant b$, then set $p = x^{-a}x^{a}x^{-r}$, and $t = x^{s}x^{-n}x^{n}$, then for $q =$

$x^r x^{-b} x^b x^{-s}$ we have:

$$pq = x^{-a}x^a x^{-r} \cdot x^r x^{-b} x^b x^{-s} = x^{-b}x^b x^{-s}$$

$$qt = x^r x^{-b} x^b x^{-s} \cdot x^s x^{-n} x^n = x^r x^{-n} x^n$$

as required. Thus edges of type 1 commute with edges of type 1.

2. **Type 1 & Type 2:**

   Take

$$(x^{-a}x^a x^{-r}, 1 = xx^{-1}, x^r x^{-n} x^n) \qquad 0 \leqslant r \leqslant a < n$$

   and

$$(x^{-n}x^n x^{-s}, 1 = xx^{-1}, x^s x^{-(b+s)} x^{b+s}) \qquad 0 \leqslant r \leqslant n,\ 0 \leqslant b \leqslant n - s$$

   then set $p = x^{-a}x^a x^{-r}$ and $t = x^s x^{-(b+s)} x^{b+s}$, then for $q = x^r x^{-n} x^n x^{-s}$ we have:

$$pq = x^{-a}x^a x^{-r} \cdot x^r x^{-n} x^n x^{-s} = x^{-n}x^n x^{-s}$$

$$qt = x^r x^{-n} x^n x^{-s} \cdot x^s x^{-(b+s)} x^{b+s} = x^r x^{-n} x^n$$

   as required. Thus edges of type 1 commute with edges of type 2.

3. **Type 1 & Type 3:**

   Take

$$(x^{-a}x^a x^{-r}, 1 = xx^{-1}, x^r x^{-n} x^n) \qquad 0 \leqslant r \leqslant a < n$$

and

$$(x^{-b}x^{b}x^{s}, 1 = xx^{-1}, x^{-s}x^{-n}x^{n}) \qquad 0 < s,\ 0 \leqslant b < n$$

Here we must consider two cases, $b \geqslant a$ and $b < a$ separately.

When $b \geqslant a$, set $p = x^{-a}x^{a}x^{-r}$, and $t = x^{-s}x^{-n}x^{n}$, then for $q = x^{r}x^{-b}x^{b}x^{s}$ we have:

$$pq = x^{-a}x^{a}x^{-r} \cdot x^{r}x^{-b}x^{b}x^{s} = x^{-b}x^{b}x^{s}$$

$$qt = x^{r}x^{-b}x^{b}x^{s} \cdot x^{-s}x^{-n}x^{n} = x^{r}x^{-n}x^{n}$$

as required, now we must consider the case when $b < a$.

In this case set $p = x^{-b}x^{b}x^{s}$ and $t = x^{r}x^{-n}x^{n}$, now for $q = x^{-s}x^{-a}x^{a}x^{-r}$ we have:

$$pq = x^{-b}x^{b}x^{s} \cdot x^{-s}x^{-a}x^{a}x^{-r} = x^{-a}x^{a}x^{-r}$$

$$qt = x^{-s}x^{-a}x^{a}x^{-r} \cdot x^{r}x^{-n}x^{n} = x^{-s}x^{-n}x^{n}$$

again as required, and so edges of type 1 commute with edges of type 3.

4. **Type 1 & Type 4:**

   Take

$$(x^{-a}x^{a}x^{-r}, 1 = xx^{-1}, x^{r}x^{-n}x^{n}) \qquad 0 \leqslant r \leqslant a < n$$

   and

$$(x^{-n}x^{n}x^{s}, 1 = xx^{-1}, x^{-s}x^{-b}x^{b}) \qquad 0 < s,\ 0 \leqslant b \leqslant n$$

then set $p = x^{-a}x^{a}x^{-r}$ and $t = x^{-s}x^{-b}x^{b}$ then for $q = x^{r}x^{-n}x^{n}x^{s}$ we have:

$$pq = x^{-a}x^{a}x^{-r} \cdot x^{r}x^{-n}x^{n}x^{s} = x^{-n}x^{n}x^{s}$$

$$qt = x^{r}x^{-n}x^{n}x^{s} \cdot x^{-s}x^{-b}x^{b} = x^{r}x^{-n}x^{n}$$

as required. Thus edges of type 1 commute with edges of type 4.

5. **Type 2 & Type 2:**

   Take

$$(x^{-n}x^{n}x^{-r}, 1 = xx^{-1}, x^{r}x^{-(b+r)}x^{(b+r)}) \qquad 0 \leqslant r \leqslant n,\ 0 \leqslant b \leqslant n-r$$

   and

$$(x^{-n}x^{n}x^{-s}, 1 = xx^{-1}, x^{s}x^{-(c+s)}x^{(c+s)}) \qquad 0 \leqslant s \leqslant n,\ 0 \leqslant c \leqslant n-s$$

   Let $c+s \leqslant b+r$ and set $p = x^{-n}x^{n}x^{-r}$ and $t = x^{s}x^{-(c+s)}x^{(c+s)}$ then for $q = x^{r}x^{-(b+r)}x^{(b+r)}x^{-s}$ we have:

$$pq = x^{-n}x^{n}x^{-r} \cdot x^{r}x^{-(b+r)}x^{(b+r)}x^{-s} = x^{-n}x^{n}x^{-s}$$

$$qt = x^{r}x^{-(b+r)}x^{(b+r)}x^{-s} \cdot x^{s}x^{-(c+s)}x^{(c+s)} = x^{r}x^{-(c+s)}x^{(c+s)}$$

   as required. Thus edges of type 2 commute with edges of type 2.

6. **Type 2 & Type 3:**

   Take

$$(x^{-n}x^{n}x^{-r}, 1 = xx^{-1}, x^{r}x^{-(b+r)}x^{(b+r)}) \qquad 0 \leqslant r \leqslant n,\ 0 \leqslant b \leqslant n-r$$

and

$$(x^{-a}x^{a}x^{s}, 1 = xx^{-1}, x^{-s}x^{-n}x^{n}) \qquad 0 < s,\ 0 \leqslant a < n$$

then set $p = x^{-a}x^{a}x^{s}$ and $t = x^{r}x^{-(b+r)}x^{(b+r)}$ the for $q = x^{-s}x^{-n}x^{n}x^{-r}$ we have:

$$pq = x^{-a}x^{a}x^{s} \cdot x^{-s}x^{-n}x^{n}x^{-r} = x^{-n}x^{n}x^{-r}$$

$$qt = x^{-s}x^{-n}x^{n}x^{-r} \cdot x^{r}x^{-(b+r)}x^{(b+r)} = x^{-s}x^{-n}x^{n}$$

as required. Thus edges of type 2 commute with edges of type 3.

7. **Type 2 & Type 4:**

Take

$$(x^{-n}x^{n}x^{-r}, 1 = xx^{-1}, x^{r}x^{-(b+r)}x^{(b+r)}) \qquad 0 \leqslant r \leqslant n,\ 0 \leqslant b \leqslant n - r$$

and

$$(x^{-n}x^{n}x^{s}, 1 = xx^{-1}, x^{-s}x^{-c}x^{c}) \qquad 0 < s,\ 0 \leqslant c \leqslant n$$

Here we must consider two cases, $b + r \geqslant c$ and $b + r < c$ separately.

When $b + r \geqslant c$ set $p = x^{-n}x^{n}x^{-r}$ and $t = x^{-s}x^{-c}x^{c}$ for $q = x^{r}x^{-(b+r)}x^{(b+r)}x^{s}$ we have:

$$pq = x^{-n}x^{n}x^{-r} \cdot x^{r}x^{-(b+r)}x^{(b+r)}x^{s} = x^{-n}x^{n}x^{s}$$

$$qt = x^{r}x^{-(b+r)}x^{(b+r)}x^{s} \cdot x^{-s}x^{-c}x^{c} = x^{r}x^{-(b+r)}x^{(b+r)}$$

as required, now we must consider the case $b + r < c$.

In this case set $p = x^{-n}x^n x^s$ and $t = x^r x^{-(b+r)} x^{(b+r)}$ so for $q = x^{-s}x^{-c}x^c x^{-r}$ we have:

$$pq = x^{-n}x^n x^s \cdot x^{-s}x^{-c}x^c x^{-r} = x^{-n}x^n x^{-r}$$

$$qt = x^{-s}x^{-c}x^c x^{-r} \cdot x^r x^{-(b+r)} x^{(b+r)} = x^{-s}x^{-c}x^c$$

as required. Thus edges of type 2 commute with edges of type 4.

8. **Type 3 & Type 3:**

Take

$$(x^{-a}x^a x^r, 1 = xx^{-1}, x^{-r}x^{-n}x^n) \qquad 0 < r,\ 0 \leqslant a < n$$

and

$$(x^{-b}x^b x^s, 1 = xx^{-1}, x^{-s}x^{-n}x^n) \qquad 0 < s,\ 0 \leqslant b < n$$

Let $b \geqslant a$, and set $p = x^{-a}x^a x^r$ and $t = x^{-s}x^{-n}x^n$, then for $q = x^{-r}x^{-b}x^b x^s$ we have:

$$pq = x^{-a}x^a x^r cdot x^{-r}x^{-b}x^b x^s = x^{-b}x^b x^s$$

$$qt = x^{-r}x^{-b}x^b x^s \cdot x^{-s}x^{-n}x^n = x^{-r}x^{-n}x^n$$

as required. Thus edges of type 3 commute with edges of type 3.

9. **Type 3 & Type 4:**

Take

$$(x^{-a}x^a x^r, 1 = xx^{-1}, x^{-r}x^{-n}x^n) \qquad 0 < r,\ 0 \leqslant a < n$$

and

$$(x^{-n}x^nx^s, 1 = xx^{-1}, x^{-s}x^{-b}x^b) \qquad 0 < s,\ 0 \leqslant b \leqslant n$$

Set $p = x^{-a}x^ax^r$ and $t = x^{-s}x^{-b}x^b$ then for $q = x^{-r}x^{-n}x^nx^s$ we have:

$$pq = x^{-a}x^ax^r \cdot x^{-r}x^{-n}x^nx^s = x^{-n}x^nx^s$$

$$qt = x^{-r}x^{-n}x^nx^s \cdot x^{-s}x^{-b}x^b = x^{-r}x^{-n}x^n$$

as required. Thus edges of type 3 commute with edges of type 4.

10. **Type 4 & Type 4:**

Take

$$(x^{-n}x^nx^r, 1 = xx^{-1}, x^{-r}x^{-a}x^a) \qquad 0 < r,\ 0 \leqslant a \leqslant n$$

and

$$(x^{-n}x^nx^s, 1 = xx^{-1}, x^{-s}x^{-b}x^b) \qquad 0 < s,\ 0 \leqslant b \leqslant n$$

Let $a \geqslant b$ and set $p = x^{-n}x^nx^r$ and $t = x^{-s}x^{-b}x^b$ then for $q = x^{-r}x^{-a}x^ax^s$ we have:

$$pq = x^{-n}x^nx^r \cdot x^{-r}x^{-a}x^ax^s = x^{-n}x^nx^s$$

$$qt = x^{-r}x^{-a}x^ax^s \cdot x^{-s}x^{-b}x^b = x^{-r}x^{-a}x^a$$

as required. Thus edges of type 4 commute with edges of type 4.

Therefore all single edges in $star(x^{-n}x^n)$ commute.

We now have a complete picture of the single edges in $star(x^{-n}x^n)$ we can consider how these map to the $star^{\bowtie}(x^{-n}x^n)$, again we can do this by considering each of the 4 edge types in turn:

1. **Type 1:**

$$\begin{array}{c} (x^{-a}x^{a}x^{-r}, 1 = xx^{-1}, x^{r}x^{-n}x^{n}) \\ \Big\downarrow \bowtie \\ (x^{-n}x^{n}x^{-r}, 1 = xx^{-1}, x^{r}x^{-n}x^{n}) \end{array} \qquad 0 \leqslant r \leqslant a < n$$

   with $0 \leqslant r < n$, and so from this collection of edges we retain n distinct edges after mapping with $\bowtie$.

2. **Type 2:**

$$\begin{array}{c} (x^{-n}x^{n}x^{-r}, 1 = xx^{-1}, x^{r}x^{-(b+r)}x^{(b+r)}) \\ \Big\downarrow \bowtie \\ (x^{-n}x^{n}x^{-r}, 1 = xx^{-1}, x^{r}x^{-n}x^{n}) \end{array} \qquad 0 \leqslant r \leqslant n,\ 0 \leqslant b \leqslant n - r$$

   with $0 \leqslant r \leqslant n$, for $0 \leqslant r < n$ we map to the same collection of edges that edges of type type 1 mapped to, but when $r = n$ we have one new edge in $star^{\bowtie}(x^{-n}x^{n})$.

3. **Type 3:**

$$\begin{array}{c} (x^{-a}x^{a}x^{r}, 1 = xx^{-1}, x^{-r}x^{-n}x^{n}) \\ \Big\downarrow \bowtie \\ (x^{-n}x^{n}x^{r}, 1 = xx^{-1}, x^{-r}x^{-n}x^{n}) \end{array} \qquad 0 < r,\ 0 \leqslant a < n$$

with $0 < r$, so we have an infinite collection of edges but now only one for each r.

4. **Type 4:**

$$\begin{array}{c} (x^{-n}x^{n}x^{r}, 1 = xx^{-1}, x^{-r}x^{-b}x^{b}) \\ \Big\downarrow \bowtie \\ (x^{-n}x^{n}x^{r}, 1 = xx^{-1}, x^{-r}x^{-n}x^{n}) \end{array} \qquad 0 < r,\ 0 \leqslant b \leqslant n$$

again with $r > 0$, this gives us exactly the same collection of edges as type 3 maps to.

So we can see that once we map to $star^{\bowtie}(x^{-n}x^{n})$ we have only two types of edges:

1. $(x^{-n}x^{n}x^{-r}, 1 = xx^{-1}, x^{r}x^{-n}x^{n})$ for $0 \leqslant r \leqslant n$, and

2. $(x^{-n}x^{n}x^{r}, 1 = xx^{-1}, x^{-r}x^{-n}x^{n})$ for $r > 0$.

We can consider the set of single edges in $star^{\bowtie}(x^{-n}x^{n})$ as the set of ordered pairs $\{(n, r) : r \geqslant -n\}$. Then we can consider the $*$-operation, since all edges in

$star^{\bowtie}(x^{-n}x^n)$ start and finish at $x^{-n}x^n$ it is clear that the $*$ operation is equal to the groupoid operation and that $star^{\bowtie}(x^{-n}x^n)$ is the free abelian group on $\{(n, r) : r \geqslant -n\}$.

We can now go on to construct the crossed-module associated with the Squier complex. Let $S^{\bowtie}$ be the Clifford semigroup $\bigsqcup_{n \geqslant 0} star^{\bowtie}(x^{-n}x^n)$, then from Theorem 7.2.8 we have:

$$S^{\bowtie} \xrightarrow{\mathbf{r}} B \rightrightarrows E$$

a free crossed B-module of groupoids.
We can consider the set of single edges in $S^{\bowtie}$ as ordered pairs,

$$\{(n, r) : n \geqslant 0,\ r \geqslant -n\}$$

as we did for each $star^{\bowtie}$, where elements of $S^{\bowtie}$ are strings $(n, r_1)(n, r_2)(n, r_3) \cdots (n, r_k)$ for $n \geqslant 0$, $r_i \geqslant -n$ and $k \geqslant 0$. Then $\mathbf{r} : S^{\bowtie} \to B$ is defined on the generators of each $star^{\bowtie}(x^{-n}x^n)$ by:

$$\mathbf{r} : (n, r) \to x^{-n}x^n$$

Then it remains to define the action of B on $S^{\bowtie}$. Every element of B can be written as $b = x^{-p}x^p x^q$, with $p \geqslant 0$ and $q \geqslant -p$. We can see that $bb^{-1} = x^{-p}x^p$, so b can only act on elements of $star^{\bowtie}(x^{-p}x^p)$. Then for $\alpha \in star^{\bowtie}(x^{-p}x^p)$, $\alpha^b \in star^{\bowtie}(b^{-1}b) = star^{\bowtie}(x^{-(p+q)}x^{(p+q)})$, and so as a string of ordered pairs we have that $\alpha^b = (\alpha_1^b \cdot \alpha_2^b \cdots \alpha_k^b) = (p+q, \_)_1(p+q, \_)_2 \cdots (p+q, \_)_k$, we still have to determine the second entries of our ordered pairs.
Let α_i be a generator for $star^{\bowtie}(x^{-p}x^p)$, with

$$\alpha_i = (x^{-p}x^p x^r, 1 = xx^{-1}, x^{-r}x^{-p}x^p)$$

with $r \geqslant -p$, then

$$\begin{aligned}
\alpha_i^b &= (b^{-1} \cdot x^{-p}x^p x^r, 1 = xx^{-1}, x^{-r}x^{-p}x^p \cdot b) \\
&= (x^{-q}x^{-p}x^p x^r, 1 = xx^{-1}, x^{-r}x^{-p}x^p x^q) \\
&= (x^{-q}x^{-p}x^p(x^q x^{-q})x^r, 1 = xx^{-1}, x^{-r}(x^q x^{-q})x^{-p}x^p x^q) \\
&= (x^{-(p+q)}x^{(p+q)}x^{-q}x^r, 1 = xx^{-1}, x^{-r}x^q x^{-(p+q)}x^{(p+q)})
\end{aligned}$$

The second entry in our ordered pair is then determind by $x^{-q}x^r$. Thus for $b = x^{-p}x^p x^q$, α^b is defined whenever $\alpha \in star^{\bowtie}(x^{-p}x^p)$, and b acts on generators as follows:

$$(x^{-p}x^p x^r, 1 = xx^{-1}, x^{-r}x^{-p}x^p) \xrightarrow{\square^b} (x^{-(p+q)}x^{(p+q)}x^{(r-q)}, 1 = xx^{-1}, x^{-(r-q)}x^{-(p+q)}x^{(p+q)}).$$

For a string $(n, r_1)(n, r_2)(n, r_3) \cdots (n, r_k) \in S^{\bowtie}$ and for $b \in \mathcal{B}$ such that $b = x^{-n}x^n x^q$,

$$((n, r_1)(n, r_2)(n, r_3) \cdots (n, r_k))^b = (n+q, r_1-q)(n+q, r_2-q)(n+q, r_3-q) \cdots (n+q, r_k-q).$$

We know from 7.2.8 that this crossed module is free on

$$R = \bigsqcup_{x \in E} R_x = \bigsqcup_{x \in E} \{(l = r, x) \in \mathcal{R} \times E : (l^{-1}r)\psi \geqslant x\psi\}$$

with

$$\begin{aligned}
\omega : R &\to \mathcal{B} \\
(l = r, x) &\mapsto xl^{-1}rx
\end{aligned}$$

We have $E(B) = \{x^{-n}x^n : n \geqslant 0\}$, and so $R_{x^{-n}x^n} = \{(1 = xx^{-1}, x^{-n}x^n)\}$. Then for $\mathcal{F}$, the free pre-crossed B-module on R, ω, we have F_e the free group on

$$(R \between B) = \{(1 = xx^{-1}, b) : b \in B, (1 = xx^{-1}, bb^{-1}) \in R_{bb^{-1}}, b^{-1}b = e\}$$

Since for any $b \in B$ $(1 = xx^{-1}, bb^{-1}) \in R_{bb^{-1}}$ exists, this essentially says that F_e is freely generated by the *costar* of e in B.

Then to move to the crossed module we need to consider the Peiffer elements, that is for $a, b \in B$:

$$(1 = xx^{-1}, a)^{-1}(1 = xx^{-1}, b)^{-1}(1 = xx^{-1}, a)(1 = xx^{-1}, b)^{a\delta}$$

Then consider

$$(1 = xx^{-1}, b)^{a\delta} = (1 = xx^{-1}, b)^{a^{-1}a} = (1 = xx^{-1}, ba^{-1}a) = (1 = xx^{-1}, bb^{-1}b) = (1 = xx^{-1}, b)$$

Thus the Peiffer identities give us exactly commutators. So $star^{\bowtie}(e)$ is the free abelian group generated by the *costar* of e in B.

From Proposition 7.2.9 $S^{\bowtie}$ is also a free $\mathfrak{L}(B)$-module on the $E(B)$ set Z in which $Z_e = \{(l = r) \in \mathcal{R} : (l^{-1}r)\psi = e\psi\}$. For $\mathcal{P}$, $\mathcal{R} = \{1 = xx^{-1}\}$, so

$$Z_e = \begin{cases} \{*\} & \text{if } e = 1 \\ \emptyset & \text{ow} \end{cases}$$

Thus the free $\mathfrak{L}(B)$-module $\mathcal{A}$ is such that A_e is the free abelian group generated by

$$B_e = \{(1 = xx^{-1}, (1, b)) : 1\psi \geqslant bb^{-1}, b^{-1}b = e\}$$

Again exactly the *costar* at e in B.

We note that since we have a trivial relation module here our presentation (7.9)

$$0 \to \bigsqcup_{e \in E(M)} \pi_e^{\bowtie} \to (S^{\bowtie})^{ab} \xrightarrow{\bar{\partial}} \mathcal{U} \to 0$$

gives us that $S^{\bowtie}$ is in fact isomorphic to the module of identities for $\mathcal{P}$.

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