STOCHASTIC DIFFERENTIAL EQUATION
FOR TWO-PHASE GROWTH MODEL

GRANITA

UNIVERSITI TEKNOLOGI MALAYSIA

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In the name Allah Subhanahu wata'ala

Specially dedicated

to my beloved husband Feri and children Taqiyyah, Rafief and Muthie'ah,

Mother, parent in law and my big family

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ABSTRACT

Most mathematical models to describe natural phenomena in ecology are models with single-phase. The models are created as such to represent the phenomena as realistic as possible such as logistic models with different types. However, several phenomena in population growth such as embryos, cells and human are better approximated by two-phase models because their growth can be divided into two phases, even more, each phase requires different growth models. Most two-phase models are presented in the form of deterministic models, since two-phase models using stochastic approach have not been extensively studied. In previous study, Zheng's two-phase growth model had been implemented in continuous time Markov chain (CTMC). It assumes that the population growth follows Yule process before the critical size, and the Prendiville process after that. In this research, Zheng's two-phase growth model has been modified into two new models. Generally, probability distribution of birth and death processes (BDPs) of CTMC is intractable; and even if its first–passage time distribution can be obtained, the conditional distribution for the second-phase is complicated to be determined. Thus, two-phase growth models are often difficult to build. To overcome this problem, stochastic differential equation (SDE) for two-phase growth model is proposed in this study. The SDE for BDPs is derived from CTMC for each phase, via Fokker-Planck equations. The SDE for two-phase population growth model developed in this study is intended to be an alternative to the two-phase models of CTMC population model, since the significance of the SDE model is simpler to construct, and it gives closer approximation to real data.

ABSTRAK

Kebanyakan model matematik untuk menggambarkan fenomena semulajadi dalam ekologi adalah dengan model fasa tunggal. Model-model dibina supaya dapat mewakili fenomena serealistik mungkin, seperti model logistik pelbagai jenis. Walau bagaimanapun, beberapa fenomena dalam pertumbuhan populasi seperti embrio, sel dan manusia adalah lebih sesuai dianggarkan menggunakan model dua fasa kerana pertumbuhan populasi boleh dibahagikan kepada dua fasa, lebih-lebih lagi setiap fasa memerlukan model pertumbuhan yang berbeza. Kebanyakan model dua fasa dibentangkan dalam bentuk model deterministik, disebabkan model dua fasa menggunakan pendekatan stokastik masih belum dikaji secara terperinci. Dalam kajian terdahulu, model pertumbuhan dua fasa Zheng telah dilaksanakan dalam rantaian Markov masa selanjar (CTMC). Ia menganggap pertumbuhan populasi mengikuti proses Yule sebelum mencapai saiz kritikal, dan seterusnya proses Prendiville. Dalam penyelidikan ini, model pertumbuhan dua fasa Zheng diubah suai menjadi dua model baharu. Secara umum, taburan kebarangkalian proses kelahiran dan kematian (BDPs) dari CTMC adalah sukar dikawal; dan walaupun taburan masa laluan pertama boleh diperolehi, taburan bersyarat untuk fasa kedua adalah rumit untuk ditentukan. Oleh itu, model pertumbuhan dua fasa adalah selalunya sukar untuk dibina. Untuk mengatasi masalah ini, persamaan pembezaan stokastik (SDE) untuk model pertumbuhan dua fasa dicadangkan dalam kajian ini. SDE untuk BDP diterbitkan dari CTMC untuk setiap fasa melalui persamaan Fokker-Planck. Model pertumbuhan populasi dua fasa SDE yang dibangunkan dalam kajian ini bertujuan sebagai alternatif kepada model populasi dua fasa CTMC, disebabkan kepentingan SDE untuk model pertumbuhan populasi dua fasa adalah lebih mudah untuk dibina, dan model ini memberi penghampiran yang lebih dekat kepada data sebenar.

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LIST OF SYMBOLS

$X(t)$	_	Random variable
t, h	_	Time
$p_{ij}(t)$	_	Transition Probability
$P(s, t)$	_	Probability generating function
$E[X(t)] = m(t)$	_	Expected value of $X(t)$ or mean function
$Var(X(t)) = v(t)$	_	Variance of $X(t)$ or variance function
$\Pr\{X\}$	_	Probability of $X(t)$
$\lambda_{n,}\ \mu_n$	_	Transition rate for state n
λ	_	Birth rate
μ	_	Death rate
α	_	Immigration rate
β	_	Emigration rate
T_L	_	First time attains L
$F(t)$	_	Cumulative distribution function for T_L
$f(t)$	_	Probability density function
$W(t)$	_	Wiener process
Σ	_	Summation

$\|x\|$	_	absolute x
Δ	_	Step size
$p(x,t)$	_	Probability density function

LIST OF ABBREVIATIONS

BDP	_	Birth and death process
SDE	_	Stochastic differential equation
ODE	_	Ordinary differential equation
FPE	_	Fokker-Planck equation
BIDE	_	Birth and death with immigration-emigration
BID	_	Birth-immigration and death
pgf	_	Probability generating function
pdf	_	Probability density function
pmf	_	Probability mass function
cdf	_	Cumulative distribution function
EM	_	Euler-Maruyama
MLE	_	Maximum likelihood estimation
RMSE	_	Root Mean Square Error

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CHAPTER 1

INTRODUCTION

1.1 Background of Study

Many populations, such as cells and humans, indicate limitations in their growth. The populations can be identified based on the birth rate change in the size range of the population, which grows rapidly in the early and later changes slowly at the end of the phase. To describe population growth, two models are commonly used, i.e. deterministic and stochastic models. Usually, stochastic modeling for population growth is based on deterministic modeling, because deterministic model has been developed by many previous studies.

To model population growth using deterministic and stochastic models, the birth and death processes in single-phase are referred. The processes are mostly modelled using logistic models [1-3]. In logistic models, many distinct biological interpretations for different applications can be extracted from them. In some cases, the models with single phase may not represent the population growth due to their lack of flexibility [4-6]. Therefore, the combination between single phase models, which the model are used in their different early and end phase, is proposed. This is because there is a growth difference at the beginning and end of the period to fit the change of the growth.

Some studies on two-phase population growth model with regard to deterministic models have been done by combining the models. Banks [2] developed some combinations of two-growth population growth models, while Meyer [7], Meyer and Ausubel [8], Meyer et al [9] proposed bi-logistic model. Later, Wong and Goh [10] made a little modification to Meyer's approach.

Faddy [4] proposed a simple two-phase population growth model to pure death process in stochastic model. Ross and Pollett [6] developed a two-phase population growth model using control regime, while Zheng [11] built a two-phase population growth model considering the existence of a critical population size. In this method, before the population attains certain prescribed critical size, the growth is modelled as Yule process. Once the population reaches its critical size, the growth is then modelled using Prendiville's logistic process. So the critical size is the junction itself. The junction is known as an inflection point. The stochastic models used in the previous studies are models of continuous time Markov chain (CTMC).

The development of stochastic models is not as fast as the deterministic model, although the stochastic models are generally accepted to model the heterogeneity of phenomena in reality. In this study, the population growth is focused on the stochastic models by using deterministic model as guide to modelling.

Among stochastic models, stochastic differential equation (SDE) model is often used besides CTMC model. Both SDE and CTMC models are Markov processes, which differ only in state. CTMC model has continuous time and discrete state, while SDE model has both continuous time and state [12]. Also, there is a close relationship between the CTMC and SDE models. This relationship is derived by the forward Kolmogorov differential (Fokker-Planck) equation in diffusion process of CTMC [13, 14]. Then the BDPs of SDE model can be derived from CTMC model. Therefore, some types of the SDE for BDPs model are studied in this study in order to find out each other behaviour.

Allen and Allen [15] studied three stochastic models with respect to persistence time. The models are discrete time Markov chain (DTMC) models, continuous time Markov chain (CTMC) models and stochastic differential equation (SDE) models. They used birth and death processes, which were set as analogue to the logistic of growth models. The study was extended with the addition of environment variability and persistence-time estimation, as shown in [16].

In sum, several natural phenomena require two-phase population growth models. Some types of two-phase population growth model are built in deterministic and stochastic models. In relation to this, the Zheng's two-phase population growth model can be still modified by changing its second phase by other birth and death processes. In stochastic model, two-phase models are only studied by using CTMC models, although in case of single phase, the SDE model may be better than CTMC models in certain conditions [15-17]. The previous models in SDE growth model have only been applied to single-phase of stochastic growth model, and stochastic differential equation model for two-phase population growth model has not been done. Thus, this research proposes to build stochastic differential equation (SDE) model for two-phase population growth model.

1.2 Problem Statement

Both CTMC and SDE models have some weaknesses which are simply inherent and inevitable to model. In this study, advantages of each model had been used to manage their weaknesses. Generally, a growth process is a special case of continuous time Markov chain, where its state represents the current size of population, occurs in discrete and continuous time, with both change and time depending only on the previous state. Meanwhile, in SDE model, there are only approximations, in which to estimate drift and diffusion coefficient from discrete experimental data for the SDE's own variable. Nevertheless, SDE nearly matches the dynamics of CTMC Model.

Unlike CTMC model, SDE model has continuous trajectories. Although the state of the SDE is a vector of real numbers, the process keeps all possession of the stochasticity related with the discrete CTMC. For simple stochastic differential equations, explicit solutions can be obtained using Ito formula, but it is generally not possible to obtain explicit solutions to SDE model. Although there is no possible explicit solution, numerical methods can still be used to approximate the sample path of SDE. To get the sample path, CTMC model requires a limiting distribution where its stationary probability distribution satisfies certain conditions. In this study, birth and death process are furthermore discussed based on the advantages of both CTMC and SDE models.

Thus a modification to Zheng's two-phase growth might be developed with respect to birth and processes. Unfortunately, some types of birth and death process are difficult to get their probability distribution, thus two-phase growth model are often difficult to build. Because generally, probability distribution of CTMC is intractable and event if it can be obtained its first –passage time distribution and the conditional distribution for the second-phase are complicated to be determined. To overcome this problem, building second-phase in SDE model is proposed in this study. In this study, two-phase deterministic model proposed by Bank [2] has been used as a guide to build the SDE model for two-phase population growth model, while the model of each phase of the two-phase growth model was derived from CTMC.

1.3 Research Objectives

Based on the research background and problem statement, the objectives of this study are:

1. To derive seven types of the birth and death processes of the continuous time Markov chain (CTMC) model for the stochastic differential equation (SDE) model.
2. To modify the second-phase of Zheng's two-phase growth model of CTMC
3. To build stochastic differential equation for two-phase growth model.
4. To apply the models to some population growth data.

1.4 Research Scope

In this study, the connection between the CTMC model and the SDE model in the growth population was modeled for only birth and death processes with constant parameters, the stochastic process which conditional probability/transition probability satisfies forward Kolmogorov/Fokker-Planck equation. Then, whooping crane population growth data and the dissolved concentration oxygen data were applied in one-phase stochastic differential equation models. Furthermore, to build the two-phase population growth model of SDE; two models had been combined in this study; the Yule process in early phase and confined exponential process or Prendiville process last phase, Pekanbaru and California population data had been applied in this study

1.5 Significance of Research

Recently, two-phase population growth models have been used to describe various fields such as biology, economies, forecasting and many other purposes. Most of these applications use the two-phase population growth models of the deterministic models to describe the phenomena, due to the influence of the environment which cannot be eliminated thoroughly in the deterministic models, where stochastic two-phase population growth models are required. In this study, the proposed model is obtained by modifying Zheng's two-phase model.

Since, there are close relationships among some stochastic models, in this study, the birth and death processes of continuous time Markov chain (CTMC) were approximated by using the stochastic differential equation via the forward Kolmogorov equations. Therefore, stochastic calculus is used for solving the problem of intractability of the transition probability of the CTMC. The transition rates of BDPs of CTMC are then used to obtain the drift and diffusion coefficient of the SDE. This approximation produces Ito SDE models for birth and death processes. By this, the combination between two SDE models is used to build the SDE model for two-phase population growth model. These models are might be applied to relative areas.

1.6 Thesis Organization

This thesis consists of five chapters. Chapter 1 provides the background, problem statement, objectives, scopes, and significance of this study. Chapter 2 presents previous works related to this study and describes the one-phase and two-phase growth population models. Chapter 3 presents explanation on the theory of continuous Markov chain for the birth and death processes and stochastic differential equation. The methodologies to connect between continuous time Markov chain and stochastic differential equation, and to build stochastic two-phase population growth model are discussed in this chapter. Results and discussion are discussed in Chapter 4 and Chapter 5. Lastly, Chapter 6 concludes this thesis and gives several suggestions for future work.

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