SYMMETRY AND DOUBLE REDUCTION FOR EXACT SOLUTIONS OF SELECTED NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

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To my beloved father and mother

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ABSTRACT

Amongst the several analytic methods available to obtain exact solutions of non-linear differential equations, Lie symmetry reduction and double reduction technique are proven to be most effective and have attracted researcher from different areas to utilize these methods in their research. In this research, Lie symmetry analysis and double reduction are used to find the exact solutions of non-linear differential equations. For Lie symmetry reduction method, symmetries of differential equation will be obtained and hence invariants will be obtained, thus differential equation will be reduced and exact solutions are calculated. For the method of double reduction, we first find Lie symmetry, followed by conservation laws using 'Multiplier' approach. Finally, possibilities of associations between symmetry with conservation law will be used to reduce the differential equation, and thereby solve the differential equation. These methods will be used on some physically very important nonlinear differential equations; such as Kadomtsev-Petviashvili equation, Boyer-Finley equation, Short Pulse Equation, and Korteweg-de Vries-Burgers equations. Furthermore, verification of the solution obtained also will be done by function of PDETest integrated in Maple or comparison to exist literature.

ABSTRAK

Antara beberapa kaedah analitis yang terdapat untuk mendapatkan penyelesaian tepat bagi persaman terbitan tidak linear, penurunan simetri Lie dan teknik penurunan dua kali ganda telah terbukti merupakan cara yang paling berkesan, dan telah menarik perhatian pengkaji dari berbeza bidang untuk menggunakan kaedah ini dalam pengkajian mereka. Dalam kajian ini, analisis simetri Lie dan penurunan dua kali ganda digunakan untuk mencari penyelesaian tepat bagi persamaan-persamaan terbitan tidak linear. Untuk kaedah penurunan simetri Lie, simetri-simetri persamaan terbitan akan didapatkan dan, maknanya koordinat berkanun akan didapati, oleh itu, persamaan terbitan akan diturunkan dan penyelesaian tepat akan dikira. Untuk kaedah penurunan dua kali ganda, kami cari simetri Lie dulu, diikuti dengan hukum-hukum keabadian dengan menggunakan pendekatan Pendarab. Akhirnya, kemungkinan kesekutuan antara simetri dan hukum keabadian akan digunakan untuk menurunkan persamaan terbitan dan dengan itu menyelesaikan persamaan terbitan tersebut. Kaedah ini akan digunakan pada sesetengah persamaan terbitan tak linear fizikal yang sangat penting; seperti persamaan Kadomtsev-Petviashvili, persamaan Boyer-Finley, Persamaan Nadi Pendek, dan Persamaan Korteweg-de Vries-Burgers. Tambahan pula, pengesahan untuk jawapan yang didapati juga akan ditentusahkan dengan fungsi PDETest yang diintegrasikan dalam Maple atau perbandingan dengan literatur yang wujud.

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CHAPTER 1

INTRODUCTION

1.1 Introduction

In this thesis, Chapter One represents the basic direction of the research. Meanwhile, Chapter Two discusses the previous researches related to the topic focusing on how symmetry and conservation laws are obtained; hence showing the symmetry reduction and double reduction. Chapter Three, Chapter Four, Chapter Five, and Chapter Six presents the methods used to solve Kadomtsev-Petviashvili equation, Boyer-Finley equation, Short Pulse Equation, and Korteweg-De Vries-Burgers equations, respectively. The final chapter, which is Chapter Seven, focuses on addressing the conclusion of this study.

This chapter contains the background of the study, problem statement, objectives, scope as well as the significance of study.

1.2 Background of the Research

Partial differential equations (PDEs) are often used in the modelling of nonlinear physical phenomena. However, nonlinearity is of a great challenge to the researchers. Several authors from different background have responded to this task and consequently, many methods (numerical, analytical) have been developed to determine the possible solutions for nonlinear equations. These types of PDEs normally involve high orders and there is no universal method to solve all nonlinear PDEs. Apparently, there are some methods that can solve certain complex PDEs under certain restrictions. One of the approaches involves the invariance of PDEs under one parameter known as the Lie group of transformations, or often called the Lie symmetries. After finding Lie symmetry of a PDE, an invariant solution of the PDE via reduction process can be made using a change of variables. Compared to other exact methods that usually require extra conditions, this method is a universal method to solve PDEs and ODEs without many obstacles. Hence, a class of more general solutions can be obtained, which are very helpful in studying the equations under different boundary conditions. Among the many literatures available on the subject are the well known fundamental references including (Ibragimov & Lie, 1994), Application of Lie Groups to Differential Equations by Olver (2000), Elementary Lie Group Analysis and Ordinary Differential Equations by Ibragimov (1999), as well as the Symmetry and Integration Methods for Differential Equations by Bluman & Anco (2010).

Another recently developed route for analysing differential equations is by finding the conservation laws. In addition to having physical interpretations like conservations of energy, momentum and volume, these laws provide a mechanism for reducing the differential equations. Many significant methods have been developed to construct conservation laws such as the Noether's theorem for variational problems (Noether, 1971; Wang *et al.*, 2014), multiplier approach (Anco & Bluman, 2002a, 2002b; Bluman & Anco, 2010), symmetry action on a known conservation law (Bluman *et al.*, 2006), partial Noether approach (Kara & Mahomed, 2006) and a new conservation method (Ibragimov, 2007). The classical approach is by Noether's theorem for variational PDEs in which a Lagrangian has to be known. There are other PDEs that belonged to the evolution type equation, which do not admit Lagrangian. To handle these equations, one can use the direct method also known as the 'multiplier' approach (Anco & Bluman, 2002a, 2002b; Bluman & Anco, 2010), which directly utilises the definition. This approach has been actively pursued recently; it involves constructing multipliers for PDEs that are then further analysed to obtain corresponding conserved vectors.

The theory of double reduction of a PDE (or systems of PDEs) is well-known for the association of conservation laws with Noether symmetries (Bluman & Anco, 2010; Cariello & Tabor, 1991). Meanwhile, the association of conservation laws with Lie Backlund symmetries (Kara & Mahomed, 2000) and non-local symmetries (Sjöberg & Mahomed 2004) led to the expansion of the theory of double reduction for PDEs with two independent variables, which do not possess Noether symmetries (Sjöberg 2007). Solving PDEs through double reduction may not be as universal as symmetry method since some PDEs do not possess any unique and non-trivial conservation laws. However, if PDEs do possess non-trivial conservation laws, the double reduction method will be able to find such exact solution that may not be obtained through symmetry method. Besides, this method is straightforward and more effective in reducing the order and variable of an equation in one step. Furthermore, this method provides a mechanism to construct more solutions with less restrictions and limitations compared to other methods. Lastly, a PDE contains more conservation laws, which means that it has high integrability considering that one can perform double reductions through conservation laws.

One way in which PDEs can be used is through models involving more than one independent variable. Plasma is a significant technology recently proposed. It is widely applied in the fields such as Biology, Physics and computing. An important equation in plasma to describe wave surface problem for an incompressible fluid with free surface and rigid horizontal bottom boundary conditions is called the Kadomtsev-Petviashvili (KP) equation (Kadomtsev & Petviashvili, 1970), which is originated from the study on Korteweg–de Vries (KdV) equations.

Data deliver through silica optic are innovated by improving the technology in telecommunication. Huge amount of data can be sent in a short period using this silica optic. The model that describes the propagation of ultra-short light pulse in silica optical fibres is the one preferred for the study. In literature, this method refers to the Short Pulse equation (SPE) (Schäfer & Wayne, 2004).

General relativity theory plays an important role in many fields; for instance, general relativity theory as a backbone of quantum theory. Explanation on general relativity theory exposes many problems in finding all real, Euclidean, self-dual spaces with one Killing vector that was reduced, which was then replaced with another equation namely Boyer-Finley equation named after the Boyer and Finley III (1982) for their contribution in developing this equation. As for the importance of reveal general theory, this equation has been also examined within this study to find the exact solution.

Finally, this study also considered the Korteweg-de Vries-Burgers (KdV-B) equation. This equation yields the famous Korteweg-de Vries (KdV) equation. KdV-B equation exists in various physical situations. Here, KdV-B equation model from the theory of ferroelectricity (Zayko, 1989) were chosen. Ferroelectricity is generally used in choosing suitable material for a capacitor. Basically, capacitors are the main components to construct battery for electronic applicants. Hence, it is worthy to study this equation to help understanding the material for the battery of electronic applicants including smartphone and laptop.

This study was conducted to study the invariance, Lie symmetries and conservation laws of the above equations namely KP, Boyer-Finley, SPE and KdV-B equations, which were mentioned in the few last paragraphs. Meanwhile, the ultimate goal is to obtain the exact solutions that are not yet reported by existing literature.

1.3 Statement of the Problem

1) To study and tackle the nonlinearity of the following four nonlinear significant equations via Lie symmetry and conservation laws:

   i) Kadomtsev-Petviashvili (KP) equation,

$$(u_t + uu_x + u_{xxx})_x + 3s^2 u_{yy} = 0,$$

where $u = u(x, y, t)$. This equation is modelled using the Euler Equation describing the wave surface problem for an incompressible fluid with free surface and rigid horizontal bottom boundary condition.

ii) Boyer-Finley equation,

$$u_{xy} = (e^u)_{tt},$$

where $u = u(x, y, t)$. This is an equation of self-dual Einstein spaces of Euclidean signature with one rotational Killing vector.

iii) Short Pulse equation (SPE),

$$u_{xt} = \alpha u + \frac{1}{3}\beta(u^3)_{xx},$$

where $u = u(x, t)$, which the unknown real function and the subscripts denote differentiation with respect to *x* and *t; α* and *β* are nonzero real parameters. This model describes the propagation of ultra-short light pulses in silica optical fibres.

iv) Korteweg-de Vries-Burgers (KdV-B) equation,

$$u_t + Auu_x + Bu_{xx} + Cu_{xxx} = 0,$$

$$A = \frac{2\alpha}{K'(v)}, \qquad B = \frac{v}{K'(v)}, \qquad C = -\frac{v^2}{K'(v)}, \qquad K'(v) = \frac{dK(v)}{dv},$$

$$K(v) = \frac{\omega_p^2 v^2}{c^2 - v^2} - \omega_0^2 - 2\alpha u_0$$

where u is the first term in expanding the series of polarisation with respect to small attenuation coefficient; $K(v) = 0$ is the dispersive equation for wave velocity; u_0 is the equilibrium value of u; ω_p, ω_0 are the frequencies of wave, which vary according to different problems; c represents the velocity of light; α is the coefficient determined by the system. These equations describe a ferroelectric system.

2) The method on the four equations, KP, Boyer-Finley equation, SPE and KdV-B equation will involve deriving the symmetry for these four differential equations with the aid of programming.
3) Application of symmetry to reduce the KP, Boyer-Finley and KdV-B equation.
4) Methods to obtain conservation laws for KP, Boyer-Finley equation, and SPE via Multiplier approach.
5) Measuring the association between Lie symmetry and conserved vector.
6) Applying the conservation laws combined with underlying associated symmetries to employ the 'double reduction' on KP equation, Boyer-Finley equation and SPE, thus calculating their solutions.

1.4 Objectives of the Study

The objectives of this study are as follow:

1. To calculate the possible Lie symmetries and conservation laws for Short Pulse Equations (SPE), Boyer-Finley equation, Kadomtsev-Petviashvili (KP) equation, and Korteweg-de Vries-Burgers (KdV-B) equation.
2. To determine the possible association between symmetry and conserved vector of SPE, Boyer-Finley equation and KP equation.
3. To utilise symmetry in reducing Boyer-Finley, KP and KdV-B equations.
4. To apply the association of conservation laws and symmetry in double reduction to the equation, and to obtain the exact solution for SPE, Boyer-Finley and KP equations.

1.5 Scope of the Study

This study concentrates on using symmetry approach and/or double reduction approach with the combination of symmetry and conservation laws to reduce non-linear differential equations namely Short Pulse Equation, Boyer-Finley Equation, Kadomtsev-Petviashvili (KP) Equation, and Korteweg-de Vries-Burgers (KdV-B) Equation. Multiplier approach was selected to find the conservation laws of these equations instead of other methods.

1.6 Significance of the Study

The method of Lie symmetry and double reduction provides a mechanism to tackle a considerable amount of nonlinear differential equations that are not easily handled by other integration methods. In some cases, even analytic methods failed to produce any results. In fact, this method is the only universal method that produces analytics solutions (Kara & Mahomed, 2000). For instance, in fluid flow problems, # mathematicians, physicists or engineers often faced many complicated nonlinear high order dimensional differential equations in complex domain with a number of unknown parameters. Nonetheless, the order of equation can be easily reduced using the Lie symmetry and conservation laws technique, thus aiding in solving or investigating the system with much ease. Moreover, as compared to numerical methods, this method is more efficient and cheaper. Besides, the exact solutions obtained by this method can serve as the benchmark for testing the algorithms and accuracy of numerical solutions. Exact solutions of differential equation may also help scientist or physicist to detect or measure the accuracy and sensitivity of various variables in their physical interest that involved in the equation.

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