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On a variation of the Erdős–Selfridge superelliptic curve

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Abstract

In a recent paper by Das, Laishram and Saradha, it was shown that if there exists a rational solution of $y^l = (x+1)\ldots(x+i-1)(x+i+1)\ldots(x+k)$ for i not too close to $k/2$ and $y \neq 0$, then $\log\ l < 3^k$. In this paper, we extend the number of terms that can be missing in the equation and remove the condition on i.

1. *Introduction*

The Erdős–Selfridge superelliptic curves are the following family of curves,

$$y^l = (x+1)\ldots(x+k). \tag{1}$$

In [**4**], it is shown to not have any solutions in positive integers x, y, k, l with $k, l \geqslant 2$. It has been conjectured by Sander [**6**] that for $l \geqslant 4$ there are no rational solutions to equation (1) with $y \neq 0$. In [**1**], for $k \geqslant 2$ a positive integer, there are at most finitely many solutions to (1) with x and y rational numbers, $l \geqslant 2$ an integer with $(k,l) \neq (2,2)$ and $y \neq 0$. Further, it is shown that if l is a prime, then all solutions satisfy $\log\ l < 3^k$.

In [**2**], by Das, Laishram and Saradha, they consider the following variation of the Erdős–Selfridge superelliptic curves,

$$y^l = (x+1)\ldots(x+i-1)(x+i+1)\ldots(x+k), \tag{2}$$

for $k \geqslant 2$ an integer, l a prime, x and $y \neq 0$ rational numbers and i an integer strictly between 1 and k. Letting q be the smallest prime greater than or equal to $k/2$, they show that if (2) holds and $2 \leqslant i \leqslant k-q$ or $q < i < k$ then $\log\ l < 3^k$. Further, they show that if (2) holds and $3 \leqslant k \leqslant 26$, then $\log\ l < 3^k$.

In this paper, we will further the results in [**2**] by removing the condition on i and also extending the terms that can be missing from the equation. For $k \geqslant 2$ an integer, l a prime, i and j integers $1 < i < j < k$ and $\epsilon_t \in \{0,1\}$ for $i < t < j$, we call the following equation the Erdős–Selfridge curve with an incomplete block,

$$y^l = \prod_{t=1}^{i}(x+t) \prod_{t=i+1}^{j-1}(x+t)^{\epsilon_t} \prod_{t=j}^{k}(x+t). \tag{3}$$

We call a solution to (3) with x and y rational numbers and $y \neq 0$ a non-trivial rational solution. We note that the case $j-i=2$ and $\epsilon_{i+1}=0$ is the same as (2).

Theorem 1. *If (x,y) is a non-trivial rational solution to equation (3) for $k \geqslant 27$ and $j-i-1 < k/18-1$, then $\log\ l < 3^k$. In particular, if $j-i=2$, then $\log\ l < 3^k$ holds for $k \geqslant 3$.*

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This will be proven by adjusting the proofs in [**1**, **2**], by adding in new identities allowing us to consider prime numbers less than $k/2$ and using a more combinatorial approach.

We will also consider a variation of the Erdős–Selfridge superelliptic curve from which terms in the product have been removed without any specification of their location in the interval $[1, k]$.

THEOREM 2. *Letting* $1 < t_1 < \ldots < t_L < k$ *and* $S = \{1, \ldots, k\} \setminus \{t_1, \ldots, t_L\}$. *If* (x, y) *is a non-trivial rational solution to*

$$y^l = \prod_{j \in S}(x + j), \tag{4}$$

for $k \geqslant 2$ *and* $L < 0.26\sqrt{\frac{k}{\log k}}$, *then* $\log\ l < 3^k$.

2. *Preliminaries*

We will assume throughout that l is prime and $l > k - 1$. We will first prove the existence of primes in the interval $[\frac{k}{3}, \frac{k}{2}]$. Following that we will look at the prime decomposition of the factors of equation (3).

LEMMA 3. *For all* $k \geqslant 22$, *there exists a prime* p *such that* $\frac{1}{3}k \leqslant p \leqslant \frac{k}{2}$.

Proof. In [**5**], it is shown that there is always a prime between z and $(1 + \frac{1}{5})z$, for $z \geqslant 25$. Hence, for $k \geqslant 75$, the result now follows, and for the other k, it follows from an explicit computation. □

Following the work of Bennett and Siksek [**1**] and of Das, Laishram and Saradha [**2**], we write the coordinates (x, y) as fractions in lowest common form, $x = n/s$ and $y = m/s'$ for $m \neq 0$, s and s' positive integers. From equation (3), we have

$$\frac{m^l}{s'^l} = \frac{\prod_{t=1}^{i}(n + ts) \prod_{t=i+1}^{j-1}(n + ts)^{\epsilon_t} \prod_{t=j}^{k}(n + ts)}{s^{k - \sum \epsilon_i}}.$$

As $gcd(n, s) = gcd(m, s') = 1$ and l is a prime greater than $k - \sum \epsilon_i$, it follows there is a positive integer d such that $s = d^l$ and $s' = d^{k - \sum \epsilon_i}$.

Hence, equation (3) can be written as

$$m^l = \prod_{t=1}^{i}(n + td^l) \prod_{t=i+1}^{j-1}(n + td^l)^{\epsilon_t} \prod_{t=j}^{k}(n + td^l), \tag{5}$$

for m, n and d integers.

We now write each term in this product as

$$n + t_1 d^l = a_{t_1} x_{t_1}^l, \tag{6}$$

such that x_{t_1} is an integer and a_{t_1} is an lth power free integer. Let p be a prime that divides a_{t_1}, then p must also divide a_{t_2} for some t_2, hence p divides $(t_1 - t_2)d^l$. If p divides d, then it must also divide n, contradicting them being co-prime, hence p divides $t_1 - t_2$. It now follows that all prime factors of a_t are bounded above by k.

We note here that the exact same reasoning applies to equation (4) giving the following equation,

$$m^l = \prod_{t=1}^{k}(n + td^l)^{\epsilon_t} \tag{7}$$

for $\epsilon_t = 1$ if $t \in S$ and zero otherwise.

Lemma 4. *For m, n and d solutions of equation (4) with $L < 0.26\sqrt{\frac{k}{\log k}}$ and $k \geqslant 22$, there exists a prime $\frac{1}{3}k \leqslant p \leqslant \frac{1}{2}k$ that either divides d or divides m.*

Proof. We can assume that no prime p in the range $[k/3, k/2]$ divides d, otherwise the result follows trivially. Such a prime must divide at least two and at most three of the terms $n + td^l$ for $t \in [1, k]$. If p does not divide m, then there are at least 2 values of t such that $\epsilon_t = 0$. We will label these as i_p and $i_p + p$. It is then clear that p is in the set of differences of the elements in $\{t_1, \ldots, t_L\}$. It is easily seen that

$$| \{t_{i'} - t_{j'} : 1 \leqslant i' < j' \leqslant L\} | \leqslant L^2 - L + 1. \tag{8}$$

It is then easily seen that if

$$L^2 - L + 1 < \pi(k/2) - \pi(k/3), \tag{9}$$

then there must be such a prime p. For $k < 181000$, we can explicitly calculate using Magma, the following bound

$$0.07\tfrac{k}{\log(k)} < \pi(k/2) - \pi(k/3). \tag{10}$$

For $k \geqslant 181000$, we use the following bounds in [3]

$$\frac{x}{\log(x) - 1} < \pi(x) \text{ for } x \geqslant 5393, \tag{11}$$

and

$$\pi(x) < \frac{x}{\log(x) - 1.1} \text{ for } x \geqslant 60184. \tag{12}$$

It is then simple algebraic manipulation to see that for $k \geqslant 181000$

$$0.17\tfrac{k}{\log(k)} < \pi(k/2) - \pi(k/3). \tag{13}$$

It is now seen that with $L < 0.26\sqrt{\frac{k}{\log k}}$, inequality (9) is true, completing the Lemma. □

3. *Fermat equation*

In this section, we will attach a solution to a Fermat equation from a solution of (3) and (4). We will then use what is known about such equations to bound the exponent l.

Lemma 5. *For $k \geqslant 27$, assume that equation (3) has a non-trivial rational point (x, y) for $j - i - 1 = L < k/18 - 1$ or $L = 1$, or equation (4) has a solution for $L < 0.26\sqrt{\frac{k}{\log k}}$. Then, there exists a prime $\frac{1}{3}k \leqslant p \leqslant \frac{1}{2}k$ such that there are non-zero integers a, b, c, u, v, w satisfying*

$$au^l + bv^l + cw^l = 0 \tag{14}$$

such that

1. *a, b, c are lth power free integers;*
2. *all prime factors of abc are less than or equal to k;*
3. *$p \nmid abc$;*
4. *p divides precisely one of u, v, w.*

Proof. We first deal with the case of equation (3). Let p be a prime as described and assume that $p \nmid d$, then p must divide m. This follows simply from the following, let j be a value in

$[1,k]$ such that $n + jd^\ell \equiv 0 \mod p$. Then, if $p \nmid m$, it follows that $j - p \leqslant 0$ and $j + p \geqslant k+1$, hence $p \geqslant (k+1)/2$ contradicting our assumption on p. It follows that p either divides d or divides exactly 1, 2 or 3 factors in the Erdős–Selfridge curve.

We first deal with $p \mid d$, then it follows that $p \nmid m$, so $p \nmid a_{t_i} x_{t_i}^\ell$. Using (6) we see that

$$d^\ell = a_t x_t^\ell - a_{t+1} x_{t+1}^\ell,$$

choosing a t such that ϵ_t and ϵ_{t+1} are non-zero that gives the desired result.

We now deal with the case that p divides exactly one factor, which we take to be $n + td^l$. We consider the identity,

$$(n + td^l) - (n + t'd^l) = (t - t')d^l,$$

for t' a positive integer less than $k+1$ such that $|t' - t| < p$. Because $L < p - 1$, it follows that there exists such a t' such that $(n + t'd^l)$ appears on the right-hand side of (5). As p must divide $n + td^l$ to an lth power, applying (6), we then get an equation satisfying the Lemma, that is,

$$a_t x_t^l - a_{t'} x_{t'}^l - (t' - t)d^l = 0.$$

We now consider the case that p divides exactly two factors, $n + td^l$ and $n + (t+p)d^l$. We consider a similar identity as before,

$$(n + td^l)(n + (t+p)d^l) - (n + (t+\alpha)d^l)(n + (t+p-\alpha)d^l) = \alpha(\alpha - p)d^{2l},$$

for α a positive integer less than p.

It is clear that for distinct α and $\alpha' \leqslant p/2$, $\{t+\alpha, t+p-\alpha\} \cap \{t+\alpha', t+p-\alpha'\} = \varnothing$. Hence, as $L < p/2 - 1$, there exists α such that both $n + (t+\alpha)d^l$ and $n + (t+p-\alpha)d^l$ appear as factors in (5). Hence, the result now follows from (6) and the same finishing argument as above.

We are left to deal with the case that p divides exactly three factors, $n + td^l$, $n + (t+p)d^l$ and $n + (t+2p)d^l$.

We point out the following identity,

$$\begin{aligned}(n + td^l)(n + (t+p)d^l)(n + (t+2p)d^l) - (n + (t+\alpha)d^l)(n + (t+p+\alpha)d^l)(n + (t+2p-2\alpha)d^l) \\ = 3\alpha(\alpha - p)\left(n + \left(t + \frac{2(p+\alpha)}{3}\right)d^l\right)d^{2l}, \qquad (15)\end{aligned}$$

defined for α a positive integer less than p with $\alpha \equiv -p \pmod 3$. For α and α' positive integers either less than $p/2$, then

$$\left\{t+\alpha, t+\frac{2(p+\alpha)}{3}, t+p+\alpha, t+2p-2\alpha\right\} \cap \left\{t+\alpha', t+\frac{2(p+\alpha')}{3}, t+p+\alpha', t+2p-2\alpha'\right\} = \varnothing.$$

This follows from some simple inequalities and calculations mod 3. Hence, it follows that there are more than $\frac{p}{6} - 1$ distinct values of α with $\alpha \equiv -p \pmod 3$, such that the terms in (15) involving α do not coincide. So, we see that we have more choices of α than terms deleted, hence at least one α will give us such an equation with all terms defined. We note that as $k \geqslant 26$, there will always be a prime greater than or equal to 13 in the permitted interval, meaning we can always take $L = 1$ for these values of k.

In the case of equation (4), we first apply Lemma 4, then follow the above argument identically. □

It is worth noting that in the third case there is also the following identity,

$$(n+td^l)(n+(t+p)d^l)(n+(t+2p)d^l) - (n+(t+2\alpha)d^l)(n+(t+p-\alpha)d^l)(n+(t+2p-\alpha)d^l)$$
$$= 3\alpha(\alpha-p)\left(n+\left(t+\frac{4p-2\alpha}{3}\right)d^l\right)d^{2l}, \tag{16}$$

defined for α a positive integer less than p with $\alpha \equiv -p \pmod 3$. In specific cases of a fixed L, the use of (15) and (16) together can give specific values of α removing the need for combinatorial arguments.

We now state a Lemma which follows from [1].

Lemma 6. *If a, b, c, u, v, w are non-zero integers satisfying*

$$au^l + bv^l + cw^l = 0, \tag{17}$$

k is a fixed integer and $\frac{1}{3}k \leqslant p \leqslant \frac{1}{2}k$ is a prime such that

(1) *a, b, c are lth power free integers;*
(2) *all prime factors of abc are less than or equal to k;*
(3) *$p \nmid abc$;*
(4) *p divides precisely one of u, v, w;*
(5) *$l > k$ is prime.*

Then, $\log\ l \leqslant \frac{(N'+1)}{6}\log(\sqrt{p}+1)$, where $N' = 2^4 Rad_2(abc)$ and $Rad_2(n)$ denotes the product of all primes dividing n, apart from 2.

Proof. This follows immediately from [1, p.4]. □

Remark 1. It is then a routine calculation, as in [1], using

$$\sum_{\substack{q \leqslant k \\ q \text{ prime}}} \log\ q < 1.000081k,$$

from [7] and $k \geqslant 26$ to conclude that

$$\log\ l < 3^k.$$

Proof of Theorem 1. For $k \geqslant 27$, this follows immediately by applying Lemma 5, Lemma 6 and the remark above. We now finish with the case of $L = 1$ and $k \leqslant 26$. If $\epsilon_{i+1} = 1$, then this follows from [1]. If however $\epsilon_{i+1} = 0$ and $k \leqslant 26$, then this is covered by [2]. □

Proof of Theorem 2. For $k \geqslant 27$, this follows identically to above, if $k < 27$, then it is clear that $L = 0$ and so follows from [1]. □

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