ELZAKI TRANSFORM HOMOTOPY PERTURBATION METHOD FOR PARTIAL DIFFERENTIAL EQUATIONS

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To my beloved parents and to all my siblings

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ABSTRACT

Partial differential equations (PDEs) occur in many applications and play a big role in engineering and applied sciences. Since some PDEs are quite difficult to solve, many new methods are introduced to the academic community. Some of them are homotopy perturbation method, variational iteration method, adomian decomposition method, differential transformation method, ELzaki transform, ELzaki transform homotopy perturbation method (ETHPM) and etc. In this study two methods are considered which is homotopy perturbation method and ELzaki transform. The two methods were introduced and examples were presented to illustrate the efficiency of both methods. It is shown that both methods can be used to solve different types of partial differential equations. Although they can be used to solve PDEs, they have their own limitations. There are certain nonlinear forms of PDEs that are quite difficult to solve using ELzaki transform, and for homotopy perturbation method, the expansion itself sometimes can be quite difficult to solve. Then, the combination of both methods was introduced and the efficiency of the method was shown by solving some applications of partial differential equations. ETHPM was used to solve some gas dynamics and Klein-Gordon equations. The results are compared with previous study to determine the efficiency of the method. The graph of each solution is illustrated by using Mathematica software. From the result, it is shown that ETHPM method produces anticipated exact solutions and the calculations is not that complicated.

ABSTRAK

Persamaan pembezaan separa memainkan peranan yang amat penting dalam menyelesaikan masalah dalam kejuruteraan dan sains gunaan. Kebanyakan persamaan pembezaan separa adalah sukar untuk diselesaikan dan terdapat pelbagai kajian yang dilakukan untuk menyelesaikan persamaan ini antaranya ialah kaedah penguraian adomian, kaedah usikan homotopi, kaedah lelaran variasi, transformasi ELzaki dan banyak lagi kaedah analitikal lain. Dalam kajian ini, kita akan menggunakan dua kaedah iaitu kaedah usikan homotopi dan transformasi ELzaki. Kedua-dua kaedah ini akan dikaji dan contoh akan diberikan untuk mengetahui keberkesanan setiap kaedah. Kajian menunjukkan bahawa kedua-dua kaedah boleh menyelesaikan masalah persamaan pembezaan separa tetapi terdapat kelemahan pada kedua-dua kaedah. Apabila menggunakan kaedah usikan homotopi, pengembangan persamaan akan menyebabkan persamaan menjadi rumit untuk diselesaikan manakala transformasi ELzaki tidak dapat menyelesaikan kebanyakan persamaan tidak linear. Kemudian, gabungan transformasi ELzaki dan kaedah usikan homotopi digunakan untuk menyelesaikan aplikasi persamaan pembezaan separa seperti persamaan dinamik gas dan persamaan Klein-Gordon. Keputusan kajian dibandingkan dengan kajian yang terdahulu untuk menentukan keberkesanan kaedah dan diilustrasi menggunakan graf dari Matematika. Kajian menunjukkan bahawa gabungan kedua-dua kaedah memberikan keputusan yang dijangka dan kaedah pengiraan persamaan menjadi lebih mudah untuk diselesaikan.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

LIST OF ABBREVIATIONS

PDE	-	Partial Differential Equations
HPM	-	Homotopy Perturbation Method
ETHPM	-	ELzaki Transform Homotopy Perturbation Method
HAM	-	Homotopy Analysis Method

LIST OF SYMBOLS

p	-	Small parameter
$O(p)$	-	Order notation
$f(r)$	-	Analytic function
L	-	Linear part of equation
N	-	Nonlinear part of equation
$H_n(u)$	-	He's polynomials
T_p	-	Particular solution of ELzaki transform
E	-	ELzaki transform
E^{-1}	-	Inverse ELzaki transform
A, B, C, D	-	Constant

CHAPTER 1

INTRODUCTION

1.1 Motivation

A partial differential equation (PDE) is an equation involving functions of more than one independent variable and their partial derivatives. They occur in many applications and play a big role in engineering and applied sciences [1]. For instance, a second order partial differential equation for the function $u(x, y)$ is

$$F(x, y, u, u_x, u_y, u_{xx}, u_{yy}, u_{xy}) = 0$$

where the function F is given.

An equation is said to be linear if the unknown function and its derivatives are linear in F. An example of a first order linear equation is

$$a(x, y)u_x + b(x, y)u_y + c(x, y)u = f(x, y)$$

where the functions $a,\ b,\ c$ and f are given. On the contrary, nonlinear partial differential equations are equations with nonlinear terms. An example is Burgers' equation,

$$u_t + uu_x = \nu u_{xx}$$

occuring in various areas of applied mathematics such as fluid mechanics, nonlinear acoustics, gas dynamics, and traffic flow.

It is interesting to study partial differential equations since they appear in many physical phenomena which in general are very hard to solve [2]. Only a few have known exact solutions. In recent years, we have seen an increase in the study of new analytical methods or approximate analytical methods. Among others are homotopy analysis method [3], homotopy perturbation method [4], variational iteration method [5], differential transformation method [6], adomian decomposition method [7], sumudu transform [8], ELzaki transform [9, 1], etc.

Of these methods, homotopy perturbation techniques and ELzaki transform caught our attention. Although the homotopy method has been around for 17 years, it is not taught formally in the graduate school. Because of this, we would like to study more on the applicability of this method in solving partial differential equations. While ELzaki transform is quite new to the academic community and has been seen attractive in transforming differential equations into simple ones. Since the method is new, it still opens to plenty of researches. Furthermore, the combination of these two methods look more promising in solving differential equations.

1.2 Background of Study

Progress in applicable mathematics has been prospered by development of many important analytical approaches and techniques. One of the techniques is perturbation theory which has a long history behind it. Typically, the method is used to solve at least one small parameter [10]. Since partial differential equations mostly did not have a small parameter, a well defined perturbation method is sought after to tackle the problem. J.H. He [11] has proposed a homotopy perturbation method (HPM) that could handle this. The method has been studied by other authors to solve many partial differential equations such as diffusion equation [2], Helmholtz equation, Fisher's equation, Boussinesq equation, singular fourth-order partial differential equation and higher-dimensional initial boundary value problems [12]. HPM is the combination of homotopy concepts in topology and perturbation techniques [11]. The method works effectively in linear and nonlinear PDEs as it gives quick convergent approximations that lead to an exact solution. HPM also can be used to solve nonlinear problems directly without linearizing the problem [2].

The Laplace transform is one of many integral transforms in applied mathematics and often used to solve differential equations. Solving some differential equations were difficult using the Laplace transform. In 1993 Watugala introduced a new transform and named it as Sumudu transform [13]. The result is that the Sumudu transform gives a more simpler solution than Laplace transform [14]. While ELzaki transform is a transform derived from the classical fourier integral which was first introduced by T.M. ELzaki [9] in 2011. The ELzaki transform was then presented when Sumudu transform and Laplace transform failed to solve some differential equations with variable coefficients [15]. ELzaki transform which is a modified general Laplace and Sumudu transforms [9] is applied to solve PDEs such as wave and heat equations. ELzaki et al. [9] and Chopade et al. [1] have shown that ELzaki transform provides powerful method for analyzing PDEs.

In this research, the main focus is to solve PDEs using the combination of homotopy perturbation method and ELzaki transform method. From the solutions, the effectiveness of both methods will be compared to previous studies solutions. In this research, we will present few examples of PDEs problems that will be solved using homotopy perturbation method and ELzaki transform. We will also demonstrate a reliable combination of homotopy perturbation method and ELzaki transform to obtain the solution of partial differential equations.

1.3 Statement of Problem

With the rapid development of analytical methods throughout the years, many methods were introduced such as homotopy perturbation method, differential transformation method, adomian decomposition method, variational iteration method, sumudu transform, ELzaki transform etc. Two methods are presented in this research which are homotopy perturbation method (HPM) and ELzaki transform. HPM is introduced in this research because of its connection with the classical perturbation method. This method was first proposed to solve the limitation that perturbation method has in which it requires a small parameter exists in the equation whereas most PDEs have larger parameters. ELzaki transform is used in this study because this method is relatively new. So, research about this method still opens to new discoveries and we would like to take a deeper look into this method. The combination of these two methods seem interesting and viable to solve PDE problems which are either difficult to solve or have not had exact solution yet.

1.4 Research Questions

This research will answer the following questions:

1. What is homotopy perturbation method and what is ELzaki transform?
2. How to solve partial differential equations using homotopy perturbation method and ELzaki transform?
3. Are homotopy perturbation method and ELzaki transform reliable for solving partial differential equations?
4. How to solve partial differential equations equations using the combination of homotopy perturbation method and ELzaki transform?
5. Is the combination of homotopy perturbation method and ELzaki transform a good approach for solving applications of partial differential equations?

1.5 Objectives of The Research

The objectives of the research are:

1. To introduce homotopy perturbation and ELzaki transform methods.
2. To solve partial differential equations using homotopy perturbation method and ELzaki transform respectively.
3. To determine whether homotopy perturbation method and ELzaki transform are reliable for solving partial differential equations.
4. To solve some applications of partial differential equations using the combination of homotopy perturbation method and ELzaki transform.

1.6 Scope of The Research

In this research, the focus is only on solving problems in partial differential equations. This is because it is more difficult to find the exact analytical solutions. Two methods are employed, specifically the homotopy perturbation method and ELzaki transform. Then we combine these methods to solve some applications of partial differential equations.

1.7 Significance of The Research

The knowledge of solving partial differential equations using homotopy perturbation method and ELzaki transform can be utilized for further research. In addition, the methods are very useful and can be used as a tool to solve actual problems in various areas in engineering and natural sciences as most of the equations involve partial differential equations. Besides that, the combination of these methods provide an alternative and efficient approach for solving partial differential equations.

REFERENCES

1. Chopade, P. P. and Devi, S. B. Applications of Elzaki Transform to Ordinary Differential Equations and Partial Differential Equations. *International Journal of Advanced Research in Computer Science and Software Engineering*, 2015. 5(3): 38–41.
2. Desai, K. R. and Pradhan, V. Solution by homotopy perturbation method of linear and nonlinear diffusion equation. *International Journal of Engineering Technology and Advanced Engineering*, 2013. 3(4): 169–175.
3. Dehghan, M., Manafian, J. and Saadatmandi, A. Solving nonlinear fractional partial differential equations using the homotopy analysis method. *Numerical Methods for Partial Differential Equations*, 2010. 26(2): 448–479.
4. Momani, S. and Odibat, Z. Homotopy perturbation method for nonlinear partial differential equations of fractional order. *Physics Letters A*, 2007. 365(5): 345–350.
5. Batiha, B., Noorani, M. S. M. and Hashim, I. Numerical solution of sine-Gordon equation by variational iteration method. *Physics Letters A*, 2007. 370(5): 437–440.
6. Keskin, Y. and Oturanc, G. Reduced differential transform method for partial differential equations. *International Journal of Nonlinear Sciences and Numerical Simulation*, 2009. 10(6): 741–750.
7. Bildik, N. and Konuralp, A. Two-dimensional differential transform method, Adomian's decomposition method, and variational iteration method for partial differential equations. *International Journal of Computer Mathematics*, 2006. 83(12): 973–987.
8. Weerakoon, S. Application of Sumudu transform to partial differential equations. *International Journal of Mathematical Education in Science and Technology*, 1994. 25(2): 277–283.
9. Elzaki, T. M. On the Connections between Laplace and Elzaki transforms. *Advances in Theoretical and Applied Mathematics*, 2011. 6(1): 1–11.
10. Johnson, R. S. *Singular perturbation theory: Mathematical and analytical*

techniques with applications to engineering. Springer Science & Business Media. 2006.

11. He, J.-H. Homotopy perturbation technique. *Computer Methods in Applied Mechanics and Engineering*, 1999. 178(3-4): 257–262.

12. Mohyud-Din, S. T. and Noor, M. A. Homotopy perturbation method for solving partial differential equations. *Zeitschrift für Naturforschung A*, 2009. 64(3-4): 157–170.

13. Watugala, G. Sumudu transform: a new integral transform to solve differential equations and control engineering problems. *Integrated Education*, 1993. 24(1): 35–43.

14. Eltayeb, H. and Kılıçman, A. A note on the Sumudu transforms and differential equations. *Applied Mathematical Sciences*, 2010. 4(22): 1089–1098.

15. Elzaki, T. M. Elzaki and Sumudu transforms for solving some differential equations. *Global Journal of Pure and Applied Mathematics.*, 2012. 8(2): 167–173.

16. Ahmed, M. E. S. *et al.* *Application of Homotopy Perturbation Method to Linear and Nonlinear Partial Differential Equations*. Ph.D. Thesis. Sudan University of Science and Technology. 2016.

17. Liao, S. Notes on the homotopy analysis method: Some definitions and theorems. *Communications in Nonlinear Science and Numerical Simulation*, 2009. 14(4): 983 – 997.

18. Momani, S. and Odibat, Z. Comparison between the homotopy perturbation method and the variational iteration method for linear fractional partial differential equations. *Computers and Mathematics with Applications*, 2007. 54(78): 910 – 919.

19. Yildirim, A. He's homotopy perturbation method for solving the space- and time-fractional telegraph equations. *International Journal of Computer Mathematics*, 2010. 87(13): 2998–3006.

20. Chun, C. and Sakthivel, R. Homotopy perturbation technique for solving two-point boundary value problems comparison with other methods. *Computer Physics Communications*, 2010. 181(6): 1021 – 1024.

21. Bera, P. and Sil, T. Homotopy perturbation method in quantum mechanical problems. *Applied Mathematics and Computation*, 2012. 219(6): 3272 – 3278.

22. Fatima, N. Homotopy Perturbation Method for Solving Boussinesq and

Fishers Type Equations. *Computational Intelligence & Communication Technology (CICT), 2016 Second International Conference on*. IEEE. 2016. 478–483.

23. Ghorbani, A. Beyond Adomian polynomials: He's polynomials. *Chaos, Solitons and Fractals*, 2009. 39(3): 1486 – 1492.

24. Elzaki, T. M. On the ELzaki Transform and System of Partial Differential Equations. *Advances in Theoretical and Applied Mathematics*, 2011. 6(1): 115–123.

25. Elzaki, T. M. The new integral transform Elzaki Transform. *Global Journal of Pure and Applied Mathematics*, 2011. 7(1): 57–64.

26. Elzaki, T. M. and Hilal, E. M. Analytical Solution for Telegraph Equation by Modified of Sumudu Transform" Elzaki Transform. *Mathematics Theory and Modeling, Vol2*, 2012. (4).

27. Pradip R. Bhadane, K. P. G. Solution of Advection-Diffusion Equation for Concentration of Pollution and Dissolved Oxygen in the River Water by Elzaki Transform. *American Journal of Engineering Research (AJER)*, 2016. 5(9): 116 – 121.

28. Lee, Y. and Kim, H. The form of solution of Dirichlet problem for the heat equation by using Elzaki transform. *Global Journal of Pure and Applied Mathematics*, 2016. 12(4): 3297–3303.

29. Kashuri, A., Fundo, A. and Kreku, M. Mixture of a new integral transform and homotopy perturbation method for solving nonlinear partial differential equations. *Advances in Pure Mathematics*, 2013. 3(03): 317.

30. Hesameddini, E. and Abdollahy, N. Homotopy perturbation and Elzaki transform for solving Sine-Gorden and Klein-Gorden equations. *Iranian Journal of Numerical Analysis and Optimization*, 2013. 3(2).

31. Elzaki, T. M. and Kim, H. The solution of Burgers equation by Elzaki homotopy perturbation method. *Applied Mathematical Sciences*, 2014. 8(59): 2931–2940.

32. Bhadane, P. K. G. and Pradhan, V. H. ELzaki Transform Homotophy Pertubation method for solving porous medium equation. *IJRET: International Journal of Research in Engineering and Technology*, 2013. (2): 116–119.

33. Elzaki, T. M. and Ezaki, S. M. Application of New Transform " Elzaki Transform " to Partial Differential Equations. 2011. 7(1): 65–70.

34. Jafarib, E. S. H. Variational iteration method: A tool for solving partial

differential equations. *The journal of Mathematics and Computer Science*, 2011. 2(2): 388–393.

35. Nourazar, S., Ramezanpour, M. and Doosthoseini, A. A new algorithm to solve the gas dynamics equation: An application of the Fourier Transform Adomian Decomposition Method. *Applied Mathematical Sciences*, 2013. 7(86): 4281–4286.

36. Aminikhah, H. and Jamalian, A. Numerical approximation for nonlinear gas dynamic equation. *International Journal of Partial Differential Equations*, 2013: 1–7.

37. Bhadane, P. K. G. and Pradhan, V. ELzaki Transform Homotopy Perturbation method for Solving Gas Dynamics Equation. *IJRET: International Journal of Research in Engineering and Technology*, 2013. 2(12): 2319–1163.

38. Matinfar, M., Saeidy, M., Mahdavi, M. and Rezaei, M. Variational iteration method for exact solution of gas dynamic equation using Hes polynomials. *Bulletin of Mathematical Analysis and Applications*, 2011. 3(3): 50–55.

39. Hemeda, A. and Alluhaibi, M. S. New iterative method for solving gas dynamic equation. *International Journal of Applied Mathematical Research*, 2014. 3(2): 190–195.

40. Alomari, A., Noorani, M. S. M. and Nazar, R. M. Approximate analytical solutions of the Klein-Gordon equation by means of the homotopy analysis method. *Journal of Quality Measurement and Analysis JQMA*, 2008. 4(1): 45–57.

41. Nam, S. and Kim, H. The representation on solutions of the Sine-Gordon and Klein-Gordon equations by Laplace transform. *Appl. Math. Sci*, 2014. 8: 4433–4440.

42. Khalid, M., Sultana, M., Zaidi, F. and Arshad, U. Solving linear and nonlinear klein-gordon equations by new perturbation iteration transform method. *TWMS Journal of Applied and Engineering Mathematics*, 2016. 6(1): 115.

43. Hesameddini, E. and Abdollahy, N. Homotopy perturbation and Elzaki transform for solving Sine-Gorden and Klein-Gorden equations. *Iranian Journal of Numerical Analysis and Optimization*, 2013. 3(2).

44. Mohyud-din, S. T. and Yildirim, A. Variational iteration method for solving Klein-Gordon equations. *Journal of Applied Mathematics, Statistics and Informatics (JAMSI)*, 2010. 6(1).

45. Yildirim, A., Mohyud-Din, S. T. and Zhang, D. Analytical solutions to the pulsed Klein–Gordon equation using modified variational iteration method (MVIM) and Boubaker polynomials expansion scheme (BPES). *Computers & Mathematics with Applications*, 2010. 59(8): 2473–2477.