Interaction of spiral waves in the general complex Ginzburg-Landau equation

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A dissertation submitted by Maria Aguareles Carrero to the Universitat Politècnica de Catalunya in partial fulfillment of the requirements for the degree of Doctor en Matemàtiques.

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Departament de Matemàtica Aplicada I
Universitat Politècnica de Catalunya

Barcelona, May of 2007

La fantasía, aislada de la razón, sólo produce monstruos imposibles. Unida a ella, en cambio, es la madre del arte y fuente de sus deseos.

Francisco de Goya y Lucientes (1746- 1828)

Aknowledgements

First of all I would like to thank my supervisor, Prof. Jonathan S. Chapman, for having the patience to teach me the "art" of asymptotic analysis and some of its so many subtleties, and also for giving me the chance to work on such a fascinating problem. I am especially grateful for guiding me and for encouraging me to continue when everything looked obscure and unclear by telling me in the most convincing way that we would solve it. My second acknowledgement goes for my tutor, Prof. Joan Solà-Morales, for encouraging me to go to Oxford in the first place, and for having always believed that a PHD thesis would come out of all the visits to OCIAM. Over these years he has always been available to listen to me when I needed it and his advice have always been very useful, discussing the mathematics of this thesis with him has also been very enlightening.

I would also like to thank the people from OCIAM who have made my visits to Oxford so pleasant. In particular I would like to thank Hilary Ockendon, the Director of OCIAM in the first year of my PHD, Sam Howison, the current Director, and John Ockendon, the current Research Director, for having always so kindly welcome me any time I was in Oxford to work on this thesis.

I also thank the people from *Departament de Matemàtica Aplicada I*. In particular I am gratefull to Tere Martinez-Seara and Neus Cónsul not only for the interesting discussions and the things they have taught me on asymptotic analysis and on partial differential equations but also for their friendship and support. Jordi Villanueva has also been extremely helpful with the numerical computations. I also thank the people from the *Research group in partial differential equations and applications*.

This research has been partially supported by the MEC grant MTM2005-07660-C02-01.

Vull també agrair als meus pares i a la meva germana el recolzament incondicional que sempre m'han donat, patint amb mi quan no veia res clar i disfrutant de les fites assolides com si fossin seves. A mi padre, mi primer profesor de Matemáticas, siempre le agradeceré la paciencia y los esfuerzos dedicados que sirvieron para que aprendiera a apreciar las matemáticas y la física desde niña. La ilusión y el cariño con que me enseñó los primeros conceptos de análisis hicieron nacer en mí la pasión por las matemáticas que me ha llevado hasta el final de esta tesis. Quiero también expresar mi gratitud a mi tío, Miguel Ángel Aguareles, por animarme a realizar una tesis y por el apoyo y los consejos que me ha dado a lo largo de estos años.

Als meus amics els vull agrair els ànims que m'han donat i la paciència que han tingut per les hores no dedicades.

Finalment, el recolzament incondicional d'en Pep ha estat per a mi el motor i l'alè necessaris per arribar fins al final d'aquesta tesi. A ell vull agrair-li la seva comprensió i la seva paciència per tants moments que li he deixat de dedicar i per tants comiats a l'aeroport que hem hagut de patir plegats.

A tots ells dedico el treball que es recull en aquesta memòria.

Contents

Chapter 1

Introduction and summary of results

Rotating spiral waves are very common in nature and they are found in various chemical, biological or physical contexts. For instance, the pictures in figure 1.1 are two examples of processes in nature where spiral waves arise. The first one is a classical one, it is a photograph of the Belousov-Zhabotinsky reaction that appeared in 1972 in the cover of the Science magazine, and that was made by Arthur Winfree, a biologist very interested in spatial and temporal patterns. This reaction, the Belousov-Zhabotinsky reaction, [46] was the first analysed example of an oscillatory reaction and it was found by Boris Pavlovich Belousov, in 1950, who was working on a solution of bromate, citric acid and ceric ions (Ce4+). He was expecting to see a smooth conversion of the yellow ceric ions Ce4+ into the colourless Ce+3. But instead the solution started oscillating changing constantly from yellow into transparent, and furthermore, he noted that, leaving the system unstirred, the solution exhibited travelling waves of yellow. Later on, in 1961, Anatol Zhabotinsky followed the studies of Belousov and found the way to catalyse the reaction and to make more clear the changes in colour as the reaction oscillates, which lead to the discovery of new and more complex spatio-temporal patterns. The second picture in figure 1.1 is a photograph of spiral waves in colonies of the mold Dictyostelium discoideum, [1]. These patterns are created when the mold forms aggregates in response to some external stress such as a lack of moisture or nutrients. Rotating spiral waves are also present in other fields like transverse patterns of high intensity light (see [31]) or in the electrical field of a heart under arrythmias (see [45]), they are thus quite ubiquitous and a very generalised phenomenon.

A property that all these systems have in common is that there is always an interplay between a process of reaction and some kind of spatial diffusion. For instance, in the case of transverse patterns of light the nonlinearity of the media may cause a self-focusing phenomenon which is in competition with the

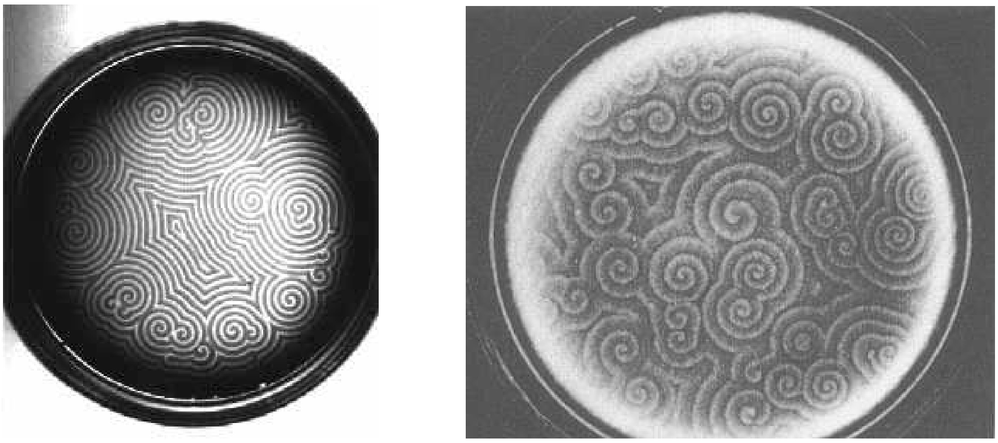

Figure 1.1: Spiral waves in the Belousov-Zhabotinsky reaction observed in a petri dish. Spiral waves in colonies of the mold Dictyostelium discoideum.

phenomenon of diffraction, and in the case of chemical reactions, the diffusion of one reactant into the other is in opposition with their reaction. This means that these processes may be modeled by a system of partial differential equation of reaction-diffusion type of the form

$$\frac{\partial u}{\partial t} = D\Delta u + F(u), \tag{1.1}$$

where D is a matrix of diffusion constants. The second thing that all these systems share is that they exhibit spatially uniform oscillating solutions. This fact is related to the structure of the nonlinear reaction term, $F(u)$, and also to the spectrum of the linearised operator corresponding to (1.1). In particular, the type of systems that we are interested in undergo a Hopf bifurcation for a given value of a parameter. In the vicinity of this bifurcation point, the dynamics defined by (1.1) may be described in terms of an envelope equation known as the general complex Ginzburg-Landau equation. The derivation of this envelope equation is performed through a multiple scale analysis and the resulting equation is expressed in terms of a complex amplitude that, together with its complex conjugate, represents the modulation of the oscillation that has just started. In [24], [26], [42] or [29] one can find derivations of the complex Giznburg-Landau equation. For a review on different physical contexts where pattern formation takes place as well as the derivation of different amplitude equations we refer to [13].

The aim of the present work is to study vortices or spiral wave solutions of the so-called general complex Ginzburg-Landau equation, that is given by

$$\frac{\partial \Psi}{\partial t} = \Psi - (1 + ia)\ |\Psi|^2\Psi + (1 + ib)\Delta\Psi, \tag{1.2}$$

where a and b are real parameters that are related to the parameters in the original equation (1.1). In particular we are interested in describing patterns where more than one spiral coexist and the way these patterns evolve in time, by obtaining the trajectories of the centres of the spirals.

The outline of the rest of this chapter is as follows. We first introduce the topic of rotating spiral waves in the complex Ginzburg-Landau equation, (1.2), by giving its definition and its main properties. We then summarise the contents of this thesis and introduce our main results. Finally, we give a brief introduction to asymptotic analysis methods, that are the tools that we will be using in this work.

1.1 Vortices and spiral waves in the general complex Ginzburg-Landau equation

In this section we introduce the object under study in this thesis, namely spiral waves or vortices in the complex Ginzburg-Landau equation. This type of waves are more commonly known as *rotating waves* since their time evolution is a rigid rotation with a constant angular velocity, ω. Thus, they are similarity solutions of the particular form

$$\Psi(\mathbf{x}, t) = \Psi(r, n\phi - \omega t), \tag{1.3}$$

where r and ϕ are the polar radius and angular variable. But in particular we are interested in solutions in the shape of (1.3) whose Brouwer degree is not zero, so the following condition holds,

$$Im \oint_C \frac{\Psi(\partial_\tau \Psi^*)}{|\Psi|^2} d\ell = 2\pi n \quad n \in \mathbb{Z} \quad n \neq 0, \tag{1.4}$$

where C is a large enough closed regular curve, ∂_τ denotes the derivative in the tangential direction of C and n is the degree or winding number of Ψ. If we now express the complex solution in terms of its modulus and phase as $\Psi = f e^{i\chi}$ we find that condition (1.4) reads

$$\oint_C \nabla\chi \cdot d\ell = 2\pi n, \tag{1.5}$$

which implies that the gradient of the phase has a pole singularity and, therefore, to obtain a continuous solution, the modulus, f, must vanish at the points were these singularities are. Actually, basic degree theory (see for instance [15]) shows that solutions with a non-zero but finite degree do vanish only at a discrete finite set of points, which means that condition (1.5) only holds if C encloses all the zeros of the solution. These type of singular solutions, that is to say rotating waves with a non-zero degree, are what we will call from now on either vortex or spiral wave solutions. We note that in a context of fluid dynamics, being $\nabla\chi$ the velocity of the fluid, equation (1.5) would be a condition on the circulation or vortex strength, but in this case it is forced to be quantised.

If we now look for single-spiral wave solutions in the general Ginzburg-Landau equation, that is solutions that only vanish at one point, they happen to have a quite simple form and, as we shall show in what follows, they can be expressed as solutions of a pair of ordinary differential equations.

To find vortex solutions in equation (1.2) it is convenient to express the solution in the rotating frame of the angular velocity, ω, by writing $\Psi = e^{-i\omega t}\psi$. This yields the system

$$(1-ib)\frac{\partial \psi}{\partial t} = \Delta\psi + \{(1+i\Omega) - (1+iq)|\psi|^2\}\psi \tag{1.6}$$

where

$$\Omega = \frac{\omega - b}{1+b\omega} \quad q = \frac{a-b}{1+ba},$$

and now rotating waves are stationary solutions of (1.6). Actually, spiral wave solutions of (1.6) have the particular form $\psi(r,\phi) = f(r)e^{i(n\phi+\Phi(r))}$, where $f(r)$ and $\Phi(r)$ satisfy the system of ordinary differential equations

$$0 = f_{rr} - f(\Phi_r)^2 + \frac{f_r}{r} - f\frac{n^2}{r^2} + (1-f^2)f, \tag{1.7}$$

$$0 = f\Phi_{rr} + 2\Phi_r f_r + f\frac{\Phi_r}{r} + \Omega f - qf^3. \tag{1.8}$$

If we now inspect the contour lines of the phase function $(n\phi + \Phi(r))$, they appear to have the form of archimedian spirals that emanate from the origin providing Φ_r is a bounded function at infinity, as it is represented in figure 1.2. Therefore, the type of single-spiral solutions that are physically relevant and that we are interested in, are stationary solutions of (1.7) and (1.8) with boundary conditions given by

$$f(0) = 0 \quad \Phi_r(0) = 0 \quad f(r) \text{ and } \Phi_r(r) \text{ bounded as } r \to \infty. \tag{1.9}$$

In the particular case that $q = 0$, the parameter Ω must also vanish in order for Φ_r to be bounded. This is seen by realising that the second equation (1.8) when $q = 0$ reads

$$\Phi_r = -\Omega\frac{\int_0^r sf^2\,ds}{rf^2}. \tag{1.10}$$

This last expression shows that the only way for Φ_r to be bounded at infinity is either by taking $\Omega = 0$ or $f \equiv 0$, but since the latest would produce a trivial

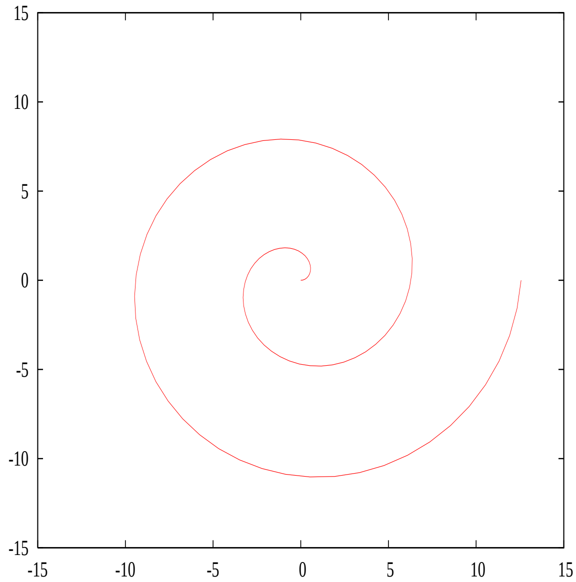

Figure 1.2: Archimedian spiral.

solution, the parameter Ω is forced to vanish. Therefore, we conclude that when $q = 0$ spiral waves are actually solutions of the form $\Psi = f(r)e^{i(n\phi - at + C)}$, that means that the frequency of oscillation happens to be exactly the value of the parameter, $\omega = a = b$. Spiral waves that correspond to this particular situation where $\Phi_r = 0$ are known as *vortices*. In that case the set of differential equations (1.7) and (1.8) reduces to the single equation

$$0 = f_{rr} + \frac{f_r}{r} - f\frac{n^2}{r^2} + (1 - f^2)f. \tag{1.11}$$

In figure 1.3 we plot the solution Ψ for different values of the arbitrary constant C. We note that the number of times that the vector rotates along a closed curve around the origin is always one, that is the winding number that we have chosen to represent. We must now point out that equation (1.2) is invariant under rotations, meaning that, being Ψ a solution to the equation, Ψe^{ic} is also a solution, where $c \in \mathbb{R}$ is an arbitrary constant. Furthermore, by inspecting the type of bounded solutions that (1.11) might have, we find that, either $f \to 0$ or $f \to 1$ as r goes to infinity. The first possibility produces the trivial solution $f \equiv 0$ and it is the second one that gives a non-trivial solution. The existence of solution of (1.11) with the boundary conditions,

$$f(0) = 0 \qquad \lim_{r\to\infty} f(r) = 1 \tag{1.12}$$

was already observed in [18] and, afterwards, existence and uniqueness of such type of solutions was well established in [7] and [11], for finite and infinite do-

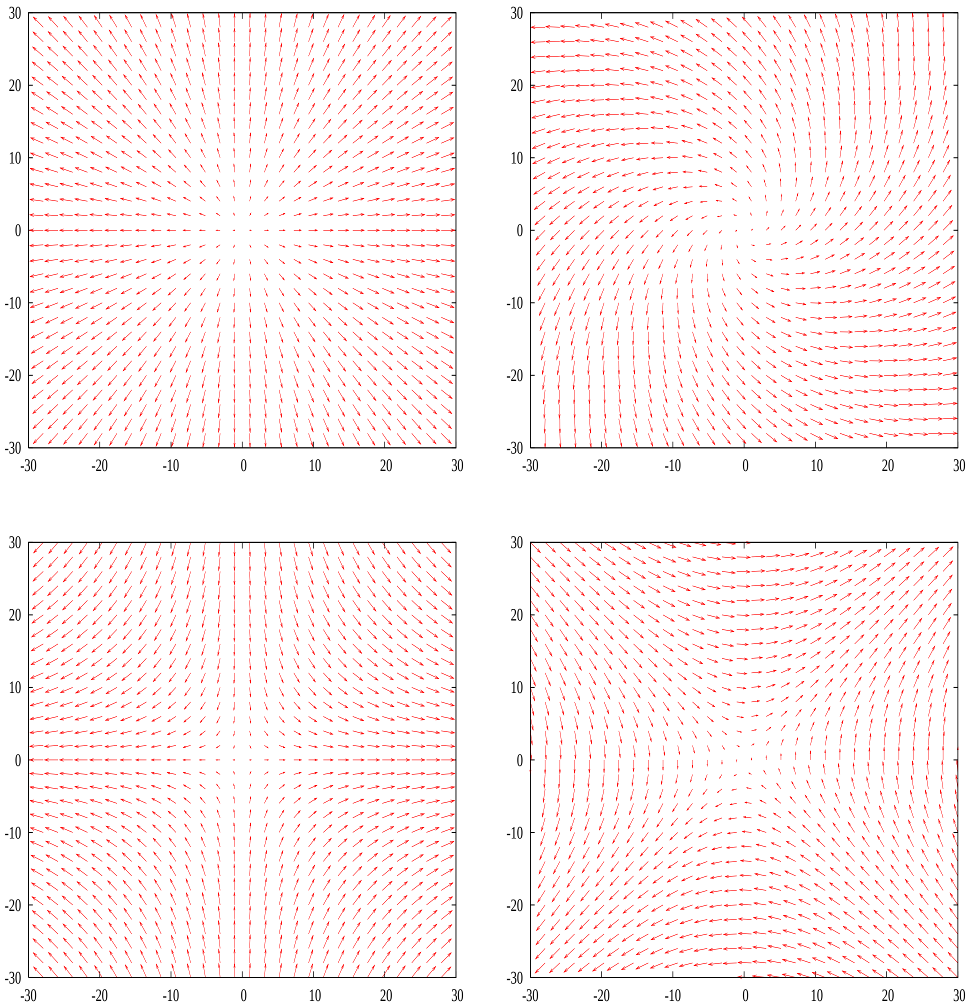

Figure 1.3: Single vortex solution for $\psi = f(r)e^{i\phi}$, $\psi = f(r)e^{i(\phi+\pi/4)}$, $\psi = f(r)e^{-i\phi}$ and $\psi = f(r)e^{i(-\phi+\pi/2)}$.

mains. This solution, which can be found numerically, is monotone increasing and it actually defines a layer at the origin, where f is small, whereas in the rest of the domain the solution is almost constantly one (see figure 1.4). In the figure, ϵ represents the width of the layer at the origin, and the solution of (1.11) has the expansion,

$$f(r) \sim 1 - \frac{n^2}{2r^2} + \mathcal{O}(\frac{1}{r^4}) \quad \text{when} \quad r \to \infty. \tag{1.13}$$

As we will show in Chapter 2, there are solutions of equation (1.6) for $q = 0$ with localised vortices that emerge from the zeros of the solution and that interact with each other.

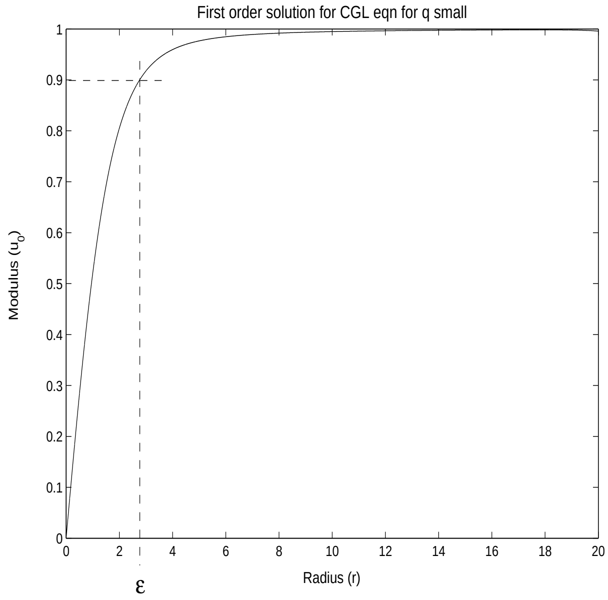

Figure 1.4: Solution of equation (1.11) in an infinite domain

For the more general case where a and b are different, that is to say, $q \neq 0$, the system of ordinary differential equations (1.7)-(1.8) with boundary conditions (1.9) has also a unique solution. Since Ω vanishes when q is zero, we can now introduce a new parameter k by writing

$$\Omega = q(1 - k^2) = \frac{\omega - b}{1 + b\omega},$$

which means that (1.6) becomes

$$(1 - i\,b)\frac{\partial \psi}{\partial t} = \Delta\psi + (1 - |\psi|^2)\psi + iq\psi(1 - k^2 - |\psi|^2). \tag{1.14}$$

The new constant, k has the role of a wavenumber at infinity. Indeed, if one writes the expansions of the stationary solutions of (1.7)-(1.8) at infinity they are found to be

$$f \sim (1 - k^2)^{1/2} - \frac{k}{2qr(1 - k^2)^{1/2}} + \mathcal{O}\left(\frac{1}{r^2}\right),$$
$$\chi_r \sim \pm k + \frac{1}{2qr} + \mathcal{O}\left(\frac{1}{r^2}\right),$$

so the phase at infinity looks like $\chi \sim n\phi \pm kr$, which shows very clearly that the role of k is that of being a wavenumber for rotating waves. In terms of figure

1.2, k represents the inverse of the separation between two neighbouring fronts in the spiral. Furthermore, as we will show in Chapter 3, only the solutions $\chi \sim n\phi - kr$ give place to a spiralling solution. We also show in Chapter 3 that, given the parameters a, b in the equation, the asymptotic wavenumber k is uniquely defined.

The book by L.M. Pismen, [39] gives a general overview of the topic of vortices and topological defects in different nonlinear systems that arise in different areas of physics. As for the general complex Ginzburg-Landau equation and its applications one can see the review by I S. Aranson and L. Kramer [2].

1.2 Summary and main results

The aim of this work is to understand the mechanisms that drive the evolution of spiral wave patterns in the complex Ginzburg-Landau equation through the use of asymptotic analysis techniques. Therefore, it has a double goal: on one hand, and from the applied analysis point of view, we show how usefull asymptotic analysis can be to discover and describe complicated solutions of partial differential equations, and, on the other hand and from the point of view of the applications in physics, chemistry or biology, we give an answer to the problem of the interaction of spiral waves that arise in oscillatory extended systems, so commonly found in nature.

Very often in the applied mathematics literature one finds papers where a simple model is considered and that serves as a paradigm to develop complicated mathematical tools, and where the main motivation lies in the mathematical techniques rather than in the specific application that the model under study may have. But we also find other works were the authors are more concerned about the applicability of the model to describe a particular phenomenon rather than on its use to deepen on some new mathematical theory, and since phenomena in physics and nature can be very difficult to describe, the equations end up being rather complicated and very difficult to handle. The main challenge is for them to extract information out of those complicated sets of equations by finding the right way to simplify them to some simpler differential equation. The present work wants to lie a bit in the middle of this two philosophies. The equation we consider, the general complex Ginzburg-Landau equation, is a very well known and widely used model for oscillatory nonlinear systems. But on the other hand it is a non-trivial partial differential equation that we use to show that asymptotic analysis can provide satisfactory answers to quite difficult problems where functional analysis techniques and even numerical methods are too hard to apply, if one does not know in advance what the answer can be.

In the rest of this section we summarise the contents of this thesis and present our main results.

Chapter 2: Vortex solutions of Ginzburg-Landau equation.

We devote this chapter to the particular case of equation (1.6) where $q = 0$ that, as we have shown, reads,

$$(1 - ib)\frac{\partial \psi}{\partial t} = \Delta\psi + (1 - |\psi|^2)\psi. \tag{1.15}$$

This equation is a generalisation of two very well-known equations: the Ginzburg-Landau equation with real coefficients, that corresponds to $b = 0$, and the Nonlinear Schrödinger equation, that is the limit equation as b tends to infinity. Both cases have been widely studied and there is an important amount of work on the subject. At the beginning of Chapter 2 we start by mentioning some of these previous works that have been relevant to our work at some point.

In the second part of the chapter we focus on many-vortex solutions of equation 1.15. The two special cases were $b = 0$ (the Ginzburg-Landau equation) and $b = \infty$ (the Nonlinear Schrödinger equation) were first studied by John Neu in [34] who found that vortices interact algebraically with a velocity that is proportional to the inverse of their separation. Furthermore, he determined that the time scale at which vortices move in Ginzburg-Landau equation is of order $\epsilon^2/|\log\epsilon|$ while vortices in the Nonlinear Schrödinger equation move at a faster rate of order ϵ^2. He also obtained asymptotic laws of motion for both cases that show that, when $b = 0$ a pair of vortices would interact along the lines of their centres either repelling themselves, if their winding numbers have the same sign, or attracting each other, if they have opposite signs in their winding numbers, while for an infinite value of b, he showed that a pair of vortices would spin around each other.

Our main contribution in this chapter is therefore to extend the results in [34] to nonzero values of b. We consider a system of N vortices that are separated by distances of order $1/\epsilon$ and we use asymptotic analysis techniques to obtain a law of motion for the more general equation (1.15). By considering b as a function of ϵ we show how its order of magnitude influence the final law of motion and the scale of the velocity as well, and the way b interpolates between the Ginzburg-Landau and the Nonlinear Schrödinger laws of motion that were presented in [34]. Therefore in Chapter 2 we obtain:

Result. *In a system of N vortices that remain well separated, with positions $\mathbf{X}_j(T) = \epsilon\mathbf{x}$, that satisfy equation* (1.15)*, their dynamics evolves with a time scale that depends on the order of magnitude of b in the following way:*

(i) if b is of order less than $|\log\epsilon|$, the time scale for the velocity is $T =$

$\epsilon^2/|\log\epsilon|t$ *and the law of motion is*

$$\frac{d\mathbf{X}_\ell}{dT} = \left(\frac{2}{n_\ell}\sum_{j\neq\ell}\frac{n_j\mathbf{e}_{rj\ell}}{|\mathbf{X}_\ell - \mathbf{X}_j|}\right)(1+o(1)),$$

(ii) if b is of order $|\log\epsilon|$, the time scale for the velocity becomes $T = \epsilon^2 t$ and the law of motion reads

$$\frac{d\mathbf{X}_\ell}{dT} = \left(\frac{2b\mu}{n_\ell}\sum_{j\neq\ell}\frac{n_j\mathbf{e}_{rj\ell}}{|\mathbf{X}_\ell - \mathbf{X}_j|} - \frac{2b^2\mu^2}{n_\ell^2+\mu b^2}\sum_{j\neq\ell}\frac{n_j\mathbf{e}_{\phi j\ell}}{|\mathbf{X}_\ell - \mathbf{X}_j|}\right)(1+o(1)),$$

where $\mu = 1/|\log\epsilon|$.

(iii) if b is of order greater than $|\log\epsilon|$, the time scale for their velocity is still $T = \epsilon^2 t$ but the law of motion is now

$$\frac{d\mathbf{X}_\ell}{dT} = \left(-2\sum_{j\neq\ell}\frac{n_j\mathbf{e}_{\phi j\ell}}{|\mathbf{X}_\ell - \mathbf{X}_j|}\right)(1+o(1)),$$

where in all the equations we have denoted by $\mathbf{e}_{rj\ell}$ and $\mathbf{e}_{\phi j\ell}$ the unitary vectors that point vortex ℓ from vortex j, using the polar radial and angular coordinates.

These laws of motion clarify the way in which, as b increases, the interaction among vortices stops being on the radial direction to include a tangential component that ends up being dominant when b becomes large enough, which is in agreement with all previous results.

In the last part of the chapter we obtain the same results as before but we then write the asymptotic expansions in a different way, that is the way that we will be using in the rest of this thesis. This would seem rather meaningless at a first glance, but on the contrary, by doing so we show that the same method can be used to tackle the problem of many spirals in the complex Ginzburg-Landau equation when q is not zero. This is indeed very important since, by inspecting the literature one realises that there seem to be different ways of understanding the case of q zero and different than zero, as if they were completely different problems that should be solved in a radically different way. Actually, if one tries to solve the problem for q not zero exactly in the same way as John Neu did in [34], the expansions, that have now a new small parameter, q, become impossible to manipulate. This is due to the fact that this is a highly singular limit and special care is needed.

Chapter 3: Symmetric spiral wave solutions.

In this chapter we consider the general Ginzburg-Landau equation (1.14) and study the family of equilibria that correspond to spiral wave solutions. These are radially symmetric solutions that have only one zero and that may be written in terms of an ordinary differential equation, as we have shown in the previous section. The existence of these type of solutions was already established by Patrick S. Hagan in [19] where he showed that spiral wave solutions do only exist with a unique asymptotic wavenumber, k. That is to say, given the parameters of the equation (1.2), a and b, or alternatively b and q, the value of k that gives place to a spiral wave solution is uniquely determined. We recall that the asymptotic wavenumber k is bound to the frequency of the spiral ω through the dispersion relation,

$$q(1-k^2) = \frac{\omega - b}{1 + b\omega},$$

which means that there is a selection mechanism that forces the frequency of the spiral to have an specific value. Furthermore, Hagan showed that, for small values of q, the asymptotic wavenumber is exponentially small in q and it is given by

$$k(q) = \frac{2}{q} e^{c_n - \gamma - \pi/(2q|n|)}(1 + o(1)),$$

where n is the degree of the spiral, c_n is a constant that depends only on the degree and γ stands for the Euler constant.

P.S. Hagan found these results by constructing spiral wave solutions through an asymptotic matching method. He therefore described these solutions using asymptotic series that approximate the solution in different regions in the spatial domain. The way these expansions are written in [19] is very different from the way J. Neu describes multiple-vortex solutions when $q = 0$. But nevertheless, everything comes from the same equation and it should be possible to write both problems in terms of similar expansions. In this work we show that there is actually a unifying way of expressing the expanded symmetric solutions of (1.14) that has its analogous expansions in the problem of many vortices when $q = 0$, where the radial symmetry is broken.

The method we use in Chapter 3 to describe symmetric spiral wave solutions is also based on an asymptotic matching method. The idea is that one wants to find heteroclinic orbits of the system of ordinary differential equation (1.7)-(1.8) that depart from the origin and reach the point where $f = \sqrt{1-k^2}$ and $\Phi_r = -k$, where k is unknown. To do so we pose an asymptotic expansion that approximates the solution close to the origin and up to a certain radius, the so-called *inner* solution, and another one that is valid for large values of r, the *outer* solution. In these two series we leave some constants unknown, that in order to represent the same solution they must be such that the outer limit of the inner expansion and the inner limit of the outer is the same. This is the process of *asymptotic matching* that determines the value of k. But as we mentioned before, the asymptotic wavenumber k is exponentially small in q and it can be understood as an eigenvalue. It is also the constant that the gradient of the phase tends to when the radius goes to infinity. This means that the gradient of the phase looks actually like k only at exponentially far distances. Using this idea, in Chapter 3 we introduce an auxiliary parameter that we name by α and that is given by $\alpha = kq/\epsilon$. This way we substitute the eigenvalue problem for k by the eigenvalue problem in terms of α. The main motivation to perform this change is that now α is an order one constant rather than an exponentially small quantity. This is very important because exponentially small quantities are 'transparent' to asymptotic expansions. Therefore, by using α the eigenvalue is determined naturally in the matching procedure. The way P.S. Hagan solves the problem in [19] is a bit different from ours, in particular he does not identify this combination of parameters as an order one magnitude so obviously as we do, although he does find different regions in space, that he calls *middle* and *final* region, where different expansions hold and that he manages to match. These two regions do actually correspond to the scales where α is either small or order one. He therefore finds that in order to find the right value of k one has to reach the *final* region, that happens to be exponentially far away from the origin. But these two regions are in a sense 'fake' regions since both of them correspond to a unique outer scale. Hence, the way we solve the problem is by using a single outer scale, $\mathbf{X} = \epsilon \mathbf{x}$, denote by α the combination of qk/ϵ, pose the asymptotic expansion $\chi \sim \chi_0 + \epsilon^2 \chi_1 + \ldots$ for the phase of the complex function ψ, and find that χ_0 satisfies the equation

$$\Delta \chi_0 + q|\nabla \chi_0|^2 - \frac{\alpha^2}{q} = 0.$$

Therefore, what has to be done is to find the solution to this equation imposing the boundary conditions given by

$$\nabla \chi_0 \to \pm k \mathbf{e}_r \quad \text{as} \quad r \to \infty,$$

where $\mathbf{e}_r$ stands for the unitary vector in the radial direction. By doing so one finds that only when the boundary condition is $-k$ the function χ_0 is strictly monotonic with the radius r and therefore, this is the solution that we are interested in. If the phase was not monotonic the isophase lines would not correspond to an archimedian spiral. Thus, after imposing this boundary condition, solving the outer equation, and comparing with the inner expansion one finds that α should have an specific value that is given by $\alpha = 2e^{c_n - \gamma}$. It is therefore an order one magnitude. On the other hand, when we match inner and outer solutions it is also found that ϵ is not arbitrary, on the contrary, there is a relation between q and ϵ that tells us how small ϵ is, or otherwise, how far the outer region is. This second condition is given by

$$qn|\log \epsilon| = \pi/2,$$

being n the winding number of the spiral. With these two conditions we can now find the asymptotic wavenumber.

The understanding of the structure of single spiral solutions is a key ingredient in order to be able to solve the many-spirals problem. Furthermore, as we mentioned before, the way we write the expansions in Chapter 3 is very close to the one used in chapter 2 and shows a common structure.

Chapters 4 and 5: Analysis of multiple-spiral systems. In these two chapters we deal with patterns of (1.14) that have several isolated zeros from which the spirals emanate. Our purpose in these chapters is to find an asymptotic law of motion for the centres of the spirals in a similar way to the one that was used in Chapter 2, for $q = 0$. One of the difficulties of this problem is that now one has to deal with four different small parameters, q, the asymptotic wavenumber k, the inverse of the separation ϵ, and the slow time scale parameter that we denote by μ. Besides, k is again unknown and one expects that it will come out in the way of obtaining an asymptotic law of motion.

Inspired by the structure of single spiral solutions that is described in Chapter 3, we start by identifying that there might be a 'special' relative separation of the spirals that corresponds to the one where α is of order one, and that we call *canonical scale*. Hence, in **Chapter 4** we solve the problem assuming α is an order one constant, but following the same pattern that we use when q is zero. The first big difference than one finds when q is not zero is that the leading order equation in the far field is now nonlinear in contrast to what happens when q vanishes. This means that a linear superposition of the effect of each spiral is now not possible. This problem is sorted by using a suitable change of functions that yields a linear equation. However, special care is needed to perform any change of function to the phase function, χ_0, since we might reach a multivalued solution for ψ and would therefore not be a valid approximation. L.M. Pismen in [36] claims that this change of functions could not be done due precisely to the fact that the solution may become multivalued. However, in Chapter 4 we show how to keep track of the angular parts of the phase to ensure that the final solution is univalued. Then, after having matched the inner and outer solutions in a quite subtle way, the following results are obtained:

Result. *The asymptotic wavenumber, k, corresponding to a system of N spirals at positions $\mathbf{X}_j$ with unitary degrees in the canonical region is given by* *the condition that the following set of N homogeneous equations ($\ell = 1, \ldots, N$)*

$$0 = \beta_{\ell 0}(c_1 - \log\alpha + \log 2 - \gamma) + \sum_{j\neq\ell}^{N} \beta_{j0} K_0(\alpha|\mathbf{X}_\ell - \mathbf{X}_j|), \tag{1.16}$$

for the N unknown constants $\beta_{\ell 0}$ has a nontrivial solution (where $\alpha = kq/\epsilon$ and K_0 stands for the modified Bessel function of the second kind).

Furthermore, the canonical length scale, ϵ, is such that $q|\log\epsilon| = \pi/2$.

Therefore, we do find a condition on the eigenvalue α in the same way as we do in the single-spiral case. For a set of two spirals, conditions (1.16) show that $\beta_{10} = \beta_{20}$ and therefore the condition on α reads

$$c_1 = \log(\alpha/2) + \gamma + K_0(\alpha|\mathbf{X}_1 - \mathbf{X}_2|).$$

Hence, as the spirals separate, since the modified Bessel function, K_0, becomes exponentially small, this eigenvalue condition tends to be the one corresponding to a single spiral with a unitary degree in complete isolation.

As for the law of motion for spirals in the canonical separation, we find that:

Result. *For a system of N spirals with unitary degrees n_ℓ that are separated by distances of order $1/\epsilon$ such that $q|\log\epsilon| = \pi/2$, the centres of the spirals move with a law of motion that reads*

$$\frac{d\mathbf{X}_\ell}{dT} = -\frac{2qn_\ell}{\beta_{\ell 0}\mu}\nabla G^{\perp}(\mathbf{X}_\ell) + \mathcal{O}(q), \tag{1.17}$$

where, being $G(\mathbf{X})$ the function defined as

$$G(\mathbf{X}) = \sum_{j\neq\ell}^{N} \beta_{j0}\alpha K_0'(\alpha|\mathbf{X} - \mathbf{X}_j|)\,\mathbf{e}_{\phi j\ell},$$

then the vector $\nabla G^{\perp}(\mathbf{X}_\ell)$ stands for the gradient of the function G at the point $\mathbf{X}_\ell$ that has been rotated $\pi/2$ counterclockwise.

The time scale is $T = \epsilon^2\mu t$, where $\mu = 1/|\log\epsilon|$.

From this law of motion one could think that a pair of spirals in the canonical scale and with the same degree would spin around each other in a periodic way since the velocity is only in the tangential component, a situation that in the physics literature is known as *bound states.* But when we compute the next order in the velocity we find that there is actually a small correction of order q that is in the radial direction.

Having obtained these laws of motion it is convenient to check if in the limit as q goes to zero we obtain the same results as in Chapter 2 when the parameter b is set to zero. But it is clear that it is not the case since in the case where q is zero the interaction is purely in the radial component. This seems to imply that we are missing some parts of the analysis when we restrict ourselves to the canonical scale. The idea is as follows: the canonical scale is defined by the relation, $q|\log\epsilon| = \pi/2$. This means that when we take the limit as q goes to zero but remain in the canonical scale, we are actually forcing ϵ to be smaller. Nevertheless, the actual limit that we want to do in order to compare with the case of $q = 0$ is to fix the spirals, that is to say, fix ϵ, and see how the law of motion changes when q is considered to be smaller. But by doing so we are implicitly forcing α to be of order less than one and the analysis stops then being valid. We therefore must repeat the calculations to obtain the law of motion but considering now that α is of small magnitude. This is done in **Chapter 5** where now, the new relative separation of the spirals we denote it by *middle scale*, coining the same term that P.S. Hagan used in [19].

In Chapter 5 we finally find a law of motion that interpolates between the one found in Chapter 2 for $q = 0$ and the one for the canonical scale.

Result. *The velocity for spirals that are separated by distances such that qk/ϵ is small is given by*

$$\frac{d\mathbf{X}_\ell}{dT} = 2\frac{q}{\mu}\frac{\cos(qn_\ell|\log\epsilon|)}{\sin(qn_\ell|\log\epsilon|)}\sum_{j\neq\ell}^{N}\Big(n_j\frac{\mathbf{e}_{rj\ell}}{|\mathbf{X}_j-\mathbf{X}_\ell|} + n_j\frac{\sin(qn_j|\log\epsilon|)}{\cos(qn_j|\log\epsilon|)}\frac{\mathbf{e}_{\phi j\ell}}{|\mathbf{X}_j-\mathbf{X}_\ell|}\Big) \tag{1.18}$$

where the time scale, $T = \epsilon^2\mu t$, is such that $\mu = 1/|\log\epsilon|$.

If we now consider the particular case where all winding numbers are either one or minus one we find that this law does actually interpolate between the case where $q = 0$ and the canonical scale.

In this middle region, though, the asymptotic wavenumber is not found during the course of determining the law of motion. On the contrary, one can find this law without even knowing the value of k and as a consequence, without knowing the real value of α which is assumed to be small. To do so we have to impose the boundary conditions at infinity. This is done in the last section of Chapter 5 by rescaling the equations using a new length scale δ that is finally found to satisfy

$$\frac{\cos(qn\log(\delta/\epsilon))}{\sin(qn\log(\delta/\epsilon))} = \frac{N}{n}\frac{\sin(q\log\epsilon)}{\cos(q\log\epsilon)}, \tag{1.19}$$

where n is the sum of the degrees of the spirals, and N is the sum of the degrees in absolute value, so in particular, for spirals with a unitary degree N is the number of spirals. Thus, redefining the eigenvalue by $\overline{\alpha} = \alpha\epsilon/\delta$ one obtains:

Result. *The value of the eigenvalue* $\overline{\alpha}$ *is given by*

$$N\frac{c_1}{n^2}\frac{\sin^2(qn\log(\delta/\epsilon)}{\cos^2(q\log\epsilon)} = \log(\overline{\alpha}/2) + \gamma, \tag{1.20}$$

that along with condition (1.19) *shows that the value of* α *remains small provided* $q|\log\epsilon| < \pi/2$.

When all the spirals have a positive winding number, that is to say, $n = N$, *this expression simplifies to*

$$\overline{\alpha} = \frac{kq}{\delta} = 2e^{c_1/n-\gamma}. \tag{1.21}$$

This last expression is actually the same expression that we found for an n-spiral in isolation, except for the constant term, c_1, that is slightly different in this case due to the fact that the spirals are indeed separated.

All these results regarding the asymptotic law of motion for systems of N spirals in the general complex Ginzburg-Landau equation answer most of the open questions that L.M. Pismen raises in [36]. In his paper he gives a partial answer to the problem of interaction of only a pair of spiral waves, while we find the solution for systems of an arbitrary number of spirals.

In **appendix A** we focus on spirals that interact in the middle scale region and present the numerically computed trajectories of a pair and three spirals with winding numbers of either plus or minus one. We also compute the same initial value problems with different values of q to see the show of q on the trajectories of the spirals.

Chapter 6: Conclusions and final remarks.

The aim of this last chapter is to give a 'global picture' of the way the interaction between spirals takes place as the parameter q moves from $q = 0$ to $q = -(\pi/2)\log\epsilon$. In this chapter we show that the role of q and the length scale ϵ are interchangeable and we explain the way this is understood. We use all this information to point out that our results are consistent with the numerically observed results when q becomes of order one.

1.3 Brief introduction to asymptotic analysis

Exact analytical solutions cannot be found for most of the models that arise in physics and applied mathematics, and even when it is possible to obtain an analytical solution explicitly, it may be useless for mathematical interpretation or numerical evaluation. Thus, in many cases, one has to rely on the possibility of obtaining some sort of approximation, or a numerical solution, or even a combination of both. *Perturbation theory* provides systematic techniques for obtaining accurate approximate solutions to differential and integral equations through the first few terms of a perturbation expansion with an error that is understood and controllable. These expansions are carried out in terms of a parameter that is very small or very large with respect to the rest of the magnitudes in the problem, the so-called *asymptotic expansions*.

Perturbation theory was first developed in the field of celestial mechanics. Actually, even before that, Isaac Newton in the *Principia*, published in 1678, he used geometric methods to find approximate solutions to algebraic equations with small parameters. But it was Poincaré in 1886 who gave a precise definition to what is nowadays known as an *asymptotic* expansion and developed the first asymptotic analysis techniques. Then, well in the twentieth century it was in the context of fluid dynamics where most of the advance was done. In this area one has to deal very often with nonlinear equations of which there were no known exact analytical solutions. The need to understand the structure of their solutions and the way the solution might depend on the parameters forced a big development in the area of asymptotic analysis and perturbation theory

Since in the rest of this work we will be using asymptotic analysis techniques, we devote some space to give the main definitions and ideas behind asymptotic analysis. For a deeper description we refer to, for instance, [25], [22], [20], [32], [44], [33], [5], or [12] for a more theoretical approach.

Asymptotic approximations

The notions of *convergent sequence* and *asymptotic sequence* are some times mixed, although they are fundamentally different:

Definition 1.1. *A series* $\sum_{n=0}^{\infty} f_n(z)$ *is said to* converge *to a function* $f(z)$ *at a fixed value of* z*, if given an arbitrary* $\epsilon > 0$ *there is a number* $N_0(z, \epsilon)$*, such that*

$$\left| \sum_{n=0}^{N} f_n(z) - f(z) \right| < \epsilon \quad \text{for all} \quad N \geq N_0.$$

Definition 1.2. *A sequence* $\{f_n(\epsilon)\}$ *is said to be* asymptotic *if, for all* $n > 1$*,*

$$\left| \frac{f_n}{f_{n-1}} \right| \to 0 \quad as \quad \epsilon \to 0.$$

Definition 1.3. *A series* $\sum_{n=0}^{\infty} f_n(\epsilon)$ *is said to be an* asymptotic expansion *or an* asymptotic approximation *to* $f(\epsilon)$ *if for all* $N > 1$*,*

$$\left| \frac{f(\epsilon) - \sum_{n=0}^{N} f_n(\epsilon)}{f_N(\epsilon)} \right| \to 0 \quad as \quad \epsilon \to 0,$$

that is to say, the remainder of the series is smaller than the terms that have been included if ϵ is sufficiently small.

This means that asymptotic expansions are not necessarily convergent. On the contrary, the accuracy of an asymptotic series is related to ϵ rather than to N. That is to say, a convergent expansion is closer to the function that represents if one takes a larger value of N so the series is truncated further. But an asymptotic series is not necessarily closer by taking more terms in the expansion. This is clear if one realises that the given definition for an asymptotic expansion allows the series to be divergent as N goes to infinity. The accuracy of an asymptotic expansion is thus reached by taking smaller values of ϵ, which means that, in particular, the first term in the expansion, the so-called *leading order term* is virtually correct as ϵ tends to zero.

The previous definitions may be rewritten in terms of the following *order symbols*, that we will use throughout the rest of this work:

Definition 1.4. *Let $x \in I \subset \mathbb{R}^n$ be fixed.*

- *We write $\phi(x,\epsilon) = \mathcal{O}(\psi(x,\epsilon))$ in the interval I and say that ϕ is of order ψ if there exists a $k(x)$ such that $|\phi(x,\epsilon)| \leq k(x)|\psi(x,\epsilon)|$ for all ϵ smaller than a given value ϵ_0. In particular, if ψ does not vanish in I the definition reduces to ϕ/ψ to be bounded in terms of ϵ.*

- *We write $\phi(x,\epsilon) = o(\psi(x,\epsilon))$ as $\epsilon \to 0$ if given any $\delta(x) > 0$, there exists a neighbourhood N_δ of 0 such that $|\phi| \leq \delta(x)|\psi|$ for all ϵ in N_δ. Here, if ψ does not vanish in N_δ the definition simplifies to $\phi/\psi \to 0$.*

- *Therefore, a sequence, $\{f_n(x,\epsilon)\}$, is asymptotic if $f_{n+1}(x,\epsilon) = o(f_n(x,\epsilon))$ as $\epsilon \to 0$. Then, we say that the series $\sum_{n=0}^{N} f_n(x,\epsilon)$ is an asymptotic expansion of the function $f(x)$ as ϵ goes to zero if it satisfies,*

$$f(x) - \sum_{n=0}^{N} f_n(x,\epsilon) = o(f_N(x,\epsilon)), \quad \textit{for all } x \in I$$

and we then write

$$f(x) \sim \sum_{n=1}^{M-1} f_n(x,\epsilon) \quad \textit{as } \epsilon \to 0.$$

It is important to observe that all these definitions are valid for a given range of the variable x, and this interval is not always obvious. We call this interval I as the region where the asymptotic series is uniformly valid since, outside I the series might rearrange and might not keep the ordering that we just defined. For this reason, very often when one wants to approximate a given function $f(x,\epsilon)$ by an asymptotic series in a given range of x it is necessary to use different series that are uniformly valid in overlapping regions that cover the whole range of x.

Multiparametric expansions The above definitions may be extended to functions that depend on more than one small parameter. If, for instance, a given function depends on two small parameters, $f(x, \mu, \epsilon)$, that are actually not independent, that is to say, $\mu = \mu(\epsilon)$. Then, one can expand first in terms of powers of one of the parameters, like for example,

$$f(x, q, \epsilon) \sim f_0(x, \mu) + f_1(x, \mu)\epsilon + \ldots + f_n(x, \mu)\epsilon^n, \tag{1.22}$$

and then expand each term of the series in terms of the other parameter,

$$f(x, q, \epsilon) \sim (f_{00} + f_{01}\mu + f_{02}\mu^2 + \ldots) + (f_{10} + f_{11}\mu + f_{12}\mu^2 + \ldots)\epsilon + \ldots. \tag{1.23}$$

But this operation is only valid provided the resulting series, (1.23), is still an asymptotic expansion, and it will be the case provided $\epsilon(\mu(\epsilon))^j \to 0$ as ϵ goes to zero for any j, positive or negative. In this work we will perform this type of multiparametric expansions very often. In particular we will use two small parameters, ϵ and $\mu = 1/|\log \epsilon|$ that satisfy this property.

Some properties of asymptotic expansions

We will use asymptotic expansions to approximate solutions of partial differential equations. This means that we will perform different operations on them and we also want to know that the resulting series approximates a unique function, otherwise it would not be a good approximation to the solution.

Manipulation: Since asymptotic expansions will be used to solve differential equations we will need to perform on them elementary operations such as addition, substraction, exponentiation, integration, differentiation and multiplication. It is thus important to make sure that the resulting expansion keeps its asymptotic character.

It is easy to justify from the definitions that *addition* and *substraction* of asymptotic expansions are in general justified, that is to say, we can add or substract them term by term and the resulting expansion is asymptotic in some interval. Therefore, *integration* is also straightforward provided the original function and the functions in the asymptotic series are integrable functions.

As for the *multiplication* of two given asymptotic series, $\sum a_n(x, \epsilon)$ and $\sum b_n(x, \epsilon)$, all products $a_n(x, \epsilon)b_m(x, \epsilon)$ occur in the product series, and it is generally not possible to arrange them so as to obtain an asymptotic sequence. But if the resulting series can be rearranged in order to give place to an asymptotic series, then the multiplication of those series is justified. An important class of such sequences are those where the dependence on ϵ comes only through its powers, that is to say, series of the form $\sum f_n(x)\epsilon^n$. This is a very common type of asymptotic series and are the type that we will consider mostly in this work.

As for *exponentiation* and *differentiation*, it is in general not justified since it leads to nonuniformities. Thus, the way to proceed is to naively perform these operations an analyse the new interval of uniformity of the resulting series. If one keeps track of the nonuniformities these operations are perfectly valid. In other words, the way to proceed is by performing these operations in a formal way and once the solution is reached one checks wether the new range of validity is the one that wanted to be studied or that was assumed to be. If it is not the case the result is not consistent with the hypothesis and should be dismissed. The self-consistency of the method is the main idea that should be kept in mind in order to accept any result that is found through asymptotic techniques.

Uniqueness: If a function possesses an asymptotic approximation in terms of an asymptotic sequence, $\delta_n(\epsilon)$, then that approximation is *unique* for that particular sequence. This means that given the existence of an approximation $f(x,\epsilon) \sim \sum a_n(x)\delta_n(\epsilon)$ in terms of a given sequence, the coefficients can be found inductively from

$$a_k(x) = \lim_{\epsilon\to 0} \frac{f(x,\epsilon) - \sum_1^{k-1} a_n(x)\delta_n(\epsilon)}{\delta_k(\epsilon)}.$$

This implies that a single function can have many approximations, each in terms of different asymptotic sequences. But to our purpose this is enough since we will pose a given asymptotic sequence and we will find an approximation to the solution of the problem in terms of that specific sequence.

However, there is another subtlety that should be kept in mind. This uniqueness property is for one given function. Thus, many different functions can share the same asymptotic approximation because they can differ by quantities that are smaller than the last term included. A paradigm of this problem is that of series of the form $\sum a_n(x)\epsilon^n$ which represent a whole set of functions that only differ by exponentially small quantities. Therefore, if one wants to catch one specific function with an specific exponentially small quantity, this series is of no use. The problem of catching exponentially small quantities is a classic one and although nowadays there are some standard tools to tackle some specific problems, it is in general still a source of difficulty that must be analysed very carefully. An example of this situation lies in the problem we address in this thesis. As we will show, the spiral waves that we analyse have an exponentially small asymptotic wavenumber and we will need to find the way to write the asymptotic expansions and the partial differential equations so that we catch the particular solution that we are interested in.

Matched asymptotic expansions method

The method of matched asymptotic expansions is used to find approximate solutions to differential equations when sharp changes in the unknown function take place. A clear example of this idea of a sharp change is found in figure 1.4, that represents the modulus of vortex solutions of the Ginzburg-Landau equation. This function is almost one in most of the domain, but jumps very fast to zero at the origin. In this type of problem, if one looks for a solution to the differential equation in the form of a straightforward asymptotic series it is found that its uniformity breaks down as the sharp change is approached. To obtain uniformly valid expansions we must then recognise scales which are different from the scale characterising the behaviour of the function outside the sharp regions, and use this new scale to find asymptotic approximations in the sharp regions.

One technique of dealing with this problem is to determine first straightforward expansions, that are called *outer expansions*, using the original variables, and to determine expansions, the *inner expansions*, describing the sharp changes using magnified scales. The outer expansions break down in the inner regions, while the inner expansions break down away from the regions of sharp change. To relate these two expansions a so-called *matching* procedure is used. The idea is that the outer expansion does only meet the boundary conditions at some parts of the domain while the inner meets the boundary conditions at the other side. Thus, these two expansions have a certain amount of undetermined constants that, in order for these series to represent the same solution, must agree as the inner variable is stretched towards the outer and the outer towards the inner. Therefore it is crucial to identify overlap areas where both the inner and the outer are valid at the same time.

In the study of interaction of spiral waves, the outer scale will correspond to the region far from the centres of all the spirals, and the inner will be the area close to each centre.

Chapter 2

Vortex solutions of Ginzburg-Landau equation

In this chapter we study the particular type of vortex solutions that arise in complex Ginzburg-Landau equation when the parameter q is zero. First we recall some previous results that are related to our study and we recall the notion of vortex solution. Next we present the asymptotic expansions technique that leads to a dynamical law for well separated vortices and that was first used by John Neu in [34]. Finally we rewrite these calculations in the same way that we do when q is not zero in order to introduce the notation and key points that will arise in this case and that will be addressed in the following chapters.

2.1 Preliminaries

We now consider complex Ginzburg-Landau equation in two spatial dimensions when $q = 0$, that is given by

$$(1 - i\,b)\frac{\partial \psi}{\partial t} = \epsilon^2 \Delta \psi + (1 - |\psi|^2)\psi, \tag{2.1}$$

being b a real parameter. In this particular case, the equation (2.1) has an associated energy functional given by

$$E_\epsilon = \frac{1}{2}\int_G |\nabla \psi|^2 + \frac{1}{4\epsilon^2}\int_G (|\psi|^2 - 1)^2, \tag{2.2}$$

where G is a two dimensional spatial domain, in the sense that

$$\frac{dE_\epsilon}{dt} = -\frac{1}{1+b^2}\int_G \left|\frac{\partial \psi}{\partial t}\right|^2 < 0,$$

that means that the solutions dissipate the energy. This condition on the energy also shows that as b tends to infinity the energy dissipation is smaller and in the limit where b is infinity the energy is conserved. When $b = \infty$ the equation is known as Nonlinear Schrödinger equation due to the fact that it is actually the nonlinear version of the Schrödinger model for quantum mechanics.

Nonlinear Schrödinger equation is written as,

$$-i\frac{\partial \psi}{\partial t} = \epsilon^2 \Delta\psi + (1 - |\psi|^2)\psi. \tag{2.3}$$

At this point we should clarify in which sense Nonlinear Schrödinger equation corresponds to the limit situation where the parameter b tends to infinity. Let us go back to equation (2.1), and let us rescale time with $t = bT$. By doing so, equation (2.1) becomes

$$(1/b - i)\frac{\partial \psi}{\partial T} = \epsilon^2 \Delta\psi + (1 - |\psi|^2)\psi$$

and it is clear that as b tends to infinity the equation approaches the Nonlinear Schrödinger equation.

The model corresponding to $b = 0$ was originally introduced by V.L Ginzburg and L. Landau in [16] in the study of phase transition problems in superconductivity (see [21]). Later on, the same type of models but with $b = \infty$, that is equation (2.3), were also used in superfluidity problems (see [17]), where the Nonlinear Schrödinger equation is more commonly denoted by the Gross-Pitaevskii equation. The Gross-Pitaevskii equation has also been used as a model in condensed matter theory and nonlinear optics. In superconductors, the function ψ is called a condensate wave function or a Riggs field, where if we express $\psi = fe^{i\chi}$, the magnitude given by f^2 is proportional to the density of superconducting electrons. On the other hand, in superfluids f^2 is proportional to the density of the superfluid while the gradient of the phase, $\nabla\chi$, is proportional to the velocity of the supercurrents of the superfluid.

For general values of b equation (2.1) plays a central role as an envelope equation in the pattern dynamics of oscillatory systems. Its behaviour for different values of b interpolates between the Ginzburg-Landau regime of superconductivity and the Nonlinear Schrödinger one.

Equation (2.1) and the energy functional (2.2) have drawn a lot of attention in the last fifty years. In mathematics and physics literature one can find thousands of works on the solutions of equation (2.1), especially in the particular case that $b = 0$. Next we present some of the most relevant results on vortex solutions and their dynamical evolution. However, it is by no means a complete summary; on the contrary, this is just a small sample of the huge amount of contributions that have been done around this topic.

2.2 Review of previous works

2.2.1 On equilibrium solutions

Bounded domains

We first recall some relevant results concerning the equilibrium solutions in bounded domains in the superconductivity case, i.e. $b = 0$.

In the work by Bethuel, Brezis and Hélein, [7], the authors focus on the solutions of the Ginzburg-Landau problem with Dirichlet boundary conditions with a non-zero degree that minimise the energy functional (2.2). They are interested in the behaviour of the limiting solution as $\epsilon \to 0$. We note that the parameter ϵ can be thought as the scaling factor of the domain, that is to say, $1/\epsilon$ tells us how big the domain is. In particular, the limit as $\epsilon \to 0$ may correspond to an unbounded domain situation.

One of the problems that one finds when dealing with vortex solutions is that the energy of vortex solutions tends to infinity as $\epsilon \to 0$ due to the gradient part. In particular, if we denote by $a = (a_1, a_2, \ldots, a_j)$ the positions of j vortices, one can see that

$$\frac{1}{2}\int_{G_\epsilon} |\nabla\psi|^2 = \pi\left(\sum_{i=1}^{n} d_i^2\right)\log(1/\epsilon) + W(a) + \mathcal{O}(\epsilon), \tag{2.4}$$

where $G\epsilon = G \setminus \cup_i B(a_i, \epsilon)$, that is to say, some balls of radius ϵ around each singularity have been subtracted from the initial domain G. The domain with such holes has an energy that becomes unbounded as the holes are shrunk to zero. Therefore, in order to compare the energy of different solutions as $\epsilon \to 0$ the authors in [7] define a *renormalised energy* by removing the unbounded part of (2.4) and using just $W(a)$. This new renormalised energy, $W(a)$ depends only on G, a_i, d_i and the boundary condition g, but not on the scaling parameter ϵ.

Two important results in [7] are: given a boundary condition with a Brouwer degree $d \neq 0$, then,

(i) the solutions as $\epsilon \to 0$ that minimise (2.2) are those with exactly d vortices with degree $+1$ each,

(ii) the location of the vortices, that is given by the set $a = (a_1, a_2, \ldots, a_d)$, is such that it minimises the renormalised energy, $W(a)$.

Later on, in the paper by Struwe [41], the author removes some of the restrictions in [7] and gives also alternative proofs for some of the results.

We note that the first result implies that the vortices with a unitary winding number are energetically more favourable. A more thorough study on the stability of vortices with unitary degrees is found in the works by Mironescu, [30] and Lieb and Loss, [27]. In these works they consider solutions of the form $\psi = f(r)e^{i\phi}$ and they find that they are stable in the sense that the quadratic form associated with the energy E_ϵ in (2.2) is positive definite.

Another recent important contribution can be found in the work by Serfaty, [40]. In this work the author extends the results by Bethuel, Brezis and Hélein, [7], to the case of Neumann boundary conditions. In particular she proves that, for both Dirichlet and Neumann boundary conditions, if the solution ψ is stable in the sense that the quadratic form of E_ϵ is nonnegative for all ϵ, then, the set $a = (a_1, a_2, \ldots, a_d)$ is a stable critical point of $W(a)$.

Unbounded domains

Equilibrium vortex solutions in unbounded domains have also been studied in several works. Since we also deal with unbounded domains, the results in this case are specially relevant to our work.

In particular, in the work by Brezis, Merle and Rivière, [11], they find that, the solutions of (2.1) with $b = 0$ in all $\mathbb{R}^2$ satisfy a quantisation condition given by

$$\int_{\mathbb{R}^2} (|\psi|^2 - 1)^2 = 2\pi d^2$$

for some integer $d = 0, 1, 2, \ldots \infty$. They also reach the following conclusions:

(i) If $d < \infty$, then $|\psi| \to 1$ as $|\mathbf{x}| \to \infty$. Thus, the degree of ψ around a ball of radius R is well-defined for any large R.

(ii) If $d = 0$, the only solutions are constant functions

(iii) For any integer $d = 0, 1, 2, \ldots \infty$, there is a solution of the form:

- **(a)** For $d = \infty$, $\psi(\mathbf{x}) = Ae^{ik\cdot\mathbf{x}}$ where A is an arbitrary constant and k is any constant vector in $\mathbb{R}^2$.
- **(b)** For $0 < d < \infty$, $\psi(r, \phi) = f(r)e^{id\phi}$.

Therefore, one of the conclusions that we obtain from this theorem is that, given a system of n vortices such that the sum of their degrees is zero, they will always have a law of motion without any equilibrium point. That is to say, they will only stop if they collide and annihilate each other. Another interesting result would be to know wether the solutions given by $\psi(r, \phi) = f(r)e^{id\phi}$ when $0 < d < \infty$ are unique. To our knowledge this is still an open problem.

Another important issue is the stability of vortex solutions in unbounded domains. Such a problem is tackled in the paper by Weinstein and Xin, [43]. They show that,

(i) all vortices are asymptotically nonlinearly stable under small radial perturbations, and,

(ii) under general perturbations, the ± 1-vortices are linearly dynamically stable in L^2.

Again it is the case of ± 1-vortices that present the most stable situation. The vortices with a higher degree are therefore more likely to split into ± 1-vortices.

2.2.2 On evolutionary solutions

In this section we focus on solutions with a set of well separated vortices in either unbounded domains or domains that are very large in comparison with the length scale of the vortex core (see Chapter 1 for a description of the vortex core and the length scale associated to it). As it was mentioned above, this type of solutions are never stationary; on the contrary, they evolve in time while the vortices remain separated.

The first paper to undertake the study of this kind of solutions for both the Ginzburg-Landau equation (with $b = 0$) and the Nonlinear Schrödinger equation (with $b = \infty$) was the one by J. Neu [34] in 1990. In this work the author shows that the solutions of these equations that have a non-vanishing degree can be described asymptotically through the evolution of their zeros. As we showed in Chapter 1, the zeros are the so called vortices and each of them has an associated winding number or degree. Therefore, although the motion of the vortices are part of the full dynamics of the system, the large time behaviour of the solution is determined only by the trajectories of the vortices and the rest of the flow is organised around them. This description of the solutions has an analogy with the physics theory of particle-field interactions.

J. Neu uses formal asymptotic analysis to conclude that, given a solution of Ginzburg-Landau or Nonlinear Schrödinger equation, with a set of zeros that remain well separated,

(i) the zeros of the solution persist in time keeping their original winding number as long as the zeros remain separated by distances of order $1/\epsilon$,

(ii) the asymptotic vortex dynamics is reduced to a set of ordinary differential equations and

(iii) the time scale for the motion of the vortices is given by $T = \epsilon^2/|\log \epsilon| t$.

Further works that extend Neu's results are found in [38], [35] or [14] among others. In the work by Pismen and Rubinstein [38] they do the three dimensional version of the work by Neu but only for the NLS equation. In $\mathbb{R}^3$ the mathematical objects that play the same role as the vortices in $\mathbb{R}^2$ are the vortex lines. Again, the vortex lines dominate the dynamics and interact with each other. In [38] they reduce the dynamics to a set of differential equations that is the law of motion for these vortex lines. Peres and Rubinstein in [35] also obtain a law of motion for vortices in the plane, but now they consider the presence of an external magnetic field, adding a new term and a new equation to the Ginzburg-Landau that we have are considering. A step forward is found in the paper by E [14]. In this work the author reproduces the results in [34] but he also considers the effect of boundaries. Furthermore, he also tackles the three dimensional case and extends the results in [38]

All these works are based on the use of asymptotic expansions techniques, that are very useful to obtain formal analytic approximations of solutions. But later on, some more works have studied more rigorously the asymptotics of the sequence of solutions generated as $\epsilon \to 0$ considered in [34] or [14]. To do so, some assumptions on the initial conditions and on the growth of the energy as $\epsilon \to 0$ need to be done. For instance, in the paper by Jerrard and Soner [23] they consider a system of vortices that do not collide, in a bounded but very large domain, with a boundary condition that forces the degree to be non-zero, and they provide a rigorous proof of some of the results that Neu found through asymptotic methods. Another example is the work by Lin and Xin [28] where the authors remove the bounded domain assumption and they tackle the problem by considering from the beginning an unbounded domain. In the work by Bethuel, Orlandi and Smets [6], they go further and analyse also collisions leading to annihilation, and in a later work by the same authors [8], they also consider multiple-degree vortices that split into unitary degree vortices.

2.3 Dynamical law of the vortices

In the rest of this chapter we focus on many-vortex solutions of equation

$$(1 - i\,b)\frac{\partial \psi}{\partial t} = \Delta\psi + (1 - |\psi|^2)\psi, \tag{2.5}$$

in unbounded domains. Our first contribution to what was already done in the work by J. Neu, [34], and other works is to extend the methods used in [34] to obtain a law of motion for the vortices for general values of b. We also use this calculation to introduce the notation that we will be using in the following chapters. As we have already shown, the parameter b interpolates between the "classical" Ginzburg-Landau equation and the Nonlinear Schrödinger equation. Therefore, it is interesting to consider that b depends on ϵ and the way its order of magnitude will influence the final law of motion. For instance we will find that for values of b of order less or equal than one the vortices move at a velocity of order $\epsilon^2/|\log\epsilon|$, but when b becomes greater than $|\log\epsilon|$, the vortices move at the faster rate of ϵ^2. In this section we show the way in which this transition is done.

So far in the literature the analysis for the $q=0$-case has been treated in a very different way from the one for nonzero values of q, and our main purpose in this thesis is to obtain a law of motion for the vortices for positive but small values of q. One may think that the methods and notation developed by J. Neu in [34] serve as a paradigm to this kind of problem, and one expects to find that when for nonzero values of the parameter q it is applicable in an straight forward way. But on the other hand, the single-spiral situation for general values of q has also been tackled through asymptotic techniques in the work by P. Hagan, [19]. But although in this second case the author also uses asymptotic matching techniques to describe the structure of these spirals, the general scheme is fundamentally different from the one used in [34]. Therefore, our main contribution in this sense is the development of an asymptotic scheme based on the one in [34] but that is capable to describe, not only the single-spiral case, but also multiple-spiral systems for any value of q greater or equal than zero. This is what we will show in the second part of this section where we will describe this new asymptotic scheme for the particular case when q vanishes. In the following chapters it will be clear in the that the case of $q=0$ is actually a singular limit and this fact makes it even harder to find the right way to work with the parameters involved. Thus, along this thesis we will show the way that the method should be written in order to unify the analysis for general values of q, even for the singular limit as $q=0$.

We start by considering equation (2.5) and expressing the complex solution ψ in its polar form as $\psi = fe^{i\chi}$, to obtain

$$f_t + bf\chi_t = \Delta f - f|\nabla\chi|^2 + (1-f^2)f, \tag{2.6}$$

$$f\chi_t - bf_t = 2\nabla\chi\cdot\nabla f + f\Delta\chi. \tag{2.7}$$

We wish to determine the law of motion for a set of well separated vortices so we assume that they are initially separated by distances of order $\mathcal{O}(\epsilon^{-1})$, with $\epsilon \ll 1$, and our analysis will be valid while this condition holds. Furthermore, since we know that the evolution of vortices might be slower than the natural diffusive time scale, we introduce an unknown parameter μ that we assume to be of order less or equal than one, rescaling time with $T=\epsilon^2\mu t$.

The standard procedure is then to define an “outer region” far from all the vortices, in which $\mathbf{x}$ is scaled by a factor of ϵ^{-1}, and an “inner region” in the vicinity of each vortex. This way we will find a simpler set of partial differential equations that approximates the solution in the ”outer region” where the contribution from all the vortices will superpose linearly. This will be the main difference with the $q\neq 0$-case where the dominant or leading order equation for the ”outer region” will be fully nonlinear. As for the ”inner region”, we will see that the leading order equation is actually the one corresponding to a vortex in complete isolation due to the fact that we suppose that the velocity of the interacting vortices is small. Thus, the fact that the vortex moves with a certain velocity will appear through a non-homogeneous term in the next order, the so-called first order correction. From the way these equations are obtained it will be clear that the homogeneous part of the first order equation is actually the linearisation of the leading order equation about the single vortex solution. Some parameters will be unknown in the process of solving all these equations, but we will determine them when matching the solutions in the "outer" and "inner region" by imposing that they actually correspond to the same solution when we expand the "inner" outwards and the "outer" inwards. Finally we proceed with a further matching by applying the Fredholm alternative to the linear operator from the first order inner equation and this will lead us to a solvability condition that should be satisfied. This condition is the one that will determine the law of motion for the vortices. Furthermore, we will show that for values of the parameter b that are of order less or equal than $|\log\epsilon|$ the vortices do actually interact at a rate of order $\mu\epsilon^2$, where μ will be determined to be $\mu = 1/|\log\epsilon|$.

2.3.1 Outer Region

To study the outer region, or far field, we rescale $\mathbf{x}$ and t by setting $\mathbf{X} = \epsilon\mathbf{x}$ and $T = \epsilon^2\mu t$. With this rescaling, equations (2.6) and (2.7) read

$$\epsilon^2\mu(f_T + bf\chi_T) = \epsilon^2(\Delta f - f|\nabla\chi|^2) + (1 - f^2)f, \tag{2.8}$$

$$\mu(f\chi_T - bf_T) = 2\nabla\chi\cdot\nabla f + f\Delta\chi. \tag{2.9}$$

At this point one must take into account that b is an arbitrary constant that is allowed to be arbitrarily large or small. In particular, when b is of order greater than one the right time scale is not the one we have just chosen and one must rescale with $T = \epsilon^2\mu/bt$. Therefore, the far field equations for large values of b read

$$\epsilon^2\mu(\frac{1}{b}f_T + f\chi_T) = \epsilon^2(\Delta f - f|\nabla\chi|^2) + (1 - f^2)f, \tag{2.10}$$

$$\mu(\frac{1}{b}f\chi_T - f_T) = 2\nabla\chi\cdot\nabla f + f\Delta\chi. \tag{2.11}$$

We note that we can compute the solutions of (2.8), (2.9) and (2.10), (2.11) all at the same time by considering the alternative system given by

$$\epsilon^2\mu(\gamma_1 f_T + \gamma_2 f\chi_T) = \epsilon^2(\Delta f - f|\nabla\chi|^2) + (1 - f^2)f, \tag{2.12}$$

$$\mu(\gamma_1 f\chi_T - \gamma_2 f_T) = 2\nabla\chi\cdot\nabla f + f\Delta\chi. \tag{2.13}$$

where γ_1 and γ_2 are now parameters that are of order equal or less than one. We will hence study the alternative system given by (2.12) and (2.13), and the actual solutions that we are looking for will correspond to, $\gamma_1 = 1$ and $\gamma_2 = b$ for values of b of order less or equal than one, and $\gamma_1 = 1/b$ and $\gamma_2 = 1$ for large values of b.

We now wish to obtain approximate solutions to these equations in the form of asymptotic expansions. We thus start by posing an ansatz for the functions f and χ in terms of the small parameter ϵ. Since ϵ does only appear in (2.12) and (2.13) in the shape of ϵ^2, one expects to be able to express the solution as a power series in ϵ^2. Hence, we pose and expansion in powers of ϵ^2 as

$$f \sim f_0 + \epsilon^2 f_1 + \ldots,$$
$$\chi \sim \chi_0 + \epsilon^2 \chi_1 + \ldots$$

By substituting these expansions into (2.12) and (2.13) and balancing powers of ϵ we find that the leading order equations are given by

$$f_0 = 1, \tag{2.14}$$
$$\gamma_1 \mu \frac{\partial \chi_0}{\partial T} = \Delta \chi_0 \tag{2.15}$$

Now expanding χ_0 for small μ as

$$\chi_0 \sim \chi_{00} + \mu \chi_{01} + \ldots$$

and substituting into (2.15) we obtain at leading order

$$\Delta \chi_{00} = 0.$$

This is a linear equation and in order to account for the existence of the N vortices with their respective topological degree, the solution that we should take is given by

$$\chi_{00} = \sum_{j=1}^{N} n_j \phi_j, \tag{2.16}$$

where ϕ_j is the polar angle measured from the position of the j th vortex $\mathbf{X}_j$, and n_j is its winding number. At the next order in μ we find

$$\Delta \chi_{01} = \gamma_1 \frac{\partial \chi_{00}}{\partial T},$$

with solution

$$\chi_{01} = -\frac{\gamma_1}{2} \sum_{j=1}^{N} n_j R_j \log R_j \, \mathbf{e}_{\phi j} \cdot \frac{d\mathbf{X}_j}{dT}, \tag{2.17}$$

where $R_j = |\mathbf{X} - \mathbf{X}_j|$ is the distance from the jth vortex and $\mathbf{e}_{\phi j}$ represents the unitary vector pointing in the direction of the angular polar variable when one chooses the vortex j as the centre of coordinates.

Continuing to $O(\mu^2)$ we find

$$\chi_{02} = \frac{\gamma_1^2}{8} \sum_{j=1}^{N} n_j R_j^2 \log R_j \left(\mathbf{e}_{\phi j} \cdot \frac{d\mathbf{X}_j}{dT} \right) \left(\mathbf{e}_{rj} \cdot \frac{d\mathbf{X}_j}{dT} \right), \tag{2.18}$$

where now $\mathbf{e}_{rj}$ is the unitary vector in the radial direction when vortex j is the origin of the coordinate system.

In general we find that each term in the sum for the N vortices in χ_{0m} is $O(R_j^m \log R_j)$ as $R_j \to 0$. We will take this fact into account when we match with the inner solution.

In figures 2.1, 2.2, 2.3, 2.4 we plot the far field for a system of two vortices with the same winding number and also with different winding numbers, and we also plot the far field corresponding to a system of three and four vortices. In these pictures one can see that the number of loops that the vectors do around all the vortices corresponds to the total sum of the winding numbers of each of them.

2.3.2 Inner Region

We rescale near the ℓth vortex by setting $\mathbf{X} = \mathbf{X}_\ell + \epsilon \mathbf{x}$ to give

$$\epsilon\mu\big(\epsilon\gamma_1 f_T - \gamma_1 \frac{d\mathbf{X}_\ell}{dT} \cdot \nabla f + \epsilon\gamma_2 f \chi_T - \gamma_2 f \frac{d\mathbf{X}_\ell}{dT} \cdot \nabla\chi\big) = \Delta f - f|\nabla\chi|^2 + (1 - f^2) f \tag{2.19}$$

$$\epsilon\mu\big(\gamma_2 \frac{d\mathbf{X}_\ell}{dT} \cdot \nabla f - \epsilon\gamma_2 f_T + \epsilon\gamma_1 f \chi_T - \gamma_1 f \frac{d\mathbf{X}_\ell}{dT} \cdot \nabla\chi\big) = 2\nabla\chi \cdot \nabla f + f\Delta\chi \tag{2.20}$$

or equivalently

$$-\epsilon\mu(\gamma_1 - i\gamma_2)(\epsilon\psi_T - \frac{d\mathbf{X}_\ell}{dT} \cdot \nabla\psi) = \psi(1 - |\psi|^2) + \Delta\psi. \tag{2.21}$$

Expanding in powers of ϵ now as $\psi \sim \psi_0 + \epsilon\psi_1 + \ldots$ we find at leading order

$$\Delta\psi_0 + \psi_0(1 - |\psi_0|^2) = 0. \tag{2.22}$$

This is just the equation for a single static vortex, with solution

$$\psi_0 = f_0(r) e^{i(n_\ell \phi + C(T))}, \tag{2.23}$$

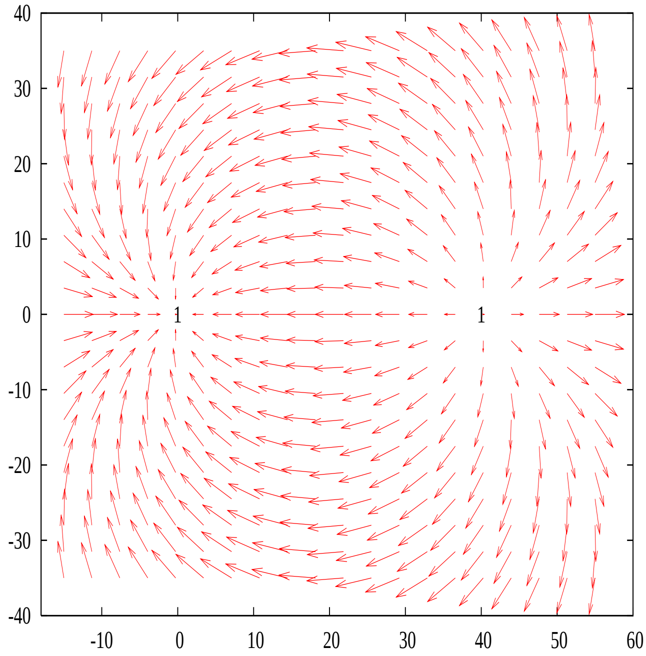

Figure 2.1: Far field for two vortices with equal winding number

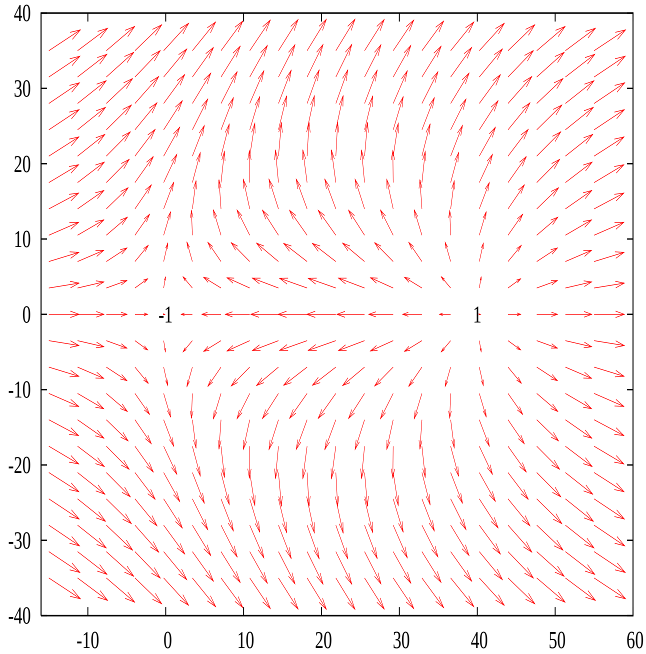

Figure 2.2: Far field for two vortices with opposite winding number

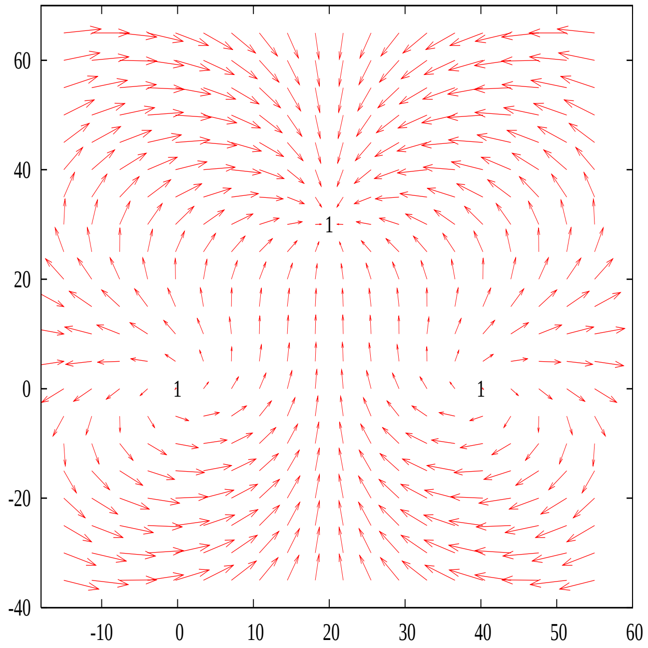

Figure 2.3: Far field for three vortices with +1 winding number

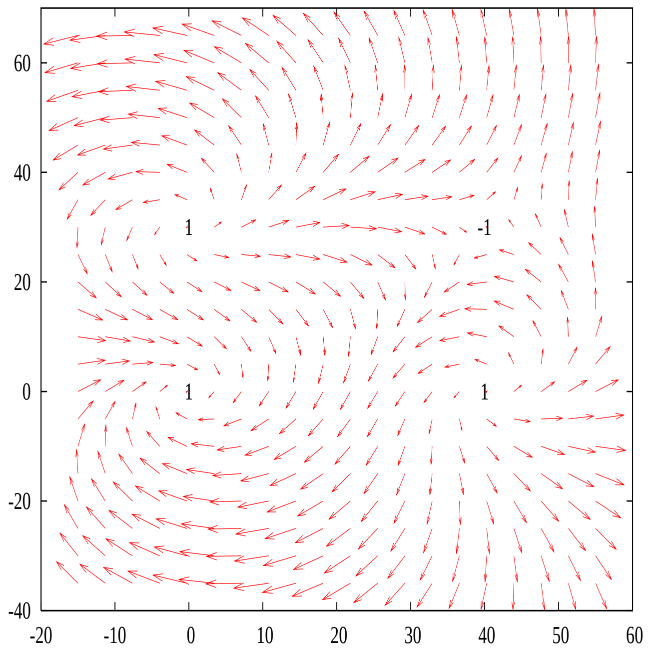

Figure 2.4: Far field for four vortices, with three +1 and one −1 degree.

where

$$\frac{d^2 f_0}{dr^2} + \frac{1}{r}\frac{df_0}{dr} - f_0\frac{n_\ell^2}{r^2} + f_0 - f_0^3 = 0,$$
$$f_0(0) = 0, \qquad f_0 \to 1 \text{ as } r \to \infty,$$

and $C(T)$ is determined by matching. As we mentioned in Chapter 1, it is well known that this equation is well-posed and has a unique increasing monotone solution.

Continuing with the expansion we find at first order that

$$\mu(i\gamma_2 - \gamma_1)\nabla\psi_0 \cdot \frac{d\mathbf{X}_\ell}{dT} = \Delta\psi_1 + \psi_1(1 - |\psi_0|^2) - \psi_0(\psi_0\psi_1^* + \psi_0^*\psi_1). \tag{2.24}$$

We note that the right hand side of this equation is just the linear operator that is obtained when linearising equation (2.22) around the basic single-vortex solution, and the non-homogeneous terms at the left hand side appear due to the effect of the interaction of vortices. Next we show that in order for (2.24) to have a solution a certain condition must hold. This condition is the one that will determine a law of motion for the vortices.

2.3.3 Solvability Condition

We recall that equation (2.24) is of the form

$$L(\psi_0)[\psi_1] = b(\psi_0, \mu, d\mathbf{X}_\ell/dT)$$

where

$$L(\psi_0)[\psi_1] = \Delta\psi_1 + \psi_1(1 - |\psi_0|^2) - \psi_0(\psi_0\psi_1^* + \psi_0^*\psi_1) \tag{2.25}$$
$$b(\psi_0, \mu, d\mathbf{X}_\ell/dT) = -(\gamma_1 - i\gamma_2)\mu\frac{d\mathbf{X}_\ell}{dT} \cdot \nabla\psi_0 \tag{2.26}$$

Let us define the following inner product

$$\big(u, v\big) = \int_D \Re\{uv*\}\, dD,$$

where D is any given ball in $\mathbb{R}^2$ and $\Re\{uv*\}$ is the real part of the product of the complex function u and the complex conjugate of v. Using the integration by parts formula

$$\int_D u\Delta v\, dD = \int_D v\Delta u\, dD + \int_{\partial D}(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n})\, dl \tag{2.27}$$

we find that the operator L is actually self-adjoin with respect to the above defined inner product. Therefore, choosing v to be the solutions to the homogeneous equation

$$L(q, \psi_0)[v] = 0,$$

and from the *Fredholm Alternative* theory for linear operators we know that in order for (2.24) to have a solution, we must assure that the solutions of the homogeneous adjoin equation are orthogonal to the non-homogeneous terms in (2.24). Therefore, we can write a solvability condition

$$-\int_D \Re\Big\{\mu(\gamma_1 - i\gamma_2)\Big(\nabla\psi_0\cdot\frac{d\mathbf{X}_\ell}{dT}\Big)\big(\nabla\psi_0\cdot\mathbf{d}\big)^*\Big\}\,dD$$
$$= \int_{\partial D} \Re\Big\{(\nabla\psi_0\cdot\mathbf{d})\frac{\partial\psi_1^*}{\partial r} - \psi_1^*\frac{\partial(\nabla\psi_0\cdot\mathbf{d})}{\partial r}\Big\}\,dl$$

where we have found this condition by studying the operator (2.25) and realising that the solutions of $L(q,\psi_0)[v] = 0$ are actually the derivatives in any direction of ψ_0, that is $v = \nabla\psi_0\cdot\mathbf{d}$, being $\mathbf{d}$ any vector in $\mathbb{R}^2$.

If we now write the solvability condition in polar coordinates we obtain

$$-\mu\pi\gamma_1\Big(\frac{d\mathbf{X}_\ell}{dT}\cdot\mathbf{d}\Big)\int_0^r \big(s(f_0')^2 + n_\ell^2\frac{f_0^2}{s}\big)\,ds + \mu\pi\gamma_2 n_\ell\Big(\frac{d\mathbf{X}_\ell}{dT}\cdot\mathbf{d}^\perp\Big)f_0^2(r)$$
$$= n_\ell\int_0^{2\pi}\left(\frac{\partial\chi_1}{\partial r} + \frac{\chi_1}{r}\right)\mathbf{e}_\phi\cdot\mathbf{d}\,d\phi \qquad (2.28)$$

being $d = (d_1, d_2)$, $d^\perp = (-d_2, d_1)$, and χ_1 is the first order term in the ϵ-expansion for the phase function χ.

2.3.4 Asymptotic matching

So far we have found asymptotic approximations to the solution in the outer and inner regions, but nevertheless, some constants still remain unknown. To determine these constants we must impose that both the inner and outer expansions do actually represent the same solution. To do so, we note that the regions where both expansions are uniformly valid have an overlap area where both approximations are valid at the same time. Therefore, in this overlap region both expansions must be the same and this will determine the unknown parameters.

We start by obtaining the *inner limit of the outer* expansion and we will find how close to the k-th vortex we must be so that the series remain uniform. Afterwards, we will proceed in the same way with the *outer limit of the inner* and compare both limits to check that they do actually have an overlap region where we will find matching conditions for the unknown constants.

Inner limit of the outer

We express the leading-order (in ϵ) outer χ_0 in terms of the inner variable $\mathbf{x}$ given by $\mathbf{X} = \mathbf{X}_\ell + \epsilon\mathbf{x}$, so that $R_\ell = \epsilon r$ and $\phi_\ell = \phi$ and write again the resulting expansion by rearranging the corresponding terms. Then upon using the expressions for χ_{00}, χ_{01} and χ_{02} derived in (2.16), (2.17) and (2.18), we find

$$\begin{aligned}\chi_0 &\sim \chi_{00} + \mu\chi_{01} + \dots \\ &\sim n_\ell\phi + G(\mathbf{X}_\ell) - n_\ell\mu\gamma_1\frac{\epsilon r}{2}\log\epsilon r\,\mathbf{e}_\phi\cdot\frac{d\mathbf{X}_\ell}{dT} + \epsilon\nabla G(\mathbf{X}_\ell)\cdot\mathbf{x} + \mathcal{O}(\epsilon^2) \qquad (2.29)\end{aligned}$$

where

$$\begin{aligned}G(\mathbf{X}) = &\sum_{j\neq\ell} n_j\phi_j - \mu\gamma_1\sum_{j\neq\ell}\frac{n_j}{2}|\mathbf{X}-\mathbf{X}_j|\log|\mathbf{X}-\mathbf{X}_j|\frac{d\mathbf{X}_j}{dT}\cdot\mathbf{e}_{\phi j\ell} \\ &+ (\mu\gamma_1)^2\sum_{j\neq\ell}\frac{n_j}{8}|\mathbf{X}-\mathbf{X}_j|^2\log|\mathbf{X}-\mathbf{X}_j|\Big(\mathbf{e}_{\phi j\ell}\cdot\frac{d\mathbf{X}_j}{dT}\Big)\Big(\mathbf{e}_{rjk}\cdot\frac{d\mathbf{X}_j}{dT}\Big) + \mathcal{O}((\mu\gamma_1)^3),\end{aligned}$$

where $\mathbf{e}_{rj\ell}$ and $\mathbf{e}_{\phi j\ell}$ represents the unitary vectors in the radial and the angular direction of vortex ℓ when seen from vortex j. Note that although G contains a full expansion in μ, the higher-order terms in the μ expansion of χ_0 do not contribute any local terms at $O(\epsilon)$. This is due to the fact that, as it is pointed out in section §2.3.1, the terms χ_{0m} in the μ expansion are locally, close to vortex ℓ, of the form $constant + R^m\log R$ and as a consequence no term of the form $\epsilon\mu^m$, being $m > 1$, can appear in the inner limit of the outer for χ_0 given in (2.29). Actually, higher order terms in the χ_0 inner expansion would always give terms like $(\mu\epsilon)^m$. This means that we have already taken all the terms in the μ-expansions which will simplify the matching procedure very much. In fact, we will see in the following chapters that this is not necessarily the case when $q \neq 0$, on the contrary, this will be one of the main difficulties that we will encounter.

Thus, if we take two terms in the inner limit of the leading order term in the outer ((2ti)(1to) in Van Dyke's matching rule notation) we find

$$\begin{aligned}\psi(\epsilon r,\phi) = e^{i(n_\ell\phi+G(\mathbf{X}_\ell))}\Big(1 - i\mu n\gamma_1\frac{\epsilon r}{2}\log(\epsilon r)\mathbf{e}_\phi\cdot\frac{d\mathbf{X}_\ell}{dT} \\ + i\epsilon\nabla G(\mathbf{X}_\ell)\cdot\mathbf{x} + \mathcal{O}\big((\gamma_1\epsilon r)^2\log(\epsilon r)\big)\Big) \qquad (2.30)\end{aligned}$$

and it is clear that this expansion is uniformly valid provided $r < \mathcal{O}(1/\gamma_1\epsilon)$.

Outer limit of the inner

The leading-order phase in the inner region is

$$\chi_0 = n_\ell\phi + C(T).$$

To proceed with the first order terms, we must compute the outer limit of the solutions of (2.24). The type of solutions that will match with the inner limit of the outer are those with the particular form

$$\psi_1 = e^{i(n_\ell\phi + C(T))}(v_1(r)\cos\phi + v_2(r)\sin\phi).$$

By equating the coefficients of $\cos\phi$ and $\sin\phi$ in (2.24) we find equations for $v_1(r)$ and $v_2(r)$,

$$\begin{aligned} v_1'' + \frac{v_1'}{r} + 2in_\ell\frac{v_2}{r^2} - (1+n_\ell^2)\frac{v_1}{r^2} + v_1(1-f_0^2) - f_0^2(v_1^* + v_1) \\ = \mu(i\gamma_2 - \gamma_1)(f_0'V_1 + i\frac{n_\ell}{r}f_0V_2), \end{aligned} \tag{2.31}$$

$$\begin{aligned} v_2'' + \frac{v_2'}{r} - 2in_\ell\frac{v_1}{r^2} - (1+n_\ell^2)\frac{v_2}{r^2} + v_2(1-f_0^2) - f_0^2(v_2^* + v_2) \\ = \mu(i\gamma_2 - \gamma_1)(f_0'V_2 - i\frac{n_\ell}{r}f_0V_1), \end{aligned} \tag{2.32}$$

where $d\mathbf{X}_\ell/dT = (V_1, V_2)$. We now write everything in terms of the outer variable $R = \epsilon r$ and we use that the outer limit of the leading order inner modulus is given by

$$f_0 = 1 - \frac{n_\ell^2}{2R^2}\epsilon^2 + \mathcal{O}(\epsilon^4)$$

to find that

$$v_1 = -i\mu\gamma_1 n_\ell V_2\frac{R}{2\epsilon}\log R + \frac{C_1}{\epsilon}R + \mathcal{O}\big(\epsilon^2\mu\log R/R\big) \tag{2.33}$$

$$v_2 = i\mu\gamma_1 n_\ell V_1\frac{R}{2\epsilon}\log R + \frac{C_2}{\epsilon}R + \mathcal{O}\big(\epsilon^2\mu\log R/R\big) \tag{2.34}$$

where C_1 and C_2 are constants to be determined when comparing with the inner limit of the outer.

If we take two terms in the inner expansion for ψ,

$$\psi \sim e^{i\chi_0}(f_0 + \epsilon(f_1 + if_0\chi_1)) + \dots,$$

but in terms of the outer variable R, we find, to leading order, that the inner solution for ψ can be expanded in terms of the outer variable as

$$\begin{aligned} \psi(R,\phi) = & e^{i(n_\ell\phi + C(T))}\Big(1 - i\mu n_\ell\gamma_1\frac{R}{2}\log(R)\mathbf{e}_\phi\cdot\frac{d\mathbf{X}_\ell}{dT} \\ & + iR(C_1\cos\phi + C_2\sin\phi) + \mathcal{O}\big(\epsilon^2/R^2 + \epsilon^2\log(R)/R\big)\Big). \end{aligned} \tag{2.35}$$

In this case we find that the series is uniformly asymptotic if $r > \mathcal{O}(1/(\gamma_1\epsilon)^{1/3})$.

Matching process in the overlap region. We find that when

$$\mathcal{O}(1/(\gamma_1\epsilon)^{1/3}) < r < \mathcal{O}(1/\gamma_1\epsilon),$$

both the outer limit of the inner and the inner limit of the outer are uniformly valid at the same time and therefore, both series must coincide in this region. We note that to perform this matching we need to take the full μ expansion of both terms. Fortunately only the expansion of G and ∇G involve infinitely many terms in μ, and these are evaluated at $\mathbf{X}_\ell$ and are therefore independent of $\mathbf{x}$.

Comparing (2.30) and (2.35) we obtain that the expansions will match if $C(T) = G(\mathbf{X})$ and $\nabla G(\mathbf{X}_\ell) = (C_1, C_2)$. We can now write χ_1 as

$$\chi_1 \sim -n_\ell\gamma_1\frac{r}{2}\log\epsilon r\,\mathbf{e}_\phi\cdot\frac{d\mathbf{X}_\ell}{dT}\mu + \nabla G(\mathbf{X}_\ell)\cdot\mathbf{x} \tag{2.36}$$

as $r \to \infty$. This is the function that we will use in the solvability condition in order to obtain a law of motion for the vortices.

2.3.5 Law of motion

We can now use expression (2.36) in the solvability condition (2.28) to find a law of motion for each vortex.

Using that $f_0^2(r) \sim 1 - n_\ell^2/r^2 + \ldots$ as $r \to \infty$, the left-hand side of (2.28) is actually

$$-\mu\pi\gamma_1 n_\ell^2(\log r + a)\,\frac{d\mathbf{X}_\ell}{dT}\cdot\mathbf{d} + \mu\pi n_\ell\gamma_2\frac{d\mathbf{X}_\ell}{dT}\cdot\mathbf{d}^\perp, \tag{2.37}$$

where

$$a = \lim_{r\to\infty}\left[\int_0^r\left(s\big(\frac{f_0'}{n_\ell}\big)^2 + \frac{f_0^2}{s}\right)ds - \log r\right],$$

is a constant independent of μ and ϵ. For example, when $n_\ell = 1$, the value of a is -0.123. Using (2.36) in the right-hand side of (2.28) gives

$$-\mu\pi\gamma_1(\frac{1}{2}+\log r)n_\ell^2\,\frac{d\mathbf{X}_\ell}{dT}\cdot\mathbf{d} - \mu\pi\gamma_1 n_\ell^2\log\epsilon\frac{d\mathbf{X}_\ell}{dT}\cdot\mathbf{d} + (k_1d_2 - d_1k_2)2n_\ell\pi, \tag{2.38}$$

where $d = (d_1, d_2)$ and $\nabla G(\mathbf{X}_\ell) = (k_1, k_2)$. Since μ is a small constant, we find that the only way to make the left and right hand sides balance is by taking $\mu = 1/|\log\epsilon|$, giving

$$\begin{aligned}-a\mu\gamma_1 n_\ell\frac{d\mathbf{X}_\ell}{dT}\cdot\mathbf{d} + \mu\gamma_2 n_\ell\frac{d\mathbf{X}_\ell}{dT}\cdot\mathbf{d}^\perp\\ = -\mu\frac{1}{2}\gamma_1 n_\ell^2\frac{d\mathbf{X}_\ell}{dT}\cdot\mathbf{d} + \gamma_1 n_\ell^2\frac{d\mathbf{X}_\ell}{dT}\cdot\mathbf{d} - 2n_\ell\nabla G(\mathbf{X}_\ell)^\perp\cdot\mathbf{d}.\end{aligned} \tag{2.39}$$

Since $\mathbf{d}$ is arbitrary this can be rearranged to give the law of motion for the vortices:

Result 2.1 (Law of motion). *A set of vortices that satisfy* (2.12) *and* (2.13) *that are separated by distances of order* $1/\epsilon$ *present a slow time evolution that is governed by the law of motion given by*

$$\begin{aligned}\frac{d\mathbf{X}_\ell}{dT} =& -\frac{2\gamma_1 n_\ell(1-\mu(1/2-a))}{\gamma_1^2 n_\ell^2(1-\mu(1/2-a))^2+(\mu\gamma_2)^2}(\nabla G(\mathbf{X}_\ell))^\perp \\ & -\frac{2n_\ell\mu\gamma_2}{\gamma_1^2 n_\ell^2(1-\mu(1/2-a))^2+(\mu\gamma_2)^2}\nabla G(\mathbf{X}_\ell)+\mathcal{O}(\epsilon)\end{aligned} \tag{2.40}$$

where

$$\begin{aligned}\mu &= \frac{1}{|\log\epsilon|}, \\ \nabla G(\mathbf{X}) &= \sum_{j\neq k}\frac{n_j\mathbf{e}_{\phi j\ell}}{|\mathbf{X}-\mathbf{X}_j|} + n_j\gamma_1\frac{\mu}{2}\log|\mathbf{X}-\mathbf{X}_j|\left(\frac{d\mathbf{X}_j}{dT}\cdot\mathbf{e}_{rj\ell}\right)\mathbf{e}_{\phi j\ell} \\ & \quad - n_j\gamma_1\frac{\mu}{2}(1+\log|\mathbf{X}-\mathbf{X}_j|)\left(\frac{d\mathbf{X}_j}{dT}\cdot\mathbf{e}_{\phi_j k}\right)\mathbf{e}_{rj\ell}+\mathcal{O}(\mu^2),\end{aligned}$$

and $(\nabla G(\mathbf{X}_\ell))^\perp$ *represents the vector orthogonal to* $\nabla G(\mathbf{X}_\ell)$ *that is obtained after rotating* $\pi/2$ *counterclockwise.*

We now analyse expression (2.40) to describe the way the law of motion changes as the parameter b in equation (2.5) is changed from zero to infinity.

2.3.6 Interpolation between the Ginzburg-Landau and the Nonlinear Schrödinger equation

As we mentioned before, the time scale for the vortices of the classical Ginzburg-Landau equation, that is when $b=0$, is $\epsilon^2/|\log(\epsilon)|$. But when b becomes infinity, the time scale accelerates to ϵ^2. The way in which this transition takes place can be deduced from the previous law of motion (2.40). We then substitute the actual values of γ_1 and γ_2 and analyse equation (2.40) to show the influence of the order of b into the velocity.

Law of motion for $\gamma_1 = 1$ and $\gamma_2 = b$: Equation (2.40) now reads,

$$\begin{aligned}\frac{d\mathbf{X}_\ell}{dT} =& -\frac{2n_\ell(1-\mu(1/2-a))}{n_\ell^2(1-\mu(1/2-a))^2+(\mu b)^2}(\nabla G(\mathbf{X}_\ell))^\perp \\ & -\frac{2n_\ell\mu b}{n_\ell^2(1-\mu(1/2-a))^2+(\mu b)^2}\nabla G(\mathbf{X}_\ell)+\mathcal{O}(\epsilon).\end{aligned} \tag{2.41}$$

If we keep only the leading order term we find:

Corolary 2.1. *For an N-vortex solution in the Ginzburg-Landau equation (2.5) where the parameter b is of order less than one, the law of motion reads*

$$\frac{d\mathbf{X}_\ell}{dT} = \frac{2}{n_\ell}\sum_{j\neq k}\frac{n_j \mathbf{e}_{rj\ell}}{|\mathbf{X}_\ell - \mathbf{X}_j|} + \mathcal{O}(\mu) \tag{2.42}$$

Therefore, the parameter b does only affect the order-μ corrections.

This means that, if we have two vortices at positions $(X_1, 0)$ and $(X_2, 0)$, the interaction is to leading order along the line of the vortices. The presence of b is introducing an order μ perturbation in the tangential direction that makes the two vortices spin around each other, as it shows the following expression,

$$\begin{aligned}\frac{d\mathbf{X}_1}{dT} = & 2\frac{n_2}{n_1}\frac{\mathbf{e}_{r21}}{|\mathbf{X}_1 - \mathbf{X}_2|} + 2\mu\frac{n_2}{n_1}\frac{(1/2-a)\ \mathbf{e}_{r21}}{|\mathbf{X}_1 - \mathbf{X}_2|} \\ & + \mu\frac{n_2}{n_1}(1+|\mathbf{X}_1 - \mathbf{X}_2|)\frac{d\mathbf{X}_2}{dT}\cdot \mathbf{e}_{\phi 21}\ \ \mathbf{e}_{\phi 21} \\ & - 2\mu\frac{bn_2}{n_1^2}\frac{\mathbf{e}_{\phi 21}}{|\mathbf{X}_1 - \mathbf{X}_2|} + \mathcal{O}(\mu^2)\end{aligned}$$

From this law of motion we also observe that vortices with the same winding number repeal each other, while those with opposite degrees attract, in the same way that charged particles in an electrical field would do.

Law of motion for $\gamma_1 = 1/b$ and $\gamma_2 = 1$: In this case equation (2.40) becomes,

$$\begin{aligned}\frac{d\mathbf{X}_\ell}{dT} = & -\frac{2}{b}\frac{n_\ell(1-\mu(1/2-a))}{(1/b)^2 n_\ell^2(1-\mu(1/2-a))^2+\mu^2}(\nabla G(\mathbf{X}_\ell))^\perp \\ & -\frac{2n_\ell\mu}{(1/b)^2 n_\ell^2(1-\mu(1/2-a))^2+\mu^2}\nabla G(\mathbf{X}_\ell) + \mathcal{O}(\epsilon)\end{aligned} \tag{2.43}$$

Now, the leading order term will be different depending on how small $1/b$ is. Therefore we obtain:

Corolary 2.2. *When $1/b$ is of order one, the law of motion for N vortices becomes*

$$\frac{d\mathbf{X}_\ell}{dT} = \frac{2}{n_\ell}\sum_{j\neq k}\frac{n_j \mathbf{e}_{rj\ell}}{|\mathbf{X}_\ell - \mathbf{X}_j|} + \mathcal{O}(\mu)$$

Hence the interaction for a system with only two vortices is once more along the line of the centres.

But if we now analyse the situation where $1/b$ is small, we will find that the order of magnitude will be important to determine the leading order term in the velocity. Thus, we start by writting $1/b = \beta\mu^\delta$ so that we can compare the order of magnitude of $1/b$ with μ. We find the following cases:

Corolary 2.3.

If $\delta < 1$, the leading order velocity is

$$\frac{d\mathbf{X}_\ell}{dT} = -\frac{2}{\beta n_\ell \mu^\delta}(\nabla G_0(\mathbf{X}_\ell))^\perp + \mathcal{O}(\mu^{1-2\delta}),$$

if $\delta = 1$, the leading order velocity is

$$\frac{d\mathbf{X}_\ell}{dT} = -\frac{2}{\mu\beta n_\ell}(\nabla G_0(\mathbf{X}_\ell))^\perp - \frac{2}{\mu(\beta^2 n_\ell^2 + 1)}\nabla G_0(\mathbf{X}_\ell) + \mathcal{O}(1),$$

and if $\delta > 1$ the law of motion is

$$\frac{d\mathbf{X}_\ell}{dT} = -\frac{2}{\mu}\nabla G_0(\mathbf{X}_\ell) + \mathcal{O}(\mu^{\delta-2}),$$

where

$$\nabla G_0(\mathbf{X}) = \sum_{j\neq k} \frac{n_j \mathbf{e}_{\phi j \ell}}{|\mathbf{X} - \mathbf{X}_j|}$$

This means that, when $\delta < 1$, the time scale for the velocity is $\epsilon^2|\log\epsilon|^{\delta-1}$, which implies that the vortices move now a bit faster than before. We recall that in all the previous cases the time scale was actually $\epsilon^2|\log\epsilon|^{-1}$. We also observe that as δ increases to become one, that is to say, b becomes logarithmically large in ϵ, the time scale of the velocity changes to ϵ^2 that is the same time scale that one finds in vortices for the Nonlinear Schrödinger equation.

We also note that, as b increases, the velocity changes its direction from having only a radial component along the lines of the centres of the vortices to an intermediate situation where both tangential and radial components are equally dominant, and finally, for large enough values of b, the dominant part of the velocity becomes only tangential. This is consistent with the fact that vortices in the Nonlinear Schrödinger equation, that corresponds to the limit as b goes to infinity, interact only tangentially.

2.3.7 Writing the $q = 0$ case as for the $q > 0$

We now show the way we would write the previous calculations following the scheme of the $q > 0$ case. For simplicity we take $b = 0$, but the extension to general values of b is straight forward. The main difference in what we present now with respect to the proceeding in [34] lies in the way we express the outer limit of the inner and the way the matching is done. For a non-vanishing value of q, the outer limit of the inner solution and the inner limit of the outer solution are found by expanding first the equations and afterwards obtaining the solution. We could also do that when $q = 0$:

Outer limit of the leading-order inner We begin by rewriting the leading-order inner equations in terms of the outer variable $R = \epsilon r$ to obtain

$$0 = \epsilon^2(\Delta f_0 - f_0|\nabla\chi_0|^2) + (1 - f_0^2)f_0, \tag{2.44}$$

$$0 = \epsilon^2\nabla \cdot (f_0^2\nabla\chi_0). \tag{2.45}$$

We now expand in powers of ϵ as

$$\chi_0 \sim \widehat{\chi}_{00} + \epsilon^2\widehat{\chi}_{01} + \cdots , \tag{2.46}$$

$$f_0 \sim \widehat{f}_{00} + \epsilon^2\widehat{f}_{01} + \cdots . \tag{2.47}$$

Substituting (2.46), (2.47) into (2.44), (2.45) gives

$$\widehat{f}_{00} = 1, \tag{2.48}$$

$$\widehat{f}_{01} = -\frac{1}{2}|\nabla\widehat{\chi}_{00}|^2, \tag{2.49}$$

$$0 = \Delta\widehat{\chi}_{00}, \tag{2.50}$$

with solution $\widehat{\chi}_{00} = n_\ell\phi$.

Outer limit of the first-order inner We write the first-order inner equations in terms of the outer variable to give

$$\begin{aligned}-\epsilon\mu\frac{d\mathbf{X}_\ell}{dT} \cdot \nabla f_0 = {}&\epsilon^2\Delta f_1 - \epsilon^2 f_1|\nabla\chi_0|^2\\ &- 2\epsilon^2 f_0\nabla\chi_0 \cdot \nabla\chi_1 + f_1 - 3f_0^2 f_1,\end{aligned} \tag{2.51}$$

$$-\mu\epsilon f_0^2\frac{d\mathbf{X}_\ell}{dT} \cdot \nabla\chi_0 = \epsilon^2\nabla \cdot (f_0^2\nabla\chi_1) + \epsilon^2\nabla \cdot (2f_0 f_1\nabla\chi_0). \tag{2.52}$$

We now expand in powers of ϵ as

$$\chi_1 \sim \frac{\widehat{\chi}_{10}}{\epsilon} + \widehat{\chi}_{11} + \cdots , \tag{2.53}$$

$$f_1 \sim \widehat{f}_{10} + \epsilon\widehat{f}_{11} + \cdots , \tag{2.54}$$

to give

$$\widehat{f}_{10} = 0, \tag{2.55}$$

$$\widehat{f}_{11} = -\nabla\widehat{\chi}_{00} \cdot \nabla\widehat{\chi}_{10}, \tag{2.56}$$

$$\begin{aligned}\Delta\widehat{\chi}_{10} = {}&-\mu\frac{d\mathbf{X}_\ell}{dT} \cdot \nabla\widehat{\chi}_{00} = -n_\ell\mu\frac{d\mathbf{X}_\ell}{dT} \cdot \frac{1}{R}\mathbf{e}_\phi\\ = {}&n_\ell\frac{\mu}{R}(V_1\sin\phi - V_2\cos\phi).\end{aligned} \tag{2.57}$$

Thus

$$\widehat{\chi}_{10} = n_\ell \frac{\mu R \log R}{2}(V_1 \sin\phi - V_2 \cos\phi) = -n_\ell \frac{\mu R \log R}{2} \frac{d\mathbf{X}_\ell}{dT} \cdot \mathbf{e}_\phi$$

plus a homogeneous solution, a harmonic function in this case, which comes from matching with the outer. We see that this solution is

$$\mathbf{X} \cdot \nabla G(\mathbf{X}_\ell).$$

Thus,

$$\widehat{\chi}_{10} = -n_\ell \frac{\mu R \log R}{2} \frac{d\mathbf{X}_\ell}{dT} \cdot \mathbf{e}_\phi + \mathbf{X} \cdot \nabla G(\mathbf{X}_\ell).$$

We remark that we have not used the $O(\mu)$ outer solution, only the leading-order outer solution. We will see in the following chapters that this way of expressing the outer limit of the inner will be the only one that will enable us to compute an outer limit of the inner without dropping any term in the q-expansion. We recall that the main difference in this chapter is that one does not have the parameter q, but in what follows q will be a small parameter that will be used to write down the expansions.

Finally, rewriting in terms of the inner variable gives

$$\chi_1 \sim -n_\ell \frac{\mu r \log \epsilon}{2} \frac{d\mathbf{X}_\ell}{dT} \cdot \mathbf{e}_\phi + \mathbf{x} \cdot \nabla G(\mathbf{X}_\ell) + \cdots ,$$

that is the same expression we found in (2.36).

The rest of the analysis would be the same, including the way we find the overlap region and the law of motion.

In the following chapters, even in the next one where we deal with single-spiral solutions, the structure we have just described will be already identified. By expressing the expansions in this way, we regard the $q = 0$-case as a particular case of $q \neq 0$, and furthermore, the single-spiral situation will also appear as the particular case of a multiple-spiral systems that arises when one considers vanishing time derivatives. It is therefore a unifying notation that is crucial in order to obtain a law of motion for many spirals when q is not zero.

Chapter 3

Symmetric spiral wave solutions: Equilibrium solutions

In this chapter we go back to the general Ginzburg-Landau equation that we already presented in Chapter 1. We recall that in a rotating frame the Ginzburg-Landau equation is written like

$$(1-ib)\psi_t = (1-|\psi|^2)\psi + iq\psi(1-k^2-|\psi|^2) + \Delta\psi. \tag{3.1}$$

In particular, we examine the family of equilibria of (3.1) such that they have a non-vanishing degree, n, which is defined through the integral

$$\oint_C \nabla\chi \cdot dl = 2\pi n, \quad n \in \mathbb{Z}, \tag{3.2}$$

being C any regular closed curved enclosing the origin. As we explained in Chapter 1, the number n is the so-called degree of the function ψ that represents the number of times that the two-dimensional vector given by the real and imaginary parts of ψ rotates when it goes along the curve C. In the following chapters we will show that equilibrium solutions have only one zero, otherwise they cannot be stationary.

We consider now equation (3.1) but we study solutions which have only one spiral with an arbitrary winding number n. These solutions are equilibrium solutions of (3.1) and were already analysed by P.S. Hagan in [19] were matching asymptotic techniques where used to construct the spirals. Furthermore, P.S. Hagan shows that the parameter k is uniquely determined by q, and he gives an asymptotic approximation to k as a function of q, for small values of q. As we will see, k represents the asymptotic wavenumber of the spiral.

The fact that k is uniquely determined by q means that, for a pair of parameters a, b, of the original equation (1.2), there is only one possible spiral wave with an specific asymptotic wavenumber and as a consequence, with a particular frequency ω. This is not the case when one studies plane wave solutions of (1.2) which are are characterised by being similarity solutions of the form $\psi(\mathbf{x}, t) = u(\mathbf{k} \cdot \mathbf{x} + \omega t)$, being k is the wavenumber and ω the corresponding frequency. These solutions are given by

$$\psi(\mathbf{x}, t) = Re^{i(\mathbf{k}\cdot\mathbf{x}+\omega t)}$$

and the following relations are satisfied

$$|\mathbf{k}|^2 = 1 - R^2, \qquad |\mathbf{x}|^2 = \frac{a-\omega}{a-b}.$$

The second one is giving the dispersion relation between ω and $\mathbf{k}$. As we see in these two expressions, for a pair of values for a and b we can choose almost any vector k. The only restriction is that in order to have a real value for R, k must be between a and b. Therefore, there exists a whole family of ω and $\mathbf{k}$ that produce plane wave solutions, while in the case of spiral waves, the dispersion relation is actually given by just one point rather than a curve or a surface.

Now we also consider the case in which q is a small parameter and we reproduce P.S. Hagan's results, but we construct the symmetric spiral solution using the same notation that we will be using to study multiple spiral solutions and that we already introduced in Chapter 2. It is important to take into account that the asymptotic wavenumber k happens to be exponentially small in q and thus the perturbation technique becomes very difficult. In this chapter we clarify some difficulties that were not pointed out in [19] and that are crucial in order to understand the dynamics of multiple spiral systems.

3.1 Outer Region

As we did before, we start by introducing a space scaling, ϵ, and a new parameter, $\alpha = qk/\epsilon$, that, as we will see, will play a very important role when we deal with the multiple spirals patterns. This new parameter was not used in any other work before and, as we will see, will clarify very much the structure of the asymptotic method.

We start by rescaling (3.1) onto the outer lengthscale, $\mathbf{X} = \epsilon\mathbf{x}$, by writing

$$0 = \epsilon^2 \Delta\psi + (1 - |\psi|^2)\psi + iq(1 - k^2 - |\psi|^2)\psi \tag{3.3}$$

Expressing the equation in terms of $\alpha = qk/\epsilon$

$$0 = \epsilon^2 \Delta\psi + (1 + iq)(1 - |\psi|^2)\psi - \frac{i\epsilon^2\alpha^2}{q}\psi \tag{3.4}$$

This can now be seen as an eigenvalue problem in α. We will find that the eigenvalue is actually given as a function of q, and this will provide the relation between k and q. One of the good points of introducing this new parameter α is that it will turn out to be an order one parameter when k is exponentially small in q. This fact shows why is so much easier working with α rather than keeping k all the way through, as it was done in [19].

Therefore, using a modulus-phase form for the solution, $\psi = fe^{i\chi}$, the solutions we are looking for have the following boundary conditions

$$\begin{aligned} &f(r) \sim \gamma r^n \quad \text{as } r \to 0 \quad \chi_r(0) = 0 \\ &f(r),\, \chi_r(r) \quad \text{bounded as } r \to \infty, \end{aligned} \tag{3.5}$$

where γ is some positive constant. The first condition will actually apply on the inner region equations, and the second one is used when deriving the following asymptotic expansions

$$\begin{aligned} f &\sim \left(1-k^2\right)^{1/2} - \frac{k}{2qr\left(1-k^2\right)^{1/2}} + \mathcal{O}\left(\frac{1}{r^2}\right) \\ \chi_r &\sim \pm k + \frac{1}{2qr} + \mathcal{O}\left(\frac{1}{r^2}\right), \end{aligned} \tag{3.6}$$

and these expressions, expressed in terms of $R = \epsilon r$, will be the boundary conditions at infinity for the outer region. In this last expression we observe that k represents the wavenumber at infinity of a wave in polar coordinates, this is why it is called the asymptotic wavenumber. Furthermore, this asymptotic expansion for χ_r will only be valid provided $k \gg 1/(qr)$, that is to say, for distances such that $r \gg 1/(qk)$, or alternatively, if we write the expressions in the outer variable R, the expansion will be valid provided α is of order one. Therefore, α can be seen as a measure of how far we are from the region where the asymptotic wavenumber dominates the expansion.

On the other hand, the solutions that we are looking for are the heteroclinic orbits of the system that start in the point (3.5) and arrive at $f = (1-k^2)^{1/2}$ and $\chi_r = \pm k$. These orbits do only exist for certain values of the asymptotic wavenumber k.

Writing $\psi = fe^{i\chi}$ and separating real and imaginary parts in (3.4) gives

$$0 = \epsilon^2 \Delta f - \epsilon^2 f |\nabla \chi|^2 + (1-f^2) f \tag{3.7}$$

$$0 = \epsilon^2 \nabla \cdot (f^2 \nabla \chi) + q f^2 (1-f^2) - \epsilon^2 \frac{\alpha^2}{q} f^2 \tag{3.8}$$

Expanding in powers of ϵ as

$$f \sim f_0(\mathbf{X}; q, \mu) + \epsilon^2 f_1(\mathbf{X}; q, \mu) + \dots \tag{3.9}$$

$$\chi \sim \chi_0(\mathbf{X}; q, \mu) + \epsilon^2 \chi_1(\mathbf{X}; q, \mu) + \dots \tag{3.10}$$

we find

$$f_0 = 1, \tag{3.11}$$

$$f_1 = -\frac{1}{2}|\nabla\chi_0|^2, \tag{3.12}$$

$$0 = \Delta\chi_0 + q|\nabla\chi_0|^2 - \frac{\alpha^2}{q}. \tag{3.13}$$

Spiral waves are solutions of the form $f = f(R)$ and $\chi = n\phi + \varphi(R)$. If we expand φ in ϵ with

$$\varphi \sim \varphi_0 + \epsilon^2\varphi_1 + \epsilon^4\varphi_2 + \dots,$$

equation (3.13) becomes

$$\varphi_0'' + \frac{\varphi_0'}{R} + q\left(\frac{n^2}{R^2} + (\varphi_0')^2\right) - \frac{\alpha^2}{q} = 0 \tag{3.14}$$

which is a Riccati equation that can be linearised through the transformation

$$\varphi_0 = \frac{1}{q}\log H_0(R), \quad \text{or} \quad \varphi_0' = \frac{H_0'(R)}{qH_0(R)}$$

to give

$$H_0'' + \frac{H_0'}{R} + H_0\left(\frac{(qn)^2}{R^2} - \alpha^2\right) = 0 \tag{3.15}$$

with the general solution

$$H_0(R) = K_{inq}(\alpha R) + \lambda I_{inq}(\alpha R) \tag{3.16}$$

where λ is an arbitrary real number, and K_{inq} and I_{inq} are the modified Bessel functions of the first and second kind. We observe that only when $\lambda = 0$ is the function φ monotone. Therefore this is the solution that gives a spiral pattern in the iso-phase contours.

The solution for φ_0' is then given by

$$\begin{aligned}\varphi_0'(R) = \frac{\partial\chi_0}{\partial R} &= -\frac{\epsilon\alpha}{q}\,\frac{\int_0^\infty e^{-\alpha R\cosh t}\cosh t\cos(nqt)\,dt}{\int_0^\infty e^{-\alpha R\cosh t}\cos(nqt)\,dt} \\ &= -k\,\frac{\int_0^\infty e^{-\alpha R\cosh t}\cosh t\cos(nqt)\,dt}{\int_0^\infty e^{-\alpha R\cosh t}\cos(nqt)\,dt},\end{aligned} \tag{3.17}$$

so the corresponding function for χ_0 is

$$\chi_0 \sim n\phi + \frac{1}{q}\log(K_{inq}(\alpha R)). \tag{3.18}$$

In this case that we only consider a single spiral we are able to find an analytic expression for φ_0 without the need of expanding it in terms of q. In general we will not be able to do such a thing when we deal with more than one spiral. Actually, upon analysing the expression given in (3.18) for χ_0 one realises that the relation between the magnitudes of q and R will be important in order to expand this expression in terms of the small parameter q.

3.2 Inner region

Since the solutions of (3.1) are invariant under translations, for simplicity we have chosen the centre of the spiral to be at the origin. We thus rescale near the origin of the spiral, by setting $\mathbf{X} = \epsilon \mathbf{x}$ to give

$$0 = \Delta f - f|\nabla \chi|^2 + (1 - f^2)f \tag{3.19}$$

$$0 = \nabla \cdot (f^2 \nabla \chi) + q(1 - f^2)f^2 - \frac{\epsilon^2 \alpha^2 f^2}{q} \tag{3.20}$$

Expanding now like

$$f \sim f_0 + \epsilon f_1 + \epsilon^2 f_2 + \dots \tag{3.21}$$

$$\chi \sim \chi_0 + \epsilon \chi_1 + \dots \tag{3.22}$$

The leading-order equations are

$$0 = \Delta f_0 - f_0|\nabla \chi_0|^2 + (1 - f_0^2)f_0, \tag{3.23}$$

$$0 = \nabla \cdot (f_0^2 \nabla \chi_0) + q(1 - f_0^2)f_0^2. \tag{3.24}$$

Imposing the topological condition that leads to a spiral shape,

$$f = f(r) \quad \chi = n\phi + \varphi(r),$$

gives

$$f_0'' + \frac{1}{r}f_0' - f_0\Big(\frac{n^2}{r^2} + (\varphi_0')^2\Big) + (1 - f_0^2)f_0 = 0, \tag{3.25}$$

$$f_0(\varphi_0'' + \frac{1}{r}\varphi_0') + 2f_0'\varphi_0' + q(1 - f_0^2)f_0 = 0. \tag{3.26}$$

Expansion for small q. If we now inspect the system (3.25)-(3.26) we realise that we can expand the leading-order solution in powers of q as

$$f_0 \sim f_{00} + f_{01}q^2 + f_{02}q^4 + \cdots , \tag{3.27}$$

$$\varphi_0 \sim \frac{\varphi_{00}}{q} + \varphi_{01}q + \varphi_{02}q^3 + \cdots , \tag{3.28}$$

and substituting these expansions into (3.25) and (3.26) and equating powers of q gives

$$\varphi_{00} = D_0, \tag{3.29}$$

$$\varphi_{01} = D_1, \tag{3.30}$$

$$0 = f_{00}'' + \frac{f_{00}'}{r} - n^2 \frac{f_{00}}{r^2} + (1 - f_{00}^2) f_{00}, \tag{3.31}$$

$$0 = f_{01}'' + \frac{f_{01}'}{r} - n^2 \frac{f_{01}}{r^2} + (1 - 3f_{00}^2) f_{01}, \tag{3.32}$$

$$\varphi_{02}'(r) = -\frac{1}{r f_{00}^2} \int_0^r s f_{00}^2 (1 - f_{00}^2)\, ds, \tag{3.33}$$

with the boundary conditions

$$f_{00}(0) = 0, \quad \lim_{r\to\infty} f_{00}(r) = 1 \quad \text{and} \tag{3.34}$$

$$f_{01}(0) = 0, \quad \lim_{r\to\infty} f_{01}(r) = 0. \tag{3.35}$$

and D_0 and D_1 are real constants to be determined by matching.

3.3 Asymptotic matching

3.3.1 Outer limit of the inner

From the expressions (3.29) to (3.33) we find that when $r \to \infty$

$$\frac{\partial \varphi_{02}}{\partial r} \sim -qn^2 \frac{\log r + c_n}{r} + \dots \tag{3.36}$$

where c_n is a constant given by

$$c_n = \lim_{r\to\infty} \frac{1}{n^2} \left(\int_0^r f_{00}^2(s) \Big(1 - f_{00}(s)^2\Big) s\, ds - n^2 \log(r) \right).$$

For instance some values of c_n are found to be $c_1 = -0.098\dots$, $c_2 = -0.998\dots$ and $c_3 = -1.3\dots$. However, in order to match with the outer expansion we need the outer limit of the whole expansion in q. This is found to be of the form

$$f_0 \sim 1 - \frac{1}{r^2} \sum_{i=0}^{N} \alpha_i \{qn^2(\log(r) + c_n)\}^{2i} + \cdots, \tag{3.37}$$

$$\frac{\partial \chi_0}{\partial r} \sim -\frac{1}{r} \sum_{i=0}^{N} \beta_i \{qn^2(\log(r) + c_n)\}^{2i+1} + \cdots, \tag{3.38}$$

where $\alpha_i > 0$ and $\beta_i > 0$ are constant values independent of q. For instance, if we compute the first two terms in each series we find that $\alpha_0 = 1/2$, $\alpha_1 = 1/2$, $\beta_0 = 1$ and $\beta_1 = 1/3n^2$. We note that these series are not convergent since if they were, their limit would be 1 and 0 respectively, and we do know that the limits at infinity are $\sqrt{1-k^2}$ and $-k$. However, expansions (3.37) and (3.38) are asymptotic provided $q(\log(r) + c_n) \ll 1$, and this fact should be taken into account when matching with the inner limit of the outer. As we did in Chapter 2, to match the outer limit of the inner with the inner limit of the outer, we must write both limits in terms of either the inner variable r or the outer R. However, if we write (3.37) and (3.38) in terms of R, we find that the q-expansions rearrange so we need to find the sum of (3.37) and (3.38) to compare with the outer. The second possibility we have is to keep r as the variable to use in the matching, write (3.17) in terms of r, expand it in terms of ϵ, and finally take the leading order of the inner limit of the outer and obtain the corresponding series in terms of q. This last series is the one we would compare to (3.38). In this particular case we can try both possibilities. But as we mentioned above, for multiple spirals we will not be able to get the whole sum in q in the outer. This means that there will not be the possibility to write the outer χ_0 in terms of r for the whole series in q. As a consequence, we will find that the only way to do the matching when we have several spirals is by summing up (3.37) and (3.38), writing them in terms of R, and then re-expanding in terms of q to compare with the outer.

Fortunately, there is a way of writing the outer limit of the inner which allows us to sum all the q terms. P.S. Hagan in [19] thinks of this as a *middle region expansion*, but it is actually just the outer limit of the inner in terms of R. The idea of the method is to start first by writing the inner equations in terms of the outer variable, expand the equations and then solve them, rather than solving first the inner equations, write them in terms of the outer variable R, and then expand the solutions in ϵ. The reader will identify this procedure with what we already did in the last section of Chapter 2.

Hence, we begin by rewriting the leading-order inner equations (3.23) and (3.24) in terms of the outer variable $R = \epsilon r$ to obtain

$$0 = \epsilon^2(\Delta f_0 - f_0|\nabla\chi_0|^2) + (1 - f_0^2)f_0 \tag{3.39}$$
$$0 = \epsilon^2\nabla\cdot(f_0^2\nabla\chi_0) + q(1 - f_0^2)f_0^2 \tag{3.40}$$

We now expand in powers of ϵ as

$$\chi_0 \sim \widehat{\chi}_{00}(q) + \epsilon^2\widehat{\chi}_{01}(q) + \cdots, \tag{3.41}$$

$$f_0 \sim \widehat{f}_{00}(q) + \epsilon^2\widehat{f}_{01}(q) + \cdots. \tag{3.42}$$

The leading-order term in this expansion, $\widehat{\chi}_{00}(q)$, is just the first term (in ϵ) in the outer expansion of the leading-order inner solution, but now it includes all the terms in q. Substituting (3.41), (3.42) into (3.39), (3.40) gives

$$\widehat{f}_{00} = 1, \tag{3.43}$$

$$\widehat{f}_{01} = -\frac{1}{2}|\nabla\widehat{\chi}_{00}|^2, \tag{3.44}$$

$$0 = \Delta\widehat{\chi}_{00} + q|\nabla\widehat{\chi}_{00}|^2. \tag{3.45}$$

We note that (3.45) is again a Riccati equation that can be linearised with the change of variable $\widehat{\chi}_{00} = (1/q)\log\widehat{h}_0$ to give

$$\Delta\widehat{h}_0 = 0.$$

Since $\widehat{\chi}_{00} = n\phi + \widehat{\varphi}(R)$ we set $\widehat{h}_0 = e^{qn\phi}e^{q\widehat{\varphi}(R)} = e^{qn\phi}\widehat{H}_0(R)$ to give

$$\widehat{H}_0'' + \frac{\widehat{H}_0'}{R} + (qn)^2\frac{\widehat{H}_0}{R^2} = 0, \tag{3.46}$$

whose solution is given by

$$\widehat{H}_0 = A(q)\epsilon^{-iqn}R^{iqn} + B(q)\epsilon^{iqn}R^{-iqn}, \tag{3.47}$$

and hence $\widehat{\chi}_{00}$ reads

$$\widehat{\chi}_{00} = n\phi + \frac{1}{q}\log(A(q)\epsilon^{-iqn}R^{iqn} + B(q)\epsilon^{iqn}R^{-iqn}) \tag{3.48}$$

where A and B are constants that depend on q and the factors $\epsilon^{\pm iqn}$ have been included to facilitate their determination by comparison with the solution in the inner variable. A and B will be determined by comparing (3.47) with the inner expansion (3.38) and also by matching with the outer. Thus, we need to write $\hat{\chi}_{00}$ in terms of r, expand in powers of q, and compare with (3.38). Expanding the constants in powers of q as

$$A(q) \sim \frac{1}{q}A_0 + A_1 + qA_2 + \cdots,$$
$$B(q) \sim \frac{1}{q}B_0 + B_1 + qB_2 + \cdots,$$

and writing $\widehat{H}_0$ in terms of r and expanding for small values of q we find

$$\begin{aligned}
\widehat{H}_0(r) &= A(q)e^{iqn\log r} + B(q)e^{-iqn\log r} \\
&\sim \left(\frac{1}{q}A_0 + A_1 + qA_2 + \cdots\right)\left(1 + iqn\log r - (qn)^2/2\log^2 r + \cdots\right) \\
&+ \left(\frac{1}{q}B_0 + B_1 + qB_2 + \cdots\right)\left(1 - iqn\log r - (qn)^2/2\log^2 r + \cdots\right) \\
&\sim \frac{A_0 + B_0}{q} + A_1 + B_1 + (A_0 - B_0)in\log r \\
&+ q\left(A_2 + B_2 + (A_1 - B_1)in\log r - n^2\frac{(A_0 + B_0)}{2}\log^2 r\right) + \cdots,
\end{aligned}$$

so that

$$\begin{aligned}\frac{\partial\widehat{\chi}_{00}}{\partial r} &= \frac{\widehat{H}_0'(r)}{q\widehat{H}_0(r)}\\ &\sim \frac{1}{qr}\frac{(A_0-B_0)in+(A_1-B_1)iqn-(A_0+B_0)qn\log r+\dots}{\frac{A_0+B_0}{q}+(A_0-B_0)in\log r+q\dots}\\ &\sim \frac{(A_0-B_0)ni}{r(A_0+B_0)}+q\left(\frac{(A_1-B_1)in}{(A_0+B_0)r}-n\frac{\log r}{r}\right.\\ &\left.+n\left(\frac{(A_0-B_0)^2}{(A_0+B_0)^2}\right)\frac{\log r}{r}+\left(\frac{-i(A_0-B_0)(A_1+B_1)}{(A_0+B_0)^2}\right)\frac{n}{r}\right)+\cdots.\end{aligned}$$

Comparing with (3.36) we see that

$$A_0-B_0=0, \tag{3.49}$$

$$\frac{(A_1-B_1)}{A_0+B_0}i=-nc_n. \tag{3.50}$$

The remaining equations determining A and B will come from matching with the outer region.

3.3.2 Inner limit of the outer

To compute the inner limit of the outer we must rewrite solution (3.16) in terms of the inner variable $\mathbf{X}=\epsilon\mathbf{x}$, expand in powers of ϵ and finally write it back in terms of $\mathbf{X}$. Furthermore, we can expand (3.16) in powers of q and then compare it with the outer limit of the inner, where we already have the whole q-expansion. On the other hand we note that we have applied the same change of functions to $\chi_0(r)$ in both the outer and the inner. Therefore, it will be simpler to proceed with the matching by comparing H_0 and $\widehat{H}_0$.

Inner limit of K_{inq} when $\alpha=\mathcal{O}(1)$ and q small The inner limit of the outer is then given by

$$\begin{aligned}H_0(R) &= K_{inq}(\alpha R)\sim K_0(\alpha R)+iq\frac{\partial K_\nu(\alpha R)}{\partial\nu}|_{\nu=0}-\frac{q^2}{2}\frac{\partial^2K_\nu(\alpha R)}{\partial\nu^2}|_{\nu=0}+\dots\\ &\sim K_0(\alpha R)-\frac{q^2}{2}\int_0^\infty t^2e^{-\alpha R\cosh t}\,dt+\dots\end{aligned}$$

and using that

$$K_0(z)=-\log\frac{z}{2}-\gamma+O(z^2), \tag{3.51}$$

where $\gamma=0.5772$ is Euler's constant, we are left with

$$H_0(R)\sim-\log\frac{R\alpha}{2}-\gamma+\mathcal{O}(q^2) \tag{3.52}$$

This should be matched to $\widehat{H}_0(R)$, and hence we have

$$\begin{aligned}\widehat{H}_0 &\sim A(q)e^{-iqn\log\epsilon}(1+iqn\log R+\ldots)(1+qn\phi+\ldots)\\ &+B(q)e^{iqn\log\epsilon}(1-iqn\log R+\ldots)(1+qn\phi+\ldots)\\ &\sim \frac{A_0e^{-iqn\log\epsilon}+B_0e^{iqn\log\epsilon}}{q}+A_1e^{-iqn\log\epsilon}+B_1e^{iqn\log\epsilon}\\ &+n\phi(A_0e^{-iqn\log\epsilon}+B_0e^{iqn\log\epsilon})\\ &+i(A_0e^{-iqn\log\epsilon}-B_0e^{iq\log\epsilon})n\log R+\mathcal{O}(q)\end{aligned}\tag{3.53}$$

Comparing (3.53) and (3.52) and using (3.49) and (3.50) we find that

$$e^{iqn\log\epsilon}+e^{-iqn\log\epsilon}=0,\tag{3.54}$$

so that

$$q|n|\log\epsilon=-\frac{\pi}{2}\tag{3.55}$$

$$i(A_1-B_1)=-\log\alpha+\log 2-\gamma\tag{3.56}$$

$$A_0=B_0=1/(2n).\tag{3.57}$$

which, on using (3.50) becomes

$$-c_n=-\log\alpha+\log 2-\gamma\qquad \alpha=2e^{c_n-\gamma},\tag{3.58}$$

so that the eigenvalue α is now determined. We remark that the meaning of expression (3.55) is that of indicating how 'far' the outer region is. Indeed, we recall that ϵ is an auxiliary variable that one does not expect to find as part of any matching condition, so at first glance it could seem that expression (3.55) does not make any sense. But if we take into account that ϵ is the lengthscale for the outer variable, the meaning of expression (3.55) is that of defining what the outer lengthscale must be in order for the unknown asymptotic wavenumber k to be the leading order term in the outer expansion for χ_r (see expansion (3.6)).

We find that α is an order one constant, for instance, $n=1$ gives $\alpha = 1.018\ldots$ and $n=2$ gives $\alpha=0.414\ldots$. Thus, using that

$$\alpha=\frac{qk}{\epsilon},\qquad \epsilon=e^{-\pi/2q|n|}\tag{3.59}$$

we find that the asymptotic wavenumber is given by

$$k(q)=\frac{2}{q}e^{c_n-\gamma-\frac{\pi}{2q|n|}}(1+o(1))\tag{3.60}$$

which agrees with the asymptotic wavenumber given by P.S. Hagan in [19]. The asymptotic wavenumber can be also obtained numerically through a collocation method for general values of q. In figure 3.1 we plot the numerically obtained curve and also the asymptotic function given in (3.60) and we observe that for values of q of around less than 0.5 the function (3.60) and the numerically obtained $k(q)$ do agree. We note that α turns out to be of order one which

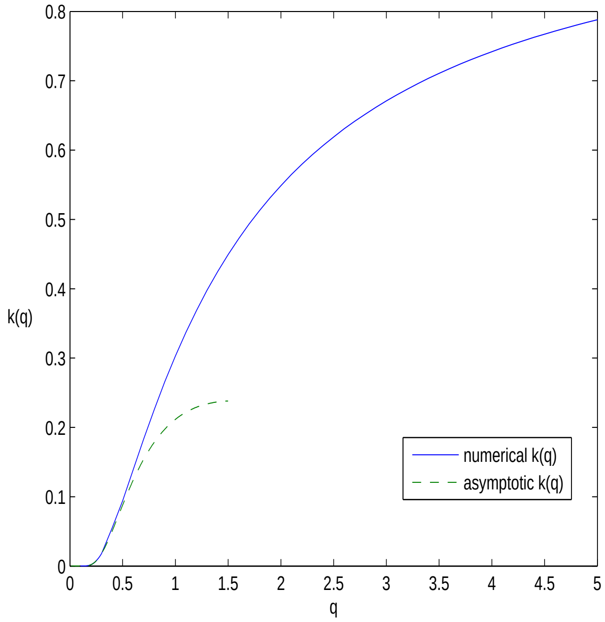

Figure 3.1: Comparison of k(q) numerically determined and the asymptotic formula (3.60).

means that when q is small we need to go exponentially far away in order to determine the asymptotic wavenumber k.

For the stationary case we have used the auxiliary parameter ϵ, that has no real meaning apart from being helpful to understand the relation between the space variables and the asymptotic wavenumber. Indeed, we could solve the whole problem without the use of ϵ, which is actually what is done in [19]. But when we deal with more than one spiral, $1/\epsilon$ represents the separation of the centres of the spirals and it is thus a real magnitude of the problem. Therefore, the relation between ϵ or the separation and the parameter q will produce different values of α that will produce different asymptotic laws of motion. In particular, what we will call in the following chapters the *canonical scaling* corresponds to the situation where the spirals are separated by distances such that α is of order one. This way, as the spirals separate or approach the value of α changes inducing a new law of motion. In this sense, it is equivalent moving the spirals apart to making q smaller.

Chapter 4

Interaction of spirals in the canonical scale

So far we have found a way to describe interacting vortices when $q = 0$ and a single spiral for general values of q. One of the key points in these two models that makes them easier than the multiple-spiral situation is the fact that in both cases one has to deal with only two small parameters (ϵ and μ for the dynamics of vortices, and k and q for the single-spiral case) that may be used to write asymptotic expansions that approximate the solutions in different regions. But multiple-spiral waves patterns when $q > 0$ present an added complication due to the fact of dealing with three small parameters, q (the given parameter in the original equation), ϵ, whose reciprocal gives a typical separation of the spirals and k, the asymptotic wavenumber, which is determined as part of the solution.

We wish to use a similar method to the one used in section 2 for when $0 < q \ll 1$. A first difficulty lies in the fact that, as we have just shown, the asymptotic wavenumber k is uniquely determined by the parameter q and therefore we cannot state a priori how large this value will be. On the other hand, the calculation in Chapter 3 shows that k is only determined at distances such that $r = \mathcal{O}(1/qk)$. It is thus plausible that for many-spirals systems there is a *canonical separation* for the spirals given by $1/qk$ at which the asymptotic wavenumber is determined to be the one for a single spiral. We will show that indeed the corresponding k for a system of spirals separated like $1/kq$ is of the same order (in q) as the one for a single spiral, but there is a different multiplying constant that accounts for the fact of having many spirals.

Hence, in this chapter we shall assume that $\alpha = kq/\epsilon$ is of order one and as we shall see, this is equivalent to assuming that, for small q, the typical spiral separation is $1/\epsilon = \mathcal{O}(e^{\pi/(2qn_\ell)})$. To simplify the calculations we consider equation (3.1) in the case where $b = 0$, so it reads,

$$\psi_t = (1 - |\psi|^2)\psi + iq\psi(1 - k^2 - |\psi|^2) + \Delta\psi. \tag{4.1}$$

We also consider spirals with unitary degree and we show that when α is of order one the spirals interact, to leading order, in the direction perpendicular to the line along the centres. Furthermore, the order of the velocity is $\epsilon^2/|\log\epsilon|$, as it happened in the $q = 0$-case. We also show that the direction in which a pair of spirals would move depends only on their own degree, and not on the one that the second spiral has.

4.1 Outer Region

As before we rescale time and space by setting $\mathbf{X} = \epsilon\mathbf{x}$, $T = \mu\epsilon^2 t$, to give

$$\epsilon^2\mu\psi_T = \epsilon^2\Delta\psi + (1 - |\psi|^2)\psi + iq(1 - k^2 - |\psi|^2)\psi, \tag{4.2}$$

which in terms of $\alpha = qk/\epsilon$ becomes

$$\epsilon^2\mu\psi_T = \epsilon^2\Delta\psi + (1 + iq)(1 - |\psi|^2)\psi - \frac{i\epsilon^2\alpha^2}{q}\psi. \tag{4.3}$$

Writing $\psi = fe^{i\chi}$ and separating real and imaginary parts in (6.5) gives

$$\mu\epsilon^2 f_T = \epsilon^2\Delta f - \epsilon^2 f|\nabla\chi|^2 + (1 - f^2)f, \tag{4.4}$$

$$\mu\epsilon^2 f^2\chi_T = \epsilon^2\nabla\cdot(f^2\nabla\chi) + qf^2(1 - f^2) - \epsilon^2\frac{\alpha^2}{q}f^2. \tag{4.5}$$

In this equations we shall take into account that the topological boundary conditions on χ are determined by the corresponding boundary condition about each spiral that is given by $n_j\phi_j$, where n_j is the degree of spiral j. So in the outer equations we must impose a topological condition of the form

$$\chi = \sum_{j=1}^{N} n_j\phi_j + F(R_1, R_2, \ldots, R_N), \tag{4.6}$$

where ϕ_j and R_j stand for the polar angle and radius variables corresponding to a reference system with the origin on the centre of spiral j.

We then expand in powers of ϵ as

$$f \sim f_0(\mathbf{X}, T; q, \mu) + \epsilon^2 f_1(\mathbf{X}, T; q, \mu) + \ldots, \tag{4.7}$$

$$\chi \sim \chi_0(\mathbf{X}, T; q, \mu) + \epsilon^2\chi_1(\mathbf{X}, T; q, \mu) + \ldots, \tag{4.8}$$

where we have only written explicitly the dependence of the functions with respect to the parameters to recall that each term in the expansion does still depend on more small parameters that we may use later to expand further each term in the series. Therefore we find

$$f_0 = 1, \tag{4.9}$$

$$f_1 = -\frac{1}{2}|\nabla\chi_0|^2, \tag{4.10}$$

$$\mu\chi_{0T} = \Delta\chi_0 + q|\nabla\chi_0|^2 - \frac{\alpha^2}{q}, \tag{4.11}$$

where in (4.11) we must impose the topological boundary condition given by (4.6). Expanding χ_0 in powers of μ as

$$\chi_0 \sim \frac{1}{q}(\chi_{00} + \mu\chi_{01} + \ldots),$$

gives, to leading order,

$$0 = \Delta\chi_{00} + |\nabla\chi_{00}|^2 - \alpha^2. \tag{4.12}$$

This expansion can be done provided χ_{0T} is not of order $1/\mu$. This means in particular that the following results hold only if the velocity we find is of order one or smaller. Equation (4.12) can be linearised through the Cole-Hopf transformation $\chi_{00} = \log h_0$ which gives

$$0 = \Delta h_0 - \alpha^2 h_0. \tag{4.13}$$

Note that we are dealing with a topologically non-trivial field. This means that special care is needed when performing this transformation since the complex function ψ may become multivalued. In [36] and [37] the authors claim that the topological condition related to each spiral invalidates the use of this transformation. Nevertheless, in what follows we will show that this transformation can actually be performed without causing the complex function ψ to become multivalued. As we will see, the key point is that in order to obtain a single valued ψ one has to introduce the right multivalueness in h.

For one vortex the solution to equation (4.13) would be

$$h_0(R,\phi) = e^{qn\phi}K_{iqn}(\alpha R),$$

(see Chapter 3, section §3.1) and this is the behaviour that we want to obtain close to each vortex. We recall that we are assuming that the spirals move slowly so that locally the spirals keep the structure of one in complete isolation, that is to say, we expect the inner solution to be exactly the same as in the equilibrium case.

In Chapter 2 the far field was dominated by a linear equation and therefore we could take the solution to be the sum of the contribution of each vortex.

In this case, we have transformed the original nonlinear equation into the linear equation given by (4.13), and hence, we are again allowed to sum the contributions coming form each vortex. Thus, for a system with N vortices, we could take a linear combination of the basic single spiral solution,

$$h_0 = \sum_{j=1}^{N} \beta_{j0}(T) e^{qn\phi_j} K_{iqn}(\alpha R_j) \tag{4.14}$$

where R_j and ϕ_j are the polar variables associated to a coordinate system related to each spiral centre and that do also depend on the slow time variable T. This function has the right type of singularities and reproduces the spiral core when we expand it locally. Nevertheless, as we mentioned before, this solution might not respect the single valueness of the complex solution ψ. However, if we take into account that q is a small parameter, we may relate it to μ by setting $\mu = \tilde{\mu} q$. Therefore, the right solution to take for h_0 is only the leading order in q given by

$$h_0 = \sum_{j=1}^{N} \beta_{j0}(T) K_0(\alpha R_j) \quad \text{and thus,}$$

$$\chi_{00} = \log \sum_{j=1}^{N} \beta_{j0}(T) K_0(\alpha R_j) \tag{4.15}$$

which is also a solution to (4.13). The weights β_{j0} will be determined when matching with the inner expansion. We remark that this is a sum of radial functions and therefore, when transforming back to χ_{00} the corresponding $\psi_0 = e^{i\chi_0} = \cos(\chi_0) + i\sin(\chi_0)$ is well defined and single valued.

The next order will be given by the solutions to the equation

$$\chi_{00T} = \Delta\chi_{01} + 2\nabla\chi_{00} \cdot \nabla\chi_{01}, \tag{4.16}$$

where we must impose that k_{01} contains the topological boundary condition given by (4.6). Therefore, we explicitly impose this boundary condition by writing the solution of this equation as a sum of three terms,

$$\chi_{01} = \frac{h_1}{h_0} + \frac{1}{\tilde{\mu}} \sum_{j=1}^{N} n_j \phi_j + \overline{\chi}_1, \tag{4.17}$$

where, upon substituting in (4.17) one finds that h_1 is the solution to the equation

$$h_{0T} = \Delta h_1 - h_1, \tag{4.18}$$

and where

$$h_{0T} = \sum_{j=1}^{N} \Big(-\beta_{j0}(T) \frac{\partial K_0(\alpha R_j)}{\partial R_j} \mathbf{e}_{R_j} \cdot \frac{d\mathbf{X}_j}{dT} + \beta'_{j0}(T) K_0(\alpha R_j) + \beta_{j0}\alpha' K'_0(\alpha R_j) \Big),$$

where we have allowed the parameter α to depend on the slow time variable T. The function h_1 would correspond to the next order in q of the function (4.14) without the angular factor $e^{qn\phi}$. In other words, upon observing the corresponding expansion for a single spiral (see Chapter 3, section §3.1) we expect h_1 to be the sum of the radial components coming from each spiral. Since (4.18) is a linear equation, we can thus solve it for each vortex by writing

$$h_{1j} = \beta_{j0} V_{1j}\, g_{1j}(R_j) \cos\phi_j + \beta_{j0} V_{2j}\, g_{2j}(R_j) \sin\phi_j + g_{3j}(R_j)$$

where g_{1j}, g_{2j} and g_{3j} satisfy the following equations

$$g''_{ij} + \frac{g'_{ij}}{R_j} - \frac{g_{ij}}{R_j^2} - \alpha^2 g_{ij} = -\frac{\partial K_0(\alpha R_j)}{\partial R_j} \tag{4.19}$$

$$g''_{3j} + \frac{g'_{3j}}{R_j} - \alpha^2 g_{3j} = \beta'_j(T) K_0(\alpha R_j) + \beta_j \alpha' K'_0(\alpha R_j) \tag{4.20}$$

where $i = 1, 2$ and the left hand side of equation (4.20) is a modified Bessel equation and again we have that the solutions of the homogeneous problem are given by $\beta_{j1} K_0(\alpha R_j)$. Hence, we reduce the partial differential equation to a system of three un-coupled ordinary differential equations.

As for the second term in (4.17), it imposes the topological condition given by the presence of N spirals. The third summand in (4.17) is therefore the single valued function that is left and that satisfies the equation

$$0 = \Delta\overline{\chi}_1 + 2\nabla\chi_{00} \cdot \nabla\overline{\chi}_1 + 2\frac{1}{\tilde{\mu}} \nabla\chi_{00} \cdot \nabla\Big(\sum_{j=1}^{N} n_j \phi_j\Big). \tag{4.21}$$

We note that by writing χ_{01} this way we make sure that it does not produce any multivalueness on ψ_0, that is therefore written as

$$\psi_0 \sim exp(i\sum_{j=1}^{N} \phi_j) exp(iF(R_1, R_2, \ldots, R_N))$$
$$= \cos\Big(\sum_{j=1}^{N} \phi_j + F(R_1, R_2, \ldots, R_N)\Big) + i \sin\Big(\sum_{j=1}^{N} \phi_j + F(R_1, R_2, \ldots, R_N)\Big)$$

4.2 Inner Region

We start by rescaling near the ℓ th vortex by setting $\mathbf{X} = \mathbf{X}_\ell + \epsilon \mathbf{x}$ to give

$$\epsilon\mu(\epsilon f_T - \frac{d\mathbf{X}_\ell}{dT} \cdot \nabla f) = \Delta f - f|\nabla\chi|^2 + (1 - f^2)f \tag{4.22}$$

$$\epsilon\mu f^2(\epsilon\chi_T - \frac{d\mathbf{X}_\ell}{dT} \cdot \nabla\chi) = \nabla \cdot (f^2\nabla\chi) + q(1 - f^2)f^2 - \frac{\epsilon^2\alpha^2 f^2}{q} \tag{4.23}$$

or equivalently

$$\epsilon\mu(\epsilon\psi_T - \frac{d\mathbf{X}_\ell}{dT} \cdot \nabla\psi) = \Delta\psi + (1 + iq)(1 - |\psi|^2)\psi - i\frac{\epsilon^2\alpha^2}{q}\psi, \tag{4.24}$$

where χ has to be of the form $\chi = n_\ell\phi_\ell$ to satisfy the topological boundary condition. Expanding in ϵ as

$$f \sim f_0(q, \mu) + \epsilon f_1(q, \mu) + \epsilon^2 f_2(q, \mu) + \ldots, \tag{4.25}$$

$$\chi \sim \chi_0(q, \mu) + \epsilon\chi_1(q, \mu) + \ldots, \tag{4.26}$$

the leading-order equations are

$$0 = \Delta f_0 - f_0|\nabla\chi_0|^2 + (1 - f_0^2)f_0, \tag{4.27}$$

$$0 = \nabla \cdot (f_0^2\nabla\chi_0) + q(1 - f_0^2)f_0^2, \tag{4.28}$$

which are the same leading-order equations that we had for a single spiral. Equations (4.27), (4.28) can be also expressed as

$$0 = \Delta\psi_0 + (1 + iq)\psi_0(1 - |\psi_0|^2). \tag{4.29}$$

Therefore, imposing that $f_0 = f_0(r, T)$ and $\chi_0 = n_\ell\phi + \varphi_0(r, T)$ gives

$$f_0'' + \frac{1}{r}f_0' - f_0\big(\frac{n_\ell}{r^2} + (\varphi_0')^2\big) + (1 - f_0^2)f_0 = 0, \tag{4.30}$$

$$f_0(\varphi_0'' + \frac{1}{r}\varphi_0') + 2f_0'\varphi_0' + q(1 - f_0^2)f_0 = 0, \tag{4.31}$$

where, to simplify the notation, we denote by ϕ the corresponding ϕ_ℓ and r the local radial variable close to vortex ℓ. As we did in Chapter 3 we pose the expansions

$$f_0 \sim f_{00} + f_{01}q^2 + f_{02}q^4 + \cdots, \tag{4.32}$$

$$\varphi_0 \sim \frac{\varphi_{00}}{q} + \varphi_{01}q + \varphi_{02}q^3 + \cdots, \tag{4.33}$$

to give

$$\varphi_{00} = \varphi_{00}(T), \tag{4.34}$$

$$0 = f_{00}'' + \frac{f_{00}'}{r} - \frac{f_{00}}{r^2} + (1 - f_{00}^2)f_{00}, \tag{4.35}$$

$$0 = f_{01}'' + \frac{f_{01}'}{r} - \frac{f_{01}}{r^2} + (1 - 3f_{00}^2)f_{01}, \tag{4.36}$$

$$\varphi_{01}'(r) = -\frac{1}{rf_{00}^2}\int_0^r sf_{00}^2(1 - f_{00}^2)\, ds, \tag{4.37}$$

with the boundary conditions

$$f_{00}(0) = 0, \quad \lim_{r\to\infty} f_{00}(r) = 1 \quad \text{and} \tag{4.38}$$

$$f_{01}(0) = 0, \quad \lim_{r\to\infty} f_{01}(r) = 0. \tag{4.39}$$

At first order in ϵ we find

$$-\mu\frac{d\mathbf{X}_\ell}{dT}\cdot\nabla\psi_0 = \Delta\psi_1 + (1 + iq)(\psi_1(1 - |\psi_0|^2) - \psi_0(\psi_0\psi_1^* + \psi_0^*\psi_1)) \tag{4.40}$$

or equivalently

$$-\mu\frac{d\mathbf{X}_\ell}{dT}\cdot\nabla f_0 = \Delta f_1 - f_1|\nabla\chi_0|^2 - 2f_0\nabla\chi_0\cdot\nabla\chi_1 + f_1 - 3f_0^2f_1, \tag{4.41}$$

$$-\mu f_0^2\frac{d\mathbf{X}_\ell}{dT}\cdot\nabla\chi_0 = \nabla\cdot(f_0^2\nabla\chi_1) + \nabla\cdot(2f_0f_1\nabla\chi_0) + 2qf_0f_1 - 4qf_0^3f_1. \tag{4.42}$$

4.3 Asymptotic matching

4.3.1 Outer limit of the inner

From the expressions (4.34)-(4.37) we find that when $r\to\infty$

$$\frac{\partial\varphi_{01}}{\partial r} \sim -qn_\ell^2\frac{\log r + c_{n_\ell}}{r} + \dots \tag{4.43}$$

where c_{n_ℓ} is a constant given by

$$c_{n_\ell} = \lim_{r\to\infty}\frac{1}{n_\ell^2}\left(\int_0^r f_0^2(s)\Big(1 - f_0(s)^2\Big)s\, ds - n_\ell^2\log(r)\right).$$

However, in order to match with the outer expansion we need the outer limit of the whole expansion in q. We notice that we only have some terms in the

q-expansion in the outer. Again, the inner expanded in q is found to be of the form

$$f_0 \sim 1 - \frac{1}{r^2} \sum_{i=0}^{N} \alpha_i \{qn_\ell^2(\log(r) + c_{n_\ell})\}^{2i} + \cdots , \tag{4.44}$$

$$\frac{\partial \chi_0}{\partial r} \sim -\frac{1}{r} \sum_{i=0}^{N} \beta_i \{qn_\ell^2(\log(r) + c_{n_\ell})\}^{2i+1} + \cdots , \tag{4.45}$$

where $\alpha_i > 0$ and $\beta_i > 0$ are constant values independent of q. As in Chapter 3, some values of the constants are given by $\alpha_0 = 1/2$, $\alpha_1 = 1/2$, $\beta_0 = 1$ and $\beta_1 = 1/3n_\ell^2$. The necessity of taking all the terms in q when matching can be seen due to the fact that the expansion in q is valid only when $q(\log(r) + c_1) \ll 1$. Since ϵ will turn out to be such that $q = O(1/|\log \epsilon|)$ in the outer region, all these terms are the same order.

However, we can use the same trick as in Chapter 3 to sum all these terms in the outer limit of the inner expansion. As before, we begin by rewriting the leading-order inner equations (4.27) and (4.28) in terms of the outer variable $R_\ell = \epsilon r$ to obtain

$$0 = \epsilon^2(\Delta f_0 - f_0|\nabla \chi_0|^2) + (1 - f_0^2)f_0, \tag{4.46}$$
$$0 = \epsilon^2 \nabla \cdot (f_0^2 \nabla \chi_0) + q(1 - f_0^2)f_0^2. \tag{4.47}$$

From now on we will denote R_ℓ by R and ϕ_ℓ by ϕ to simplify the notation. We now expand in powers of ϵ as

$$\chi_0 \sim \widehat{\chi}_{00}(q) + \epsilon^2 \widehat{\chi}_{01}(q) + \cdots , \tag{4.48}$$
$$f_0 \sim \widehat{f}_{00}(q) + \epsilon^2 \widehat{f}_{01}(q) + \cdots . \tag{4.49}$$

The leading order term in this expansion $\widehat{\chi}_{00}(q)$ is just the first term (in ϵ) in the outer expansion of the leading order inner solution, including all the terms in q. Substituting (4.48), (4.49) into (4.46), (4.47) gives

$$\widehat{f}_{00} = 1, \tag{4.50}$$
$$\widehat{f}_{01} = -\frac{1}{2}|\nabla \widehat{\chi}_{00}|^2, \tag{4.51}$$
$$0 = \Delta \widehat{\chi}_{00} + q|\nabla \widehat{\chi}_{00}|^2. \tag{4.52}$$

Since (4.52) is a Riccati equation it can be linearised with the change of variable $\widehat{\chi}_{00} = (1/q) \log \widehat{h}_0$ to give

$$\Delta \widehat{h}_0 = 0.$$

With the topological condition $\widehat{\chi}_{00} = n_\ell \phi + \widehat{\varphi}(R)$, we set $\widehat{h}_0 = e^{qn_\ell \phi} e^{q\widehat{\varphi}(R)} = e^{qn_\ell \phi} H_0(R)$, to give

$$H_0'' + \frac{H_0'}{R} + (qn_\ell)^2 \frac{H_0}{R^2} = 0 \tag{4.53}$$

that has the solution

$$H_0 = A_\ell(q)\epsilon^{-iqn_\ell}R^{iqn_\ell} + B_\ell(q)\epsilon^{iqn_\ell}R^{-iqn_\ell}, \tag{4.54}$$

where A_ℓ and B_ℓ are constants that depend on q which may be different at each vortex, and the factors $\epsilon^{\pm iqn_\ell}$ have been included to facilitate their determination by comparison with the solution in the inner variable. To determine A_ℓ and B_ℓ we need to write $\hat{\chi}_{00}$ in terms of r, expand in powers of q, and compare with (4.43). Expanding the constants in powers of q as

$$\begin{aligned} A_\ell(q) &\sim \frac{1}{q}A_{\ell 0} + A_{\ell 1} + qA_{\ell 2} + \cdots , \\ B_\ell(q) &\sim \frac{1}{q}B_{\ell 0} + B_{\ell 1} + qB_{\ell 2} + \cdots , \end{aligned}$$

and writing H_0 in terms of r we find

$$\begin{aligned} H_0(r) &= A_\ell(q)e^{iqn_\ell \log r} + B_\ell(q)e^{-iqn_\ell \log r} \\ &\sim \left(\frac{1}{q}A_{\ell 0} + A_{\ell 1} + qA_{\ell 2} + \cdots\right)\left(1 + iqn_\ell \log r - (qn_\ell)^2/2\log^2 r + \cdots\right) \\ &+ \left(\frac{1}{q}B_{\ell 0} + B_{\ell 1} + qB_{\ell 2} + \cdots\right)\left(1 - iqn_\ell \log r - (qn_\ell)^2/2\log^2 r + \cdots\right) \\ &\sim \frac{A_{\ell 0} + B_{\ell 0}}{q} + A_{\ell 1} + B_{\ell 1} + (A_{\ell 0} - B_{\ell 0})in_\ell \log r \\ &+ q\left(A_{\ell 2} + B_{\ell 2} + (A_{\ell 1} - B_{\ell 1})in_\ell \log r - \frac{(A_{\ell 0} + B_{\ell 0})}{2}n_\ell^2 \log^2 r\right) + \cdots , \end{aligned}$$

so that

$$\begin{aligned} \frac{\partial \widehat{\chi}_{00}}{\partial r} &= \frac{H_0'(r)}{qH_0(r)} \\ &\sim \frac{n_\ell}{qr}\frac{(A_{\ell 0} - B_{\ell 0})i + (A_{\ell 1} - B_{\ell 1})iq - (A_{\ell 0} + B_{\ell 0})qn_\ell \log r + \ldots}{\frac{A_{\ell 0}+B_{\ell 0}}{q} + A_{\ell 1} + B_{\ell 1} + (A_{\ell 0} - B_{\ell 0})in_\ell \log r + q\ldots} \\ &\sim \frac{n_\ell(A_{\ell 0} - B_{\ell 0})i}{r(A_{\ell 0} + B_{\ell 0})} + q\left(\frac{(A_{\ell 1} - B_{\ell 1})}{(A_{\ell 0} + B_{\ell 0})}\frac{n_\ell i}{r} - n_\ell^2\frac{\log r}{r}\right. \\ &+ \left.\left(n_\ell^2\frac{(A_{\ell 0} - B_{\ell 0})^2}{(A_{\ell 0} + B_{\ell 0})^2}\right)\frac{\log r}{r} - \left(\frac{i(A_{\ell 0} - B_{\ell 0})(A_{\ell 1} + B_{\ell 1})}{(A_{\ell 0} + B_{\ell 0})^2}\right)\frac{n_\ell}{r}\right) + \cdots \end{aligned}$$

Comparing with (4.43) we see that

$$A_{\ell 0} - B_{\ell 0} = 0, \tag{4.55}$$

$$\frac{(A_{\ell 1} - B_{\ell 1})}{A_{\ell 0} + B_{\ell 0}}i = -\, n_\ell c_{n_\ell} \quad \text{for } k = 1, \ldots, N. \tag{4.56}$$

The remaining equations determining A_k and B_k will be fixed when matching with the outer region.

Outer limit of the first-order inner We play the same game with the first-order inner solution. We first write equation (4.41)-(4.42) in terms of the outer variable to give

$$-\epsilon\mu\frac{d\mathbf{X}_\ell}{dT}\cdot\nabla f_0 = \epsilon^2\Delta f_1 - \epsilon^2 f_1|\nabla\chi_0|^2 - 2\epsilon^2 f_0\nabla\chi_0\cdot\nabla\chi_1 + f_1 - 3f_0^2 f_1,$$
$$-\mu\epsilon f_0^2\frac{d\mathbf{X}_\ell}{dT}\cdot\nabla\chi_0 = \epsilon^2\nabla\cdot(f_0^2\nabla\chi_1) + \epsilon^2\nabla\cdot(2f_0 f_1\nabla\chi_0) + 2qf_0 f_1 - 4qf_0^3 f_1.$$

We now expand in powers of ϵ as

$$\chi_1 \sim \frac{\widehat{\chi}_{10}(q)}{\epsilon} + \widehat{\chi}_{11}(q) + \cdots, \tag{4.57}$$
$$f_1 \sim \widehat{f}_{10}(q) + \epsilon\widehat{f}_{11}(q) + \cdots, \tag{4.58}$$

to give

$$\widehat{f}_{10} = 0, \tag{4.59}$$
$$\widehat{f}_{11} = -\nabla\widehat{\chi}_{00}\cdot\nabla\widehat{\chi}_{10}, \tag{4.60}$$
$$-\mu\frac{d\mathbf{X}_\ell}{dT}\cdot\nabla\widehat{\chi}_{00} = \Delta\widehat{\chi}_{10} - 2q\widehat{f}_{11} = \Delta\widehat{\chi}_{10} + 2q\nabla\widehat{\chi}_{00}\cdot\nabla\widehat{\chi}_{10}. \tag{4.61}$$

Motivated by the transformation we applied to $\widehat{\chi}_{00}$ we write

$$\widehat{\chi}_{10} = \frac{\widehat{h}_1}{q\widehat{h}_0} = \frac{\widehat{h}_1 e^{-q\widehat{\chi}_{00}}}{q},$$

and (4.61) becomes

$$\begin{aligned}-\mu\frac{d\mathbf{X}_\ell}{dT}\cdot\nabla\widehat{\chi}_{00} &= \frac{e^{-q\widehat{\chi}_{00}}}{q}\Big(\Delta\widehat{h}_1 - 2q\nabla\widehat{h}_1\cdot\nabla\widehat{\chi}_{00} - q\widehat{h}_1\Delta\widehat{\chi}_{00} + q^2\widehat{h}_1|\nabla\widehat{\chi}_{00}|^2 \\ &\qquad +2q\nabla\widehat{\chi}_{00}\cdot\nabla\widehat{h}_1 - 2q^2\widehat{h}_1|\nabla\widehat{\chi}_{00}|^2\Big) \\ &= \frac{e^{-q\widehat{\chi}_{00}}}{q}\Delta\widehat{h}_1.\end{aligned}$$

Writing $\widehat{\chi}_{00}$ in terms of $\widehat{h}_0$ gives

$$-\mu\frac{d\mathbf{X}_\ell}{dT}\cdot\nabla\widehat{h}_0 = \nabla^2\widehat{h}_1. \tag{4.62}$$

Writing the velocity like

$$\frac{d\mathbf{X}_\ell}{dT} = (V_1, V_2)$$

and recalling that $\widehat{h}_0 = e^{qn_\ell\phi}H_0(R)$, the left hand side of (4.62) gives

$$\begin{aligned}
& -\mu\frac{d\mathbf{X}_\ell}{dT}\cdot\left(\frac{qn_\ell e^{qn_\ell\phi}H_0(R)}{R}\mathbf{e}_\theta + H'(R)e^{qn_\ell\phi}\mathbf{e}_R\right) \\
= \;& \mu e^{qn_\ell\phi}\sin\phi\left(qn_\ell V_1\frac{H_0(R)}{R} - V_2H'(R)\right) \\
& -\mu e^{qn_\ell\phi}\cos\phi\left(qn_\ell V_2\frac{H_0(R)}{R} + V_1H'(R)\right) \\
= \;& \mu e^{qn_\ell\phi}qn_\ell\sin\phi\big(R^{iqn_\ell-1}A_\ell\epsilon^{-iqn_\ell}(V_1-iV_2) + R^{-iqn_\ell-1}B_\ell\epsilon^{iqn_\ell}(V_1+iV_2)\big) \\
- \;& \mu e^{qn_\ell\phi}qn_\ell\cos\phi\big(R^{iqn_\ell-1}A_\ell\epsilon^{-iqn_\ell}(V_2+iV_1) - R^{-iqn_\ell-1}B_\ell\epsilon^{iqn_\ell}(V_2-iV_1)) \\
= \;& \mu e^{qn_\ell\phi}\frac{(e^{i\phi}-e^{-i\phi})}{2i}R^{iqn_\ell-1}qn_\ell A_\ell\epsilon^{-iqn_\ell}\left(V_1-iV_2\right) \\
& +\mu e^{qn_\ell\phi}\frac{(e^{i\phi}-e^{-i\phi})}{2i}R^{-iqn_\ell-1}qn_\ell B_\ell\epsilon^{iqn_\ell}\left(V_1+iV_2\right) \\
& -\mu e^{qn_\ell\phi}\frac{(e^{i\phi}+e^{-i\phi})}{2}R^{iqn_\ell-1}qn_\ell A_\ell\epsilon^{-iqn_\ell}\left(V_2+iV_1\right) \\
& -\mu e^{qn_\ell\phi}\frac{(e^{i\phi}+e^{-i\phi})}{2}R^{-iqn_\ell-1}qn_\ell B_\ell\epsilon^{iqn_\ell}\left(V_2-iV_1\right) = \\
- \;& \frac{\mu qn_\ell e^{qn_\ell\phi}}{R}\big(e^{i\phi}R^{iqn_\ell}A_\ell\epsilon^{-iqn_\ell}(V_2+iV_1) - e^{-i\phi}R^{-iqn_\ell}B_\ell\epsilon^{iqn_\ell}(V_2-iV_1)\big),
\end{aligned}$$

since

$$\begin{aligned}
\frac{H_0(R)}{R} &= A_\ell(q)\epsilon^{-iqn_\ell}R^{iqn_\ell-1} + B_\ell(q)\epsilon^{iqn_\ell}R^{-iqn_\ell-1}, \\
H'(R) &= iqn_\ell A_\ell(q)\epsilon^{-iqn_\ell}R^{iqn_\ell-1} - iqn_\ell B_\ell(q)\epsilon^{iqn_\ell}R^{-iqn_\ell-1}.
\end{aligned}$$

Writing

$$\widehat{h}_1 = -\mu qn_\ell A_\ell\epsilon^{-iqn_\ell}(V_2+iV_1)g_1(R)\,e^{(qn_\ell+i)\phi} - \mu qn_\ell B_\ell\epsilon^{iqn_\ell}(V_2-iV_1)g_2(R)\,e^{(qn_\ell-i)\phi},$$

gives

$$\begin{aligned}
g_1'' + \frac{g_1'}{R} + \frac{(qn_\ell+i)^2g_1}{R^2} &= R^{iqn_\ell-1}, \\
g_2'' + \frac{g_2'}{R} + \frac{(qn_\ell-i)^2g_2}{R^2} &= R^{-iqn_\ell-1},
\end{aligned}$$

with the general solution

$$\begin{aligned}
g_1 &= \frac{R^{iqn_\ell+1}}{4iqn_\ell} + \frac{\gamma_1R^{1-iqn_\ell}}{4iqn_\ell} + \gamma_3R^{-1+iqn_\ell}, \\
g_2 &= -\frac{R^{-iqn_\ell+1}}{4iqn_\ell} - \frac{\gamma_2R^{1+iqn_\ell}}{4iqn_\ell} + \gamma_4R^{-1-iqn_\ell},
\end{aligned}$$

where γ_1, γ_2, γ_3 and γ_4 are arbitrary constants. We first note that this solution does not agree with the inner solution unless $\gamma_3 = \gamma_4 = 0$. As for γ_1 and γ_2, they will be determined by matching to the inner limit of the outer. Hence, we have that the first order outer limit of the inner is given by

$$\begin{aligned}\widehat{h}_1 \;=\; & -\frac{\mu A_\ell \epsilon^{-iqn_\ell}(V_1 - iV_2)}{4}(R^{iqn_\ell+1} + \gamma_1 R^{1-iqn_\ell})\, e^{(qn_\ell+i)\phi} \\ & -\frac{\mu B_\ell \epsilon^{iqn_\ell}(V_1 + iV_2)}{4}(R^{-iqn_\ell+1} + \gamma_2 R^{1+iqn_\ell})\, e^{(qn_\ell-i)\phi}.\end{aligned}$$

4.3.2 Inner limit of the outer

To compute the inner limit of the outer we rewrite solution (4.15) in terms of the inner variable $\mathbf{X} = \mathbf{X}_\ell + \epsilon\mathbf{x}$ and expand in powers of ϵ

$$h_0 \sim \sum_{j=1}^{N} \beta_{j0}(T) K_0(\alpha|\mathbf{X}_\ell + \epsilon\mathbf{x} - \mathbf{X}_j|) + \ldots \tag{4.63}$$

$$\sim \beta_{\ell 0} K_0(\alpha\epsilon r) + \sum_{j\neq\ell}^{N} \beta_{j0} K_0(\alpha|\mathbf{X}_\ell - \mathbf{X}_j|) + \epsilon\mathbf{x}\cdot\nabla G_0(\mathbf{X}_\ell) + \ldots \tag{4.64}$$

where

$$G_0(\mathbf{X}) = \sum_{j\neq\ell}^{N} \beta_{j0}(T) K_0(\alpha|\mathbf{X} - \mathbf{X}_j|).$$

Since the modified Bessel function satisfies

$$K_0(z) = -\log\frac{z}{2} - \gamma + O(z^2) \tag{4.65}$$

as $z \to 0$, where γ is the Euler constant, we find that the leading order in q and ϵ is written like

$$h_0 \sim -\beta_{\ell 0}\log\frac{\alpha\epsilon r}{2} - \beta_{\ell 0}\gamma + \sum_{j\neq\ell}^{N} \beta_{j0} K_0(\alpha|\mathbf{X}_\ell - \mathbf{X}_j|) + \epsilon\mathbf{x}\cdot\nabla G_0(\mathbf{X}_\ell) + \ldots \tag{4.66}$$

Leading order matching, (1ti)(1to)=(1to)(1ti). We now match the leading order in ϵ of the inner limit of the outer with the leading order outer limit of the inner, that is given by (4.54). We recall that we have the whole q-expansion of the outer limit of the leading order inner, while we only have some terms in the inner limit of the outer. As we explained above this is the reason why we must compare both series in terms of the outer variable R.

The first term in the q-expansion of the leading order inner limit of the outer is thus

$$\begin{aligned}\chi_0 &\sim \frac{1}{q}\log\big(\beta_{\ell 0}K_0(\alpha\epsilon r)+G_0(\mathbf{X}_\ell)\big)+\dots\\ &\sim \frac{1}{q}\log\big(-\beta_{\ell 0}\log\frac{\alpha\epsilon r}{2}-\beta_{\ell 0}\gamma+G_0(\mathbf{X}_\ell)\big)+\dots\end{aligned} \tag{4.67}$$

We now compare with the leading order outer limit of the inner which looks like

$$\begin{aligned}\widehat{\chi}_{00} &= n_\ell\phi+\frac{1}{q}\log H_0(R)\\ &= n_\ell\phi+\frac{1}{q}\log\big(A_\ell(q)\epsilon^{-iqn_\ell}R^{iqn_\ell}+B_\ell(q)\epsilon^{iqn_\ell}R^{-iqn_\ell}\big).\end{aligned} \tag{4.68}$$

If we now expand $H_0(R)$ in powers of q we find

$$\begin{aligned}H_0 &\sim A_k(q)e^{-iqn_\ell\log\epsilon}(1+iqn_\ell\log R+\dots)\\ &\quad+B_k(q)e^{iqn_\ell\log\epsilon}(1-iqn_\ell\log R+\dots)\\ &\sim \frac{A_{\ell 0}e^{-iqn_\ell\log\epsilon}+B_{\ell 0}e^{iqn_\ell\log\epsilon}}{q}+A_{\ell 1}e^{-iqn_\ell\log\epsilon}+B_{\ell 1}e^{iqn_\ell\log\epsilon}\\ &\quad+in_\ell(A_{\ell 0}e^{-iqn_\ell\log\epsilon}-B_{\ell 0}e^{iqn_\ell\log\epsilon})\log R+\dots,\end{aligned}$$

and therefore, the leading order in q for the outer limit of the inner is

$$\begin{aligned}\widehat{\chi}_{00} \sim \frac{1}{q}\log\Big(&\frac{A_{\ell 0}e^{-iqn_\ell\log\epsilon}+B_{\ell 0}e^{iqn_\ell\log\epsilon}}{q}+A_{\ell 1}e^{-iqn_\ell\log\epsilon}+B_{\ell 1}e^{iqn_\ell\log\epsilon}\\ &+in_\ell(A_{\ell 0}e^{-iqn_\ell\log\epsilon}-B_{\ell 0}e^{iqn_\ell\log\epsilon})\log R\Big)+\dots\end{aligned} \tag{4.69}$$

Comparing (4.67) and (4.69) and using (4.55) (that states that $A_{\ell 0}=B_{\ell 0}$) and (4.56) we find that the $O(1/q)$ terms in (4.69) imply

$$e^{iqn_\ell\log\epsilon}+e^{-iqn_\ell\log\epsilon}=0, \tag{4.70}$$

so that

$$q|n_\ell|\log\epsilon=-\frac{\pi}{2},\quad k=1..N \tag{4.71}$$

This is the condition on q for α to be of order one, that we recall it is equivalent to assuming that the typical vortex separation is $1/\epsilon=\mathcal{O}(e^{\pi/(2qn_\ell)})$.

From this last expression we note that all β_ℓ are of order one provided the winding number of the vortices is $|n_\ell|=1$. In the case of having a vortex with a non unitary winding number, then we should reconsider the calculations above allowing some of the constants $\beta_{\ell 0}$ to be zero. We now analyse the case of unitary winding numbers where, as we will show, all $\beta_{\ell 0}$ are non zero.

The coefficient of $\log R$ then gives

$$\beta_{\ell 0} = (A_{\ell 0} + B_{\ell 0})/|n_\ell|, \tag{4.72}$$

which together with (4.55) implies

$$A_{\ell 0} = B_{\ell 0} = \beta_{\ell 0}/2|n_\ell|. \tag{4.73}$$

Finally, the $O(1)$ terms of H_0 give

$$i\, n_\ell(A_{\ell 1} - B_{\ell 1}) = -\beta_{\ell 0}(\log\alpha - \log 2 + \gamma) + \sum_{j\neq\ell}^{N} \beta_{j0} K_0(\alpha|\mathbf{X}_\ell - \mathbf{X}_j|), \tag{4.74}$$

which, on using (4.56) becomes

$$-c_1\beta_{\ell 0} = -\beta_{\ell 0}(\log\alpha - \log 2 + \gamma) + \sum_{j\neq\ell}^{N} \beta_{j0} K_0(\alpha|\mathbf{X}_\ell - \mathbf{X}_j|). \tag{4.75}$$

Since (4.75) holds for each spiral this is a system of N homogeneous equations for the unknown weights β_{j0}. Therefore, to have a nontrivial solution the determinant of the system must vanish: this is the eigenvalue condition for α in the case of multiple vortices:

Result 4.1 (Asymptotic wavenumber). *The asymptotic wavenumber corresponding to a system of N spirals in the canonical separation evolve at the same time scale as the centres of the spirals. Furthermore, it is given by the condition that the following set of N homogeneous equations ($\ell = 1, \dots, N$)*

$$0 = \beta_{\ell 0}(c_1 - \log\alpha + \log 2 - \gamma) + \sum_{j\neq\ell}^{N} \beta_{j0} K_0(\alpha|\mathbf{X}_\ell - \mathbf{X}_j|),$$

for the N unknown constants β_{j0} has a nontrivial solution.

As an example let us consider a system with two vortices, in which case the equations (4.75) become

$$-c_1\beta_{10} = -\,\beta_{10}(\log\alpha - \log 2 + \gamma) + \beta_{20} K_0(\alpha|\mathbf{X}_1 - \mathbf{X}_2|) \tag{4.76}$$

$$-c_1\beta_2 = -\,\beta_{20}(\log\alpha - \log 2 + \gamma) + \beta_{10} K_0(\alpha|\mathbf{X}_1 - \mathbf{X}_2|) \tag{4.77}$$

from where we see that $\beta_{10} = \beta_{20}$ and the eigenvalue condition is thus

$$-c_1 = -(\log\alpha - \log 2 + \gamma) + K_0(\alpha|\mathbf{X}_1 - \mathbf{X}_2|), \tag{4.78}$$

and hence

$$\alpha = 2e^{c_1-\gamma+K_0(\alpha|\mathbf{X}_1-\mathbf{X}_2|)}.$$

From this last expression we observe that if the distance between the spirals becomes larger, the value of α becomes closer to the one corresponding to a single spiral solution that was given in (3.58), and therefore, the asymptotic wavenumber tends also to the single spiral one. This is seen by the fact that $K_0(z)$ becomes exponentially small as z becomes larger.

We also conclude that, since α is a function of the separation, it will depend on the slow time variable T, so that the asymptotic wavenumber k is also slowly varying. The rate of variation is still unknown and, as it happened when $q = 0$, it will be found to be like $1/|\log \epsilon|$.

First order matching (2ti)(1to)=(1to)(2ti). We could have followed the matching of the leading order terms by comparing the next order in q for the inner and the outer, but we would not find any further condition on the eigenvalue; on the contrary, the matching is automatically satisfied.

To find the law of motion for the spirals we know from the $q = 0$ case that we need to match the first order in ϵ. In the inner limit expansion of the outer χ_0 we find that there are no order ϵ terms in the leading order term in q. It is thus clear that we must take the next term, χ_{01}, and find its inner limit. The plan will be similar to what we did for χ_{00}: we find the inner limit, take the order ϵ terms that are left, and match to the first term in the outer limit of the inner. This will actually give the matching of two terms in the inner expansion with one term in the outer but only to leading order in q.

Hence, we go back to equation (4.16) that was simplified by expressing the solution as

$$\chi_{01} = \frac{h_1}{h_0} + \frac{1}{\tilde{\mu}} \sum_{j=1}^{N} n_j \phi_j + \overline{\chi}_1,$$

and solve it locally close to each vortex by scaling with $\mathbf{X} = \mathbf{X}_\ell + \epsilon \mathbf{x}$ and expanding in powers of ϵ.

We start by computing the inner limit for h_1, that satisfies equation (4.18). As we showed, computing for each vortex, the solution is of the particular form

$$h_{1\ell} = \beta_{\ell 0} V_{1\ell}\, g_{1\ell}(R) \cos\phi + \beta_{\ell 0} V_{2\ell}\, g_{2\ell}(R) \sin\phi + g_{3\ell}(R)$$

where $g_{1\ell}$, $g_{2\ell}$ and $g_{3\ell}$ satisfy equations (4.19) and (4.20). In order to solve these equations locally we expand h_{0T} to find

$$\begin{aligned}\frac{\partial h_0}{\partial T} \sim \frac{\beta_{\ell 0}}{\epsilon r}\frac{d\mathbf{X}_\ell}{dT} \cdot \mathbf{e}_r - \beta'_{\ell 0}(\log(\alpha \epsilon r/2) - \gamma) \\ - \frac{\beta_{\ell 0}\alpha'}{\epsilon r} + \frac{\partial (G_0(\mathbf{X}_\ell) + \epsilon x \cdot \nabla G_0(\mathbf{X}_\ell))}{\partial T}.\end{aligned} \tag{4.79}$$

Therefore, to find the inner limit we just have to solve the equations given by

$$\begin{aligned}\frac{g_1''}{\epsilon^2}+\frac{g_1'}{\epsilon^2 r}-\frac{g_1}{\epsilon^2 r^2}-\alpha^2 g_1 =& \frac{1}{\epsilon r},\\ \frac{g_3''}{\epsilon^2}+\frac{g_3'}{\epsilon^2 r}-\alpha^2 g_3 =& -\beta_{\ell 0}'\big(\log\frac{\epsilon r\alpha}{2}-\gamma\big)-\beta_{\ell 0}\frac{\alpha'}{r}\\ &+\frac{\partial(G_0(\mathbf{X}_j)+\epsilon x\cdot\nabla G_0(\mathbf{X}_\ell))}{\partial T}.\end{aligned}$$

With the ansatz $g_1(r)\sim g_{10}(r)\epsilon+\ldots$ and $g_3(r)\sim \epsilon^2\log\epsilon\, g_{20}(r)+\ldots$ we obtain

$$\begin{aligned}g_1(r) &\sim \frac{\epsilon r}{2}\log\epsilon r+C_1\epsilon r+C_2\epsilon\frac{1}{r}+\ldots,\\ g_3(r) &\sim \frac{\epsilon^2 r^2}{4}\log\epsilon r+\frac{\beta_{\ell 1}}{\tilde\mu}K_0(\alpha\epsilon r)+\frac{1}{\tilde\mu}G_1(\mathbf{X}_\ell)+\frac{1}{\tilde\mu}\epsilon\mathbf{x}\cdot\nabla G_1(\mathbf{X}_\ell)+\ldots,\end{aligned}$$

where

$$G_1(\mathbf{X}_\ell)=\sum_{j\neq\ell}^{N}\beta_{j1}K_0(\alpha|\mathbf{X}_j-\mathbf{X}_\ell|).$$

Hence, in terms of R, as R becomes small,

$$\begin{aligned}h_1 \sim\, &\beta_{\ell 1}K_0(\alpha R)+G_1(\mathbf{X}_\ell)+\mathbf{X}\cdot\nabla G_1(\mathbf{X}_\ell)\\ &+\beta_{\ell 0}\frac{R}{2}\log R\frac{d\mathbf{X}_\ell}{dT}\cdot\mathbf{e}_r+\mathcal{O}(R^2).\end{aligned}\tag{4.80}$$

We continue and obtain the inner limit of $\overline{\chi}_1$ close to vortex ℓ. The equation to solve locally is equation (4.21) that was written as

$$\Delta\overline{\chi}_1+2\nabla\chi_{00}\cdot\nabla\overline{\chi}_1=-2\frac{1}{\tilde\mu}\nabla\chi_{00}\cdot\nabla\big(\sum_{j=1}^{N}n_j\phi_j\big).$$

We use that

$$\nabla\chi_{00}=\frac{\nabla H_0}{H_0}=\frac{\beta_\ell K_0'(\alpha\epsilon r)\mathbf{e}_r+\nabla G_0(\mathbf{X}_\ell)}{\beta_\ell K_0(\alpha\epsilon r)+G(\mathbf{X}_\ell)}\tag{4.81}$$

where G_0 is the constant term that arises when we expand the leading-order term close to vortex ℓ.

We recall that $K_0'(\alpha\epsilon r)=\mathcal{O}(1/R)$ and $K_0(\alpha\epsilon r)=\mathcal{O}(\log R)$ and the rest in expression (4.81) are order one terms. Then, putting (4.81) into equation (4.21) and taking the leading order terms in (4.81) yields

$$\begin{aligned}\Delta\overline{\chi}_1+&\frac{2\beta_{\ell 0}K_0'(\alpha R)}{\beta_{\ell 0}K_0(\alpha R)+G_0(\mathbf{X}_\ell)}\frac{\partial\overline{\chi}_1}{\partial R}=\\ &-\frac{2}{\tilde\mu}\frac{\beta_{\ell 0}K_0'(\alpha R)\nabla\Phi^*\cdot\mathbf{e}_r+(n_\ell/R)\nabla G_0(\mathbf{X}_\ell)\cdot\mathbf{e}_\phi}{\beta_{\ell 0}K_0(\alpha R)+G_0(\mathbf{X}_\ell)}\end{aligned}$$

where

$$\nabla\Phi^* = \sum_{j\neq\ell}^{N}\Big(n_j\frac{\mathbf{e}_{\phi j\ell}}{|\mathbf{X}-\mathbf{X}_j|}\Big).$$

With the change of function

$$\overline{\chi}_1 = \frac{\overline{h}_1}{\beta_\ell K_0(\alpha R) + G_0(\mathbf{X}_\ell)},$$

we are left with the equation

$$\Delta\overline{h}_1 - \overline{h}_1\alpha^2\beta_{\ell 0}K_0(\alpha R) = -\frac{2}{\tilde{\mu}}\Big(\beta_{\ell 0}K_0'(\alpha R)\nabla\Phi^*\cdot\mathbf{e}_r + \frac{n_\ell}{R}\nabla G_0(\mathbf{X}_\ell)\cdot\mathbf{e}_\phi\Big) \quad (4.82)$$

Writing everything in terms of r and expanding $K_0'(\alpha R)$ we find

$$\frac{1}{\epsilon^2}\Delta\overline{h}_1 - \overline{h}_1\alpha^2\beta_{\ell 0}K_0(\alpha\epsilon r) = -\frac{2}{\epsilon\tilde{\mu}}\Big(-\frac{\beta_{\ell 0}}{r}\nabla\Phi^*\cdot\mathbf{e}_r + \frac{n_\ell}{r}\nabla G_0(\mathbf{X}_\ell)\cdot\mathbf{e}_\phi\Big),$$

and therefore, with the ansatz $\overline{h}_1 \sim \epsilon\overline{h}_{10} + \epsilon^2\overline{h}_{11} + \dots$ we find the simpler equation

$$\Delta\overline{h}_{10} = -\frac{2}{\tilde{\mu}}\big(-\frac{\beta_{\ell 0}}{r}\nabla\Phi^*\cdot\mathbf{e}_r + \frac{n_\ell}{r}\nabla G_0(\mathbf{X}_\ell)\cdot\mathbf{e}_\phi\big), \quad (4.83)$$

that has the solution

$$\overline{h}_{10} = -\frac{r}{\tilde{\mu}}\log\epsilon r\big(-\beta_{\ell 0}\nabla\Phi^*\cdot\mathbf{e}_r + n_\ell\nabla G_0(\mathbf{X}_\ell)\cdot\mathbf{e}_\phi\big). \quad (4.84)$$

This way we have found that the first order term for χ_0 in the μ-expansion reads

$$\begin{aligned}\chi_{01} \sim\ & \frac{1}{\overline{h}_0}\big(\beta_{\ell 1}K_0(\alpha R) + G_1(\mathbf{X}_\ell) \\ & + \mathbf{X}\cdot\nabla G_1(\mathbf{X}_\ell) + \beta_{\ell 0}\frac{R}{2}\log R\frac{d\mathbf{X}_\ell}{dT}\cdot\mathbf{e}_r\big) + \frac{1}{\tilde{\mu}}\sum_{j=1}^{N}n_j\phi_j \\ & - \frac{r}{\tilde{\mu}\overline{h}_0}\log\epsilon r\big(-\beta_{\ell 0}\nabla\Phi^*\cdot\mathbf{e}_r + n_\ell\nabla G_0(\mathbf{X}_\ell)\cdot\mathbf{e}_\phi\big) + \dots,\end{aligned} \quad (4.85)$$

where we have used the notation $\overline{h}_0 = \beta_{\ell 0}K_0(\alpha R) + G_0(\mathbf{X}_\ell)$. Therefore putting together the inner limit of the outer for χ_{00} and χ_{01}, the inner limit of the outer close to vortex ℓ reads,

$$\begin{aligned}\chi_0^{outer} \sim &\frac{1}{q}\log\overline{h}_0 + n_\ell\phi + \sum_{j\neq\ell}^{N} n_j\phi_{j\ell} + \frac{\mathbf{X}\cdot\nabla G_0(\mathbf{X}_\ell)}{q\overline{h}_0} + \frac{1}{\overline{h}_0}\beta_{\ell 1}K_0(\alpha R)\\ &+ \frac{1}{\overline{h}_0}G_1(\mathbf{X}_\ell) + \frac{1}{\overline{h}_0}\mathbf{X}\cdot\nabla G_1(\mathbf{X}_\ell) + \frac{\tilde{\mu}\beta_{\ell 0}}{2\overline{h}_0}R\log R\frac{d\mathbf{X}_\ell}{dT}\cdot\mathbf{e}_r\\ &+ \frac{1}{\overline{h}_0}R\log R\big(\beta_{\ell 0}\nabla\Phi^*\cdot\mathbf{e}_r - n_\ell\nabla G_0(\mathbf{X}_\ell)\cdot\mathbf{e}_\phi\big)\\ &+ \mathbf{X}\cdot\nabla\big(\sum_{j\neq\ell}^{N} n_j\mathbf{e}_{\phi j\ell}\big) + \mathcal{O}(q). \end{aligned}\tag{4.86}$$

We observe that we have found the order one terms in q that we would use in the leading order matching. To match the first order terms in ϵ we must keep the order R terms in (4.86). Using the expression for $\overline{h}_0$, the order R terms in the inner limit of the outer are written as

$$\begin{aligned}\chi_{0\ \mathcal{O}(R)}^{outer} \sim &\frac{\mathbf{X}\cdot\nabla G_0(\mathbf{X}_\ell)}{q\overline{h}_0} + \frac{\tilde{\mu}\beta_{\ell 0}}{2\overline{h}_0}R\log R\frac{d\mathbf{X}_\ell}{dT}\cdot\mathbf{e}_r\\ &-\frac{n_\ell}{\overline{h}_0}R\log R\,\nabla G_0(\mathbf{X}_\ell)\cdot\mathbf{e}_\phi + \frac{1}{\overline{h}_0}\mathbf{X}\cdot\nabla G_1(\mathbf{X}_\ell)\\ &+\frac{-\beta_{\ell 0}(\log\alpha/2+\gamma) + G_0(\mathbf{X}_\ell)}{\overline{h}_0}\mathbf{X}\cdot\nabla\big(\sum_{j\neq\ell}^{N} n_j\mathbf{e}_{\phi j\ell}\big) + \ldots,\end{aligned}\tag{4.87}$$

where we have used the following expansion for the term regarding the gradient of the angular terms,

$$\begin{aligned}\mathbf{X}\cdot\nabla\big(\sum_{j\neq\ell}^{N} n_j\mathbf{e}_{\phi j\ell}\big) =&\frac{\beta_{\ell 0}K_0(\alpha R) + G_0(\mathbf{X}_\ell)}{\overline{h}_0}\sum_{j\neq\ell}^{N}\frac{n_j}{\mathbf{X}_\ell - \mathbf{X}_j}\mathbf{e}_{\phi j\ell}\\ \sim&\frac{-\beta_{\ell 0}(\log(\alpha/2)+\gamma) + G_0(\mathbf{X}_\ell)}{\overline{h}_0}\sum_{j\neq\ell}^{N}\frac{n_j}{\mathbf{X}_\ell - \mathbf{X}_j}\mathbf{e}_{\phi j\ell}\\ &-\frac{\beta_{\ell 0}\log R}{\overline{h}_0}\sum_{j\neq\ell}^{N}\frac{n_j}{\mathbf{X}_\ell - \mathbf{X}_j}\mathbf{e}_{\phi j\ell}.\end{aligned}$$

As for the first order in the outer limit of the inner, the order R terms do only appear in the leading order outer limit of the inner of χ_1 and they are given by

$$\begin{aligned}\chi_{10\ \mathcal{O}(R)}^{inner} =& -\frac{\mu A_\ell\epsilon^{-iqn_\ell}(V_1 - iV_2)e^{i\phi}}{4q(A_\ell R^{iqn}\epsilon^{-iqn_\ell} + B_\ell R^{-iqn}\epsilon^{iqn_\ell})}R(R^{iqn_\ell} + \gamma_1 R^{-iqn})\\ &-\frac{\mu B_\ell\epsilon^{iqn_\ell}(V_1 + iV_2)e^{-i\phi}}{4q(A_\ell R^{iqn}\epsilon^{-iqn_\ell} + B_\ell R^{-iqn}\epsilon^{iqn_\ell})}R(R^{-iqn_\ell} + \gamma_2 R^{iqn}).\end{aligned}\tag{4.88}$$

As it happened in the leading order matching, we now have the whole q-expansion in the outer limit of the inner, but only the first two terms in the inner limit of the outer. Again, this implies that the right way to proceed with the matching is by comparing both series in terms of the outer variable, that is to say, we will now expand expression (4.88) in powers of q and afterwards we will compare the first two terms to those in (4.87).

The denominator in (4.88) is expanded as

$$\begin{aligned}
&A_{\ell 1}e^{-iqn_\ell \log\epsilon} + B_{\ell 1}e^{iqn_\ell \log\epsilon} + (A_{\ell 0}e^{-iqn_\ell \log\epsilon} - B_{\ell 0}e^{iqn_\ell \log\epsilon})in_\ell \log R \\
&+ q\big(A_{\ell 2}e^{-iqn_\ell \log\epsilon} + B_{\ell 2}e^{iqn_\ell \log\epsilon} \\
&+ in_\ell(A_{\ell 1}e^{-iqn_\ell \log\epsilon} - B_{\ell 1}e^{iqn_\ell \log\epsilon})\log R)\big) + \dots,
\end{aligned}$$

and we have already found that

$$\begin{aligned}
\overline{h}_0 =&A_{\ell 1}e^{-iqn_\ell \log\epsilon} + B_{\ell 1}e^{iqn_\ell \log\epsilon} \\
&+(A_{\ell 0}e^{-iqn_\ell \log\epsilon} - B_{\ell 0}e^{iqn_\ell \log\epsilon})in_\ell \log R
\end{aligned}$$

Hence, the expansion of (4.88) reads

$$\begin{aligned}
\chi^{inner}_{10\ \mathcal{O}(R)} \sim &- \tilde{\mu}\frac{A_{\ell 0}e^{-iqn_\ell \log\epsilon}e^{i\phi}(V_1 - iV_2)(1+\gamma_1)}{4q\overline{h}_0}R \\
&-\tilde{\mu}\frac{A_{\ell 0}e^{iqn_\ell \log\epsilon}e^{-i\phi}(V_1 + iV_2)(1+\gamma_2)}{4\overline{h}_0}R \\
&-\tilde{\mu}\frac{A_{\ell 0}e^{-iqn_\ell \log\epsilon}e^{i\phi}(V_1 - iV_2)(1-\gamma_1)}{4\overline{h}_0}in_\ell R\log R \\
&-\tilde{\mu}\frac{A_{\ell 0}e^{iqn_\ell \log\epsilon}e^{-i\phi}(V_1 + iV_2)(-1+\gamma_2)}{4\overline{h}_0}in_\ell R\log R \\
&-\tilde{\mu}\frac{A_{\ell 1}e^{-iqn_\ell \log\epsilon}e^{i\phi}(V_1 - iV_2)(1+\gamma_1)}{4\overline{h}_0}R \\
&-\tilde{\mu}\frac{B_{\ell 1}e^{iqn_\ell \log\epsilon}e^{-i\phi}(V_1 + iV_2)(1+\gamma_2)}{4\overline{h}_0}R \\
&+\tilde{\mu}\frac{A_{\ell 0}e^{-iqn_\ell \log\epsilon}e^{i\phi}(V_1 - iV_2)(1+\gamma_1)}{4q\overline{h}_0}\frac{A_{\ell 1}+B_{\ell 1}}{2A_{\ell 0}}R \\
&+\tilde{\mu}\frac{A_{\ell 0}e^{iqn_\ell \log\epsilon}e^{-i\phi}(V_1 + iV_2)(1+\gamma_2)}{4\overline{h}_0}\frac{A_{\ell 1}+B_{\ell 1}}{2A_{\ell 0}}R + \mathcal{O}(q). \qquad (4.89)
\end{aligned}$$

Comparing (4.87) and (4.89) and expanding the unknown constants in q as

$$\gamma_1 \sim \gamma_{10} + q\gamma_{11} + \dots \qquad \gamma_2 \sim \gamma_{20} + q\gamma_{21} + \dots$$

gives the following conditions on the constants γ_{10} and γ_{20},

$$\frac{\tilde{\mu}\beta_{\ell 0}}{8}(-V_1(\gamma_{10}+\gamma_{20})+iV_2(\gamma_{10}-\gamma_{20})) = \frac{\tilde{\mu}\beta_{\ell 0}}{4}V_1 + \frac{A_1}{2}, \tag{4.90}$$

$$\frac{\tilde{\mu}\beta_{\ell 0}}{8}(-V_2(\gamma_{10}+\gamma_{20})-iV_1(\gamma_{10}-\gamma_{20})) = \frac{\tilde{\mu}\beta_{\ell 0}}{4}V_2 + \frac{A_2}{2}, \tag{4.91}$$

where $A_1\cos\phi + A_2\sin\phi = 2n_\ell\,\nabla G_0(\mathbf{X}_\ell)\cdot\mathbf{e}_\phi$. Solving (4.90) and (4.91) and substituting A_1 and A_2,

$$\gamma_{10} = -1 - \frac{4n_\ell}{\tilde{\mu}\beta_{\ell 0}}\frac{(d\mathbf{X}_\ell/dT)\cdot\nabla G((\mathbf{X}_\ell)^\perp}{|d\mathbf{X}_\ell/dT|^2} - \frac{4n_\ell}{i\tilde{\mu}\beta_{\ell 0}}\frac{(d\mathbf{X}_\ell/dT)\cdot\nabla G((\mathbf{X}_\ell)}{|d\mathbf{X}_\ell/dT|^2},$$

$$\gamma_{20} = -1 - \frac{4n_\ell}{\tilde{\mu}\beta_{\ell 0}}\frac{(d\mathbf{X}_\ell/dT)\cdot\nabla G((\mathbf{X}_\ell)^\perp}{|d\mathbf{X}_\ell/dT|^2} + \frac{4n_\ell}{i\tilde{\mu}\beta_{\ell 0}}\frac{(d\mathbf{X}_\ell/dT)\cdot\nabla G((\mathbf{X}_\ell)}{|d\mathbf{X}_\ell/dT|^2}.$$

After having determined all these constants we can write the first order outer limit of the inner in terms of r, that is actually the term that will give the law of motion,

$$\widehat{\chi}_{10} \sim -\frac{\tilde{\mu}r}{2}\left(V_1\cos\phi + V_2\sin\phi\right) + \frac{n_\ell r}{\beta_{\ell 0}}\nabla G(\mathbf{X}_\ell)\cdot\mathbf{e}_\phi + \mathcal{O}(q). \tag{4.92}$$

If we write the corresponding expansion for ψ we have

$$\psi \sim e^{i(n_\ell\phi + \frac{\log H_0}{q})}\big(1 - i\frac{\tilde{\mu}\epsilon r}{2}\left(V_1\cos\phi + V_2\sin\phi\right) + i\frac{n_\ell\epsilon r}{\beta_{\ell 0}}\nabla G(\mathbf{X}_\ell)\cdot\mathbf{e}_\phi + \mathcal{O}(1/r^2)\big)(1+\mathcal{O}(q)), \tag{4.93}$$

which shows that the outer limit of the inner is only valid provided $r > \mathcal{O}(1/\epsilon^{1/3})$. Putting this together with the fact that the inner limit of the outer is valid when $R < \mathcal{O}(1)$, that is $r < \mathcal{O}(1/\epsilon)$, gives that the overlap region is defined by

$$\frac{1}{\epsilon^{1/3}} < r < \frac{1}{\epsilon} \quad \text{or alternatively} \quad \frac{1}{\epsilon^{\pi/6q}} < r < \frac{1}{\epsilon^{\pi/2q}}.$$

This implies that the outer limit of the inner in the form of series (4.45) breaks its asymptoticeness in the overlap region and is not valid at these distances. It is hence now more clear the need to find the sum of series (4.45) to compare it with the outer.

Solvability Condition

When we analysed the inner region we found that the first order equation (4.40) was a non homogeneous linear equation. As a consequence, in order to have a solution, a certain condition on the non homogeneous term must be satisfied. This solvability condition is provided by the *Fredholm Alternative* for linear operators.

We recall that equation (4.40) is of the form

$$L(q,\psi_0)[\psi_1] = b(\psi_0, q, \mu, d\mathbf{X}_\ell/dT)$$

where

$$L(q,\psi_0)[\psi_1] = (1-iq)\Delta\psi_1 + \psi_1(1-|\psi_0|^2) - \psi_0(\psi_0\psi_1^* + \psi_0^*\psi_1) \tag{4.94}$$

$$b(\psi_0, q, \mu, d\mathbf{X}_\ell/dT) = -\mu\frac{d\mathbf{X}_\ell}{dT}\cdot\nabla\psi_0 \tag{4.95}$$

Let us define the following inner product

$$\big(u,v\big) = \int_D \Re\{uv*\}\,dD,$$

where D is any given ball in $\mathbb{R}^2$ and $\Re\{uv*\}$ is the real part of the product of the complex function u and the complex conjugate of v. Using the integration by parts formula

$$\int_D u\Delta v\,dD = \int_D v\Delta u\,dD + \int_{\partial D}(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n})\,dl \tag{4.96}$$

we can compute the adjoin of L and it is given by

$$\overline{L}(q,\psi_0)[v] = (1+iq)\Delta v + v(1-|\psi_0|^2) - \psi_0(\psi_0 v^* + \psi_0^* v) \tag{4.97}$$

And comparing the linear operators (4.94) and (4.97) we find that

$$\overline{L}(q,\psi_0)[v] = L(-q,\psi_0)[v] \tag{4.98}$$

Choosing v to be the solutions to the homogeneous equation

$$L(-q,\psi_0)[v] = 0$$

and using the Fredholm Alternative we obtain the following solvability condition

$$-\int_D \Re\{\mu\frac{d\mathbf{X}_\ell}{dT}\cdot\nabla\psi_0 v^*\}dD = \int_{\partial D}\Re\{(1-iq)\big(v^*\frac{\partial\psi_1}{\partial n} - \frac{\partial v^*}{\partial n}\psi_1\big)\}dl \tag{4.99}$$

By studying the operator (4.94) it is clear that the solutions of $L(q,\psi_0)[v] = 0$ are actually the derivatives in any direction of ψ_0, that is $\nabla\psi_0\cdot\mathbf{d}$, being $\mathbf{d}$ any vector in $\mathbb{R}^2$. Then, in order to compute v we just need to differentiate ψ_0 and substitute q by $-q$. We note that this condition is independent of the size of

q, so it would still be valid if we analysed the problem for order-one values of q.

To simplify the solvability condition we use that the left-hand side is $O(\mu)$ while the right-hand side is $O(1)$. Thus, to leading order, the solvability condition is

$$0 = \int_{\partial D} \Re\{(\mathbf{d} \cdot \nabla \psi_0) \frac{\partial \psi_1^*}{\partial n} - \frac{\partial (\mathbf{d} \cdot \nabla \psi_0)}{\partial n} \psi_1^*\}$$

We write it in polar coordinates and choose a ball D such that its radius lies in the overlap region where we have matched the inner and the outer and the condition to leading order becomes

$$0 = \int_0^{2\pi} \Re\{\left(v^* \frac{\partial \psi_1}{\partial r} - \frac{\partial v^*}{\partial r} \psi_1\right)\} r \, d\phi. \tag{4.100}$$

Now in the inner region as $r \to \infty$ we have

$$\nabla \chi_0 \sim \frac{n_\ell}{r} \mathbf{e}_\phi - \frac{q(\log r + C_1)}{r} \mathbf{e}_r + \cdots \tag{4.101}$$

$$f_0 \sim 1 + O(r^{-2}), \tag{4.102}$$

$$f_1 = -\frac{1}{2} |\nabla \chi_0|^2 \sim -\frac{1}{2r^2} \tag{4.103}$$

$$\begin{aligned} \chi_1 &\sim -\frac{\tilde{\mu} r}{2} \left(V_1 \cos\phi + V_2 \sin\phi\right) + \frac{n_\ell r}{\beta_{\ell 0}} \mathbf{e}_\phi \cdot \nabla G(\mathbf{X}_\ell) \\ &= -\frac{\mu}{2q} \frac{d\mathbf{X}_\ell}{dT} \cdot \mathbf{x} + \frac{n_\ell r}{\beta_{\ell 0}} \mathbf{e}_\phi \cdot \nabla G(\mathbf{X}_\ell) \end{aligned} \tag{4.104}$$

$$v \sim i\left(\frac{n_\ell}{r} \mathbf{e}_\phi + \frac{q(\log r + C_1)}{r} \mathbf{e}_r + \cdots\right) \cdot \mathbf{d} \, e^{i\phi} \tag{4.105}$$

$$\frac{\partial v}{\partial r} \sim -i \frac{n_\ell}{r^2} \mathbf{e}_\phi \cdot \mathbf{d} e^{i\phi} \tag{4.106}$$

$$\begin{aligned} \psi_1 &= (f_1 + i f_0 \chi_1) e^{i\chi_0} \\ &\sim i \chi_1 e^{i\phi}. \end{aligned} \tag{4.107}$$

When we substitute all these expressions in (4.100) we obtain, to leading order, the simpler equation

$$\begin{aligned} rhs &\sim \Re\left\{\int_0^{2\pi} \left(\frac{1}{r} \mathbf{e}_\phi \cdot \mathbf{d} \frac{\partial \chi_1}{\partial r} + \frac{1}{r^2} \mathbf{e}_\phi \cdot \mathbf{d} \chi_1\right) r \, d\phi\right\} \\ &= \int_0^{2\pi} (\mathbf{e}_\phi \cdot \mathbf{d}) \left(\frac{\partial \chi_1}{\partial r} + \frac{\chi_1}{r}\right) d\phi \end{aligned} \tag{4.108}$$

4.4 Law of motion

Finally, to obtain the law of motion we use expression (4.104),

$$\chi_1 \sim -\frac{\mu}{2q}\frac{d\mathbf{X}_\ell}{dT}\cdot\mathbf{x} + \frac{n_\ell r}{\beta_{\ell 0}}\mathbf{e}_\phi\cdot\nabla G(\mathbf{X}_\ell)$$

where

$$G(\mathbf{X}_\ell) = \sum_{j\neq\ell}^{N}\beta_{j0}(T)K_0(\alpha|\mathbf{X}-\mathbf{X}_j|),$$

and hence,

$$\nabla G(\mathbf{X}_\ell) = \sum_{j\neq\ell}^{N}\beta_{j0}\alpha K_0'(\alpha|\mathbf{X}-\mathbf{X}_j|)\,\mathbf{e}_{rj\ell}.$$

Substituting (4.104) in (4.108) in the same way that we did in Chapter 2 we find:

Result 4.2 (Law of motion). *For a system of N spirals, with winding numbers of either plus or minus one, that are separated by distances of order $1/\epsilon$ such that the parameter $\alpha = kq/\epsilon$ is of order one, the centres of the spirals move with a law of motion that reads*

$$\begin{aligned}\frac{d\mathbf{X}_\ell}{dT} &= -\frac{2qn_\ell}{\beta_{\ell 0}\mu}\nabla G^{\perp}(\mathbf{X}_\ell) + \mathcal{O}(q)\\ &= -\frac{2qn_\ell}{\beta_{\ell 0}\mu}\sum_{j\neq\ell}^{N}\beta_{j0}\alpha K_0'(\alpha|\mathbf{X}-\mathbf{X}_j|)\,\mathbf{e}_{\phi j\ell} + \mathcal{O}(q), \end{aligned} \tag{4.109}$$

where the time scale is $T = \epsilon^2\mu t$ where

$$\mu = \frac{1}{|\log\epsilon|}.$$

We note that this is the same time scale as in the case of $q = 0$. This is what one would expect to happen because in the limit $q \to 0$ one expects to meet the results for $q = 0$. But we now show that although the time scale agrees with the $q = 0$ case, the law of motion itself does not agree with the one we obtained in Chapter 2. In fact, if we observe the law of motion in (4.109) we realise that the interaction takes place in the direction orthogonal to the line along the centres of the spirals, which is in clear contradiction with the law of motion for $q = 0$. Let us show how this works by considering the simple example of two interacting spirals.

Law of motion for two spirals For two spirals at positions $(X_1, 0)$ and $(X_2, 0)$, with $X_1 < X_2$,

$$\nabla G(X_1, 0) \sim (\beta_{20}\alpha K_0'(\alpha(X_2 - X_1)), 0),$$

and, as we showed, $\beta_{10} = \beta_{20}$. Using $\mu = 1/|\log\epsilon|$ and $q = \pi/(2|\log\epsilon|)$, the law of motion is

$$\frac{dX_1}{dT} = -\pi n_1 K_0'(\alpha(X_2 - X_1))\mathbf{e}_{\phi 2} + \mathcal{O}(q). \tag{4.110}$$

When the spirals are close and $X_2 - X_1 \ll 1$ this is approximately

$$\frac{dX_1}{dT} = \Big(0, \frac{-n_1\pi}{X_2 - X_1}\Big). \tag{4.111}$$

In figure 4.1 we represent the direction in which a pair of spirals would move, which depends only on the own degree and not on the degree of the other spiral. This is exactly the opposite to what happens in fluid dynamics vortices. Fluid vortices move depending on the vorticity of the surrounding vortices and it does not matter what the own vorticity is. In this case we have the opposite situation: spirals move in a direction that is determined by their own winding number, regardless of what the other spiral's winding number is. On the other hand, we note that only when the spirals have the same winding number they may rotate around each other composing an equilibrium or a bound state, but this would never happen for oppositely charged spirals. However, as we will show in Chapter 5, when the spirals are closer there is a component along the lines of the centre that could correspond to an order q term in the law of motion in the canonical scale.

The singular limit as $q \to 0$ Equation (4.111) has been found by taking closer the two spirals that are interacting, but keeping the parameter α of order one. But when we considered $q = 0$ we did not put any restriction on the distances of separation apart from requiring ϵ to be small. This means that since there are not q's left in equation (4.111), it should be the same law of motion as the one obtained when $q = 0$. It is obvious that this is not the case, on the contrary, the spirals move now perpendicularly to the line along the centres. But this is not a real inconsistency and we will show that both laws are actually correct. The key point lies on the size of α and we must consider now the situation where α is of order less than one. This is done in Chapter 5 where we reach a different law of motion that will interpolate between (4.109) and (2.42).

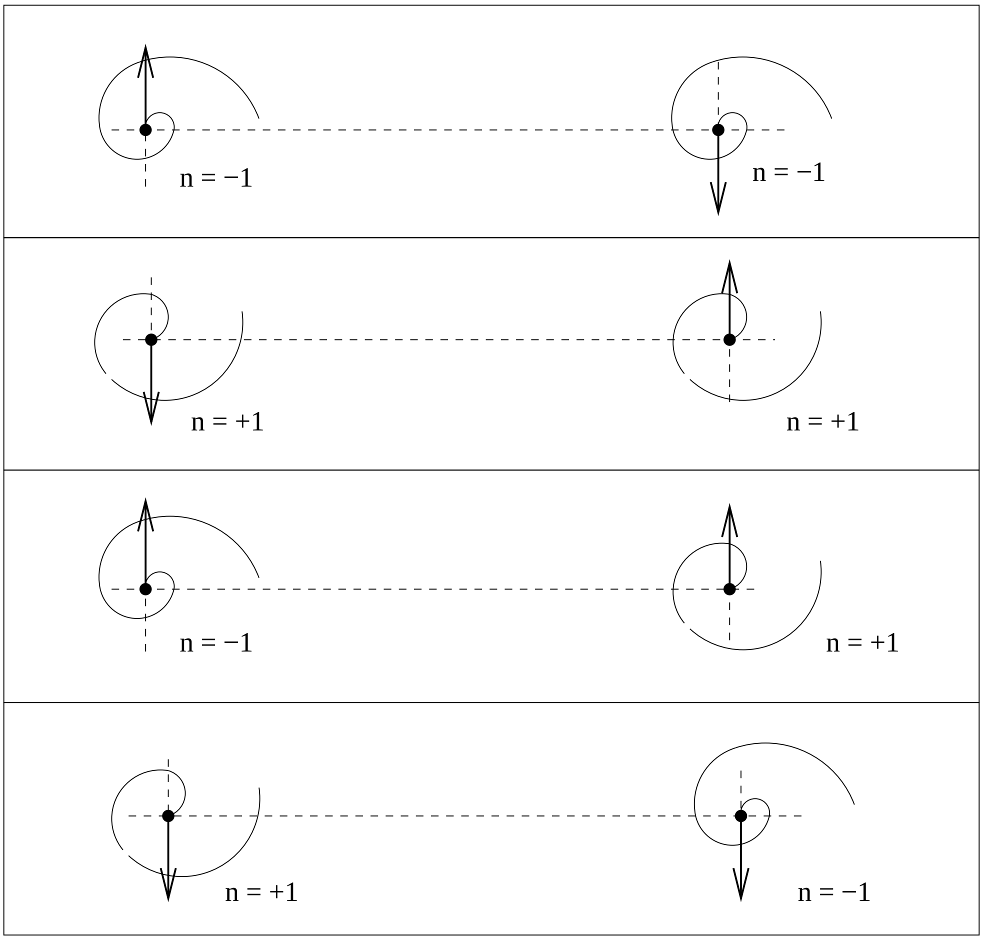

Figure 4.1: Direction of the velocity for a pair of spirals in the canonical scaling with positive and negative winding numbers.

Chapter 5

Interaction of spirals in the middle scale

In the previous chapter we derived the spiral law of motion in the case where q is small and for typical vortex separation $1/\epsilon = \mathcal{O}(e^{\pi/(2q|n_\ell|)})$, that we called the *canonical separation*; the latter turns out to be equivalent to assuming that the parameter α is of order one. But we also realised that in the limit of q tending to zero, this law did not agree with the law of motion for spirals when $q = 0$ precisely because of the imposed vortex separation. When we then consider the limit as q tends to zero we are therefore taking the spirals to even larger distances so it makes sense to expect that the interaction will become weaker rather than algebraic as it is when $q = 0$. This is quite subtle and what we must do now is to take the limit as q goes to zero but keep ϵ the same, that is to say, in this chapter we will consider smaller vortex separations by taking α itself to be small. By doing so we will find a new law of motion which in the limit as $q \to 0$ does agree with the one when $q = 0$ and, on the other hand, as $|q \log \epsilon| \to \pi/2$ we find again the law that we obtained in the previous chapter. Furthermore, we will now find that the asymptotic wavenumber when the spirals are closer than the canonical separation is almost like the one of a spiral in isolation with a degree that is the sum of the degrees of the interacting spirals, except of a multiplying constant that accounts for the fact that there are many spirals. We will find that in the limit as the spirals come closer we actually have the asymptotic wavenumber corresponding to a single spiral. As for the time scale for the velocity, again it will be logarithmic, that is to say, the spirals move at a rate of order $\epsilon^2/|\log \epsilon|$.

Hence, we now consider equation (3.1) in the limit in which the separation of spirals $1/\epsilon$ is such that $\alpha = qk/\epsilon \ll 1$. As before we rescale time and space by setting $\mathbf{X} = \epsilon \mathbf{x}$ and $T = \mu \epsilon^2 t$, to give

$$\epsilon^2 \mu \psi_T = \epsilon^2 \Delta \psi + (1 - |\psi|^2)\psi + iq(1 - k^2 - |\psi|^2)\psi. \tag{5.1}$$

Expressing the equation in terms of $\alpha = qk/\epsilon$

$$\epsilon^2 \mu \psi_T = \epsilon^2 \Delta\psi + (1 + iq)(1 - |\psi|^2)\psi - \frac{i\epsilon^2\alpha^2}{q}\psi. \tag{5.2}$$

5.1 Outer Region

Again we start by writing $\psi = fe^{i\chi}$ and separating real and imaginary parts in (5.2) to give

$$\mu\epsilon^2 f_T = \epsilon^2 \Delta f - \epsilon^2 f|\nabla\chi|^2 + (1 - f^2)f, \tag{5.3}$$

$$\mu\epsilon^2 f^2 \chi_T = \epsilon^2 \nabla \cdot (f^2 \nabla\chi) + qf^2(1 - f^2) - \epsilon^2 \frac{\alpha^2}{q} f^2. \tag{5.4}$$

Expanding in powers of ϵ as

$$f \sim f_0(q, \mu) + \epsilon^2 f_1(q, \mu) + \ldots \tag{5.5}$$

$$\chi \sim \chi_0(q, \mu) + \epsilon^2 \chi_1(q, \mu) + \ldots \tag{5.6}$$

we find

$$f_0 = 1, \tag{5.7}$$

$$f_1 = -\frac{1}{2}|\nabla\chi_0|^2, \tag{5.8}$$

$$\mu\chi_{0T} = \Delta\chi_0 + q|\nabla\chi_0|^2, \tag{5.9}$$

where we are assuming that now α is of order less than q or μ. As for the parameters q and μ are both small parameters, we may therefore write $\mu = \tilde{\mu}q$. If we now expand χ_0 as

$$\chi_0 \sim \frac{1}{q}(\chi_{00} + \mu\,\chi_{01} + \mu^2\,\chi_{02} + \ldots),$$

and assume that the the time derivative χ_{0T} is of order one or less, we find the following equations

$$0 = \Delta\chi_{00} + |\nabla\chi_{00}|^2, \tag{5.10}$$

$$\chi_{00T} = \Delta\chi_{01} + 2\nabla\chi_{00} \cdot \nabla\chi_{01}, \tag{5.11}$$

$$\chi_{01T} = \Delta\chi_{02} + 2\nabla\chi_{00} \cdot \nabla\chi_{02} + |\nabla\chi_{01}|^2. \tag{5.12}$$

By inspecting the asymptotics of a single spiral and also the calculations in the previous chapter we expect the first equation to have a constant as a solution that will be determined when we compare the outer and inner solutions. We thus take

$$\chi_{00} = C_1(T). \tag{5.13}$$

As for equation (5.11), since it is the order one term in the outer expansion, it is the one that should account for the presence of N spirals with its corresponding topological singularities. We then choose as a solution to (5.11) for a system of N spirals,

$$\chi_{01} = \sum_{j=1}^{N} \big(\frac{q}{\mu} n_j \phi_j + C_{2j} \log R_j + C_{3j}\big), \tag{5.14}$$

where C_{2j} and C_{3j} are also unknown constants at this stage that will be found when matching with the inner solution. As before, we use the notation R_j and ϕ_j to represent the polar coordinates centred at the spiral number j.

Finally, to compute the solution of (5.12) it is very useful to take advantage of the fact that it is linear to decompose the solution in two parts, $\chi_{02} = v_1 + v_2$ where each part satisfies

$$\chi_{01T} = \Delta v_1, \tag{5.15}$$
$$0 = \Delta v_2 + |\nabla \chi_{01}|^2. \tag{5.16}$$

Equation (5.15) has the solution

$$v_1 = -\frac{1}{2} \sum_{j=1}^{N} \Big(R_j \log R_j \big(\frac{q}{\mu} n_j \frac{d\mathbf{X}_j}{dT} \cdot \mathbf{e}_{\phi j} + C_{2j} \frac{d\mathbf{X}_j}{dT} \cdot \mathbf{e}_{rj}\big) + C_{4j} \log R_j \Big). \tag{5.17}$$

Equation (5.16) does not have an explicit solution in the outer region, but it will be solvable in the limit as we approach one of the spirals. This will be enough to match the outer solution with the inner.

By the way it is written the outer so far it is clear that the corresponding function ψ is single valued.

5.2 Inner Region

We rescale near the ℓth vortex by setting $\mathbf{X} = \mathbf{X}_\ell + \epsilon \mathbf{x}$ to give

$$\epsilon\mu(\epsilon f_T - \frac{d\mathbf{X}_\ell}{dT} \cdot \nabla f) = \Delta f - f|\nabla \chi|^2 + (1 - f^2) f \tag{5.18}$$
$$\epsilon\mu f^2(\epsilon \chi_T - \frac{d\mathbf{X}_\ell}{dT} \cdot \nabla \chi) = \nabla \cdot (f^2 \nabla \chi) + q(1 - f^2) f^2 - \frac{\epsilon^2 \alpha^2 f^2}{q} \tag{5.19}$$

or equivalently

$$\epsilon\mu(\epsilon \psi_T - \frac{d\mathbf{X}_\ell}{dT} \cdot \nabla \psi) = \Delta \psi + (1 + iq)(1 - |\psi|^2)\psi - i\frac{\epsilon^2 \alpha^2}{q} \psi \tag{5.20}$$

Expanding

$$f \sim f_0(q,\mu) + \epsilon f_1(q,\mu) + \epsilon^2 f_2(q,\mu) + \dots, \quad \text{and} \tag{5.21}$$

$$\chi \sim \chi_0(q,\mu) + \epsilon \chi_1(q,\mu) + \dots, \tag{5.22}$$

gives the leading-order equations

$$0 = \Delta f_0 - f_0|\nabla\chi_0|^2 + (1-f_0^2)f_0, \tag{5.23}$$

$$0 = \nabla\cdot(f_0^2\nabla\chi_0) + q(1-f_0^2)f_0^2. \tag{5.24}$$

Assuming that $f_0 = f_0(r,T)$ and $\chi_0 = n_\ell\phi + \varphi_0(r,T)$ gives

$$f_0'' + \frac{1}{r}f_0' - f_0\big(\frac{n_\ell^2}{r^2} + (\varphi_0')^2\big) + (1-f_0^2)f_0 = 0, \tag{5.25}$$

$$f_0(\varphi_0'' + \frac{1}{r}\varphi_0') + 2f_0'\varphi_0' + q(1-f_0^2)f_0 = 0. \tag{5.26}$$

Equations (5.23), (5.24) can be also expressed as

$$0 = \Delta\psi_0 + (1+iq)\psi_0(1-|\psi_0|^2). \tag{5.27}$$

We now pose the expansion

$$f_0 \sim f_{00} + f_{01}q^2 + f_{02}q^4\cdots, \tag{5.28}$$

$$\varphi_0 \sim \frac{\varphi_{00}}{q} + \varphi_{01}q + \varphi_{02}q^3 + \cdots. \tag{5.29}$$

Substituting these expansions into (5.25) and (5.26) and equating powers of q gives

$$\varphi_{00} = \varphi_{00}(T), \tag{5.30}$$

$$f_{00}'' + \frac{f_{00}'}{r} - n_\ell^2\frac{f_{00}}{r^2} + (1-f_{00}^2)f_{00} = 0, \tag{5.31}$$

$$f_{01}'' + \frac{f_{01}'}{r} - n_\ell^2\frac{f_{01}}{r^2} + (1-3f_{00}^2)f_{01} = 0, \tag{5.32}$$

$$\varphi_{01}'(r) = -\frac{1}{rf_{00}^2}\int_0^r sf_{00}^2(1-f_{00}^2)\,ds, \tag{5.33}$$

with the boundary conditions

$$\begin{aligned} f_{00}(0) &= 0, \\ f_{00}(\infty) &= 1. \end{aligned}$$

At first order in ϵ we find

$$-\mu\frac{d\mathbf{X}_\ell}{dT}\cdot\nabla\psi_0 = \nabla^2\psi_1 + (1+iq)(\psi_1(1-|\psi_0|^2) - \psi_0(\psi_0\psi_1^* + \psi_0^*\psi_1)), \tag{5.34}$$

or equivalently

$$-\mu\frac{d\mathbf{X}_\ell}{dT}\cdot\nabla f_0 = \Delta f_1 - f_1|\nabla\chi_0|^2 - 2f_0\nabla\chi_0\cdot\nabla\chi_1 + f_1 - 3f_0^2 f_1, \quad (5.35)$$

$$-\mu f_0^2\frac{d\mathbf{X}_\ell}{dT}\cdot\nabla\chi_0 = \nabla\cdot(f_0^2\nabla\chi_1) + \nabla\cdot(2f_0f_1\nabla\chi_0) + 2qf_0f_1 - 4qf_0^3f_1. \quad (5.36)$$

5.3 Asymptotic matching

5.3.1 Outer limit of the inner

From the expressions (5.30)-(5.33) we find that when $r\to\infty$

$$\frac{\partial\varphi_{01}}{\partial r} \sim -qn_\ell^2\,\frac{\log r + c_{n_\ell}}{r} + \dots \quad (5.37)$$

where c_{n_ℓ} is a constant given by

$$c_{n_\ell} = -\lim_{r\to\infty}\frac{1}{n_\ell^2}\left(\int_0^r f_0^2(s)\Big(f_0(s)^2-1\Big)s\,ds + \log(r)\right).$$

However, in order to match with the outer expansion we need the outer limit of the whole expansion in q. This can be found to be of the form

$$f_0 \sim 1 + \frac{1}{r^2}\sum_{i=0}^{N}\alpha_i\{qn_\ell^2(\log(r)+c_1)\}^{2i} + \cdots, \quad (5.38)$$

$$\frac{\partial\chi_0}{\partial r} \sim -\frac{1}{r}\sum_{i=0}^{N}\beta_i\{qn_\ell^2(\log(r)+c_1)\}^{2i+1} + \cdots, \quad (5.39)$$

where $\alpha_i > 0$ and $\beta_i > 0$ are constant values independent of q and n_ℓ^2. The necessity of taking all the terms in q when matching can be seen, since the expansion in q is valid only when $q(\log(r) + C_1) \ll 1$.

Again we can use the same trick as in Chapter 3 which allows us to sum all these terms in the outer limit of the inner expansion. We note that the outer limit of the inner looks the same for a single spiral, for spirals in the canonical separation and in this case. Thus, we just rewrite what we have done before for the sake of clarity.

We begin then by rewriting the leading-order inner equations (5.23) and (5.24) in terms of the outer variable $R_\ell = \epsilon r$ to obtain

$$0 = \epsilon^2(\Delta f_0 - f_0|\nabla\chi_0|^2) + (1-f_0^2)f_0 \quad (5.40)$$

$$0 = \epsilon^2\nabla\cdot(f_0^2\nabla\chi_0) + q(1-f_0^2)f_0^2 \quad (5.41)$$

We now expand in powers of ϵ as

$$\chi_0 \sim \widehat{\chi}_{00}(q) + \epsilon^2 \widehat{\chi}_{01}(q) + \cdots , \tag{5.42}$$

$$f_0 \sim \widehat{f}_{00}(q) + \epsilon^2 \widehat{f}_{01}(q) + \cdots . \tag{5.43}$$

The leading-order term in this expansion $\widehat{\chi}_{00}(q)$ is just the first term (in ϵ) in the outer expansion of the leading-order inner solution, including all the terms in q. Substituting (5.42), (5.43) into (5.40), (5.41) gives

$$\widehat{f}_{00} = 1, \tag{5.44}$$

$$\widehat{f}_{01} = -\frac{1}{2}|\nabla \widehat{\chi}_{00}|^2, \tag{5.45}$$

$$0 = \Delta \widehat{\chi}_{00} + q|\nabla \widehat{\chi}_{00}|^2. \tag{5.46}$$

Since (5.46) is a Riccati equation it can be linearised with the change of variable $\widehat{\chi}_{00} = (1/q)\log \widehat{h}_0$ to give

$$\Delta \widehat{h}_0 = 0.$$

Since $\widehat{\chi}_{00} = n_\ell \phi + \widehat{\varphi}(R)$ we set $\widehat{h}_0 = e^{qn_\ell\phi} e^{q\widehat{\varphi}(R)} = e^{qn_\ell\phi} H_0(R)$ to give

$$H_0'' + \frac{H_0'}{R} + (qn_\ell)^2 \frac{H_0}{R^2} = 0 \tag{5.47}$$

with solution

$$H_0 = A_\ell(q)\epsilon^{-iqn_\ell} R^{iqn_\ell} + B_\ell(q)\epsilon^{iqn_\ell} R^{-iqn_\ell}, \tag{5.48}$$

where A_ℓ and B_ℓ are constants that depend on q which may be different at each vortex, and the factors $\epsilon^{\pm iqn_\ell}$ have been included to facilitate their determination by comparison with the solution in the inner variable. To determine A_ℓ and B_ℓ we need to write $\hat{\chi}_{00}$ in terms of r, expand in powers of q, and compare with (5.37). Expanding the constants in powers of q as

$$\begin{aligned} A_\ell(q) &\sim \frac{1}{q}A_{\ell 0} + A_{\ell 1} + qA_{\ell 2} + \cdots , \\ B_\ell(q) &\sim \frac{1}{q}B_{\ell 0} + B_{\ell 1} + qB_{\ell 2} + \cdots , \end{aligned}$$

and writing H_0 in terms of r we find

$$\begin{aligned} H_0(r) &= A_\ell(q)e^{iqn_\ell \log r} + B_\ell(q)e^{-iqn_\ell \log r} \\ &\sim \left(\frac{1}{q}A_{\ell 0} + A_{\ell 1} + qA_{\ell 2} + \cdots\right)\left(1 + iqn_\ell \log r - (qn_\ell)^2/2 \log^2 r + \cdots\right) \\ &+ \left(\frac{1}{q}B_{\ell 0} + B_{\ell 1} + qB_{\ell 2} + \cdots\right)\left(1 - iqn_\ell \log r - (qn_\ell)^2/2 \log^2 r + \cdots\right) \\ &\sim \frac{A_{\ell 0} + B_{\ell 0}}{q} + A_{\ell 1} + B_{\ell 1} + (A_{\ell 0} - B_{\ell 0})in_\ell \log r \\ &+ q\left(A_{\ell 2} + B_{\ell 2} + (A_{\ell 1} - B_{\ell 1})in_\ell \log r - \frac{(A_{\ell 0} + B_{\ell 0})}{2}n_\ell^2 \log^2 r\right) + \cdots , \end{aligned}$$

so that

$$\begin{aligned}
\frac{\partial\widehat{\chi}_{00}}{\partial r} &= \frac{H_0'(r)}{qH_0(r)} \\
&\sim \frac{n_\ell}{qr}\frac{(A_{\ell 0}-B_{\ell 0})i+(A_{\ell 1}-B_{\ell 1})iq-(A_{\ell 0}+B_{\ell 0})qn_\ell\log r+\ldots}{\frac{A_{\ell 0}+B_{\ell 0}}{q}+A_{\ell 1}+B_{\ell 1}+(A_{\ell 0}-B_{\ell 0})in_\ell\log r+q\ldots} \\
&\sim \frac{n_\ell(A_{\ell 0}-B_{\ell 0})i}{r(A_{\ell 0}+B_{\ell 0})}+q\left(\frac{(A_{\ell 1}-B_{\ell 1})}{(A_{\ell 0}+B_{\ell 0})}\frac{n_\ell i}{r}-n_\ell^2\frac{\log r}{r}\right. \\
&+ \left.\left(n_\ell^2\frac{(A_{\ell 0}-B_{\ell 0})^2}{(A_{\ell 0}+B_{\ell 0})^2}\right)\frac{\log r}{r}-\left(\frac{i(A_{\ell 0}-B_{\ell 0})(A_{\ell 1}+B_{\ell 1})}{(A_{\ell 0}+B_{\ell 0})^2}\right)\frac{n_\ell}{r}\right)+\cdots
\end{aligned}$$

Comparing with (5.37) we see that

$$A_{\ell 0}-B_{\ell 0}=0, \tag{5.49}$$

$$\frac{(A_{\ell 1}-B_{\ell 1})}{A_{\ell 0}+B_{\ell 0}}i=-c_{n_\ell}n_\ell \quad \text{for } k=1,\ldots,N. \tag{5.50}$$

The remaining equations determining A_ℓ and B_ℓ will come from matching with the outer region.

Outer limit of the first-order inner We play the same game with the first-order inner solution. We first write equation (5.35)-(5.36) in terms of the outer variable to give

$$\begin{aligned}
-\epsilon\mu\frac{d\mathbf{X}_\ell}{dT}\cdot\nabla f_0 &= \epsilon^2\nabla^2 f_1-\epsilon^2 f_1|\nabla\chi_0|^2-2\epsilon^2 f_0\nabla\chi_0\cdot\nabla\chi_1+f_1-3f_0^2f_1, \\
-\mu\epsilon f_0^2\frac{d\mathbf{X}_\ell}{dT}\cdot\nabla\chi_0 &= \epsilon^2\nabla\cdot(f_0^2\nabla\chi_1)+\epsilon^2\nabla\cdot(2f_0f_1\nabla\chi_0)+2qf_0f_1-4qf_0^3f_1.
\end{aligned}$$

We now expand in powers of ϵ as

$$\chi_1 \sim \frac{\widehat{\chi}_{10}(q)}{\epsilon}+\widehat{\chi}_{11}(q)+\cdots, \tag{5.51}$$

$$f_1 \sim \widehat{f}_{10}(q)+\epsilon\widehat{f}_{11}(q)+\cdots, \tag{5.52}$$

to give

$$\widehat{f}_{10}=0, \tag{5.53}$$

$$\widehat{f}_{11}=-\nabla\widehat{\chi}_{00}\cdot\nabla\widehat{\chi}_{10}, \tag{5.54}$$

$$-\mu\frac{d\mathbf{X}_\ell}{dT}\cdot\nabla\widehat{\chi}_{00}=\Delta\widehat{\chi}_{10}-2q\widehat{f}_{11}=\Delta\widehat{\chi}_{10}+2q\nabla\widehat{\chi}_{00}\cdot\nabla\widehat{\chi}_{10}. \tag{5.55}$$

Motivated by the transformation we applied to $\widehat{\chi}_{00}$ we write

$$\widehat{\chi}_{10}=\frac{\widehat{h}_1}{q\widehat{h}_0}=\frac{\widehat{h}_1e^{-q\widehat{\chi}_{00}}}{q},$$

so that if we had written for the outer limit of the inner $\chi = 1/q \log \widehat{h}$ then $\widehat{h}$ has the expansion $\widehat{h} \sim \widehat{h}_0 + \epsilon \widehat{h}_1 + \cdots$. In (5.55) this gives

$$\begin{aligned}
-\mu \frac{d\mathbf{X}_\ell}{dT} \cdot \nabla \widehat{\chi}_{00} &= \frac{e^{-q\widehat{\chi}_{00}}}{q} \Big(\Delta \widehat{h}_1 - 2q \nabla \widehat{h}_1 \cdot \nabla \widehat{\chi}_{00} - q \widehat{h}_1 \Delta \widehat{\chi}_{00} + q^2 \widehat{h}_1 |\nabla \widehat{\chi}_{00}|^2 \\
&\qquad + 2q \nabla \widehat{\chi}_{00} \cdot \nabla \widehat{h}_1 - 2q^2 \widehat{h}_1 |\nabla \widehat{\chi}_{00}|^2 \Big) \\
&= \frac{e^{-q\widehat{\chi}_{00}}}{q} \Delta \widehat{h}_1.
\end{aligned}$$

Writing $\widehat{\chi}_{00}$ in terms of $\widehat{h}_0$ gives

$$-\mu \frac{d\mathbf{X}_\ell}{dT} \cdot \nabla \widehat{h}_0 = \nabla^2 \widehat{h}_1. \tag{5.56}$$

Denoting

$$\frac{d\mathbf{X}_\ell}{dT} = (V_1, V_2)$$

and recalling that $\widehat{h}_0 = e^{qn_\ell \phi} H_0(R)$, equation (5.56) becomes

$$\begin{aligned}
& -\mu \frac{d\mathbf{X}_\ell}{dT} \cdot \left(\frac{qn_\ell e^{qn_\ell \phi} H_0(R)}{R} \mathbf{e}_\phi + H'(R) e^{qn_\ell \phi} \mathbf{e}_r \right) \\
= \; & \mu e^{qn_\ell \phi} \sin\phi \left(qn_\ell V_1 \frac{H_0(R)}{R} - V_2 H'(R) \right) \\
& - \mu e^{qn_\ell \phi} \cos\phi \left(qn_\ell V_2 \frac{H_0(R)}{R} + V_1 H'(R) \right) \\
= \; & \mu e^{qn_\ell \phi} qn_\ell \sin\phi \big(R^{iqn_\ell - 1} A_\ell \epsilon^{-iqn_\ell} (V_1 - iV_2) + R^{-iqn_\ell - 1} B_\ell \epsilon^{iqn_\ell} (V_1 + iV_2) \big) \\
- \; & \mu e^{qn_\ell \phi} qn_\ell \cos\phi \big(R^{iqn_\ell - 1} A_\ell \epsilon^{-iqn_\ell} (V_2 + iV_1) - R^{-iqn_\ell - 1} B_\ell \epsilon^{iqn_\ell} (V_2 - iV_1)) \\
= \; & \mu e^{qn_\ell \phi} \frac{(e^{i\phi} - e^{-i\phi})}{2i} R^{iqn_\ell - 1} qn_\ell A_\ell \epsilon^{-iqn_\ell} \, (V_1 - iV_2) \\
& + \mu e^{qn_\ell \phi} \frac{(e^{i\phi} - e^{-i\phi})}{2i} R^{-iqn_\ell - 1} qn_\ell B_\ell \epsilon^{iqn_\ell} \, (V_1 + iV_2) \\
& - \mu e^{qn_\ell \phi} \frac{(e^{i\phi} + e^{-i\phi})}{2} R^{iqn_\ell - 1} qn_\ell A_\ell \epsilon^{-iqn_\ell} \, (V_2 + iV_1) \\
& - \mu e^{qn_\ell \phi} \frac{(e^{i\phi} + e^{-i\phi})}{2} R^{-iqn_\ell - 1} qn_\ell B_\ell \epsilon^{iqn_\ell} \, (V_2 - iV_1) = \\
- \; & \frac{\mu qn_\ell e^{qn_\ell \phi}}{R} \big(e^{i\phi} R^{iqn_\ell} A_\ell \epsilon^{-iqn_\ell} (V_2 + iV_1) - e^{-i\phi} R^{-iqn_\ell} B_\ell \epsilon^{iqn_\ell} (V_2 - iV_1) \big),
\end{aligned}$$

since

$$\begin{aligned}
\frac{H_0(R)}{R} &= A_\ell(q) \epsilon^{-iqn_\ell} R^{iqn_\ell - 1} + B_\ell(q) \epsilon^{iqn_\ell} R^{-iqn_\ell - 1}, \\
H'(R) &= iqn_\ell A_\ell(q) \epsilon^{-iqn_\ell} R^{iqn_\ell - 1} - iqn_\ell B_\ell(q) \epsilon^{iqn_\ell} R^{-iqn_\ell - 1}.
\end{aligned}$$

Writing
$\widehat{h}_1 = -\mu q n_\ell A_\ell \epsilon^{-iqn_\ell}(V_2+iV_1)g_1(R)\, e^{(qn_\ell+i)\phi} - \mu q n_\ell B_\ell \epsilon^{iqn_\ell}(V_2-iV_1)g_2(R)\, e^{(qn_\ell-i)\phi}$,
gives

$$\begin{aligned} g_1'' + \frac{g_1'}{R} + \frac{(qn_\ell+i)^2 g_1}{R^2} &= R^{iqn_\ell-1}, \\ g_2'' + \frac{g_2'}{R} + \frac{(qn_\ell-i)^2 g_2}{R^2} &= R^{-iqn_\ell-1}, \end{aligned}$$

with solution

$$\begin{aligned} g_1 &= \frac{R^{iqn_\ell+1}}{4iqn_\ell} + \frac{\gamma_1 R^{1-iqn_\ell}}{4iqn_\ell} + \gamma_3 R^{-1+iqn_\ell}, \\ g_2 &= -\frac{R^{-iqn_\ell+1}}{4iqn_\ell} - \frac{\gamma_2 R^{1+iqn_\ell}}{4iqn_\ell} + \gamma_4 R^{-1-iqn_\ell}, \end{aligned}$$

where γ_1, γ_2, γ_3 and γ_4 are arbitrary constants. We first note that this solution does not agree with the inner solution unless $\gamma_3 = \gamma_4 = 0$. As for γ_1 and γ_2, they will be determined by matching to the inner limit of the outer. Hence, we have that the first order outer limit of the inner is given by

$$\begin{aligned} \widehat{h}_1 &= -\frac{\mu A_\ell \epsilon^{-iqn_\ell}(V_1 - iV_2)}{4}(R^{iqn_\ell+1} + \gamma_1 R^{1-iqn_\ell})\, e^{(qn_\ell+i)\phi} \\ &\quad -\frac{\mu B_\ell \epsilon^{iqn_\ell}(V_1 + iV_2)}{4}(R^{-iqn_\ell+1} + \gamma_2 R^{1+iqn_\ell})\, e^{(qn_\ell-i)\phi}. \end{aligned}$$

5.3.2 Inner limit of the outer

To compute the inner limit of the outer we rewrite solutions (5.13), (5.14) and (5.17) in terms of the inner variable $\mathbf{X} = \mathbf{X}_\ell + \epsilon \mathbf{x}$ and expand in powers of ϵ. This gives

$$\chi_{01} \sim \frac{q}{\mu} n_\ell \phi + C_{2\ell} \log R + C_{3\ell} + G(\mathbf{X}_\ell) + \nabla G_0(\mathbf{X}_\ell) \cdot \mathbf{X} + \ldots, \tag{5.57}$$

$$\begin{aligned} \chi_{02} \sim &-\frac{1}{2} R \log R \big(\frac{q}{\mu} n_\ell \frac{d\mathbf{X}_\ell}{dT} \cdot \mathbf{e}_\phi + C_{2\ell} \frac{d\mathbf{X}_\ell}{dT} \cdot \mathbf{e}_r\big) - G_1(\mathbf{X}_\ell) \\ &+ C_{4\ell} \log R + v_2(R/\epsilon) + \ldots, \end{aligned} \tag{5.58}$$

where

$$\begin{aligned} G_0(\mathbf{X}_\ell) &= \sum_{j\neq\ell}^{N} \frac{q}{\mu} n_j \phi_{j\ell} + C_{2j} \log(|\mathbf{X}_\ell - \mathbf{X}_j|) + C_{3j}, \\ G_1(\mathbf{X}_\ell) &= \sum_{j\neq\ell}^{N} \Big(\frac{|\mathbf{X}_\ell - \mathbf{X}_j|}{2} \log|\mathbf{X}_\ell - \mathbf{X}_j| \big(\frac{qn_\ell}{\mu} \frac{d\mathbf{X}_j}{dT} \cdot \mathbf{e}_{\phi j\ell} + C_{2j} \frac{d\mathbf{X}_j}{dT} \cdot \mathbf{e}_{rj\ell}\big) \\ &\quad - -C_{4j} \log|\mathbf{X}_\ell - \mathbf{X}_j| \Big) \end{aligned}$$

As for v_2, we can find its value locally, close to vortex ℓ. We thus rewrite equation (5.16) in terms of the inner variable r,

$$\epsilon^{-2}\Delta v_2 \sim -\epsilon^{-2}\frac{n_\ell^2}{r^2} - \epsilon^{-2}\frac{C_{2\ell}^2}{r^2} - \epsilon^{-1}n_\ell\frac{\mathbf{e}_\phi\cdot\nabla G_0(\mathbf{X}_\ell)}{r} - \epsilon^{-1}C_{2\ell}\frac{\mathbf{e}_r\cdot\nabla G_0(\mathbf{X}_\ell)}{r} + \ldots,$$

and by expanding $v_2 \sim v_{20} + \epsilon v_{21} + \epsilon^2 v_{22} + \ldots$ we find that

$$\begin{aligned} v_2 \sim &-\frac{1}{2}((q/\mu)^2 n_\ell^2 + C_{2\ell}^2)\log\epsilon r \\ &-(n_\ell(q/\mu)\mathbf{e}_\phi\cdot\nabla G_0(\mathbf{X}_\ell) + C_{2\ell}\mathbf{e}_r\cdot\nabla G_0(\mathbf{X}_\ell)\epsilon r\log\epsilon r + \mathcal{O}(\epsilon^2). \end{aligned}$$

With all this, the inner limit of the outer, in terms of R, reads

$$\begin{aligned} \chi_0^{outer} \sim &\frac{C_1}{q} + n_\ell\phi + \frac{\mu}{q}C_{2\ell}\log R + \frac{\mu}{q}C_{3\ell} + \frac{\mu}{q}G(\mathbf{X}_\ell) + \frac{\mu}{q}\nabla G_0(\mathbf{X}_\ell)\cdot\mathbf{X} \\ &-\frac{\mu^2}{q}\frac{1}{2}R\log R\big(\frac{q}{\mu}n_\ell\frac{d\mathbf{X}_\ell}{dT}\cdot\mathbf{e}_\phi + C_{2\ell}\frac{d\mathbf{X}_\ell}{dT}\cdot\mathbf{e}_r\big) - \frac{\mu^2}{q}G_1(\mathbf{X}_\ell) \\ &-\frac{\mu^2}{q}\frac{1}{2}((q/\mu)^2 n_\ell^2 + C_{2\ell}^2)\log R + \frac{\mu^2}{q}C_{4\ell}\log R \\ &-\frac{\mu^2}{q}(\frac{q}{\mu}n_\ell\mathbf{e}_\phi\cdot\nabla G_0(\mathbf{X}_\ell) + C_{2\ell}\mathbf{e}_r\cdot\nabla G_0(\mathbf{X}_\ell)R\log R + \mathcal{O}(R^2). \end{aligned} \tag{5.59}$$

We can now match this inner limit of the outer with the outer limit of the inner, given by (5.48). As it happened before, for a full matching we need to take infinitely many terms in both logarithmic series. However, we will then write the series in terms of one variable and compare their terms. Thus we can match providing we have all the terms in just one series and we write both expansions in terms of the other variable. In our case we have the full logarithmic expansion in the inner region, but only some terms in the logarithmic expansion of the outer region. We must therefore write both expansion in terms of the outer variable before comparing terms.

Expanding (5.48) in powers of q and writing the corresponding $\widehat{\chi}_{00}$,

$$\begin{aligned} \widehat{\chi}_{00} \sim &\frac{\log(A_{\ell 0}e^{-iqn_\ell\log\epsilon} + B_{\ell 0}e^{iqn_\ell\log\epsilon})}{q} + n_\ell\phi \\ &+\frac{A_{\ell 1}e^{-iqn_\ell\log\epsilon} + B_{\ell 1}e^{iqn_\ell\log\epsilon}}{A_{\ell 0}e^{-iqn_\ell\log\epsilon} + B_{\ell 0}e^{iqn_\ell\log\epsilon}} + in_\ell\frac{A_{\ell 0}e^{-iqn_\ell\log\epsilon} - B_{\ell 0}e^{iqn_\ell\log\epsilon}}{A_{\ell 0}e^{-iqn_\ell\log\epsilon} + B_{\ell 0}e^{iqn_\ell\log\epsilon}}\log R \\ &+q\Big(\frac{A_{\ell 2}e^{-iqn_\ell\log\epsilon} + B_{\ell 2}e^{iqn_\ell\log\epsilon}}{A_{\ell 0}e^{-iqn_\ell\log\epsilon} + B_{\ell 0}e^{iqn_\ell\log\epsilon}} + in_\ell\frac{A_{\ell 1}e^{-iqn_\ell\log\epsilon} - B_{\ell 1}e^{iqn_\ell\log\epsilon}}{A_{\ell 0}e^{-iqn_\ell\log\epsilon} + B_{\ell 0}e^{iqn_\ell\log\epsilon}}\log R \\ &-\frac{1}{2}n_\ell^2\log^2 R - \frac{1}{2}\frac{(A_{\ell 1}e^{-iqn_\ell\log\epsilon} + B_{\ell 1}e^{iqn_\ell\log\epsilon})^2}{(A_{\ell 0}e^{-iqn_\ell\log\epsilon} + B_{\ell 0}e^{iqn_\ell\log\epsilon})^2} \\ &-\frac{in_\ell}{2}\frac{(A_{\ell 0}e^{-iqn_\ell\log\epsilon} - B_{\ell 0}e^{iqn_\ell\log\epsilon})^2}{(A_{\ell 0}e^{-iqn_\ell\log\epsilon} + B_{\ell 0}e^{iqn_\ell\log\epsilon})^2}\log R\Big) + \mathcal{O}(q^2). \end{aligned} \tag{5.60}$$

Leading order matching (1ti)(1to)=(1to)(1ti). We can now match the leading order terms by comparing (5.60) with the leading order terms in ϵ in the expression for the outer (5.59). Then,

$$A_{\ell 0}e^{-iqn_\ell \log\epsilon} + B_{\ell 0}e^{iqn_\ell \log\epsilon} = e^{C_1} \tag{5.61}$$

$$\frac{A_{\ell 1}e^{-iqn_\ell \log\epsilon} + B_{\ell 1}e^{iqn_\ell \log\epsilon}}{A_{\ell 0}e^{-iqn_\ell \log\epsilon} + B_{\ell 0}e^{iqn_\ell \log\epsilon}} = \frac{\mu}{q}(C_3 + G_0(\mathbf{X}_\ell)) \tag{5.62}$$

$$in_\ell \frac{A_{\ell 0}e^{-iqn_\ell \log\epsilon} - B_{\ell 0}e^{iqn_\ell \log\epsilon}}{A_{\ell 0}e^{-iqn_\ell \log\epsilon} + B_{\ell 0}e^{iqn_\ell \log\epsilon}} = \frac{\mu}{q}C_{2\ell} \tag{5.63}$$

$$\begin{aligned}&\frac{A_{\ell 1}e^{-iqn_\ell \log\epsilon} - B_{\ell 1}e^{iqn_\ell \log\epsilon}}{A_{\ell 0}e^{-iqn_\ell \log\epsilon} + B_{\ell 0}e^{iqn_\ell \log\epsilon}} in_\ell \\ &- \frac{A_{\ell 1}e^{-iqn_\ell \log\epsilon} + B_{\ell 1}e^{iqn_\ell \log\epsilon}}{A_{\ell 0}e^{-iqn_\ell \log\epsilon} + B_{\ell 0}e^{iqn_\ell \log\epsilon}} \cdot \frac{A_{\ell 0}e^{-iqn_\ell \log\epsilon} - B_{\ell 0}e^{iqn_\ell \log\epsilon}}{A_{\ell 0}e^{-iqn_\ell \log\epsilon} + B_{\ell 0}e^{iqn_\ell \log\epsilon}} in_\ell \\ &= \frac{\mu^2}{q^2}C_{4\ell}\end{aligned} \tag{5.64}$$

Using these conditions the $\log^2 R$ terms do automatically match and they do not introduce any other condition on the constants.

First order matching (2ti)(1to)=(1to)(2ti). To match the ϵ terms in the outer and in the inner we must compare the terms in (5.59) that have a multiplying R with the first order outer limit of the inner that is given by

$$\begin{aligned}\widehat{\chi}_{10} = &- \frac{\mu}{4q}\frac{A_\ell e^{-iqn_\ell \log\epsilon}(V_1 - iV_2)R(R^{iqn_\ell} + \gamma_1 R^{-iqn_\ell})e^{i\phi}}{A_\ell R^{iqn_\ell}e^{-iqn_\ell \log\epsilon} + B_\ell R^{-iqn_\ell}e^{iqn_\ell \log\epsilon}} \\ &- \frac{\mu}{4q}\frac{B_\ell e^{iqn_\ell \log\epsilon}(V_1 + iV_2)R(R^{-iqn_\ell} + \gamma_2 R^{iqn_\ell})e^{-i\phi}}{A_\ell R^{iqn_\ell}e^{-iqn_\ell \log\epsilon} + B_\ell R^{-iqn_\ell}e^{iqn_\ell \log\epsilon}}.\end{aligned}$$

This last expression may be expanded as

$$\widehat{\chi}_{10} \sim -\frac{\mu}{4q}\frac{N_0/q + N_1 + N_2 q + \dots}{D_0/q + D_1 + D_2 q + \dots} \sim -\frac{\mu}{4q}\Big(\frac{N_0}{D_0} + q\big(\frac{N_1}{D_0} - \frac{N_0}{D_0}\frac{D_1}{D_0}\big)\Big),$$

where, by expanding the velocity like $V_i \sim V_{i0} + qV_{i1} + \dots$, for $i = 1, 2$ and the parameters γ_1 and γ_2 as $\gamma_i \sim \gamma_{i0} + q\gamma_{i1} + \dots$,

$$\begin{aligned}N_0 =& A_{\ell 0}e^{-iqn_\ell \log\epsilon}(V_{10} - iV_{20})R(1 + \gamma_{10})e^{i\phi} \\ &+ B_{\ell 0}e^{iqn_\ell \log\epsilon}(V_{10} + iV_{20})R(1 + \gamma_{20})e^{-i\phi}, \\ D_0 =& A_{\ell 0}e^{-iqn_\ell \log\epsilon} + B_{\ell 0}e^{iqn_\ell \log\epsilon}, \\ D_1 =& A_{\ell 1}e^{-iqn_\ell \log\epsilon} + B_{\ell 1}e^{iqn_\ell \log\epsilon} + in_\ell(A_{\ell 0}e^{-iqn_\ell \log\epsilon} - B_{\ell 0}e^{iqn_\ell \log\epsilon}) \log R,\end{aligned}$$

and

$$\begin{aligned} N_1 =& A_{\ell 0} e^{-iqn_\ell \log\epsilon}(V_{11} - iV_{21})R(1+\gamma_{10})e^{i\phi} \\ &+ B_{\ell 0} e^{iqn_\ell \log\epsilon}(V_{11} + iV_{21})R(1+\gamma_{20})e^{-i\phi} \\ &+ A_{\ell 1} e^{-iqn_\ell \log\epsilon}(V_{10} - iV_{20})R(1+\gamma_{10})e^{i\phi} \\ &+ B_{\ell 1} e^{iqn_\ell \log\epsilon}(V_{10} + iV_{20})R(1+\gamma_{20})e^{-i\phi} \\ &+ A_{\ell 0} e^{-iqn_\ell \log\epsilon}(V_{10} - iV_{20})Riqn_\ell(1-\gamma_{10})\log R e^{i\phi} \\ &- B_{\ell 0} e^{iqn_\ell \log\epsilon}(V_{10} + iV_{20})Riqn_\ell(1-\gamma_{20})\log R e^{-i\phi} \\ &+ A_{\ell 0} e^{-iqn_\ell \log\epsilon}(V_{10} - iV_{20})R\gamma_{11}e^{i\phi} \\ &+ B_{\ell 0} e^{iqn_\ell \log\epsilon}(V_{10} + iV_{20})R\gamma_{21}e^{-i\phi}. \end{aligned}$$

Therefore, matching the terms of order $1/q$ the parameters γ_{10} and γ_{20} are fixed and satisfy

$$k_1 - ik_2 = -\frac{1}{4\cos(qn_\ell \log\epsilon)} e^{-iqn_\ell \log\epsilon}(V_{10} - iV_{20}(1+\gamma_{10})), \tag{5.65}$$

$$k_1 + ik_2 = -\frac{1}{4\cos(qn_\ell \log\epsilon)} e^{iqn_\ell \log\epsilon}(V_{10} + iV_{20}(1+\gamma_{20})), \tag{5.66}$$

where we have used that $G_0(\mathbf{X}_\ell)$ is a two dimensional vector with components (k_1, k_2). Using this parameters and also the matching conditions found to leading order the coefficients of $\log R$ in the inner and the outer do match.

Finally we are ready to write $\widehat{\chi}_{10}$ back in terms of the inner variable r to give

$$\begin{aligned} \widehat{\chi}_{10} \quad = \quad & -r\frac{\mu}{4q}\frac{d\mathbf{X}_\ell}{dT}\cdot \mathbf{e}_r + r\frac{\mu}{4q}\big(V_1\cos(\phi - 2qn_\ell\log\epsilon) + V_2\sin(\phi - 2qn_\ell\log\epsilon)\big) \\ & + r\frac{\mu}{4q}\cos(qn_\ell\log\epsilon)\big(k_1\cos(\phi - qn_\ell\log\epsilon) + k_2\sin(\phi - qn_\ell\log\epsilon)\big). \end{aligned}$$

To find the overlap region where both the inner limit of the outer and outer limit of the inner are valid we write the corresponding outer limit of the inner for ψ that is

$$\begin{aligned} \psi \sim\ & e^{i(n_\ell\phi + \frac{\log C_1}{q})}\big(1 - i\epsilon r\frac{\mu}{4q}\frac{d\mathbf{X}_\ell}{dT}\cdot\mathbf{e}_r + i\epsilon r\frac{\mu}{4q}\big(V_1\cos(\phi - 2qn_\ell\log\epsilon) \\ & + V_2\sin(\phi - 2qn_\ell\log\epsilon) + i\epsilon r\frac{\mu}{4q}\cos(qn_\ell\log\epsilon)\big(k_1\cos(\phi - qn_\ell\log\epsilon) \\ & + k_2\sin(\phi - qn_\ell\log\epsilon)\big) + \mathcal{O}(1/r^2)\big)(1+\mathcal{O}(q)), \end{aligned} \tag{5.67}$$

which together with the inner limit of the outer shows that the overlap region is defined by

$$\frac{1}{\epsilon^{1/3}} < r < \frac{1}{\epsilon}.$$

Solvability Condition

In the same way that we did in Chapters 2 and 4, the first order equation, that is a linear equation, provides a solvability condition that forces the velocity to have a particular form.

We recall that equation (5.34) is of the form

$$L(q,\psi_0)[\psi_1] = b(\psi_0, q, \mu, d\mathbf{X}_\ell/dT),$$

where

$$L(q,\psi_0)[\psi_1] = (1-iq)\Delta\psi_1 + \psi_1(1-|\psi_0|^2) - \psi_0(\psi_0\psi_1^* + \psi_0^*\psi_1) \tag{5.68}$$

$$b(\psi_0, q, \mu, d\mathbf{X}_\ell/dT) = -\mu\frac{d\mathbf{X}_\ell}{dT}\cdot\nabla\psi_0. \tag{5.69}$$

Again, we define the inner product

$$\big(u, v\big) = \int_D \Re\{uv*\}\, dD, \tag{5.70}$$

where D is any given ball in $\mathbb{R}^2$. Using the integration by parts formula

$$\int_D u\nabla^2 v\, dD = \int_D v\nabla^2 u\, dD + \int_{\partial D}(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n})\, dl, \tag{5.71}$$

we can compute the adjoin of L and it is given by

$$\overline{L}(q,\psi_0)[v] = (1+iq)\Delta v + v(1-|\psi_0|^2) - \psi_0(\psi_0 v^* + \psi_0^* v). \tag{5.72}$$

Comparing the linear operators (5.68) and (5.72) we find that

$$\overline{L}(q,\psi_0)[v] = L(-q,\psi_0)[v]. \tag{5.73}$$

Choosing v to be the solution to the homogeneous equation

$$L(-q,\psi_0)[v] = 0,$$

and using the Fredholm Alternative, we obtain the same solvability condition as in Chapter 4, that is

$$\int_D \Re\{\mu\frac{dX_\ell}{dT}\cdot\nabla\psi_0 v^*\}dD = \int_{\partial D}\Re\{(1-iq)\big(v^*\frac{\partial\psi_1}{\partial n} - \frac{\partial v^*}{\partial n}\psi_1\big)\}dl. \tag{5.74}$$

Now, the solutions of $L(q,\psi_0)[v] = 0$ are actually the derivatives in any direction of ψ_0, that is $\nabla\psi_0\cdot\mathbf{d}$, being $\mathbf{d}$ any vector in $\mathbb{R}^2$. Then, in order to compute v we just need to differentiate ψ_0 and substitute q by $-q$.

To simplify the equation we can use that the left-hand side is $O(\mu)$ while the right-hand side is $O(1)$. Thus, to leading order, the solvability condition is

$$0 = \int_{\partial D} \Re\{(\mathbf{d}\cdot\nabla\psi_0)\frac{\partial\psi_1^*}{\partial n} - \frac{\partial(\mathbf{d}\cdot\nabla\psi_0)}{\partial n}\psi_1^*\}.$$

Using polar coordinates and choosing D to be a ball such that its radius lies in the overlap region, the solvability condition to leading order becomes

$$rhs = \int_0^{2\pi} \Re\{(1+iq)\big(v^*\frac{\partial\psi_1}{\partial r} - \frac{\partial v^*}{\partial r}\psi_1\big)\}r\,d\phi. \tag{5.75}$$

Now in the inner solution as $r\to\infty$ we have

$$\nabla\chi_0 \sim \frac{n_\ell}{r}\mathbf{e}_\phi - \frac{qn_\ell^2(\log r + C_1)}{r}\mathbf{e}_r + \cdots \tag{5.76}$$

$$f_0 \sim 1 + O(r^{-2}), \tag{5.77}$$

$$f_1 = -\frac{1}{2}|\nabla\chi_0|^2 \sim -\frac{n_\ell^2}{2r^2} \tag{5.78}$$

$$\begin{aligned}\chi_1 \sim &-r\frac{\mu}{4q}\frac{d\mathbf{X}_\ell}{dT}\cdot\mathbf{e}_r + r\frac{\mu}{4q}\big(V_1\cos(\phi - 2qn_\ell\log\epsilon)\\ &+ V_2\sin(\phi - 2qn_\ell\log\epsilon)\big)\\ &+ r\frac{\mu}{4q}\cos(qn_\ell\log\epsilon)\big(k_1\cos(\phi - qn_\ell\log\epsilon)\\ &+ k_2\sin(\phi - qn_\ell\log\epsilon)\big)\end{aligned} \tag{5.79}$$

$$v \sim i(\frac{n_\ell}{r}\mathbf{e}_\phi + \frac{n_\ell^2 q(\log r + C_1)}{r}\mathbf{e}_r + \cdots)\cdot\mathbf{d}\,e^{i\phi} \tag{5.80}$$

$$\frac{\partial v}{\partial r} \sim -i\frac{n_\ell}{r^2}\mathbf{e}_\phi\cdot\mathbf{d}e^{i\phi} \tag{5.81}$$

$$\psi_1 = (f_1 + if_0\chi_1)e^{i\chi_0} \tag{5.82}$$

$$\sim i\chi_1 e^{i\phi}. \tag{5.83}$$

If we substitute (5.76)-(5.83) in (5.75) we obtain, to leading order

$$\begin{aligned} rhs &\sim \Re\left\{\int_0^{2\pi}\left(\frac{1}{r}\mathbf{e}_\phi\cdot\mathbf{d}\frac{\partial\chi_1}{\partial r} + \frac{1}{r^2}\mathbf{e}_\phi\cdot\mathbf{d}\chi_1\right) r\,d\phi\right\}\\ &= \int_0^{2\pi}(\mathbf{e}_\phi\cdot\mathbf{d})\left(\frac{\partial\chi_1}{\partial r} + \frac{\chi_1}{r}\right)d\phi. \end{aligned} \tag{5.84}$$

5.4 Law of motion

Now, to obtain the law of motion we put expression (5.79) in the equation (5.84) from where we find that the velocity is given by

$$\frac{d\mathbf{X}_\ell}{dT} = 2\frac{\cos(qn_\ell\log\epsilon)}{\sin(qn_\ell\log\epsilon)}\nabla G_0(\mathbf{X}_\ell)^\perp, \tag{5.85}$$

where

$$\nabla G_0(\mathbf{X}) = \sum_{j\neq\ell}^{N}\Big(n_j\frac{q}{\mu}\frac{\mathbf{e}_{\phi j\ell}}{|\mathbf{X}_j-\mathbf{X}_\ell|} + C_{2j}\frac{\mathbf{e}_{rj\ell}}{|\mathbf{X}_j-\mathbf{X}_\ell|}\Big).$$

Therefore, using the matching conditions to substitute C_{2j} we find that:

Result 5.1 (Law of motion). *The velocity for spirals that are separated by distances such that qk/ϵ is small is given by*

$$\begin{aligned}\frac{d\mathbf{X}_\ell}{dT} =& 2\frac{q}{\mu}\frac{\cos(qn_\ell|\log\epsilon|)}{\sin(qn_\ell|\log\epsilon|)}\sum_{j\neq\ell}^{N}\Big(n_j\frac{\mathbf{e}_{rj\ell}}{|\mathbf{X}_j-\mathbf{X}_\ell|}\\ &+ n_j\frac{\sin(qn_j|\log\epsilon|)}{\cos(qn_j|\log\epsilon|)}\frac{\mathbf{e}_{\phi j\ell}}{|\mathbf{X}_j-\mathbf{X}_\ell|}\Big)\end{aligned} \tag{5.86}$$

where the time scale, $T=\epsilon^2\mu t$, is such that $\mu = 1/|\log\epsilon|$.

In appendix A we consider systems with two and three spirals in the middle scale and integrate numerically equations (5.86) to represent the trajectories of the spirals. We also analyse the effect of changing the value of q while remaining in the middle region.

Interpolation between $q = 0$ and the canonical separation. If we now take the limit as q tends to zero and use that

$$\sin(qn_j|\log\epsilon|)\sim qn_j|\log\epsilon| \quad\text{and}\quad \cos(qn_j|\log\epsilon|)\sim 1,$$

and also that $\mu = 1/|\log\epsilon|$, it is clear that we obtain the corresponding law for vortices when $q = 0$ (see equation (2.42)). The condition on the magnitude of μ is still not obvious, but it will be clear in the following section that $1/|\log\epsilon|$ is the right value of μ in order for the law to be self consistent.

On the other hand, for spirals with unitary winding number the law of motion is simply

$$\begin{aligned}\frac{d\mathbf{X}_\ell}{dT} =& 2n_\ell\frac{q}{\mu}\sum_{j\neq\ell}^{N}\Big(n_j\frac{\cos(q|\log\epsilon|)}{\sin(q|\log\epsilon|)}\frac{\mathbf{e}_{rj\ell}}{|\mathbf{X}_j-\mathbf{X}_\ell|}\\ &+\frac{\mathbf{e}_{\phi j\ell}}{|\mathbf{X}_j-\mathbf{X}_\ell|}\Big),\end{aligned}$$

and taking the limit as $q|\log\epsilon|$ goes to $\pi/2$ we obtain the law of motion that we found for α of order one (see equation (4.109)). This means that the law in (5.85) interpolates between the two regimes and it is actually the perturbation to the case of $q = 0$.

In what follows we will show that the law of motion given in (5.86) is only valid provided $q\log|\epsilon| < \pi/2$, for spirals with unitary degree, and as a consequence μ must be $1/|\log\epsilon|$ to ensure that q/μ is small in all the range where the law (5.86) is valid. This agrees with the idea that once the spirals reach the canonical separation, the law of motion has to be the one given in the previous section. Not only that, we will show that the asymptotic wavenumber is slowly changing and also becomes the one given in the previous chapter, when the spirals reach the canonical separation.

5.5 Boundary conditions and the asymptotic wavenumber

So far we have not found the corresponding asymptotic wavenumber for a system of N spirals in the middle region where α is of order less than one. To do so we must impose the boundary conditions at infinity, that we have not used yet. Thus, to impose the boundary conditions at infinity what we will do is find the right scale that place us "far enough" to impose them. In other words, we have assumed that α was a small parameter, but if we start the problem from the beginning by rescaling in such a way that the corresponding new value of $\overline{\alpha}$ becomes of order one, then all the centres of the spirals look as if they were at the origin. In this region, as we will show, we have the contribution of all of them in the same way as if we only had one spiral in the centre.

We then start by rescaling with a small unknown parameter δ such that $\overline{\alpha} = kq/\delta$ becomes of order one. We will see that this is actually the condition that is satisfied when we are far enough to "see" the corresponding asymptotic wavenumber.

Hence we rescale with $\overline{\mathbf{X}} = \delta\mathbf{x}$ and $\overline{T} = \mu\delta^2 t$. Writing $\psi = fe^{i\chi}$ and separating real and imaginary parts gives

$$\mu\delta^2 f_{\overline{T}} = \delta^2\Delta f - \delta^2 f|\nabla\chi|^2 + (1-f^2)f \tag{5.87}$$

$$\mu\delta^2 f^2\chi_{\overline{T}} = \delta^2\nabla\cdot(f^2\nabla\chi) + qf^2(1-f^2) - \delta^2\frac{\overline{\alpha}^2}{q}f^2 \tag{5.88}$$

Expanding in powers of δ as

$$f \sim f_0(q,\mu) + \delta^2 f_1(q,\mu) + \dots \tag{5.89}$$

$$\chi \sim \chi_0(q,\mu) + \delta^2\chi_1(q,\mu) + \dots \tag{5.90}$$

we find

$$f_0 = 1, \tag{5.91}$$

$$f_1 = -\frac{1}{2}|\nabla\chi_0|^2, \tag{5.92}$$

$$\mu\chi_{0T} = \nabla^2\chi_0 + q|\nabla\chi_0|^2 - \frac{\overline{\alpha}^2}{q}. \tag{5.93}$$

Again we expand χ_0 as

$$\chi_0 \sim \frac{1}{q}\chi_{00} + \frac{\mu}{q}\chi_{01} + \frac{\mu^2}{q}\chi_{02} + \dots,$$

and the leading order term satisfies

$$0 = \Delta\chi_{00} + |\nabla\chi_{00}|^2 - \overline{\alpha}^2,$$

that has the solution

$$\chi_0 \sim \frac{1}{q}\log(-\log(\overline{\alpha}/2) - \gamma - \log\overline{R}) + \mathcal{O}(\overline{R}^2), \tag{5.94}$$

as $\overline{R}$ becomes smaller. This expression should agree with what we find when we express the previous outer solution in terms of this new variable $\overline{R}$.

Leading order outer solution in terms of $\overline{R}$ The leading order outer equation that we had before was given by

$$\mu\chi_{0T} = \Delta\chi_0 + q|\nabla\chi_0|^2, \tag{5.95}$$

that expanded in terms of μ gives, to leading order

$$0 = \Delta\chi_{00} + |\nabla\chi_{00}|^2.$$

Now, the solution to this equation far from all the vortices, that is to say, in terms of the new variable $\overline{R}$ reads

$$\chi_0 \sim \frac{1}{q}\log\left(\overline{A}\big(\frac{\epsilon\overline{R}}{\delta}\big)^{iqn} + \overline{B}\big(\frac{\epsilon\overline{R}}{\delta}\big)^{-iqn}\right) + \dots. \tag{5.96}$$

If we now allow the unknown constants $\overline{A}$ and $\overline{B}$ to depend on q as

$$\overline{A} \sim \frac{1}{q}\overline{A}_0 + \overline{A}_1 + q\overline{A}_2 + \dots \quad \overline{B} \sim \frac{1}{q}\overline{B}_0 + \overline{B}_1 + q\overline{B}_2 + \dots,$$

and expand the solution in terms of q we find

$$\begin{aligned}\chi_0 \sim & \frac{1}{q}\log\Big(\frac{\overline{A}_0 e^{-iqn\log(\delta/\epsilon)} + \overline{B}_0 e^{iqn\log(\delta/\epsilon)}}{q} \\ & + in\log\overline{R}(\overline{A}_0 e^{-iqn\log(\delta/\epsilon)} - \overline{B}_0 e^{iqn\log(\delta/\epsilon)}) \\ & + \overline{A}_1 e^{-iqn\log(\delta/\epsilon)} + \overline{B}_1 e^{iqn\log(\delta/\epsilon)}\Big) + \dots. \end{aligned} \tag{5.97}$$

Comparing (5.94) and (5.97), we find the following conditions on the constants,

$$\overline{A}_0 e^{-iqn\log(\delta/\epsilon)} + \overline{B}_0 e^{iqn\log(\delta/\epsilon)} = 0, \tag{5.98}$$

$$in\log\overline{R}(\overline{A}_0 e^{-iqn\log(\delta/\epsilon)} - \overline{B}_0 e^{iqn\log(\delta/\epsilon)}) = -1, \tag{5.99}$$

$$\overline{A}_1 e^{-iqn\log(\delta/\epsilon)} + \overline{B}_1 e^{iqn\log(\delta/\epsilon)} = -\log(\overline{\alpha}/2) - \gamma. \tag{5.100}$$

By solving together equations (5.98) and (5.99), we find that

$$\overline{A}_0 = -\frac{1}{2in} e^{iqn\log(\delta/\epsilon)} \quad \overline{B}_0 = \frac{1}{2in} e^{-iqn\log(\delta/\epsilon)}. \tag{5.101}$$

Matching 'inwards' It is clear that the equations in (5.98), (5.99) and (5.100) are not enough to solve the eigenvalue problem for $\overline{\alpha}$. Thus, the next thing to do is compare the expression (5.96) with the outer solution in terms of R. This way we will relate $\overline{A}$ and $\overline{B}$ to the inner constants A_ℓ and B_ℓ and therefore, the matching conditions between the inner and the outer will also play a role here and will actually allow us to find the equation that $\overline{\alpha}$ satisfies. This, in turn, will provide the corresponding asymptotic wavenumber.

We then take (5.96), write it in terms of the outer variable R, and expand in terms of q to find

$$\begin{aligned}\chi_0 \sim & \frac{1}{q}\log\Big(\frac{\overline{A}_0+\overline{B}_0}{q}\Big) + in\log R\frac{\overline{A}_0-\overline{B}_0}{\overline{A}_0+\overline{B}_0} + \frac{\overline{A}_1+\overline{B}_1}{\overline{A}_0+\overline{B}_0} \\ & + q\Big(\frac{\overline{A}_2+\overline{B}_2}{\overline{A}_0+\overline{B}_0} + in\log R\frac{\overline{A}_1-\overline{B}_1}{\overline{A}_0+\overline{B}_0} - \frac{n^2}{2}\log^2 R + \frac{n^2}{2}\Big(\frac{\overline{A}_0-\overline{B}_0}{\overline{A}_0+\overline{B}_0}\Big)^2\log^2 R \\ & - \frac{1}{2}\Big(\frac{\overline{A}_1+\overline{B}_1}{\overline{A}_0+\overline{B}_0}\Big)^2 - in\log R\frac{\overline{A}_0-\overline{B}_0}{\overline{A}_0+\overline{B}_0}\frac{\overline{A}_1+\overline{B}_1}{\overline{A}_0+\overline{B}_0}\Big) + n\phi + \mathcal{O}(q^2).\end{aligned}$$

If we write the leading terms in the outer expansion for large values of all R_j, we realise that all R_j become the same and we call them just by R. The same happens with the angular variables that, to leading order, they are all equal and we will denote all of them by ϕ. By comparing this last expression with the leading terms in the outer for large values of R,

$$\chi_0 \sim \frac{C_1}{q} + \sum_{j=1}^{N} n_j\phi + \frac{\mu}{q}\sum_{j=1}^{N} C_{2j}\log R + \frac{\mu}{q}\sum_{j=1}^{N} C_{3j}$$
$$+ \frac{\mu^2}{q}\sum_{j=1}^{N} C_{4j}\log R - \frac{\mu^2}{2q}\big(\frac{q^2}{\mu^2}(\sum_{j=1}^{N} n_j)^2 + (\sum_{j=1}^{N} C_{2j})^2\big)\log^2 R + \dots,$$

we find that

$$\sum_{j=1}^{N} n_j = n, \tag{5.102}$$
$$\overline{A}_0 + \overline{B}_0 = qe^{C_1}, \tag{5.103}$$
$$in\frac{\overline{A}_0 - \overline{B}_0}{\overline{A}_0 + \overline{B}_0} = \frac{\mu}{q}\sum_{j=1}^{N} C_{2j}, \tag{5.104}$$
$$\frac{\overline{A}_1 + \overline{B}_1}{\overline{A}_0 + \overline{B}_0} = \frac{\mu}{q}\sum_{j=1}^{N} C_{3j}, \tag{5.105}$$
$$in\frac{\overline{A}_1 - \overline{B}_1}{\overline{A}_0 + \overline{B}_0} - in\frac{\overline{A}_0 - \overline{B}_0}{\overline{A}_0 + \overline{B}_0}\frac{\overline{A}_1 + \overline{B}_1}{\overline{A}_0 + \overline{B}_0} = \frac{\mu^2}{q^2}\sum_{j=1}^{N} C_{4j}. \tag{5.106}$$

and the $\log R$ terms do not introduce any other restriction on the constants. The first equation, (5.102) shows that the system of N spirals is seen at infinity as if there was only one spiral with a winding number that is the sum of the degrees of all of them. We now consider the case were the sum of the degrees is not zero, and we will study this other situation separately afterwards.

The matching conditions that we found for the inner and the outer were

$$A_{\ell 0}e^{-iqn_\ell \log\epsilon} + B_{\ell 0}e^{iqn_\ell \log\epsilon} = e^{C_1}, \tag{5.107}$$
$$\frac{A_{\ell 1}e^{-iqn_\ell \log\epsilon} + B_{\ell 1}e^{iqn_\ell \log\epsilon}}{A_{\ell 0}e^{-iqn_\ell \log\epsilon} + B_{\ell 0}e^{iqn_\ell \log\epsilon}} = \frac{\mu}{q}(C_3 + G_0(\mathbf{X}_\ell)), \tag{5.108}$$
$$in_\ell\frac{A_{\ell 0}e^{-iqn_\ell \log\epsilon} - B_{\ell 0}e^{iqn_\ell \log\epsilon}}{A_{\ell 0}e^{-iqn_\ell \log\epsilon} + B_{\ell 0}e^{iqn_\ell \log\epsilon}} = \frac{\mu}{q}C_{2\ell}, \tag{5.109}$$
$$\frac{A_{\ell 1}e^{-iqn_\ell \log\epsilon} - B_{\ell 1}e^{iqn_\ell \log\epsilon}}{A_{\ell 0}e^{-iqn_\ell \log\epsilon} + B_{\ell 0}e^{iqn_\ell \log\epsilon}}in_\ell$$
$$- \frac{A_{\ell 1}e^{-iqn_\ell \log\epsilon} + B_{\ell 1}e^{iqn_\ell \log\epsilon}}{A_{\ell 0}e^{-iqn_\ell \log\epsilon} + B_{\ell 0}e^{iqn_\ell \log\epsilon}} \cdot \frac{A_{\ell 0}e^{-iqn_\ell \log\epsilon} - B_{\ell 0}e^{iqn_\ell \log\epsilon}}{A_{\ell 0}e^{-iqn_\ell \log\epsilon} + B_{\ell 0}e^{iqn_\ell \log\epsilon}}in_\ell$$
$$= \frac{\mu^2}{q^2}C_{4\ell}, \tag{5.110}$$

and by comparing with the inner we already new that

$$A_{\ell 0} - B_{\ell 0} = 0, \tag{5.111}$$

$$\frac{(A_{\ell 1} - B_{\ell 1})}{A_{\ell 0} + B_{\ell 0}} i = - c_{n_\ell} n_\ell. \tag{5.112}$$

We first solve the problem for the simpler case where all $n_j = \pm 1$. In this case, we see from (5.111) and (5.107) that $A_{\ell 0}$ have the same value for all the spirals and it is given by

$$A_{\ell 0} = \frac{2qe^{C_1}}{\cos(q \log \epsilon)} = A_0.$$

From (5.108) we also have that the same happens for $C_{2\ell}$,

$$\frac{\mu}{q} C_{2\ell} = \frac{\sin(q \log \epsilon)}{\cos(q \log \epsilon)} = C_2.$$

Now, putting together (5.102), (5.104) and (5.107) and using the expressions we have just found for A_0 and C_2 we find that the following condition must hold,

$$\frac{\cos(qn \log(\delta/\epsilon))}{\sin(qn \log(\delta/\epsilon))} = \frac{N}{n} \frac{\sin(q \log \epsilon)}{\cos(q \log \epsilon)} \tag{5.113}$$

where $N = \sum_{j=1}^{N} |n_j|$, that in this case that all n_j are unitary is the same as saying that N stands for the number of vortices.

In the particular case where $n = N$, that is to say, all the spirals have the same plus one degree, the condition (5.113) is equivalent to

$$\cos(qn \log(\delta/\epsilon) + q \log \epsilon) = 0 \quad \text{and hence} \quad qn \log(\delta/\epsilon) + q \log \epsilon = -\frac{\pi}{2}.$$

Now, equation (5.100) along with what we know of A_{j1} and B_{j1} will give us the equation for the eigenvalue $\overline{\alpha}$. Therefore, we must find the way to express $\overline{A}_1$ and $\overline{B}_1$ in terms of the constants A_{j1} and B_{j1}. We thus use (5.105) and (5.108) to find

$$\overline{A}_1 + \overline{B}_1 = -\mu e^{C_1} \sum_{j=1}^{N} G_0(\mathbf{X}_j) + \sum_{j=1}^{N} (A_{j1} e^{-iqn_j \log \epsilon} + B_{j1} e^{iqn_j \log \epsilon}). \tag{5.114}$$

And upon using equation (5.106), (5.110), and (5.114) we obtain

$$\begin{aligned} in(\overline{A}_1 - \overline{B}_1) = -N\tan(q\log\epsilon)\mu e^{C_1}\sum_{j=1}^{N} G_0(\mathbf{X}_j) \\ + (N-1)\tan(q\log\epsilon)\sum_{j=1}^{N}(A_{j1}e^{-iqn_j\log\epsilon} + B_{j1}e^{iqn_j\log\epsilon}) \\ + \sum_{j=1}^{N}(A_{j1}e^{-iqn_j\log\epsilon} - B_{j1}e^{iqn_j\log\epsilon}). \end{aligned} \tag{5.115}$$

From these two equations we find that

$$\begin{aligned} 2in\overline{A}_1 = -\mu e^{C_1}(N\tan(q\log\epsilon) + in)\sum_{j=1}^{N} G_0(\mathbf{X}_j) \\ + ((N-1)\tan(q\log\epsilon) + in)\sum_{j=1}^{N}(A_{j1}e^{-iqn_j\log\epsilon} + B_{j1}e^{iqn_j\log\epsilon}) \\ + \sum_{j=1}^{N}(A_{j1}e^{-iqn_j\log\epsilon} - B_{j1}e^{iqn_j\log\epsilon}), \end{aligned} \tag{5.116}$$

$$\begin{aligned} 2in\overline{B}_1 = \mu e^{C_1}(N\tan(q\log\epsilon) - in)\sum_{j=1}^{N} G_0(\mathbf{X}_j) \\ - ((N-1)\tan(q\log\epsilon) - in)\sum_{j=1}^{N}(A_{j1}e^{-iqn_j\log\epsilon} + B_{j1}e^{iqn_j\log\epsilon}) \\ - \sum_{j=1}^{N}(A_{j1}e^{-iqn_j\log\epsilon} - B_{j1}e^{iqn_j\log\epsilon}). \end{aligned} \tag{5.117}$$

Determination of the eigenvalue $\overline{\alpha}$. Now, if we put these expressions into (5.100) we finally obtain:

Result 5.2. *The value of the eigenvalue $\overline{\alpha}$ is given by*

$$N\frac{c_1}{n^2}\frac{\sin^2(qn\log(\delta/\epsilon)}{\cos^2(q\log\epsilon)} = \log(\overline{\alpha}/2) + \gamma, \tag{5.118}$$

where c_1 is the constant in (5.112) when $n_\ell = 1$. Therefore $\overline{\alpha}$ depends on the separation of the spirals. When all the spirals have a positive unitary winding number, that is to say, $n = N$, this expression simplifies to

$$\overline{\alpha} = \frac{kq}{\delta} = 2e^{c_1/n-\gamma}, \tag{5.119}$$

where the scale parameter δ *is given by* (5.113).

This last expression is actually the same expression that we found for an n-spiral in isolation, except for the constant term c_1 that is slightly different in this case due to the fact that the spirals are indeed separated. We also note that $\overline{\alpha}$ is an order one magnitude, as we had assumed at the beginning.

The asymptotic wavenumber is therefore defined by (5.113) along with (5.119). We note that the relation between α and $\overline{\alpha}$ is given by $\alpha = \overline{\alpha}\delta/\epsilon$. Therefore, our assumption on α to be of order less than one is only valid provided δ/ϵ is also of order less than one. By inspecting the expression (5.113) we find that this is only true provided $q|\log\epsilon| < \pi/2$.

In the particular case that $n = N$, the asymptotic wavenumber can also be found explicitly and it is given by

$$k(q) = \frac{2}{q}\epsilon^{1-\frac{1}{n}}e^{-\frac{\pi}{2qn}+\frac{c_1}{n}-\gamma}(1+o(1)), \tag{5.120}$$

while the relation between δ and ϵ is $\delta = \epsilon^{1-\frac{1}{n}}e^{-\frac{\pi}{2qn}}$. This shows that

$$\frac{\delta}{\epsilon} = e^{-\frac{\pi}{2qn}-\frac{1}{n}\log\epsilon},$$

and therefore, it is also clear that δ is much smaller than ϵ only if $q|\log\epsilon| < \pi/2$.

Asymptotic wavenumber when $n = 0$ We now consider the situation where the total sum of the winding numbers of all the spirals is zero. In this case, the solution in terms of the new variables $\overline{X}$ and $\overline{T}$ is again

$$\chi_0 \sim \frac{1}{q}\log(-\log(\overline{\alpha}/2) - \gamma - \log\overline{R}) + \dots. \tag{5.121}$$

The difference lies in the outer solution when it is written in terms of $\overline{R}$ that it should be expressed as

$$\chi_0 = \frac{1}{q}\log\big(\overline{A}\log(\epsilon\overline{R}/\delta) + \overline{B}\big). \tag{5.122}$$

Again we allow the unknown constants to depend on q like

$$\overline{A} \sim \frac{1}{q}\overline{A}_0 + \overline{A}_1 + q\overline{A}_2 + \dots \quad \overline{B} \sim \frac{1}{q}\overline{B}_0 + \overline{B}_1 + q\overline{B}_2 + \dots,$$

and we must remember that we are assuming $q\log(\epsilon/\delta)$ to be an order one magnitude. Therefore, the expansion of (5.122) now reads

$$\begin{aligned}\chi_0 \sim &\frac{1}{q}\log\big(\frac{\overline{B}_0}{q} + \overline{A}_1\log(\epsilon/\delta) + \frac{\overline{A}_0}{q}\log(\epsilon/\delta) + \overline{A}_1\log\overline{R}\\ &+ \overline{B}_1 + \overline{A}_2 q\log(\epsilon/\delta) + \dots\big).\end{aligned} \tag{5.123}$$

Thus, by comparing (5.121) with (5.123) we find that

$$\overline{A}_0 = 0, \tag{5.124}$$

$$\overline{B}_0 + \overline{A}_1 q \log(\epsilon/\delta) = 0, \tag{5.125}$$

$$\overline{A}_1 = -1, \tag{5.126}$$

$$\overline{B}_1 + \overline{A}_2 q \log(\epsilon/\delta) = -\gamma - \log(\overline{\alpha}/2). \tag{5.127}$$

We now compare (5.122) with the expression for the outer that we have in terms of R. Thus, we take (5.122), write it in terms of R and expand again in q to find

$$\begin{aligned}\chi_0 \sim & \frac{1}{q}\log\big(\frac{\overline{B}_0}{q}\big) + \frac{\overline{A}_1}{\overline{B}_0}\log R + \frac{\overline{B}_1}{\overline{B}_0} \\ & + q\Big(\frac{\overline{A}_2}{\overline{B}_0}\log R + \frac{\overline{B}_2}{\overline{B}_0} - \frac{1}{2}\big(\frac{\overline{A}_1}{\overline{B}_0}\log R + \frac{\overline{B}_1}{\overline{B}_0}\log(R)\big)^2\Big).\end{aligned} \tag{5.128}$$

Finally, we take the outer as we had it before, write it in terms of $\overline{R}$, expand again in ϵ, and write the expansion back in R to obtain

$$\begin{aligned}\chi_0 \sim & \frac{C_1}{q} + \frac{\mu}{q}\sum_{j=1}^{N} C_{2\ell}\log R + \frac{\mu}{q}\sum_{j=1}^{N} C_{3\ell} \\ & + \frac{\mu^2}{q}\sum_{j=1}^{N} C_{4\ell}\log R - \frac{\mu^2}{q}\big(\sum_{j=1}^{N} C_{2\ell}\big)\log^2 R + \ldots,\end{aligned} \tag{5.129}$$

which gives the following relations

$$C_1 = \log\Big(\frac{\overline{B}_0}{q}\Big), \tag{5.130}$$

$$\frac{\overline{A}_1}{\overline{B}_0} = \frac{\mu}{q}\sum_{j=1}^{N} C_{2\ell}, \tag{5.131}$$

$$\frac{\overline{B}_1}{\overline{B}_0} = \frac{\mu}{q}\sum_{j=1}^{N} C_{3\ell}, \tag{5.132}$$

$$\frac{\overline{A}_2}{\overline{B}_0} - \frac{\overline{A}_1}{\overline{B}_0}\frac{\overline{B}_1}{\overline{B}_0} = \frac{\mu^2}{q^2}\sum_{j=1}^{N} C_{4\ell}, \tag{5.133}$$

and considering again the case of $n_j = \pm 1$, equations (5.133) and (5.134) simplify very much and we find that

$$1 = N\tan(q\log\epsilon) q \log(\delta/\epsilon), \tag{5.134}$$

which gives us the relation between ϵ and the new parameter δ and again shows that if $q\log\epsilon$ becomes $-\pi/2$, then $\delta = \epsilon$. Using now the matching conditions (5.107)-(5.110) we find that

$$\overline{B}_1 = \sum_{j=1}^{N}(A_{\ell 1}e^{-iqn_\ell \log\epsilon} + B_{\ell 1}e^{iqn_\ell \log\epsilon}) - \mu e^{C_1}\sum_{j=1}^{N} G(\mathbf{X}_\ell), \tag{5.135}$$

$$\begin{aligned}\overline{A}_2 =& N\tan(q\log\epsilon)\overline{B}_1 - \tan(q\log\epsilon)\sum_{j=1}^{N}(A_{\ell 1}e^{-iqn_\ell \log\epsilon} + B_{\ell 1}e^{iqn_\ell \log\epsilon}) \\ &+ \sum_{j=1}^{N}(A_{\ell 1}e^{-iqn_\ell \log\epsilon} - B_{\ell 1}e^{iqn_\ell \log\epsilon}) i n_\ell. \end{aligned} \tag{5.136}$$

If we now substitute $\overline{A}_2$ and $\overline{B}_1$ into (5.127) and use also (5.134) we find that the eigenvalue $\overline{\alpha}$ satisfies

$$\frac{c_1}{(N\sin(q\log\epsilon))^2} = \gamma + \log(\overline{\alpha}/2). \tag{5.137}$$

and using that $\overline{\alpha} = kq/\delta$, we find that the asymptotic wavenumber is given by

$$k(q) = \frac{2\epsilon}{q} e^{-\frac{1}{Nq}\frac{\cos(q\log\epsilon)}{\sin(q\log\epsilon)} - \gamma + \frac{c_1}{(N\sin(q\log\epsilon))^2}}. \tag{5.138}$$

Analysis when not all the winding numbers are unitary. For the most general situation of having a set of well separated spirals with arbitrary degrees, we can still compute the equation that the eigenvalue $\overline{\alpha}$ satisfies. The main difference lies in the fact that it is not true now that $n_\ell \sin(qn_\ell \log\epsilon) = \sin(q\log\epsilon)$ or $\cos(qn_\ell \log\epsilon) = \cos(q\log\epsilon)$. As a consequence, the relation between δ and ϵ, for $n \neq 0$, is now given by

$$\frac{\cos(qn\log(\delta/\epsilon))}{\sin(qn\log(\delta/\epsilon))} = \frac{1}{n}\sum_{j=1}^{N} n_j \tan(qn_j \log\epsilon), \tag{5.139}$$

where again

$$\sum_{j=1}^{N} n_j = n.$$

As a consequence the relation between $\overline{A}_1$, $\overline{B}_1$ and $A_{\ell 0}$, $B_{\ell 0}$ reads now

$$\overline{A}_1 + \overline{B}_1 = \mu e^{C_1} \sum_{j=1}^{N} G(\mathbf{X}_j) + \sum_{j=1}^{N} (A_{j1} e^{-iqn_j \log \epsilon} + B_{j1} e^{-iqn_j \log \epsilon}) \tag{5.140}$$

$$\begin{aligned} in(\overline{A}_1 - \overline{B}_1) = & (\overline{A}_1 + \overline{B}_1) \sum_{j=1}^{N} n_j \tan(qn_j \log \epsilon) \\ & + \sum_{j=1}^{N} (A_{j1} e^{-iqn_j \log \epsilon} - B_{j1} e^{-iqn_j \log \epsilon}) in_j \\ & - \sum_{j=1}^{N} n_j \tan(qn_j \log \epsilon)(A_{j1} e^{-iqn_j \log \epsilon} + B_{j1} e^{-iqn_j \log \epsilon}) \end{aligned} \tag{5.141}$$

and the eigenvalue is now given by

$$\log(\overline{\alpha}/2) + \gamma = \frac{1}{n^2} \sum_{j=1}^{N} n_j^2 c_{nj} \tan^2(qn_j \log(\delta/\epsilon)). \tag{5.142}$$

We observe that in particular, for a system with a single spiral with a winding number of n_j, we obtain again the same relation that we had found in Chapter 3 when we dealt with single spirals in isolation.

On the other hand, when $n = 0$, we find that the relation between δ and ϵ reads

$$q \log(\delta/\epsilon) \sum_{j=1}^{N} n_j \tan(qn_j \log \epsilon) = 1, \tag{5.143}$$

and the eigenvalue now is

$$\log(\overline{\alpha}/2) + \gamma = \sum_{j=1}^{N} n_j^2 c_{nj} \frac{(q \log(\epsilon/\delta))^2}{\cos^2(qn_j \log(\delta/\epsilon))}. \tag{5.144}$$

Finally, if we inspect (5.139) and (5.143) we observe that the values of ϵ that guarantee that the spirals are in the middle region where α is of order less than one are not so simply defined as before. On the contrary, the condition on the distances of separation of the spirals is now given by ϵ to be such that

$$\sum_{j=1}^{N} n_j \tan(qn_j \log \epsilon) \neq 0.$$

As soon as this value vanishes it is clear from expressions (5.139) and (5.143) that $\delta = \epsilon$ and therefore we have reached the canonical scale.

Chapter 6

Some conclusions and final remarks

The aim of this last chapter is to give a 'global picture' of the way the interaction between spirals takes place as the parameter q moves from $q = 0$, that is the Ginzburg-Landau equation with real coefficients, to $q = -\pi/(2\log\epsilon)$, and also to show that the results in this thesis are also valid when the original parameter b is not zero and we also discuss about the changes that a non-vanishing value of b would introduce in higher order terms in the spiral's law of motion.

We will first go back to the expressions that we found for the leading order phase function, χ_0, and we will use them to show what happens when we take larger distances of separation for the spirals. We will see that we can interchange the role of the parameter q and the length scale ϵ.

6.1 Equivalence between the parametric problem in q and the distance of separation of the spirals

In the previous chapters we introduced the new parameter α as a function of the unknown asymptotic wavenumber k in the shape of $\alpha = kq/\epsilon$. In terms of this new parameter, the equation for the leading order phase, χ_0, reads

$$\chi_{0\mu} = \Delta\chi_0 + q|\nabla\chi_0|^2 - \frac{\alpha^2}{q}. \tag{6.1}$$

In this equation we had already shown that α is actually an unknown parameter that depends on q and that, given a configuration of vortices, it is uniquely defined. In the so called *canonical scale* (see Chapter 4) in which the typical spiral separation is $1/\epsilon = \mathcal{O}(e^{\pi/(2q|n_\ell)})$, the parameter α was supposed to be an order one parameter and was indeed found to be

$$\alpha = 2e^{c_1-\gamma+K_0(\alpha|\mathbf{X}_1-\mathbf{X}_2|)}(1+o(1)),$$

for a system with two vortices with a degree of either plus or minus one. The consistency of the result gives its validity since it is in fact an order one parameter, as it was assumed to be at the beginning. On the other hand we found that in this scale ϵ takes the specific value of

$$q|\log\epsilon| = \frac{\pi}{2}.$$

At a first glance this condition may be a bit surprising since one usually does not expect to find such a restricting condition on ϵ. But actually, if the spirals approach in such a way that this relation stops being true, say for instance that we are in a situation where

$$q|\log\epsilon| = \frac{\pi}{2} + a,$$

the effect of this change on the final value of the corresponding α is such that not only α stops being small but it becomes exponentially small in q. This is seen by recalling some results from what we have named the *middle scale* (see Chapter 5). We started by assuming that α was small to find that it was indeed given by

$$\alpha = 2e^{c_1-\gamma}e^{-\frac{a}{qn}}(1+o(1)),$$

so α is now exponentially small in q. This shows that the transition from the canonical to the middle scale is somehow singular due to the fact that if we think of α as a function of ϵ, there is a critical ϵ^* where α jumps from being almost zero to become an order one constant (see figure 6.1). Furthermore, this critical length scale, ϵ^*, depends on q so that given $q_1 > q_2$, the corresponding critical lengths are given by $\epsilon_1^* > \epsilon_2^*$, as it is also shown in figure 6.1. This means that in the limit where $q \to 0$ the corresponding critical length would be at infinity. In other words, when we consider the real Ginzburg-Landau equation ($q = 0$), the corresponding canonical scale would be at infinity. It is now clear that it is equivalent to study the parametric problem of changing q and once q is fixed to take the spirals further apart which would mean that we are changing ϵ.

6.2 Change of the isophase lines when moving from the middle to the canonical scale

One important feature of the problem that has enabled us to find a law of motion for the spirals is the fact that the inner region equations happen to

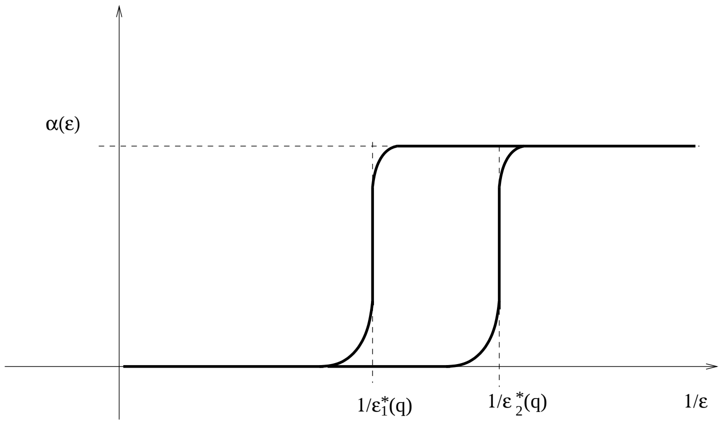

Figure 6.1: Representation of α as a function of ϵ

be exactly the same for the middle and canonical scale. Actually, the leading-order inner equation is the same one that we find when considering a spiral in complete isolation. This is due to the fact that when the spirals are far apart they only interact weakly producing only small perturbations on the core of each spiral.

Restricting ourselves to the case of unitary winding numbers, figure 6.2 show how the inner solution vector, ψ, looks like close to spirals of degree plus and minus one respectively. Now, if we analyse the structure of the far field

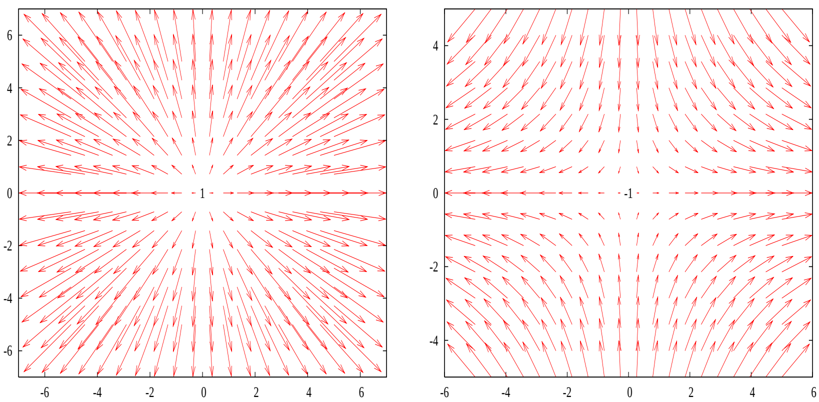

Figure 6.2: Inner solution for a spiral of degree $+1$ (left) and -1 (right)

for the three different possibilities:

$$\chi_0 \sim \sum_{j=1}^{N}(n_j\phi_j) \quad \text{when } q = 0, \tag{6.2}$$

$$\chi_0 \sim \sum_{j=1}^{N}(n_j\phi_j + \tan(q\log\epsilon)\log R_j) \quad \text{in the middle scale} \tag{6.3}$$

$$\chi_0 \sim \frac{1}{q}\log\big(\sum_{j=1}^{N}(\beta_j K_0(\alpha R_j))\big) \quad \text{in the canonical scale} \tag{6.4}$$

we realise that the weights of the radial and angular parts switch from balancing, in the middle region, to a dominance of the angular part, when $q = 0$, and to a dominance of the radial part, in the canonical scale. To understand how this affects the contour lines in the far field, we have plotted the outer solution for $q = 0$ in picture 6.3, and the far fields in the canonical and middle region in figures 6.5 and 6.4. These plots are representations of the contour lines for the far field phase solutions, that is to say, the isophase lines where χ_0 takes a constant value. The figures show that the middle scale is very similar to the $q = 0$, it is actually the perturbation to the Ginzburg-Landau vortices. But on the contrary and roughly speaking we could say that when the spirals sit in the canonical scale (figure 6.5) the spiral branch has completely 'bent' and the wavelength is now 'noticeable' to the other vortex.

If we now consider pairs of spirals in the middle and canonical scale, we note that when we are in the canonical scale the isophase lines arrange in such a way that one can guess that some sort of "ridge line" starts to appear in the line that lies in the middle of the two vortices (see figures 6.6 and 6.7). This is actually what happens for values of q that are of order one, as it is explained in [10]. Indeed, if we consider larger values of q, the corresponding critical ϵ^* that indicates the beginning of the canonical scale becomes eventually too large, which means that the spirals are too close and our analysis stops being valid. But nevertheless, from our analysis one can guess that as q becomes larger the middle region may no longer make sense and the spirals will actually be even "further" than our canonical scale. This seems to imply that for values of q of order one the spirals would interact in an exponentially weak way leading to metastable patterns, like the ones presented in figure 6.12 [1], which correspond to the numerically integrated equation for order one values of q. This phenomenon of metastability of the many-spiral solutions for order one values of q has already been observed in numerical experiments (see [10]), but, to our knowledge, the problem of the interaction of spirals for general values of q remains still not well understood. Nevertheless, some authors (see for

[1] Simulation obtained from $www - chaos.umd.edu/gallery/pattern.html$

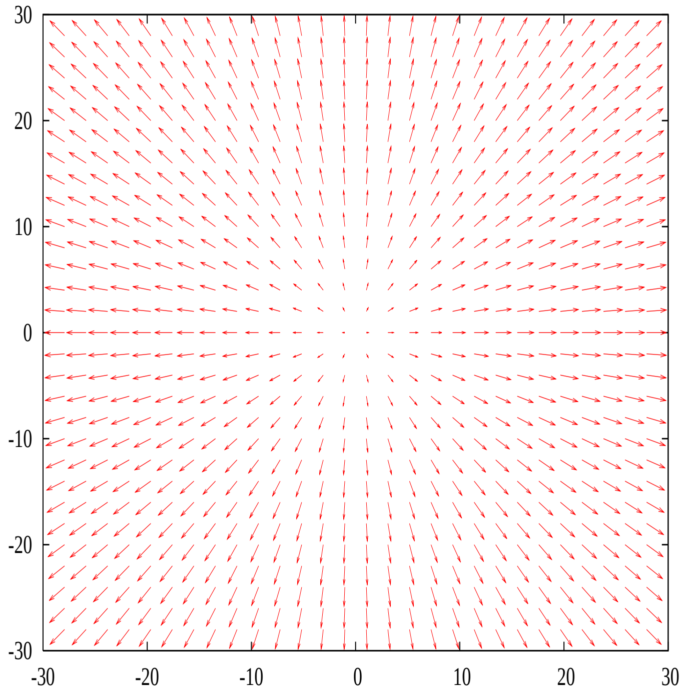

Figure 6.3: Single spiral isophase lines for the Ginzburg-Landau equation

instance, [9], [3] and [4]) have contributed to this problem by giving partial answers, using asymptotic analysis techniques, to the problem of interaction of spiral waves in the complex Ginzburg-Landau equation and comparing their results with the numerically obtained solutions. However, since the validity of asymptotic analysis results rely so much on the fact of being self-consistent, partial answers are quite likely to be not completely correct. On the other hand, if we inspect the far field solution when the two spirals lie in the middle region we notice that the isophase lines are only a small deformation of the ones that we had when we considered the Ginzburg-Landau equation ($q = 0$). In figures 6.8 and 6.9 we plot the isophase lines and the far field solution for a pair of spirals with the same degree, while in figures 6.10 and 6.11 we plot spirals with opposite winding number.

6.3 About the extension of all previous results to general values of the parameter b

For the sake of clarity in Chapters 4 and 5 we have taken the parameter b that was initially in equation (3.1) to be zero. But the extension to general values of b is quite straight forward and in fact it does not introduce any change to

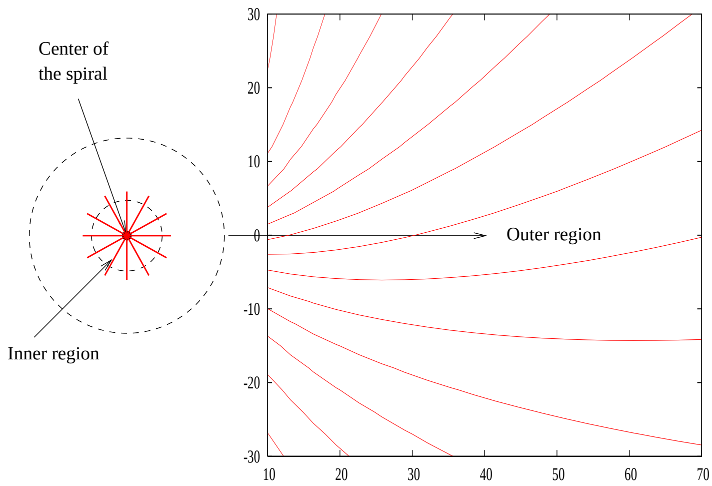

Figure 6.4: Isophase lines for a spiral in the middle region

the results in this thesis. In what follows we show the effect of a non-vanishing value of b on the calculations.

The outer equation including the parameter b reads

$$\epsilon^2(1-ib)\mu\psi_T = \epsilon^2\Delta\psi + (1+iq)(1-|\psi|^2)\psi - \frac{i\epsilon^2\alpha^2}{q}\psi. \tag{6.5}$$

and in terms of modulus f and phase χ becomes

$$\mu\epsilon^2(f_T + bf\chi_T) = \epsilon^2\Delta f - \epsilon^2 f|\nabla\chi|^2 + (1-f^2)f, \tag{6.6}$$

$$\mu\epsilon^2(f^2\chi_T - bff_T) = \epsilon^2\nabla\cdot(f^2\nabla\chi) + qf^2(1-f^2) - \epsilon^2\frac{\alpha^2}{q}f^2. \tag{6.7}$$

Upon expanding χ and f in powers of ϵ in the same way that we did in Chapters 4 and 5 we find that the leading order terms satisfy the equations

$$f_0 = 1,$$
$$\mu\chi_{0T} = \Delta\chi_0 + q|\nabla\chi_0|^2 - \alpha^2/q.$$

where the parameter b does not appear anymore and therefore the rest of the outer calculation remains the same.

As for the inner equations close to the ℓ-th vortex when $b \neq 0$ they are

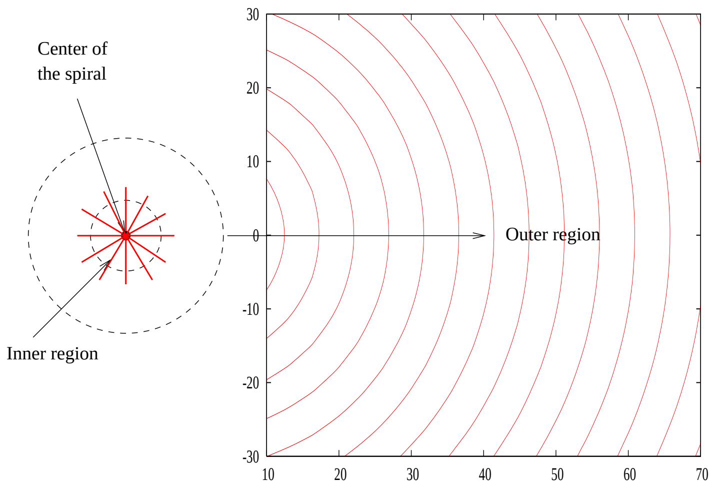

Figure 6.5: Isophase lines for a spiral in the canonical region

given by

$$\epsilon\mu\big(\epsilon f_T - \frac{d\mathbf{X}_\ell}{dT}\cdot\nabla f + \epsilon b f\chi_T - bf\frac{d\mathbf{X}_\ell}{dT}\cdot\nabla\chi\big),$$
$$= \Delta f - f|\nabla\chi|^2 + (1-f^2)f \tag{6.8}$$
$$\epsilon\mu\big(b\frac{d\mathbf{X}_\ell}{dT}\cdot f\nabla f - \epsilon b f f_T + \epsilon f^2\chi_T - f^2\frac{d\mathbf{X}_\ell}{dT}\cdot\nabla\chi\big)$$
$$= \nabla\cdot(f^2\nabla\chi) + q(1-f^2)f - \frac{\epsilon^2\alpha^2}{q}, \tag{6.9}$$

and expanding in powers of ϵ we obtain, to leading order, the equation for ψ_0

$$0 = \Delta\psi_0 + (1+iq)\psi_0(1-|\psi_0|^2), \tag{6.10}$$

and for the first order term ψ_1 we have now the new equation

$$-\mu(1-ib)\frac{d\mathbf{X}_\ell}{dT}\cdot\nabla\psi_0 = \Delta\psi_1 + (1+iq)(\psi_1(1-|\psi_0|^2) - \psi_0(\psi_0\psi_1^* + \psi_0^*\psi_1)), \tag{6.11}$$

that expressed in terms of the expanded modulus and phase terms f_1 and χ_1 gives the set of equations

$$-\mu\frac{d\mathbf{X}_\ell}{dT}\cdot(\nabla f_0 + bf_0\nabla\chi_0) = \Delta f_1 - f_1|\nabla\chi_0|^2 - 2f_0\nabla\chi_0\cdot\nabla\chi_1$$
$$+ f_1 - 3f_0^2 f_1, \tag{6.12}$$
$$-\mu\frac{d\mathbf{X}_\ell}{dT}\cdot(f_0^2\nabla\chi_0 - bf_0\nabla f_0) = \nabla\cdot(f_0^2\nabla\chi_1) + \nabla\cdot(2f_0 f_1\nabla\chi_0) \tag{6.13}$$
$$+ 2qf_0f_1 - 4qf_0^3 f_1.$$

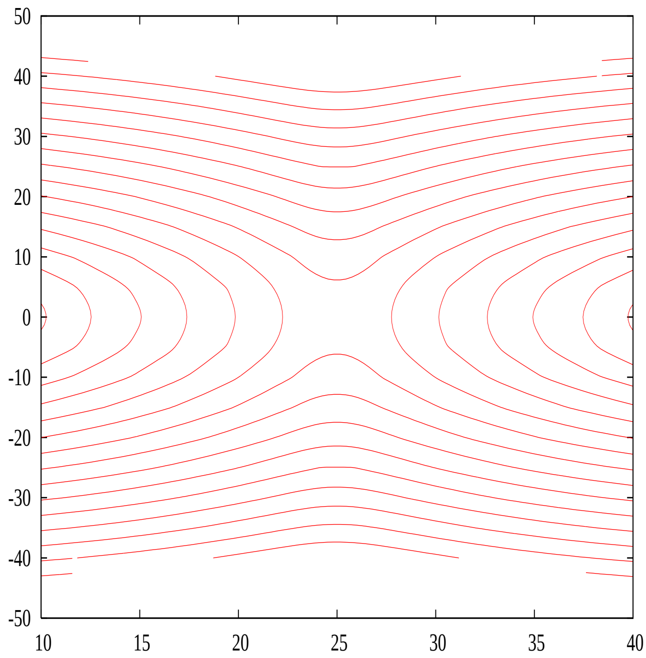

Figure 6.6: Isophase lines for two spirals in the canonical region

Upon expanding in ϵ as

$$\chi_1 \sim \frac{\widehat{\chi}_{10}(q)}{\epsilon} + \widehat{\chi}_{11}(q) + \cdots , \tag{6.14}$$

$$f_1 \sim \widehat{f}_{10}(q) + \epsilon \widehat{f}_{11}(q) + \cdots , \tag{6.15}$$

we find the equation for $\widehat{\chi}_{10}$ given by

$$-\mu \frac{d\mathbf{X}_\ell}{dT} \cdot \nabla \widehat{\chi}_{00} = \Delta \widehat{\chi}_{10} - 2q \widehat{f}_{11} = \Delta \widehat{\chi}_{10} + 2q \nabla \widehat{\chi}_{00} \cdot \nabla \widehat{\chi}_{10}, \tag{6.16}$$

where again b has disappeared due to the fact that it was multiplying some higher order term. This implies that the whole process of matching stays the same for general values of b.

Finally we must derive the law of motion when b is not zero. To do so we start by defining the same inner product as before and consider the first order linear operator given in (6.11) that is of the form

$$L(q, \psi_0)[\psi_1] = w(\psi_0, q, \mu, b, d\mathbf{X}_\ell/dT)$$

where L does not depend on b but where the non-homogeneous term is now of the form

$$w(\psi_0, q, \mu, b, d\mathbf{X}_\ell/dT) = -\mu(1 - ib)\frac{d\mathbf{X}_\ell}{dT} \cdot \nabla \psi_0.$$

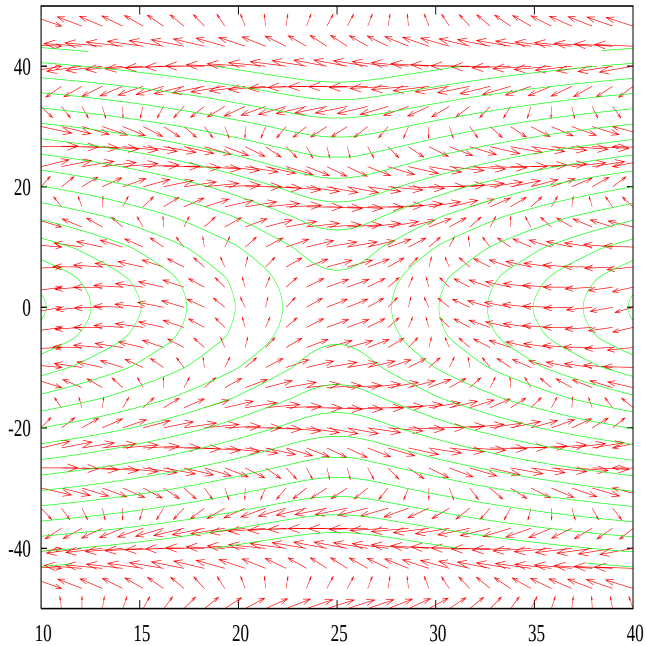

Figure 6.7: Far field for two spirals in the canonical region

Since the linear operator L is the same that was in Chapters 4 and 5 the only place where b has an effect is in the solvability condition itself that is obtained through the Fredholm Alternative that now reads

$$-\int_D \Re\{\mu(1-ib)\frac{d\mathbf{X}_\ell}{dT}\cdot\nabla\psi_0 v^*\}dD = \int_{\partial D} \Re\{(1-iq)\big(v^*\frac{\partial\psi_1}{\partial n}-\frac{\partial v^*}{\partial n}\psi_1\big)\}dl. \qquad (6.17)$$

Since b is only part of the left hand side that is a term of order μ it will not be present in the leading order terms coming out of the solvability condition unless b depends on ϵ. But if we consider that it is not the case, that b is just a constant of the problem, the leading order velocity is not affected by the value of b and only the next term in the velocity will have a term coming from the left hand side of equation (6.17) with b as a multiplying constant.

If b depends on ϵ then essentially the same analysis that was done in Chapter 2 holds. This means that only when b is of order greater or equal than $\log\epsilon$ does the leading order law of motion change.

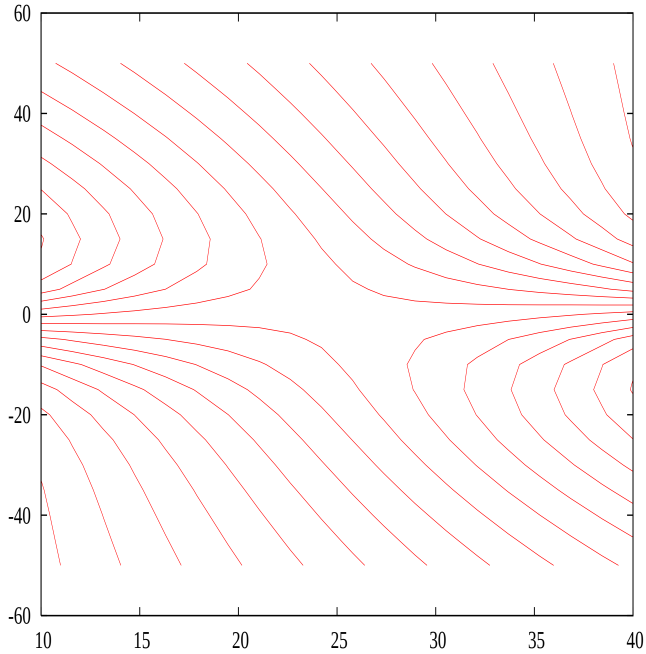

Figure 6.8: Isophase lines for two spirals in the middle region when both have +1 degree

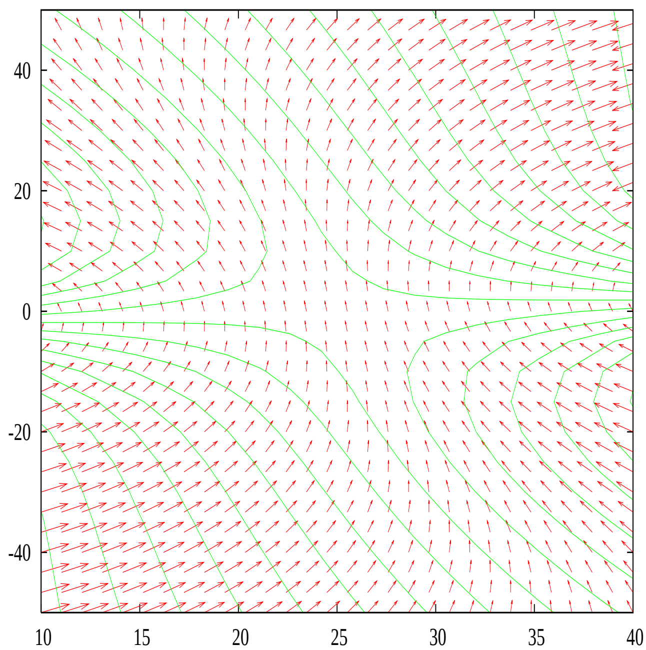

Figure 6.9: Isophase lines for spirals in the middle region when both have +1 degree

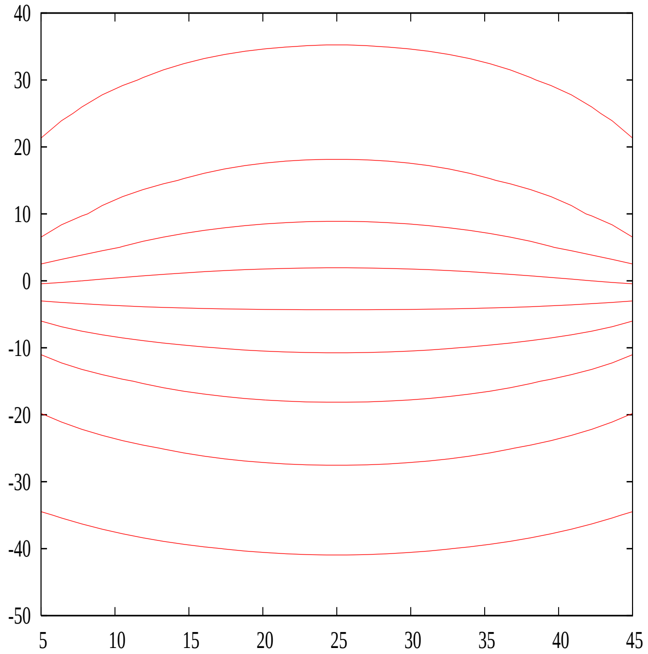

Figure 6.10: Isophase lines for two spirals in the middle region when the left one has $+1$ degree and the right -1

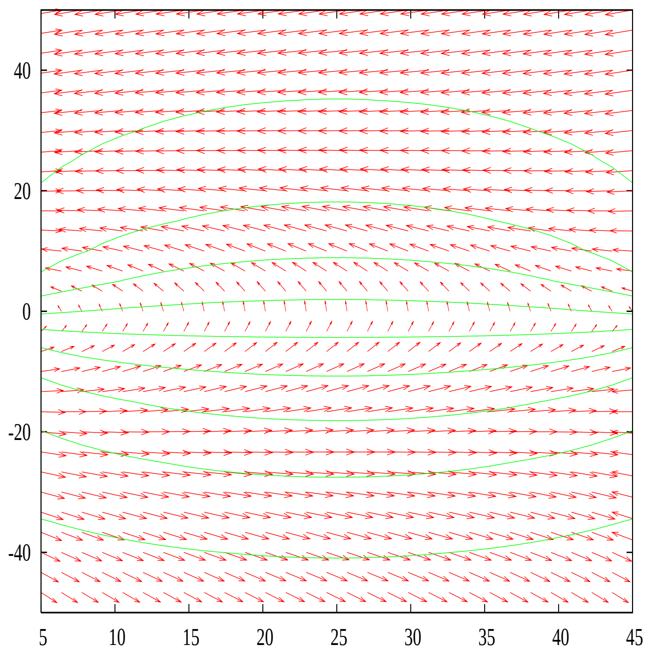

Figure 6.11: Isophase lines for spirals in the middle region when the left one has $+1$ degree and the right -1

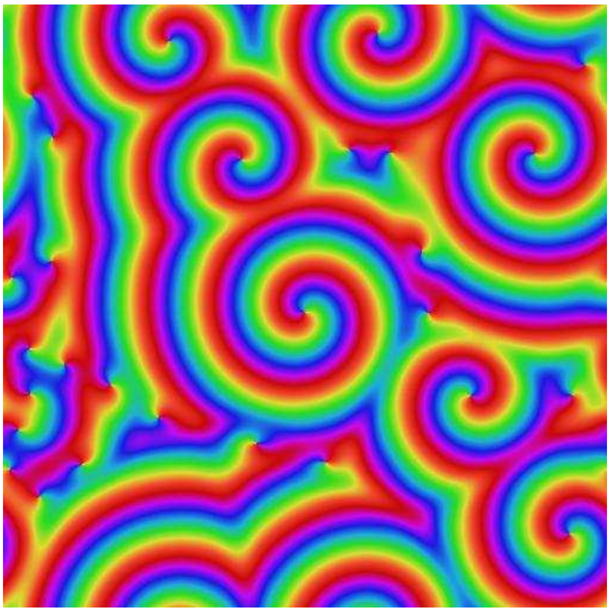

Figure 6.12: Phase field resulting from a simulation of the complex Ginzburg-Landau equation

Appendix A

Trajectories of the spirals in the middle scale

In this appendix we focus on spirals that interact in the middle scale region. We present the numerically computed trajectories of a pair and three spirals with winding numbers of either plus or minus one. The numerical method that has been used is based on a Runge-Kutta algorithm of order seven and eight with automatic stepsize control.

A.1 Trajectories of a pair of spirals

As we showed in Chapter 5, spirals that are separated by distances such that the auxiliary parameter α is of order less than one, have a law of motion that is given by

$$\frac{d\mathbf{X}_k}{dT} = 2n_k \frac{q}{\mu} \sum_{j\neq k}^{N} \Big(n_j \frac{\cos(q|\log\epsilon|)}{\sin(q|\log\epsilon|)} \frac{\mathbf{e}_{rjk}}{|\mathbf{X}_j - \mathbf{X}_k|} + \frac{\mathbf{e}_{\phi jk}}{|\mathbf{X}_j - \mathbf{X}_k|} \Big), \tag{A.1}$$

when their degrees are either plus or minus one. We recall that the parameter μ is actually $1/|\log\epsilon|$, and $1/\epsilon$ stands for the separation of the spirals, and therefore it also depends on the spirals' position, $\mathbf{X}_k$.

An interesting question regarding this equation is wether or not periodic solutions are possible. If periodic orbits were possible, spirals could eventually reach them and stay forever in this middle scale. If we take for instance the particular case of a pair of spirals, the differential equation (A.1) reads

$$\frac{d\mathbf{X}_1}{dT} = 2n_1 \frac{q}{\mu} \Big(n_2 \frac{\cos(q|\log\epsilon|)}{\sin(q|\log\epsilon|)} \frac{\mathbf{e}_{r21}}{|\mathbf{X}_2 - \mathbf{X}_1|} + \frac{\mathbf{e}_{\phi 21}}{|\mathbf{X}_2 - \mathbf{X}_1|} \Big) \tag{A.2}$$

where $\mathbf{e}_{r21}$ and $\mathbf{e}_{\phi 21}$ are the polar vectors corresponding to the position of the spiral 1 when the centre of spiral 2 is taken as the origin of coordinates. By symmetry, the second spiral has the same equation but we must interchange the 1 and 2 subindexes.

The most simple periodic solutions that we could find in equation (A.2) are those were the two spirals would spin around each other in a circle. This would correspond to a distance of separation such that the radial component of (A.2) vanishes and only the tangential component of the velocity survives, that is to say,

$$\cos(q|\log\epsilon|) = 0, \quad \text{and thus} \quad q|\log\epsilon| = \frac{\pi}{2}n, \quad , n \in \mathbb{Z}. \tag{A.3}$$

But if the spirals are separated by distances such that $q|\log\epsilon| = \pi/2$, this means that they have already left the middle scale and are now in the canonical region where equation (A.2) is no longer valid. This seems to imply that periodic solutions, that are also called *bound states* in some physics context, do not exist in this middle region. Actually, the equations for two spirals are simple enough to see clearly that bound states will indeed never appear, regardless of the degree of the spirals. If for instance we start with a pair of spirals with the same degree, we observe that they rotate around each other at the same time that they separate in a spiralling way. As one would expect by analysing equation (A.2), spirals rotate counterclockwise when their degrees are +1, as it is seen in figures A.1 and A.2, and clockwise when their winding number is −1, see figures A.3 and A.4. Another interesting issue is that, when we take larger values of q, the rate of separation clearly slows down. Nevertheless, and as we mentioned above, they will never stop separating, on the contrary, they will slowly reach the canonical scaling.

As for the case of spirals with opposite degrees, they attract each other rather than repel and they would eventually coalesce, although we must say that the middle region equations stop being valid when the separation of spirals become of order one, which means that this equations do not model the interaction of spirals when they are too close. In figures A.5, A.6, A.7 and A.8 we start with spirals in positions on the x-axe, and it is clear that they approach each other in different directions depending on which spiral has the positive and negative degree. It is also noticeable that this approach is faster as q becomes larger.

A.2 Trajectories for systems of three spirals

If we now consider a system of three spirals, again the simplest periodic solutions would arise at distances such that $q|\log\epsilon| = \pi/2$ where three spirals with symmetric initial conditions would rotate on a circle. But as we mentioned above this distance corresponds to the canonical scale and equation (A.1) does not hold. However, for three spirals we have a system of six nonlinear ordinary differential equations whose dynamics can be very complicated. To have a feeling of the kind of trajectories that a system of three spirals would have we have numerically integrated equations (A.1) with different initial conditions and changing the values of q.

We first consider a set of three spirals that are initially aligned on the x-axis (see figures A.9 and A.10), and we find that if we take a bigger value of q it looks as if the first spiral, the one starting at the origin, tends to remain almost at the same place while the other two seem to move around this one. We recall that as we increase q spirals are closer to the canonical scale, but nevertheless, when we have three spirals it is perfectly possible that one of them separates from the other two so that this first one starts satisfying the canonical scaling equations while the other two remain close enough to be one inside the middle scale with respect to the other, but both of them interact with the third in the canonical regime. This situation has not been considered in this work but the behaviour of the three spirals in A.9 can as well be the limit to this situation of *asymmetric separation* that we have just described.

The next pair of plots, A.11 and A.12 correspond to three spirals that are initially on a equilateral triangle. The symmetry of the initial conditions and of the equations with respect to the centre of the triangle makes the vortices separate at a constant rate. As we increase q again the solutions rotate in a more abrupt way but they still keep separating, as it is expected. In figures A.13 and A.14 we have chosen initial conditions that are not symmetric and as a consequence the spirals still separate but the distance of separation between them is now not constant. In figures A.15 and A.16 we have changed the winding number to minus one for the three spirals and we find that the spirals now approach each other while spinning in the same direction as before. Finally, the plot in A.17 shows that if we consider three spirals where one has a positive degree and the other two have a negative degree, the two spirals with opposite degree that are closer start attracting and would eventually coalesce.

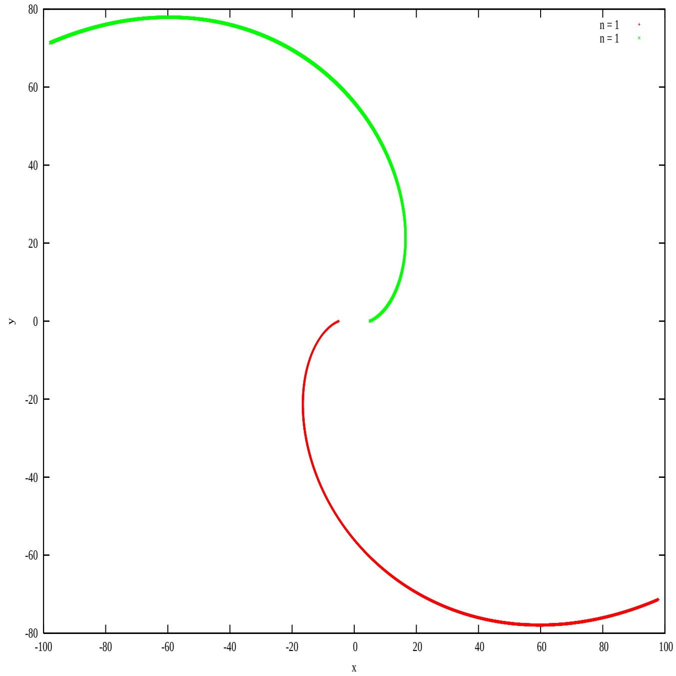

Figure A.1: Spirals in the middle region with degrees of +1, when $q = 0.2$.

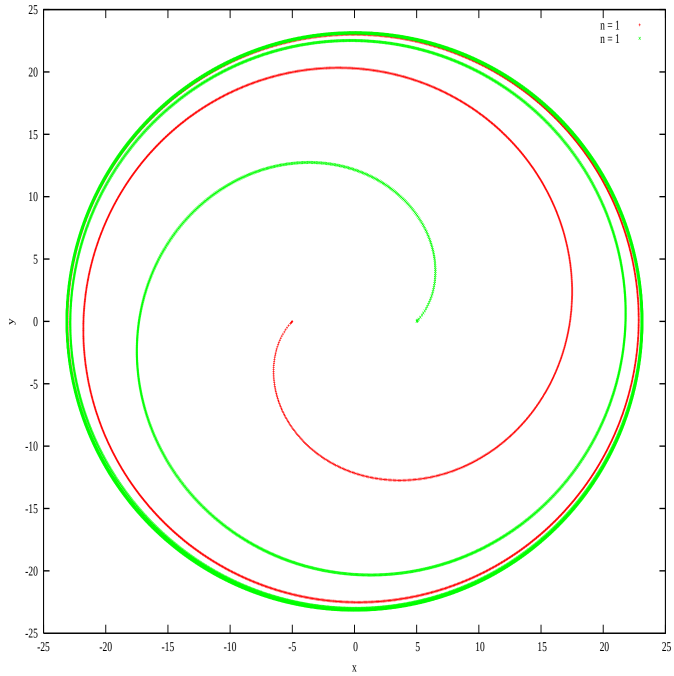

Figure A.2: Spirals in the middle region with degrees of +1, when $q = 0.5$.

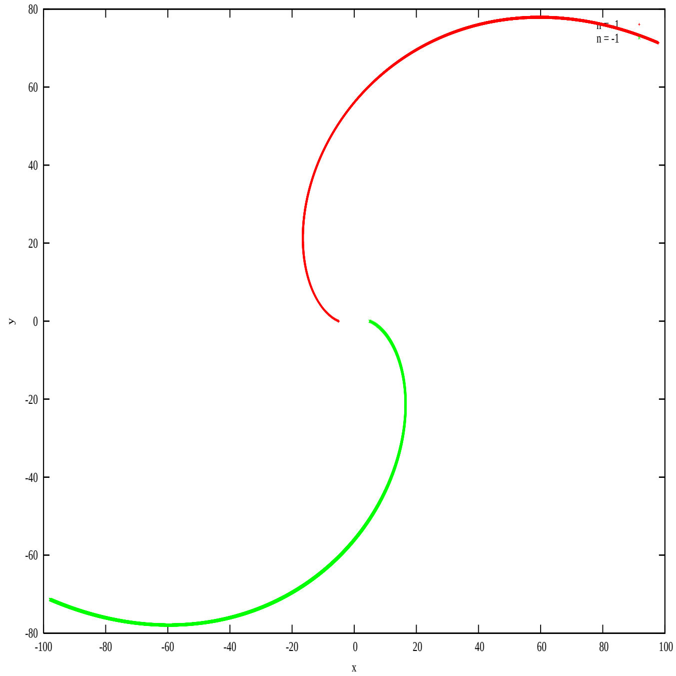

Figure A.3: Spirals in the middle region with degrees of -1, when $q = 0.2$.

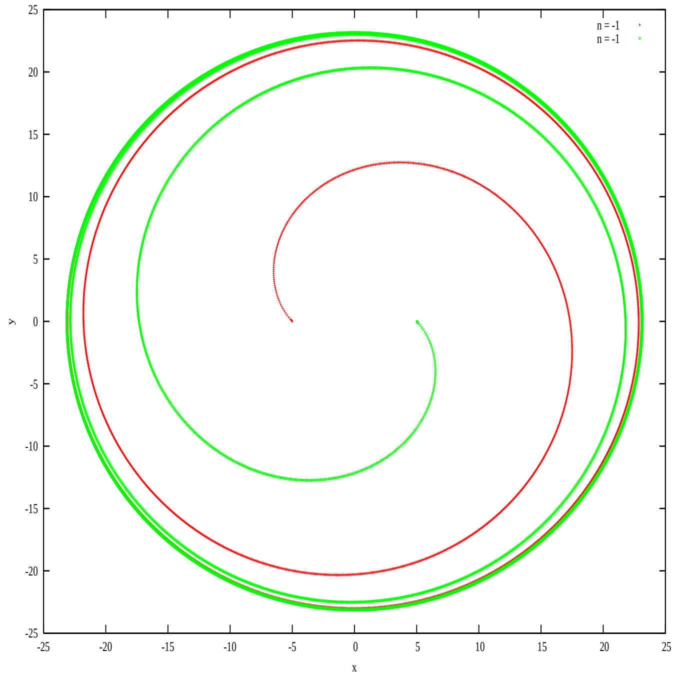

Figure A.4: Spirals in the middle region with degrees of -1, when $q = 0.5$.

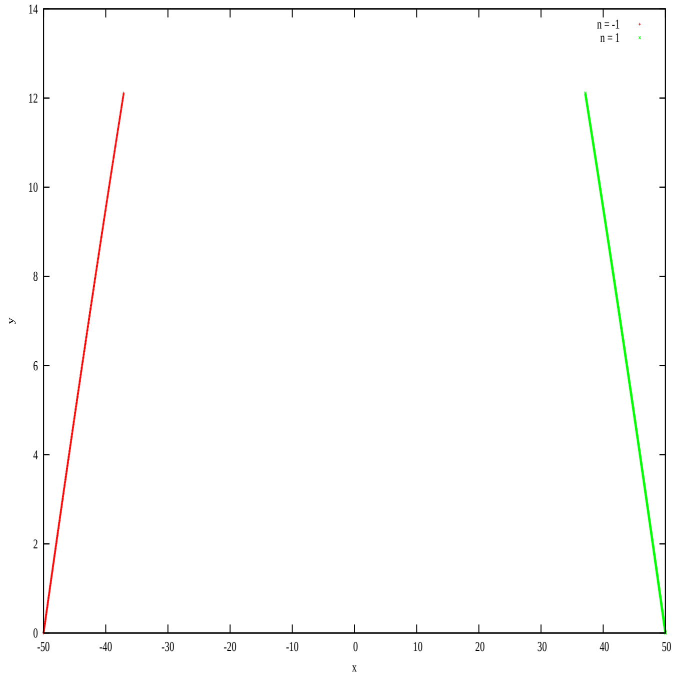

Figure A.5: Spirals in the middle region with degrees of -1 and $+1$, when $q = 0.2$.

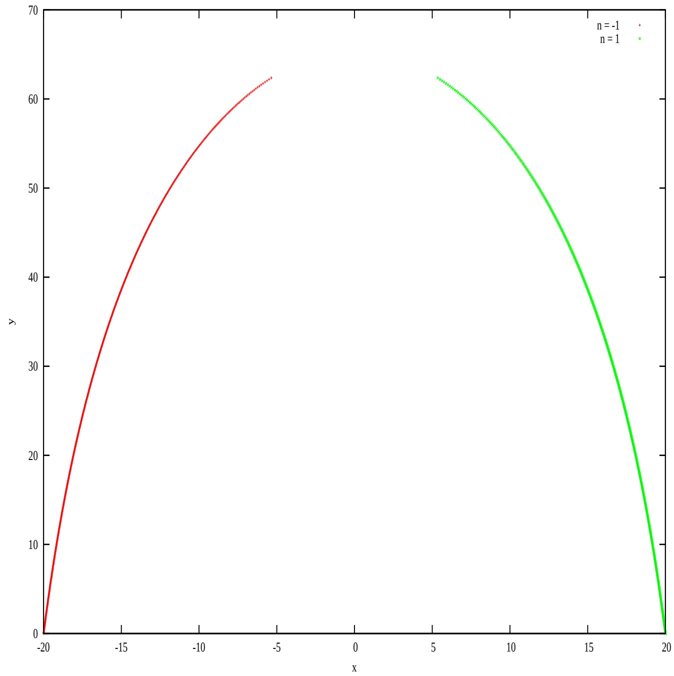

Figure A.6: Spirals in the middle region with degrees of -1 and $+1$, when $q = 0.5$.

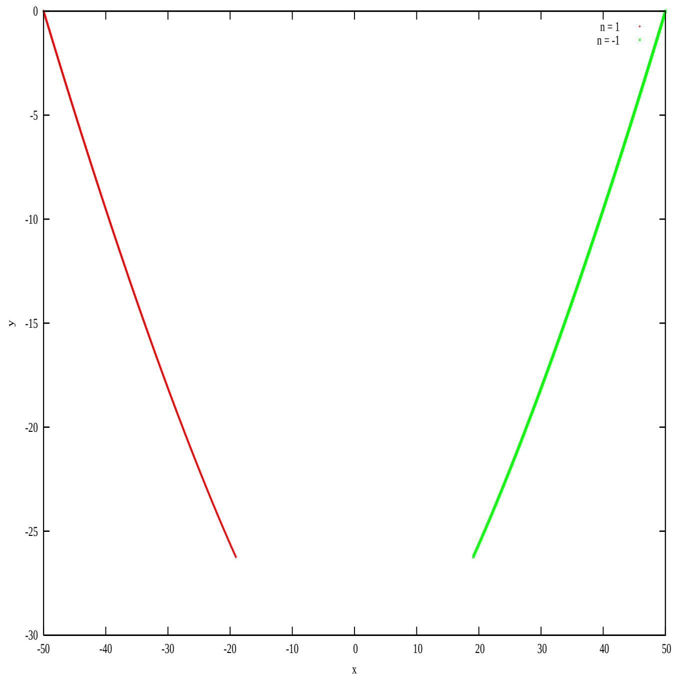

Figure A.7: Spirals in the middle region with degrees of $+1$ and -1, when $q = 0.2$.

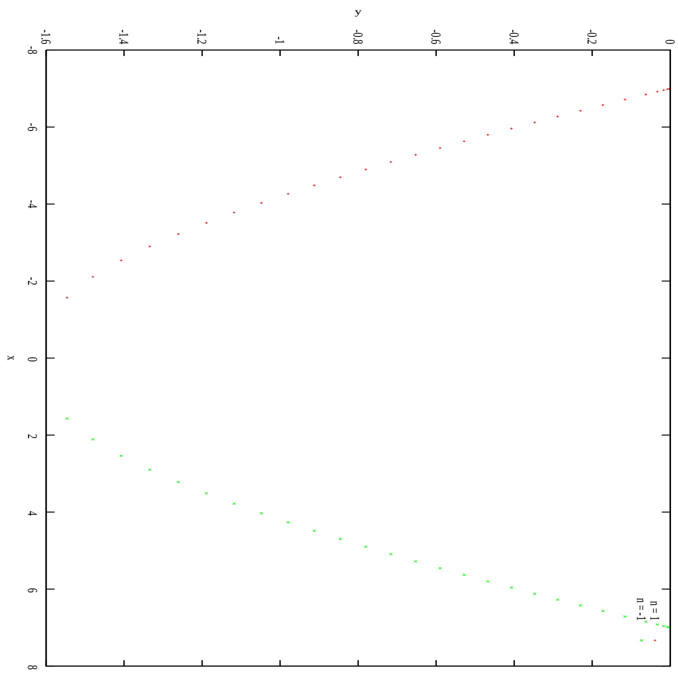

Figure A.8: Spirals in the middle region with degrees of $+1$ and -1, when $q = 0.5$.

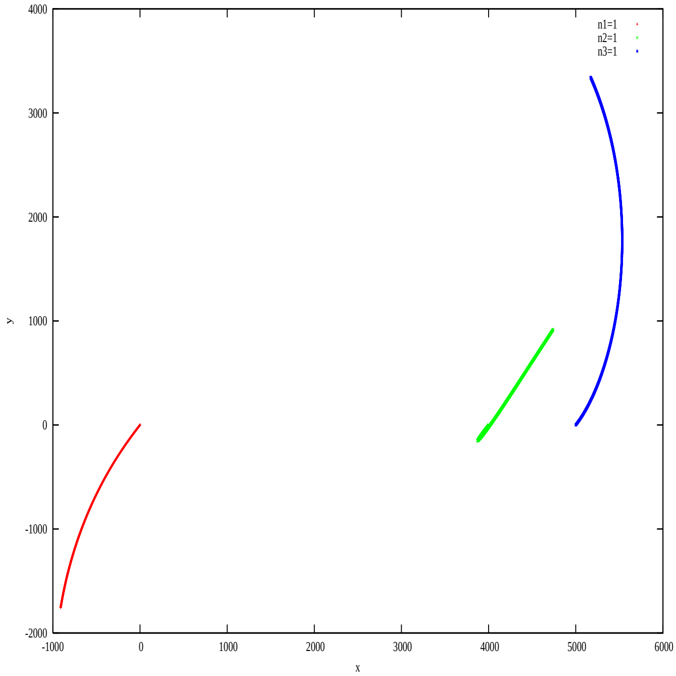

Figure A.9: Three spirals in the middle scale with degrees of +1 and with initial positions on the x-axis when $q = 0.1$.

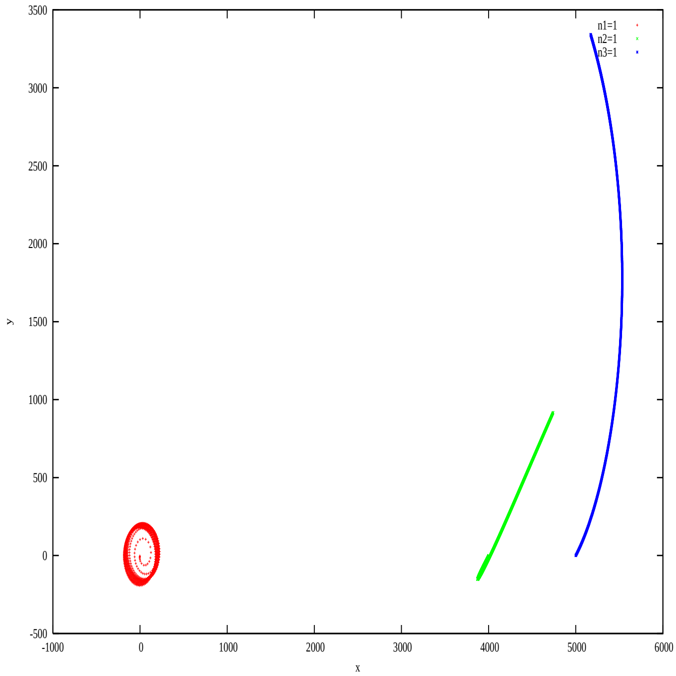

Figure A.10: Three spirals in the middle scale with degrees of +1 and with initial positions on the x-axis when $q = 0.3$.

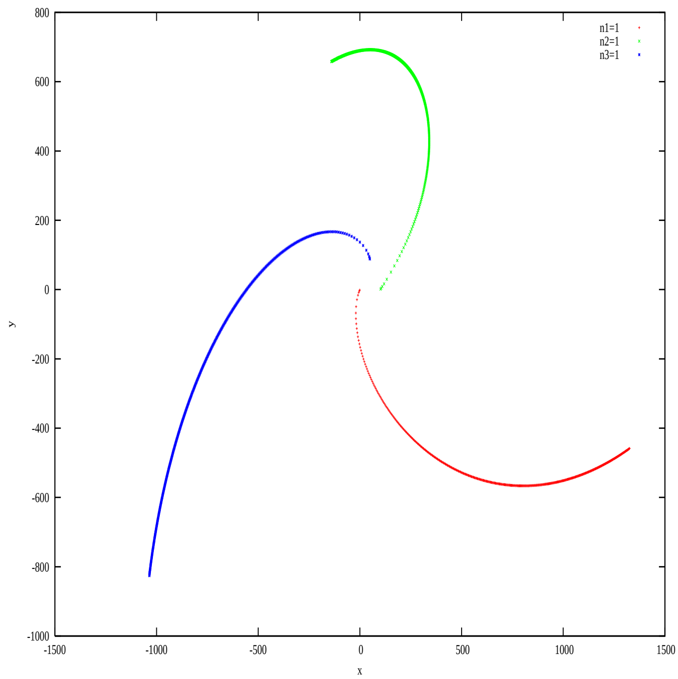

Figure A.11: Three spirals in the middle scale with degrees +1 and with initial conditions on an equilateral triangle when $q = 0.1$.

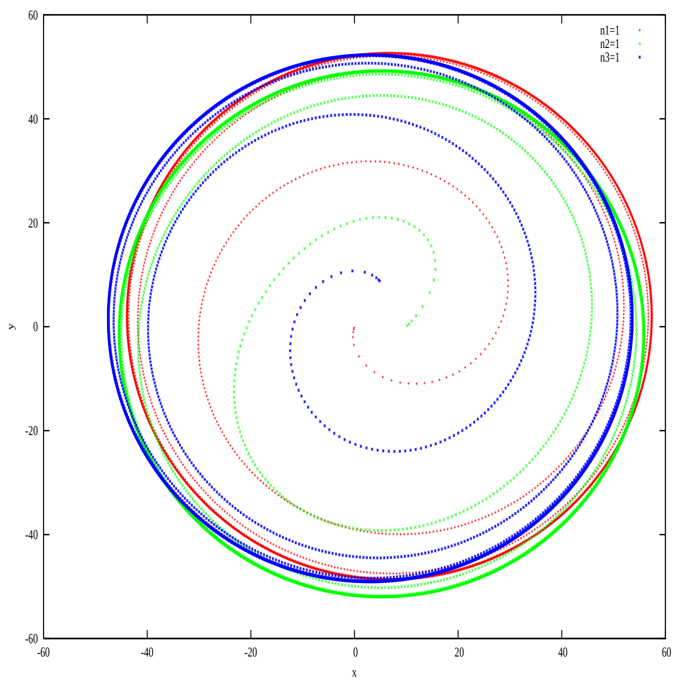

Figure A.12: Three spirals in the middle scale with degrees +1 and with initial conditions on an equilateral triangle when $q = 0.3$.

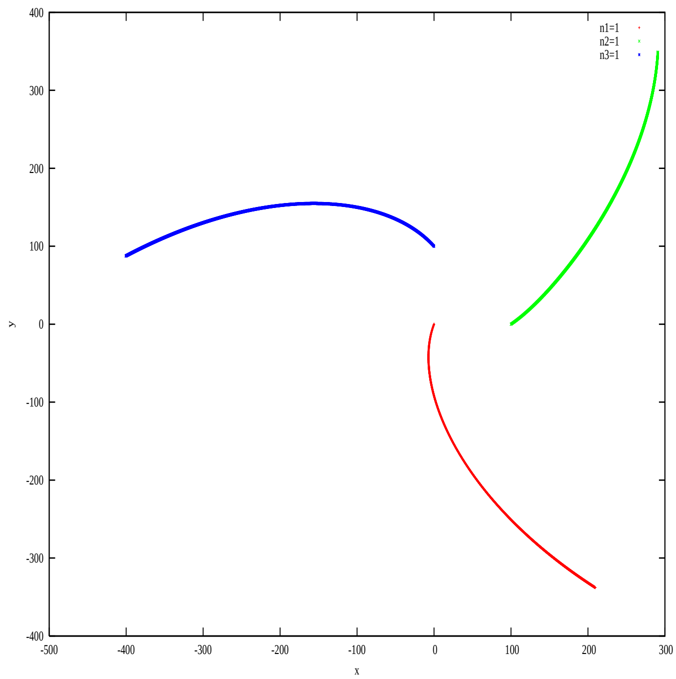

Figure A.13: Three spirals in the middle scale with degrees +1 and with initial conditions on a non equilater triangle when $q = 0.1$.

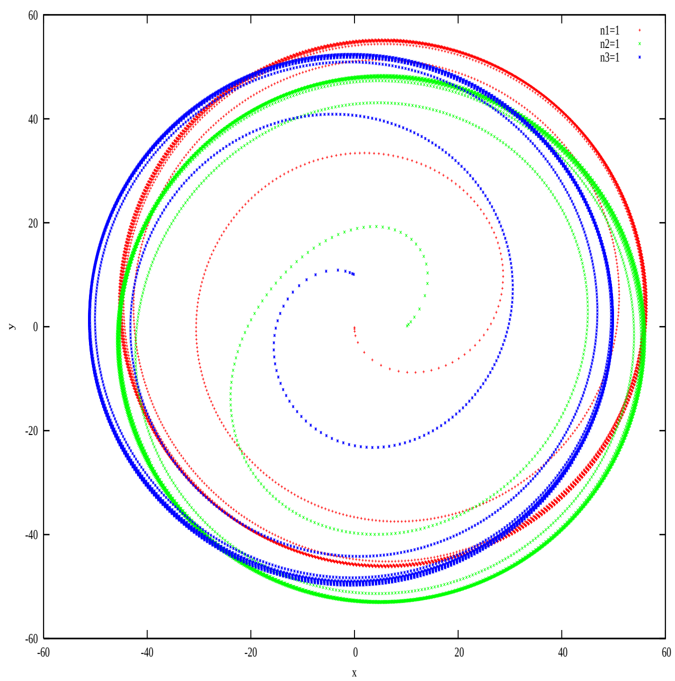

Figure A.14: Three spirals in the middle scale with degrees +1 and with initial conditions on a non equilater triangle when $q = 0.3$.

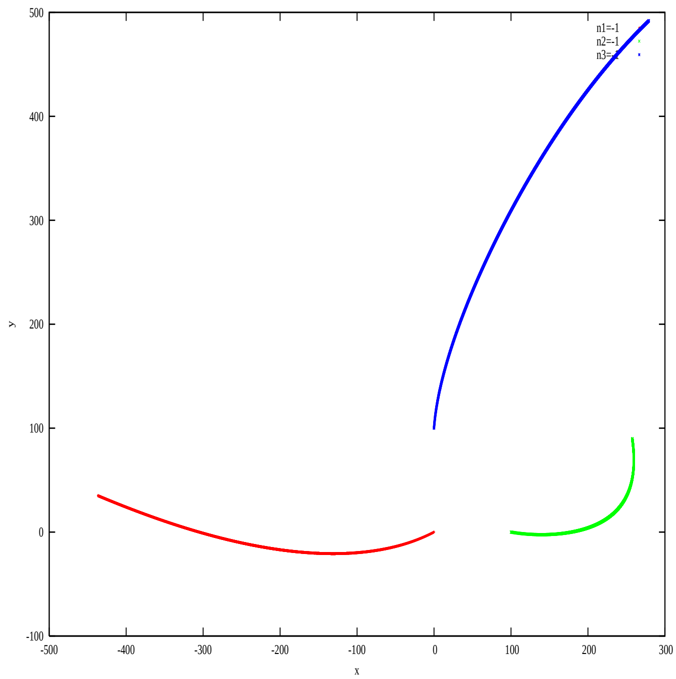

Figure A.15: Three spirals in the middle scale with degrees -1 and with initial conditions on a non equilater triangle when $q = 0.1$.

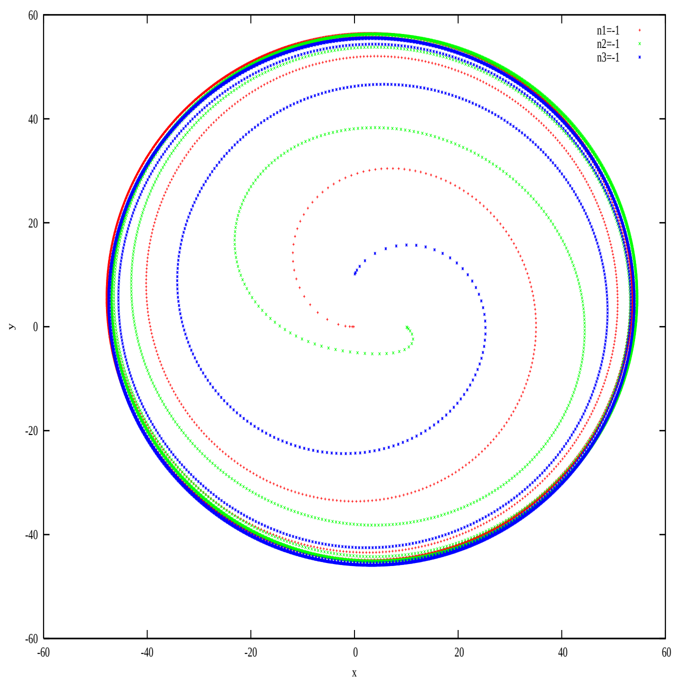

Figure A.16: Three spirals in the middle scale with degrees -1 and with initial conditions on a non equilater triangle when $q = 0.3$.

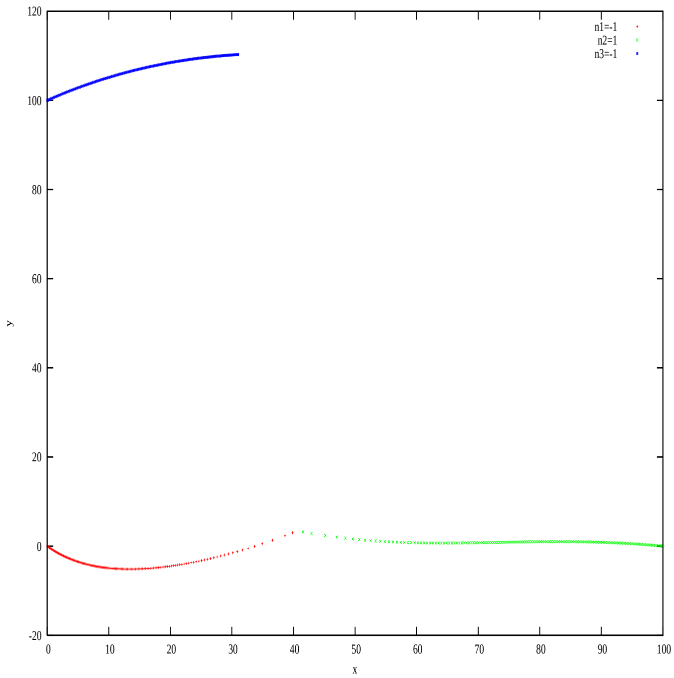

Figure A.17: Three spirals in the middle scale with degrees −1, +1 and −1 and with initial conditions on a non equilater triangle when $q = 0.1$.

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