FUZZY DATA ENVELOPMENT ANALYSIS AND ITS APPLICATIONS FOR AGGREGATING PREFERENCE RANKING

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by

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DEDICATED

…to my

wife Afsaneh,

my father,

and my wife's family

Acknowledgments

I would like to express my heartiest thanks to the people who contributed, either directly or indirectly, to the completion of this study. Thanks to Dr. Adli Mustafa and Dr. Ali Emrouznejad for their encouragement and guidance throughout my research career. Without their support, the research culminated in this thesis would not have been possible.

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Table of Contents

List of Tables

List of Figures

ANALISIS PENYAMPULAN DATA KABUR DAN PENGGUNAANNYA UNTUK MENGAGREGAT PENARAFAN KEUTAMAAN

Abstrak

Dalam tempoh dua dekad yang lepas, Analisis Penyampulan Data (DEA) telah muncul sebagai satu kaedah penting dalam bidang pengukuran kecekapan. DEA telah digunakan untuk membanding pelbagai Unit Pembuatan Keputusan (*DMU*) seperti cawangan bank, hospital dan pusat jualan, yang menggunakan satu atau lebih input bukan homogen untuk mengeluarkan satu atau lebih output bukan homogen. *DMU* biasanya menggunakan input yang sama dan mengeluarkan output yang sama tetapi pada tahap yang berbeda. Salah satu daripada ciri utama DEA adalah kepekaan terhadap data. Yakni, data yang kurang tepat mungkin memesong keputusan analisis kecekapan daripada keadaan sebenar. Namun demikian, pengukuran data yang tepat, di dalam banyak masalah nyata, adalah mustahil disebabkan oleh ketiadaan alat pengukur yang tepat ataupun disebabkan oleh sifat kualitatif semulajadi fenomenon yang berlaku. Maklumat yang sebegini, sebenarnya boleh diwakilkan sebagai nombor kabur ataupun istilah linguistik.

Salah satu daripada objektif tesis ini adalah untuk meneroka penggunaan pengukur kabur dan pengaturcaraan matematik kabur dalam model DEA. Dua prosedur pengaturcaraan linear kabur yang dapat menyelesaikan model DEA kabur diperkenalkan. Versi kabur suatu model bentuk nisbah, dikenal sebagai model CCR (Charnes et al. 1978) dibentuk dan beberapa kaedah penyelesaian model berkenaan diberikan. Berasaskan kepada corak sebegini, satu model penarafan dicadangkan. Kemudian, konsep kabur dan DEA digunakan untuk memperkenalkan kaedah

matematik baru untuk memilih alternatif terbaik dalam pembuatan keputusan berkumpulan. Model yang diperkenalkan adalah fungsi objektik-berganda yang diubah menjadi model pengaturcaraan linear objektif-berganda, yang kemudiannya diselesaikan.

Skop kajian ini, dari sudut pandangan teori, adalah untuk menyediakan satu platform bagi menerokai pelbagai model DEA klasik dalam keadaan kabur. Dari sudut pandangan praktikal pula, banyak masalah yang melibatkan faktor kualitatif kini dapat dikendalikan. Model yang diperkenalkan ini dapat digunakan dalam analisis kecekapan personel, kumpulan, kualiti barangan dan sebagainya yang banyak faktor di dalamnya adalah sememangnya kabur.

FUZZY DATA ENVELOPMENT ANALYSIS AND ITS APPLICATIONS FOR AGGREGATING PREFERENCE RANKING

Abstract

Over the past two decades, Data Envelopment Analysis (DEA) has appeared as an important tool in the field of efficiency measurement. DEA is used to compare Decision Making Units (*DMU*s) such as bank branches, hospitals, sales outlets, which consume one or more non-homogenous inputs to produce one or more non-homogenous outputs. The *DMU*s consume the same inputs and produce the same outputs but generally at varying levels. One of the main characteristics of DEA is its sensitivity to data. That is, inaccurate data may divert effectively the results of efficiency analysis from its actual value. But accurate measurement in many real world problems, due to either non-availability of sophisticated measurement tools or qualitative nature of the phenomena may not be possible. This kind of information can be represented as fuzzy numbers or linguistic terms.

One of the objectives in this thesis is to explore the use of fuzzy measures and fuzzy mathematical programs in the DEA models. Two procedures for fuzzy linear programming are presented which are able to solve fuzzy DEA models. The fuzzy version of a ratio form known as the CCR model (Charnes et al. 1978) is developed and some methodologies for the solution of this model are provided. Based on this pattern, a ranking model is suggested. Then, the fuzzy concepts and Data Envelopment Analysis are used to introduce new mathematical methods for

selecting the best alternative in a group decision making environment. The introduced models are multi-objective functions which are converted into multi-objective linear programming models from which the optimal solutions are obtained.

The scope of this study, from theoretical point of view, is to provide a platform for exploring different classical DEA models in fuzzy environment. From practical point of view, many problems pertaining to qualitative factors may now be handled. (These models may be applied in the efficiency analysis of personnel, groups, quality of goods, etc. where many of the factors may be inherently fuzzy).

Chapter 1

Introduction

Data Envelopment Analysis (DEA) is increasingly the centre of many research and applications for measuring efficiency and productivity of decision making unit aimed at improving organizational efficiency. However, despite the importance of efficiency measurement in public and private services, it is only recently that the more advanced fuzzy logic and fuzzy mathematical programming concepts are applied to DEA.

On the other hand, in recent years, researchers have used Data Envelopment Analysis technique in various applications.

1.1 Data Envelopment Analysis

Data Envelopment Analysis is a recognized modern approach to the assessment of performance of organizations and their functional units. DEA extends the boundaries of several academic areas including management science, operational research, economics and mathematics. DEA is a non-parametric technique in order to measure the relative efficiencies of a set of decision making units (*DMU*s) which use multiple inputs to product multiple outputs. This technique which was initially proposed by Charnes et al. (1978) (CCR model) and was improved by other scholars, especially Banker et al. (1984) (BCC model), evaluates the relative efficiency of a set of homogenous decision making units (*DMU*s) by using a ratio of the weighted sum of outputs to the weighted sum of inputs. It generalizes the usual efficiency measurement from a single-input, single-output ratio to a multiple-input, multiple-output ratio.

Let inputs x_{ij} $(i = 1, 2, ..., m)$ and outputs y_{rj} $(r = 1, 2, ..., s)$ be given for DMU_j $(j = 1, 2, ..., n)$.

The fractional programming statement for the CCR model is formulated as follows:

$$\max \quad \frac{\sum_{r=1}^{s} u_r y_{rp}}{\sum_{i=1}^{m} v_i x_{ip}} \tag{1.1}$$

$$\text{s.t.}$$

$$\frac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{i=1}^{m} v_i x_{ij}} \le 1 \qquad \forall j$$

$$u_r, v_i \ge 0 \qquad \forall r, i$$

where v_i and u_r are the weight variables for ith and rth input and output, respectively.

Model (1.1) is transformed to the following linear programming problem by some substitutions:

$$\begin{aligned} \max \quad & \sum_{r=1}^{s} u_r y_{rp} \\ \text{s.t.} \quad & \\ & \sum_{i=1}^{m} v_i x_{ip} = 1 \\ & \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \le 0 \qquad \forall j \\ & u_r, v_i \ge 0 \qquad \forall r, i \end{aligned} \tag{1.2}$$

At the turn of the present century, reducing complex real-world systems into precise mathematical models was the main trend in science and engineering. Unfortunately, real-world situations are frequently not deterministic. Thus precise mathematical models are not enough to tackle all practical problems. In practice there are many problems in which, all (or some) input–output levels are fuzzy numbers. It is difficult to evaluate *DMU*s in an accurate manner to measure the efficiency. Fuzzy Data Envelopment Analysis (FDEA) is a powerful tool for evaluating the performance of a set of organizations or activities under uncertaint environment.

1.1.1 Fuzzy data envelopment analysis

Due to lack of complete knowledge and information, precise mathematics is not sufficient to model a complex system. Although, in real world situations, decisions are based on qualitative as well as quantitative data, a fuzzy approach is able to deal with such problems. The CCR model with fuzzy coefficients is given in Equation (1.3).

$$
\begin{aligned}
\max \quad & \sum_{r=1}^{s} u_r \tilde{y}_{rp} \\
s.t. \quad & \\
& \sum_{i=1}^{m} v_i \tilde{x}_{ip} = 1 \\
& \sum_{r=1}^{s} u_r \tilde{y}_{rj} - \sum_{i=1}^{m} v_i \tilde{x}_{ij} \leq 0 \qquad \forall j \\
& u_r, v_i \geq 0 \qquad \forall r, i
\end{aligned} \tag{1.3}
$$

where $\tilde{x}_{ij}$ and $\tilde{y}_{rj}$ are the ith fuzzy input and rth fuzzy output of DMU_j, respectively.

DEA researchers have begun using fuzzy concept for measuring efficiency and productivity of decision making units since 1992. One of the first literatures on fuzzy DEA is by Sengupta (1992a). He exerted principles of fuzzy set theory developed by Bellman and Zadeh (1970) and Zimmermann (1976) to evaluate DMUs with fuzzy inputs and fuzzy outputs in data envelopment analysis. He considered the objective function and the right-hand side vector of the conventional DEA model developed by Charnes et al. (1978), as fuzzy numbers. However, Sengupta did not provide an application roadmap of his proposed framework to measure efficiency using fuzzy DEA. After that, other researchers suggested various approaches to solve fuzzy DEA problem, but these methods have some shortcomings that will be mentioned in the next sections.

1.2 Assumptions

Among the various types of fuzzy numbers, triangular and trapezoidal fuzzy numbers are most commonly used, because they have intuitive appeal and can be easily specified by the decision maker. In this thesis, the inputs and outputs of *DMU*s are considered as triangular fuzzy numbers. This assumption on inputs and outputs can be easily extended to trapezoidal fuzzy numbers.

1.3 Problem Statements and Objectives

The assumption underlying DEA is that all the data are in the form of specific numerical values. However, the data are sometimes observed with a noise and/or with inaccuracy. For example, in evaluating operation efficiencies of airlines, seat-kilometers available, cargo-kilometers available, fuel and labor are regarded as the inputs and passenger-kilometers as the output. It is common that these inputs and outputs can easily change because of weather, season, operating state and so on. To deal with imprecise data like these, the notion of fuzziness has been introduced. However in evaluating *DMU*s using fuzzy DEA, some problems arise. Among them are:

- **a)** the lost of many information on uncertainty in most of the existing approaches, and
- **b)** the computational inefficiency of the other approaches that try to retain as much as possible the information on uncertainty.

The objectives of this thesis are:

a) to improve the existing approaches of solving fuzzy DEA by proposing methods that are able to retain as much as possible information on uncertainty and at the same time are computationally efficient, and

b) to use the concept of DEA and fuzzy DEA in selecting the best alternative in a group decision making environment.

1.4 Outline of the Thesis

This thesis consists of four contributions utilizing fuzzy concepts; two methods for evaluating DMUs with fuzzy inputs and outputs, and then two procedures using the aforementioned methods for aggregating preference ranking.

The remaining chapters are organized as follows. Chapter 2 is dedicated to introduce DEA, fuzzy DEA and voting system. Some approaches for solving fuzzy DEA models are also introduced in this chapter.

Based on a non-radial measure which allows non-proportional reductions in positive inputs or augmentations in positive outputs, a method for solving fuzzy non-radial model is presented in Chapter 3. Chapter 4 presents a new approach which is referred to as discrete approach. First, using local *α-level* concept and applying discrete data, a nonlinear multi-objective programming model is formulated, and then, the problem is converted to a linear multi-objective programming model. A model for aggregating preference ranking with fuzzy concept, which is an application of fuzzy DEA, is presented in Chapter 5. In Chapter 6 the concept of Data Envelopment Analysis is used to introduce a new mathematical method for selecting the best alternative in a group decision making environment. The introduced model is a multi-objective function which is converted into a multi-objective linear programming model from which the optimal solution is obtained. Some examples are given to demonstrate the implementation of each model in Chapters 3 to 6. Conclusion and discussion on some future research directions are presented in Chapter 7.

Chapter 2

Literature Review

This chapter provides a synoptic survey of DEA, fuzzy DEA and the aggregating preference ranking methods. First, a literature review on traditional DEA, including a theoretical analysis on the construction of the CCR model is presented. Then, literature review on fuzzy DEA which includes discussion on applications, advantages and shortcomings of the existing approaches are presented. The discussion then focuses on the preferential voting system. Special attention was given to the Cook and Kress (1990) method because it is the closest to the suggested methods in this thesis. In their approach, DEA model of Charnes et al. (1978) which utilizes the weighted composite of place standings of each candidate was used.

2.1 Research Background on DEA

Measurement and evaluation of efficiency has been progressing in management science. A production function defines the relationship between the outputs and inputs of a production technology. Mathematically, a production function relates the amount of output (Y) as a function of the amount of input (X) used to generate that output. Technical efficiency is assumed for a production function i.e., every feasible combination of inputs generates the maximum possible output (from an output oriented-view) or all outputs are produced using the minimum feasible combination of inputs (output an oriented-view). Efficiency can then be measured relative to the frontier defined by the production function. DEA is a nonparametric approach, and unlike parametric approach such as the methods based on regression analysis, there is no need to assume the form of the production function relating inputs and outputs. It evaluates the efficiency of each *DMU* relative to similar *DMU*s. Thus, it provides an efficient frontier or envelop for all considered *DMU*s rather than fitting a regression plane through the center of the data.

2.1.1 Data envelopment analysis models

This section specifies all the symbols used for describing DEA and also presents a theoretical discussion for constructing DEA models.

The definitions of the notations are as follows:

Notations:

i = the subscript of inputs $(i = 1, 2, ..., m)$,

j = the subscript of *DMU*s $(j = 1, 2, ..., n)$,

r = the subscript of outputs $(r = 1, 2, ..., s)$,

p = a specific *DMU* to be measured $1 \le p \le n$,

X_j = input vector of the j th *DMU*,

Y_j = output vector of the j th *DMU*,

x_{ij} = the i th input of the j th *DMU*,

y_{rj} = the r th output of the j th *DMU*,

s_i^- = the slack variable for the i th input,

s_i^+ = the slack variable for the r th output,

λ_j = a nonnegative value related to the j th DMU. The vector $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n)^t$ is used to construct a hull that covers all the data points.

v_i = the weighting variable for the i th input,

u_r = the weighting variable for the r th output,

w_p = DEA efficiency score of Saati's (2001),

θ = DEA efficiency score of CCR model,

T_c = the production possibility set corresponding to constant returns to scale,

T_v = the production possibility set corresponding to variable returns to scale,

ε = a positive non-Archimedean infinitesimal.

In order to review the DEA models, the following two general assumptions are specified:

a) There are n DMUs denoted by $j \in J$, each of which produces a nonzero output vector $Y_j = (y_{1j}, y_{2j}, ..., y_{sj})^t \geq 0$ using a nonzero input vector $X_j = (x_{1j}, x_{2j}, ..., x_{mj})^t \geq 0$, where the superscript $'t'$ indicates the transpose of a vector. Here, the symbol $'\geq'$ indicates that at least one component of X_j or Y_j is positive while the remaining X_j's or Y_j's are nonnegative.

b) There is no *DMU* in J whose data domain can be proportionally expressed by that of another *DMU*.

Definition 2.1. Given the (empirical) points $(X_j, Y_j), j = 1, 2, ..., n,$ the Production Possibility Set (PPS) is defined as follows:

$$T = \{(X_t, Y_t) \mid \text{output } Y_t \text{ can be produced by input } X_t\}$$

Definition 2.2.

a) The production possibility (X_t, Y_t) is a frontier point (input-oriented) if $(\alpha X_t, Y_t) \in T$ implies $\alpha \geq 1$.

b) Production possibility (X_t, Y_t) is a frontier point (output-oriented) if $(X_t, \beta Y_t) \in T$ implies $\beta \leq 1$.

To construct the production possibility set, the following postulates are assumed:

1) (Ray Unboundedness) If $(X_t, Y_t) \in T$ then $(\gamma X_t, \gamma Y_t) \in T$ for $\gamma > 0$.

2) (Convexity) If $(X_t, Y_t) \in T$ and $(X_u, Y_u) \in T$ then

$$(\lambda X_t + (1 - \lambda) X_u, \lambda Y_t + (1 - \lambda) Y_u) \in T \text{ for all } \lambda \in [0,1].$$

3) (Monotonicity) If $(X_t, Y_t) \in T$, $X_u \geq X_t$ and $Y_u \leq Y_t$ then $(X_u, Y_u) \in T$.

4) (Inclusion of Observation) All the observations belong to production possibility set.

5) (Minimum Extrapolation) If T' be a set different from T which satisfies the mentioned above postulates, then $T \subseteq T'$.

The production possibility set corresponding to constant return to scale constructed with the aforementioned postulates will be as follows:

$$T_c = \left\{ (X_t, Y_t) \mid X_t = \sum_{j=1}^{n} \lambda_j X_j, Y_t = \sum_{j=1}^{n} \lambda_j Y_j, \lambda_j \geq 0, j = 1, 2, ..., n \right\} \tag{2.1}$$

Constant return to scale (CRS) means that an increase in the amount of inputs consumed leads to a proportional increase in the amount of outputs produced and if this increase is culminated in larger or smaller than proportional increase in the amount of outputs, return to scale will be increasing (IRS) or decreasing (DRS), respectively.

CCR model (input-oriented)

To evaluate efficiency corresponding to set T_c, consider the following model.

$$\begin{aligned} \theta_p^* &= \min \theta_p \\ &\text{s.t.} \\ &\quad (\theta_p X_p, Y_p) \in T \end{aligned} \tag{2.2}$$

CCR model (input-oriented) for evaluating the efficiency of DMU_p, is written as follows:

$$\theta_p^* = \min \theta_p \tag{2.3}$$

$$
\begin{aligned}
& s.t. \\
& \qquad \sum_{j=1}^{n} \lambda_j X_j \le \theta_p X_p \\
& \qquad \sum_{j=1}^{n} \lambda_j Y_j \ge Y_p \\
& \qquad \lambda_j \ge 0 \qquad\qquad j = 1, 2, ..., n \\
& \qquad \theta \text{ free}
\end{aligned}
$$

Dual (Multiplier) form of CCR model (input-oriented) for DMU_p is then written as:

(Multiplier)

$$\max \quad \sum_{r=1}^{s} u_r y_{rp} \tag{2.4}$$

$$
\begin{aligned}
& \text{s.t.} \\
& \qquad \sum_{i=1}^{m} v_i x_{ip} = 1 \\
& \qquad \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \le 0 \qquad \forall i, r \\
& \qquad u_r , v_i \ge \varepsilon \qquad\qquad\qquad \forall r, i
\end{aligned}
$$

By adding the following revised constraints of returns to scale:

$$\sum_{j=1}^{n} \lambda_j \begin{pmatrix} = 1 \\ \le 1 \\ \ge 1 \end{pmatrix} \tag{2.5}$$

to (2.3), the revised models with variables that are increasing and decreasing returns to scale are obtained.

Based on (2.1), three of the DEA models, which form the framework of this thesis, are introduced and developed. These models are CCR, Andersen and Petersen's (1993) and Saati et al. (2001).

The revision of DEA models by omitting the corresponding column of *DMU* under consideration in the technological matrix has been proposed by Andersen and Petersen (1993) (AP model). Their envelopment model is as follows:

$$
\begin{aligned}
\min \quad & \theta \\
\text{s.t.} \quad & \\
& \sum_{\substack{j=1 \\ j \neq p}}^{n} \lambda_j x_{ij} \leq \theta x_{ip} \qquad \forall i \\
& \sum_{\substack{j=1 \\ j \neq p}}^{n} \lambda_j y_{rj} \geq y_{rp} \qquad \forall r \\
& \lambda_j \geq 0 \qquad \forall j \\
& \theta \text{ free}
\end{aligned}
\tag{2.6}
$$

Reformulating the linear programs for ranking DMUs in DEA by omitting the corresponding column in the technological matrix causes some theoretical and applied difficulties. One of them is that the corresponding problem might becomes infeasible.

We can call the envelopment DEA models as radial efficiency measures, because these models optimize all inputs or outputs of a DMU at a certain proportion. Färe and Lovell (1978) introduced a non-radial measure which allows non-proportional reductions in positive inputs or augmentations in positive outputs. Saati et al. (2001) suggested a non radial model to remove the difficulties about infeasibility of AP model. The primal linear programming statement for the model is:

$$\begin{aligned}
\min \quad & W_p = w_p + 1 \qquad\qquad (2.7)\\
\text{s.t.} \quad & \\
& \sum_{\substack{j=1 \\ j\neq p}}^{n} \lambda_j x_{ij} \le x_{ip} + w_p 1 \quad \forall i \\
& \sum_{\substack{j=1 \\ j\neq p}}^{n} \lambda_j y_{rj} \ge y_{rp} - w_p 1 \quad \forall r \\
& \lambda_j \ge 0 \quad \forall j \\
& w_p \text{ free}
\end{aligned}$$

such that w_p is a free variable and measures the efficiency of DMU_p.

This model projects the DMU under evaluation on the frontier by decreasing the inputs and increasing the outputs.

2.2 Literature Review on Fuzzy DEA

In this section, first some fuzzy notations and definitions are introduced and then, after introducing the fuzzy CCR model, some approaches in fuzzy DEA together with their shortcomings and advantages are discussed.

Notations and Definitions

$\tilde{X}_j$ = fuzzy input vector of the j th *DMU*,

$\tilde{Y}_j$ = fuzzy output vector of the j th *DMU*,

$\tilde{x}_{ij}$ = the i th fuzzy input of the j th *DMU*,

$\tilde{y}_{rj}$ = the r th fuzzy output of the j th *DMU*.

Since terms like fuzzy sets, membership functions, fuzzy numbers and α-cut from fuzzy set theory will be used several times in the coming discussion, we shall consider a few necessary definitions.

Definition 2.3. (Zimmermann, 1976) If X is a collection of objects denoted generically by x, then a fuzzy set A in X is a set of ordered pairs:

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) \mid x \in X\}$$

$\mu_{\tilde{A}}(x)$ is called the membership function which associates with each $x \in X$ a number in $[0,1]$ indicating to what degree x is a member of A.

Definition 2.4. A trapezoidal fuzzy number $\tilde{A} = (p,q,l,u)$ (for simplification, $\tilde{A} = (a^p, a^q, a^l, a^u)$ is presented as $\tilde{A} = (p,q,l,u)$) is a fuzzy subset of R, such that its membership function $\mu_{\tilde{A}}$ is:

1. a continuous mapping from R to the closed interval $[0, w]$, $0 \le w \le 1$,

2. constant on $R - (l,u)$, $\mu_{\tilde{A}}(x) = 0$ for $x \in R - (l,u)$,

3. strictly increasing linear function on (l, p),

4. constant on $[p,q]$; $\mu_{\tilde{A}}(x) = 1$ for $p \le x \le q$,

5. strictly decreasing linear function on (q,u),

where p is the left main value and q is the right main value with the complete membership, l is the lowest value and u is the upper value. $p - l$ and $u - q$ are called left and right spreads, respectively (see Figure 2.1).

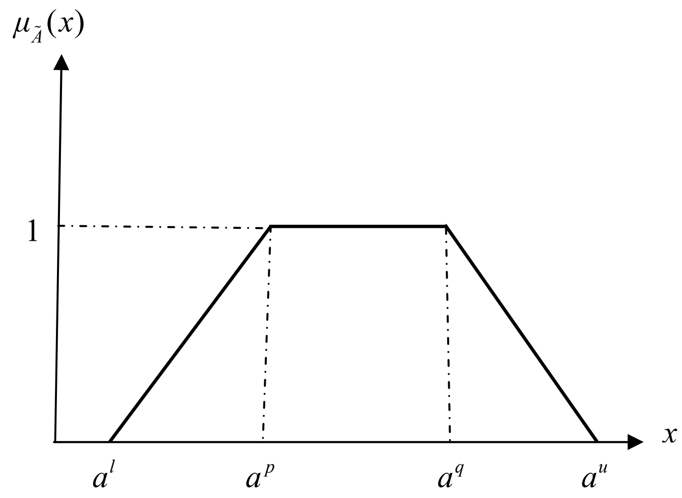

Figure 2.1 Membership function of trapezoidal fuzzy number $\tilde{A}$

If $p = q = m$, then the number $\tilde{A} = (m,l,u)$ is called triangular fuzzy number (see Figure 2.2).

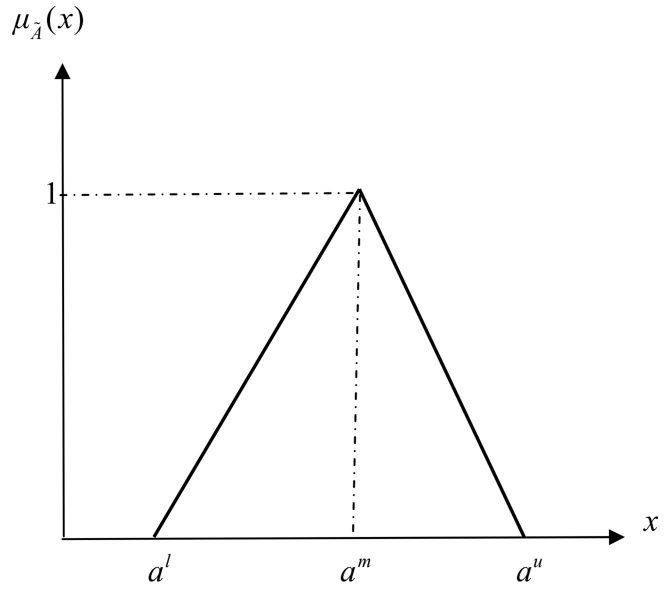

Figure 2.2 Membership function of triangular fuzzy number $\tilde{A}$

Definition 2.5. (*α-level* set) Let $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) \mid x \in X\}$ be a fuzzy set representing a fuzzy event. *α -level* sets or *α–cut*s shall be denoted by A_α and defined as:

$$A_\alpha = \{x \in X \mid \mu_{\tilde{A}}(x) \geq \alpha\}$$

Definition 2.6. (Normal fuzzy set) A fuzzy set $\tilde{A}$ is normal if

$$\sup_{x \in X}(\mu_{\tilde{A}}(x)) = 1$$

Definition 2.7. (Convex fuzzy set)A fuzzy set $\tilde{A}$ is convex if

$$\mu_{\tilde{A}}(\lambda x_1 + (1 - \lambda)x_2) \geq \min(\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2)) \text{ for all } x_1, x_2 \in X \;\; \lambda \in [0,1].$$

Definition 2.8. (Lai and Hwang, 1992) A fuzzy linear program can be stated as:

$$\begin{aligned} \max \quad & \sum_{j=1}^{n} \tilde{c}_j x_j \\ \text{s.t.} \quad & \\ & \sum_{j=1}^{n} \tilde{a}_{ij} x_j \le \tilde{b}_i \qquad i = 1,\dots,m \\ & x_j \ge 0 \qquad\qquad\quad j = 1,\dots,n \end{aligned} \tag{2.8}$$

where, $\tilde{c}_j \ (j = 1,2,\dots,n)$; $\tilde{b}_i \ (i = 1,2,\dots,m)$ and $\tilde{a}_{ij} (i = 1,2,\dots,m, \ j = 1,2,\dots,n)$ may be imprecise with membership function.

2.2.1 Fuzzy CCR model

Due to lack of complete knowledge and information, precise mathematics is not sufficient to model a complex system. Although, in real world, decisions are based on qualitative as well as quantitative data, a fuzzy approach seems fit to deal with such problems.

CCR model has its production frontier spanned by the linear combination of the existing *DMU*s. But, production frontier in CCR and fuzzy CCR model are different. The frontiers of the CCR model have no flexibility and have linear characteristics, while those of the fuzzy are flexible.

Figure 2.3 illustrates the efficiency frontier of fuzzy CCR model in the simplest case of single input and single output, respectively. In Figure 2.3, *DMU* consumes fuzzy input $\tilde{x} = (x^m, x^l, x^u)$ to produces output y_o.

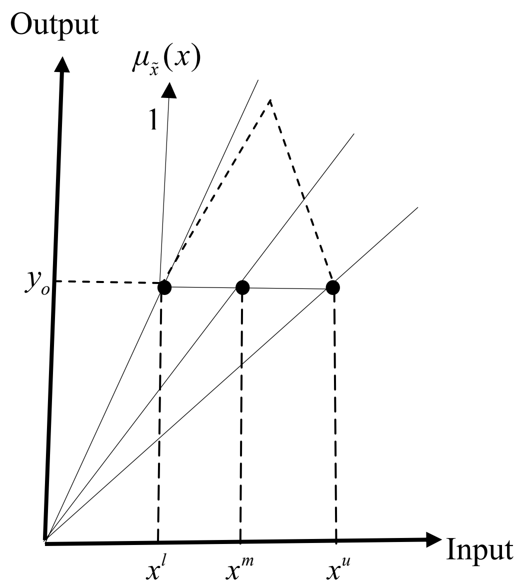

Figure 2.3 Production frontier of fuzzy CCR model

The CCR dual (multiplier) and primal (envelopment) models with fuzzy coefficients are given in Equations (2.9) and (2.10).

$$\max \quad \sum_{r=1}^{s} u_r \tilde{y}_{rp} \tag{2.9}$$

s.t.

$$\sum_{i=1}^{m} v_i \tilde{x}_{ip} = 1$$

$$\sum_{r=1}^{s} u_r \tilde{y}_{rj} - \sum_{i=1}^{m} v_i \tilde{x}_{ij} \le 0 \quad \forall i, r$$

$$u_r, v_i \ge \varepsilon \qquad \forall r, i$$

$$\min \quad \theta_p \tag{2.10}$$

s.t.

$$\sum_{j=1}^{n} \lambda_j \tilde{x}_{ij} \le \theta_p \tilde{x}_{ip}$$

$$\sum_{j=1}^{n} \lambda_j \tilde{y}_{rj} \ge \tilde{y}_{rp}$$

$$\lambda_j \ge 0$$

$$\theta_p \text{ free}$$

2.2.2. A chronicle development of fuzzy in measurement of efficiency

DEA researchers have begun using fuzzy concept for measuring efficiency and productivity of decision making units since 1992. This section gives a chronicle development of fuzzy concept in measurement of efficiency with Data Envelopment Analysis.

1992

Seaver and Triantis (1992) developed a method of fuzzy clustering approach in evaluating technical efficiency measures in manufacturing.

Sengupta (1992a) explored the use of fuzzy set-theoretic measures in the context of Data Envelopment Analysis, which utilizes a nonparametric approach to measure efficiency. He employed three types of fuzzy statics e.g., fuzzy mathematical programming, fuzzy regression and fuzzy entropy, to illustrate the types of decisions and solutions that are achievable, when the data are vague and prior information is inexact and imprecise.

Sengupta (1992b) developed methods of measuring economic efficiency of input-output systems by employing a fuzzy statistical approach using DEA. He illustrated fuzzy measures in the context of a two-person game theory model.

1994

Morita and Nose (1994) introduced fuzzy categorical variables in Data Envelopment Analysis. They considered ambiguous data and propose a DEA model for a non-controllable fuzzy categorical input variable. As claimed by them, their model gives a reasonable efficiency score and has robustness against the change of the boundaries of categorization.

1995

Morita (1995) further developed his earlier work on fuzzy DEA to address the uncertainty issue of the input and output data such as an observational disturbance and subjective data in DEA using fuzzy approach.

1997

Triantis (1997) and Triantis and Girod (1997) introduced a fuzzy non-radial DEA measures of technical efficiency. They replaced the notion of a radial distance measure with the concept of a non-radial distance measure. Also, the assumption of crisp production plans was substituted with the assumption of fuzzy production plans. Their paper merged these concepts and evaluated the efficiency performance of a newspaper preprint insertion production line.

Uemura (1997) focused on satisfactional method by introducing the concept of a fuzzy goal in DEA. The main focus of the paper is to address the case where they obtain DEA efficiency for *DMU*s, from looking at only one output. This means that DEA analyzes one output by some inputs and ignores other outputs.

1998

Chai and Ho (1998) dealt with ordinal data in DEA using fuzzy criteria. Their main objective was to use multiple criteria decision model for resource allocation with a case study in an electric utility company. In their paper the decision situation is characterized by a large number of projects competing for limited funding, the presence of fuzzy criteria, and the available data being ordinal in nature. The large number of projects and the nature of the data preclude the use of utility-theoretic approaches. The ordinal-DEA decision model is used as a screening tool in a partially automated decision process.

Kahraman and Tolga (1998) used DEA in fuzzy environments, which allow flexibility in constraints and nonlinear programming. In their paper, assuming that the values of inputs and outputs in DEA are not known with certainty, a fuzzy mathematical programming is proposed in which the objective function and the constraints are represented by using their degrees of membership in DEA. The main advantage of this solution is that the decision maker is not forced into a precise formulation for mathematical reasons.

Karsak (1998) introduced a two-phase robot selection procedure which in phase 1, DEA is used as a means to determine the technically efficient robot alternatives, considering cost and technical performance parameters. In the second phase, a fuzzy robot selection algorithm is used so as to rank the technically efficient robots according to both predetermined objective criteria and additional vendor-related subjective criteria. The algorithm presented in the paper is based on calculating fuzzy suitability indices for the technically efficient robot alternatives, and then, ranking the fuzzy indices to select the best robot alternative. The algorithm proposed in the paper is also applicable to a broader area of decision problems, e.g. facility site selection in order to determine the best CNC machine or flexible manufacturing system among a set of mutually exclusive alternatives.

Meada et al. (1998) built their research on fuzzy DEA with interval efficiency. With interval efficiency there exist two phases of efficiency evaluation with respect to the upper limit and the lower limit. From these viewpoints, they defined two extreme points of efficiency. As a result, interval efficiency for each decision-making unit can be obtained. They also formulated the interval cross-efficiency.

Tanaka et al. (1998) explored the possibility data analysis with rough sets concept. Hence they dealt with the upper and lower approximation models for