Robust dynamic bus controls considering delay disturbances and passenger demand uncertainty

Shukai Li[a], Ronghui Liu[b], Lixing Yang[a], Ziyou Gao[a]

[a]*State Key Laboratory of Rail Traffic Control and Safety, Beijing Jiaotong University, Beijing, 100044, China*
[b]*Institute for Transport Studies, University of Leeds, Leeds LS2 9JT, U.K.*

Abstract

This paper proposes a robust dynamic control mechanism for bus transit system, taking account of variations in congestion delays and passenger demand, and combines bus holding and operating speed control strategies. By using a prespecified uncertainty set, we propose a state space model for bus motion with delay disturbances and passenger demand uncertainties. According to the Lyapunov function analysis method, we design a robust dynamic control based on the state-feedback scheme as the bus control to achieve the robust stability of the bus transit system, which effectively reduces the bus bunching phenomenon. Furthermore, we formulate a nonlinear optimal control problem to design the robust optimal bus control, which not only reduces the bus bunching, but also improves the schedule adherence and headway regularity of bus service lines. To handle the complexity of the nonlinear optimal control problem with uncertain parameters and disturbances, we reduce it to a convex optimization problem by the minimization of an upper bound on the objective function. The problem is solved in a polynomial time and satisfies the practical real time requirement. Numerical examples are presented to validate the effectiveness of the model and control methods.

Keywords: Bus bunching, Dynamic control, Robust stability, Convex optimization, Demand uncertainty

1. Introduction

1.1. Motivation

Bus bunching is a common phenomenon especially along high-frequency bus lines due to disturbances to bus running time and uncertainties in passenger arrival flow (Osuna and Newell, 1972; Newell, 1974; Daganzo, 2009; Daganzo and Pilachowski, 2011; Fonzone et al., 2015; Schmocker et al., 2016). When a bus is delayed from scheduled timetable, the headway with its leading bus becomes large and then more passengers are needed to be transported at downstream stops, which leads to the delays of this bus once again. At the same time, the next bus collects fewer passengers at stops, which needs less stop time, and then the bus bunching phenomenon is created. Bus bunching reduces headway regularity and thereby increases the passengers waiting time. The conventional schedule-based strategies by holding buses were designed to cope with this problem. However, these methods require adding slack time to the schedule, which slow buses. In particular, with availability of automatic vehicle location (AVL), the real-time updated bus

*Corresponding author.
Email address: R.Liu@its.leeds.ac.uk (Ronghui Liu)

operation data can be obtained for bus transit system. This provides an opportunity to apply the dynamic control method based on the state feedback information in real time. Dynamic control methods based on real-time information have been shown to alleviate these problems and improve resiliency alongside of schedule-improvement methods (Daganzo, 2009; Muñoz et al., 2013; Sánchez-Martínez et al., 2016).

It is well known that delay disturbances such as traffic congestion, traffic accident and other events, along with the uncertainties of passenger arrival flow lead to the bus bunching phenomenon. Thus, it is essential to take account of uncertainties of the passenger demand and the delay disturbances for the bus control design problem (Daganzo, 2009; Xuan et al., 2011). The robust optimization problems for the bus schedule with uncertainties have been investigated by (Yan et al., 2012; Wu et al., 2016). Moreover, under the dynamic control framework, the corresponding bus motion can be regarded as a discrete dynamic system with uncertain parameters for the passenger demand and delay disturbances, and robust control theory is proper to cope with the dynamic system with uncertain parameters and disturbances (Zhou et al., 1996; Bertsimas and Sim, 2004; Haddad and Shraiber, 2014). This motivates us to investigate the robust dynamic bus control method to reduce bus bunching. The robust dynamic bus control can well combine the holding and the operating speed control. Therefore, the aim of this study is to design robust dynamic control strategy to alleviate the effect of uncertainties on bus bunching and improve schedule adherence of bus service lines.

1.2. Literature review

To reduce bus bunching and improve headway regularity, real-time bus holding controls have attracted considerable attentions in the literature (Osuna and Newell, 1972; Newell, 1974; Zhao et al., 2006). Most of these earlier studies did not consider the real time feedback data and ignored dwell time effects on the bus headway. With availability of automatic vehicle location (AVL), dynamic control methods based on real-time feedback information have been proposed to alleviate these problems (Eberlein et al., 2001; Dessouky et al., 2003; Shorter et al., 2005; Daganzo, 2009; Delgado et al., 2009; Hernandez et al., 2015; Andres et al., 2017).

The real time bus control strategies mainly includes two classes: (1) Station control, including bus holding; (2) Interstation control, as bus overtaking and operating speed control (Eberlein et al., 2001; Muñoz et al., 2013).

The dynamic bus holding have attracted lots of attentions from the researchers. Based on the real-time information, Eberlein et al. (2001) proposed a rolling horizon strategy for the bus holding problem, in which an efficient solution algorithm is designed. Zhao et al. (2003) designed a distributed control method for the bus holding problem, which found that the designed method is robust to a wide range of transport environments. Daganzo (2009) developed a new dynamic holding method with the headway dynamic information, which effectively reduces the slack time due to the use of the real-time information. The proposed method needs less slack time than the schedule-based method, where buses can travel more faster. Base on the bus arrival deviations information, a general control method was proposed by (Xuan et al., 2011). With this method, buses can not only close to a given schedule, but also maintain a regular headway with less slack. By abandoning the idea of the target schedule and headway, Bartholdi and Eisenstein (2012) developed a method with the coordination of all the considered buses so as to achieve a better service. With the proposed method, bus headways are dynamically self-equalizing. By using the real-time information, (Berrebi et al., 2015) designed a real-time holding strategy to dispatch buses on a loop route, which can effectively reduce the passengers average waiting time and meanwhile improve the system resiliency. Sánchez-Martínez et al. (2016) developed a bus holding control optimization model with the dynamic demand and running time, which revealed that the proposed control method outperforms its static equivalent in a high demand environment.

As for the bus operating speed control, stop-skipping is a widely used strategy. Sun and Hickman (2005) studied a stop-skipping policy for real-time operations of service disruption with varying durations and occurrences along the route, where the boarding and alighting passengers are assumed to satisfy random distributions. A nonlinear integer programming problem is proposed to determine the stop-skipping strategy, which is solved based on an exhaustive search means. By considering the random bus travel time, Liu et al. (2013) investigated a bus stop-skipping strategy to minimize the weighted sum of the in-vehicle travel time, the passenger waiting time and the service cost. Moreover, combining bus holding and stop-skipping actions, Cortés et al. (2010) presented a hybrid predictive controller to optimize real-time bus operations by taking into account uncertain passenger demand. Two control policies were proposed by (Delgado et al., 2012) with vehicle holding limitation and boarding limitation, where the limits of the number of boarding passengers can effectively increase bus operational speed and reduce the cycle time. To deal with large disturbances, an adaptive control scheme is designed by (Daganzo and Pilachowski, 2011), where the bus cruising speed is adjusted in real-time according to the expected passenger demand and the distance information of the current bus and its neighboring buses. The proposed adaptive control can effectively prevent the bus bunching, especially for the large disturbances. With the consideration of the bus overtaking case and the distributed passenger boarding behaviour, Wu et al. (2017) investigated the bus holding control scheme to further reduce the bus bunching.

In summary, considerable research over the past decade have be done on the bus control strategies using real-time information. Most of these studies are based on a deterministic control, where system parameters and disturbances of the bus line are considered to be deterministic or satisfy a given random probabilistic distribution (Eberlein et al., 2001; Hickman, 2001; Daganzo, 2009; Xuan et al., 2011). However, for the practical bus operating, the system parameters of bus line (e.g., passenger arrival flows) and bus travel times are naturally uncertain and varying with time. The probability distribution assumptions are difficult to satisfy in practice, as the distributions themselves may vary over time and space. To overcome this, in this paper, we adopt an uncertainty model for the system parameters that can be described as deterministic and set-based. The uncertainty set does no require to know the specific probability distribution of the uncertain data. In addition, the optimization problem with the uncertainty set becomes a robust optimization problem, which constructs a solution that is feasible for any realization of the uncertainty in a given set, instead of seeking to immunize the solution in some probabilistic sense to stochastic uncertainty. Based on this, we investigate the robust bus control strategies concerning uncertain system parameters and disturbances.

1.3. Proposed approach and contributions

Instead of using a probabilistic distribution to describe the system uncertainties, we adopt a pre-specified uncertainty set to describe uncertainties of passenger arrival flow and a bounded variable to represent the delay disturbances. The running time and passenger demand are both time-dependent variables, where the dynamic passenger demand is descried by a uncertainty set and the time-varying running time is reflected by a bounded variable. To the best of our knowledge, few works pay attention to design the robust bus control strategies to reduce bus bunching. The difficulty faced by the robust bus control problem is the robust stability analysis of the bus motion with uncertainties and disturbances. The Lyapunov function analysis method paves a powerful way to cope with the stability problems of many complex systems (Boyd et al., 1994; Kothare et al., 1996; Haddad and Shraiber, 2014). Following the bus control framework of

(Daganzo, 2009; Xuan et al., 2011), this study further investigates the robust dynamic control based on a Lyapunov function analysis method to improve schedule/headway regularity for a bus line with uncertain system parameters and delay disturbances. The bus control strategy combines both the holding and the operating speed control. The main contributions of this study are given as follows.

(1) Using a prespecified uncertainty set to represent the uncertainties of passenger demand and a bounded variable to represent the delay disturbances, a state space model for the bus motion with uncertain passenger demand and delay disturbances is proposed on the basis of the dynamic bus model in (Daganzo, 2009; Xuan et al., 2011). The model does not need to estimate the probability distributions for the uncertainties, only needs the upper and lower bounds of uncertainties, which explicitly incorporates the time-variability nature of running times and passenger demand.

(2) Within a Lyapunov function analysis framework, the robust bus feedback control strategies are designed to guarantee the robust stability of the bus motion. As an extension to the determined bus control of (Daganzo, 2009; Xuan et al., 2011), the proposed robust bus control ensures that the bus motion is robustly stable for the admissible system uncertainties, i.e., the bus bunching can be effectively reduced and the schedule adherence can be improved for all the admissible delay disturbances and uncertain passenger arrival flow.

(3) A robust optimal bus control scheme to jointly reduce bunching and improve schedule adherence is formulated as a nonlinear optimal control problem, which contains an infinite number of constraints related to the uncertain parameters of the bus motion model and is hard to be solved using traditional dynamic programming. To address this, we reduce the original problem to a convex optimization problem with the minimization of an upper bound on the given objective, which is solved efficiently in a polynomial time.

Table 1: The comparison with other existing results.

Characteristics	Objective	Passenger flow	Running time	Control strategy	Solution methodology	Stability property
Eberlein et al. (2001)	Minimize waiting time	Deterministic	Deterministic and stationary	Bus holding	Quadratic program	Not
Daganzo (2009)	Headway regularity	Deterministic	Stochastic and dynamic	Bus holding	Adaptive control	Verify
Cortés et al. (2010)	Minimize waiting time	Stochastic	Deterministic and dynamic	Hybrid strategy	Genetic algorithms	Not
Xuan et al. (2011)	Schedule adherence and headway regularity	Deterministic	Stochastic and dynamic	Bus holding	Optimal control	Verify
Delgado et al. (2012)	Minimize waiting time and travel time	Deterministic	Deterministic and dynamic	Hybrid strategy	Mathematical programming	Not
Sánchez-Martínez et al. (2016)	Minimize waiting time and in-vehicle delay	Deterministic and time-varying	Deterministic and dynamic	Bus holding	Optimization-based control	Not
Wu et al. (2017)	Headway regularity	Stochastic	Stochastic and dynamic	Bus holding	Adaptive control	Not
This paper	Schedule adherence and headway regularity	Uncertain and time-varying	Uncertain and dynamic	Hybrid strategy	Robust optimal control	Robust stability

The main features of our paper, as compared to existing literatures, are summarized in Table 1. The rest of this paper is organized as follows. In section 2, the bus motion with uncertainties and disturbances is constructed. In section 3, the robust and optimal bus control problems are formulated. In section 4, the robust stability for bus motion is proved, and the robust bus control strategy is developed. In section 5, numerical examples are presented to validate the effectiveness of the bus model and control methods. The conclusion of this paper is given in section 6.

2. A bus motion with delay disturbance and demand uncertainty

2.1. Problem description

Let us consider a single bus transit line that has M stations along a looped structure, where a set of buses are operating along the line with to a regular schedule. Figure 1 presents the structure of the bus line, where buses start their run at a terminal defined station 1, visit all stations downstream $(2, 3, \ldots, M)$ and loop around after arriving at the terminal (station 1) with a new trip.

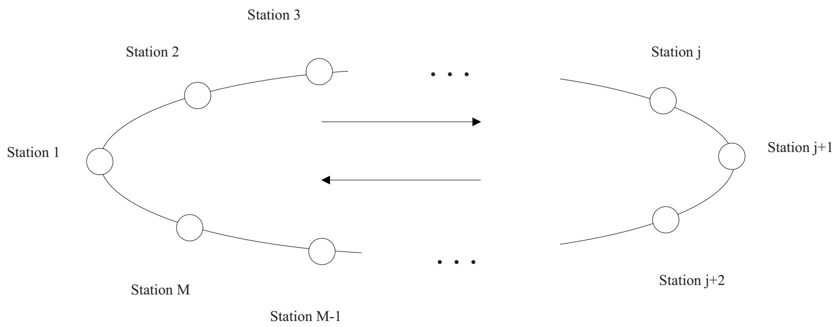


Figure 1. The structure of bus transit system.

The following assumptions are needed to formulate the problem of this paper.

(1) Bus capacity is unlimited;

(2) Buses do not pass each other;

(3) Buses stop at every stop;

(4) Dwell time is an affine function of the number of boarding passengers;

(5) Passenger arrival rate lies in a pre-specified uncertainty set;

(6) A bounded variable is used to represent the delay disturbances.

(1) and (2) are common assumptions for a well designed bus system (Xuan et al., 2011). In the following formulation, a representation of the uncertain set for the realised value of demand rate can make this assumption less restrictive, in the sense that if the waiting passenger number exceeds the remaining capacity constrained by the maximum bus capacity, a smaller realised value of demand rate can take effect to denote the influence of the actual boarding passenger number to the dwell time. Assumption (3) prevents skip-stop operations even in the case of sudden service disruption. Assumption (4) is appropriate for bus systems with the fact that the boarding and alighting are occurring concurrently and in most cases the alighting time is less than the boarding time. The dwell time might be influenced more by the alighting passengers than by the boarding passengers, especially towards the end of the trip, for a linear route with distinctive terminus stops (such as one along a radial corridor towards the city centre). In this study, we consider a circular bus route, akin to a route along a ring road. With such a looped bus path, we may assume that the boarding passengers (and boarding time) is the dominate factor affecting the bus dwell time. In addition, the disturbance term in the following formulation in a way can be used to denote the effect of a more influence of the alighting passengers to the dwell time with the fact that the alighting time per passenger is much smaller than the boarding time per passenger. This will make this assumption less restrictive. Assumption (5) incorporates demand uncertainty in a general demand model. Assumption (6) uses a bounded variable to represent the delay disturbances, which is also more general than a certain probability distribution.

Bus bunching usually happens because of the inevitable disturbance to running time and the uncertainties in the passenger arrival flow. When a bus is delayed, the headway with its front bus is enlarged, and then more passengers need to be picked up at the downstream stops, the bus is delayed further more. Meanwhile, the following bus transports only fewer passengers at stops that needs less dwelling time, and will gradually catch up with the bus in front, cause bus bunching. The robust dynamic holding strategy is to alleviate the effect of the delay disturbances and demand uncertainties to the operations of buses, so as to reduce bus bunching. The dynamic control strategies mainly include two types: station control and interstation control. In this study, we will simultaneously consider station and interstation controls for the robust control design problem.

The model variables and parameters are shown in Table 2.

Table 2: Variables and parameters.

$i = 1, 2 \ldots, N$: indices of bus on transit line;
$j = 1, 2, \ldots$: indices of stations on transit line;
t_j^i: the scheduled arrival time of bus i at station j;
a_j^i: the actual arrival time of bus i at station j;
e_j^i: the deviation from scheduled arrival time of bus i at station j;
H: the scheduled headway;
c_j: the average running time of the bus from station j to station $j+1$;
u_j^i: the running time or dwell time adjustment for bus i from station j;
w_j^i: the disturbance delay for the running time of bus i between stations j and $j+1$;
β_j: a fraction of headway to the dwell time of bus at station j;
$\tilde{\beta}_j$: the realized value of β_j;
e_j: the state vector at station j;
u_j: the control vector at station j;
w_j: the delay disturbance vector at station j;
A: the system parameter matrix;
ΔA: the uncertain system parameter matrix;
K: the control parameter to be designed;
Q, R, S: the positive definite weighted matrices;
γ: the robustness index.

2.2. Bus motion with delay disturbances and demand uncertainties

Consider a bus transit line operating according to a regular schedule. To investigate the bus bunching problem, we firstly take into account a scheduled arrival time of the bus. The transfer equation of the scheduled arrival time is formulated as

$$t_{j+1}^i = t_j^i + \beta_j H + c_j + d_j, \quad i = 1, 2 \ldots, N, j = 1, 2 \ldots, \tag{1}$$

where $H = t_j^i - t_j^{i-1}$ is the scheduled headway, c_j is the average trip time between station j to $j+1$, d_j is the slack time builded in the plan. The station number j will be increased at the next new trip. β_j is a dimensionless variable representing the demand rate (i.e. the ratio between the arrival and boarding rates of passengers) at station j, which denotes a fraction of headway to the dwell time of bus at station j and satisfies that $0 < \beta_j < 1$ (Xuan et al., 2011), $\beta_j H$ denotes the dwell time of bus to serve passengers at station j. In practice, the parameter β_j is affected by many factors, such as exogenous passenger demand and endogenous load on the bus. To reflect the complex effects of these factors, we will later adopt an uncertainty interval as the realized value of β_j.

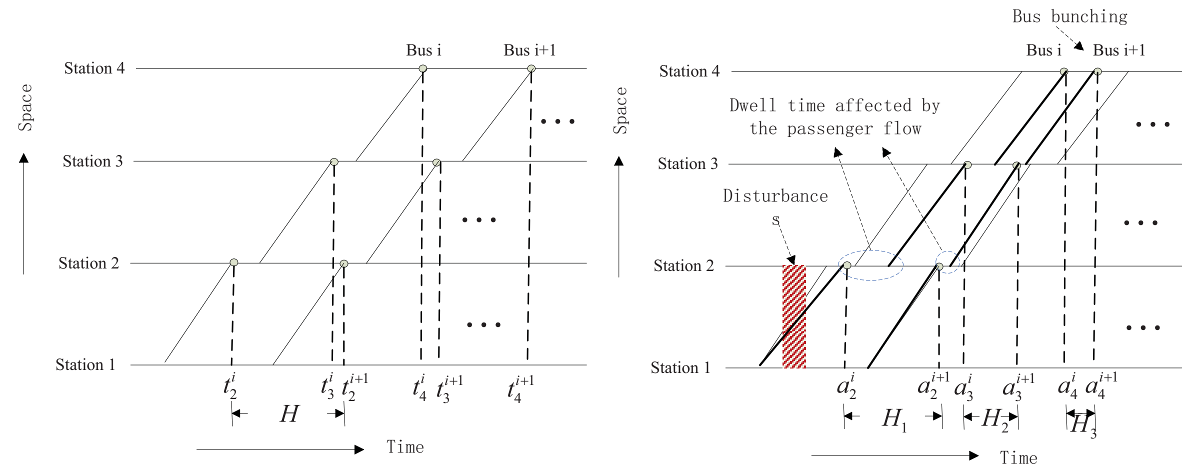
Consider delay disturbances and apply bus control, the transfer equation of the actual arrival time is given by

$$a_{j+1}^i = a_j^i + \beta_j(a_j^i - a_j^{i-1}) + u_j^i + c_j + w_j^i, \tag{2}$$

where u_j^i is the running time adjustment of bus i from station j to $j+1$ or the dwell time adjustment of bus i at station j (i.e. holding time), thus it represents a combined control strategy: $u_j^i > 0$ indicates it is a holding strategy, while $u_j^i < 0$ denotes that an inter-station control strategy is applied to increase the operating speed. In practical applications, the slack times are often built into the plan to allow schedule recovery. Therefore, with the slack time, the buses do not have to go as fast as they can, and it gives room for speed-up controls. In addition, considering that the shortening dwell times will deny boardings, the dwell times shortening strategy is not allowable in the proposed control strategy. The term w_j^i is the disturbance delay for the running time and the dwell time of bus i between stations j and $j+1$. Here we use a bounded variable w_j^i to denote the delay disturbances, where $\sum_{k=1}^{\infty}(w_k^i)^2 < \infty$. This specific term of the sum of squares for the bounded disturbances is in accordance with the L_2 norm of the following formulated robustness index.

In the bus motion model (2), the bus dwell time (the second term on the left) is represented as a linear function of bus arrival time and the boarding passengers only: $\beta_j(a_j^i - a_j^{i-1})$, where β_j is the ratio between the arrival and boarding rates of passengers to stop j. This linear formulation of a discrete dynamic system enables the application of real-time state feedback information on the design of dynamic bus control from a control-oriented perspective. In the literature, there have been studies that explicitly consider the alighting passengers and bus capacity constraint on bus dwell times (Sánchez-Martínez et al., 2016; Wu et al., 2017). There, it would require either prior knowledge of passenger OD movements (boarding and alighting stops) (Sánchez-Martínez et al., 2016), or make assumptions on the alighting patterns. For example, Wu et al. (2017) assumes a fixed proportion of passengers currently on-board the bus to alight at each stop. Then, to take account of bus capacity in model (2), the formulation of minimization between the waiting passengers and the remaining capacity subject to the maximum capacity, on-board and alighting passengers needs to be taken to represent the dwell time. This results in a nonlinear term for the dwell time and thus a nonlinear bus motion model of (2), which changes the structure of the original linear discrete dynamic system framework in this study and it becomes more cumbersome to give the rigorous proof of robust stability for the nonlinear system. Additionally, to further take account of the combination of boarding and alighting to the dwell time, the dwell time can be denoted either as the greater of the boarding and alighting times if the alighting and boarding are happening in parallel, or as the sum of boarding and alighting time if alighting and boarding are in series. Either way, the alighting number needs to be further modelled and the dynamic evolution model for the number of on-board passengers should also be further constructed. The problem is usually formulated as a nonlinear programming problem, and it becomes more difficult to design the feedback regulators from a control perspective. The formulated nonlinear programming problem is usually solved in a rolling horizon scheme to satisfy the real-time requirement, where it however needs the future demands to be predicted in real-time, and thus more effort is needed for online use. By comparison, the following proposed feedback regulator from the control-oriented perspective is based on the real-time state feedback information, which is quite robust to moderate parameter value changes.

It should be pointed that the bus motion of (2) is unstable, where the headway increasingly deviates from the regular headway with the time, and then the buses bunch is created (Daganzo, 2009). The parameter β_j acts like forces that repel buses when the headway is longer than the regular one, otherwise attract buses when the headway is shorter. Figure 2 depicts the evolution of the bus traffic dynamics, according to the scheduled bus timetable (Figure 2(a)) and the actual bus trajectories under delay disturbances (Figure 2(b)), where the thin black line denotes the scheduled bus timetable and the thick black line shows the actual stop visit times. In Figure 2 (b), bus i is affected by the disturbance from station 1 to 2, which delays its arrival at station 2. Meanwhile, passenger demand at station is increased due to the delay of bus 1, which further increases the dwell time of bus i at station 2. Thus the actual headway H_1 between bus i and bus $i+1$ at station 2 is decreased. When the following bus $i+1$ is arriving at station 2, the dwell time of bus $i+1$ is reduced since the accumulated passenger demand has been carried by bus i, which further reduces the headway H_2 at station 3, and similarly, at station 4, the headway H_3 of buses i and $i+1$ is further reduced, which creates the bus bunching. The bus bunching phenomenon not only leads to the bus delays, but also reduces the headway regularity, which increases the passenger waiting time.



(a) The scheduled timetable (b) The actual timetable

Figure 2. An illustration of the bus traffic dynamics.

Let $e_j^i = a_j^i - t_j^i$ be the time deviation from the scheduled arrival time of bus i at station j. Subtracting equation (2) from (1), we then get the error dynamic model of bus motion with the following form.

$$e_{j+1}^i = e_j^i + \beta_j(e_j^i - e_j^{i-1}) + \bar{u}_j^i + w_j^i, \tag{3}$$

where $\bar{u}_j^i = u_j^i - d_j$ is the control variable to be designed.

Most of the existing literature on bus motion modelling assumes a uniform demand rate (e.g., the demand rates are deterministic or satisfy a random distribution (Eberlein et al., 2001; Hickman, 2001; Daganzo, 2009; Xuan et al., 2011)). In practice, the demand rate (defined as the ratio between the arrival and boarding rates of passengers) β_j is uncertain and can be highly variable over time. This is determined by complex effects of the exogenous passenger demand and endogenous load on the bus to the bus dwell time. Figure 3 shows the observed demand rates for 10 buses at one station (denoted by station 5) of Beijing bus Line 16 over one hour morning peak period; it shows the highly dynamic demand rates.

In this paper, we consider demand uncertainty and define $\tilde{\beta}_j$ as the realized value of β_j. Traditionally, the passenger demand is assumed to satisfy a certain probability distribution, such as Poisson process, and bus control with a probabilistic distribution is formulated as a stochastic optimization problem. The performance of stochastic formulations is very sensitive to the accuracy of assumptions regarding the probability distribution of uncertain data. In this paper, the variable demand is represented as an uncertainty set and the control problem as a robust optimisation formulation which constructs a solution that is feasible for any realization of the uncertainty in a given set. In this way, the robust optimisation formulation reduces the sensitivity of the solutions to the specific probability distribution. The uncertainty interval provides a simple and effective approximation to reflect the complex effects of exogenous passenger demand and endogenous load on the bus to the bus dwell time. The use of uncertainty set will lead to conservative solutions for the control strategies because it needs to be protected for any realization within this set. In practice, the choice of the uncertainty interval is based on the historical statistics data, the risk preferences and tractability.

We assume that the realized values are varying in a range with a known nominal value β and a given half-length r. Specially, the realized values lie in a prespecified uncertainty set (see Figure 3), which is given by

$$\tilde{\beta}_j = \beta + \Delta\beta_j, \tag{4}$$

where $\Delta\beta_j = \alpha_j^i r$ and $-1 \leq \alpha_j^i \leq 1$ can be variable with time and between stations (such as uncertainties at major activity centers).

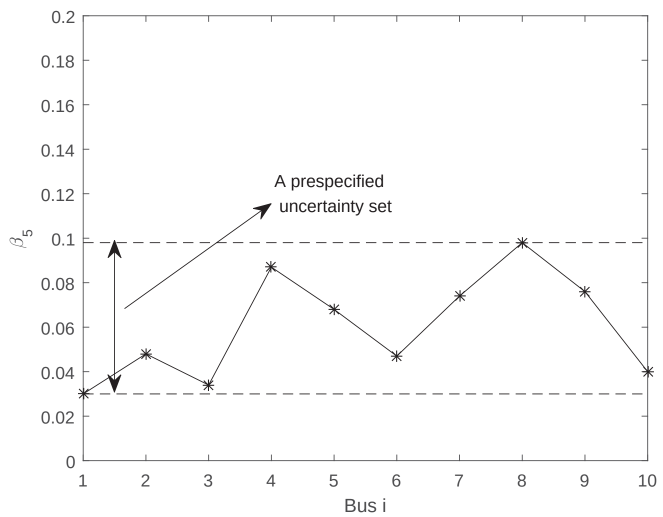


Figure 3. The measure for the demand rate β_j of one station in Beijing bus line 16 for bus 1 to 10.

In practice, a small increase in β_j leads to a significant difference in bus headway deviation, which will further affect the accuracy of design of the bus control strategy. Therefore, for the uncertainties of the time-varying parameter β_j, it is more accurate to adopt a pre-specified uncertainty set as $\tilde{\beta}_j$ and design the so-called robust bus control method to reduce the bus bunching.

By combining (1)–(4), we can easily get the traffic model for the bus motion with disturbances and uncertainties as

$$e_{j+1}^i = e_j^i + (\beta + \Delta\beta_j)(e_j^i - e_j^{i-1}) + \bar{u}_j^i + w_{j+1}^i, \tag{5}$$

which reveals the deviation of bus motion from the schedule under delay disturbances and demand uncertainties. For convenience, the traffic model (5) for the bus motion can be further rewritten in a matrix form.

$$e_{j+1} = (A + \Delta A)e_j + u_j + w_j, \ j = 1, 2, \ldots \tag{6}$$

where $e_j = [e_j^1, e_j^2, \ldots, e_j^N]^T$ is the state vector, $u_j = [\bar{u}_j^1, \bar{u}_j^2, \ldots, \bar{u}_j^N]^T$ is the control vector, and $w_j = [w_j^1, w_j^2, \ldots, w_j^N]^T$ is the disturbance vector. We use a simple example of a bus line with $N = 6$ buses and $M = 5$ stations to illustrate the changing of the adopted state variable e_j with the stage j. Figure 4 depicts the dynamic changing of the state from e_1 to e_5, where station j represents stage j of system (6).

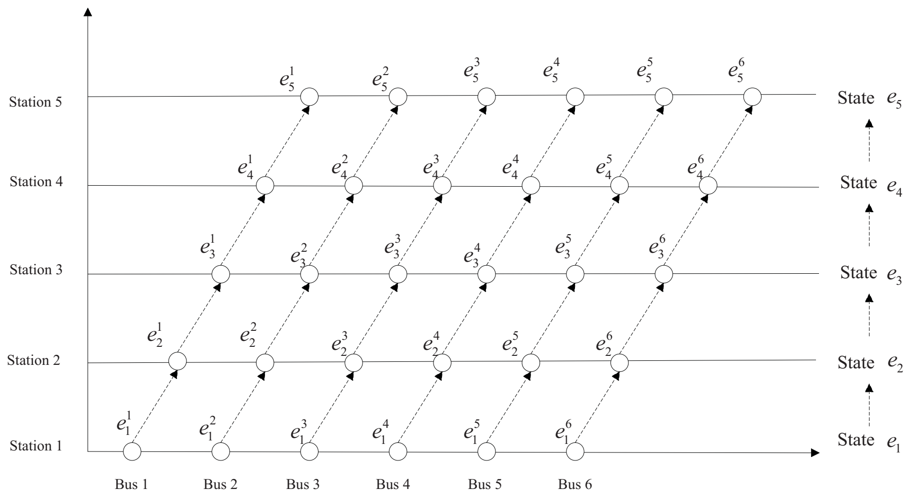


Figure 4. The dynamic changing from state e_1 to state e_5.

According to (5), the system parameter matrix A of (6) is presented as follows.

$$A = \begin{bmatrix} 1+\beta & 0 & 0 & 0 & \cdots \\ -\beta & 1+\beta & 0 & 0 & \cdots \\ & \cdots & \cdots & \cdots & \\ 0 & \cdots & 0 & -\beta & 1+\beta \end{bmatrix}_{N\times N},$$

and the uncertain system parameter matrix ΔA can be rewritten as $\Delta A = IE_1F_1 + IE_2F_2$ based on (4), where

$$F_1 = \begin{bmatrix} r & 0 & 0 & \cdots \\ 0 & r & 0 & \cdots \\ & \cdots & \cdots & \\ 0 & & \cdots & r \end{bmatrix}_{N\times N}, \ F_2 = \begin{bmatrix} 0 & 0 & 0 & \cdots \\ r & 0 & 0 & \cdots \\ & \cdots & \cdots & \\ 0 & \cdots & r & 0 \end{bmatrix}_{N\times N}, \ E_1 = \begin{bmatrix} \alpha_j^1 & 0 & 0 & \cdots \\ 0 & \alpha_j^2 & 0 & \cdots \\ & \cdots & \cdots & \\ 0 & \cdots & 0 & \alpha_j^N \end{bmatrix}_{N\times N},$$

and $E_2 = \begin{bmatrix} -\alpha_j^1 & 0 & 0 & \cdots \\ 0 & -\alpha_j^2 & 0 & \cdots \\ & \cdots & \cdots & \\ 0 & \cdots & 0 & -\alpha_j^N \end{bmatrix}_{N\times N}$.

In addition, for the uncertain matrixes E_1 and E_2, we have the following conditions.

$$E_1^T E_1 \leq I, \ E_2^T E_2 \leq I. \tag{7}$$

Thus, the matrix form of bus model (6) is an uncertain discrete dynamic system with delay disturbances w_j and demand uncertain parameter matrix ΔA. The dimension for this system is N, which also equals to the number of the operating buses, and the system stage j denotes the station j of the bus line. The bus model (6) captures the dynamic evolution of the arrival times of the buses at the stations. The model enables us to apply the real-time state feedback information to design the bus dynamic control scheme.

3. The robust and optimal control problems

Concerning the uncertain linear discrete dynamic system (6) for the bus line, the following proposition shows the instability of the bus transit system without control u_j.

Proposition 3.1. *The uncertain linear discrete system (6) under $u_j = 0$ is unstable, i.e., the bounded disturbance w_j will produce unbounded errors.*

Proof. It can be easily derived from (6) that all the eigenvalues of the system matrix $(A + \Delta A)$ are larger than one, thus, the uncertain linear discrete system (6) under $u_j = 0$ is unstable. □

Proposition 3.1 suggests that, without control u_j, the bounded disturbance w_j produces unbounded deviations of the practical bus arrival time from the nominal one. It also indicates the bus bunching phenomenon. The robust control theory is appropriate to deal with the dynamic system with uncertain parameters and disturbances. Therefore, it is essential to develop the robust control method to achieve the stability of the bus motion with delay disturbances and demand uncertainties. In this section, we present the robust control and optimal control problems for the bus motions to reduce the bus bunching and improve the the operating speed of bus service lines.

3.1. The robust bus control problem

We first consider the robust bus control strategy based on a state feedback scheme as follows

$$u_j = Ke_j, \tag{8}$$

where K is the feedback control matrix to be determined, which is the state-feedback control parameter. The control (8) is in fact on-line control: the control for each bus i at station j is a linear combination of deviations for the arrival times of all buses at station j. The structure of the matrix K determines which bus' feedback information at station j should be used for the bus control strategy, which will be discussed in the next section. The bus control u_j is designed to be applied to each bus, which can combine the holding strategy and the operating speed control strategy.

Specifically, for the practical applications, the bus control strategies combining with speed control strategy and holding strategy are illustrated in Figure 5, where the dotted and solid lines denote the scheduled timetable and actual stop visit times, respectively. From Figure 5, we can observe that, when one bus is delayed by some disturbances to its running time, the speed control strategy is applied to this bus to track the scheduled timetable. Meanwhile, for the following bus, we adopt the holding strategy at the station to keep the scheduled headway with the front bus. Therefore, under the proposed bus control u_j with both dwell time variance strategy and the operating speed control strategy, the schedule adherence and headway regularity of bus line can be effectively increased.

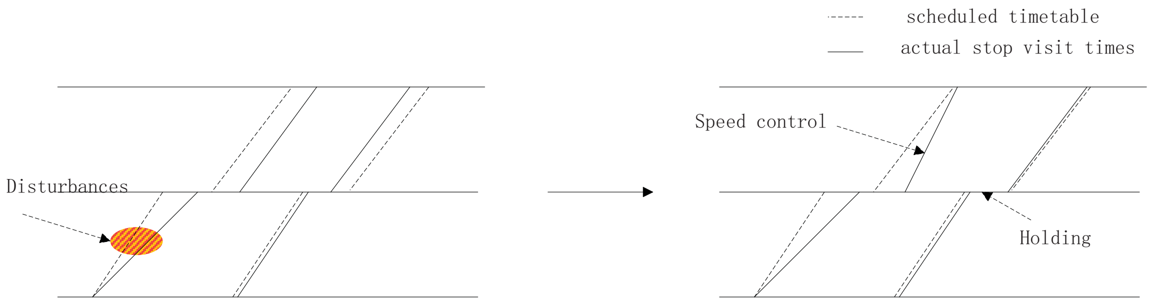


Figure 5. The bus control strategies combining with speed control and holding.

Then, the robust control problem for reducing bus bunching can be converted to a robust stability problem of the bus motion (6), i.e., the bounded disturbance w_j can not produce unbounded deviations of the practical bus arrival time from the nominal one. Due to the uncertainties and disturbances, the robust stability analysis becomes more complex in contrast to the stability analysis of a deterministic system. Then, we propose a definition for the robust stability of bus transit system (robust control problem) as follows.

Definition 3.1. *For the bus system (6) with a given robustness index $\gamma > 0$, get the feedback control parameter K such that the error states satisfy the following condition*

$$\sum_{j=1}^{\bar{M}}(e_j^T e_j)^{1/2} \leq \gamma \sum_{j=1}^{\bar{M}}(w_j^T w_j)^{1/2}, \tag{9}$$

for all the admissible uncertainties (4) and external disturbance under the zero initial condition $e_1 = 0$, where $\bar{M}$ represents the end decision stage, which refers to the considered time horizon. $\sum_{j=1}^{\bar{M}}(e_j^T e_j)^{1/2}$ is L_2 norm of the error vector $(e_1, e_2, \ldots, e_{\bar{M}})$ and $\sum_{j=1}^{\bar{M}}(w_j^T w_j)^{1/2}$ is L_2 norm of the disturbance vector $(w_1, w_2, \ldots, w_{\bar{M}})$. Then the bus system (6) is robustly stable, i.e., the bounded disturbance w_j can not produce unbounded errors.

According to the state transfer equation (6), for a given initial state e_1, the states e_j $(j > 1)$ are determined by the state-feedback control $u_j = Ke_j$. Thus, the left side of (9) implicitly contains the control parameter K. With Definition 3.1, the robust stability of the bus motion is ensured under the proposed bus control method. In particular, by Definition 3.1, we can also get that

$$\frac{\sum_{j=1}^{\bar{M}}(e_j^T e_j)^{1/2}}{\sum_{j=1}^{\bar{M}}(w_j^T w_j)^{1/2}} \leq \gamma. \tag{10}$$

It is shown from (10) that γ indexes an upper bound on the effect of bus scheduled deviations with respect to the disturbances. In the practical applications, a smaller value of γ implies a less accumulated bus error under the maximum delay disturbance, which reveals the robustness of the control results. We can choose a smaller γ to achieve a high robustness of the control system. In addition, the use of interval set for the system parameters and disturbance ignores the distribution, which may lead to biased measurement of robustness in some extend. However, in contrast with robust optimization formulations, the performance of formulations based on stochastic optimization techniques is very sensitive to the accuracy of assumptions regarding the probability distribution of uncertain data. In this way, the use of interval seems to be a good alternative. Compared to the existing results for the stability of bus system (Daganzo, 2009), definition 3.1 further defines the robust stability of bus system, which can explicitly express the effect of disturbances and uncertainties to the stability of bus system, so as to provide a general control method to reduce the bus bunching.

3.2. The robust optimal control problem

To further improve both schedule adherence and headway regularity of the bus system, we design a robust optimal bus control to optimize the performances of the bus system. At first, the following objective function is considered.

$$J = \sum_{i=1}^{N} \sum_{j=1}^{\bar{M}} \left\{ q^i {e_j^i}^T e_j^i + r^i (e_j^{i+1} - e_j^i)^T (e_j^{i+1} - e_j^i) + s^i {u_j^i}^T u_j^i \right\}, \tag{11}$$

where the index i from 1 to N denotes the considered buses and the index j from 1 to $\bar{M}$ represents the considered stations. This equation covers the operations horizon of all the buses from station 1 to $\bar{M}$, which includes the past and future events of buses. q^i, r^i and s^i are pre-determined positive weighted coefficients, whose values are set according to the practical requirements and we set $r^N = 0$. The first term in (11) denotes the accumulated deviations from schedule, equivalently the schedule punctuality. The second term represents the accumulated headway deviation of buses, which corresponds to the headway regularity. In particular, the second term penalizes the deviations of the time intervals between buses, which is therefore related to the average waiting time for the passengers and the congestion of buses (Van Breusegem et al., 1991). The minimization of the deviations of the time intervals between bus implies to reduce the average waiting time for the passengers (Mannino and Mascis, 2009; Fernandez et al., 2006; Li et al., 2017). The third term is used to penalize control actions for the practical limits to the holding strategy and the operating speed control strategy. Thus, the minimization of the objective function (11) implies the improvement of schedule adherence and headway regularity for the bus system. In addition, by considering that the station number j increases at the next new trip, the number of $\bar{M}$ can be larger than the actual number M of stations. Under this case, the objective equation (11) covers buses looping around, where all buses are operating through the terminal of their current trip.

Moreover, according to (6), the above objective function can be represented with the vector and matrix form as follows.

$$J = \sum_{j=1}^{\bar{M}} \left\{ e_j^T Q e_j + e_j^T W^T R W e_j + u_j^T S u_j \right\}, \tag{12}$$

where Q, R and S are three given weighted matrices, which corresponds to the weighted coefficients q^i, r^i and s^i in (11), and $W = \begin{bmatrix} -1 & 1 & 0 & 0 & \cdots \\ 0 & -1 & 1 & 0 & \cdots \\ & \cdots & \cdots & \cdots & \\ 0 & \cdots & -1 & 1 & 0 \\ 0 & 0 & \cdots & -1 & 1 \end{bmatrix}_{(N-1)\times N}$.

Consider the bus system (6) with the objective function (12). The robust optimal bus control problem is formulated to solve an optimal control problem with disturbances and uncertainties under the robust stability constraint (9) as follows.

$$\begin{aligned} &\min_{u_j} \max_{\Delta A} \sum_{j=1}^{\bar{M}} \{e_j^T Q e_j + e_j^T W^T R W e_j + u_j^T S u_j\} \\ &\text{s.t.} \quad e_{j+1} = (A + \Delta A) e_j + u_j + w_j, \\ &\qquad \sum_{j=1}^{\bar{M}} (e_j^T e_j)^{1/2} \leq \gamma \sum_{j=1}^{\bar{M}} (w_j^T w_j)^{1/2}. \end{aligned} \tag{13}$$

The formulated optimization problem (13) is in fact an optimal control problem that contains disturbances w_j and uncertain matrix ΔA. The first constraint denotes the state constraint, and the second constraint is to guarantee the robustly stability of the bus motion. According to the optimization problem (13), we can obtain the robust optimal bus control u_j such that the bus motion is robustly stable and meanwhile the bus system has an optimal performance. This optimization problem includes an infinite number of constraints due to the constraints with uncertain parameters that has to be satisfied for any feasible values. The different value in the uncertain set ΔA represents one constraint. By considering that the values for the uncertain set ΔA is continues and infinite, the state constraints $e_{j+1} = (A + \Delta A)e_j + u_j + w_j$ have an infinite number of constraints. The delay disturbances and demand uncertainties make the traditional dynamic programming method hard to solve this problem. The formulated optimization problem is not tractable in general, and needs to be relaxed to another minimization problem. In the formulated optimization problem, the disturbances w_j represents the variants of the running time that lead to bus delays, and the uncertain matrix ΔA denotes the variants of the passenger demand. The larger value of the disturbances w_j or the larger value for the demand rate in uncertain matrix ΔA will reduce the performance of the control strategies and result in the conservativeness for the solution under the proposed strategies. In the next section, we introduce practical methods to solve the above defined robust optimal control problems.

4. Robust dynamic control strategies

4.1. Robust stability condition

According to Lyapunov function analysis method, we first present the robust stability condition for the bus transit system with disturbances and uncertainties as the following theorem.

Theorem 4.1. *For the bus motion (6) with disturbances w_j and a given robustness level γ, if there are positive scalars ε_1, ε_2, positive definite matrix P, and any matrix parameter K such that the following matrix inequality holds*

$$\begin{bmatrix} -P & 0 & A^T + K^T & F_1^T & F_2^T & I \\ 0 & -\gamma^2 I & I & 0 & 0 & 0 \\ A + K & I & -P^{-1} + \varepsilon_1 I + \varepsilon_2 I & 0 & 0 & 0 \\ F_1 & 0 & 0 & -\varepsilon_1 I & 0 & 0 \\ F_2 & 0 & 0 & 0 & -\varepsilon_2 I & 0 \\ I & 0 & 0 & 0 & 0 & -I \end{bmatrix} < 0, \tag{14}$$

then a state feedback control $u_j = Kx_j$ is obtained such that the dynamic system (6) is robustly stable, that is, under the zero initial condition $e_1 = 0$, the error states satisfy that

$$\sum_{j=1}^{\bar{M}} (e_j^T e_j)^{1/2} \leq \gamma \sum_{j=1}^{\bar{M}} (w_j^T w_j)^{1/2}.$$

for all the admissible uncertainties (4) and disturbances.

Proof. See Appendix A. □

With Theorem 4.1, we can obtain the control $u_j = Ke_j$ based on the state feedback information as the robust control such that the uncertain linear discrete system (6) is robustly stability. The condition (14) in Theorem 4.1 is given in the form of matrix, where the matrixes A, F_1, F_2 are the bus system matrixes, which essentially describe the dynamic evolution characteristic of bus system, and the matrixes P, K are variables, which are used to determine the feedback control parameter K to achieve the robust stability of bus system. In addition, the positive scalars ε_1, ε_2 are used for the uncertainties for the passengers demand. By solving the matrix inequality (14), the state feedback control $u_j = Ke_j$ can be obtained as the robust dynamic bus control strategy to guarantee the robust stability of the bus motion (6) and thus reduce the bus bunching.

4.2. Robust optimal control design

We reduce the robust optimal control problem (13) to an optimization problem by the minimization of an upper bound of the objective function (13). The following proposition presents an upper bound of the objective function (13).

Proposition 4.1. *Consider the bus system (6) with the objective function (12). For a given initial condition e_1 and robustness level γ, if there exist positive scalars α, ε_1, ε_2, and positive definite matrix P, and any matrix K with appropriate dimension such that the following matrix inequalities holds,*

$$\begin{bmatrix} -P + Q + W^T RW + K^T SK^T & 0 & A^T + K^T & F_1^T & F_2^T & I \\ 0 & -\gamma^2 I & I & 0 & 0 & 0 \\ A + K & I & -P^{-1} + \varepsilon_1 I + \varepsilon_2 I & 0 & 0 & 0 \\ F_1 & 0 & 0 & -\varepsilon_1 I & 0 & 0 \\ F_2 & 0 & 0 & 0 & -\varepsilon_2 I & 0 \\ I & 0 & 0 & 0 & 0 & -I \end{bmatrix} < 0, \quad (15)$$

$$e_1^T P e_1 < \alpha, \quad (16)$$

then the bus system (6) is robustly stable, and meanwhile the positive scale α can be regarded as an equivalently upper bound on the objective function of (12) for all admissible uncertainties and disturbances.

Proof. First, the inequality (15) implies that the inequality (14) holds, so the inequality (15) ensures that the bus system (6) is robustly stable.

Moreover, according to the Schur complement, the inequality (15) is equivalent to

$$\begin{bmatrix} -P + \Theta^T P\Theta + I + Q + W^T RW + K^T SK^T & \Theta^T P \\ P\Theta & P - \gamma^2 I \end{bmatrix} < 0, \quad (17)$$

where Θ takes the same form in (27) of Appendix A.

Then, it is directly derived from (17) that

$$\begin{bmatrix} e_j \\ w_j \end{bmatrix}^T \begin{bmatrix} -P+\Theta^T P\Theta & \Theta^T P \\ P\Theta & P-\gamma^2 I \end{bmatrix} \begin{bmatrix} e_j \\ w_j \end{bmatrix}$$
$$< \begin{bmatrix} e_j \\ w_j \end{bmatrix}^T \begin{bmatrix} -Q-W^T RW-K^T SK^T & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} e_j \\ w_j \end{bmatrix}. \tag{18}$$

By summing both sides of (18) from $j=1$ to $j=\bar{M}$ and recalling that $V(e_j)=e_j^T Pe_j$, it follows from condition (16) that

$$\begin{aligned}
&\sum_{j=1}^{\bar{M}} \left\{ e_j^T Qe_j + e_j^T W^T RWe_j + u_j^T Su_j \right\} \\
&< V(e_1) - V(e_{M+1}) + \gamma^2 \sum_{j=1}^{\bar{M}} w_j^T w_j \\
&< V(e_1) + \gamma^2 \sum_{j=1}^{\bar{M}} w_j^T w_j \\
&< \alpha + \gamma^2 \sum_{j=1}^{\bar{M}} w_j^T w_j.
\end{aligned} \tag{19}$$

For the given robustness level γ and bounded disturbance w_j, the term $\gamma^2 \sum_{j=1}^{\bar{M}} w_j^T w_j$ is a constant. Thus, by minimizing the variable α, one can optimize an upper bound of the objective function $\sum_{j=1}^{\bar{M}} \left\{ e_j^T Qe_j + e_j^T W^T RWe_j + u_j^T Su_j \right\}$. Therefore, α can be regarded as an equivalently upper bound on the objective function of (13) for all admissible uncertainties and disturbances. □

Based on the result in Proposition 4.1, the optimal control problem (13) is reduced by solving an optimization problem, which is presented as follows.

$$\begin{aligned}
&\min_{(\alpha>0,\varepsilon_1>0,\varepsilon_2>0,P>0,K)} \alpha \\
&\text{s.t. } (15)-(16).
\end{aligned} \tag{20}$$

In (20), the objective function in (13) is converted to an upper bounded of α, and the constraints of (13) are presented with the form of matrix inequality. Instead of directly solving the original optimal control problem of (13) with infinite number of constraints, the transformation of (13) to (20) facilitates the design of efficient numerical solution algorithms. If there exists a set of solution $(\alpha, \varepsilon_1, \varepsilon_2, P, K)$, the robust optimal bus control is then obtained as $u_j = Ke_j$ with the optimal upper bound as α.

Note that the condition (15) is nonlinear on matrix variables P and K, which lead to a nonlinear optimal control problem of (20) that can not be solved easily. We show below that (20) can be further converted to a relaxed convex optimization problem. It can be solved efficiently in a short time.

Theorem 4.2. *For the nonlinear optimal control problem (20), it can be converted to the following relaxed convex optimization problem taking the form of linear matrix inequalities:*

$$\min_{(\alpha>0,\tilde{\varepsilon}_1>0,\ \tilde{\varepsilon}_2>0,\ X>0,\ Y)} \alpha \tag{21}$$

$$\text{s.t. (i)} \begin{bmatrix} -X & 0 & XA^T+Y^T & XF_1^T & XF_2^T & X & X & XW^T & Y^T \\ 0 & -\alpha\gamma^2 I & \alpha I & 0 & 0 & 0 & 0 & 0 & 0 \\ AX+Y & \alpha I & -X+\tilde{\varepsilon}_1 I+\tilde{\varepsilon}_2 I & 0 & 0 & 0 & 0 & 0 & 0 \\ F_1X & 0 & 0 & -\tilde{\varepsilon}_1 I & 0 & 0 & 0 & 0 & 0 \\ F_2X & 0 & 0 & 0 & -\tilde{\varepsilon}_2 I & 0 & 0 & 0 & 0 \\ X & 0 & 0 & 0 & 0 & -I & 0 & 0 & 0 \\ X & 0 & 0 & 0 & 0 & 0 & -\alpha Q^{-1} & 0 & 0 \\ WX & 0 & 0 & 0 & 0 & 0 & 0 & -\alpha R^{-1} & 0 \\ Y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\alpha S^{-1} \end{bmatrix} < 0, \quad (22)$$

$$\text{(ii)} \begin{bmatrix} 1 & e_1^T \\ e_1 & X \end{bmatrix} \geq 0. \quad (23)$$

If there are a set of solutions $(\alpha, \tilde{\varepsilon}_1, \tilde{\varepsilon}_2, X, Y)$, then the robust optimal control $u_k = YX^{-1}e_k$ is obtained as robust optimal bus control to minimize the upper bound on the objective (13).

Proof. Let $X = \alpha P^{-1}$, $Y = KX$, $\tilde{\varepsilon}_1 = \alpha\varepsilon_1$, $\tilde{\varepsilon}_2 = \alpha\varepsilon_2$. Pre- and post-multiplying both sides of (22) by diag$\{\alpha^{1/2}P^{-1}, \alpha^{1/2}I, \alpha^{1/2}I, \alpha^{1/2}I, \alpha^{1/2}I, \alpha^{1/2}I, \alpha^{1/2}I\}\alpha^{1/2}I, \alpha^{1/2}I\}$, it is easily obtained that inequality (22) is equivalent to (15). Moreover, we can get that the inequality (23) is equivalent to (16). Finally, (20) is equivalently converted to solving a convex optimization problem (21) taking the form of linear matrix inequalities. If we calculate a set of solution $(\alpha, \tilde{\varepsilon}_1, \tilde{\varepsilon}_2, X, Y)$, an optimal feedback control $u_j = YX^{-1}e_j$ can be obtained as robust optimal bus control, under which, the bus motion is robustly stability with the minimization of the upper bound on the objective function. □

With Theorem 4.2, the nonlinear optimal control problem (13) with disturbances and uncertainties can be converted to a convex optimization problem taking the form of linear matrix inequalities. Consider the fact that the linear matrix inequalities-based optimization problem is solved in a polynomial time. The formulated optimization problem (21) has a low computational complexity that is solved easily. The dimension of the matrix decision variable is the number of considered buses. The number of buses in operation during a certain time period is limited, so is the dimension of the matrix decision variable. Therefore, it is effective to implement the proposed control method to an actual bus transit system in real-time.

4.3. Robust structured control design

The robust bus control form (8) provides a general control method based on the arrival time of all the buses at each stage, namely, all the arrival time information of buses for each stage are required for the bus control strategy. However, in practical implement, for the control strategy of bus i at station $j+1$, the arrival time for the subsequent bus $i+2, i+3, \ldots, N$ at station j may not directly available (although they can be estimated). So it is reasonable to use only the available local information as the control strategy for bus i at station $j+1$. This motivates us to further design the robust structured control $u_j = Ke_j$, equivalently the structured control parameter K, so as to use the local information of e_j.

Without loss of generality, we take into account the case whereby, for each bus i at station $j+1$, the information from current bus i and its precede bus $i-1$ and backward bus $i+1$ at station j is available.

Then the corresponding structured control parameter K has the following form:

$$K = \begin{bmatrix} k_{11} & k_{12} & 0 & 0 & \cdots \\ k_{21} & k_{22} & k_{23} & 0 & \cdots \\ 0 & k_{32} & k_{33} & k_{34} & \cdots \\ & \cdots & \cdots & \cdots & \\ 0 & \cdots & 0 & k_{N-1\times N} & k_{N\times N} \end{bmatrix}_{N\times N}, \tag{24}$$

which shows that the special structured control parameter K determines the structured controller $u_j = Ke_j$.

For the optimization problem (21) in Theorem 4.2, to satisfy the structured control $u_j = Ke_j$ with the control parameter K of the form (24), we should consider the constraints for the matrix variables X and Y recalling that $K = YX^{-1}$. Based on the result in Theorem 4.2, we give the following corollary for the robust structured bus control design with local information.

Corollary 4.1. *For the optimization problem (21) with the constraints (22)-(23) and the additional constraints that the matrix variable X takes the form of $X = \text{dig}\{x_{11}, x_{22}, \ldots, x_{NN}\}$ and the matrix variable Y takes the form*

$$Y = \begin{bmatrix} y_{11} & y_{12} & 0 & 0 & \cdots \\ y_{21} & y_{22} & y_{23} & 0 & \cdots \\ 0 & y_{32} & y_{33} & y_{34} & \cdots \\ & \cdots & \cdots & \cdots & \\ 0 & \cdots & 0 & y_{N-1\times N} & y_{N\times N} \end{bmatrix}_{N\times N}, \tag{25}$$

if there are a set of solutions $(\alpha, \tilde{\varepsilon}_1, \tilde{\varepsilon}_2, X, Y)$, then the robust optimal structured control $u_k = YX^{-1}e_k$ is obtained as bus control strategy to minimize the upper bound on the objective function of (13).

Corollary 4.1 presents a sufficient condition to determine the optimal structured control $u_j = Ke_j$ with the control parameter K of the form (24). Although some additional constraints are added for the matrix variable X and Y, the formulated optimization problem in Corollary 4.1 is still a convex optimization problem with the form linear matrix inequalities that are computationally tractable. By solving the above linear matrix inequalities, a robust structured bus control can be obtained, i.e., the control strategy for bus i at station $j+1$ is based on the information from current bus i and its precede bus $i-1$ and backward bus $i+1$ at station j. As for a general case, we can design the different structured controls as dynamic bus control strategy according to the practical requirement by setting different form of the matrix variables X and Y.

5. Numerical Examples

To validate the effectiveness of the proposed bus motion and control strategies, we apply the bus motion model with delay disturbances and demand uncertainties, and the proposed robust control strategies to a test bus line abstracting from a real bus line 16 in beijing city of China. There are totally 22 stations along this bus line. The structure of this bus line is with a looped bus path, where buses loop around after arriving at the terminal with a new trip. Buses do not pass each other and stop at every stop. The number of the segment is $M = 22$. Without loss of generality, we assume that the scheduled headway for this bus line is $H = 5$min. There are 20 buses in operation during the peak period, and they are indexed as

$i = 1, 2 \ldots, 20$, i.e., the system dimension is $N = 20$. Let s_j is the target dwell time. To satisfy the equation $NH = \sum_{j=1}^{M} (c_j + d_j + s_j)$, we consider that the average running time c_j of the bus between two stations is between 4min to 4.3min and the slack time d_j is chosen as 10s to absorb variability. The target dwell time s_j varies from 0.2min to 0.6min with different demand rate β_j. To reflect the passenger demand variability in the actual operation, we choose the uncertain set $[0.01, 0.15]$ for the passenger demand rate β_j based on the real survey data for one station in Beijing bus line 16 as Figure 3 and also the typical value for the passenger demand rate in the existing literature (Daganzo, 2009). In the numerical examples, we use the deviations of actual bus arrival time from the nominal one and the deviations of actual bus headway from the nominal one to evaluate the control performance (schedule adherence and headway regularity), where smaller deviations of actual bus arrival time means better performance for schedule adherence and smaller deviations of actual bus headway implies better performance for headway regularity.

5.1. Performance of the control strategies

We compare the proposed robust control methods with the headway-based control method. We consider the variations in congestion delays to the bus running time from station 8 to 30. Let buses $i (i = 6, 7, \ldots, 12)$ approaching stations $j (j = 8, 9, \ldots, 30)$ experience traffic delay disturbances w_j, represented by an uncertain range $[-5\text{s}, 30\text{s}]$, typically experience in bus services in Beijing. The number $j > 22$ represents a new trip of buses after arriving at the terminal. For the dynamic changing passenger arrival flow, we choose the a pre-specified uncertainty set for the demand rate of passenger arrival flow at station j as $\tilde{\beta}_j = [0.02, 0.08]$. Based on the real survey data for the demand variability and run time variability, the robust bus control method is evaluated compared to the previous strategy based on the headway control. Then, we consider the following three control strategies:

1) Case 1: The headway-based control method. To reduce bus bunching caused by the disturbances, a heuristic headway-based holding control is adapted from (Sánchez-Martínez et al., 2016; Wu et al., 2017), where a bus is held if the headway to the preceding bus is less than the minimum allowable headway relative to the planned headway, where the minimum allowable headway is chosen as 4.8 min. For this method, the time margins are introduced at the terminal to recover bus delays, which will require more buses in standby at the terminus. Here we choose the time margin at the terminal as 5 min and the number of bused is increased to 21 in order to ensure the periodicity of the nominal schedule.

2) Case 2: Robust control method with full information. By solving the optimization problem (21) with Theorem 4.2, we can obtain the robust control $u_k = YX^{-1}e_k$ to adjust bus motions, which is based the arrival times of all buses at station j for the control of bus i at station $j+1$. Here, we choose a given robustness level $\gamma = 2.8$, and the same weighted parameters Q, R of the objective are set as $Q = \text{diag}\{0.01, 0.01, \ldots, 0.01\}$ and $R = \text{diag}\{0.01, 0.01, \ldots, 0.01\}$, under which there is the same requirement of the schedule adherence and headway regularity. The weighted parameter S is set as $S = \text{diag}\{1, 1, \ldots, 1\}$, which is relatively big to satisfy the practical constraints for the controller. Here matrixes Q, R, and S are weighted matrixes for the objective function, which represent performances of schedule adherence, headway regularity and control force, respectively. In practice, we can choose different values for matrices Q, R, and S to realise a tradeoff among the performances of schedule adherence, headway regularity and control force according to the practical requirement. Under the control method, there is no time margin at the terminal.

3) Case 3: Robust control method with local information. In this case, based on the results in Corollary 4.1, we can calculate a robust structured control that uses only current bus i and its precede bus $i - 1$ and

backward bus $i+1$ at station j for the control of bus i at station $j+1$. The robustness level γ and the weighted parameters Q, R, S are the same to those in case 2. There is no time margin at the terminal.

Following the bus motion model of (6) and applying each of the above three control strategies to this test line, we can calculate the deviations of actual bus arrival time from the scheduled one and the actual headway according to the dynamic evolution equation of (6). The results are shown in Figures 6-8. Figure 6 shows the results under the headway-based control method for buses 8-11. From Figure 6(a), we can find that, due to the frequent delay disturbances, the deviations of the actual arrival time for all buses are increased from one station to the next one with the uncertain demand rates, which reduces the operating speed of bus system. The time margin is allowed to recover the nominal schedule at the terminal (station 22), which however requires more numbers of buses. The headway is plotted in Figure 6(b), which shows that headway-based control method makes the actual headway can approach the scheduled headway 300s. It improves the headway regularity and therefore reduces the bus bunching. Although the headway-based control method can effectively improve the headway regularity, it reduces the schedule adherence.

Figures 7 and 8 show the results under the robust control as case 2 and case 3 respectively. Compare with Figure 6, we can see that the robust control methods can significantly reduce the deviations of actual bus arrival time and therefore improve the schedule adherence. With full information, the robust control (case 2 and Figure 7) is able to keep schedule deviations within 80s at all the stations, as compared to over 130s for schedule deviations in case 1. Figure 7(b) depicts the headway under the robust control with full information, which shows that the headway can also be effectively controlled around the schedule headway 300s and the proposed control method increases the headway regularity and reduces the bus bunching, although the deviations of the actual headway are a little larger than that in case 1 with headway control method. The results suggest that, the proposed method based on the real time information combining the holding strategy and speed control strategy can significantly improve the control efficiency to reduce bus bunching and meanwhile improve the schedule adherence. After the disturbances disappear from station 30, the deviations of actual bus arrival time and the actual headway are all stabilized at the nominal state, which shows the robust stability of bus transit system under the proposed control despite the uncertain parameters for the demand rate of passenger arrival flow. Additionally, even with only the local information as the structured robust control (case 3 and Figure 8), the maximum schedule deviation is kept within 80s and the actual headway is around the scheduled headway. Compared to Figure 7, the obtain results from local information control are similar to that from full information control.

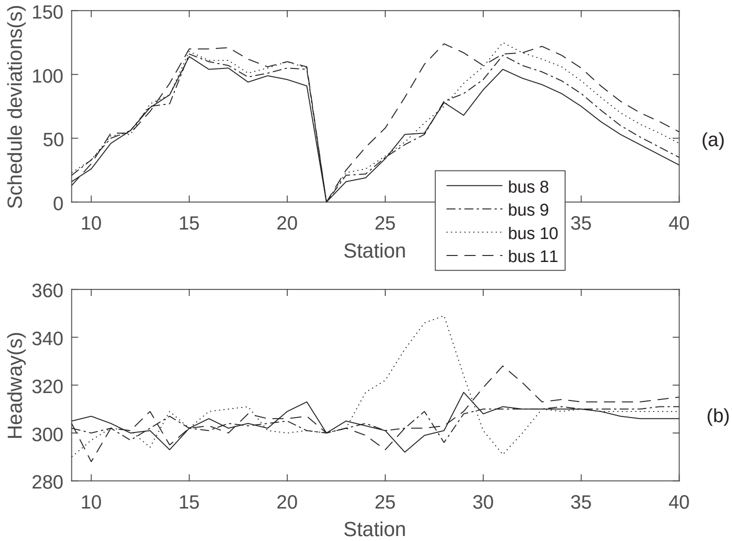


Figure 6. The schedule deviations (a) and headway (b) in case 1.

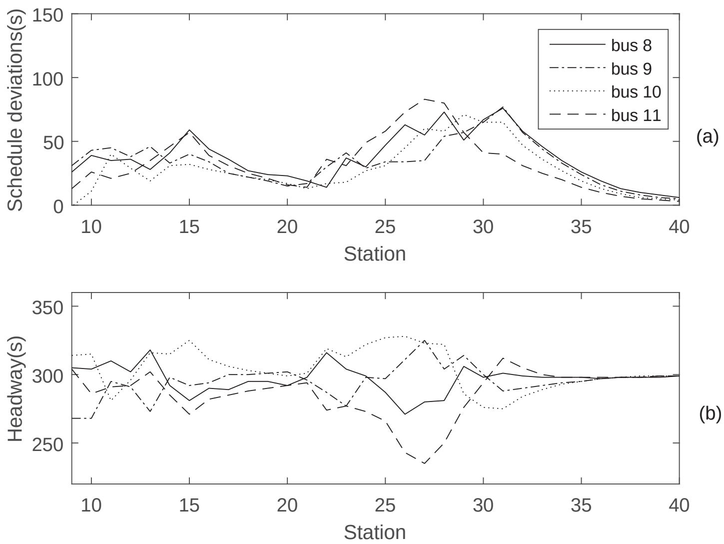


Figure 7. The schedule deviations (a) and headway (b) in case 2.

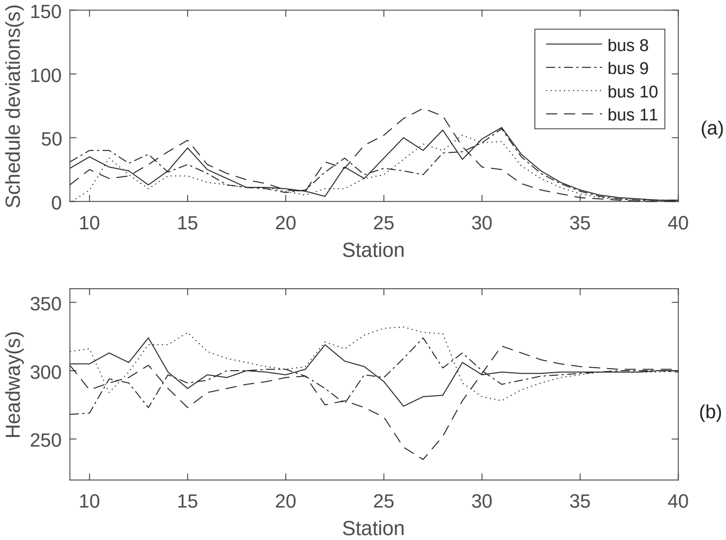


Figure 8. The schedule deviations (a) and headway (b) in case 3.

Let $\sqrt{e_j^T e_j}$ be the performance index for bus schedule deviation (SD) at station j, and $\sqrt{e_j^T W^T W e_j}$ be the performance index for the bus headway deviation (HD) at station j. The comparison results for the schedule adherence and headway regularity performances from stations 9-17 of case 1-3 are summarised in Table 3. From Table 3, we can clearly observe that the schedule deviations in cases 2-3 are evidently smaller than that in case 1, which means that the proposed robust control method can generate a better schedule adherence performance than the headway-based control method. In addition, from station 8 to 13, the headway deviation in case 1 of the headway-based control method is smaller than that in cases 2-3 of the proposed robust control method, which means that the headway-based control method can keep a better headway regularity than the proposed robust control methods in the earlier stations. However, with the increase number of station, the headway deviation in case 1 begins to be larger than that in cases 2-3, which indicates that the headway regularity performance for case 1 is reduced due to the bus delay prorogation effect in the subsequent stations. The headway-based control method can well keep the headway regularity in the previous stations, but it evidently reduces the schedule adherence performance and the headway regularity performance is also reduced with the increase number of stations.

Table 3: The comparisons of schedule deviation (SD) and headway deviation (HD).

	Station	9	10	11	12	13	14	15	16	17
SD	Case 1	85.8	101.3	222.1	226.5	311.9	333.9	467.5	459.1	447.4
	Case 2	50.0	78.2	95.0	99.6	101.6	104.5	133.7	104.0	83.0
	Case 3	50.0	72.6	81.9	80.9	80.2	77.5	103.1	68.5	48.7
HD	Case 1	19.0	22.7	37.7	59.4	64.0	75.9	86.6	87.5	89.6
	Case 2	45.5	47.8	57.3	65.2	77.2	58.1	73.4	50.7	39.0
	Case 3	45.5	46.7	55.4	63.1	78.2	55.0	66.6	42.5	29.6

In addition, the control forces for the proposed robust control methods in case 2 and case 3 can be calculated according to (8). The results are summarized in Table 4, where the positive value represents the holding strategy and the negative value denotes the speeding control strategy. The hybrid strategy combining both the holding strategy and the speeding control strategy improves the bus control efficiency. From Table 4, we can observe that the maximum holding time is 13s and the maximum time reduction is -10s. Such control forces can be readily implemented in practice.

Table 4: The control forces (s) from station 5 to station 12.

	Station	9	10	11	12	13	14	15	16
Case 2	Bus 8	3	0	3	3	7	0	-5	-1
	Bus 9	-1	-4	-3	-1	-5	2	0	1
	Bus 10	13	10	-1	3	8	3	4	4
	Bus 11	6	1	3	0	-4	-9	-2	0
Case 3	Bus 8	-1	-4	0	1	5	0	-7	0
	Bus 9	-3	-7	-6	-2	-5	1	-2	1
	Bus 10	11	7	-4	2	7	2	3	4
	Bus 11	5	0	3	2	-2	-6	-10	-2

Table 5: The sensitivity on the variant of run times.

Disturbances	[-3,5]	[-3,10]	[-3,15]	[-3,20]	[-3,25]	[-3,30]	[-3,35]	[-3,40]
SD	125.4	229.0	385.0	543.1	703.6	862.8	1021.1	1183.2
HD	104.6	167.0	248.2	331.3	414.4	498.8	586.4	672.1

Table 6: The sensitivity on the variant of passenger demand.

Demand rate	[0.02,0.04]	[0.04,0.06]	[0.06,0.08]	[0.08,0.10]	[0.10,0.12]	[0.12,0.14]
SD	864.4	868.8	876.0	882.5	891.6	900.7
HD	484.9	503.5	520.9	541.9	568.4	597.3

In the formulated optimization problem, the disturbance w_j represents the variant of the running time that leads to bus delays, and the uncertain matrix ΔA denotes the variant of the passenger demand. To investigate their impact on the performance of the proposed robust control method (case 2), we further conduct sensitivity test on the two variants. First, we consider the different ranges of disturbances to test the sensitivity on the variant of run times. Under different ranges, the sum of the schedule deviation $\sqrt{e_j^T e_j}$ and headway deviation $\sqrt{e_j^T W^T W e_j}$ from stations 1-20 are calculated and presented in Table 5. The results show that the control performance of the schedule adherence and headway regularity decreases with the increase range of the disturbances. In particular, the headway deviation increases from 104.6 to 672.1 while the schedule deviation increase from 125.4 to 1183.2, which indicates that the headway regularity is less sensitive than schedule adherence with respect to the variants of running time under the proposed robust control method. Second, we chose different ranges of demand rate $\tilde{\beta}_j$ to test the sensitivity on the variant of the passenger demand, and the results are summarised in Table 6. We can observe that the control performance of the schedule adherence and headway regularity also decreases with the increase values of the demand rate. The schedule deviation increases from 864.4 to 900.7 while the headway deviation increase from 484.9 to 597.3, which indicates that the headway regularity is more sensitive than schedule adherence with respect to the variants of passenger demand under the proposed control strategy.

5.2. Trade-off between schedule adherence and headway regularity

For objective (12), the first term means the schedule adherence while the second is the headway regularity. The improvement of schedule adherence means to make the buses closely adhere to a published schedule so as improving the operating speed, while the improvement of headway regularity is to reduce bus bunching phenomenon and thus reduce the awaiting time of passengers. In practice, for high frequency services, the objective tends to keep headway regularity, while for long-headway services, keeping the buses to scheduled times are more important. There is therefore a trade-off between schedule adherence and headway regularity, and we can design different robust dynamic control by choosing different matrixes Q and R to satisfy the practical requirements for the schedule adherence and headway regularity. In this test, we will consider the following three cases for different Q and R of the objective (12), in which $Q = \text{diag}\{q, q, \ldots, q\}$ and $R = \text{diag}\{r, r, , \ldots, r\}$. We choose a prespecified uncertainty set for the measure of the passenger demand rate as $\tilde{\beta}_j = [0.07, 0.13]$.

Suppose that at the initial stage, the schedule deviations of of all the buses at station 5 is given as

$$e_5 = [40, 1, 5, 2, 5, 90, 8, 60, 6, 50, 5, 40, 4, 0, 0, 0, 0, 0, 0, 0],$$

which means that at station 5, buses 1, 6, 8, 10 and 12 are affected by large delays 40s, 90s, 60s, 50s and 40s, respectively. For the large delays, we choose the robustness level as $\gamma = 7.07$. Under the large delays during the peak hours with shorter scheduled headway, it is desirable to improve headway regularity over schedule adherence. Based on this, we consider three cases with different weights of Q and R to improve the headway regularity of bus line system.

1) Case 1: $q = 0.01$, $r = 0.01$. In this case, the weight for the schedule adherence and headway regularity is equivalent.

2) Case 2: $q = 0.01$, $r = 0.1$. The weight for the headway regularity is larger than schedule adherence, which means to improve headway regularity rather than schedule adherence.

3) Case 3: $q = 0.01$, $r = 1$. The weight for the headway regularity is greatly larger than schedule adherence, which also means to improve headway regularity rather than schedule adherence.

Under the different cases for the weight Q and R, by solving the optimization problem (21) according to Theorem 4.2, we can obtain the corresponding robust control $u_k = YX^{-1}e_k$ to adjust bus motions. The smaller bus headway deviation indicates the better bus headway regularity. Under the proposed robust control strategy, the bus headway deviations $\sqrt{e_j^T W^T W e_j}$ for cases 1-3 from station 6 to 12 are plotted in Figure 9. We can observe that the bus headway deviations at case 2 is smaller than that at case 1 from station 6 to 12, and the bus headway deviations of case 3 is smaller than that at case 2 from station 6 to 12. It is shown that with the increase of the weight of headway regularity from case 1 to case 3, the bus headway deviations from 6 to 12 decrease from case 1 to case 3, which illustrates that increasing the weight of R improves the headway regularity of bus system. Therefore, in practice, a tradeoff between the headway regularity and schedule adherence of bus line can be achieved by the adjustment of the different weights Q and R.

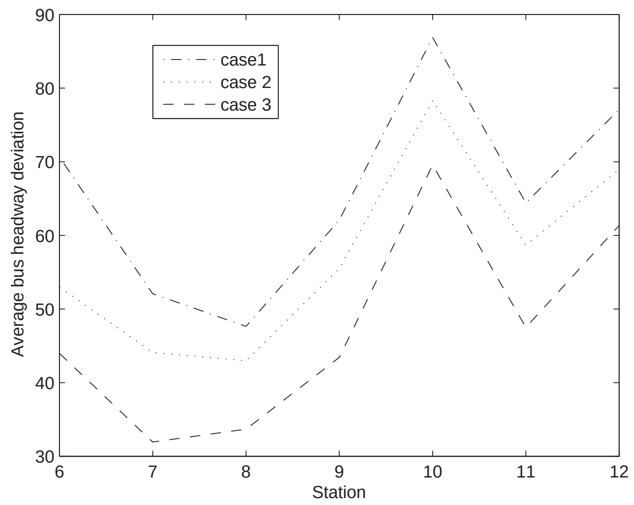


Figure 9. The bus headway deviations under different weights.

5.3. Robustness of the proposed control method

In this part, the robustness of the proposed control methods is further examined. First, for dynamic changing of the passenger demand with the time and day, we conduct simulation experiments with varying level of realized value $\tilde{\beta}_j$ to evaluate the robustness of the proposed control method to demand uncertainty parameters. The realized value of $\tilde{\beta}_j$ is supposed to be fixed at each station j. Through Monte Carlo simulations, a set of ten cases of the realized value $\tilde{\beta}_j$ for all the stations are generated in a range [0.05,0.15], which are shown in Appendix B. The traffic disturbances w_j are chosen as same that in 5.1. The initial state are given randomly from range [0,10]. Then, under the proposed bus control strategy as Theorem 4.2, the corresponding bus schedule deviation $\sqrt{e_j^T e_j}$ at stations 1-10 from case 1 to case 10 with the different parameters are calculated in Table 7, which indicates that, for the changing passenger demand from case 1 to case 10, the bus schedule deviations at each station are approximately equal where the standard deviation (Std) for each station only changes from 0 to 0.2111, which indicates the robustness of the proposed method to the uncertain parameters for improving the schedule adherence.

Let $\sqrt{e_j^T W^T W e_j}$ be the index for the bus headway deviation at station j. The corresponding bus headway deviations at stations 1-10 from case 1 to case 10 with the different parameters are calculated in Table 7, where the standard deviation for each station changes from 0 to 0.4762, which shows that the bus headway deviations at each station are also approximately equal. Thus, the propose control method is robust for the improvement of the headway regularity with respect to the uncertain parameters.

Table 7: The bus schedule deviations (SD) and headway deviations(HD) in 10 different cases.

	Station	1	2	3	4	5	6	7	8	9	10
	Case 1	24.6	21.3	19.7	25.3	23.9	24.0	19.8	22.9	19.6	23.0
	Case 2	24.6	21.7	19.9	25.3	24.1	24.0	19.8	22.9	19.5	22.7
	Case 3	24.6	21.6	19.7	25.3	24.1	24.4	19.8	22.9	19.5	23.0
	Case 4	24.6	21.3	19.7	25.0	24.0	24.0	20.1	23.0	19.8	23.0
SD	Case 5	24.6	21.6	19.7	25.0	24.1	24.0	20.2	22.7	19.6	22.7
	Case 6	24.6	21.7	19.9	25.3	23.8	24.2	20.1	23.0	19.5	23.0
	Case 7	24.6	21.3	19.9	25.3	24.4	24.4	20.2	23.1	19.5	23.0
	Case 8	24.6	21.7	19.9	25.0	24.4	24.0	20.2	23.1	19.6	23.0
	Case 9	24.6	21.3	19.7	25.3	23.8	24.0	20.2	22.9	19.2	23.0
	Case 10	24.6	21.7	19.9	25.3	24.1	24.0	20.2	22.9	19.5	23.0
	Mean	24.60	21.52	19.80	25.21	24.07	24.10	20.06	22.94	19.53	22.94
	Std	0	0.1932	0.1054	0.1449	0.2111	0.1700	0.1838	0.1174	0.1494	0.1265
	Case 1	19.8	9.6	12.9	13.2	10.2	12.3	11.5	14.2	9.4	14.1
	Case 2	19.8	9.7	12.8	13.2	10.2	12.3	11.5	14.2	10.0	13.8
	Case 3	19.8	10.0	12.9	13.2	10.2	12.4	11.5	14.2	10.4	14.1
	Case 4	19.8	9.6	12.9	12.9	10.7	11.7	10.7	14.4	10.0	14.1
HD	Case 5	19.8	10.0	12.9	12.6	10.2	11.7	11.5	14.4	9.7	13.8
	Case 6	19.8	10.0	12.9	12.6	10.2	11.7	11.5	14.4	9.7	13.8
	Case 7	19.8	9.6	12.8	13.2	10.7	12.4	11.5	14.3	10.4	14.1
	Case 8	19.8	9.7	12.8	12.6	10.7	12.3	11.5	14.3	9.4	14.1
	Case 9	19.8	9.6	12.9	13.2	10.3	12.3	11.5	14.2	9.2	14.1
	Case 10	19.8	9.7	12.8	13.2	10.2	12.3	11.5	14.2	10.4	14.1
	Mean	19.8	9.69	12.85	13.05	10.41	12.14	11.34	14.28	9.93	14.04
	Std	0	0.1853	0.0527	0.2550	0.2514	0.3062	0.3373	0.0919	0.4762	0.1265

In addition, considering that the delay disturbances w_j to the bus operating, we will further investigate the robustness of the proposed method to the disturbances. Suppose that the traffic disturbances w_j satisfy a random noise with a mean of 20s and the standard of 10s. By Monte Carlo simulations, we generate 10 cases for uncertain disturbances w_j for all the stations. Let $\sqrt{w^T w}$, $\sqrt{e^T e}$, and $\sqrt{y^T y}$ be the indexes for the total disturbances, scheduled deviations and headway deviations for all the buses of all the stations, respectively. Then, under the proposed bus control method as Theorem 4.2, the corresponding values of $\sqrt{w^T w}$, $\sqrt{e^T e}$, and $\sqrt{y^T y}$ of different ten cases with uncertain disturbances are calculated as Table 8, which shows that for the different disturbances of cases 1-10, the total disturbances, scheduled and headway deviations are different. However, the total scheduled and headway deviations are all smaller than the total disturbances, which indicate the effective of the proposed control method. The proposed method suppresses the propagation of the disturbance to the bus operation and reduce the bus bunching phenomenon. Typically, the proportion of the total scheduled deviations to the total disturbances is controlled to 0.97 and 0.98 for different cases, where the standard deviation is only 0.0053. In addition, the proportion of the total headway deviations to the total disturbances is controlled between 0.88 and 0.96, where the standard deviation is only 0.0245. The proportion of the total scheduled deviations and headway deviations to the total disturbances are approximately equal, respectively, which further indicates the robustness of the control method to the uncertain disturbances.

Table 8: The ten cases with different disturbances.

	$\sqrt{w^T w}$	$\sqrt{e^T e}$	$\sqrt{y^T y}$	$\sqrt{e^T e}/\sqrt{w^T w}$	$\sqrt{y^T y}/\sqrt{w^T w}$
Case 1	1431.0	1400.8	1386.3	0.97	0.96
Case 2	1360.8	1336.6	1258.3	0.98	0.92
Case 3	1448.6	1409.5	1348.6	0.97	0.93
Case 4	1402.4	1380.6	1275.8	0.98	0.90
Case 5	1489.3	1450.1	1341.2	0.97	0.90
Case 6	1445.3	1415.0	1282.6	0.97	0.88
Case 7	1471.5	1432.7	1311.3	0.97	0.89
Case 8	1443.9	1427.0	1348.6	0.98	0.93
Case 9	1416.5	1388.8	1309.4	0.98	0.92
Case 10	1393.4	1374.0	1320.0	0.98	0.94
Mean	1430.2	1401.5	1318.2	0.9750	0.9170
Std	40.4411	32.9962	39.1526	0.0053	0.0245

6. Conclusion

To effectively reduce bus bunching caused by the inevitable delay disturbance and the uncertainties of the passenger arrival activities, this paper provides a practical robust dynamic control framework which simultaneously considers delay disturbances and passenger demand uncertainty. Specially, we propose a new robust dynamic control combining both bus holding strategy and operating speed control strategy, which has been shown to significantly improve the bus control efficiency. By using a pre-specified uncertainty set to describe the uncertainties of passenger demand, we propose a state space model for the bus motion with disturbances and uncertainties. Based on a Lyapunov function analysis method, we designed a robust bus control strategy that is determined by matrix inequalities, under which, the robust stability of the bus motion is achieved to reduce the bus bunching. It extends the stability results of the bus system with the determined bus control.

Moreover, we formulated a nonlinear optimal control problem to determine the robust optimal bus control strategy with the improvement of both schedule adherence and headway regularity of bus system. The formulated optimal control problem is reduced to a relaxed convex optimization problem with the minimization of an upper bound on the objective, which can be solved in polynomial time and satisfy the real time requirement. In the framework of the proposed robust control method, the passenger demand and the delay disturbance to running time are described by uncertainty set, which avoids the need to estimate the exact probability distributions of uncertain data. Once the historical data of the passenger demand and the delay disturbance to running time for one bus line are available in practice, we can easily choose a proper uncertainty set to cover these historical data and the proposed robust bus control can be applied to the bus line operation with respect to uncertainties. The proposed robust bus control takes the form of linear state-feedback form, which is easy to implement in the real-time bus control system design.

Numerical examples show that the proposed robust control method not only significantly improves the control efficiency to reduce bus bunching, but also improves the schedule adherence of bus system. A tradeoff between the headway and schedule regularity of bus line can also be achieved by adjusting the different weights of the objective according to the practical requirement. The numerical examples show the robustness of the control method to the bus system with uncertain parameters and disturbances, which provides insights of critical importance to the bus service management. The bus capacity is assumed to be infinite in the proposed model, which does not consider high-demand situations where vehicles fill up. For the high-demand situations, the robust stability analysis of the bus motion with capacity constraints becomes more difficult, which needs to be investigated in our future work. The passenger numbers affected by the bus holding are not considered in the objective function of the formulated optimal control model. As a future research, it is interesting to further consider the passengers affected by holding for the robust optimal bus control problem. Moreover, to accurately describe the number of alighting and boarding passengers to the dwell time, we should further construct a more complex coupled bus and passenger dynamic model, which is also the future research direction. Additionally, with the availability of the full data for the real-time passenger flow information including the alighting number and on-board number, it can pave a new way to design the joint bus control and passenger boarding limit control strategy in a control perspective, which is another interesting topic needing to be investigated in the future research.

Acknowledgements

This work is supported by the National Science Foundation of China (Nos. 71771017, 71621001, 71825004), the Royal Academy of Engineering Newton Fund project UK–CIAPP\286, and the Research Foundation of State Key Laboratory of Rail Traffic Control and Safety, Beijing Jiaotong University (No. RCS2018ZZ001).

Appendix A. Proof of Theorem 4.1

For the uncertain linear discrete system (6), we choosing the following Lyapunov function

$$V(e_j) = e_j^T P e_j \tag{26}$$

with P is a positive-definite matrix.

The difference of the Lyapunov function along the system is calculated as

$$\begin{aligned} \Delta V(e_j) &= V(e_{j+1}) - V(e_j) \\ &= e_{k+1}^T P e_{k+1} - e_k^T P e_k \\ &= [(A+\Delta A)e_j + Ke_j + w_j]^T P[(A+\Delta A)e_j + Ke_j + w_j] - e_j^T P e_j \\ &= \begin{bmatrix} e_j \\ w_j \end{bmatrix}^T \begin{bmatrix} -P + \Theta^T P \Theta & \Theta^T P \\ P\Theta & P \end{bmatrix} \begin{bmatrix} e_j \\ w_j \end{bmatrix}, \end{aligned} \tag{27}$$

where $\Theta = A + IE_1F_1 + IE_2F_2 + K$.

Then, based on (27), one can further obtain that

$$\begin{aligned} & e_j^T e_j - \gamma^2 w_j^T w_j + \Delta V(e_j) \\ = & \begin{bmatrix} e_j \\ w_j \end{bmatrix}^T \begin{bmatrix} -P + \Theta^T P \Theta + I & \Theta^T P \\ P\Theta & P - \gamma^2 I \end{bmatrix} \begin{bmatrix} e_j \\ w_j \end{bmatrix}. \end{aligned} \tag{28}$$

Moreover, we have

$$\begin{bmatrix} -P + \Theta^T P \Theta + I & \Theta^T P \\ P\Theta & P - \gamma^2 I \end{bmatrix} = \begin{bmatrix} -P + I & 0 \\ 0 & -\gamma^2 I \end{bmatrix} + \begin{bmatrix} \Theta^T \\ I \end{bmatrix} P \begin{bmatrix} \Theta & I \end{bmatrix}. \tag{29}$$

According to the Schur complement (Boyd et al., 1994), it can be derived that the following matrix inequality

$$\begin{bmatrix} -P+\Theta^T P\Theta + I & \Theta^T P \\ P\Theta & P-\gamma^2 I \end{bmatrix} < 0 \tag{30}$$

is equivalent to

$$\begin{bmatrix} -P+I & 0 & \Theta^T \\ 0 & -\gamma^2 I & I \\ \Theta & I & -P^{-1} \end{bmatrix} < 0, \tag{31}$$

which can rewritten as

$$\begin{bmatrix} -P+I & 0 & A^T+K^T \\ 0 & -\gamma^2 I & I \\ A+K & I & -P^{-1} \end{bmatrix} + 2\begin{bmatrix} F_1^T \\ 0 \end{bmatrix} E_1 \begin{bmatrix} 0 & I \end{bmatrix} + 2\begin{bmatrix} F_2^T \\ 0 \end{bmatrix} E_2 \begin{bmatrix} 0 & I \end{bmatrix} < 0, \tag{32}$$

By the Cauchy inequality that $A^T B + B^T A \leq \varepsilon^{-1} A^T A + \varepsilon B^T B, \forall\ \varepsilon > 0$, and matrices A and B with appropriate dimensions, we can get from condition (7) that

$$2\begin{bmatrix} F_1^T \\ 0 \end{bmatrix} E_1 \begin{bmatrix} 0 & I \end{bmatrix} \leq \varepsilon_1 \begin{bmatrix} 0 \\ I \end{bmatrix} \begin{bmatrix} 0 & I \end{bmatrix} + \varepsilon_1^{-1} \begin{bmatrix} F_1^T \\ 0 \end{bmatrix} \begin{bmatrix} F_1 & 0 \end{bmatrix}, \tag{33}$$

$$2\begin{bmatrix} F_2^T \\ 0 \end{bmatrix} E_2 \begin{bmatrix} 0 & I \end{bmatrix} \leq \varepsilon_2 \begin{bmatrix} 0 \\ I \end{bmatrix} \begin{bmatrix} 0 & I \end{bmatrix} + \varepsilon_2^{-1} \begin{bmatrix} F_2^T \\ 0 \end{bmatrix} \begin{bmatrix} F_2 & 0 \end{bmatrix}. \tag{34}$$

Thus, the following inequality

$$\begin{aligned} &\begin{bmatrix} -P+I & 0 & A^T+K^T \\ 0 & -\gamma^2 I & I \\ A+K & I & -P^{-1} \end{bmatrix} + \varepsilon_1 \begin{bmatrix} 0 \\ I \end{bmatrix} \begin{bmatrix} 0 & I \end{bmatrix} + \varepsilon_1^{-1} \begin{bmatrix} F_1^T \\ 0 \end{bmatrix} \begin{bmatrix} F_1 & 0 \end{bmatrix} \\ &+\varepsilon_2 \begin{bmatrix} 0 \\ I \end{bmatrix} \begin{bmatrix} 0 & I \end{bmatrix} + \varepsilon_2^{-1} \begin{bmatrix} F_2^T \\ 0 \end{bmatrix} \begin{bmatrix} F_2 & 0 \end{bmatrix} < 0 \end{aligned} \tag{35}$$

implies the inequality (32) holds.

It can be derived from the Schur complement that, the inequality (35) is equivalent to the inequality (14) in Theorem 4.1. Thus the inequality (14) means that

$$e_j^T e_j - \gamma^2 w_j^T w_j + \Delta V(e_j) < 0. \tag{36}$$

By summing (36) from $j=1$ to $j=\bar{M}$, one has

$$\begin{aligned} \sum_{j=1}^{\bar{M}} e_j^T e_j &< V(e_1) - V(e_{\bar{M}+1}) + \gamma \sum_{j=1}^{\bar{M}} w_j^T w_j \\ &< V(e_1) + \gamma \sum_{j=1}^{\bar{M}} w_j^T w_j. \end{aligned} \tag{37}$$

Therefore, under the zero initial condition $e_1 = 0$, the error states satisfy

$$\sum_{j=1}^{\bar{M}} (e_j^T e_j)^{1/2} \leq \gamma \sum_{j=1}^{\bar{M}} (w_j^T w_j)^{1/2}. \tag{38}$$

for all the admissible uncertainties (4) and disturbance. The proof is complete.

Appendix B. The realized value $\tilde{\beta}_j$ from station 1 to station 22 for different cases

Case 1	0.0676	0.0769	0.1148	0.0959	0.1082	0.1045	0.0905	0.1196	0.0848	0.1236	0.1322
	0.1290	0.1014	0.1034	0.1351	0.1113	0.1239	0.1269	0.1052	0.0647	0.0624	0.1448
Case 2	0.1222	0.1266	0.1179	0.1162	0.1041	0.1147	0.0948	0.0594	0.0650	0.0895	0.0930
	0.1449	0.1384	0.0590	0.1061	0.1490	0.1086	0.1081	0.1130	0.0689	0.0990	0.0582
Case 3	0.0973	0.0689	0.1136	0.1270	0.1370	0.1044	0.0866	0.1025	0.1086	0.1183	0.1388
	0.0828	0.1088	0.0612	0.1430	0.1028	0.0747	0.1428	0.0532	0.0543	0.1353	0.0606
Case 4	0.0653	0.0787	0.1445	0.0850	0.0765	0.1221	0.1264	0.1030	0.0762	0.1204	0.0891
	0.1171	0.0655	0.0636	0.1197	0.0980	0.1166	0.1080	0.1115	0.1135	0.1374	0.0642
Case 5	0.0841	0.0591	0.0709	0.1162	0.0818	0.1022	0.1128	0.1361	0.0544	0.0942	0.1269
	0.0939	0.0700	0.1179	0.1083	0.1301	0.0583	0.0517	0.0862	0.0782	0.0770	0.0666
Case 6	0.1107	0.1076	0.1209	0.0916	0.0619	0.1494	0.1272	0.0985	0.1255	0.0520	0.0897
	0.1334	0.0907	0.0995	0.1315	0.0728	0.1126	0.0621	0.0550	0.1039	0.0708	0.1121
Case 7	0.0692	0.1183	0.0736	0.1342	0.1440	0.0719	0.1433	0.0893	0.0743	0.0831	0.1309
	0.1269	0.1249	0.0690	0.1379	0.0998	0.1161	0.1363	0.0990	0.1195	0.1065	0.1074
Case 8	0.1238	0.1047	0.0619	0.1333	0.1146	0.0606	0.1473	0.1171	0.0942	0.0924	0.1255
	0.0667	0.1326	0.0995	0.1489	0.1401	0.1230	0.0984	0.0693	0.0999	0.1140	0.0552
Case 9	0.0743	0.0926	0.1107	0.0756	0.0979	0.0610	0.0692	0.1241	0.1188	0.0770	0.0877
	0.1362	0.1290	0.0648	0.0501	0.1075	0.1391	0.1345	0.0623	0.1036	0.0917	0.1431
Case 10	0.1417	0.1144	0.0950	0.1113	0.1139	0.0564	0.0639	0.1020	0.0859	0.0697	0.0716
	0.1490	0.0819	0.0555	0.1365	0.1345	0.1482	0.0709	0.0705	0.0945	0.0706	0.1229

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