SYMPLECTIC GEOMETRY OF A MODULI SPACE OF FRAMED HIGGS BUNDLES

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ABSTRACT. Let X be a compact connected Riemann surface and D an effective divisor on X. Let $\mathcal{N}_H(r, d)$ denote the moduli space of D–twisted stable Higgs bundles (a special class of Hitchin pairs) on X of rank r and degree d. It is known that $\mathcal{N}_H(r, d)$ has a natural holomorphic Poisson structure which is in fact symplectic if and only if D is the zero divisor. We prove that $\mathcal{N}_H(r, d)$ admits a natural enhancement to a holomorphic symplectic manifold which is called here $\mathcal{M}_H(r, d)$. This $\mathcal{M}_H(r, d)$ is constructed by trivializing, over D, the restriction of the vector bundles underlying the D-twisted Higgs bundles; such objects are called here as framed Higgs bundles. We also investigate the symplectic structure on the moduli space $\mathcal{M}_H(r, d)$ of framed Higgs bundles as well as the Hitchin system associated to it.

1. INTRODUCTION

Since their inception in [Hi], Hitchin systems have been a rich source of examples of algebraically completely integrable systems. In [Hi], Hitchin proved that the moduli space of Higgs bundles on a compact Riemann surface is a holomorphic symplectic variety that admits an algebraically completely integrable structure which is provided by what is known as the Hitchin fibration. The symplectic structure of the moduli space of Higgs bundles arises from the natural Liouville symplectic structure on the total space of the cotangent bundle of the moduli space of vector bundles; this cotangent bundle is in fact a Zariski open dense subset of the moduli space of Higgs bundles.

Let X be a compact connected Riemann surface. Fix an effective divisor D on X. A Hitchin pair, or more precisely a D-twisted Higgs bundle, on X is a pair of the form (E, θ), where E is a holomorphic vector bundle on X and θ is a holomorphic section of the vector bundle $\text{End}(E) \otimes K_X \otimes \mathcal{O}_X(D)$ with K_X being the holomorphic cotangent bundle of X. Let $\mathcal{N}_H(r, d)$ be the moduli space of stable D-twisted Higgs bundles on X of rank r and degree d. It is known that $\mathcal{N}_H(r, d)$ carries a natural holomorphic Poisson structure [Bo], [Mak]. This Poisson structure coincides with the symplectic structure on the moduli space of stable Higgs bundles of rank r and degree d (constructed in [Hi]) when D is the zero divisor. It is also known that this Poisson structure is not symplectic when the divisor D is nonzero.

Given a holomorphic vector bundle E on X of rank r, a framing of E over D is a trivialization of the vector bundle $E|_D$ over D, meaning a holomorphic isomorphism δ of

$E|_D$ with the trivial vector bundle $\mathcal{O}_D^{\oplus r}$. A framed bundle is a holomorphic vector bundle equipped with a framing. A framed Higgs bundle is a triple (E, δ, θ), where (E, δ) is a framed bundle and θ is a holomorphic section of $\mathrm{End}(E) \otimes K_X \otimes \mathcal{O}_X(D)$ as before. Let $\mathcal{M}_H(r, d)$ be the moduli space of semistable framed Higgs bundles on X of rank r and degree d. This is a smooth, irreducible quasi-projective variety.

Also Simpson considered moduli spaces of framed Higgs bundles [Si1, Si2]. The difference between these moduli spaces and the ones hereby considered is the twist of the Higgs fields (K_X versus $K_X(D)$). Nonetheless, the construction of the quot-scheme found in [Si1, §4] remains valid in our case. Combining his arguments (cf. [Si1, Theorem 4.1]) with Nitsure's [Ni] for arbitrary twistings, we may deduce that $\mathcal{M}_H(r, d)$ is an irreducible quasi-projective variety.

It should be mentioned that a frame is also known as a D-level structure as studied by Seshadri in [Se]. Markman in [Mak] studied a Poisson structure on the cotangent space to the moduli space of vector bundles with D-level structures constructed by Seshadri. The difference between the space considered by Markman and ours is that the stability condition we use is stronger.

We investigate the local structure of $\mathcal{M}_H(r, d)$. The tangent space to $\mathcal{M}_H(r, d)$ at a point $(E, \delta, \theta) \in \mathcal{M}_H(r, d)$ is given by the first hypercohomology of the complex

$$\mathcal{C}_\bullet : \mathcal{C}_0 = \mathrm{End}(E) \otimes \mathcal{O}_X(-D) \xrightarrow{f_\theta} \mathcal{C}_1 = \mathrm{End}(E) \otimes K_X \otimes \mathcal{O}_X(D),$$

where $f_\theta(s) = \theta \circ s - s \circ \theta$ (see Corollary 2.9).

It turns out that $\mathcal{M}_H(r, d)$ has a holomorphic symplectic structure, and moreover the symplectic form is exact (Theorem 3.4). We prove that the forgetful map from the stable locus $\mathcal{M}_H^s(r, d)$ of $\mathcal{M}_H(r, d)$ to $\mathcal{N}_H(r, d)$,

$$(E, \delta, \theta) \longmapsto (E, \theta),$$

is compatible with the Poisson structures on $\mathcal{M}_H^s(r, d)$ and $\mathcal{N}_H(r, d)$ (see Theorem 4.1). This means that the pullback, by this forgetful map, of the Poisson bracket $\{f, g\}$ of two locally defined holomorphic functions f and g on $\mathcal{N}_H(r, d)$ coincides with the Poisson bracket of the pullbacks of f and g.

Finally, we study the complete integrability properties of the Hitchin system h on $\mathcal{M}_H(r, d)$ with respect to that of $\mathcal{N}_H(r, d)$ (denoted by $\widetilde{h}$) when D is reduced. The generic fibers of h are torsors over Jacobian varieties of spectral curves (cf. Proposition 5.4).The symplectic form becomes degenerate when restricted to the largest abelianizable subsystem. The fibers of the latter system are semiabelian varieties (Proposition 5.11), corresponding to a (local) completion, in the sense that it provides a set of Poisson commuting functions consisting of those in the Hitchin system of $\mathcal{M}_H(r, d)$ together with some functions transversal to $\widetilde{h}$ (Proposition 5.12).

2. Framed Higgs bundles and their deformations

2.1. **Framed Higgs bundles and their moduli spaces.** Let X be a compact connected Riemann surface. The genus of X will be denoted by g_X. Let K_X denote the holomorphic

cotangent bundle of X. Fix a nonzero effective divisor

$$D = \sum_{i=1}^{s} n_i x_i \,, \tag{2.1}$$

where $\{x_i\}_{i=1}^{s}$ are distinct points of X with $n_i > 0$ for all i, and $s \geq 1$.

We shall identify D with the subscheme of X with structure sheaf $\mathcal{O}_D = \mathcal{O}_X/\mathcal{O}_X(-D)$. Similarly, for any coherent analytic sheaf F on X, its restriction to D will be denoted by F_D and, from now on we will denote the tensor products $F \otimes_{\mathcal{O}_X} \mathcal{O}_X(-D)$ and $F \otimes_{\mathcal{O}_X} \mathcal{O}_X(D)$ by $F(-D)$ and $F(D)$ respectively.

We shall identify a holomorphic vector bundle with the coherent analytic sheaf given by its locally defined holomorphic sections. In particular, for a holomorphic vector bundle F on X, the above restriction F_D will also mean the restriction of the holomorphic vector bundle F to the subscheme D. For any vector bundle F on X, its slope is defined to be

$$\mu(F) := \frac{\deg(F)}{\mathrm{rk}(F)} \in \mathbb{Q} \,.$$

Definition 2.1. Let E be a holomorphic vector bundle on X of rank r. A *frame* on E is an isomorphism of $\mathcal{O}_D$ modules

$$\delta \,:\, E_D \longrightarrow \mathcal{O}_D^{\oplus r} \,.$$

A vector bundle with a frame will be called a *framed bundle*.

Definition 2.2. A *Higgs field* on a framed bundle (E, δ) is a holomorphic section of the vector bundle $\mathrm{End}(E) \otimes K_X(D)$. A *framed Higgs bundle* is a triple (E, δ, θ), where (E, δ) is a framed bundle and θ is a Higgs field on (E, δ).

A framed bundle (E, δ) will be called *stable* (respectively, *semistable*) if the underlying vector bundle E is *stable* (respectively, *semistable*). A framed Higgs bundle (E, δ, θ) will be called *stable* (respectively, *semistable*) if for every subbundle $0 \neq F \subsetneq E$ with $\theta(F) \subset F \otimes K_X(D)$, the inequality

$$\mu(F) < \mu(E) \ \Big(\text{respectively}, \ \mu(F) \leq \mu(E)\Big)$$

holds.

Simpson showed that the semistable framed bundles (respectively, semistable framed Higgs bundles) of rank r and degree d have a fine moduli space denoted by $\mathcal{M}(r, d)$ (respectively, $\mathcal{M}_H(r, d)$), which is a smooth quasiprojective irreducible variety [Si1], [Si2].

We shall now see that all semistable framed Higgs bundles are simple. The frame thus provides a rigidification of the moduli problem.

Lemma 2.3. *Let (E, θ) and (E', θ') be semistable Higgs bundles on X with*

$$\mu(E) = \mu(E') \,. \tag{2.2}$$

Let $h \,:\, E \longrightarrow E'$ be a homomorphism such that

- *$\theta' \circ h = (h \otimes Id_{K_X(D)}) \circ \theta$ as homomorphisms from E to $E' \otimes K_X(D)$, and*
- *there is a point $x_0 \in X$ such that $h(x_0) = 0$.*

Then h vanishes identically.

Proof. Assume that h is not identically zero. Let I denote the image of h; note that I is torsion-free because it is a subsheaf of E'. Since (E, θ) and (E', θ') are semistable, we have

$$\mu(E') \geq \mu(I) \geq \mu(E) \,,$$

so from (2.2) it follows that $\mu(I) = \mu(E')$. Now, let I' be the subbundle of E' generated by I; this means that $I' \subset E'$ is the inverse image, under the quotient map $E' \longrightarrow E'/I$, of the torsion part of E'/I. Then

$$\mu(I') \leq \mu(E') = \mu(I) \,.$$

On the other hand, $\mu(I') \geq \mu(I)$, because I is subsheaf of I' with the quotient I'/I being a torsion sheaf. Since $\mu(I') = \mu(I)$, it follows that $\deg(I') = \deg(I)$, because $\mathrm{rank}(I') = \mathrm{rank}(I)$. Hence we have $I' = I$. But this is a contradiction, because $h(x_0) = 0$ and $I \neq 0$ implying that x_0 lies in the support of I'/I. Therefore, we conclude that $h = 0$. $\qquad\qquad\square$

An isomorphism between framed Higgs bundles (E, δ, θ) and (E', δ', θ') is a holomorphic isomorphism $h : E \longrightarrow E'$ of vector bundles such that

- $h \circ \delta = \delta'$, and
- $\theta' \circ h = (h \otimes \mathrm{Id}_{K_X(D)}) \circ \theta$.

Corollary 2.4. *A semistable framed Higgs bundle (E, δ, θ) does not admit any nontrivial automorphism.*

Proof. For any automorphism h of (E, δ, θ), consider $h - \mathrm{Id}_E$. It vanishes on D. Since the effective divisor D is nonzero, from Lemma 2.3 it follows that $h - \mathrm{Id}_E = 0$. $\qquad\square$

2.2. **Infinitesimal deformations.** The following results are analogous to those of [Mak], [Bo, Section 3] and [BR, Section 2]. In our situation the moduli space is built using a stability condition different from that in [Mak].

Let $X(\epsilon) := X \times \mathrm{Spec}(\mathbb{C}[\epsilon]/\epsilon^2)$. An infinitesimal deformation of a framed bundle (E, δ) (respectively, framed Higgs bundle (E, δ, θ)) is given by an isomorphism class of a pair $(E_\epsilon, \delta_\epsilon)$ (respectively, triple $(E_\epsilon, \delta_\epsilon, \theta_\epsilon)$) such that

- $E_\epsilon \longrightarrow X(\epsilon)$ is a holomorphic vector bundle,
- $\delta_\epsilon : E_\epsilon|_{D \times \mathrm{Spec}(\mathbb{C}[\epsilon]/\epsilon^2)} \longrightarrow \mathcal{O}^{\oplus r}_{D \times \mathrm{Spec}(\mathbb{C}[\epsilon]/\epsilon^2)}$ is an isomorphism,
- (in case of framed Higgs bundles) $\theta_\epsilon \in H^0(X(\epsilon), \mathrm{End}(E_\epsilon) \otimes q_0^* K_X(D))$, where q_0 is the natural projection of $X(\epsilon)$ to X,
- $(E_\epsilon, \delta_\epsilon)|_{X \times \{0\}} \cong (E, \delta)$ (respectively, $(E_\epsilon, \delta_\epsilon, \theta_\epsilon)|_{X \times \{0\}} \cong (E, \delta, \theta)$), where $0 \in \mathrm{Spec}(\mathbb{C}[\epsilon]/\epsilon^2)$ denotes the closed point.

Lemma 2.5. *The space of all infinitesimal deformations of a framed bundle (E, δ) is identified with $H^1(X, \mathrm{End}(E)(-D))$.*

Proof. Recall that the space of all infinitesimal deformations of E is $H^1(X, \text{End}(E))$. Now, deformations of (E, δ) are deformations of E together with the framing. Take a covering of X by affine open subsets $\{U_j\}_{j=1}^d$. A 1-cocycle $\{s_{j,k}\}_{j,k=1}^d$ of $\text{End}(E)$ gives an infinitesimal deformation of E via the vector bundle E_ϵ on $X(\epsilon)$ defined by the transition functions $\text{Id} + \epsilon \cdot s_{j,k}$ for the local trivializations $(q_0^* E)|_{U_j \times \text{Spec}(\mathbb{C}[\epsilon]/\epsilon^2)}$ over $U_j \times \text{Spec}(\mathbb{C}[\epsilon]/\epsilon^2)$. Clearly, these preserve a frame if and only if $s_{j,k}$ are sections of $\text{End}(E)(-D)$. Next note that a 1-coboundary for $\text{End}(E)(-D)$ gives a trivial infinitesimal deformation of the framed bundle (E, δ), as it is a trivial deformation of E which preserves the framing. Now the lemma is straight-forward. $\qquad\square$

Remark 2.6. Lemma 2.5 is analogous to [Mak, Proposition 6.1], the difference being that the latter considers a different moduli space, the moduli space of D-level structures, where the stability condition, known as $\overline{\delta}$-stability, is as follows.

Let D an effective divisor and $\overline{\delta} = \deg(D)$, a framed bundle (E, δ) with framing over D is $\overline{\delta}$-stable if and only if for all subbundle $F \subset E$ it satisfies the inequality

$$\mu(F) < \mu(E) + \overline{\delta}\left(\frac{1}{\text{rk}(F)} - \frac{1}{\text{rk}(E)}\right).$$

Given a framed Higgs bundle (E, δ, θ) on X, consider the following complex:

$$\mathscr{C}_\bullet : \quad \mathscr{C}_0 = \text{End}(E)(-D) \xrightarrow{f_\theta} \mathscr{C}_1 = \text{End}(E) \otimes K_X(D) \tag{2.3}$$
$$s \longmapsto \theta \circ s - s \circ \theta.$$

It may be noted that $f_\theta(\text{End}(E)(-D)) \subset \text{End}(E) \otimes K_X \subset \text{End}(E) \otimes K_X(D)$.

Lemma 2.7. *The infinitesimal deformations of a framed Higgs bundle (E, δ, θ) are parametrized by the first hypercohomology $\mathbb{H}^1(\mathscr{C}_\bullet)$ of the complex $\mathscr{C}_\bullet$ in (2.3).*

Proof. Its proof is very similar to the proof of [BR, Theorem 2.3] (also given in [Bo], [Mak]).

Proceeding as in the proof of [BR, Theorem 2.3], we check that if $\{U_j\}_{j=1}^d$ is an affine open covering of X such that

- each point of $\{x_i\}_{i=1}^s$ lies in exactly one of these open subsets $\{U_j\}_{j=1}^d$, and
- $E|_{U_j} \cong \mathcal{O}_{U_j}^{\oplus r}$ for every j,

then $\{U_j(\epsilon) := U_j \times \text{Spec}(\mathbb{C}[\epsilon]/\epsilon^2)\}_{j=1}^d$ is an affine open covering of $X(\epsilon)$ satisfying similar conditions. From Lemma 2.5 we know that the isomorphism class of E_ϵ is determined by the element of $H^1(X, \text{End}(E)(-D))$ defined by a 1-cocycle η_{jk}. The Higgs field θ_ϵ is then given by $\theta + \gamma_j \epsilon$ on $U_j(\epsilon)$, where

$$\gamma_j \in H^0(U_j, \text{End}(E) \otimes K_X(D)|_{U_j}).$$

The compatibility condition for η_{jk} and γ_j coincides with the condition that (η_{jk}, γ_j) defines an element of $\mathbb{H}^1(\mathscr{C}_\bullet)$; see [BR, Theorem 2.3], [Bi, Theorem 2.5] for details. $\qquad\square$

The dual of the complex $\mathscr{C}_\bullet$ in (2.3) is the complex of coherent sheaves on X

$$\mathscr{C}^\bullet : \mathscr{C}^0 := (\text{End}(E) \otimes K_X(D))^* \otimes K_X \xrightarrow{(f_\theta)^* \otimes \text{Id}_{K_X}} \mathscr{C}^1 := \text{End}(E)(-D)^* \otimes K_X \tag{2.4}$$

To clarify, $\mathscr{C}^0$ and $\mathscr{C}^1$ are at the 0-th and first position respectively.

Since $\mathrm{End}(E) = \mathrm{End}(E)^*$, we have the following:

Lemma 2.8. *There is an isomorphism of complexes*

$$\mathscr{C}_\bullet \cong \mathscr{C}^\bullet \tag{2.5}$$

given by the commutative diagram

$$
\begin{array}{ccc}
\mathrm{End}(E)(-D) & \xrightarrow{\;f_\theta\;} & \mathrm{End}(E) \otimes K_X(D) \\
\| & & \| \\
(\mathrm{End}(E) \otimes K_X(D))^* \otimes K_X & \xrightarrow{(f_\theta)^* \otimes \mathrm{Id}_{K_X}} & \mathrm{End}(E)(-D)^* \otimes K_X.
\end{array}
$$

Corollary 2.9. *Let $(E,\,\delta,\,\theta) \in \mathscr{M}_H(r,d)$. Deformations of $(E,\,\delta,\,\theta)$ are unobstructed. The tangent space of $\mathscr{M}_H(r,d)$ at $(E,\,\delta,\,\theta)$ is identified with $\mathbb{H}^1(\mathscr{C}_\bullet)$.*

Proof. In view of Lemma 2.7, it suffices to show that there is no obstruction to the infinitesimal deformations of a framed Higgs bundle. From Lemma 2.3 it follows immediately that

$$\mathbb{H}^0(\mathscr{C}_\bullet) = 0\,, \tag{2.6}$$

where $\mathscr{C}_\bullet$ is the complex in (2.3). Serre duality says that $\mathbb{H}^2(\mathscr{C}_\bullet) = \mathbb{H}^0(\mathscr{C}^\bullet)^*$. Hence using Lemma 2.8 and (2.6) it follows that

$$\mathbb{H}^2(\mathscr{C}_\bullet) = 0\,. \tag{2.7}$$

Hence there is no obstruction to deformations. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

Lemma 2.10. *Let $\mathscr{M}(r,d)$ be the moduli space of stable framed bundles of rank r and degree d. There is a tautological embedding*

$$\iota\;:\;T^*\mathscr{M}(r,d) \;\hookrightarrow\; \mathscr{M}_H(r,d) \tag{2.8}$$

whose image is a Zariski open dense subset.

Proof. From Lemma 2.5 it follows that $T_{(E,\delta)}\mathscr{M}(r,d) = H^1(X, \mathrm{End}(E)(-D))$. Also, we have $H^1(X, \mathrm{End}(E)(-D))^* = H^0(X, \mathrm{End}(E) \otimes K_X(D))$ (Serre duality). Consequently, the total space of the cotangent bundle $T^*\mathscr{M}(r,d)$ embeds into $\mathscr{M}_H(r,d)$; this embedding will be denoted by ι. From openness of the semistability condition, [May, p. 635, Theorem 2.8(B)], it follows that the image of ι is a Zariski open subset of $\mathscr{M}_H(r,d)$. This also implies that the image of ι is Zariski dense in $\mathscr{M}_H(r,d)$, because $\mathscr{M}_H(r,d)$ is irreducible. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

The dimension of the moduli space $\mathscr{M}_H(r,d)$ can now be calculated.

Proposition 2.11. *Let $n = \sum_{i=1}^s n_i$. The dimension of $\mathscr{M}_H(r,d)$ is $2r^2(g_X + n - 1)$, where g_X as before is the genus of X.*

Proof. By Corollary 2.9

$$\dim \mathscr{M}_H(r,d) = \dim \mathbb{H}^1(\mathscr{C}_\bullet)\,.$$

Now, by (2.6) and (2.7)

$$\mathbb{H}^0(\mathscr{C}_\bullet) = 0 = \mathbb{H}^2(\mathscr{C}_\bullet)\,.$$

Hence we have

$$\dim \mathbb{H}^1(\mathscr{C}_\bullet) = -\chi(\mathscr{C}_\bullet) = -\chi(\text{End}(E)(-D)) + \chi(\text{End}(E) \otimes K_X(D)).$$

By Riemann–Roch, $\chi(\text{End}(E) \otimes K_X(D)) = -\chi(\text{End}(E)(-D)) = r^2(g_X + n - 1)$. Therefore, it follows that $\dim \mathbb{H}^1(\mathscr{C}_\bullet) = 2r^2(g_X + n - 1)$. $\qquad\square$

3. SYMPLECTIC GEOMETRY

3.1. **A symplectic form on $\mathscr{M}_H(r, d)$.** In this section we construct a symplectic form on $\mathscr{M}_H(r, d)$ following analogous arguments to those in [Bi, Bo, Mak]. In view of Corollary 2.9, we shall start by constructing a nondegenerate antisymmetric bilinear form on $\mathbb{H}^1(\mathscr{C}_\bullet)$.

The isomorphism of complexes $\mathscr{C}_\bullet \xrightarrow{\sim} \mathscr{C}^\bullet$ in (2.5) produces an isomorphism of hypercohomologies

$$B_\theta : \mathbb{H}^1(\mathscr{C}_\bullet) \xrightarrow{\sim} \mathbb{H}^1(\mathscr{C}^\bullet). \tag{3.1}$$

On the other hand, Serre duality gives an isomorphism

$$\mathbb{H}^1(\mathscr{C}^\bullet) \xrightarrow{\sim} \mathbb{H}^1(\mathscr{C}_\bullet)^*. \tag{3.2}$$

We recall the explicit description of Serre duality in this case. First note that in view of the isomorphism in (3.1), the isomorphism in (3.2) is uniquely determined by the corresponding isomorphism

$$\mathbb{H}^1(\mathscr{C}_\bullet) \xrightarrow{\sim} \mathbb{H}^1(\mathscr{C}_\bullet)^* \tag{3.3}$$

constructed using (3.1). To construct the isomorphism in (3.3), consider the tensor product of complexes $\mathscr{C}_\bullet \otimes \mathscr{C}_\bullet$:

$$(\mathscr{C}_\bullet \otimes \mathscr{C}_\bullet)_0 = \text{End}(E)(-D) \otimes \text{End}(E)(-D) \xrightarrow{(f_\theta \otimes \text{Id}) + (\text{Id} \otimes f_\theta)} (\mathscr{C}_\bullet \otimes \mathscr{C}_\bullet)_1$$

$$= (\text{End}(E) \otimes K_X(D) \otimes \text{End}(E)(-D)) \oplus (\text{End}(E)(-D) \otimes \text{End}(E) \otimes K_X(D))$$

$$\xrightarrow{(\text{Id} \otimes f_\theta) - (f_\theta \otimes \text{Id})} (\mathscr{C}_\bullet \otimes \mathscr{C}_\bullet)_2 = (\text{End}(E) \otimes K_X(D)) \otimes (\text{End}(E) \otimes K_X(D)).$$

We also have the homomorphism

$$\rho : (\text{End}(E) \otimes K_X(D) \otimes \text{End}(E)(-D)) \oplus (\text{End}(E)(-D) \otimes \text{End}(E) \otimes K_X(D))$$

$$\longrightarrow K_X, \quad (a_1 \otimes b_1) \oplus (a_2 \otimes b_2) \longmapsto \text{Tr}(a_1 \circ b_1 + a_2 \circ b_2).$$

In the above, a_1 and b_2 are sections of $\text{End}(E) \otimes K_X(D)$ and a_2 and b_1 are sections of $\text{End}(E)(-D)$. These give the homomorphism of complexes

$$\begin{array}{ccccc}
(\mathscr{C}_\bullet \otimes \mathscr{C}_\bullet)_0 & \longrightarrow & (\mathscr{C}_\bullet \otimes \mathscr{C}_\bullet)_1 & \longrightarrow & (\mathscr{C}_\bullet \otimes \mathscr{C}_\bullet)_2 \\
\downarrow & & \downarrow{\scriptstyle\rho} & & \downarrow \\
0 & \longrightarrow & K_X & \longrightarrow & 0
\end{array} \tag{3.4}$$

Now we have the composition of homomorphisms of hypercohomologies

$$\mathbb{H}^1(\mathscr{C}_\bullet) \otimes \mathbb{H}^1(\mathscr{C}_\bullet) \longrightarrow \mathbb{H}^2(\mathscr{C}_\bullet \otimes \mathscr{C}_\bullet) \xrightarrow{\rho'} \mathbb{H}^2(0 \to K_X \to 0) = H^1(X, K_X) = \mathbb{C}, \tag{3.5}$$

where ρ' is given by the homomorphism of complexes in (3.4). Let

$$\Psi_\theta : \mathbb{H}^1(\mathscr{C}_\bullet) \otimes \mathbb{H}^1(\mathscr{C}_\bullet) \longrightarrow \mathbb{C} \tag{3.6}$$

be the bilinear form constructed in (3.5).

The earlier mentioned Serre duality in (3.3) is given by the bilinear form Ψ_θ in (3.6). Note that Ψ_θ is nondegenerate because the homomorphism in (3.3) is an isomorphism. Also, the anti-symmetry of Ψ_θ comes from its construction and it is related to the graded symmetry of the cup-product in cohomology.

Therefore, we have the following:

Proposition 3.1. *There is a nondegenerate holomorphic two form Ψ on $\mathcal{M}_H(r,d)$ whose evaluation at any point $(E, \delta, \theta) \in \mathcal{M}_H(r,d)$ is Ψ_θ in (3.6).*

Proof. The above pointwise construction of Ψ_θ clearly works for families of framed Higgs bundles. Recall from Corollary 2.9 that $\mathbb{H}^1(\mathscr{C}_\bullet)$ is the tangent space to $\mathcal{M}_H(r,d)$ at the point (E, δ, θ). $\qquad\square$

Before exploring Ψ_θ further, we introduce a one form on $\mathcal{M}_H(r,d)$.

Consider the short exact sequence of complexes of coherent sheaves on X

$$
\begin{array}{ccccc}
 & & 0 & & 0 \\
 & & \downarrow & & \downarrow \\
\mathscr{D}_\bullet : & & 0 & \longrightarrow & \mathrm{End}(E) \otimes K_X(D) \\
 & & \downarrow & & \downarrow \\
\mathscr{C}_\bullet : & \mathrm{End}(E)(-D) & \xrightarrow{f_\theta} & \mathrm{End}(E) \otimes K_X(D) \\
 & & \downarrow & & \downarrow \\
\mathscr{E}_\bullet : & \mathrm{End}(E)(-D) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow \\
 & & 0 & & 0
\end{array}
$$

In the above diagram, both of the complexes $\mathscr{D}_\bullet$ and $\mathscr{E}_\bullet$ have only one nonzero term. Hence their hypercohomologies are just (shifted) cohomologies of the single nonzero term. Thus we have an associated long exact sequence of hypercohomologies

$$0 \longrightarrow \mathbb{H}^0(\mathscr{C}_\bullet) \longrightarrow \mathbb{H}^0(\mathscr{E}_\bullet) = H^0(X, \mathrm{End}(E)(-D)) \longrightarrow \mathbb{H}^1(\mathscr{D}_\bullet)$$

$$= H^0(X, \mathrm{End}(E) \otimes K_X(D)) \xrightarrow{a} \mathbb{H}^1(\mathscr{C}_\bullet) \xrightarrow{b} \mathbb{H}^1(\mathscr{E}_\bullet) = H^1(X, \mathrm{End}(E)(-D)) \quad (3.7)$$

$$\longrightarrow \mathbb{H}^2(\mathscr{D}_\bullet) = H^1(X, \mathrm{End}(E) \otimes K_X(D)) \longrightarrow \mathbb{H}^2(\mathscr{C}_\bullet) \longrightarrow 0.$$

The homomorphism b in (3.7) is the forgetful map that sends an infinitesimal deformation of (E, δ, θ) to the infinitesimal deformation of (E, δ) obtained from it by simply forgetting the Higgs field. The homomorphism a in (3.7) is the one that sends a section

$$u \in H^0(X, \mathrm{End}(E) \otimes K_X(D))$$

to the infinitesimal deformation of (E, δ, θ) defined by $t \longmapsto (E, \delta, \theta + tu)$.

Since $H^0(X, \mathrm{End}(E) \otimes K_X(D)) = H^1(X, \mathrm{End}(E)(-D))^*$ (Serre duality), we have a homomorphism

$$\Phi_\theta : \mathbb{H}^1(\mathscr{C}_\bullet) \longrightarrow \mathbb{C}, \quad v \longmapsto \theta(b(v)), \qquad (3.8)$$

where b is the homomorphism in (3.7).

The above construction produces the following:

Proposition 3.2. *There is a holomorphic one-form Φ on $\mathcal{M}_H(r,d)$ whose evaluation at any point $(E, \delta, \theta) \in \mathcal{M}_H(r,d)$ is Φ_θ in (3.8).*

Proof. The above pointwise construction of Φ_θ clearly works for families of framed Higgs bundles. $\square$

Using the arguments in [BR, Theorem 4.6] it can be shown that Ψ_θ defined in (3.6) is the evaluation at (E, δ, θ) of the form $d\Phi$. We hereby give a somewhat simplified proof (see Theorem 3.4).

3.2. **Relating to the Liouville symplectic form.** For any complex manifold N, the total space T^*N of the holomorphic cotangent bundle of N is equipped with the Liouville holomorphic 1-form, which will be denoted by η_N. The holomorphic 2-form $d\eta_N$ is the Liouville symplectic form on T^*N.

Recall from Lemma 2.10 that we have a Zariski dense open subset

$$\iota \,:\, T^*\mathcal{M}(r,d) \,\subset\, \mathcal{M}_H(r,d)\,.$$

In this subsection we will prove that $\iota^*\Phi$ is the Liouville 1-form and $\iota^*\Psi$ is the Liouville symplectic form on the cotangent bundle $T^*\mathcal{M}(r,d)$.

Proposition 3.3. *The Liouville 1-form on $T^*\mathcal{M}(r,d)$ coincides with the pullback $\iota^*\Phi$, where ι is the embedding in (2.8) and Φ is the 1-form in Proposition 3.2.*

The pullback $\iota^*\Psi$ is the Liouville symplectic form on $T^*\mathcal{M}(r,d)$, meaning $\iota^*\Psi = d\iota^*\Phi$, where Ψ is the 2-form in Proposition 3.1.

Proof. Let

$$p \,:\, T^*\mathcal{M}(r,d) \,\longrightarrow\, \mathcal{M}(r,d)$$

be the natural projection. For any $z := (E, \delta, \theta) \in T^*\mathcal{M}(r,d) \subset \mathcal{M}_H(r,d)$, let

$$dp(z) \,:\, T_z T^*\mathcal{M}(r,d) \,\longrightarrow\, T_{p(z)}\mathcal{M}(r,d)$$

be the differential of p at z. We noted earlier that the homomorphism b in (3.7) is the forgetful map that sends an infinitesimal deformation of (E, δ, θ) to the infinitesimal deformation of (E, δ) obtained from it by simply forgetting the Higgs field. This means that b coincides with the above homomorphism $dp(z)$. Now from the definition of the Liouville 1-form on $T^*\mathcal{M}(r,d)$, and the construction of Φ_θ in (3.8), it follows immediately that $\iota^*\Phi$ is the Liouville 1-form on $T^*\mathcal{M}(r,d)$.

The fact that $\iota^*\Psi = d\iota^*\Phi$ follows from a standard argument (see [BR, Theorem 4.3], [Bi, Proposition 7.3], [Bo, Theorem 4.5.1], [Mak, Corollary 7.15]). We omit the details. $\square$

Theorem 3.4. *The holomorphic 2-form Ψ on $\mathcal{M}_H(r,d)$ is symplectic. Moreover,*

$$\Psi \,=\, d\Phi$$

on $\mathcal{M}_H(r,d)$.

Proof. We know that Ψ is nondegenerate and antisymmetric. So it suffices to prove that $\Psi = d\Phi$. By Proposition 3.3, the form Ψ coincides with $d\Phi$ on the dense open subset $T^*\mathcal{M}(r,d)$ of $\mathcal{M}_H(r,d)$. This implies that $\Psi = d\Phi$ on the entire $\mathcal{M}_H(r,d)$. $\square$

4. POISSON MAPS

In this section we compare the Poisson structure on $\mathcal{M}_H(r,d)$ with the Poisson structure on the moduli space $\mathcal{N}_H(r,d)$ of stable Hitchin pairs.

For any holomorphic symplectic manifold (M,ω), we may construct a Poisson bracket of any two locally defined holomorphic functions $f, g \in \mathcal{O}_M$, by means of the Hamiltonian vector fields X_f and X_g associated to them, that is,

$$\{f, g\} := X_f g = -X_g f = -\{g, f\};$$

the Hamiltonian vector field X_h for a function h is defined by the equation $dh(v) = \omega(X_h, v)$, where v is any vector field on M. The above pairing $\{\cdot, \cdot\}$ provides $\mathcal{O}_M$ with a Lie algebra structure which satisfies the Leibniz rule which says that $\{f, gh\} = \{f, g\}h + g\{f, h\}$; therefore this pairing produces a Poisson structure on M.

Let (Y_1, ω_1) and (Y_2, ω_2) be two holomorphic Poisson manifolds, where

$$\omega_1 : T^*Y_1 \longrightarrow TY_1 \quad \text{and} \quad \omega_2 : T^*Y_2 \longrightarrow TY_2$$

are the holomorphic homomorphisms giving the Poisson structures. A holomorphic map

$$\beta : Y_1 \longrightarrow Y_2$$

is said to be compatible with the Poisson structures if

$$\{f, g\}_2 \circ \beta = \{f \circ \beta, g \circ \beta\}_1$$

for all locally defined holomorphic functions f, g on Y_2, where $\{-, -\}_1$ and $\{-, -\}_2$ are the Poisson brackets on Y_1 and Y_2 respectively.

Let

$$d\beta : TY_1 \longrightarrow TY_2 \tag{4.1}$$

be the differential of the map β. It is straight-forward to check that the map β is compatible with the Poisson structures if and only if

$$d\beta(\omega_1(x)((d\beta)^*(x)(u))) = \omega_2(u) \tag{4.2}$$

for all $u \in T_y^*Y_2$ and $x \in \beta^{-1}(y)$, where $(d\beta)^*(x) : T_y^*Y_2 \longrightarrow T_x^*Y_1$ is the dual of the differential $d\beta(x)$ in (4.1). Note that both sides of (4.2) are elements of T_yY_2.

Let $\mathcal{N}_H(r,d)$ be the moduli space of stable Hitchin pairs (E, θ), where

- E is a holomorphic vector bundle on X of rank r and degree d, and
- $\theta \in H^0(X, \operatorname{End}(E) \otimes K_X(D))$.

The choice of a section s of $\mathcal{O}_X(D)$ determines a holomorphic Poisson structure on $\mathcal{N}_H(r,d)$ [Bo, Theorem 4.5.1] (see also [Hi], [La]); we will take s to be the section given by the constant function 1. The moduli space $\mathcal{N}_H(r,d)$ is also investigated in [BGL]. The construction of the Poisson structure is recalled in the proof of Theorem 4.1.

Let

$$\widetilde{\Psi}' \,:\, T^* \mathcal{N}_H(r,d) \,\longrightarrow\, T\mathcal{N}_H(r,d) \tag{4.3}$$

be this Poisson structure on $\mathcal{N}_H(r,d)$. Let

$$\widetilde{\Psi} \,:\, T^* \mathcal{M}_H(r,d) \,\xrightarrow{\sim}\, T\mathcal{M}_H(r,d) \tag{4.4}$$

be the Poisson structure on $\mathcal{M}_H(r,d)$ given by the symplectic form Ψ in Theorem 3.4.

Let $\mathcal{M}_H^s(r,d) \,\subset\, \mathcal{M}_H(r,d)$ be the stable locus; it is a Zariski open dense subset. Let

$$Q \,:\, \mathcal{M}_H^s(r,d) \,\longrightarrow\, \mathcal{N}_H(r,d), \quad (E,\,\delta,\,\theta) \,\longmapsto\, (E,\,\theta) \tag{4.5}$$

be the forgetful map that simply forgets the framing.

Theorem 4.1. *The map Q in (4.5) is compatible with the Poisson structures $\widetilde{\Psi}'$ and $\widetilde{\Psi}$ on $\mathcal{N}_H(r,d)$ and $\mathcal{M}_H^s(r,d)$ respectively.*

Proof. We will check the criterion in (4.2).

Consider the complex of coherent sheaves on X

$$\mathscr{C}_\tau^\bullet \,:\, \mathscr{C}_\tau^0 \,:=\, (\mathrm{End}(E) \otimes K_X(D))^* \otimes K_X \xrightarrow{(f_\theta)^* \otimes \mathrm{Id}_{K_X}} \mathscr{C}_\tau^1 \,:=\, \mathrm{End}(E)^* \otimes K_X. \tag{4.6}$$

The identity map of $(\mathrm{End}(E) \otimes K_X(D))^* \otimes K_X$ and the natural inclusion of $\mathrm{End}(E)^* \otimes K_X$ in $\mathrm{End}(E)^* \otimes K_X(D)$ (recall that the divisor D is effective) together produce a homomorphism of complexes

$$\xi \,:\, \mathscr{C}_\tau^\bullet \,\longrightarrow\, \mathscr{C}^\bullet$$

(the complex $\mathscr{C}^\bullet$ is constructed in (2.4)); in other words, the commutative diagram

$$\begin{array}{ccc}
(\mathrm{End}(E) \otimes K_X(D))^* \otimes K_X & \xrightarrow{(f_\theta)^* \otimes \mathrm{Id}_{K_X}} & \mathrm{End}(E)^* \otimes K_X \\
\| & & \downarrow \\
(\mathrm{End}(E) \otimes K_X(D))^* \otimes K_X & \xrightarrow{(f_\theta)^* \otimes \mathrm{Id}_{K_X}} & \mathrm{End}(E)(-D)^* \otimes K_X = \mathrm{End}(E)^* \otimes K_X(D)
\end{array}$$

defines ξ. Let

$$\xi_* \,:\, \mathbb{H}^1(\mathscr{C}_\tau^\bullet) \,\longrightarrow\, \mathbb{H}^1(\mathscr{C}^\bullet) \tag{4.7}$$

be the homomorphism of hypercohomologies induced by the above homomorphism ξ of complexes.

We note that $T_{Q(E,\delta,\theta)}^* \mathcal{N}_H(r,d) \,=\, T_{(E,\theta)}^* \mathcal{N}_H(r,d) \,=\, \mathbb{H}^1(\mathscr{C}_\tau^\bullet)$ [Bo, Proposition 3.1.10], [Mak, Corollary 7.5], [BR, Theorem 2.3]. On the other hand,

$$\mathbb{H}^1(\mathscr{C}^\bullet) \,=\, \mathbb{H}^1(\mathscr{C}_\bullet)^* \,=\, T_{(E,\delta,\theta)}^* \mathcal{M}_H^s(r,d)$$

(Corollary 2.9 and (3.2)). Take any

$$w \,\in\, \mathbb{H}^1(\mathscr{C}_\tau^\bullet) \,=\, T_{Q(E,\delta,\theta)}^* \mathcal{N}_H(r,d).$$

We will show that

$$(dQ)^*(E,\delta,\theta)(w) \,=\, \xi_*(w), \tag{4.8}$$

where ξ_* is constructed in (4.7).

To prove (4.8), consider the complex of coherent sheaves on X

$$\mathscr{C}_\bullet^\tau : \mathscr{C}_0^\tau = \operatorname{End}(E) \xrightarrow{f_\theta} \mathscr{C}_1^\tau = \operatorname{End}(E) \otimes K_X(D). \qquad (4.9)$$

We have

$$T_{Q(E,\delta,\theta)}\mathcal{N}_H(r,d) = T_{(E,\theta)}\mathcal{N}_H(r,d) = \mathbb{H}^1(\mathscr{C}_\bullet^\tau)$$

[Bo], [Mak], [BR]. Next we note that the identity map of $\operatorname{End}(E) \otimes K_X(D)$ and the natural inclusion of $\operatorname{End}(E)(-D)$ in $\operatorname{End}(E)$ together produce a homomorphism of complexes

$$\zeta : \mathscr{C}_\bullet \longrightarrow \mathscr{C}_\bullet^\tau \qquad (4.10)$$

($\mathscr{C}_\bullet$ is constructed in (2.3)); in other words, we have the commutative diagram

$$
\begin{array}{ccc}
\operatorname{End}(E)(-D) & \xrightarrow{f_\theta} & \operatorname{End}(E) \otimes K_X(D) \\
\downarrow & & \| \\
\operatorname{End}(E) & \xrightarrow{f_\theta} & \operatorname{End}(E) \otimes K_X(D)
\end{array}
$$

that defines ζ. Let

$$\zeta_* : \mathbb{H}^1(\mathscr{C}_\bullet) \longrightarrow \mathbb{H}^1(\mathscr{C}_\bullet^\tau) \qquad (4.11)$$

be the homomorphism of hypercohomologies induced by the above homomorphism ζ of complexes. It can be shown that in terms of the identifications

$$T_{Q(E,\delta,\theta)}\mathcal{N}_H(r,d) = \mathbb{H}^1(\mathscr{C}_\bullet^\tau) \quad \text{and} \quad T_{(E,\delta,\theta)}\mathcal{M}_H^s(r,d) = \mathbb{H}^1(\mathscr{C}_\bullet),$$

the differential

$$dQ(E,\delta,\theta) : T_{(E,\delta,\theta)}\mathcal{M}_H^s(r,d) \longrightarrow T_{Q(E,\delta,\theta)}\mathcal{N}_H(r,d)$$

of dQ at $(E,\,\delta,\,\theta) \in \mathcal{M}_H^s(r,d)$ coincides with the homomorphism ζ_* constructed in (4.11). Indeed, this follows immediately from the constructions of the map Q and the homomorphism ζ_*. Finally, (4.8) follows from the isomorphism in (3.2) and the definition of the homomorphism $(dQ)^*(E,\delta,\theta)$ as the dual homomorphism.

At this point, we shall recall the construction of the Poisson structure $\widetilde{\Psi}'$ in (4.3). For this, consider the complexes $\mathscr{C}_\tau^\bullet$ and $\mathscr{C}_\bullet^\tau$ constructed in (4.6) and (4.9) respectively. We have a homomorphism of complexes

$$\varpi : \mathscr{C}_\tau^\bullet \longrightarrow \mathscr{C}_\bullet^\tau \qquad (4.12)$$

defined by the following diagram of homomorphisms:

$$
\begin{array}{ccc}
(\operatorname{End}(E) \otimes K_X(D))^* \otimes K_X = \operatorname{End}(E)(-D) & \xrightarrow{(f_\theta)^* \otimes \operatorname{Id}_{K_X}} & \operatorname{End}(E)^* \otimes K_X \\
\downarrow & & \downarrow \\
\operatorname{End}(E) & \xrightarrow{f_\theta} & \operatorname{End}(E) \otimes K_X(D)
\end{array}
$$

note that here it is used that

- $\operatorname{End}(E)^* = \operatorname{End}(E)$, and
- the sheaves $\mathcal{O}_X(-D)$ and $\mathcal{O}_X$ are contained in $\mathcal{O}_X$ and $\mathcal{O}_X(D)$ respectively.

Let

$$\varpi_* \; : \; \mathbb{H}^1(\mathscr{C}_\tau^\bullet) \; \longrightarrow \; \mathbb{H}^1(\mathscr{C}_\bullet^\tau) \tag{4.13}$$

be the homomorphism of hypercohomologies induced by the homomorphism of complexes ϖ in (4.12). Since

$$\mathbb{H}^1(\mathscr{C}_\tau^\bullet) \; = \; T^*_{(E,\theta)}\mathcal{N}_H(r,d) \quad \text{and} \quad \mathbb{H}^1(\mathscr{C}_\bullet^\tau) \; = \; T_{(E,\theta)}\mathcal{N}_H(r,d)\,,$$

the above homomorphism ϖ_* produces a homomorphism

$$T^*_{(E,\theta)}\mathcal{N}_H(r,d) \; \longrightarrow \; T_{(E,\theta)}\mathcal{N}_H(r,d)\,.$$

This homomorphism coincides with $\widetilde{\Psi}'(E,\theta)$ in (4.3).

Finally, consider the isomorphism $\widetilde{\Psi}$ in (4.4). We note that $\widetilde{\Psi}$ coincides with the homomorphism of hypercohomologies induced by the isomorphism of complexes in Lemma 2.8. Using this and (3.3) it follows that the composition

$$\widetilde{\Psi} \circ (dQ)^*(E,\delta,\theta) \; : \; T^*_{Q(E,\delta,\theta)}\mathcal{N}_H(r,d) \; = \; \mathbb{H}^1(\mathscr{C}_\tau^\bullet) \; \longrightarrow \; \mathbb{H}^1(\mathscr{C}_\bullet) \; = \; T_{(E,\delta,\theta)}\mathcal{M}_H^s(r,d)$$

coincides with the homomorphism of hypercohomologies associated to the following natural homomorphism of complexes:

$$
\begin{array}{ccc}
(\operatorname{End}(E) \otimes K_X(D))^* \otimes K_X & \xrightarrow{(f_\theta)^* \otimes \operatorname{Id}_{K_X}} & \operatorname{End}(E)^* \otimes K_X \\
\| & & \downarrow \\
\operatorname{End}(E)(-D) & \xrightarrow{\;f_\theta\;} & \operatorname{End}(E) \otimes K_X(D)
\end{array}
$$

(as before, we use that $\operatorname{End}(E)^* = \operatorname{End}(E)$ and that $\mathcal{O}_X$ is contained in $\mathcal{O}_X(D)$). From the above description of $\widetilde{\Psi} \circ (dQ)^*(E,\delta,\theta)$, and the earlier observation that $dQ(E,\delta,\theta)$ coincides with the homomorphism ζ_* in (4.11), it follows that the composition

$$(dQ) \circ \widetilde{\Psi} \circ (dQ)^*(E,\delta,\theta) \; : \; T^*_{Q(E,\delta,\theta)}\mathcal{N}_H(r,d) \; = \; \mathbb{H}^1(\mathscr{C}_\tau^\bullet) \; \longrightarrow \; \mathbb{H}^1(\mathscr{C}_\bullet^\tau) \; = \; T_{Q(E,\delta,\theta)}\mathcal{N}_H(r,d) \tag{4.14}$$

coincides with the homomorphism ϖ_* in (4.13) of hypercohomologies associated to the homomorphism ϖ of complexes in (4.12). Consequently, the homomorphism in (4.14) coincides with $\widetilde{\Psi}'$ in (4.3). This completes the proof. $\qquad\square$

We end this section by studying the structure of the map Q.

Proposition 4.2. *Assume that the divisor D is reduced, thus $n_i = 1$ for all $1 \le i \le s$ and $n = s$. Consider the diagonal embedding of the center $\mathbb{C}^\times$ of $\operatorname{GL}(r,\mathbb{C})$ in $\operatorname{GL}(r,\mathbb{C})^n$ The morphism Q turns $\mathcal{M}_H^s(r,d)$ into a $\operatorname{GL}(r,\mathbb{C})^n/\mathbb{C}^\times$-torsor over $\mathcal{N}_H^s(r,d)$.*

Proof. Take any $(E,\theta) \in \mathcal{N}_H(r,d)$. The group

$$\operatorname{Aut}(\mathcal{O}_D^{\oplus r}) \; \xrightarrow{\;\sim\;} \; \prod_{i=1}^n \operatorname{Aut}(\mathcal{O}_{x_i}^{\oplus r}) \; = \; \operatorname{GL}(r,\mathbb{C})^n$$

(the above isomorphism holds because D is reduced) acts on the space of framings on E (note that $n = s$); see Definition 2.1. This action factors through the quotient group $\operatorname{GL}(r,\mathbb{C})^n/\mathbb{C}^\times$ because scalars act as automorphisms of (E, θ). So $\operatorname{GL}(r,\mathbb{C})^n/\mathbb{C}^\times$ acts on the fiber of Q over the point $(E, \theta) \in \mathcal{N}_H(r,d)$; this action is evidently transitive. Since

(E, θ) is stable, it is simple, hence the above action of $\mathrm{GL}(r, \mathbb{C})^n / \mathbb{C}^\times$ on the fiber of Q over (E, θ) is free.

In the proof of Theorem 4.1 it was observed that the differential $dQ(E, \delta, \theta)$ coincides with the homomorphism ζ_* in (4.11). Note that ζ is injective, and the quotient of the homomorphism ζ in (4.10) is a torsion sheaf at the 0-th position. As the first cohomology of a torsion sheaf on X vanishes, the first hypercohomology of the quotient complex vanishes. Therefore, from the corresponding long exact sequence of hypercohomologies associated to ζ it follows that the homomorphism ζ_* is surjective. Consequently, the map Q is a submersion. This completes the proof. $\square$

5. The Hitchin integrable system

Recall from the previous section that, for any holomorphic symplectic manifold (M, ω), we denote by $\{\cdot, \cdot\}$ the associated Poisson bracket on $\mathcal{O}_M$.

Two functions $f, g \in \mathcal{O}_M$ are said to Poisson commute if

$$\{f, g\} = 0.$$

An algebraically completely integrable system on M consists of functions $f_1, \ldots, f_d \in \mathcal{O}_M$ with $d = \frac{1}{2} \dim M$, such that

- $\{f_i, f_j\} = 0$ for all $1 \leq i, j \leq d$,
- the corresponding Hamiltonian vector fields $X_{f_1}, \cdots, X_{f_d}$ are linearly independent at the general point, and
- the general fiber of the map $(f_1, \cdots, f_d) : M \longrightarrow \mathbb{C}^d$ is an open subset of an abelian variety such that the vector fields $X_{f_1}, \cdots, X_{f_d}$ are linear on it.

When the number of the Poisson commuting functions satisfying the above three conditions is less than half the dimension of M we call it a complex partially integrable system.

5.1. The Hitchin fibration. We assume that $\mathrm{genus}(X) + n = \mathrm{genus}(X) + \sum_{i=1}^{s} n_i \geq 2$.

The Hitchin fibration for the moduli space of framed Higgs bundles is defined to be the map

$$h : \mathcal{M}_H(r, d) \longrightarrow \mathcal{H} := \bigoplus_{i=1}^{r} H^0(X, K_X^i(iD)) \tag{5.1}$$

given by the characteristic polynomial of the Higgs field, that is

$$h(E, \delta, \theta)(x) = \det(k1_{E|_x} - \theta|_x) = k^r + a_1(\theta)(x)k^{r-1} + \cdots + a_r(\theta)(x)$$

where $k \in K_X(D)|_x$ and $a_1, \ldots, a_r$ are a conjugation invariant basis of homogeneous polynomial functions on $\mathfrak{gl}(r)$. Equivalently, they are elementary symmetric polynomials on the Cartan subalgebra $\mathbb{C}^r$.

Since $H^1(X, K_X^i(iD)) = 0$, for all nonnegative i (recall that genus$(X) + n \geq 2$) the dimension of the vector space $\mathscr{H}$ is

$$N := \dim \mathscr{H} = r^2(g_X - 1) + \frac{r(r+1)}{2}n, \tag{5.2}$$

by the Riemann-Roch theorem. Hence $h = (h_1, \cdots, h_N)$, where h_i is a polynomial function of degree i.

Theorem 5.1. *The above functions $h_1, \cdots, h_N$ Poisson commute.*

Proof. Notice that the frame does not play any role in local arguments, therefore the proof follows analogously as in [Bo, Proposition 4.7.1]. $\square$

As a corollary we recover the following result due to Bottacin [Bo, Proposition 4.7.1].

Corollary 5.2. *Let*
$$\widetilde{h} = (\widetilde{h}_1, \cdots, \widetilde{h}_N) : \mathcal{N}_H(r, d) \longrightarrow \mathscr{H} \tag{5.3}$$
be the Hitchin map, defined in [Hi], [Bo], [Ni], which sends any pair (E, θ) to the characteristic polynomial of θ. Then the functions $\widetilde{h}_i$ Poisson commute.

Proof. This follows from Theorem 5.1 and Theorem 4.1 after observing that
$$\widetilde{h}(Q(E, \delta, \theta)) = h(E, \delta, \theta).$$
for the forgetful map Q in (4.5). $\square$

Remark 5.3. Conversely, Corollary 5.2 and Theorem 4.1 together imply Theorem 5.1.

We next study the fibers of the map h. Henceforth, we will assume the divisor D to be reduced.

Let $|K_X(D)|$ denote the total space of the line bundle $K_X(D)$. This surface admits a natural morphism
$$\pi : |K_X(D)| \longrightarrow X.$$
First, for each $b = (b_1, \ldots, b_r) \in \mathscr{H}$, we define a divisor $X_b \subset |K_X(D)|$, called the spectral curve, as the vanishing locus of the section
$$\lambda^r + \pi^*b_1\lambda^{r-1} + \cdots + \pi^*b_r \in H^0(|K_X(D)|, \pi^*K_X(D)^{\otimes r}),$$
where $\lambda \in H^0(|K_X(D)|, \pi^*K_X(D))$ is the tautological section.

Proposition 5.4. *The generic fiber of h, the Hitchin map for $\mathcal{M}_H(r, d)$, is a $\mathrm{GL}(r, \mathbb{C})^n/\mathbb{C}^\times$ torsor over the Jacobian of the spectral curve X_b. More precisely, the torsor whose fiber over $L \in \mathrm{Jac}(X_b)$ is*
$$\mathrm{Aut}((\pi_*L)_D)/\mathbb{C}^\times. \tag{5.4}$$

Proof. Let $\widetilde{h} : \mathcal{N}_H(r, d) \longrightarrow \mathscr{H}$ be the Hitchin map for Higgs pairs. The forgetful map Q defined in (4.5) satisfies that
$$\widetilde{h}(Q(E, \delta, \theta)) = h(E, \delta, \theta).$$
Thus Q takes fibers to fibers and, generically, $\widetilde{h}^{-1}(b)$ is the Jacobian of the spectral curve X_b [BNR, Proposition 3.6]. The proposition now follows from Proposition 4.2. $\square$

5.2. **An integrable subsystem.** There are two main problems that prevent h, the Hitchin map in (5.1), from being an algebraic completely integrable system:

(1) the fibers are not abelian groups hence they are not abelian varieties (Proposition 4.2);
(2) the dimension of fibers is too large as seen from Proposition 2.11 and the formula in (5.2).

In this section we define a smaller subsystem

$$h : \mathcal{M}_H^\Delta(r,d) \longrightarrow \mathcal{H}$$

whose fibers are semiabelian varieties, and whose dimension is twice the dimension of $\mathcal{H}$. The price for it is the loss of the symplectic structure, which becomes degenerate.

In order to do this, consider the family of spectral curves parametrized by $\mathcal{H}$:

$$\mathcal{X} \longrightarrow \mathcal{H}. \tag{5.5}$$

This is the subscheme of $|K_X(D)| \times \mathcal{H}$ consisting of points (k,b) satisfying

$$\lambda^r(k) + \pi^* b_1 \lambda^{r-1}(k) + \cdots + \pi^* b_r(k) = 0.$$

Using [BNR, Proposition 3.6], it follows that the generic Hitchin fiber $\tilde{h}^{-1}(b)$ is isomorphic to $\mathrm{Jac}(X_b)$ the Jacobian of the spectral curve X_b. This isomorphism is called the spectral correspondence.

We can restate the above in terms of relative Jacobians. Consider the relative Jacobian $\mathrm{Jac}_{\mathcal{H}}(\mathcal{X})$, by which we mean stable relative line bundles. Stability is defined so that $\mathrm{Jac}_{\mathcal{H}}(\mathcal{X})$ parametrizes, via the spectral correspondence, stable Hitchin pairs. The spectral correspondence gives a morphism

$$s \; : \; \mathrm{Jac}_{\mathcal{H}}(\mathcal{X}) \longrightarrow \mathcal{N}_H(r,d). \tag{5.6}$$

Let $\widetilde{D} := \pi^* D \subset |K_X(D)|$ and consider relative framed line bundles on $\mathcal{X}$, that is, pairs (L, δ') for $L \in \mathrm{Jac}_{\mathcal{H}}(\mathcal{X})$, such that

$$L_b|_{\widetilde{D}_b} \cong \mathcal{O}_{\widetilde{D}_b},$$

where $X_b := \mathcal{X}|_{\{b\}}$, $L_b = L|_{X_b}$ and $\widetilde{D}_b = \widetilde{D} \cap X_b$. We will say that such a framed line bundle (L, δ') is stable if the underlying line bundle is so, or in other words, if L yields a stable framed Higgs bundle under the spectral correspondence.

Proposition 5.5. *Let $\mathcal{H}_{nr} \subset \mathcal{H}$ denote the subset of the Hitchin base corresponding to smooth spectral curves which are unramified over D. Then, relative framed line bundles on $\mathcal{X}|_{\mathcal{H}_{nr}}$ are parametrized by a $(\mathbb{C}^\times)^{nr-1}$ torsor $\mathscr{P}$ over $\mathrm{Jac}_{\mathcal{H}}(\mathcal{X}|_{\mathcal{H}_{nr}})$.*

Proof. The proof that this defines a torsor is very similar to that of Proposition 4.2 and is thus omitted.   □

We also get:

Lemma 5.6. *The spectral correspondence induces a morphism*

$$\sigma : \mathscr{P} \to \mathcal{M}_H(r,d). \tag{5.7}$$

Definition 5.7. The image $\sigma(\mathscr{P}) \subset \mathcal{M}_H(r,d)$, under the spectral correspondence (5.7), will be called the moduli of *diagonally framed Higgs pairs*, and it will be denoted by $\mathcal{M}_H^\Delta(r,d)$.

Next we study the subvariety $\mathcal{M}_H^\Delta(r,d)$ infinitesimally. Given $(E,\delta,\theta) \in \mathcal{M}_H^\Delta(r,d)$, we have that over each $x_i \in D$, there are distinguished one dimensional subspaces L_j^i of E_{x_i} such that $\delta = \pi_* \delta'$ is diagonal for the identification $E_{x_i} \cong \oplus_j L_j^i$. Let $\pi_j : E_{x_i} \twoheadrightarrow L_j^i$ be the projection. Let $Diag_E \subset \mathrm{End}(E)|_D$ be the subset of endomorphisms which are diagonal in the preferred reference, namely, $\{L_j^i : j = 1,\ldots,r\}$ over x_i. Let

$$\pi : \mathrm{End}(E) \twoheadrightarrow Diag_E$$

be the morphism associating to f the map

$$\pi(f) : v \mapsto (\pi_1(f(\pi_1(v))),\ldots,\pi_r(f(\pi_r(v))))$$

where $v \in E_{x_i}$.

Consider the complex

$$\mathscr{C}_\bullet^\Delta : \mathrm{Ker}(\pi) \xrightarrow{[\theta,\cdot]} \mathrm{End}(E) \otimes K_X(D).$$

Lemma 5.8. *Infinitesimal deformations of (E,δ,θ) along $\mathcal{M}_H^\Delta(r,d)$ are parametrized by $\mathbb{H}^1(\mathscr{C}_\bullet^\Delta)$.*

In particular, the dimension of $\mathcal{M}_H^\Delta(r,d)$ is

$$\dim \mathcal{M}_H^\Delta(r,d) = 2 \dim \mathscr{H}.$$

Proof. The proof that the complex parametrizes the right deformations follows the same arguments as Lemma 2.7.

We see that $\mathbb{H}^0(\mathscr{C}_\bullet^\Delta) = 0$ just as in Corollary 2.4. If the underlying bundle is stable, we have moreover that $\mathbb{H}^2(\mathscr{C}_\bullet^\Delta) = 0$ (as the dual complex has no zero hypercohomology).

So the tangent space at a point with underlying stable bundle is identified with $\mathbb{H}^1(\mathscr{C}_\bullet^\Delta)$.

To compute dimensions, by vanishing of all hypercohomology groups other than the first one, it follows that

$$\dim \mathbb{H}^1(\mathscr{C}_\bullet^\Delta) = \chi(\mathrm{End}(E) \otimes K_X(D)) - \chi(\mathrm{Ker}(\pi)).$$

By definition

$$\chi(\mathrm{Ker}(\pi)) = \chi(\mathrm{End}(E)) - \chi(Diag_E) = -r^2(g-1) - nr.$$

Also:

$$\chi(\mathrm{End}(E) \otimes K_X(D)) = r^2(g-1+n).$$

So

$$\dim \mathbb{H}^1(\mathscr{C}_\bullet^\Delta) = r^2(2g-2) + nr(r+1)$$

and the statement follows. $\qquad\square$

Remark 5.9. The dimension of $\mathcal{M}_H^\Delta(r,d)$ can also be computed using that $\mathcal{M}_H^\Delta(r,d)$ is a $(\mathbb{C}^\times)^{nr-1}$ torsor over $\mathcal{N}_H(r,d)$. Indeed

$$\dim \mathcal{M}_H^\Delta(r,d) = \dim \mathscr{P} = nr - 1 + \dim \operatorname{Jac}_{\mathscr{H}}(\mathfrak{X}).$$

Also, by [Bo, (3.1.8)]

$$\dim \operatorname{Jac}_{\mathscr{H}} = \dim \mathcal{N}_H(r,d) = r^2(2g - 2 + n) + 1.$$

Finally,

$$\dim \mathcal{M}_H^\Delta(r,d) = 2\left(r^2(g-1) + n\frac{r(r+1)}{2}\right) = 2\dim\mathscr{H}.$$

Remark 5.10. Note that as non-degeneracy of Ψ is a consequence of Serre duality

$$H^1(\operatorname{End}(E)(-D)) \cong H^0(\operatorname{End}(E) \otimes K_X(D))^*,$$

and $h^1(\operatorname{Ker}(\pi)) < h^1(\operatorname{End}(E)(-D))$ whenever E is stable, it follows that $\Psi|_{\mathcal{M}_H^\Delta}$ may degenerate on a subspace.

Proposition 5.11. *Let*

$$h^\Delta : \mathcal{M}_H^\Delta(r,d) \longrightarrow \mathscr{H}$$

be the restriction of the Hitchin map. Then:

(1) h^Δ *is surjective,*
(2) *the generic fibers are semiabelian varieties, i.e. an extension of an abelian variety by a torus. More precisely*

$$(h^\Delta)^{-1}(b) = \{(L,\gamma) \ : \ L \in \operatorname{Jac}(X_b), \gamma = (\gamma_1, \ldots, \gamma_r); \gamma_i^j : L_{y_i^j} \cong \mathcal{O}_{y_i^j}, y_i^j \in \pi_b^{-1}(x_i)\}/\mathbb{C}^\times.$$

Proof. Surjectivity follows immediately from that of $\widetilde{h}$. As for semiabelian-ness of the fibers, it follows from Proposition 5.5, together with [BSU, Proposition 7.2.1]. $\quad\square$

5.3. **Comparison of two integrable systems: completing the Hitchin system.** In this section we compare the Hitchin systems h and $\widetilde{h}$ defined in (5.1) and (5.3) respectively.

First of all, note that the symplectic structure on $\mathcal{M}_H(r,d)$ is compatible with the Poisson structure of $\mathcal{N}_H(r,d)$. Now, in what follows we argue that this is done somewhat perpendicularly to the Hitchin fibers of $\mathcal{N}_H(r,d)$. In particular, in order to obtain extra action angle coordinates to (locally) recover, for example, the semiabelian subvarieties appearing in Proposition 5.11, new functions need to be added to the system.

By Proposition 4.2, we have that locally

$$\mathcal{O}_{\mathcal{M}_{\mathscr{H}}} = \mathcal{O}_{\mathcal{N}_H} \otimes_\mathbb{C} \mathcal{O}_{\operatorname{GL}(r,\mathbb{C})/\mathbb{G}_m}.$$

Consider the embedding

$$\operatorname{GL}(r,\mathbb{C})^n/\mathbb{C}^\times \hookrightarrow \operatorname{PGL}(rn,\mathbb{C}). \tag{5.8}$$

Proposition 5.12. *Let* $f \in \mathcal{O}_{\operatorname{GL}(r,\mathbb{C})/\mathbb{G}_m}$ *be a local function on the fibers of* Q. *Then*

$$\{f, h_i\} = 0$$

for all $i = 1, \ldots, N$. In particular, consider the (local) coordinate functions x_i on the maximal torus of an affine open subset of $\mathrm{GL}(r, \mathbb{C})^n/\mathbb{C}^\times$ (defined for instance via the embedding (5.8)). Then the following set of local functions Poisson commutes

$$\{x_i, h_k\} = 0$$

for $1 \leq i \leq nr - 1, k = 1, \ldots, N$.

Moreover, the associated Hamiltonian vectors are linear on the fibers of the local map

$$(h_1, \ldots, h_N, x_1, \ldots, x_{nr-1}) \; : \; \mathcal{M}_H(r, d) \longrightarrow \mathbb{C}^{r^2(g-1) + \frac{r(r+1)}{2}n + nr - 1}.$$

Proof. Since the computation is local, we may assume the torsor (4.5) is trivial.

Consider the following commutative diagram:

$$
\begin{array}{ccc}
 & & H^0(\mathrm{End}(E)|_D)/\mathbb{C} \qquad (5.9) \\
 & & \downarrow{\scriptstyle i} \\
H^0(\mathrm{End}(E) \otimes K_X(D)) \longrightarrow \mathbb{H}^1(\mathscr{C}_\bullet) \longrightarrow H^1(\mathrm{End}(E)(-D)) \\
\Big\| \qquad\qquad\qquad\quad {\scriptstyle dQ}\Big\downarrow \qquad\qquad\qquad \Big\downarrow \\
H^0(\mathrm{End}(E) \otimes K_X(D)) \longrightarrow \mathbb{H}^1(\mathscr{C}_\bullet^\tau) \longrightarrow H^1(\mathrm{End}(E))
\end{array}
$$

The second and third lines give a decomposition of the tangent spaces at a point $(E, \delta, \theta) \in \mathcal{M}_H(r, d)$ and $Q(E, \delta, \theta) = (E, \theta) \in \mathcal{N}_H(r, d)$ respectively, while the first row is the tangent space of $Q^{-1}(E, \theta)$.

Take $v \in T_{(E,\delta,\theta)}\mathcal{M}_H$ and represent it as (μ_{ij}, η_i). Note that

$$\mu_{ij} = (\sigma_i - \sigma_j, \rho_{ij}),$$

where $\sigma_i \in H^0(\mathrm{End}(E)(-D))$ are local functions with the same restriction to D, namely $\sigma_i - \sigma_j \in \mathrm{Im}\,\{H^0(\mathrm{End}(E)|_D) \longrightarrow H^1(\mathrm{End}(E)(-D))\}$, and $\rho_{ij} \in H^1(\mathrm{End}(E))$.

Now, clearly, for $f \in \mathcal{O}_{\mathrm{GL(r,\mathbb{C})}/\mathbb{G}_\mathrm{m}}$, we have

$$df(\mu_{ij}, \eta_i) = df(\sigma_i - \sigma_j, \eta_i) = (df(\sigma_k|_D), 0).$$

In order to assign to df an element of $\mathbb{H}^1(\mathscr{C}^\bullet)$, consider the diagram dual to (5.9):

$$
\begin{array}{ccc}
V & & \qquad (5.10) \\
\uparrow & & \\
H^0(\mathrm{End}(E) \otimes K_X(D)) \longrightarrow \mathbb{H}^1(\mathscr{C}^\bullet) \longrightarrow H^1(\mathrm{End}(E)(-D)) \\
\uparrow \qquad\qquad\qquad\quad {\scriptstyle dQ^t}\Big\uparrow \qquad\qquad\qquad \Big\| \\
H^0(\mathrm{End}(E) \otimes K_X) \longrightarrow \mathbb{H}^1(\mathscr{C}_\bullet^\tau)^* \longrightarrow H^1(\mathrm{End}(E)(-D))
\end{array}
$$

where $V \subset H^0(\mathrm{End}(E) \otimes K_X(D)|_D)$ is the image of the restriction map

$$H^0(\mathrm{End}(E) \otimes K_X(D)) \to H^0(\mathrm{End}(E) \otimes K_X(D)|_D).$$

Note that Serre duality pairing between $H^0(\text{End}(E) \otimes K_X(D))$ and $H^1(\text{End}(E)(-D))$ identifies

$$H^0(\text{End}(E)|_D) \xrightarrow{\sim} H^0(\text{End}(E) \otimes K_X(D)|_D)^*, \; H^0(\text{End}(E) \otimes K)) \xrightarrow{\sim} H^1(\text{End}(E))^*.$$

So df is represented by an element $(\xi_i, 0)$ where $\xi_i \in V \subset H^0(\text{End}(E) \otimes K_X(D)|_D)$. We have $(\xi_i, 0) \in \mathbb{H}^1(\mathscr{C}^\bullet)$, as $\xi_i - \xi_j = [0, \theta]$.

Now, dh_k is represented by $(0, k\theta^{k-1})$, where $k\theta^{k-1}$ is interpreted as a 1-cocycle of $\text{End}(E)(-D)$ as follows: let $\{U_i\}_{i=1}^d$ be a open covering of X, choose a point $p \in X$ with local coordinate z on U_{ij}, the intersection of U_i and U_j, such that $K_X(D)|_{U_{ij}} \cong \mathcal{O}_{U_{ij}}$. Let $U'_{ij} = U_{ij} \setminus \{p\}$. Then

$$\theta^{k-1}|_{U'_{ij}} \in H^0(U'_{ij}, \text{End}(E)).$$

Moreover, $K|_{U_{ij}} \cong \mathcal{O}_{U_{ij}}(-D)$, so the meromorphic form dz/z generates $H^1(X, K_X)$ and moreover $\theta^k dz/z \in \Gamma(U_{ij}, \text{End}(E)(-D))$. We may check that this defines a 1-cocycle.

Now, on U_{ij}, we have $K \cong \mathcal{O}(-D)$, and the pairing of $\theta^k dz/z$ with any element of V will be identically zero. It thus follows that

$$\{f, h_k\} = 0.$$

Finally, since Serre duality restricts on the fibers of Q to the Killing form, the functions on the moduli space, given by the coordinate functions of the maximal torus, Poisson commute.

The last statement follows because of (a) linearity of X_{h_k} on the Jacobian of the spectral curve, (b) vanishing of X_{h_k} on the fibers of Q , (c) linearity of X_{x_i} on $(\mathbb{C}^\times)^{nr-1}$ and (d) vanishing of X_{x_i} on $\text{Jac}(X_b)$. $\qquad\qquad \square$

ACKNOWLEDGEMENTS

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