# Model selection, updating and prediction of fatigue crack propagation using nested sampling algorithm

**A. BEN ABDESSALEM**[a]

a. Dynamics Research Group, Department of Mechanical Engineering, University of Sheffield, Mappin Street, Sheffield S1 3JD, United Kingdom, a.b.abdessalem@sheffield.ac.uk

## Abstract

*Mathematical models are often used to interpret experimental data, estimate the parameters and then predictions can be made. In practice, and in several applications, it is common that often more than one model could be used to describe the dynamics of a given phenomenon. Modelling and prediction of fatigue crack growth (FCG) is one of the engineering problems where a number of models with different levels of complexities exist and the selection of the most suitable one is always a challenging task. In this study, model selection, updating and prediction of fatigue crack propagation is carried out under a Bayesian framework. The nested sampling algorithm is selected to estimate the evidence of each competing model using an experimental data set of Aluminum 2024-T3. The obtained results are very encouraging and show the efficiency of the proposed approach when dealing with model selection, updating and prediction issues.*

**Keywords: Fatigue crack propagation, model selection, Bayesian method, nested sampling, model averaging**

## 1 Introduction

Model selection, updating and prediction of fatigue crack growth (FCG) have been extensively studied by researchers during the last decade see for example [1, 2] and references therein. In the literature, several models have been proposed to model fatigue crack propagation but there is no universally accepted model. All of them are based on relating a damage condition and load/geometry configuration to crack propagation growth rate. The selection of the most suitable model for a specific material is always a challenging task and the definition of a formal way to deal with this problem seems of great importance in practice. The Bayesian approach has been extensively adopted to deal with model selection, updating and prediction in general and for fatigue model selection in particular [1, 3]. In FCG several candidate models proposed in the literature could be used and the choice of the most suitable one requires a deep understanding of the dynamics of fatigue crack propagation. The complexity of model selection issue is increased by the uncertainties inherent to the fatigue crack propagation which has been widely discussed in the literature (see [4] and the references therein). Several approaches could be used to deal with model selection and the Bayesian methods are arguably the most popular. Guan et al. [1] have used the rejection-jump Markov chain Monte Carlo proposed by [5]. Model selection statistics, such as the Bayesian

evidence is a promising alternative which could be applied to assign weights to the competing models of different levels of complexities. In the present study, the *nested sampling* (NS) algorithm which has been specifically designed to estimate the model evidence is used. The algorithm has proven its efficiency in different domains ranging from cosmology [6] to structural dynamics [7]. The application of the NS algorithm to address model selection in fatigue crack propagation has not been studied yet and will be investigated in this work. The present research primarily focuses on fatigue model selection using the NS algorithm. Then, Bayesian updating and prediction of fatigue life based on only a few updating measurements obtained at the early stage of fatigue crack propagation is investigated. In this contribution, two candidate models will be considered for simplicity and the same scheme could be extended when multiple models needed to be considered.

The paper is structured as follows. Section 2 introduces the competing fatigue models used in this study. Section 3 presents the experimental data and a brief statistical analysis. Section 4 provides the basics of the Bayesian theory and introduces the nested sampling algorithm. Section 5 presents the main results and Section 6 concludes the paper.

# 2 Fatigue crack propagation models

As mentioned earlier, several models have been proposed in the literature to describe the dynamics of fatigue crack propagation. The Paris model [8] given by Eq. (1) is one of the widely used models for steel and aluminum materials.

$$\frac{\mathrm{d}a}{\mathrm{d}N} = \mathrm{C}(\Delta K)^{\mathrm{m}} \tag{1}$$

where $\frac{\mathrm{d}a}{\mathrm{d}N}$ denotes the crack growth rate ($a$ is the crack length and $N$ indicates the number of fatigue load cycles), $C$ and $m$ are the material properties, $\Delta K$ denotes the amplitude of the stress intensity factor which can be generally expressed as

$$\Delta K = \beta \Delta S \sqrt{\pi a} \tag{2}$$

where $\beta$ is the geometric correction factor and $\Delta\sigma = \sigma_{\max} - \sigma_{\min}$ represents the amplitude of cyclic stress ($\sigma_{\max}$ and $\sigma_{\max}$ are the maximum and minimum of the fatigue stress, respectively).

The second model used in this paper is proposed by Forman et al. [9] and given by Eq. (3). This model includes the stress ratio and fracture toughness

$$\frac{\mathrm{d}a}{\mathrm{d}N} = \frac{\mathrm{C}(\Delta K)^{\mathrm{m}}}{(1-R)k_c - \Delta K} \tag{3}$$

where $R$ is the stress ratio $R = \frac{\sigma_{min}}{\sigma_{max}}$ and $k_c$ is the fracture toughness expressed in (MPa$\sqrt{m}$).

For simplicity let us denote by $\Theta_{\mathrm{P}} = [m_p, C_p]$ and $\Theta_{\mathrm{F}} = [m_f, C_f, k_c]$ the parameter vectors associated with the Paris amd Forman models, respectively.

# 3 Experimental data and data fitting

To perform fatigue model selection, updating and prediction, a set of experimental fatigue tests available in the literature has been used. The experimental data set consists of 68 crack propagation trajectories obtained from a series of fatigue tests performed on identical center-cracked 2024-T3 aluminum plates [10] (specimen geometry is illustrated in Fig. 1a). The tested specimens have the following dimensions: length $\ell = 558.80$ mm, width $w = 152.40$ mm, thickness $t = 2.54$ mm and a crack of initial size

$2a = 18$ mm located at the center of specimen. The tests were performed under load control using a sinusoidal input at 20 Hz producing a constant stress range of 48.28 MPa with a stress ratio equal to 0.2. The experimental trajectories are shown in Fig. 1b where only a set of 30 fatigue crack trajectories have been used to quantify the uncertainty for each model's parameters and one interdependent test specimen is selected to illustrate the ability of the NS algorithm to update the model and make future predictions. Fig. 1c shows the data in the crack growth rates versus stress intensity factor space where one can observe a noticeable scatter.

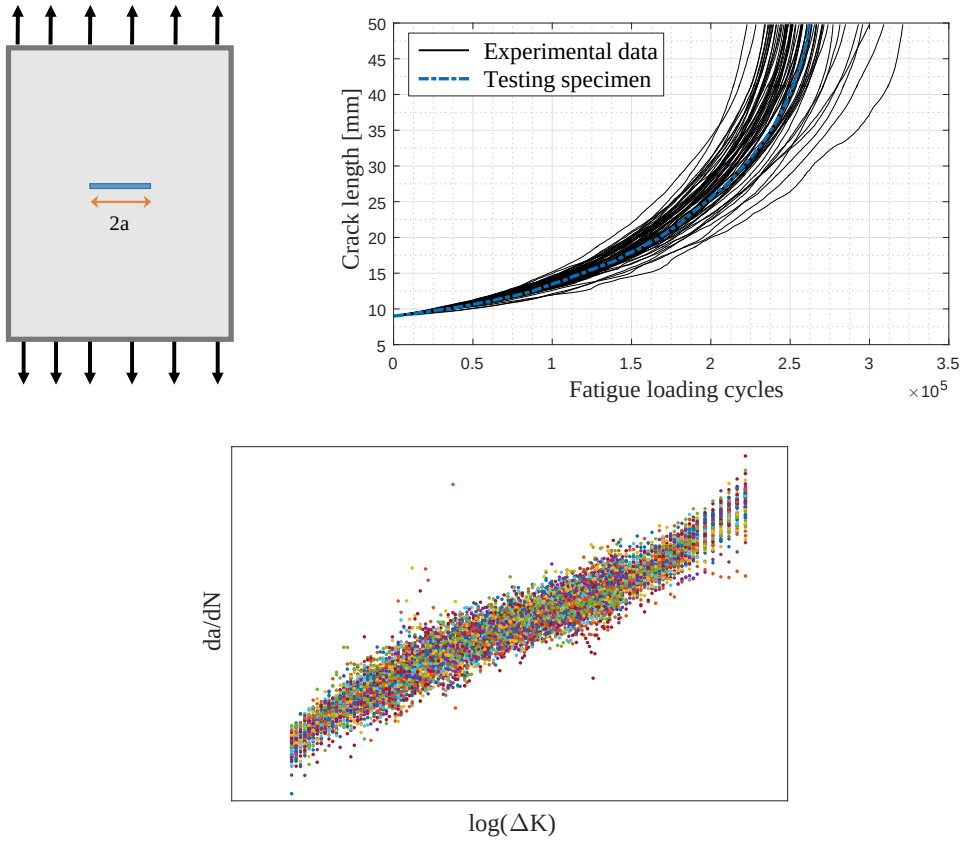



Figure 1: (a, left) Specimen geometry, (b, right) experimental $a - N$ curves, (c, bottom) variability of the fatigue crack growth rates.

The first step in the proposed approach consists of searching the optimal parameters needed to properly fit the experimental fatigue trajectories and thus obtaining the prior statistical distributions for each candidate set of model parameters. The cost function given by Eq. (4) is used to find the optimal parameters.

$$\Delta = \sum_{i=1}^{164} \left( a_i^{\text{exp}} - a_i^{\text{sim}}(\Theta) \right)^2 \tag{4}$$

where $\Delta$ is the cost function to be minimised, $a^{\text{exp}}$ and $a^{\text{sim}}$ represent the experimental and numerical predictions of crack lengths given by the model $\mathcal{M}$ in each increment (the total number of increments is 164). It should be noted that the fatigue models given by Eqs. (1) and (3) are numerically integrated using the Runge-Kutta method.

The simulated annealing algorithm has been used to determine the optimal parameters for each model. Table 1 shows the statistical properties (mean values, covariance for each vector and covariance matrix)

of the optimal parameters for both models. It should be noted that both models fit reasonably well to the empirical crack trajectories. The best and worst fittings for each competing model is shown in Figs. 2.

| | Paris model | | | Forman model | | |
|---|---|---|---|---|---|---|
| | $\mu$ | COV [%] | $\Sigma\ (m, \log C)$ | $\mu$ | COV [%] | $\Sigma\ (m, \log C, k_c)$ |
| $m$ | 2.9131 | 4.3530 | $\begin{bmatrix} 0.0161 & -0.0957 \\ -0.0957 & 0.5704 \end{bmatrix}$ | 2.1559 | 5.8796 | $\begin{bmatrix} 0.0161 & -0.095 & 0.0216 \\ -0.095 & 0.5622 & -0.1223 \\ 0.0216 & -0.1223 & 0.1277 \end{bmatrix}$ |
| $\log C$ | -26.3940 | 2.8614 | | -15.3809 | 4.8747 | |
| $k_c$ (MPa$\sqrt{m}$) | | | | 42.7829 | 0.3573 | |

Table 1: Statistical properties of the input priors for both models.

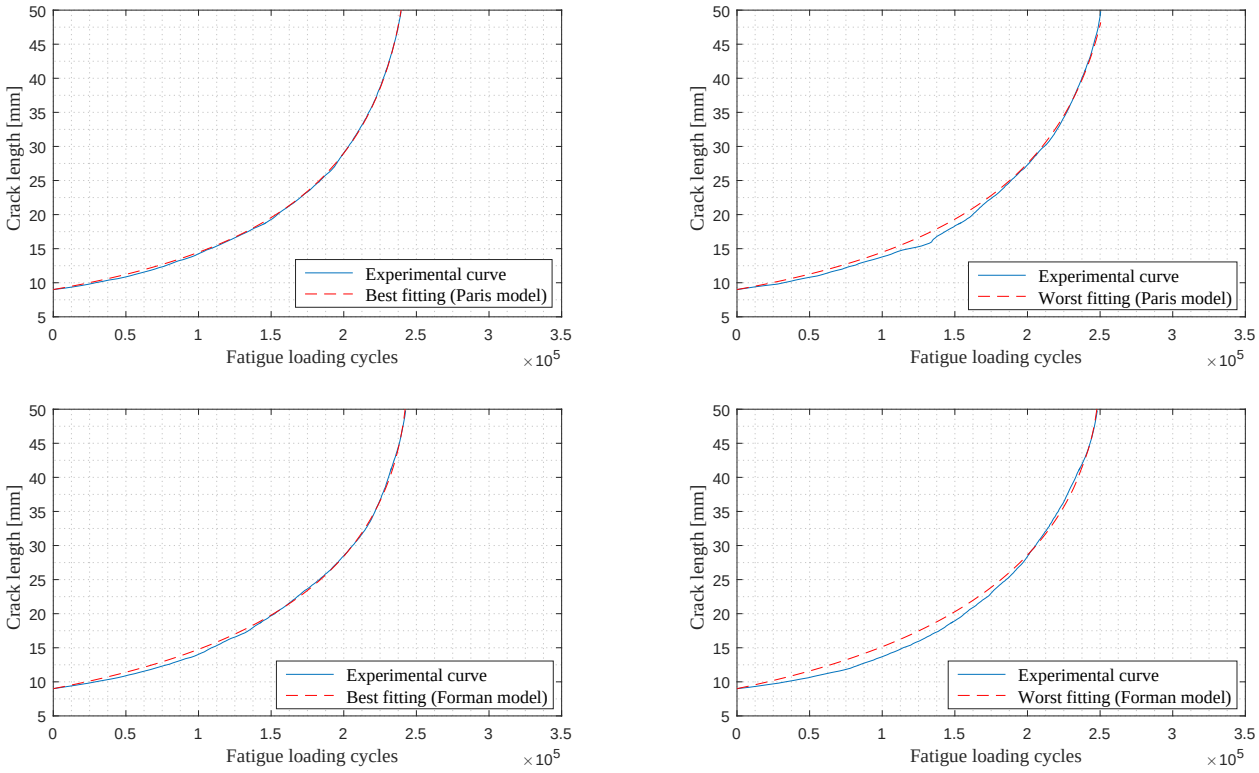



Figure 2: Best and worst fittings obtained using Paris and Forman models.

# 4 Bayesian model selection

## 4.1 Formulation of the likelihood function

Bayesian inference requires the formulation of a likelihood function to measure the similarity between the available measurements and simulated data from the model. Given a set of observed data of length $n_d$, $\mathbf{d}_{1:n_d}$ and a vector of model parameters $\Theta$, the likelihood function can be defined as

$$\mathcal{L}(\Theta) = f(\mathbf{d}_{1:n_d}|\Theta) \tag{5}$$

For statistically independent and identical Gaussian error $\varepsilon$ distributed with mean $\mu_\varepsilon = 0$ and variance

$\sigma_\varepsilon^2$, the likelihood function can be expressed as:

$$\mathcal{L}(\Theta) = f(\mathbf{d}_{1:n_d}|\Theta) = (2\pi\sigma_\varepsilon)^{-n_d/2} \exp\left(-\frac{1}{2\sigma_\varepsilon^2}\sum_{i=1}^{n_d}\left(a_i^{\text{obs}} - a_i^{\text{sim}}\right)\right) \tag{6}$$

where $a^{\text{obs}}$ is the observed crack lengths and $a^{\text{sim}}$ is the crack lengths estimated from the model.

## 4.2 Parameter estimation using Bayesian inference

Bayesian inference provides a robust procedure of parameter estimation by updating the prior knowledge of $\Theta$ using the information contained in the measurements data $\mathbf{d}_{1:n_d}$. The updated knowledge of $\Theta$ is represented by the joint posterior probability density function (PDF) $p(\Theta|\mathbf{d}_{1:n_d}, \mathcal{M})$ obtained as:

$$p(\Theta|\mathbf{d}_{1:n_d}, \mathcal{M}) = \frac{p(\mathbf{d}_{1:n_d}|\Theta, \mathcal{M})p(\Theta|\mathcal{M})}{p(\mathbf{d}_{1:n_d}|\mathcal{M})} \tag{7}$$

where $p(\mathbf{d}_{1:n_d}|\Theta, \mathcal{M})$ is the likelihood function, $p(\Theta|\mathcal{M})$ is the prior PDF and $p(\mathbf{d}_{1:n_d}|\mathcal{M})$ is the marginal likelihood or the evidence expressed by:

$$p(\mathbf{d}_{1:n_d}|\mathcal{M}) = \int_\Theta p(\mathbf{d}_{1:n_d}|\Theta, \mathcal{M})p(\Theta|\mathcal{M})\mathrm{d}\Theta \tag{8}$$

## 4.3 Bayesian model selection

Bayesian model selection exploits Bayes' theorem to calculate the posterior model probability $p(\mathcal{M}|\mathbf{d}_{1:n_d}, \boldsymbol{\mathcal{M}})$ of any model $\mathcal{M}$ in a candidate model set $\boldsymbol{\mathcal{M}}$ as:

$$p(\mathcal{M}|\mathbf{d}_{1:n_d}, \boldsymbol{\mathcal{M}}) = \frac{\overbrace{p(\mathbf{d}_{1:n_d}|\mathcal{M})}^{\mathcal{Z},\ \text{evidence}}\, p(\mathcal{M}|\boldsymbol{\mathcal{M}})}{p(\mathbf{d}_{1:n_d}|\boldsymbol{\mathcal{M}})} \tag{9}$$

where $p(\mathcal{M}|\boldsymbol{\mathcal{M}})$ is the prior model probability, $p(\mathbf{d}_{1:n_d}|\mathcal{M})$ is the evidence and $p(\mathbf{d}_{1:n_d}|\boldsymbol{\mathcal{M}})$ is a constant for a given model set $\boldsymbol{\mathcal{M}}$. The evidence $p(\mathbf{d}_{1:n_d}|\mathcal{M})$ in the context of model selection allows to guarantee the trade-off between the goodness-of-fit and the information gained from the available data.

## 4.4 Nested sampling

Nested sampling (NS) is a Bayesian sampling algorithm that can simultaneously sample from the posterior distribution and estimate the evidence for model selection. It has been applied effectively in different domains found in literature. In this subsection, a brief introduction of the principle of the NS algorithm is given, for further details the reader is referred to [11, 12]. NS is an adaptive algorithm that utilises a set of samples from the prior PDF (an active set) to explore the posterior PDF. This active set is adaptively evolved to high-likelihood regions of the parameters space by replacing the sample with the lowest likelihood value in the active set by a new sample at each NS iteration. The introduced new sample is a sample from the prior and has to satisfy a likelihood constraint (i.e. have a higher likelihood than the discarded sample). This iterative evolution of samples enables a decomposition of the volume under the posterior PDF into one-dimensional slices for numerical integration. The final output of the NS algorithm is a set of samples with associated weights computed using Eq. (10) which can be resampled to obtain the posterior distribution. These samples can then be used to derive statistics of posterior parameters, such

as means, standard deviations, covariances or to construct marginalised posterior distributions.

$$\nu_j = \frac{\mathcal{L}_j \omega_j}{\mathcal{Z}} \tag{10}$$

where the sample index $j$ runs from 1 to $\mathrm{T} = M + N$, the total number of sampled points.

**Algorithm 1** NESTED SAMPLING ALGORITHM

**Input**: size of the active set $N$, number of iterations $M$, model $\mathcal{M}$, prior distribution $p(\Theta|\mathcal{M})$, $\mathcal{D}$ available data

**Output**: Model evidence, posterior distribution

**Set** $\mathcal{Z}_0 = 0$, $X_0 = 1$, $\mathcal{S}_{\text{active}} = \{\ \}$, $\mathcal{S}_{\text{posterior}} = \{\ \}$
**Sample** $\Theta_i \sim p(\cdot|\mathcal{M})$ for $i = 1 : N$ ▷ generate active set
**Set** $\mathcal{S}_{\text{active}} \leftarrow \{\Theta_i\}_{i=1}^{N}$
**for** $j = 1, \ldots, M$ **do**
  **Find** $\Theta_{\text{worst}} = \min\{\mathcal{L}(\Theta_i)\}_{i=1}^{N}$
  **Set** $\mathcal{L}_j = p(\mathcal{D}|\Theta_{\text{worst}}, \mathcal{M})$
  **Set** $X_j = \exp(-j/N)$
  **Set** $w_j = \frac{1}{2}(X_{j-1} - X_j)$
  **Set** $\mathcal{Z}_j = \mathcal{Z}_{j-1} + w_j \mathcal{L}_j$
  **Set** $\mathcal{S}_{\text{posterior}} \leftarrow \mathcal{S}_{\text{posterior}} \cup \{(\Theta_{\text{worst}}, w_i \mathcal{L}_j)\}$
  **Sample** $\Theta_{\text{new}} \sim p(\cdot|\mathcal{M})$ such that $p(\mathcal{D}|\Theta_{\text{new}}, \mathcal{M}) > \mathcal{L}_j$ ▷ constrained sampling
  **Set** $\mathcal{S}_{\text{active}} \leftarrow (\mathcal{S}_{\text{active}} \backslash \Theta_{\text{worst}}\}) \cup \{\Theta_{\text{new}}\}$ ▷ update the active set
**Set** $\mathcal{Z} = \mathcal{Z}_M + (X_M/N)(\mathcal{L}(\Theta_1) + \ldots + \mathcal{L}(\Theta_N))$
**Set** $\mathcal{S}_{\text{posterior}} \leftarrow \mathcal{S}_{\text{posterior}} \cup \{(\Theta_i, (X_M/N)\mathcal{L}_i)\}$ $\forall \Theta_i \in \mathcal{S}_{\text{active}}$

# 5 Results and discussion

Table 2 shows the estimated evidence associated to each model considering a different number of updating measurements following the scheme summarised in Algorithm 1. Jeffrey's scale [14] is used to determine the significance of the obtained model evidences and Table 3 shows the interpretation of this measure in this paper. This measure is calculated from the ratio of model evidences, also known as Bayes factors. Based on Jeffreys's scale, one can see that the Paris model is weakly favoured considering different number of updating measurements as it has a higher evidence. This result suggests that probably both models could be used to make future predictions as the difference is not significant. This may be explained by the fact that each of the competing models performs better in a specific regime of propagation (stable and unstable regimes of FCG) as was demonstrated in [15] where the FCG is modelled as a piecewise-deterministic Markov process. In the same study, it has been shown that the Forman model is better able to capture the asymptotic behaviour at the end of crack propagation. Moreover, it has been shown that combining Paris and Forman models allows one to estimate the FCG rates precisely while the Paris model underestimates considerably those values. As a result and based on the interpretation of the Bayes factors obtained by considering different updating measurements, it would be better to use both models to predict the remaining fatigue life by averaging the predictions provided by the competing models. Figs. 3 show the prediction provided by each model and the average prediction considering 5 updating measurements. Figs. 4 (left and right) show the sequence of samples obtained during the NS iterative process and in red the samples used to estimate the posterior statistical distributions for Paris and Forman models.

| Model/Updating measurements | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| Paris, $\mathcal{Z}_{\mathrm{P}}$ | -5.4992 | -7.3049 | -7.6095 | -8.1782 | -8.1689 |
| Forman, $\mathcal{Z}_{\mathrm{F}}$ | -6.7047 | -7.8112 | -9.3694 | -9.9529 | -10.0813 |
| $\log(\mathcal{Z}_{\mathrm{F}}/\mathcal{Z}_{\mathrm{P}})$ | **1.22** | **1.07** | **1.23** | **1.21** | **1.23** |

Table 2: Estimation of the evidence considering different number of updating measurements for both models.

| $\mathcal{Z}_j/\mathcal{Z}_i$ | $\log(\mathcal{Z}_j/\mathcal{Z}_i)$ | Evidence against model $\mathcal{M}_i$ |
|---|---|---|
| $1-3.2$ | $0-1.2$ | Weak |
| $3.2-10$ | $1.2-2.3$ | Substantial |
| $10-100$ | $2.3-4.6$ | Strong |
| $>100$ | $>4.6$ | Decisive |

Table 3: The significance ratios used in Jeffreys's scale.

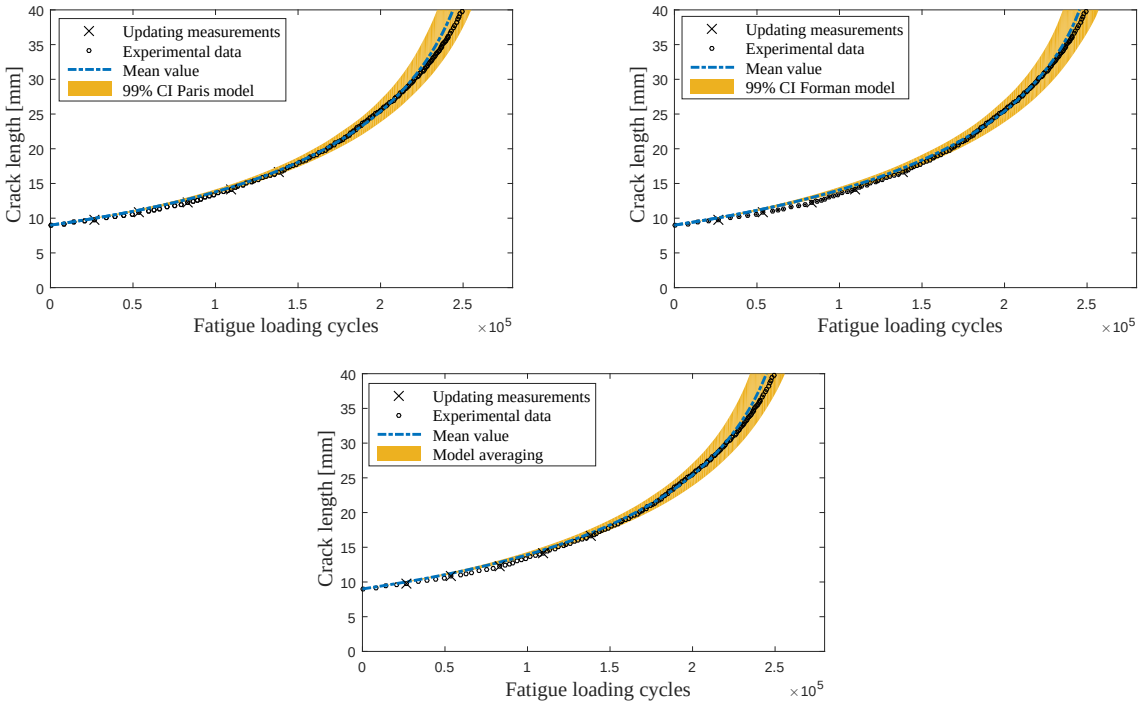
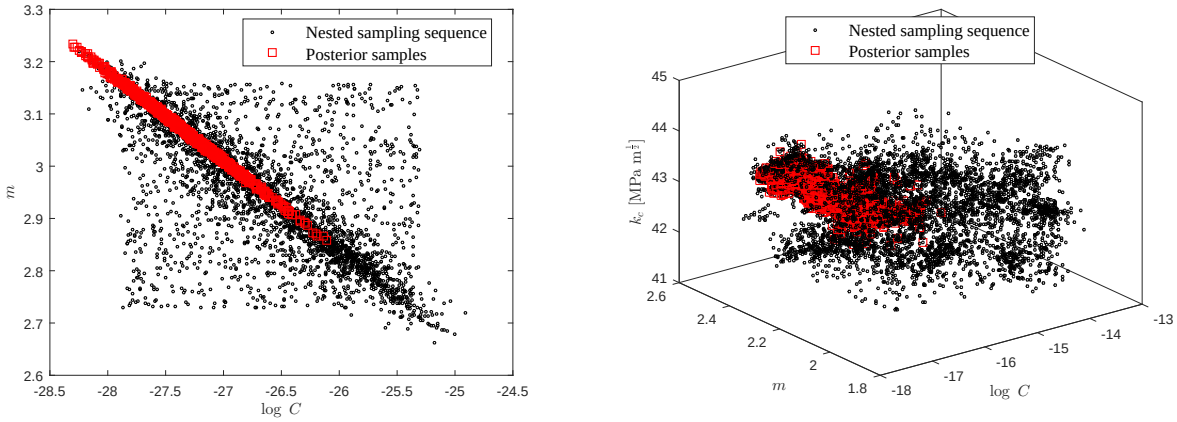



Figure 3: Prediction of fatigue crack propagation with 5 updating measurements using Paris model, Forman model and model averaging.

# 6 Conclusion

In this work, the efficiency of the NS algorithm to deal with model selection, updating and prediction in fatigue crack propagation has been demonstrated. It has been shown that the NS algorithm could be a good alternative to quantitatively estimate the evidence associated to each of the competing models and therefore provide a comparison between all of them in a robust way. In this study, it has been shown that none of the considered competing models is strongly favoured as each of them is more suitable for a specific regime of propagation. As it was shown, the NS algorithm allows model averaging by combining the models to make future predictions. This result is coherent with the findings shown in




(a) (b)

Figure 4: NS sequence after 30 000 iterations and posterior samples for (a) Paris and (b) Forman models.

[15] where it has been demonstrated that the combination of both models allows a better description of the dynamics of fatigue crack propgation and a precise estimate of the fatigue crack growth rates during the stable and unstable regimes of propagation.

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