

A HYBRID RESULTANT MATRIX ALGORITHM BASED ON
THE SYLVESTER-BÉZOUT FORMULATION

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To
my beloved husband, Mohd Hanip Aspar
and my lovely children
Nurul Hidayah, Nurfatihah, Muhammad Aflah Saifuddin, and Nur Aishaturradhiah.

Specially dedicated to
my loving and supportive parents, Hj. Ahmad and Hjh Sa'adiyah.

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ABSTRACT

The resultant of a system of polynomial equations is a factor of the determinant of the resultant matrix. The matrix is said to be optimal when its determinant equals exactly the resultant. Other factors besides the determinant are known as extraneous factors and it has been the major interest among researchers to seek for a determinantal resultant formula that gives optimal resultant matrix whose determinant exactly equals the resultant. If such determinantal formula does not exist, a formulation that reduces the existence of these extraneous factors is sought. This thesis focuses on the construction and implementations of determinantal formulas that gives exact resultant for certain classes of multihomogeneous multivariate polynomial equations. For the class of multigraded polynomial systems, a Sylvester type formula giving exact resultant can be derived out of certain degree vectors. The first part of this thesis implements the Sylvester type formula for determining the entries and dimension of the Sylvester type matrix of multigraded systems by applying the properties of certain linear maps and permutations of groups of variables. Even though the Sylvester type formula gives exact resultants for multigraded systems, this approach does not take advantage of the sparseness conditions when considering sparse polynomials. Sparse systems can be utilized by considering the underlying properties of its Newton polytopes, the convex hull of the support of the system. Preliminary observations on the properties of mixed volumes of the polytopes in comparison to the degree of the resultant of polynomial systems derived from Sylvester type matrices are used in the determination of whether the resultant matrix is optimal. This research proceeds to construct and implement a new hybrid resultant matrix algorithm based on the Sylvester-Bézout formulation. The basis of this construction applies some related concepts and tools from algebraic geometry such as divisors, fans and cones, homogeneous coordinate rings and the projective space. The major tasks in the construction are determining the degree vector of the homogeneous variables known as homogeneous coordinates and solving a set of linear inequalities. In this work, the method of solving these equations involves a systematic procedure or combinatorial approach on the set of exponent vectors of the monomials. Two new rules are added as a termination criterion for obtaining the unique solutions for the Bézout matrix. The implementation of the new algorithm on certain class of unmixed multigraded systems of bivariate polynomial equations with some coefficients being zero suggests conditions that can produce exact resultant. From the results, some theorems on these conditions and properties are proven. An application of the hybrid resultant matrix to solving the multivariate polynomial equations in three variables is discussed. Upon completion of this research two new computer algebra packages have been developed, namely the Sylvester matrix package for multivariate polynomial equations and the hybrid Sylvester-Bézout matrix package for computing the resultant of bivariate polynomial equations.

ABSTRAK

Hasil suatu sistem persamaan polinomial ialah faktor bagi penentu matriks hasilan. Apabila penentu matriks hasilan bersamaan tepat dengan hasilan, matriks tersebut dikatakan sebagai optimum. Faktor-faktor lain selain penentu yang dikenali sebagai faktor-faktor luaran telah menjadi kepentingan utama di kalangan penyelidik untuk mencari formula matriks hasilan penentuan yang optimum di mana penentunya bersamaan tepat dengan hasilan. Jika formula penentuan tersebut tidak wujud, maka formula bagi mengurangkan kewujudan faktor luaran adalah diperlukan. Tesis ini ditumpukan kepada pembinaan dan perlaksanaan formula penentuan yang memberikan hasilan tepat kepada persamaan polinomial multivariat bermultihomogen untuk kelas tertentu. Sistem polinomial berkelas multigred boleh menghasilkan formula jenis Sylvester yang dapat memberi hasilan tepat daripada vektor darjah tertentu. Bahagian pertama tesis ini melaksanakan formula jenis Sylvester untuk menentukan kemasukan dan dimensi matriks jenis Sylvester bagi sistem multigred dengan menggunakan pemetaan linear dan pilihatur kumpulan. Walaubagaimanapun, pendekatan ini tidak melibatkan kelebihan sifat kejarangan sesuatu sistem apabila sistem polinomial jarang dipertimbangkan. Sistem jarang boleh digunakan dengan mengambil kira ciri yang berasaskan politop Newton, iaitu badan cembung sokongan sistem. Pemerhatian awal terhadap sifat isipadu campuran politop dibandingkan dengan darjah hasilan sistem polinomial yang diperolehi daripada matriks jenis Sylvester digunakan untuk menentukan samaada matriks hasilan adalah optimum. Penyelidikan ini diteruskan untuk membina dan melaksana satu algoritma matriks hasilan hibrid berasaskan formula Sylvester-Bézout. Asas pembinaan algoritma matriks hibrid perlu menggunakan konsep geometri algebra seperti pembahagi, kipas dan kon, gelanggang koordinat homogen dan ruang unjuran. Kesukaran utama dalam pembinaan matriks hasilan hibrid adalah untuk menentukan vektor darjah bagi pembolehubah homogen yang merupakan koordinat homogen dan mendapatkan penyelesaian bagi suatu sistem ketaksamaan linear. Kaedah penyelesaian sistem ketaksamaan ini melibatkan tatacara bersistem atau pendekatan kombinatorial terhadap set vektor eksponen bagi semua monomial. Dua tatacara baharu ditambah sebagai kriteria penamatan bagi mendapatkan penyelesaian unik untuk matriks Bézout. Perlaksanaan algoritma baharu terhadap kelas tertentu bagi sistem persamaan multigred dua pembolehubah tak bercampur dengan sebahagian koefisiennya kosong, mencadangkan syarat yang boleh memberikan hasilan tepat. Beberapa teorem berkaitan syarat hasilan tepat dibuktikan. Satu aplikasi hibrid matriks hasilan untuk menyelesaikan persamaan polinomial multivariat dengan tiga pembolehubah dibincangkan. Sebagai melengkapkan penyelidikan ini, dua pakej komputer algebra telah dibangunkan, iaitu pakej matriks Sylvester bagi persamaan polinomial multivariat dan pakej matriks hibrid Sylvester-Bézout untuk pengiraan hasilan persamaan polinomial dua pembolehubah.

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LIST OF SYMBOLS

| | | |
|---|---|---|
| $\mathcal{A}_i \subseteq \mathbb{Z}^n$ | - | support of f_i |
| $A + B = \{a + b a \in A, b \in B\}$ | - | Minkowski sum of sets A and B |
| \mathbb{C} | - | set of complex numbers |
| $\mathbb{C}[y_1, y_2, \dots, y_s]$ | - | homogeneous coordinate ring |
| $D = \sum_{i=1}^s a_i D_i$ | - | Weil divisor |
| $F_0 = F_1 = \dots = F_n = 0$ | - | homogeneous polynomials |
| $f_1 = f_2 = \dots = f_m = 0$ | - | polynomial equations |
| $f_i = \sum_{\alpha \in \mathcal{A}_i} C_{i,\alpha} x^\alpha$ | - | terms in each polynomial with coefficients $C_{i,\alpha} \in \mathbb{Z}^n$ |
| $H^{q=p-v}(X, m - pd) \binom{n+1}{p}$ | - | cohomology |
| $K_v(m)$ | - | terms in complexes |
| $(l_1, \dots, l_r; k_1, \dots, k_r)$ | - | type of multihomogeneous system |
| $MV(\mathcal{Q})$ | - | mixed volume of Newton polytope |
| \mathbb{P} | - | projective space |
| $\mathbb{P}^n(K) = (K^{n+1} - \{0\})$ | - | n -dimensional projective space |
| $\mathcal{Q}_i = \text{Conv}(\mathcal{A}_i) \subseteq X^n$ | - | Newton polytopes (convex hull of support \mathcal{A}_i) |
| \mathbb{R} | - | set of real numbers |
| $\mathcal{R}(f_0, \dots, f_n)$ | - | resultant |
| $S = \mathbb{C}[y_1, y_2, \dots, y_s]$ | - | polynomial ring |
| $[uvw]$ | - | entries of Bézout matrix |
| $[uvw]y^{u+v+w-\alpha-\omega}$ | - | monomial in homogeneous polynomial ring |
| $V(F) \subset \mathbb{P}^n$ | - | projective variety of \mathbb{P}^n for homogeneous polynomials F |
| y_1, y_2, \dots, y_s | - | facet variables |

| | | |
|--|---|---|
| \mathbb{Z} | - | set of integer numbers |
| $\alpha = \{\alpha_1, \dots, \alpha_s\} \subseteq \mathbb{Z}^n$ | - | exponent vectors of x |
| δ_k | - | defect vectors for multihomogeneous systems |
| Σ_Q | - | fan |
| σ_s | - | cones |
| \simeq | - | approximately equal |
| $\phi_Q : \mathbb{Z}^n \rightarrow \mathbb{Z}^s$ | - | homogenization map |
| \sim | - | similarity or equivalent |
| $0 \longrightarrow K_1 \xrightarrow{\Phi} K_2 \longrightarrow 0$ | - | Koszul complex |

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CHAPTER 1

INTRODUCTION

1.1 Preface

Research on polynomial systems has a wide range of applications in science and engineering. For example, Bézier-Bernstein splines have been used to model mechanical components in computer aided design (CAD) and computer aided manufacturing (CAM) systems [1]. In robotics, when a robot moves, it needs to detect whether it will collide with an obstacle. So, the movement of the robot and the obstacle can be represented as polynomial systems and the collision detection is then reduced to solving a polynomial system. Another example is in modeling geometric and kinematic constraints whereby the algebraic problem representing the motion of a camera is reduced to a well constrained system of polynomial equations.

Such problems are in real life applications which are very seldom modeled as dense polynomials. Generally sparse polynomials are more suitable for real applications representations. The sparseness of such system can be utilized by considering the combinatorial structure of the polynomials represented as the **Newton polytopes**, which is the convex set of the exponent vectors of the polynomials' variables.

Consequently, for analysing and solving various polynomial systems, mathematicians have developed effective and efficient tools such as resultants. As

revealed in the works of [2–6], the resultants are the most powerful of the computational techniques for solving systems of polynomial.

Computing resultants has many applications in geometrical modeling for computer aided geometric design (CAGD) [7, 8] and robot kinematics [2, 5, 9]. These applications can be thought as fundamental problems in geometric computations and crucial or challenging problems in algorithmic algebra. For some problems, the computational complexity of the method involved, makes it very difficult to solve the problem.

An introduction to the research area on the resultant of multivariate polynomial systems is first presented in the following section.

1.2 Research Background

In this section some important concepts which are related to the research topic are discussed. Some notations and basic properties for homogeneous polynomials are also introduced. The discussion leads to the problem formulation for the research.

A system of polynomial equations is a set of n polynomials with coefficients over an arbitrary field. For $i = 1, 2, \dots, n$, let

$$f_i := \sum_{\alpha \in \mathcal{A}} C_{i\alpha} \mathbf{x}^\alpha \quad (1.1)$$

be the polynomial with coefficient $C_{i\alpha}$ and monomials \mathbf{x}^α . If f_i is a polynomial in m variables then $\mathbf{x} = x_1^{\alpha_1}, x_2^{\alpha_2}, \dots, x_m^{\alpha_m}$ and $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_m)$ is the exponent vector of \mathbf{x} . The set of all exponent vectors of the monomials of f_i is called the support of f_i and is denoted as \mathcal{A}_i . The system of polynomials (1.1) is an unmixed system if its supports are identical, otherwise it is called a mixed system.

If \mathcal{A} is the support of a polynomial f then the convex hull of \mathcal{A} is called the Newton polytope which is denoted as $\mathcal{Q} = \text{conv}(\mathcal{A})$.

The following examples are to describe the notion of mixed or unmixed supports of a multivariate polynomial system.

Example 1.1 Consider a polynomial system of two variables of mixed supports,

$$f := a_1 + a_2xy + a_3x^2; \quad g := b_1x + b_2x^2 + b_3y + b_4x^2y. \quad (1.2)$$

The system has support $\mathcal{A}_1 = \{(0, 0), (1, 1), (2, 0)\}$, $\mathcal{A}_2 = \{(1, 0), (2, 0), (0, 1), (2, 1)\}$. Observe that both sets of supports are not identical.

Example 1.2 [10] Consider three polynomials in two variables of a mixed system below,

$$\begin{aligned} f_1 &:= C_{11} + C_{12}x_1 + C_{13}x_1x_2 + C_{14}x_1^2x_2, \\ f_2 &:= C_{21}x_1 + C_{22}x_2 + C_{23}x_1^2x_2 + C_{24}x_1^2x_2^2, \\ f_3 &:= C_{31} + C_{32}x_1 + C_{33}x_2 + C_{34}x_1x_2. \end{aligned}$$

The supports of this mixed system are $\mathcal{A}_1 = \{(0, 0), (1, 0), (1, 1), (2, 1)\}$; $\mathcal{A}_2 = \{(1, 0), (0, 1), (2, 1), (2, 2)\}$; $\mathcal{A}_3 = \{(0, 0), (1, 0), (0, 1), (1, 1)\}$ and they are also not identical.

Example 1.3 [4] Consider an unmixed polynomial system in two variables below,

$$f_i := C_{i1} + C_{i2}x_1 + C_{i3}x_2 + C_{i4}x_1x_2 + C_{i5}x_1^2 + C_{i6}x_2^2, i = 1, 2, 3.$$

The supports of this unmixed system are identical which are $A_i = \{(0, 0), (1, 0), (0, 1), (1, 1), (2, 0), (0, 2)\}$, $i = 1, 2, 3$.

The number of variables in a system of polynomial equations has a relation with classes of polynomials systems. Figure 1.1 shows the classes of polynomials systems and resultant matrix approach.

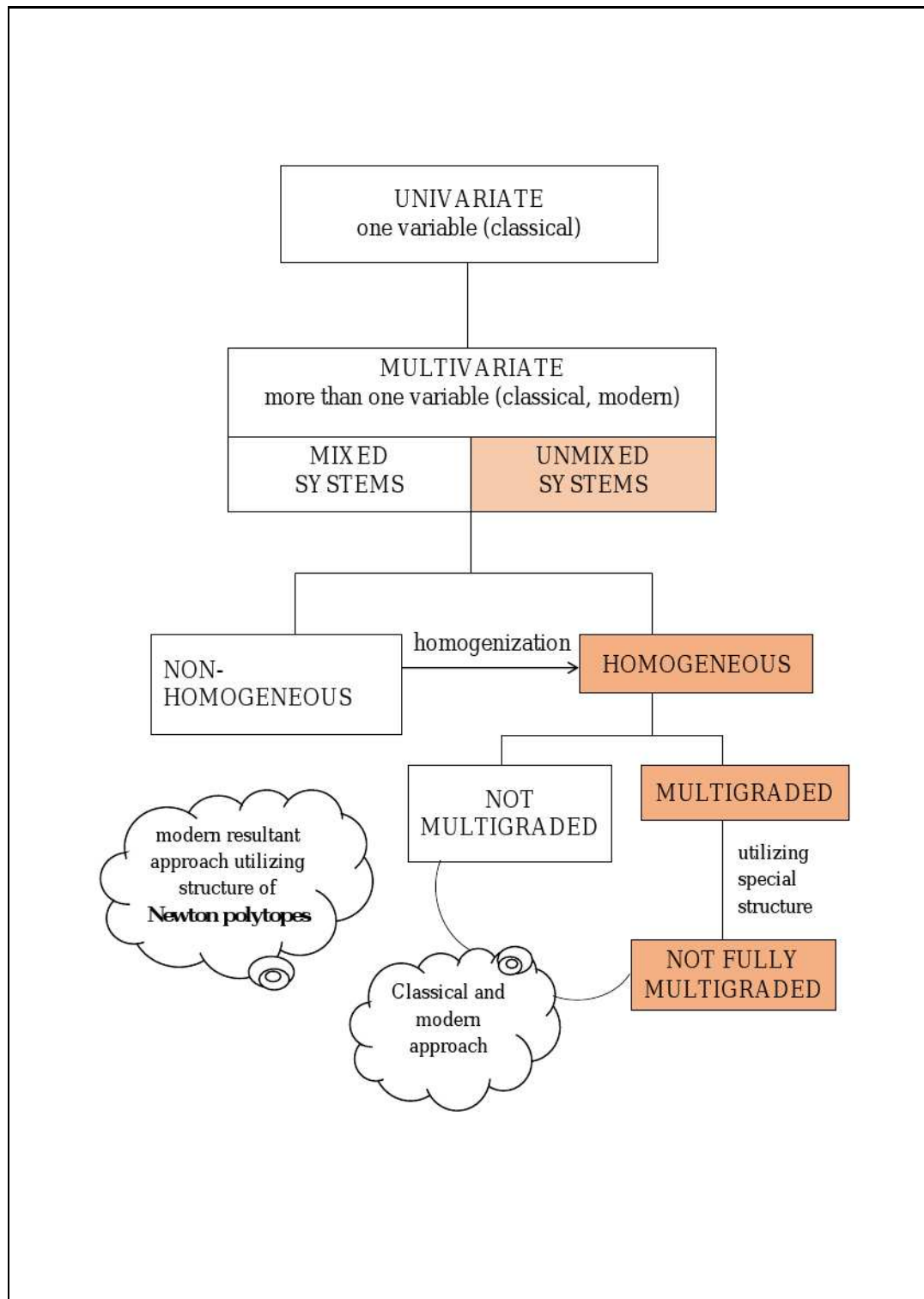


Figure 1.1 Classes of polynomials systems and resultant matrix approach

Classical resultant approach is for univariate or multivariate polynomials, but modern resultant approach only for multivariate polynomials. In addition, modern resultant approach utilizing structure of Newton polytope. The research starts from studying the univariate polynomial system and going down to multivariate system which is either mixed or unmixed systems. Types of polynomial systems are non-homogeneous and homogeneous systems. It is known that all non-homogeneous polynomial systems can be homogenized by certain technique. Then the homogeneous systems may fall under a subclass of multihomogeneous that are *not multigraded* systems and *multigraded* systems. For sparseness criteria, by utilizing special structure the multigraded systems can be formed to not fully multigraded. This thesis focuses to the coloured boxes as shown in Figure 1.1.

Let

$$F_i = \sum_{|\alpha|=d_i} u_{i,\alpha} x^\alpha, \quad i = 0, \dots, n \quad (1.3)$$

be $n + 1$ homogeneous polynomials in $n + 1$ variables over the real number field of degree d_0, \dots, d_n , where for each monomial x^α of degree d_i , its coefficients are the variables $u_{i,\alpha}$. Let N be the total number of these variables, so that \mathbb{C}^N is an affine space with coordinates $u_{i,\alpha}$ for all $i = 0, \dots, n$ and $|\alpha| = d_i$. A point of \mathbb{C}^N will be written $(c_{i,\alpha})$. Therefore, in order to describe a polynomial in the coefficients of the F_i , each variable of $u_{i,\alpha}$ is replaced with the corresponding coefficient $c_{i,\alpha}$ when F_i is evaluated at $(c_{i,\alpha}) \in \mathbb{C}^N$. For instance in the special case of a linear system, $n = 3, d_1 = d_2 = d_3 = 1$, which has a nonzero solution if and only if the determinant of 3×3 -determinant is zero [11]. The determinant may be regarded as a cubic polynomial in the ring of polynomials with integer coefficients in variables u_i , i.e., $\mathbb{Z}[u_0, u_1, \dots, u_n]$. Theorem 1.1 describes the properties of resultant.

Theorem 1.1 [11] Let d_1, \dots, d_n positive degrees, then a unique (up to sign) irreducible polynomial $Res \in \mathbb{Z}[u]$ has the following properties:

- (i) If $f_1, \dots, f_n \in \mathbb{C}[x_1, \dots, x_n]$ are homogeneous of degree d_1, \dots, d_n then the equations of polynomial have a nonzero solution in \mathbb{C}^n if and only if $\mathcal{R}(f_1, \dots, f_n) = 0$.

(ii) \mathcal{R} is irreducible, even when regarded as a polynomial in $\mathbb{C}[u]$.

The elements of $\mathbb{C}[u]$ are polynomial functions on the affine space \mathbb{C}^N and regards $x = (x_1, \dots, x_n)$ as variables for the complex projective space \mathbb{P}^{n-1} . This gives (u, x) is the coordinates on the product variety. The polynomial \mathcal{R} is called the (multivariate) resultant of the given system of polynomials. Next, the degree of the resultant is provided. This theorem has appeared in several references, for instance in [11] and Chapter 13 in [12]. The generalization of this theorem is the theorem on mixed volumes [13]. In the following, the underlying background knowledge involves in the construction of the Sylvester matrix from $n + 1$ homogeneous polynomials in $n + 1$ variables is presented.

Theorem 1.2 The resultant $\mathcal{R}(F_0, F_1, \dots, F_n)$ is a homogeneous polynomial in the coefficients of each form F_i of degree $d_0 d_1 \dots d_{i-1} d_{i+1} \dots d_n$.

Consider the homogeneous system

$$F_0 = F_1 = \dots = F_n = 0. \quad (1.4)$$

In general, for most values of the coefficients of F_i the homogeneous system with homogeneous coordinates, x , has no nontrivial solutions in the complex projective space $\mathbb{P}^n(\mathbb{C})$ or \mathbb{P}^n , while for certain special values of the coefficients, the solutions exist. In fact the condition for the existence of a solution for the system in \mathbb{P}^n is a condition on the coefficients of the F_i 's. If these coefficients are undetermined variables, then there is a polynomial \mathcal{R} , known as the multipolynomial resultant for the system, in the coefficients of F_i 's which vanishes if and only if the system has a complex solution in \mathbb{P}^n .

In the next section, the motivation for this research is further described.

1.3 Motivation

Sylvester type formulas give Sylvester type matrices whose non zero entries are coefficients of the polynomials when a determinantal formula exists. The determinant of the matrix equals precisely the resultant and the resultant matrix is said to be optimal.

An empirical implementation of Dickenstein and Emiris search algorithm [14] via the Maple package “mhomo.mpl” shows that for every multihomogeneous system with zero defect vector, a determinantal Sylvester type formula can be derived from certain degree vectors. A multigraded system which is an unmixed system of multihomogeneous system of type $(l_1, \dots, l_r; d_1, \dots, d_r)$, such that either l_i or d_i is 1 for each $i = 1, 2, \dots, r$ belongs to this class.

In finding the resultant matrix, a determinantal formula is first sought. If the determinantal formula does not exist, a reasonably small square matrix whose determinant is a nonzero multiple of the resultant is then the next best solution. In the original work of Sturmfels and Zelevinsky [15], the formulation of the Sylvester type formula for fully multigraded system is derived. However if the system is not fully multigraded, the size of the Sylvester matrix remains the same even though some or most of the coefficients can be zero, since the matrix is optimal in the Sylvester form. In this thesis, a formulation that takes advantage of the sparseness property of such system shall be determined using a hybrid construction from the Sylvester-Bézout formulation.

In [14] a hybrid matrix is proposed by suggesting the dimension of the submatrices of the matrix from the dimension of the Koszul complexes and the multihomogeneous representation of the system. The terms in the complexes gives the dimension of the matrix whereby some facts from cohomology theory are used to

calculate the dimension of the matrix ($\dim K_1 = \dim K_0$), while the entries of the matrix have not been explicitly determined.

On the other hand, Khetan [6] had proposed a Sylvester-Bézout hybrid matrix formulation by applying exterior algebra concepts and bracket variables for the construction of the Bézout matrix entries. This formulation is investigated, determining the methods of homogenizing variables which applies the concepts from toric varieties and homogeneous coordinate rings involving projective vector spaces. The underlying algebraic geometry tools such as the theories on divisors and fans as a collection of cones are presented in order to construct and implement the Sylvester-Bézout formulation leading to a new hybrid resultant matrix algorithm which can be implemented using the computer.

1.4 Problem Statement

An optimal resultant matrix can be derived from every generic multihomogeneous polynomial system, in particular from an unmixed system, known as fully multigraded system. Sturmfels and Zelevinsky [15] have formulated a determinantal formula of Sylvester type for fully multigraded system that gives exact resultant which can be derived from certain degree vectors. The formula is characterized by showing a bijection with the permutation of $\{1, \dots, r\}$ and is implemented in Maple package “m homo” by Dickenstein and Emiris [14]. However, the implementation only focuses on the dimension of the matrix and the Sylvester type formula does not take advantage of the sparseness conditions in cases when the system is not fully multigraded with some coefficients being zero. This means that the size of the Sylvester type matrix remains the same for both fully and not fully multigraded system in [14, 15].

In addition, there also exists system of equations whose resultant matrix of Sylvester-type has a determinant equal to zero. This implies that either the resultant

is zero or the extraneous factors equal zero. This phenomenon leads to the problem for finding solutions to the system since there exists the problem of extracting the resultant from the determinant of the resultant matrix when the determinant equals to zero. Certain polynomial systems in two variables from [4, 6] are found to have this problem. In such cases a hybrid resultant matrix is sought.

A hybrid matrix of Sylvester-Bézout type had been proposed in [14] by suggesting the dimension of the submatrices from cohomology theories on the terms in the complexes, using multihomogeneous representation of the system. Even though the size of the hybrid matrix is smaller, the entries of the matrix have not been explicitly determined. Meanwhile, the hybrid matrix of Sylvester-Bézout type by Khetan [6] had used the techniques of exterior algebra and toric varieties intensively in the construction of the Bézout block for the hybrid matrix. In addition, this hybrid matrix can give exact resultant and its entries can be determined explicitly. The formulation of this hybrid matrix involved algebraic geometry objects such as toric varieties, Newton polytope and divisors. These objects are very important for the formulation of the set of inequalities that was formulated by [6] which contribute to the construction of the Bézout block in the hybrid Sylvester-Bézout resultant matrix.

Therefore, in this research, an explicit formula for each entry of the Sylvester type matrix of the multigraded systems is constructed and implemented. In the case of not fully multigraded systems, this research seek to combine two types of matrices following Khetan's resultant hybrid matrix of Sylvester-Bézout type by considering sparseness due to the elimination of certain coefficients of the polynomials. In this research, the Bézout block is constructed and implemented by solving the 5-Rule through an efficient algorithm which is developed in C++ by using combinatoric approach. The results of the implementation are analysed and conditions for optimality conditions for this hybrid formulation are observed and presented.

1.5 Research Objectives

The objectives of this research are:

1. To construct explicit formula for each entry of the Sylvester type matrix of multigraded systems.
2. To develop an efficient algorithm for solving the 5-Rule equations.
3. To formulate and implement the Sylvester-Bézout hybrid matrix algorithm for certain classes of bivariate polynomial systems.
4. To investigate its application in solving multivariate polynomial equations.

The first objective is achieved by applying linear maps and permutations of groups of variables. Meanwhile the second objective is achieved by determining the homogenizing variables by using technique of homogeneous coordinate ring, solve the set of inequalities in 5-Rule and determine the entries of the Bézout block. For the third objective, the Sylvester-Bézout matrix algorithm for unmixed bivariate sparse polynomial systems is constructed and implemented on several cases of the multigraded systems and the empirical results are observed into several cases are made. Then the conditions that give optimal matrix are generalized. For the completion of the research, the fourth objective is added.

Overall research work in this thesis is summarized in Figure 1.2. The coloured boxes are the objectives of the research. This research started from the resultant matrix whose determinant equals resultant, or multiple of resultants or zero. Then, the related works from previous researchers have been studied, to find the gap and the solution for the problems.

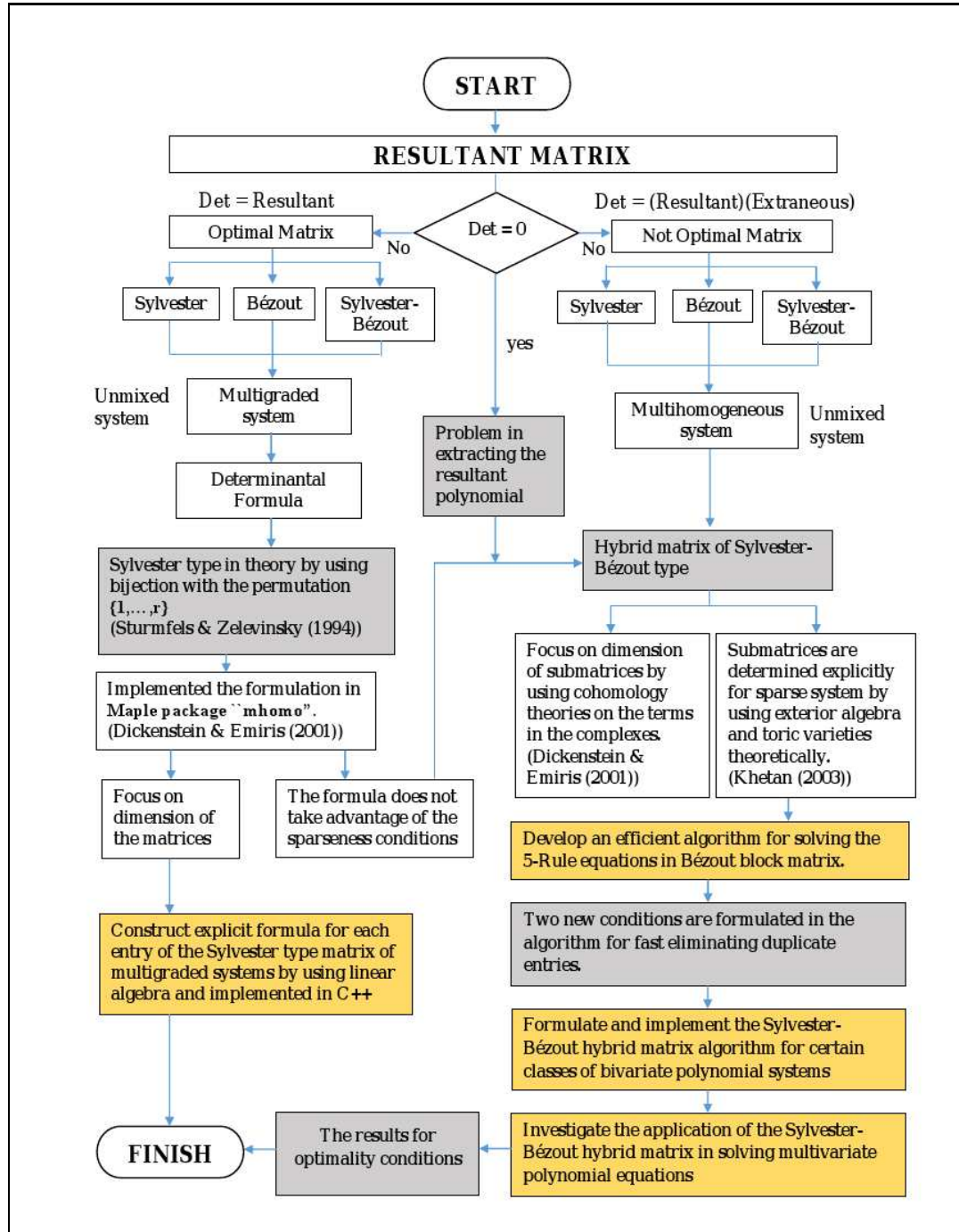


Figure 1.2 Research framework

1.6 Scope of the Study

The research focuses on systems of three unmixed sparse polynomial equations with two variables for constructing and implementing Sylvester-Bézout hybrid matrix. Appropriate algebraic geometry tools or combinatorics are applied in determining possible dimensions of the resultant matrix. The hybrid construction also gives an alternative practical construction to multihomogeneous systems which are not fully multigraded with some coefficients equal zero.

1.7 Significance of Findings

Problems in solving polynomials equations have been reduced to linear algebra problem with the use of resultant matrix. This can be solved numerically with the help of new algorithms and package produced from the research.

1.8 Research Methodology

There are two approaches used for the construction of Sylvester type resultant matrices. In the classical Sylvester matrix approach a multihomogeneous system is first obtained by introducing homogenizing variables that homogenize each equation of the system. On the other hand, the modern approach to sparse resultant matrix is a generalization of the coefficient matrix of a certain linear system via the Sylvester's and Macaulay's matrix formulation. The Macaulay's matrix formulation generalizes the Sylvester's formulation for the univariate case to multivariate systems where the polynomials have more than one variable.

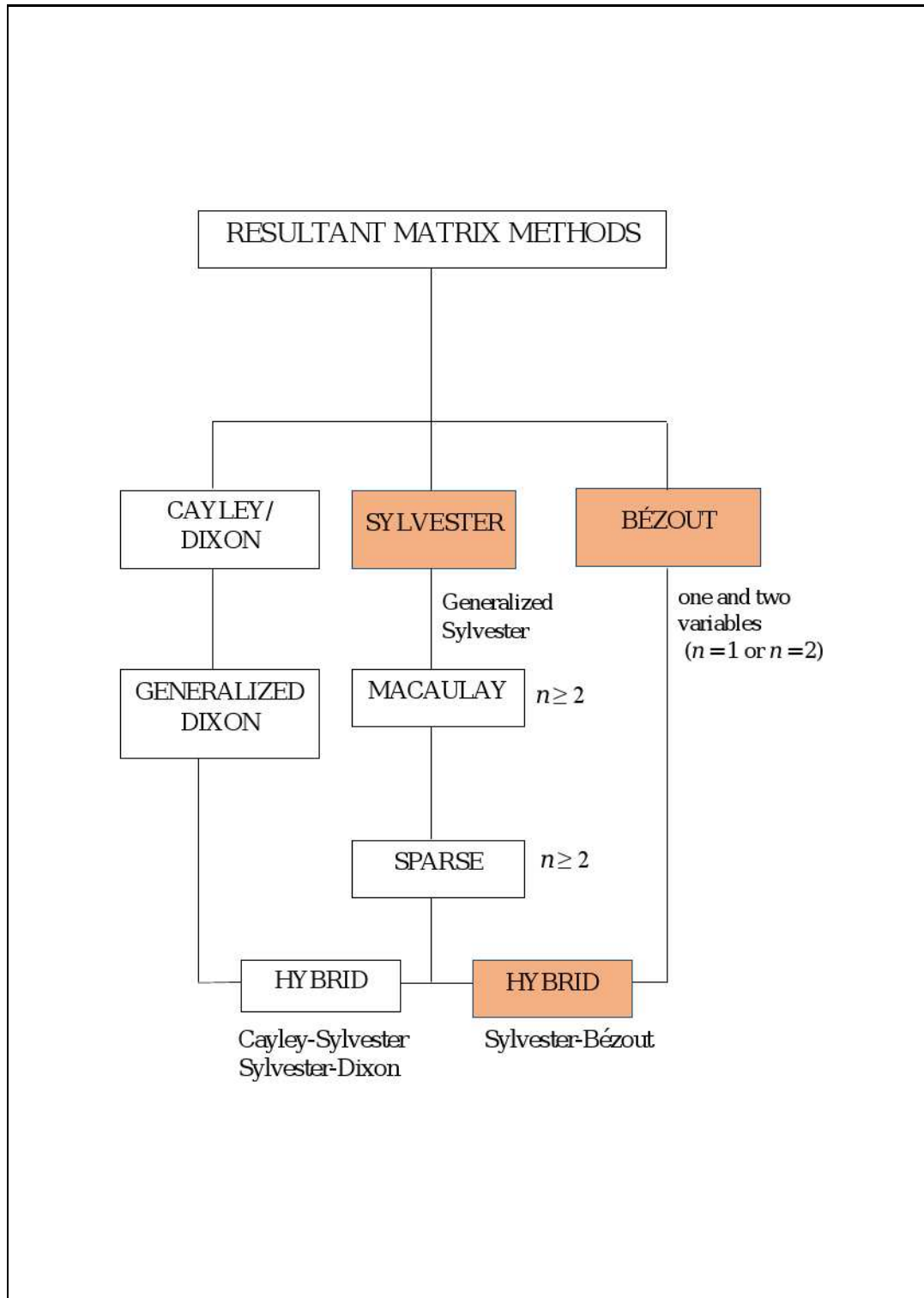


Figure 1.3 Resultant matrix methods

Figure 1.3 shows resultant matrix methods, and the coloured boxes are the methods involve in this research. The main resultant matrix method in this thesis is hybrid Sylvester-Bézout.

1.9 Thesis Organization

The first chapter serves as an introduction to the whole thesis. This chapter introduces the concept of resultants. Chapter 1 also includes motivation of this research, problem statements, research objectives, scope of the study as well as significant of findings.

Chapter 2 presents the literature review of this research. This chapter is divided into two parts. Firstly is on various works by different researchers regarding the development or construction of resultant matrices ranging from classical work to modern approach are presented. Secondly, this chapter presents the preliminary research work on resultant and mixed volume. In order to determine whether a smaller possible matrix or an optimal matrix have been obtained, computation of the mixed volume and the determinant of the resultant matrix is applied. Mixed volume computation has relation with algebraic geometry and combinatoric basis. The “multires” package in Maple software is used to compare the size of sparse resultant matrices with the mixed volume, so the optimal or smallest matrix can be determined. Based on the literature review and preliminary empirical observations of existing methods, it is important to utilize the structure of the Newton polytopes of the polynomial system to produce the desired matrix with smallest possible dimension. Moreover the mixed volume of the Minkowski sum of the convex polytopes computation is very efficient to predict the number of roots for such systems.

The determinantal formulae of Sylvester-type for multigraded systems having defect vectors ranging between zero to two and not in this range are implemented in Chapter 3. A Sylvester matrix package that can display the Sylvester matrix and

determines the corresponding linear map and dimension of the matrix is developed. This extends the "mhomo" package which only computes the matrix dimension without explicitly computing the entries of the resultant matrix.

Chapter 4 presents the formulation of the Bézout matrix for sparse polynomials. Some theoretical basis of algebraic geometry and the combinatorial approach used for the construction of the Sylvester-Bézout hybrid matrix, in particular for the entries of Bézout matrices are presented. The notion of toric varieties such as cones and fans, projective space and divisor $D = \sum a_i D_i$ that determines a convex polytope are applied in the formulation.

Chapter 5 provides the details of the Sylvester-Bézout hybrid matrix algorithm for unmixed bivariate sparse polynomial systems which belongs to a class of multigraded system. The algorithm comprises four phases. The important subroutines are presented in terms of pseudo codes. In addition, the complexity of the algorithm is also discussed in this chapter.

The results of implementation of the hybrid Sylvester-Bézout resultant matrix algorithm are presented in Chapter 6. In this chapter, the unmixed bivariate polynomial equations are categorized into several cases, so that the results can be analysed and generalized. In addition an application of the algorithm to the problem of optimization a variable in the unmixed bivariate polynomial system and to determine the solutions of such system is presented. Besides the C++ package, Maple software is also used to assist the computations in this chapter.

Chapter 7 presents the conclusion which highlights the contribution of the thesis and include suggestions to further work.

The thesis organization is summarized in Figure 1.4. Next chapter is the literature review for this research topic.

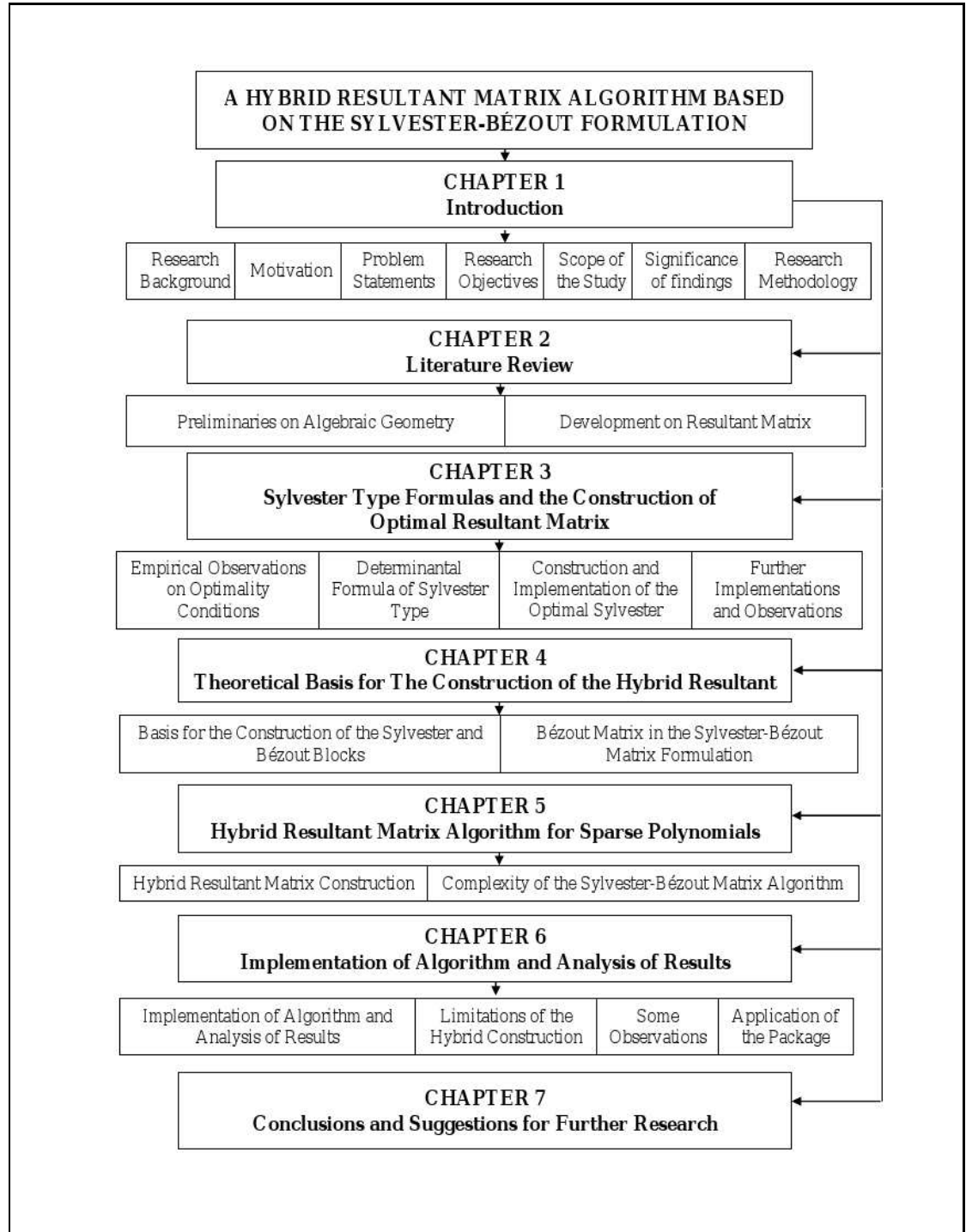


Figure 1.4 Thesis organization

type matrix. A Sylvester-type matrix based on the permutation on the set $\{1, 2, \dots, r\}$ which forms the row and column spaces of certain linear maps of the resultant matrix have been constructed. The work in [14] only displays the dimensions of the Sylvester-type resultant matrix for multihomogeneous systems and is implemented in the Maple package. The Sylvester-type matrix algorithm in this thesis is implemented in C++ and determines the vector spaces that defines the linear map and the Sylvester-type resultant matrix for the class of multigraded systems. Thus this research applies an alternative method of constructing optimal matrices extending the work in [14] by providing the matrix, focusing on the multigraded systems.

Secondly, the efficient algorithm is developed for solving the 5-Rule equations such that the systematic procedures are formulated based on Theorem 4.2. Besides the determination of the homogenizing variables given by Definition 4.5, the 5-Rule is the main component and basis in developing the algorithm for constructing the Bézout block in the hybrid Sylvester-Bézout resultant matrix. The theoretical concepts of toric varieties such as hyperplane, cones, fan and divisors in the formulation of the 5-Rule have been investigated. The formulation of the 5-Rule is due to [6] and this work proceed to solve the inequalities given by the rule and determines the entries of Bézout matrix. Two conditions have been proposed in the algorithm for effectiveness and efficiency eliminating duplicate elements. The algorithm is also implemented in C++. Thus the application and implementation of Theorem 4.2 have been realized with the development of the Sylvester-Bézout matrix algorithm and C++ package.

Thirdly, the results of the implementation of the Sylvester-Bézout resultant matrix algorithm have been analyzed and generalized. The analysis is based on several cases of unmixed polynomial equations and conditions that give an optimal matrix in the Sylvester-Bézout formulation have been derived and proven.

Finally, for significant finding purposes, one application in solving multivariate polynomial equations is investigated by applying the new implemented hybrid matrix of Sylvester-Bézout type.

7.3 Suggestions for Further Research

This study is an application of algebraic geometry. However there are still many problems to be resolved further as listed below.

- Systems that do not fall in the class of multigraded systems may not have a determinantal formula. Therefore, optimal matrix for such system cannot be derived. This observation is illustrated in Section 3.5 by presenting three systems which are not multigraded, which do not have a Sylvester-type formula. For such cases, Dickenstein and Emiris [14] proposed the hybrid matrix for one of the examples whereby the system is not multigraded of type $(3,2;2,3)$. However the proposed hybrid matrix is not shown explicitly. [14] only gives the dimension of the submatrices that can lead to a hybrid matrix construction. The study on cohomologies and its complexes can be further investigated in order to derive the hybrid resultant matrix proposed in [14].
- In this thesis the new hybrid Sylvester-Bézout matrix algorithm cannot be applied to some cases of multigraded system that have a Newton polytope such that the edges of the Newton polytope are parallel to xy -axis. The algorithm fails to produce the Bézout block and is due the number of partitions of the fan associated with the system. The method of partition is mainly based on Khetan's formulation in [6]. The theoretical bases of this formulation needs further investigation to be able to handle these cases in future.
- When reducing the matrix, the complexity of computing is also reduced. Therefore, investigate on the possibility of further reducing the size of the optimal matrix obtained from the hybrid Sylvester-Bézout algorithm have to move on. Smaller sized matrix will reduce the complexity of computing the resultant and is of interest in any work related to resultant formulation and computations.

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