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Comparison of Differential Operators with Lie Derivative of Three-Dimensional Real Hypersurfaces in Non-Flat Complex Space Forms

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Received: 29 March 2018; Accepted: 14 May 2018; Published: 20 May 2018

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Abstract: In this paper, three-dimensional real hypersurfaces in non-flat complex space forms, whose shape operator satisfies a geometric condition, are studied. Moreover, the tensor field $P = \phi A - A\phi$ is given and three-dimensional real hypersurfaces in non-flat complex space forms whose tensor field P satisfies geometric conditions are classified.

Keywords: k-th generalized Tanaka–Webster connection; non-flat complex space form; real hypersurface; lie derivative; shape operator

2010 Mathematics Subject Classification: 53C15; 53B25

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1. Introduction

A *real hypersurface* is a submanifold of a Riemannian manifold with a real co-dimensional one. Among the Riemannian manifolds, it is of great interest in the area of Differential Geometry to study real hypersurfaces in complex space forms. A *complex space form* is a Kähler manifold of dimension n and constant holomorphic sectional curvature c. In addition, complete and simply connected complex space forms are analytically isometric to complex projective space $\mathbb{C}P^n$ if $c > 0$, to complex Euclidean space $\mathbb{C}^n$ if $c = 0$, or to complex hyperbolic space $\mathbb{C}H^n$ if $c < 0$. The notion of non-flat complex space form refers to complex projective and complex hyperbolic space when it is not necessary to distinguish between them and is denoted by $M_n(c), n \geq 2$.

Let J be the Kähler structure and $\tilde{\nabla}$ the Levi–Civita connection of the non-flat complex space form $M_n(c), n \geq 2$. Consider M a connected real hypersurface of $M_n(c)$ and N a locally defined unit normal vector field on M. The Kähler structure induces on M an *almost contact metric structure* (ϕ, ξ, η, g). The latter consists of a tensor field of type (1, 1) ϕ called *structure tensor field*, a one-form η, a vector field ξ given by $\xi = -JN$ known as the *structure vector field* of M and g, which is the induced Riemannian metric on M by G. Among real hypersurfaces in non-flat complex space forms, the class of *Hopf hypersurfaces* is the most important. A Hopf hypersurface is a real hypersurface whose structure vector field ξ is an eigenvector of the shape operator A of M .

Takagi initiated the study of real hypersurfaces in non-flat complex space forms. He provided the classification of homogeneous real hypersurfaces in complex projective space $\mathbb{C}P^n$ and divided them into five classes (A), (B), (C), (D) and (E) (see [1–3]). Later, Kimura proved that homogeneous real hypersurfaces in complex projective space are the unique Hopf hypersurfaces with constant principal curvatures, i.e., the eigenvalues of the shape operator A are constant (see [4]). Among the above real hypersurfaces, the three-dimensional real hypersurfaces in $\mathbb{C}P^2$ are geodesic hyperspheres of radius r, $0 < r < \frac{\pi}{2}$, called real hypersurfaces of type (A) and tubes of radius r, $0 < r < \frac{\pi}{4}$, over the complex quadric called real hypersurfaces of type (B). Table 1 includes the values of the constant principal curvatures corresponding to the real hypersurfaces above (see [1,2]).

Table 1. Principal curvatures of real hypersurfaces in $\mathbb{C}P^2$.

Type	α	λ_1	ν	m_α	m_{λ_1}	m_ν
(A)	$2\cot(2r)$	$\cot(r)$	-	1	2	-
(B)	$2\cot(2r)$	$\cot(r-\frac{\pi}{4})$	$-\tan(r-\frac{\pi}{4})$	1	1	1

The study of Hopf hypersurfaces with constant principal curvatures in complex hyperbolic space $\mathbb{C}H^n, n \geq 2$, was initiated by Montiel in [5] and completed by Berndt in [6]. They are divided into two types: type (A), which are open subsets of horospheres (A_0), geodesic hyperspheres $(A_{1,0})$, or tubes over totally geodesic complex hyperbolic hyperplane $\mathbb{C}H^{n-1}$ $(A_{1,1})$ and type (B), which are open subsets of tubes over totally geodesic real hyperbolic space $\mathbb{R}H^n$. Table 2 includes the values of the constant principal curvatures corresponding to above real hypersurfaces for $n = 2$ (see [6]).

Table 2. Principal curvatures of real hypersurfaces in $\mathbb{C}H^2$.

Type	α	λ	ν	m_α	m_λ	m_ν
(A_0)	2	1	-	1	2	-
$(A_{1,1})$	$2\coth(2r)$	$\coth(r)$	-	1	2	-
$(A_{1,2})$	$2\coth(2r)$	$\tanh(r)$	-	1	2	-
(B)	$2\tanh(2r)$	$\tanh(r)$	$\coth(r)$	1	1	1

The Levi–Civita connection $\tilde{\nabla}$ of the non-flat complex space form $M_n(c), n \geq 2$ induces on M a Levi–Civita connection ∇. Apart from the last one, Cho in [7,8] introduces the notion of the *k-th generalized Tanaka–Webster connection* $\hat{\nabla}^{(k)}$ on a real hypersurface in non-flat complex space form given by

$$\hat{\nabla}^{(k)}_X Y = \nabla_X Y + g(\phi AX, Y)\xi - \eta(Y)\phi AX - k\eta(X)\phi Y, \tag{1}$$

for all X, Y tangent to M , where k is a nonnull real number. The latter is an extension of the definition of *generalized Tanaka–Webster connection* for contact metric manifolds given by Tanno in [9] and satisfying the relation

$$\hat{\nabla}_X Y = \nabla_X Y + (\nabla_X \eta)(Y)\xi - \eta(Y)\nabla_X \xi - \eta(X)\phi Y.$$

The following relations hold:

$$\hat{\nabla}^{(k)}\eta = 0, \quad \hat{\nabla}^{(k)}\xi = 0, \quad \hat{\nabla}^{(k)}g = 0, \quad \hat{\nabla}^{(k)}\phi = 0.$$

In particular, if the shape operator of a real hypersurface satisfies $\phi A + A\phi = 2k\phi$, the generalized Tanaka–Webster connection coincides with the Tanaka–Webster connection.

The k-th Cho operator on M associated with the vector field X is denoted by $\hat{F}^{(k)}_X$ and given by

$$\hat{F}^{(k)}_X Y = g(\phi AX, Y)\xi - \eta(Y)\phi AX - k\eta(X)\phi Y, \tag{2}$$

for any Y tangent to M. Then, the torsion of the k-th generalized Tanaka–Webster connection $\hat{\nabla}^{(k)}$ is given by

$$T^{(k)}(X,Y) = \hat{F}^{(k)}_X Y - \hat{F}^{(k)}_Y X,$$

for any X, Y tangent to M. Associated with the vector field X, the *k-th torsion operator* $T^{(k)}_X$ is defined and given by

$$T_X^{(k)}Y = T^{(k)}(X,Y),$$

for any Y tangent to M.

The existence of Levi–Civita and k-th generalized Tanaka–Webster connections on a real hypersurface implies that the covariant derivative can be expressed with respect to both connections. Let K be a tensor field of type (1, 1); then, the symbols ∇K and $\hat{\nabla}^{(k)}K$ are used to denote the covariant derivatives of K with respect to the Levi–Civita and the k-th generalized Tanaka–Webster connection, respectively. Furthermore, the Lie derivative of a tensor field K of type (1, 1) with respect to Levi–Civita connection $\mathcal{L}K$ is given by

$$(\mathcal{L}_X K)Y = \nabla_X(KY) - \nabla_{KY}X - K\nabla_X Y + K\nabla_Y X, \tag{3}$$

for all X, Y tangent to M . Another first order differential operator of a tensor field K of type (1, 1) with respect to the k-th generalized Tanaka–Webster connection $\hat{\mathcal{L}}^{(k)}K$ is defined and it is given by

$$(\hat{\mathcal{L}}_X^{(k)}K)Y = \hat{\nabla}_X^{(k)}(KY) - \hat{\nabla}_{KY}^{(k)}X - K(\hat{\nabla}_X^{(k)}Y) + K(\hat{\nabla}_Y^{(k)}X), \tag{4}$$

for all X, Y tangent to M .

Due to the existence of the above differential operators and derivatives, the following questions come up

1. Are there real hypersurfaces in non-flat complex space forms whose derivatives with respect to different connections coincide?
2. Are there real hypersurfaces in non-flat complex space forms whose differential operator $\hat{\mathcal{L}}^{(k)}$ coincides with derivatives with respect to different connections?

The first answer is obtained in [10], where the classification of real hypersurfaces in complex projective space $\mathbb{C}P^n$,$n \geq 3$, whose covariant derivative of the shape operator with respect to the Levi–Civita connection coincides with the covariant derivative of it with respect to the k-th generalized Tanaka–Webster connection is provided, i.e., $\nabla_X A = \hat{\nabla}_X^{(k)}A$, where X is any vector field on M. Next, in [11], real hypersurfaces in complex projective space $\mathbb{C}P^n$, $n \geq 3$, whose Lie derivative of the shape operator coincides with the operator $\hat{\mathcal{L}}^{(k)}$ are studied, i.e., $\mathcal{L}_X A = \hat{\mathcal{L}}_X^{(k)}A$, where X is any vector field on M. Finally, in [12], the problem of classifying three-dimensional real hypersurfaces in non-flat complex space forms $M_2(c)$, for which the operator $\hat{\mathcal{L}}^{(k)}$ applied to the shape operator coincides with the covariant derivative of it, has been studied, i.e., $\hat{\mathcal{L}}_X^{(k)}A = \nabla_X A$, for any vector field X tangent to M.

In this paper, the condition $\mathcal{L}_X A = \hat{\mathcal{L}}_X^{(k)}A$, where X is any vector field on M is studied in the case of three-dimensional real hypersurfaces in $M_2(c)$.

The aim of the present paper is to complete the work of [11] in the case of three-dimensional real hypersurfaces in non-flat complex space forms $M_2(c)$. The equality $\mathcal{L}_X A = \hat{\mathcal{L}}_X^{(k)}A$ is equivalent to the fact that $T_X^{(k)}A = AT_X^{(k)}$. Thus, the eigenspaces of A are preserved by the k-th torsion operator $T_X^{(k)}$, for any X tangent to M . First, three-dimensional real hypersurfaces in $M_2(c)$ whose shape operator A satisfies the following relation:

$$\hat{\mathcal{L}}_X^{(k)}A = \mathcal{L}_X A, \tag{5}$$

for any X orthogonal to ξ are studied and the following Theorem is proved:

Theorem 1. *There do not exist real hypersurfaces in $M_2(c)$ whose shape operator satisfies relation (5).*

Next, three-dimensional real hypersurfaces in $M_2(c)$ whose shape operator satisfies the following relation are studied:

$$\hat{\mathcal{L}}_{\xi}^{(k)} A = \mathcal{L}_{\xi} A, \tag{6}$$

and the following Theorem is provided.

Theorem 2. *Every real hypersurface in $M_2(c)$ whose shape operator satisfies relation (6) is locally congruent to a real hypersurface of type (A).*

As an immediate consequence of the above theorems, it is obtained that

Corollary 1. *There do not exist real hypersurfaces in $M_2(c)$ such that $\hat{\mathcal{L}}_X^{(k)} A = \mathcal{L}_X A$, for all $X \in TM$.*

Next, the following tensor field P of type (1, 1) is introduced:

$$PX = \phi AX - A\phi X,$$

for any vector field X tangent to M. The relation $P = 0$ implies that the shape operator commutes with the structure tensor ϕ. Real hypersurfaces whose shape operator A commutes with the structure tensor ϕ have been studied by Okumura in the case of $\mathbb{C}P^n$, $n \geq 2$, (see [13]) and by Montiel and Romero in the case of $\mathbb{C}H^n$, $n \geq 2$ (see [14]). The following Theorem provides the above classification of real hypersurfaces in $M_n(c)$, $n \geq 2$.

Theorem 3. *Let M be a real hypersurface of $M_n(c)$, $n \geq 2$. Then, $A\phi = \phi A$, if and only if M is locally congruent to a homogeneous real hypersurface of type (A). More precisely:*
In the case of $\mathbb{C}P^n$

- (A_1) *a geodesic hypersphere of radius r , where $0 < r < \frac{\pi}{2}$,*
- (A_2) *a tube of radius r over a totally geodesic $\mathbb{C}P^k$,$(1 \leq k \leq n-2)$, where $0 < r < \frac{\pi}{2}$.*

In the case of $\mathbb{C}H^n$,

- (A_0) *a horosphere in $\mathbb{C}H^n$, i.e., a Montiel tube,*
- (A_1) *a geodesic hypersphere or a tube over a totally geodesic complex hyperbolic hyperplane $\mathbb{C}H^{n-1}$,*
- (A_2) *a tube over a totally geodesic $\mathbb{C}H^k$ $(1 \leq k \leq n-2)$.*

Remark 1. *In the case of three-dimensional real hypersurfaces in $M_2(c)$, real hypersurfaces of type (A_2) do not exist.*

It is interesting to study real hypersurfaces in non-flat complex spaces forms, whose tensor field P satisfies certain geometric conditions. We begin by studying three-dimensional real hypersurfaces in $M_2(c)$ whose tensor field P satisfies the relation

$$(\hat{\mathcal{L}}_X^{(k)} P)Y = (\mathcal{L}_X P)Y, \tag{7}$$

for any vector fields X, Y tangent to M.

First, the following Theorem is proved:

Theorem 4. *Every real hypersurface in $M_2(c)$ whose tensor field P satisfies relation (7) for any X orthogonal to ξ and $Y \in TM$ is locally congruent to a real hypersurface of type (A).*

Next, we study three-dimensional real hypersurfaces in $M_2(c)$ whose tensor field P satisfies relation (7) for $X = \xi$, i.e.,

$$(\hat{\mathcal{L}}_{\xi}^{(k)} P)Y = (\mathcal{L}_{\xi} P)Y, \tag{8}$$

for any vector field Y tangent to M. Then, the following Theorem is proved:

Theorem 5. *Every real hypersurface in $M_2(c)$ whose tensor field P satisfies relation (8) is a Hopf hypersurface. In the case of $\mathbb{C}P^2$, M is locally congruent to a real hypersurface of type (A) or to a real hypersurface of type (B) with $\alpha = -2k$ and in the case of $\mathbb{C}H^2$ M is a locally congruent either to a real hypersurface of type (A) or to a real hypersurface of type (B) with $\alpha = \dfrac{4}{k}$.*

This paper is organized as follows: in Section 2, basic relations and theorems concerning real hypersurfaces in non-flat complex space forms are presented. In Section 3, analytic proofs of Theorems 1 and 2 are provided. Finally, in Section 4, proofs of Theorems 4 and 5 are given.

2. Preliminaries

Throughout this paper, all manifolds, vector fields, etc. are considered of class C^∞ and all manifolds are assumed to be connected.

The non-flat complex space form $M_n(c)$, $n \geq 2$ is equipped with a Kähler structure J and G is the Kählerian metric. The constant holomorphic sectional curvature c in the case of complex projective space $\mathbb{C}P^n$ is $c = 4$ and in the case of complex hyperbolic space $\mathbb{C}H^n$ is $c = -4$. The Levi–Civita connection of the non-flat complex space form is denoted by $\overline{\nabla}$.

Let M be a connected real hypersurface immersed in $M_n(c)$, $n \geq 2$, without boundary and N be a locally defined unit normal vector field on M. The shape operator A of the real hypersurface M with respect to the vector field N is given by

$$\overline{\nabla}_X N = -AX.$$

The Levi–Civita connection ∇ of the real hypersurface M satisfies the relation

$$\overline{\nabla}_X Y = \nabla_X Y + g(AX, Y)N.$$

The Kähler structure of the ambient space induces on M an almost contact metric structure (ϕ, ξ, η, g) in the following way: any vector field X tangent to M satisfies the relation

$$JX = \phi X + \eta(X)N.$$

The tangential component of the above relation defines on M a skew-symmetric tensor field of type $(1,1)$ denoted by ϕ known as *the structure tensor*. The structure vector field ξ is defined by $\xi = -JN$ and the 1-form η is given by $\eta(X) = g(X, \xi)$ for any vector field X tangent to M. The elements of the almost contact structure satisfy the following relation:

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{9}$$

for all tangent vectors X, Y to M. Relation (9) implies

$$\phi\xi = 0, \quad \eta(X) = g(X, \xi).$$

Because of $\overline{\nabla} J = 0$, it is obtained

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi \quad \text{and} \quad \nabla_X \xi = \phi AX$$

for all X, Y tangent to M. Moreover, the Gauss and Codazzi equations of the real hypersurface are respectively given by

$$R(X,Y)Z = \frac{c}{4}[g(Y,Z)X - g(X,Z)Y + g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y \\ -2g(\phi X,Y)\phi Z] + g(AY,Z)AX - g(AX,Z)AY, \quad (10)$$

and

$$(\nabla_X A)Y - (\nabla_Y A)X = \frac{c}{4}[\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X,Y)\xi], \quad (11)$$

for all vectors X, Y, Z tangent to M, where R is the curvature tensor of M.

The tangent space T_pM at every point $p \in M$ is decomposed as

$$T_pM = span\{\xi\} \oplus \mathbb{D}, \quad (12)$$

where $\mathbb{D} = \ker\eta = \{X \in T_pM : \eta(X) = 0\}$ and is called (*maximal*) *holomorphic distribution* (*if* $n \geq 3$).

Next, the following results concern any non-Hopf real hypersurface M in $M_2(c)$ with local orthonormal basis $\{U, \phi U, \xi\}$ at a point p of M.

Lemma 1. *Let M be a non-Hopf real hypersurface in $M_2(c)$. The following relations hold on M:*

$$\begin{aligned} &AU = \gamma U + \delta\phi U + \beta\xi, && A\phi U = \delta U + \mu\phi U, && A\xi = \alpha\xi + \beta U, \\ &\nabla_U\xi = -\delta U + \gamma\phi U, && \nabla_{\phi U}\xi = -\mu U + \delta\phi U, && \nabla_\xi\xi = \beta\phi U, \\ &\nabla_U U = \kappa_1\phi U + \delta\xi, && \nabla_{\phi U}U = \kappa_2\phi U + \mu\xi, && \nabla_\xi U = \kappa_3\phi U, \\ &\nabla_U\phi U = -\kappa_1 U - \gamma\xi, && \nabla_{\phi U}\phi U = -\kappa_2 U - \delta\xi, && \nabla_\xi\phi U = -\kappa_3 U - \beta\xi, \end{aligned} \quad (13)$$

where $\alpha, \beta, \gamma, \delta, \mu, \kappa_1, \kappa_2, \kappa_3$ are smooth functions on M and $\beta \neq 0$.

Remark 2. *The proof of Lemma 1 is included in [15].*

The Codazzi equation for $X \in \{U, \phi U\}$ and $Y = \xi$ implies, because of Lemma 1, the following relations:

$$\xi\delta = \alpha\gamma + \beta\kappa_1 + \delta^2 + \mu\kappa_3 + \frac{c}{4} - \gamma\mu - \gamma\kappa_3 - \beta^2, \quad (14)$$

$$\xi\mu = \alpha\delta + \beta\kappa_2 - 2\delta\kappa_3, \quad (15)$$

$$(\phi U)\alpha = \alpha\beta + \beta\kappa_3 - 3\beta\mu, \quad (16)$$

$$(\phi U)\beta = \alpha\gamma + \beta\kappa_1 + 2\delta^2 + \frac{c}{2} - 2\gamma\mu + \alpha\mu, \quad (17)$$

and for $X = U$ and $Y = \phi U$

$$U\delta - (\phi U)\gamma = \mu\kappa_1 - \kappa_1\gamma - \beta\gamma - 2\delta\kappa_2 - 2\beta\mu. \quad (18)$$

The following Theorem refers to Hopf hypersurfaces. In the case of complex projective space $\mathbb{C}P^n$, it is given by Maeda [16], and, in the case of complex hyperbolic space $\mathbb{C}H^n$, it is given by Ki and Suh [17] (see also Corollary 2.3 in [18]).

Theorem 6. *Let M be a Hopf hypersurface in $M_n(c)$, $n \geq 2$. Then,*

(i) $\alpha = g(A\xi, \xi)$ *is constant.*
(ii) *If W is a vector field, which belongs to $\mathbb{D}$ such that $AW = \lambda W$, then*

$$(\lambda - \frac{\alpha}{2})A\phi W = (\frac{\lambda\alpha}{2} + \frac{c}{4})\phi W.$$

(iii) *If the vector field W satisfies $AW = \lambda W$ and $A\phi W = \nu\phi W$, then*

$$\lambda\nu = \frac{\alpha}{2}(\lambda + \nu) + \frac{c}{4}. \tag{19}$$

Remark 3. *Let M be a three-dimensional Hopf hypersurface in $M_2(c)$. Since M is a Hopf hypersurface relation $A\xi = \alpha\xi$, it holds when $\alpha = constant$. At any point $p \in M$, we consider a unit vector field $W \in \mathbb{D}$ such that $AW = \lambda W$. Then, the unit vector field ϕW is orthogonal to W and ξ and relation $A\phi W = \nu\phi W$ holds. Therefore, at any point $p \in M$, we can consider the local orthonormal frame $\{W, \phi W, \xi\}$ and the shape operator satisfies the above relations.*

3. Proofs of Theorems 1 and 2

Suppose that M is a real hypersurface in $M_2(c)$ whose shape operator satisfies relation (5), which because of the relation of k-th generalized Tanaka-Webster connection (1) becomes

$$\begin{aligned} &g((A\phi A + A^2\phi)X, Y)\xi - g((A\phi + \phi A)X, Y)A\xi + k\eta(AY)\phi X + \eta(Y)A\phi AX \\ &-\eta(AY)\phi AX - k\eta(Y)A\phi X = 0, \end{aligned} \tag{20}$$

for any $X \in \mathbb{D}$ and for all $Y \in TM$.

Let $\mathcal{N}$ be the open subset of M such that

$$\mathcal{N} = \{p \in M : \beta \neq 0, \text{ in a neighborhood of } p\}.$$

The inner product of relation (20) for $Y = \xi$ with ξ due to relation (13) implies $\delta = 0$ and the shape operator on the local orthonormal basis $\{U, \phi U, \xi\}$ becomes

$$A\xi = \alpha\xi + \beta U, \quad AU = \gamma U + \beta\xi \text{ and } A\phi U = \mu\phi U. \tag{21}$$

Relation (20) for $X = Y = U$ and $X = \phi U$ and $Y = \xi$ due to (21) yields, respectively,

$$\gamma = k \text{ and } \mu = 0. \tag{22}$$

Differentiation of $\gamma = k$ with respect to ϕU taking into account that k is a nonzero real number implies $(\phi U)\gamma = 0$. Thus, relation (18) results, because of $\delta = \mu = 0$, in $\kappa_1 = -\beta$. Furthermore, relations (14)–(17) due to $\delta = 0$ and relation (22) become

$$\alpha k + \frac{c}{4} = 2\beta^2 + k\kappa_3, \tag{23}$$

$$\kappa_2 = 0, \tag{24}$$

$$(\phi U)\alpha = \beta(\alpha + \kappa_3), \tag{25}$$

$$(\phi U)\beta = \alpha k - \beta^2 + \frac{c}{2}. \tag{26}$$

The inner product of Codazzi equation (11) for $X = U$ and $Y = \xi$ with U and ξ implies because of $\delta = 0$ and relation (21),

$$U\alpha = U\beta = \xi\beta = \xi\gamma = 0. \tag{27}$$

The Lie bracket of U and ξ satisfies the following two relations:

$$\begin{aligned} [U, \xi]\beta &= U(\xi\beta) - \xi(U\beta), \\ [U, \xi]\beta &= (\nabla_U\xi - \nabla_\xi U)\beta. \end{aligned}$$

A combination of the two relations above taking into account relations of Lemma 1 and (27) yields

$$(k - \kappa_3)[(\phi U)\beta] = 0.$$

Suppose that $k \neq \kappa_3$, then $(\phi U)\beta = 0$ and relation (26) implies $\alpha k + \frac{c}{2} = \beta^2$. Differentiation of the last one with respect to ϕU results, taking into account relation (25), in $\kappa_3 = -\alpha$. The Riemannian curvature satisfies the relation

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

for any X, Y, Z tangent to M. Combination of the last relation with Gaussian Equation (10) for $X = U$, $Y = \phi U$ and $Z = U$ due to relation (22) and relation (24), $\kappa_1 = -\beta$, $\kappa_3 = -\alpha$ and $(\phi U)\beta = 0$ implies $c = 0$, which is a contradiction.

Therefore, on M, relation $k = \kappa_3$ holds. A combination of $R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$ with Gauss Equation (10) for $X = U$, $Y = \phi U$ and $Z = U$ because of relations (22) and (26) and $\kappa_1 = -\beta$ yields

$$k^2 = -\alpha k - \frac{3c}{2}.$$

A combination of the latter with relation (23) implies

$$\beta^2 + k^2 = -\frac{5c}{8}.$$

Differentiation of the above relation with respect to ϕU gives, due to relation (26) and $k^2 = -\alpha k - \frac{3c}{2}$,

$$\beta^2 + k^2 = -\frac{c}{2}.$$

If the ambient space is the complex projective space $\mathbb{C}P^2$ with $c = 4$, then the above relation leads to a contradiction. If the ambient space is the complex hyperbolic space $\mathbb{C}H^2$ with $c = -4$, combination of the latter relation with $\beta^2 + k^2 = -\frac{5c}{8}$ yields $c = 0$, which is a contradiction.

Thus, $\mathcal{N}$ is empty and the following proposition is proved:

Proposition 1. *Every real hypersurface in $M_2(c)$ whose shape operator satisfies relation (5) is a Hopf hypersurface.*

Since M is a Hopf hypersurface, Theorem 6 and remark 3 hold. Relation (20) for $X = W$ and for $X = \phi W$ implies, respectively,

$$(\lambda - k)(\nu - \alpha) = 0 \text{ and } (\nu - k)(\lambda - \alpha) = 0. \tag{28}$$

Combination of the above relations results in

$$(\nu - \lambda)(\alpha - k) = 0.$$

If $\lambda \neq \nu$, then $\alpha = k$ and relation $(\lambda - k)(\nu - \alpha) = 0$ becomes

$$(\lambda - \alpha)(\nu - \alpha) = 0.$$

If $\nu \neq \alpha$, then $\lambda = \alpha$ and relation (19) implies that ν is also constant. Therefore, the real hypersurface is locally congruent to a real hypersurface of type (B). Substitution of the values of eigenvalues in relation $\lambda = \alpha$ leads to a contradiction. Thus, on M, relation $\nu = \alpha$ holds. Following similar steps to the previous case, we are led to a contradiction.

Therefore, on M, we have $\lambda = \nu$ and the first of relations (28) becomes

$$(\lambda - k)(\lambda - \alpha) = 0.$$

Supposing that $\lambda \neq k$, then $\lambda = \nu = \alpha$. Thus, the real hypersurface is totally umbilical, which is impossible since there do not exist totally umbilical real hypersurfaces in non-flat complex space forms [18].

Thus, on M relation $\lambda = k$ holds. Relation (20) for $X = W$ and $Y = \phi W$ implies, because of $\lambda = \nu = k$, $\lambda = \alpha$. Thus, $\lambda = \nu = \alpha$ and the real hypersurface is totally umbilical, which is a contradiction and this completes the proof of Theorem 1.

Next, suppose that M is a real hypersurface in $M_2(c)$ whose shape operator satisfies relation (6), which, because of the relation of the k-th generalized Tanaka-Webster connection (1), becomes

$$(A\phi - \phi A)AX - g(\phi A\xi, AX)\xi + \eta(AX)\phi A\xi + k\phi AX + g(\phi A\xi, X)A\xi \\ -\eta(X)A\phi A\xi - kA\phi X = 0, \tag{29}$$

for any $X \in TM$.

Let $\mathcal{N}$ be the open subset of M such that

$$\mathcal{N} = \{p \in M : \beta \neq 0, \text{ in a neighborhood of } p\}.$$

The inner product of relation (29) for $X = U$ with ξ implies, due to relation (13), $\delta = 0$ and the shape operator on the local orthonormal basis $\{U, \phi U, \xi\}$ becomes

$$A\xi = \alpha\xi + \beta U, \quad AU = \gamma U + \beta\xi \text{ and } A\phi U = \mu\phi U. \tag{30}$$

Relation (29) for $X = \xi$ yields, taking into account relation (30), $\gamma = k$. Finally, relation (29) for $X = \phi U$ implies, due to relation (30) and the last relation,

$$(\mu^2 - 2k\mu + k^2) + \beta^2 = 0.$$

The above relation results in $\beta = 0$, which implies that $\mathcal{N}$ is empty. Thus, the following proposition is proved:

Proposition 2. *Every real hypersurface in $M_2(c)$ whose shape operator satisfies relation (6) is a Hopf hypersurface.*

Due to the above Proposition, Theorem 6 and Remark 3 hold. Relation (29) for $X = W$ and for $X = \phi W$ implies, respectively,

$$(\lambda - k)(\lambda - \nu) = 0 \text{ and } (\nu - k)(\lambda - \nu) = 0.$$

Suppose that $\lambda \neq \nu$. Then, the above relations imply $\lambda = \nu = k$, which is a contradiction.

Thus, on M, relation $\lambda = \nu$ holds and this results in the structure tensor ϕ commuting with the shape operator A, i.e., $A\phi = \phi A$ and, because of Theorem 3 M , is locally congruent to a real hypersurface of type (A), and this completes the proof of Theorem 2.

4. Proof of Theorems 4 and 5

Suppose that M is a real hypersurface in $M_2(c)$ whose tensor field P satisfies relation (7) for any $X \in \mathbb{D}$ and for all $Y \in TM$. Then, the latter relation becomes, because of the relation of the k-th generalized Tanaka-Webster connection (1) and relations (3) and (4),

$$g(\phi AX, PY)\xi - \eta(PY)\phi AX - g(\phi APY, X)\xi + k\eta(PY)\phi X - g(\phi AX, Y)P\xi \\ +\eta(Y)P\phi AX + g(\phi AY, X)P\xi - k\eta(Y)P\phi X = 0, \tag{31}$$

for any $X \in \mathbb{D}$ and for all $Y \in TM$.

Let $\mathcal{N}$ be the open subset of M such that

$$\mathcal{N} = \{p \in M : \beta \neq 0, \text{ in a neighborhood of } p\}.$$

Relation (31) for $Y = \xi$ implies, taking into account relation (13),

$$\beta\{g(AX, U) + g(A\phi U, \phi X)\}\xi + P\phi AX + \beta^2 g(\phi U, X)\phi U - kP\phi X = 0, \tag{32}$$

for any $X \in \mathbb{D}$.

The inner product of relation (32) for $X = \phi U$ with ξ due to relation (13) yields $\delta = 0$. Moreover, the inner product of relation (32) for $X = \phi U$ with ϕU, taking into account relation (13) and $\delta = 0$, results in

$$\beta^2 + k(\gamma - \mu) = \mu(\gamma - \mu). \tag{33}$$

The inner product of relation (32) for $X = U$ with U gives, because of relation (13) and $\delta = 0$,

$$(\gamma - k)(\gamma - \mu) = 0.$$

Suppose that $\gamma \neq k$, then the above relation implies $\gamma = \mu$ and relation (33) implies $\beta = 0$, which is impossible.

Thus, relation $\gamma = k$ holds and relation (33) results in

$$\beta^2 + (\gamma - \mu)^2 = 0.$$

The latter implies $\beta = 0$, which is impossible.

Thus, $\mathcal{N}$ is empty and the following proposition has been proved:

Proposition 3. *Every real hypersurface in $M_2(c)$ whose tensor field P satisfies relation (7) is a Hopf hypersurface.*

As a result of the proposition above, Theorem 6 and remark 3 hold. Thus, relation (31) for $X = W$ and $Y = \xi$ and for $X = \phi W$ and $Y = \xi$ yields, respectively,

$$(\lambda - k)(\lambda - \nu) = 0 \text{ and } (\nu - k)(\lambda - \nu) = 0.$$

Supposing that $\lambda \neq \nu$, the above relations imply $\lambda = \nu = k$, which is a contradiction.

Therefore, relation $\lambda = \nu$ holds and this implies that $A\phi = \phi A$. Thus, because of Theorem 3, M is locally congruent to a real hypersurface of type (A) and this completes the proof of Theorem 4.

Next, we study three-dimensional real hypersurfaces in $M_2(c)$ whose tensor field P satisfies relation (8). The last relation becomes, due to relation (2),

$$F_{\xi}^{(k)}PY - PF_{\xi}^{(k)}Y + \phi APY - P\phi AY = 0, \tag{34}$$

for any Y tangent to M.

Let $\mathcal{N}$ be the open subset of M such that

$$\mathcal{N} = \{p \in M : \beta \neq 0, \text{ in a neighborhood of } p\}.$$

The inner product of relation (34) for $Y = \xi$ implies, taking into account relation (13), $\beta = 0$, which is impossible. Thus, $\mathcal{N}$ is empty and the following proposition has been proved

Proposition 4. *Every real hypersurface in $M_2(c)$ whose tensor field P satisfies relation (8) is a Hopf hypersurface.*

Since M is a Hopf hypersurface, Theorems 6 and 3 hold. Relation (34) for $Y = W$ implies, due to $AW = \lambda W$ and $A\phi W = \nu\phi W$,

$$(\lambda - \nu)(\nu + \lambda - 2k) = 0.$$

We have two cases:

Case I: Supposing that $\lambda \neq \nu$, then the above relation implies $\nu + \lambda = 2k$. Relation (19) implies, due to the last one, that λ, ν are constant. Thus, M is locally congruent to a real hypersurface with three distinct principal curvatures. Therefore, it is locally congruent to a real hypersurface of type (B).

Thus, in the case of $\mathbb{C}P^2$, substitution of the eigenvalues of real hypersurface of type (B) in $\nu + \lambda = 2k$ implies $\alpha = -2k$. In the case of $\mathbb{C}H^2$, substitution of the eigenvalues of real hypersurface of type (B) in $\nu + \lambda = 2k$ yields $\alpha = \frac{4}{k}$.

Case II: Supposing that $\lambda = \nu$, then the structure tensor ϕ commutes with the shape operator A, i.e., $A\phi = \phi A$ and, because of Theorem 3, M is locally congruent to a real hypersurface of type (A) and this completes the proof of Theorem 5.

As a consequence of Theorems 4 and 5, the following Corollary is obtained:

Corollary 2. *A real hypersurface M in $M_2(c)$ whose tensor field P satisfies relation (7) is locally congruent to a real hypersurface of type (A).*

5. Conclusions

In this paper, we answer the question if there are three-dimensional real hypersurfaces in non-flat complex space forms whose differential operator $\mathcal{L}^{(k)}$ of a tensor field of type (1, 1) coincides with the Lie derivative of it. First, we study the case of the tensor field being the shape operator A of the real hypersurface. The obtained results complete the work that has been done in the case of real hypersurfaces of dimensions greater than three in complex projective space (see [11]). In Table 3 all the existing results and also provides open problems are summarized.

Table 3. Results on condition $\hat{\mathcal{L}}_X^{(k)} A = \mathcal{L}_X A$.

Condition	$M_2(c)$	$\mathbb{C}P^n, n \geq 3$	$\mathbb{C}H^n, n \geq 3$
$\hat{\mathcal{L}}_X^{(k)} A = \mathcal{L}_X A, X \in \mathbb{D}$	does not exist	does not exist	open
$\hat{\mathcal{L}}_\xi^{(k)} A = \mathcal{L}_\xi A$	type (A)	type (A)	open
$\hat{\mathcal{L}}_X^{(k)} A = \mathcal{L}_X A, X \in TM$	does not exist	does not exist	open

Next, we study the above geometric condition in the case of the tensor field being $P = A\phi - \phi A$, which is introduced here. In Table 4, we summarize the obtained results.

Table 4. Results on condition $\hat{\mathcal{L}}_X^{(k)} P = \mathcal{L}_X P$.

Condition	$\mathbb{C}P^2$	$\mathbb{C}H^2$
$\hat{\mathcal{L}}_X^{(k)} P = \mathcal{L}_X P, X \in \mathbb{D}$	type (A)	type (A)
$\hat{\mathcal{L}}_\xi^{(k)} P = \mathcal{L}_\xi P$	type (A) and type (B) with $\alpha = -2k$	type (A) and type (B) with $\alpha = \frac{4}{k}$
$\hat{\mathcal{L}}_X^{(k)} P = \mathcal{L}_X P, X \in TM$	type (A)	type (A)

Author Contributions: All authors contributed equally to this research.

Acknowledgments: The authors would like to express their gratitude to the referees for valuable comments on improving the paper.

Conflicts of Interest: The authors declare no conflict of interest.

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