Energetic approach coupled with analytic solutions for the evaluation of residual stress.

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Résumé

Cet article contribue plus généralement à une stratégie de calcul mixte analytique/numérique visant à calculer les contraintes résiduelles apparaîssant dans les bandes métaliques après le bobinage. Au cours du refroidissement des bobines on assiste à des transformations multi-phases et à de la plasticité de transformation induite. Ainsi, chaque spire de l'enroulement est soumise à une déformation libre globale qui peut être suffisante pour générer des déformations plastiques macroscopiques. Pour chaque spire on dérive une solution du problème d'un tube cylindrique soumis à une déformation libre arbitraire. Les dévelopements mathématiques reposent sur l'équation linéaire inhomogène de Navier en traitant la plasticité par l'introduction d'une déformation plastique déviatorique inconnue. Une solution analytique est obtenue sous forme de dévelopement en série pour chaque déformation plastique testée. Enfin, un principe énergétique permet de déterminer la déformation plastique solution du problème. En pratique, un optimisation numérique est menée directement sur les coefficients du développement en série de la déformation plastique.

Abstract

This paper is part of a more general mixed analytical/numerical strategy aiming at computing residual stresses of metallic strips after coiling process. Multiphase transitions and transformation induced plasticity occur during coil cooling. Thus, each layer of coil is subjected to an overall eigenstrain that can be sufficient to generate macroscopic plastic deformations. For each layer, a solution of the problem of an elastic-plastic hollow cylinder undergoing an arbitrary eigenstrain is derived. Mathematical developments relies on the linear inhomogeneous Navier equation by dealing with plasticity through the introduction of a deviatoric unknown plastic strain. An analytical solution is obtained in the form of series expansion, for any trial plastic strain. Then, an energetic principle enables to determine the plastic strain chosen as a solution of the problem. Practically, a numerical optimization procedure is performed directly on coefficients of the plastic strain series expansion.

Key words :

1 Introduction

The current dynamic of steel manufacturing is to regularly develop new stronger grades enabling users to reduce strips or profile thicknesses and thus reduce produced tonnages, which participates to the energy efficiency by minimizing for instance the total mass of vehicles etc. One of the major issues related to this evolution of steel production is the forming processes that lead to serious residual stress problems which in turn can result in instabilities such as strip buckling during rolling process or coils collapsing on themselves. In addition to heterogeneous plastic deformations, irreversible deformations responsible for these residual stresses are due to different phase transitions under applied loads that occur during most forming processes. In order to establish technological strategies aimed at a better control of residual stress fields, it is essential to understand and to simulate accurately these processes.

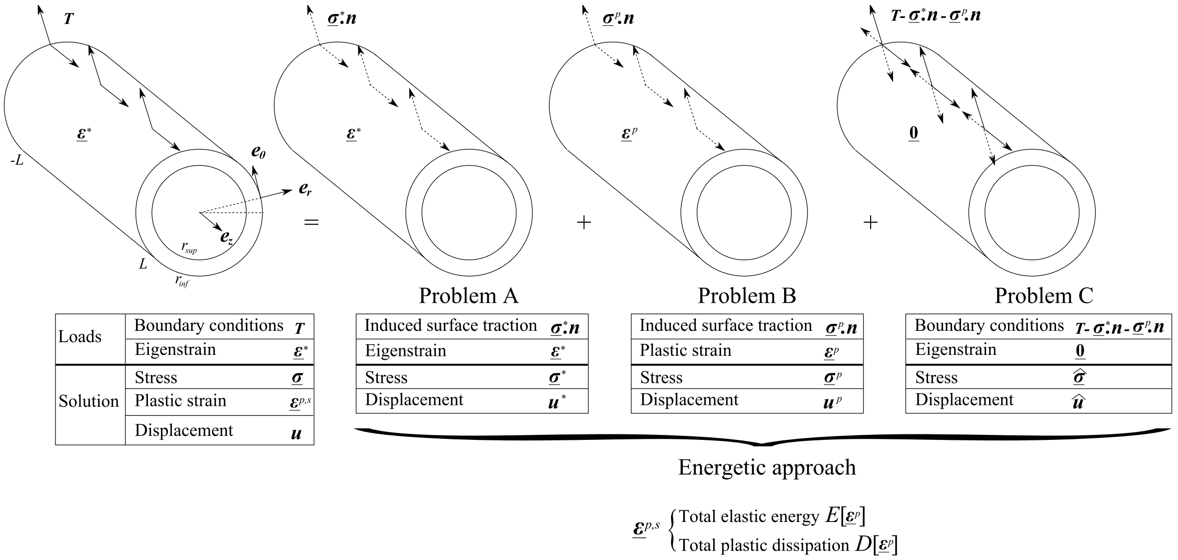
This paper is situated within the frame work of numerical simulation of the coiling process of steel. Plastic deformations along with multi-phase transitions are responsible for large irreversible strain leading to major residual stress issues. A non-linear mixed analytical/numerical approach has been recently proposed [1] in order to compute residual stresses generated by different contributions of inelastic eigenstrain occurring during the coiling process (including both the winding phase and the cooling phase). In particular, transformation induced plasticity has been taken into account following the recent work [2] based on the classical Leblond's model [3, 4, 5, 6]. The mixed analytical/numerical approach [1] consists for each time step in applying the overall inelastic eigenstrain (depending on the previous time step) by solving analytically the inhomogeneous Navier equation in each layer of the coil. Contact pressures are updated by numerical optimization. The analytic solution relies on a series expansion of the right side term of the inhomogeneous Navier equation. A specific function basis has been introduced obtain simple identification of the solution. The homogeneous solution is more classically obtained by using harmonic potential theory as exposed in [7] and bi-harmonic potentials as in [8].

Even though non-linear contributions such as microscopic plasticity have been taken into account, macroscopic plasticity has been neglected, that is to say that the macroscopic von Mises equivalent stress does not reach the macroscopic yield stress. However, macroscopic plasticity may occur if the yield stress has already been reached during the winding phase of the process, for rather thick strips for instance. Thus, this paper is an attempt to introduce, under simplifying assumptions, macroscopic plasticity by adding an unknown deviatoric contribution to the imposed eigenstrain. Then, an energetic approach is used to identify this plastic contribution. It consists in minimizing the total stored elastic energy plus the plastic dissipation associated to the plastic strain tensor introduced in the eigenstrain. This plastic contribution can be interpreted as a distance (or a cost) between different states [9, 10]. Thus, the energetic approach consists in seeking the lowest energy state by taking into account the cost by terms of dissipation distance. The proposed solution combines analytical developments for the inhomogeneous Navier equation and numerical optimizations for the identification of the plastic strain tensor.

2 Decomposition of the problem

In this contribution only one layer is considered, the problem of the determination of contact pressures being addressed in [1]. For each time step the body is subjected to an imposed eigenstrain computed on the basis of the extended Leblond's model [2]. This contribution focuses only on one time step and solves semi-analytically the problem of an elastic-plastic tube subjected to an arbitrary eigenstrain under axi-symmetrical assumption as proposed in [1]. Even though a non-linear behavior is considered, the proposed strategy relies on linear solutions. Indeed, the linear inhomogeneous Navier equation is solved for any unknown trial plastic strain $\underset{\sim}{\varepsilon}^p$ (where p stands for *plastic*). The latter is determined in the end by minimizing the sum of the elastic energy $E\left[\underset{\sim}{\varepsilon}^p\right]$ and the plastic dissipation $D\left[\underset{\sim}{\varepsilon}^p\right]$ associated to $\underset{\sim}{\varepsilon}^p$. Therefore, the problem can be decomposed into sub-problems as shown in figure 1 and the plastic strain considered as solution denoted by $\underline{\varepsilon}^{p,s}$ (where s stands for *solution*) is numerically determined by:

$$\underline{\varepsilon}^{p,s} = \underset{\underline{\varepsilon}^p,\ \mathrm{tr}\,\underline{\varepsilon}^p=0}{\text{argmin}} \; E\left[\underline{\varepsilon}^p\right] + D\left[\underline{\varepsilon}^p\right] \tag{1}$$



Loads	Boundary conditions	$\boldsymbol{T}$
	Eigenstrain	$\underline{\boldsymbol{\varepsilon}}^*$
Solution	Stress	$\underline{\boldsymbol{\sigma}}$
	Plastic strain	$\underline{\boldsymbol{\varepsilon}}^{p,s}$
	Displacement	$\boldsymbol{u}$

Induced surface traction	$\underline{\boldsymbol{\sigma}}^*.\boldsymbol{n}$
Eigenstrain	$\underline{\boldsymbol{\varepsilon}}^*$
Stress	$\underline{\boldsymbol{\sigma}}^*$
Displacement	$\boldsymbol{u}^*$

Induced surface traction	$\underline{\boldsymbol{\sigma}}^p.\boldsymbol{n}$
Plastic strain	$\underline{\boldsymbol{\varepsilon}}^p$
Stress	$\underline{\boldsymbol{\sigma}}^p$
Displacement	$\boldsymbol{u}^p$

Boundary conditions	$\boldsymbol{T}-\underline{\boldsymbol{\sigma}}^*.\boldsymbol{n}-\underline{\boldsymbol{\sigma}}^p.\boldsymbol{n}$
Eigenstrain	$\underline{\boldsymbol{0}}$
Stress	$\hat{\underline{\boldsymbol{\sigma}}}$
Displacement	$\hat{\boldsymbol{u}}$

Energetic approach

$$\underline{\boldsymbol{\varepsilon}}^{p,s} \begin{cases} \text{Total elastic energy } E[\underline{\boldsymbol{\varepsilon}}^p] \\ \text{Total plastic dissipation } D[\underline{\boldsymbol{\varepsilon}}^p] \end{cases}$$

Figure 1: Decomposition

3 Mathematical preliminaries

This section deals with mathematical definitions needed for the proposed solution. The following functions are proposed for expanding several functions arising in the paper into series:

$$G_m^{(\alpha,\beta)}(r) = J_\alpha\left(x_m^{(\beta)}\frac{r}{r_{sup}}\right)Y_\beta\left(x_m^{(\beta)}\frac{r_{inf}}{r_{sup}}\right) - J_\beta\left(x_m^{(\beta)}\frac{r_{inf}}{r_{sup}}\right)Y_\alpha\left(x_m^{(\beta)}\frac{r}{r_{sup}}\right) \tag{2}$$

where J_α and Y_α are the Bessel functions of the α-th order of the first and second kind respectively and $x_m^{(\beta)}$ are successive positive roots (indexed by $m \in \mathbb{N}$) of

$$x \mapsto J_\beta\left(x\right)Y_\beta\left(x\frac{r_{inf}}{r_{sup}}\right) - J_\beta\left(x\frac{r_{inf}}{r_{sup}}\right)Y_\beta\left(x\right) \tag{3}$$

Introducing the following scalar product:

$$\langle f, g\rangle = \int_{r_{inf}}^{r_{sup}} rf(r)g(r)\mathrm{d}r \tag{4}$$

One obtains the following orthogonality relations:

$$\left\langle G_m^{(\alpha,\alpha)}, G_l^{(\alpha,\alpha)}\right\rangle = \begin{cases} \left\langle G_m^{(\alpha,\alpha)}, G_l^{(\alpha,\alpha)}\right\rangle & \text{if} \quad m = l \\ 0 & \text{if} \quad m \neq l \end{cases} \tag{5}$$

In the following, some functions will be projected on the linear span of functions $G_m^{(\alpha,\alpha)}$ denoted by span $\left(G_m^{(\alpha,\alpha)}(r), 1 \leq m \leq M\right)$ where M is a fixed integer. However, it should be noted that $G_m^{(\alpha,\alpha)}$

vanishes at $r = r_{inf}$ and r_{sup}. Thus, if functions that do not vanish at these points are considered one should add for instance $J_\alpha\left(r/r_{sup}\right)$ and $Y_\alpha\left(r/r_{sup}\right)$ to the vector space in order to have non-vanishing values at $r = r_{inf}$ and $r = r_{sup}$. Thus the vector space on which most functions arising in the following will be projected reads:

$$\mathcal{A}^{(\alpha)} = \text{span}\left(J_\alpha\left(\frac{r}{r_{sup}}\right), Y_\alpha\left(\frac{r}{r_{sup}}\right), G_m^{(\alpha,\alpha)}(r), 1 \leq m \leq M\right) \tag{6}$$

Let $f : (r, z) \in [r_{inf}, r_{sup}] \times [-L, L] \mapsto f(r, z)$ be a function sufficiently regular so that scalar products (4) are well defined, one can write for all $\alpha \in \mathbb{R}$ by first expanding $f(r, z)$ into a Fourier series along the z-direction and then by projecting all r-dependent Fourier coefficients denoted by $f_k(r)$ on the vector space $\mathcal{A}^{(\alpha)}$:

$$f(r, z) \simeq \sum_{k=-K}^{K}\left[A_k J_\alpha\left(\frac{r}{r_{sup}}\right) + B_k Y_\alpha\left(\frac{r}{r_{sup}}\right) + \sum_{m=1}^{M} C_{m,k} G_m^{(\alpha,\alpha)}(r)\right]\exp\left(\frac{ik\pi}{L}z\right) \tag{7}$$

where the Fourier coefficients according to the z-direction are:

$$f_k(r) = \frac{1}{2L}\int_{-L}^{L} f(r, z)\exp\left(\frac{-ik\pi}{L}z\right)\mathrm{d}z \tag{8}$$

where the projection reads:

$$C_{m,k} = \frac{\left\langle G_m^{(\alpha,\alpha)}, f_k(r)\right\rangle}{\left\langle G_m^{(\alpha,\alpha)}, G_m^{(\alpha,\alpha)}\right\rangle} \tag{9}$$

and where values at $r = r_{inf}$ and $r = r_{sup}$ gives:

$$\begin{pmatrix} A_k \\ B_k \end{pmatrix} = \begin{pmatrix} J_\alpha\left(\frac{r_{inf}}{r_{sup}}\right) & Y_\alpha\left(\frac{r_{inf}}{r_{sup}}\right) \\ J_\alpha(1) & Y_\alpha(1) \end{pmatrix}^{-1} \cdot \begin{pmatrix} f_k(r_{inf}) \\ f_k(r_{sup}) \end{pmatrix} \tag{10}$$

4 Inhomogeneous problem A

The first problem is a tube subjected to the imposed eigenstrain $\underline{\varepsilon}^*$. Boundary conditions are not specified, that is to say that only a particular solution of the inhomogeneous Navier equation is sought regardless of surface traction $\boldsymbol{T}$. The obtained stress field is denoted by $\underline{\boldsymbol{\sigma}}^*$ and the traction vector $\underline{\boldsymbol{\sigma}}^*.\boldsymbol{n}$ will be corrected by adding the homogenous solution of the problem C. The solution of problem A is already addressed in [1]. Main results are stated here for sake of clarity. The inhomogeneous Navier equation reads:

$$\mu\boldsymbol{\Delta}\,\boldsymbol{u}^* + (\lambda + \mu)\boldsymbol{\nabla}\text{div}\,\boldsymbol{u}^* = \textbf{div}\left(\lambda\text{tr}\left(\underline{\varepsilon}^*\right)\underline{\boldsymbol{I}} + 2\mu\underline{\varepsilon}^*\right) \tag{11}$$

Hence, considering the axi-symmetrical assumption:

$$\left\{\begin{array}{l} (\lambda + 2\mu)\left(\dfrac{\partial^2 u_r^*}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u_r^*}{\partial r} - \dfrac{u_r^*}{r^2}\right) + \mu\dfrac{\partial^2 u_r^*}{\partial z^2} + (\lambda + \mu)\dfrac{\partial^2 u_z^*}{\partial r\partial z} = f_r^*(r, z) \\ (\lambda + 2\mu)\dfrac{\partial^2 u_z^*}{\partial z^2} + \mu\left(\dfrac{\partial^2 u_z^*}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u_z^*}{\partial r}\right) + (\lambda + \mu)\left(\dfrac{\partial^2 u_r^*}{\partial r\partial z} + \dfrac{1}{r}\dfrac{\partial u_r^*}{\partial z}\right) = f_z^*(r, z) \end{array}\right. \tag{12}$$

where:

$$\left\{\begin{array}{l} f_r^*(r,z) = (\lambda+2\mu)\dfrac{\partial \varepsilon_{rr}^*}{\partial r} + \lambda\left(\dfrac{\partial \varepsilon_{\theta\theta}^*}{\partial r} + \dfrac{\partial \varepsilon_{zz}^*}{\partial r}\right) + 2\mu\left(\dfrac{\varepsilon_{rr}^*}{r} - \dfrac{\varepsilon_{\theta\theta}^*}{r} + \dfrac{\partial \varepsilon_{rz}^*}{\partial z}\right) \\ f_z^*(r,z) = (\lambda+2\mu)\dfrac{\partial \varepsilon_{zz}^*}{\partial z} + \lambda\left(\dfrac{\partial \varepsilon_{rr}^*}{\partial z} + \dfrac{\partial \varepsilon_{\theta\theta}^*}{\partial z}\right) + 2\mu\left(\dfrac{\partial \varepsilon_{rz}^*}{\partial r} + \dfrac{\varepsilon_{rz}^*}{r}\right) \end{array}\right. \tag{13}$$

Then by using the procedure described in section 3, following series expansions are considered:

$$\left\{\begin{array}{l} f_r^*(r,z) = \displaystyle\sum_{k=-K}^{K}\left[A_k^r J_1\left(\frac{r}{r_{sup}}\right) + B_k^r Y_1\left(\frac{r}{r_{sup}}\right) + \sum_{m=1}^{M} C_{m,k}^r G_m^{(1,1)}(r)\right]\exp\left(\frac{ik\pi}{L}z\right) \\ f_z^*(r,z) = \displaystyle\sum_{k=-K}^{K}\left[A_k^z J_0\left(\frac{r}{r_{sup}}\right) + B_k^z Y_0\left(\frac{r}{r_{sup}}\right) + \sum_{m=1}^{M} C_{m,k}^z G_m^{(0,1)}(r) + D_k^z\right]\exp\left(\frac{ik\pi}{L}z\right) \end{array}\right. \tag{14}$$

where coefficients A_k^r, A_k^z, B_k^r, B_k^z, $C_{m,k}^r$, $C_{m,k}^z$, and D_k^z are explicitly computed as functions of the imposed eigenstrain $\underline{\varepsilon}^*$ following the procedure detailed in section 3. It should be noted that $f_r^*(r,z)$ and the partial derivative of $f_z^*(r,z)$ with respect to r have been expanded. Since the link between $\underline{\varepsilon}^*$ and $f_r^*(r,z), f_z^*(r,z)$ is not essential in this contribution, details are omitted and the reader is simply referred to [1]. However, it should be mentioned that numerical derivations of the imposed eigenstrain are necessitated. A particular solution of the inhomogeneous Navier equation is sought as follows:

$$\left\{\begin{array}{l} u_r^*(r,z) = \displaystyle\sum_{k=-K}^{K}\left[U_k^r J_1\left(\frac{r}{r_{sup}}\right) + V_k^r Y_1\left(\frac{r}{r_{sup}}\right) + \sum_{m=1}^{M} u_{m,k}^r G_m^{(1,1)}(r)\right]\exp\left(\frac{ik\pi}{L}z\right) \\ u_z^*(r,z) = \displaystyle\sum_{k=-K}^{K}\left[U_k^z J_0\left(\frac{r}{r_{sup}}\right) + V_k^z Y_0\left(\frac{r}{r_{sup}}\right) + \sum_{m=1}^{M} u_{m,k}^z G_m^{(0,1)}(r) + W_k^z(r)\right]\exp\left(\frac{ik\pi}{L}z\right) \end{array}\right. \tag{15}$$

where:

$$W_k^z(r) = \left\{\begin{array}{ll} \dfrac{D_k^z r^2}{4\mu} & \text{if} \quad k=0 \\ -\dfrac{D_k^z}{(k\pi/L)^2(\lambda+2\mu)} & \text{if} \quad k\neq 0 \end{array}\right. \tag{16}$$

By plugging (15) into (12) and identifying:

$$\begin{pmatrix} U_k^r \\ U_k^z \end{pmatrix} = \underline{\boldsymbol{S}}_k \cdot \begin{pmatrix} A_k^r \\ A_k^z \end{pmatrix} \text{ and } \begin{pmatrix} V_k^r \\ V_k^z \end{pmatrix} = \underline{\boldsymbol{S}}_k \cdot \begin{pmatrix} B_k^r \\ B_k^z \end{pmatrix} \text{ and } \begin{pmatrix} u_{m,k}^r \\ u_{m,k}^z \end{pmatrix} = \underline{\boldsymbol{S}}_{m,k}^{(1)} \cdot \begin{pmatrix} C_{m,k}^r \\ C_{m,k}^z \end{pmatrix} \tag{17}$$

where:

$$\underline{\boldsymbol{S}}_k = \begin{pmatrix} -(\lambda+2\mu)\left(\frac{1}{r_{sup}}\right)^2 - \mu\left(\frac{k\pi}{L}\right)^2 & -(\lambda+\mu)\frac{1}{r_{sup}}\frac{ik\pi}{L} \\ (\lambda+\mu)\frac{1}{r_{sup}}\frac{ik\pi}{L} & -\mu\left(\frac{1}{r_{sup}}\right)^2 - (\lambda+2\mu)\left(\frac{k\pi}{L}\right)^2 \end{pmatrix}^{-1} \tag{18}$$

and:

$$\underline{\boldsymbol{S}}_{m,k}^{(\beta)} = \begin{pmatrix} -(\lambda+2\mu)\left(\frac{x_m^{(\beta)}}{r_{sup}}\right)^2 - \mu\left(\frac{k\pi}{L}\right)^2 & -(\lambda+\mu)\frac{ik\pi}{L}\frac{x_m^{(\beta)}}{r_{sup}} \\ (\lambda+\mu)\frac{x_m^{(\beta)}}{r_{sup}}\frac{ik\pi}{L} & -\mu\left(\frac{x_m^{(\beta)}}{r_{sup}}\right)^2 - (\lambda+2\mu)\left(\frac{k\pi}{L}\right)^2 \end{pmatrix}^{-1} \tag{19}$$

A particular displacement field u_r^*, u_z^*, solution of the inhomogeneous Navier equation (12) has been established. Therefore the associated stress field $\underline{\boldsymbol{\sigma}}^*$ can be computed as well, using the isotropic behavior. More precisely, a Fourier series expansion of $\underline{\boldsymbol{\sigma}}^*$ is obtained since the displacement field is known as a Fourier series expansion.

5 Inhomogeneous problem B

The second problem is similar to the problem B. A tube subjected to the unknown deviatoric plastic strain $\underline{\boldsymbol{\varepsilon}}^p$ is considered. For the previous problem A, the relationship between $f_r^*(r,z), f_z^*(r,z)$ and the imposed eigenstrain $\underline{\boldsymbol{\varepsilon}}^*$ is purely numerical, since $\underline{\boldsymbol{\varepsilon}}^*$ is known. One could directly consider the series expansion of $f_r^*(r,z), f_z^*(r,z)$ without referring to $\underline{\boldsymbol{\varepsilon}}^*$. However, in this section the plastic strain $\underline{\boldsymbol{\varepsilon}}^p$ is unknown and should be determined through (1) in the end. The proposed minimization is done directly on $\underline{\boldsymbol{\varepsilon}}^p$ and not on the right side term of the inhomogeneous Navier equation denoted by $f_r^p(r,z), f_z^p(r,z)$. Therefore one should consider a series expansion of $\underline{\boldsymbol{\varepsilon}}^p$ instead of $f_r^p(r,z), f_z^p(r,z)$ and solve the inhomogeneous equation.

The unknown plastic strain $\underline{\boldsymbol{\varepsilon}}^p$ is deviatoric. Therefore, it remains three independent components, namely ε_{rr}^p, $\varepsilon_{\theta\theta}^p$ and ε_{rz}^p. In this contribution, an approximate solution is obtained by introducing an assumption so that the number of independent components is reduced to two. Considering applications to coiling process, shear stresses are much smaller than other components and can be neglected in the plastic strain, thus $\varepsilon_{rz}^p \simeq 0$. Therefore considering deviatoric plastic strain (i.e., $\varepsilon_{\theta\theta}^p = -\varepsilon_{rr}^p - \varepsilon_{zz}^p$), the inhomogeneous Navier equation associated to the problem B reads:

$$\left\{\begin{array}{l} (\lambda+2\mu)\left(\dfrac{\partial^2 u_r^p}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial u_r^p}{\partial r}-\dfrac{u_r^p}{r^2}\right)+\mu\dfrac{\partial^2 u_r^p}{\partial z^2}+(\lambda+\mu)\dfrac{\partial^2 u_z^p}{\partial r\partial z}=f_r^p(r,z) \\ (\lambda+2\mu)\dfrac{\partial^2 u_z^p}{\partial z^2}+\mu\left(\dfrac{\partial^2 u_z^p}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial u_z^p}{\partial r}\right)+(\lambda+\mu)\left(\dfrac{\partial^2 u_r^p}{\partial r\partial z}+\dfrac{1}{r}\dfrac{\partial u_r^p}{\partial z}\right)=f_z^p(r,z) \end{array}\right. \tag{20}$$

where:

$$\left\{\begin{array}{l} f_r^p(r,z)=2\mu\left(\dfrac{\partial\varepsilon_{rr}^p}{\partial r}+2\dfrac{\varepsilon_{rr}^p}{r}+\dfrac{\varepsilon_{zz}^p}{r}\right) \\ f_z^p(r,z)=2\mu\dfrac{\partial\varepsilon_{zz}^p}{\partial z} \end{array}\right. \tag{21}$$

Following ideas developed in section 3 the right side term of the Navier equation is expanded into series:

$$\left\{\begin{array}{l} f_r^p(r,z)=\displaystyle\sum_{k=-K}^{K}\left[A_k^{p,r}J_1\left(\frac{r}{r_{sup}}\right)+B_k^{p,r}Y_1\left(\frac{r}{r_{sup}}\right)+\sum_{m=1}^{M}C_{m,k}^{p,r}G_m^{(1,1)}(r)\right]\exp\left(\frac{ik\pi}{L}z\right) \\ f_z^p(r,z)=\displaystyle\sum_{k=-K}^{K}\left[A_k^{p,z}J_0\left(\frac{r}{r_{sup}}\right)+B_k^{p,z}Y_0\left(\frac{r}{r_{sup}}\right)+\sum_{m=1}^{M}C_{m,k}^{p,z}G_m^{(0,1)}(r)+D_k^{p,z}\right]\exp\left(\frac{ik\pi}{L}z\right) \end{array}\right. \tag{22}$$

where $G_m^{(\alpha,\beta)}$ is defined by (2) and where $f_r^p(r,z)$ and the partial derivative of $f_z^p(r,z)$ with respect to r. As mentioned above, series expansion of the plastic strain should be obtained in order to write the plastic dissipation. Following expressions are solutions of (21) considering that the right side terms are given by (22):

$$\varepsilon_{zz}^{p} = \sum_{k=-K}^{K} \left[a_k^{p,z} J_0\left(\frac{r}{r_{sup}}\right) + b_k^{p,z} Y_0\left(\frac{r}{r_{sup}}\right) + \sum_{m=1}^{M} c_{m,k}^{p,z} G_m^{(0,1)}(r) + d_k^{p,z} \right] \exp\left(\frac{ik\pi}{L} z\right) \tag{23}$$

and:

$$\begin{aligned} \varepsilon_{rr}^{p} &= \sum_{k=-K}^{K} \left[a_k^{p,r} J_2\left(\frac{r}{r_{sup}}\right) + b_k^{p,r} Y_2\left(\frac{r}{r_{sup}}\right) + \sum_{m=1}^{M} c_{m,k}^{p,r} G_m^{(2,1)}(r) \right] \exp\left(\frac{ik\pi}{L} z\right) \\ &- \sum_{k=-K}^{K} \left[\frac{r_{sup}}{r} \left(a_k^{p,z} J_1\left(\frac{r}{r_{sup}}\right) + b_k^{p,z} Y_1\left(\frac{r}{r_{sup}}\right) + \sum_{m=1}^{M} \frac{c_{m,k}^{p,z}}{x_m^{(1)}} G_m^{(1,1)}(r) \right) + \frac{d_k^{p,z}}{2} \right] \exp\left(\frac{ik\pi}{L} z\right) \end{aligned} \tag{24}$$

with:

$$\left\{ \begin{aligned} A_k^{p,r} &= \frac{2\mu a_k^{p,r}}{r_{sup}} \\ B_k^{p,r} &= \frac{2\mu b_k^{p,r}}{r_{sup}} \\ C_{m,k}^{p,r} &= \frac{2\mu c_{m,k}^{p,r} x_m^{(1)}}{r_{sup}} \end{aligned} \right. \quad \text{and} \quad \left\{ \begin{aligned} A_k^{p,z} &= 2\mu a_k^{p,z} \frac{ik\pi}{L} \\ B_k^{p,z} &= 2\mu b_k^{p,z} \frac{ik\pi}{L} \\ C_{m,k}^{p,z} &= 2\mu c_{m,k}^{p,z} \frac{ik\pi}{L} \\ D_k^{p,z} &= 2\mu d_k^{p,z} \frac{ik\pi}{L} \end{aligned} \right. \tag{25}$$

Right side terms $f_r^p(r,z)$ and $f_z^p(r,z)$ are expressed as Bessel functions of the first and zero orders respectively alike $f_r^*(r,z)$ and $f_z^*(r,z)$ in (14). It should be noted that coefficients $a_k^{p,r}, b_k^{p,r}, c_{m,k}^{p,r}$ and $a_k^{p,z}, b_k^{p,z}, c_{m,k}^{p,z}$ are determined in section 7 through a minimization procedure according to (1). Displacements u_r^p and u_z^p are sought in the form:

$$\left\{ \begin{aligned} u_r^p(r,z) &= \sum_{k=-K}^{K} \left[U_k^{p,r} J_1\left(\frac{r}{r_{sup}}\right) + V_k^{p,r} Y_1\left(\frac{r}{r_{sup}}\right) + \sum_{m=1}^{M} u_{m,k}^{p,r} G_m^{(1,1)} \right] \exp\left(\frac{ik\pi}{L} z\right) \\ u_z^p(r,z) &= \sum_{k=-K}^{K} \left[U_k^{p,z} J_0\left(\frac{r}{r_{sup}}\right) + V_k^{p,z} Y_0\left(\frac{r}{r_{sup}}\right) + \sum_{m=1}^{M} u_{m,k}^{p,z} G_m^{(0,1)}(r) + W_k^{p,z} \right] \exp\left(\frac{ik\pi}{L} z\right) \end{aligned} \right. \tag{26}$$

Where for $k > 0$ (clearly $D_0^{p,z} = 0$ hence $W_0^{p,z} = 0$):

$$W_k^{p,z} = -\frac{D_k^{p,z}}{(k\pi/L)^2(\lambda + 2\mu)} \tag{27}$$

By inserting (26) into (20) and identifying, one obtains:

$$\begin{pmatrix} U_k^{p,r} \\ U_k^{p,z} \end{pmatrix} = \underline{\boldsymbol{S}}_k \cdot \begin{pmatrix} A_k^{p,r} \\ A_k^{p,z} \end{pmatrix} \text{ and } \begin{pmatrix} V_k^{p,r} \\ V_k^{p,z} \end{pmatrix} = \underline{\boldsymbol{S}}_k \cdot \begin{pmatrix} B_k^{p,r} \\ B_k^{p,z} \end{pmatrix} \text{ and } \begin{pmatrix} u_{m,k}^{p,r} \\ u_{m,k}^{p,z} \end{pmatrix} = \underline{\boldsymbol{S}}_{m,k}^{(1)} \cdot \begin{pmatrix} C_{m,k}^{p,r} \\ C_{m,k}^{p,z} \end{pmatrix} \tag{28}$$

where $\underline{\boldsymbol{S}}_k$ and $\underline{\boldsymbol{S}}_{m,k}^{(\beta)}$ are defined by (18) and (19). A particular displacement field u_r^p, u_z^p, solution of the inhomogeneous Navier equation (20) has been established. Therefore the associated stress field $\underline{\boldsymbol{\sigma}}^p$ can be computed as well, using the isotropic behavior. More precisely, a Fourier series expansion of $\underline{\boldsymbol{\sigma}}^p$ is obtained since the displacement field is known as a Fourier series expansion.

6 Homogeneous problem C

In this section additional homogenous solutions are derived in order to verify boundary conditions. Indeed only particular solutions of the inhomogeneous Navier equation have been exhibited so far, regardless of the surface traction denoted by $\boldsymbol{T}$. Thus the surface traction considered in this section is $\boldsymbol{T}^{tot} = \boldsymbol{T} - \underline{\boldsymbol{\sigma}}^{*}.\boldsymbol{n} - \underline{\boldsymbol{\sigma}}^{p}.\boldsymbol{n}$ as shown in figure 1. This part of the solution is classically obtained by using harmonic potential theory and bi-harmonic potentials. The problem being solved in this section is a simple tube subjected to the surface traction $\boldsymbol{T}^{tot}(r_{inf})$ and $\boldsymbol{T}^{tot}(r_{sup})$ at the inner and outer radii respectively with neither body forces nor imposed eigenstrain. This part of the solution is identical to those derived in [1] and similar to those presented in [8]. Classic harmonic and bi-harmonic potentials are used as exposed in [7]. One can mention that an hypercomplex potential formulation [11] could also have been used. Details are not exposed in this contribution since the originality of the present work relies more on the determination of the unknown plastic strain by minimization procedures. Thus, displacement and stress fields $\widehat{\boldsymbol{u}}$ and $\widehat{\underline{\boldsymbol{\sigma}}}$ are assumed to be known for each set of tested coefficients $a_k^{p,r}, b_k^{p,r}, c_{m,k}^{p,r}$ and $a_k^{p,z}, b_k^{p,z}, c_{m,k}^{p,z}$.

7 Energetic approach

The unknown plastic strain $\underline{\boldsymbol{\varepsilon}}^p$ or more precisely $\boldsymbol{C}^p = \left(a_k^{p,r}, b_k^{p,r}, c_{m,k}^{p,r}, a_k^{p,z}, b_k^{p,z}, c_{m,k}^{p,z}\right)$ are determined in this section using energetic arguments. The total elastic stored energy $E\left[\boldsymbol{C}^p\right]$ plus the dissipated energy associated to the plastic strain $D\left[\boldsymbol{C}^p\right]$ should be minimized and the problem reduces to (1). It should be noted that the minimization process contains all non-linear aspects of the problem. Indeed, the Navier equation remains linear even with an elastic-plastic behavior, only the determination of the plastic strain is not a linear procedure. The total stored energy reads:

$$E\left[\boldsymbol{C}^p\right] = \frac{1}{2}\int_V \underline{\boldsymbol{\sigma}} : \underline{\varepsilon}\mathrm{d}V \tag{29}$$

where V denotes the volume of the tube, $\underline{\boldsymbol{\sigma}} = \underline{\boldsymbol{\sigma}}^{*} + \underline{\boldsymbol{\sigma}}^{p} + \widehat{\underline{\boldsymbol{\sigma}}}$ is the total stress considering all the three problems A,B and C and $\underline{\varepsilon}$ is the associated strain with the isotropic behavior of Lamé's coefficients λ, μ. The dependence of the elastic energy on the plastic strain is not explicit in (29), however, $\underline{\boldsymbol{\sigma}}^{p}$ obviously depends on $\underline{\boldsymbol{\varepsilon}}^p$ as well as $\widehat{\underline{\boldsymbol{\sigma}}}$ because of the surface traction $\boldsymbol{T} - \underline{\boldsymbol{\sigma}}^{*}.\boldsymbol{n} - \underline{\boldsymbol{\sigma}}^{p}.\boldsymbol{n}$.

The equivalent plastic strain rate (or the cumulative plastic strain rate) is defined by:

$$\dot{\varepsilon}^{eq} = \sqrt{\frac{2}{3}\dot{\underline{\boldsymbol{\varepsilon}}}^p : \dot{\underline{\boldsymbol{\varepsilon}}}^p} \tag{30}$$

The dissipated energy is defined as:

$$D\left[\boldsymbol{C}^p\right] = \int_V \varepsilon^{eq}\sigma^Y \mathrm{d}V \tag{31}$$

where σ^Y denotes the yield stress and ε^{eq} depends explicitly on $\boldsymbol{C}^p$ through (24). Thus the minimization problem (1) enabling to determine the solution (denoted with the superscript s) reads:

$$\boldsymbol{C}^{p,s} = \underset{\boldsymbol{C}^p}{\text{argmin}}\ E\left[\boldsymbol{C}^p\right] + D\left[\boldsymbol{C}^p\right] \tag{32}$$

In practice the gradient free Nelder-Mead algorithm base on simplex updates and included in the free software Scilab [12] has been used in order to solve (32).

8 Results

In this section a numerical test is performed in order to validate the proposed approach. An rather arbitrary eigenstrain given by (33) and presented in figure 2 is imposed with free surface traction (i.e., $\boldsymbol{T} = 0$). Comparisons with a Finite Element computation, performed using Castem [13] are proposed. Even though the proposed approach is intended to coil modeling for which layers are very thin compared to their width (usual thicknesses are less than 1 mm for widths greater than 1 m) a very thick hollow cylinder is considered in this paper in order to avoid long computation times for the FEM. Indeed, a very thin layer would have necessitated a refined mesh. Geometrical parameters are listed in table 1. Practically an axi-symmetrical Finite Element simulation has been performed with linear $N_r \times N_z$ quadrangular elements.

$$\underline{\underline{\varepsilon}}^* = 0.2\frac{(r - r_{sup})(r - r_{inf})}{r_{sup} r_{inf}} \cos\left(\pi \frac{z}{L}\right) (\boldsymbol{e}_r \otimes \boldsymbol{e}_r - \boldsymbol{e}_\theta \otimes \boldsymbol{e}_\theta) \tag{33}$$

Table 1: Geometrical parameters

L	40	(mm)
r_{inf}	15	(mm)
r_{sup}	20	(mm)
N_r	20	
N_z	40	

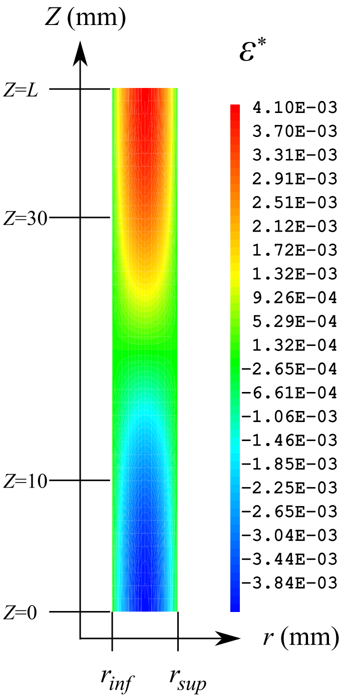
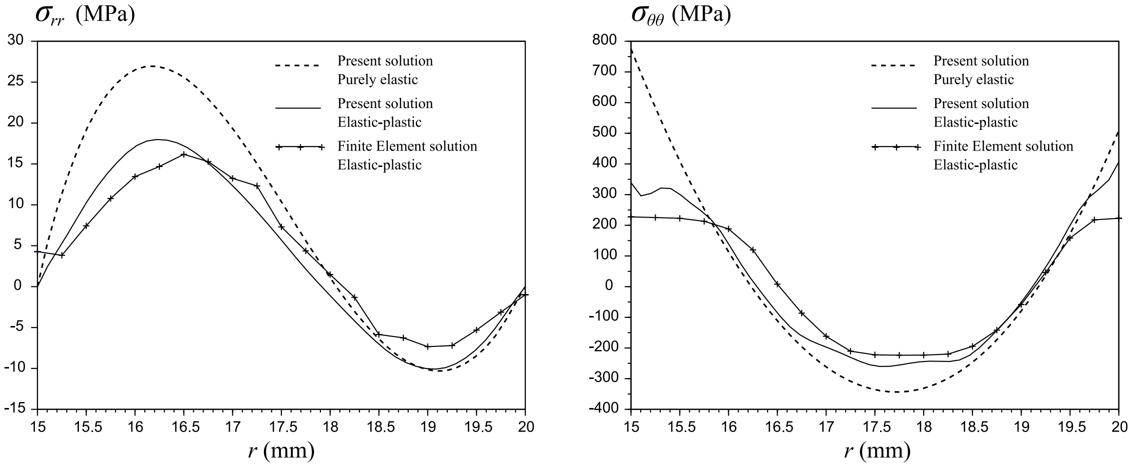
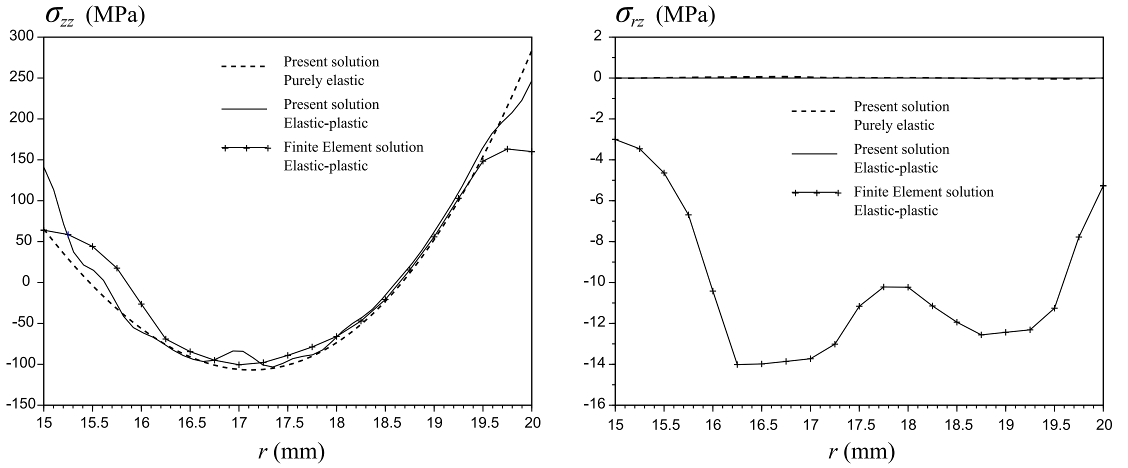


Figure 2: Imposed eigenstrain

Comparisons are presented for three axial positions $Z = 0$ mm and $Z = 10$ and 30 mm in figures 3 and 5. A good agreement is observed between the presented solution and the Finite Element computation. The purely elastic solution is added in order to show the effect of plasticity. It should be mentioned that boundary conditions are correctly verified by the proposed approach and not correctly verified by the Finite Element computation, indeed σ_{rr} and σ_{rz} should vanish at $r = r_{inf}$ and $r = r_{sup}$ and σ_{rz} should vanish at $Z = 0$ considering the symmetry of the problem. This aspect may explain the discrepancy observed for σ_{rz} in figures 3d and 5d. Moreover the equivalent plastic strain (or the cumulative plastic strain) is quite well predicted by the proposed model as shown in figures 4 and 6.



(a) σ_{rr} at $Z = 0$ (b) $\sigma_{\theta\theta}$ at $Z = 0$



(c) σ_{zz} at $Z = 0$ (d) σ_{rz} at $Z = 0$

Figure 3: Test 1: Stress comparision with Finite Element simulation

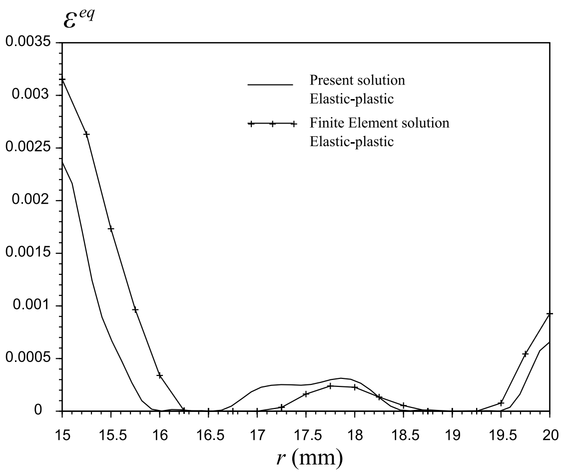
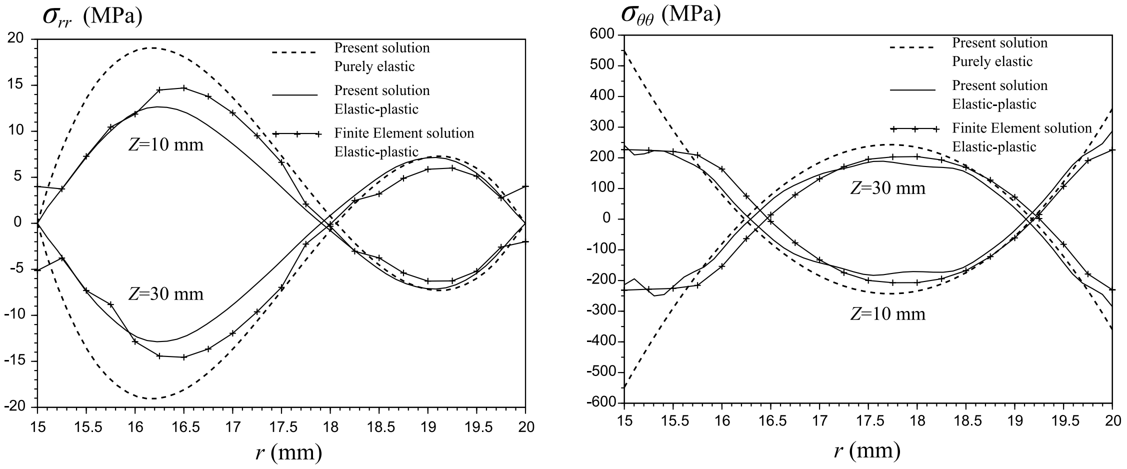
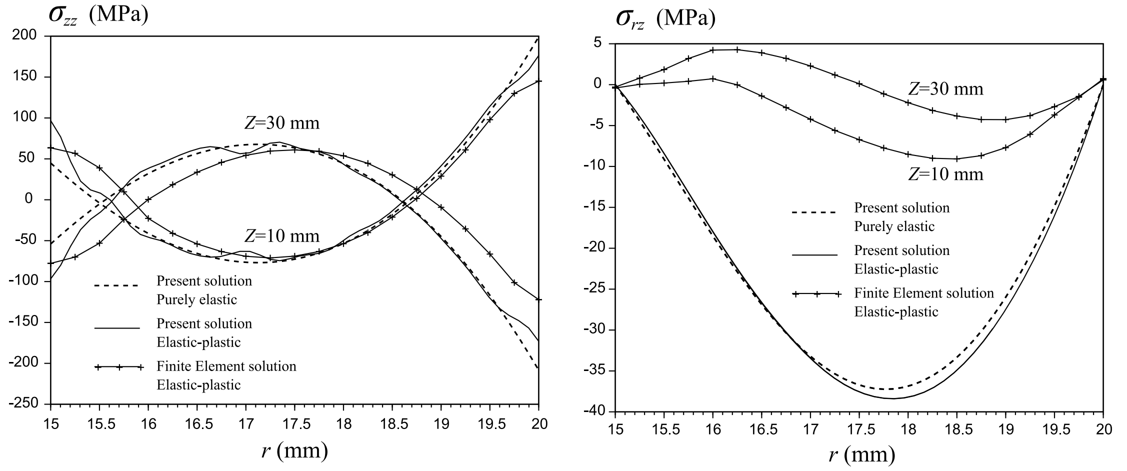


Figure 4: ε^{eq} at $Z = 0$



(a) σ_{rr} at $Z = 10$ mm and $Z = 30$ mm (b) $\sigma_{\theta\theta}$ at $Z = 10$ mm and $Z = 30$ mm



(c) σ_{zz} at $Z = 10$ mm and $Z = 30$ mm (d) σ_{rz} at $Z = 10$ mm and $Z = 30$ mm

Figure 5: Stress comparision with Finite Element simulation

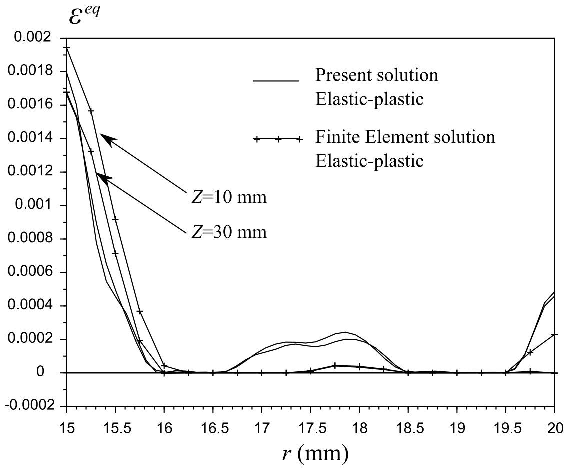


Figure 6: Test 1: ε^{eq} at $Z = 10$ mm and $Z = 30$ mm

9 Conclusion

A mixed analytical/numerical method has been developed to solve the problem of an hollow cylinder subjected to arbitrary eigenstrain and surface traction. An unknown deviatoric plastic strain is added to the imposed eigenstrain and the linear inhomogeneous Navier equation is solved analytically by introducing specific series expansions. The unknown plastic strain is then identified eventually by minimizing the sum of the elastic energy and the dissipated energy associated to the unknown plastic strain. This approach enables to use conveniently the linearity of the Navier equation even though plasticity is considered, non-linear aspects being encompassed in the energetic formulation. A comparison with a Finite Element computation has been proposed and good agreement is observed. This paper is part of a more general numerical strategy consisting in modeling residual stresses (generated by phase transitions, transformation induced plasticity etc...) in coils.

References

[1] Daniel Weisz-Patrault. Residual stress generated by transformation induced plasticity. application to stability of coils. *International Journal of Solids and Structures*, 2017. submission.

[2] Daniel Weisz-Patrault. Multiphase model for transformation induced plasticity. extended leblond's model. *Journal of the Mechanics and Physics of Solids*, 2017. submission.

[3] Jean-Baptiste Leblond, G Mottet, and JC Devaux. A theoretical and numerical approach to the plastic behaviour of steels during phase transformations i. derivation of general relations. *Journal of the Mechanics and Physics of Solids*, 34(4):395–409, 1986.

[4] Jean-Baptiste Leblond, G Mottet, and JC Devaux. A theoretical and numerical approach to the plastic behaviour of steels during phase transformations ii. study of classical plasticity for ideal-plastic phases. *Journal of the Mechanics and Physics of Solids*, 34(4):411–432, 1986.

[5] Jean-Baptiste Leblond, J Devaux, and JC Devaux. Mathematical modelling of transformation plasticity in steels i: case of ideal-plastic phases. *International journal of plasticity*, 5(6):551–572, 1989.

[6] Jean-Baptiste Leblond. Mathematical modelling of transformation plasticity in steels ii: coupling with strain hardening phenomena. *International journal of plasticity*, 5(6):573–591, 1989.

[7] J.R. Barber. *Elasticity*, volume 107 of *Solid Mechanics and Its Applications*. Springer, Berlin, 2003.

[8] Daniel Weisz-Patrault, Alain Ehrlacher, and Nicolas Legrand. Evaluation of contact stress during rolling process, by three dimensional analytical inverse method. *International Journal of Solids and Structures*, 50(20-21):3319 – 3331, 2013.

[9] B Fedelich and A Ehrlacher. An analysis of stability of equilibrium and of quasi-static transformations on the basis of the dissipation function. *European journal of mechanics. A. Solids*, 16(5):833–855, 1997.

[10] Alexander Mielke. Energetic formulation of multiplicative elasto-plasticity using dissipation distances. *Continuum Mechanics and Thermodynamics*, 15(4):351–382, 2003.

[11] Daniel Weisz-Patrault, Sebastian Bock, and Klaus Gürlebeck. Three-dimensional elasticity based on quaternion-valued potentials. *International Journal of Solids and Structures*, 51(19):3422–3430, 2014.

[12] Scilab. Scilab: Free and open source software for numerical computation. *Scilab Enterprises, Orsay, France*, 2012.

[13] CEA. Cast3m, 2011. Commissariat A l'Energie Atomique, http://www-cast3m.cea.fr/.