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Preservation of Choice Principles under Realizability

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Abstract

Especially nice models of intuitionistic set theories are realizability models $V(\mathcal{A})$, where $\mathcal{A}$ is an applicative structure or partial combinatory algebra. This paper is concerned with the preservation of various *choice principles* in $V(\mathcal{A})$ if assumed in the underlying universe V, adopting Constructive Zermelo-Fraenkel, **CZF**, as background theory for all of these investigations.

Examples of choice principles are the axiom schemes of countable choice, dependent choice, relativized dependent choice and the presentation axiom. It is shown that any of these axioms holds in $V(\mathcal{A})$ for every applicative structure $\mathcal{A}$ if it holds in the background universe.[1]

It is also shown that a weak form of the countable axiom of choice, $\mathbf{AC}^{\omega,\omega}$, is rendered true in any $V(\mathcal{A})$ regardless of whether it holds in the background universe. The paper extends work by McCarty [16] and Rathjen [19].

Keywords: Intuitionistic, Constructive Zermelo-Fraenkel set theory, axioms of choice, realizability, applicative structure
MSC2000: 03F50; 03F25; 03E55; 03B15; 03C70

1 Background

In 1945, Kleene developed realizability semantics for intuitionistic arithmetic and later for other theories. Kreisel and Troelstra [15] gave a definition of realizability for higher order Heyting arithmetic which was extended to type theories and systems of set theory by Myhill [17] and later by Friedman [13]. Realizability models for several non-extensional set theories were studied by Beeson [8, 9]. The extensional version of this realizability, already indicated by Beeson, was worked out by McCarty [16]. [16] is mainly concerned with realizability for intuitionistic Zermelo-Fraenkel set theory, **IZF**. As this approach employs transfinite iterations of the powerset operation through all the ordinals in defining the realizability (class) structure $V(\mathcal{A})$ for any applicative structure $\mathcal{A}$, it was not clear whether this semantics could be developed internally in Constructive Zermelo-Fraenkel set theory, **CZF**. Moreover, in addition to the powerset axiom, [16] also uses the unrestricted separation axioms. As **CZF** lacks the powerset axiom and has only bounded separation it was not clear whether **CZF** was sufficient as background theory. The development of this kind of realizability on the basis of **CZF** was carried out in [19].

[1]This is analogous to the well-known result from forcing, that if **AC** holds in V then **AC** also holds in any generic extension $V[G]$. However, in the the context of realizability this holds only for special forms of the axiom of choice as unfettered **AC** implies the law of excluded middle for atomic formulas (see [5, Section 10]) and thus is not realizable.

Preservation of several choice principles was shown for the special case of the first Kleene algebra, $\mathcal{K}_1$, assuming a classical background theory by McCarty [16], and on the basis of **CZF** in [19]. Especially it was shown that **CZF** augmented by the presentation axiom suffices to validate that the presentation axiom holds in $V(\mathcal{K}_1)$ whereas [16] uses the full axiom of choice.

A question not addressed in [16, 19] was whether these choice principles also propagate to $V(\mathcal{A})$ when $\mathcal{A}$ is an arbitrary applicative structure. The purpose of this paper is to revisit the proofs in [16, 19] dealing with $V(\mathcal{K}_1)$ in order to show that they can be amended to also work for $V(\mathcal{A})$.

In what follows we shall be drawing on [19] but will briefly explain terminology in the rest of this section.

1.1 Constructive Zermelo-Fraenkel Set Theory CZF

CZF is an intuitionistic set theory that is closely related to Martin-Löf type theory (see [1, 2, 3]) and provides an important framework for developing and formalizing constructive mathematics (see [4, 5]). **CZF** theory has the same first order language as **ZF**, where the only non-logical symbol is $\in$ though it is based on intuitionistic logic.
Its axioms are *Extensionality*, *Pairing*, *Union*, *Set Induction Scheme* and *Infinity* in their usual forms and the following axiom schemas:
Bounded Separation Scheme

$$\forall x \exists y \forall a[a \in y \leftrightarrow a \in x \wedge \phi(a)]$$

for any bounded formula ϕ, where ϕ is bounded if all quantifiers occurring in ϕ are bounded.
Subset Collection Scheme

$$\begin{aligned}&\forall x \forall y \exists z \forall u[\forall a \in x \exists b \in y \psi(a,b,u) \rightarrow \\ &\qquad \exists c \in z(\forall a \in x \exists b \in c \psi(a,b,u) \wedge \forall b \in c \exists a \in x \psi(a,b,u))]\end{aligned}$$

for any formula ψ.
Strong Collection Scheme

$$\forall x[\forall a \in x \exists b \phi(a,b) \rightarrow \exists y[\forall a \in x \exists b \in y \phi(a,b) \wedge \forall b \in y \exists a \in x \phi(a,b)]]$$

for any formula ϕ.

1.2 Axioms of Choice

In many texts on constructive mathematics, the axioms of countable choice and dependent choice are adopted as constructive principles. The weakest choice axiom we shall consider, denoted by $\mathbf{AC}^{\omega,\omega}$, asserts that there is a function $f:\omega\rightarrow\omega$ with $\forall i \in \omega\, \theta(i, f(i))$ whenever $\forall i \in \omega \exists j \in \omega\, \theta(i,j)$, where θ is an arbitrary set-theoretic formula.

The axiom scheme of *Countable Choice*, $\mathbf{AC}_\omega$, asserts that if $\forall i \in \omega \exists x\, \theta(i,x)$ holds for any formula θ, then there is a function f with domain ω such that $\forall i \in \omega\, \theta(i, f(i))$. Obviously, $\mathbf{AC}_\omega$ implies $\mathbf{AC}^{\omega,\omega}$.

The axiom scheme of *Dependent Choice*, **DC**, asserts that for any formula ϕ, whenever $(\forall a \in x)(\exists b \in x)\phi(a,b)$ and $x_0 \in x$, then there exists a function $f:\omega \rightarrow x$ such that $f(0) = x_0$ and $(\forall n \in \omega)\, \phi(f(n), f(n+1))$.

A very useful extension of **DC** is *Relativized Dependent Choice*, **RDC**, which states that for arbitrary formulas ϕ and ψ, if

$$\forall x[\phi(x) \rightarrow \exists y(\phi(y) \wedge \psi(x,y))]$$

and $\phi(a_0)$, then there exists a function f whose domain is ω with $f(0) = a_0$ and

$$(\forall n \in \omega)[\phi(f(n)) \wedge \psi(f(n), f(n+1))].$$

On the basis of **CZF**, $\mathbf{AC}_\omega$ follows from **DC** and **RDC** implies **DC** (see [4, Proposition 8.3] and [5, Section 10]).

Another very interesting choice principle is the *Presentation Axiom*, **PAx**, which has a categorical flavor when expressed in terms of projective sets. Let $\mathcal{C}$ be a category and let P be an object in $\mathcal{C}$. Then, P is called *projective* in $\mathcal{C}$ if for any objects A, B in $\mathcal{C}$ and morphisms $A \xrightarrow{f} B$, $P \xrightarrow{g} B$ with f an epimorphism, there is a morphism $P \xrightarrow{h} A$ such that the diagram below commutes.

$$\begin{tikzcd}
A \arrow[r, two heads, "f"] & B \\
P \arrow[u, dashed, "h"] \arrow[ur, "g"'] &
\end{tikzcd}$$

Now, taking $\mathcal{C}$ to be the category of sets, then it follows easily that a set P is projective if for all P-indexed family $(X_i)_{i \in P}$ of inhabited sets X_i there is a function f with domain P such that for all $i \in P$, $f(i) \in X_i$.
The *presentation axiom* **PAx** asserts that each set is the surjective image of a projective set. Projective sets are often called *bases*. **PAx** implies **DC** in **CZF** (see [4, Proposition 8.6]).

1.3 Applicative structures

In order to define a realizability interpretation we must be given a notion of realizing functions. A particularly general and elegant approach to realizability builds on structures which have been variably called *partial combinatory algebras*, *applicative structures*, or *Schönfinkel algebras*. These structures are best described as the models of a theory **APP**. The following presents the main features of **APP**; for full details cf. [11, 12, 9, 22]. The language of **APP** is a first-order language with a ternary relation symbol App, a unary relation symbol N (for a copy of the natural numbers) and equality, $=$, as primitives. The language has an infinite collection of variables, denoted $x, y, z, \ldots$, and nine distinguished constants: $\mathbf{0}, \mathbf{s}_N, \mathbf{p}_N, \mathbf{k}, \mathbf{s}, \mathbf{d}, \mathbf{p}, \mathbf{p_0}, \mathbf{p_1}$ for, respectively, zero, successor on N, predecessor on N, the two basic combinators, definition by cases, pairing and the corresponding two projections. There is no arity associated with the various constants. The *terms* of **APP** are just the variables and constants. We write $t_1 t_2 \simeq t_3$ for $\mathrm{App}(t_1, t_2, t_3)$.

Formulae are then generated from atomic formulae using the propositional connectives and the quantifiers.

In order to facilitate the formulation of the axioms, the language of **APP** is expanded definitionally with the symbol $\simeq$ and the auxiliary notion of an *application term* is introduced. The set of application terms is given by two clauses:

1. all terms of **APP** are application terms; and

2. if s and t are application terms, then (st) is an application term.

For s and t application terms, we have auxiliary, defined formulae of the form:

$$s \simeq t \quad := \quad \forall y(s \simeq y \leftrightarrow t \simeq y),$$

if t is not a variable. Here $s \simeq a$ (for a a free variable) is inductively defined by:

$$s \simeq a \quad \text{is} \quad \begin{cases} s = a, & \text{if } s \text{ is a term of } \mathbf{APP}, \\ \exists x, y[s_1 \simeq x \wedge s_2 \simeq y \wedge \mathrm{App}(x, y, a)] & \text{if } s \text{ is of the form } (s_1 s_2). \end{cases}$$

Some abbreviations are $t_1 t_2 \ldots t_n$ for $((...(t_1 t_2)...)t_n)$; $t \downarrow$ for $\exists y(t \simeq y)$ and $\phi(t)$ for $\exists y(t \simeq y \wedge \phi(y))$.

Some further conventions are useful. Systematic notation for *n-tuples* is introduced as follows: (t) is t, (s,t) is $\mathbf{p}st$, and $(t_1, \ldots, t_n)$ is defined by $((t_1, \ldots, t_{n-1}), t_n)$. In this paper, the **logic** of **APP** is assumed to be that of intuitionistic predicate logic with identity. **APP**'s **non-logical axioms** are the following:

Applicative Axioms

1. $\mathrm{App}(x, y, z_1) \wedge \mathrm{App}(x, y, z_2) \rightarrow z_1 = z_2$.
2. $(\mathbf{k}xy) \downarrow \wedge \mathbf{k}xy \simeq x$.
3. $(\mathbf{s}xy) \downarrow \wedge \mathbf{s}xyz \simeq xz(yz)$.
4. $(\mathbf{p}x_0x_1) \downarrow \wedge (\mathbf{p_0}x) \downarrow \wedge (\mathbf{p_1}x) \downarrow \wedge \mathbf{p_i}(\mathbf{p}x_0x_1) \simeq x_i$ for $i = 0, 1$.
5. $N(z_1) \wedge N(z_2) \wedge z_1 = z_2 \rightarrow \mathbf{d}xyz_1z_2 \downarrow \wedge \mathbf{d}xyz_1z_2 \simeq x$.
6. $N(z_1) \wedge N(z_2) \wedge z_1 \neq z_2 \rightarrow \mathbf{d}xyz_1z_2 \downarrow \wedge \mathbf{d}xyz_1z_2 \simeq y$.
7. $N(x) \rightarrow \big[\mathbf{s}_N x \downarrow \wedge \mathbf{s}_N x \neq \mathbf{0} \wedge N(\mathbf{s}_N x)\big]$.
8. $N(\mathbf{0})$ and $N(x) \wedge x \neq \mathbf{0} \rightarrow \big[\mathbf{p}_N x \downarrow \wedge \mathbf{s}_N(\mathbf{p}_N x) = x\big]$.
9. $N(x) \rightarrow \mathbf{p}_N(\mathbf{s}_N x) = x$.
10. $\varphi(\mathbf{0}) \wedge \forall x\big[N(x) \wedge \varphi(x) \rightarrow \varphi(\mathbf{s}_N x)\big] \rightarrow \forall x\big[N(x) \rightarrow \varphi(x)\big]$.

Let $\mathbf{1} := \mathbf{s}_N\,\mathbf{0}$. The applicative axioms entail that $\mathbf{1}$ is an application term that evaluates to an object falling under N but distinct from $\mathbf{0}$, i.e., $\mathbf{1}\downarrow$, $N(\mathbf{1})$ and $\mathbf{0} \neq \mathbf{1}$.

Remark 1.1. The theory **APP** comprises that basic part of Feferman's theory of explicit mathematics (see [11, 12]) which is concerned only with its recursion-theoretic aspects. The acronym **APP** is used in [22, 9.3.1] while in [9, VI.6.4] the same theory is called **EON**. The fragment of **APP** (or **EON**) without the last axiom scheme (10) is referred to by the acronym $\mathbf{PCA}^+$ in [9, VI.6.3]. In [16, Ch. 2, Sec. 2], APP refers to yet another theory which closely resembles a fragment of Beeson's $\mathbf{PCA}^+$, where the predicate N and the pertaining combinators $\mathbf{s}_N$ and $\mathbf{p}_N$ are omitted but an axiom $\forall x \forall y\ \mathbf{p}xy \neq \mathbf{0}$ is added. As it turns out, all of these differences are not very important. The common core of these theories is described by the fragment **PCA** (see [9, VI.2.1]) which has only the constants $\mathbf{k}$ and $\mathbf{s}$ and whose axioms are the first three of **APP** plus the axiom $\mathbf{s} \neq \mathbf{k}$ (that is deducible in $\mathbf{PCA}^+$ and the theory APP of [16]). A model of **PCA** is said to be a *partial combinatory algebra* (pca). Now, every pca can be expanded to a model of **APP** (as well as McCarty's APP) in a uniform way (see [9, VI.2.9]) (provably so in **CZF**). Thus one can say that a partial combinatory algebra is already as rich a structure as an applicative structure.

Employing the axioms for the combinators $\mathbf{k}$ and $\mathbf{s}$ one can deduce an abstraction lemma yielding λ-terms of one argument. This can be generalized using n–tuples and projections.

Lemma 1.2. (cf. [11]) (**Abstraction Lemma**) *For each application term t there is a new application term t^* such that the parameters of t^* are among the parameters of t minus $x_1, \ldots, x_n$ and such that*

$$\mathbf{APP} \vdash t^*\downarrow \ \wedge \ t^*(x_1, \ldots, x_n) \simeq t.$$

$\lambda(x_1, \ldots, x_n).t$ is written for t^*.

The most important consequence of the Abstraction Lemma is the Recursion Theorem. It can be derived in the same way as for the λ–calculus (cf. [11], [12], [9], VI.2.7). Actually, one can prove a uniform version of the following in **APP**.

Corollary 1.3. (Recursion Theorem)

$$\forall f \exists g \forall x_1 \ldots \forall x_n \, g(x_1, \ldots, x_n) \simeq f(g, x_1, \ldots, x_n).$$

The "standard" applicative structure is Kl in which the universe $|Kl|$ is ω and $\mathrm{App}^{Kl}(x, y, z)$ is Turing machine application:

$$\mathrm{App}^{Kl}(x, y, z) \quad \text{iff} \quad \{x\}(y) \simeq z.$$

The primitive constants of **APP** are interpreted over $|Kl|$ in the obvious way.

2 Realizability for intuitionistic set theories

In this section we define a realizability universe $V(\mathcal{A})$ for every applicative structure $\mathcal{A}$ and the pertaining notion of realizability. Our background theory will be **CZF**. So we will have to make sure that all definitions can formalized in this theory. For the remainder of this section we fix an applicative structure $\mathcal{A}$. $|\mathcal{A}|$ denotes the carrier set of $\mathcal{A}$ but sometimes we will overload notation and write just $\mathcal{A}$ for $|\mathcal{A}|$.

Definition 2.1. Ordinals are transitive sets with transitive elements. Lower case Greek letters will be used to range over ordinals. Define

$$\begin{aligned} V(\mathcal{A})_\alpha &= \bigcup_{\beta \in \alpha} \mathcal{P}(\mid \mathcal{A} \mid \times V(\mathcal{A})_\beta) \\ V(\mathcal{A}) &= \bigcup_{\alpha} V(\mathcal{A})_\alpha. \end{aligned}$$

On the face of it, it is not clear whether the definition of $V(\mathcal{A})$ can be formalized in **CZF**, since the power set axiom is not among its axioms. That this can be done is shown in [19, Lemma 3.4]. However, let us point out that the levels $V(\mathcal{A})_\alpha$ of this hierarchy are merely definable classes (uniformly in α, A) in **CZF**, and in general cannot be shown to be sets in this theory.

We also have the following result (provably in **CZF**).

Lemma 2.2. *(i) For any $\beta \in \alpha \quad V(\mathcal{A})_\beta \subseteq V(\mathcal{A})_\alpha$.*

(ii) For any set U, if $U \subseteq \mid \mathcal{A} \mid \times V(\mathcal{A})$ then $U \in V(\mathcal{A})$.

Proof. [19, Lemma 3.5]. □

Next we define for $r \in \mid \mathcal{A} \mid$ what it means for r to realize a set-theoretic sentence ϕ with parameters in $V(\mathcal{A})$, which is written as $r \Vdash \phi$.

For $r \in \mid \mathcal{A} \mid$, we also write $(r)_0$ instead of $\mathbf{p}_0 r$ and $(r)_1$ instead of $\mathbf{p}_1 r$. $\langle x, y \rangle$ denotes the usual set-theoretic ordered pair of x and y.

Definition 2.3. Bounded and unbounded quantifiers are syntactically considered as different types of quantifiers. If $r \in\mid \mathcal{A} \mid$ and $a, b \in V(\mathcal{A})$ then $r \Vdash \phi$ for a sentence ϕ with parameters in $\mathtt{V}(\mathcal{A})$ is defined inductively on the complexity of ϕ as follows:

(i) $r \Vdash a \in b \iff \exists c[\langle (r)_0, c\rangle \in b \wedge (r)_1 \Vdash a = c]$.

(ii) $r \Vdash a = b \iff \forall g, f[(\langle g, f\rangle \in a \rightarrow (r)_0 g \Vdash f \in b) \wedge (\langle g, f\rangle \in b \rightarrow (r)_1 g \Vdash f \in a)]$.

(iii) $r \Vdash \phi \wedge \psi \iff (r)_0 \Vdash \phi$ and $(r)_1 \Vdash \psi$.

(iv) $r \Vdash \phi \vee \psi \iff [(r)_0 = \mathbf{0} \wedge (r)_1 \Vdash \phi] \vee [(r)_0 = \mathbf{1} \wedge (r)_1 \Vdash \psi]$.

(v) $r \Vdash \neg\phi \iff \forall k \in\mid \mathcal{A} \mid \quad \neg k \Vdash \phi$.

(vi) $r \Vdash \phi \rightarrow \psi \iff \forall k \in\mid \mathcal{A} \mid [k \Vdash \phi \rightarrow rk \Vdash \psi]$.

(vii) $r \Vdash \forall x \in a\phi \iff \forall \langle k, h\rangle \in a \quad rk \Vdash \phi[x/h]$.

(viii) $r \Vdash \exists x \in a\phi \iff \exists h(\langle (r)_0, h\rangle \in a \wedge (r)_1 \Vdash \phi[x/h])$.

(ix) $r \Vdash \forall x\phi(x) \iff \forall x \in V(\mathcal{A}) \quad r \Vdash \phi(x)$.

(x) $r \Vdash \exists x\phi(x) \iff \exists x \in V(\mathcal{A}) \quad r \Vdash \phi(x)$.

Note that the statement $r \Vdash \phi$ contains a hidden reference to $\mathcal{A}$. If we want to make this dependence explicit we shall write $r \Vdash_{\mathcal{A}} \phi$.

Notice that *(i)* and *(ii)* are definitions by transfinite recursion. In particular, the (class) functions

$$\begin{aligned} F_{\in}(x, y) &= \{r \in |\mathcal{A}| \,:\, r \Vdash x \in y\} \\ G_{=}(x, y) &= \{r \in |\mathcal{A}| \,:\, r \Vdash x = y\} \end{aligned}$$

can be (simultaneously) defined on $\mathtt{V} \times \mathtt{V}$ by recursion on the relation

$$\langle c, d\rangle \lhd \langle a, b\rangle \quad \Leftrightarrow \quad \big(c = a \,\wedge\, d \in \mathbf{TC}(b)\big) \,\vee\, \big(d = b \,\wedge\, c \in \mathbf{TC}(a)\big)$$

where $\mathbf{TC}(x)$ is the transitive closure of a set x. Definitions by transfinite recursion on $\lhd$ are legitimate in **CZF** (see [21] Lemma 7.1).

Definition 2.4. More often than not, realizers do not depend on parameters nor on the particular applicative structure. This uniformity can be nicely expressed with the aid of application terms. If t is a closed application term we define $t \Vdash \phi$ to mean that there exists a $b \in |\mathcal{A}|$ such that $b \Vdash \phi$ and $\mathcal{A} \models \exists x[t \simeq x \wedge x = b]$. Note that such b is necessarily unique.

Lemma 2.5. *There are closed application terms $\mathbf{i}_r, \mathbf{i}_s, \mathbf{i}_t, \mathbf{i}_0, \mathbf{i}_1$ that do not depend on $\mathcal{A}$ (i.e. they are the same for all $\mathcal{A}$) such that for all $x, y, z \in \mathtt{V}(\mathcal{A})$ the following hold.*

(i) $\mathbf{i}_r \Vdash x = x$.

(ii) $\mathbf{i}_s \Vdash x = y \rightarrow y = x$.

(iii) $\mathbf{i}_t \Vdash (x = y \wedge y = z) \rightarrow x = z$.

(iv) $\mathbf{i}_0 \Vdash (x = y \wedge y \in z) \rightarrow x \in z$.

(v) $\mathbf{i}_1 \Vdash (x = y \wedge z \in x) \rightarrow z \in y$.

Furthermore, for every **CZF**-*formula* $\phi(u, v_1, ..., v_n)$ *with* $FV(\phi) \subseteq u, v_1, ..., v_n$, *there is a closed application term* $\mathbf{i}_\phi$ *not depending on* $\mathcal{A}$ *such that:*

$$\forall x, y, z_1, ..., z_n[\mathbf{i}_\phi \Vdash \phi(x, \vec{z}) \wedge x = y \rightarrow \phi(y, \vec{z})] \textit{ where } \vec{z} = z_1, ..., z_n\,.$$

Proof. [19, Lemma 4.2] or better [16] chapter 2, sections 5 and 6. □

Theorem 2.6. *Let* $\mathcal{P}$ *be a proof of a* **CZF**-*formula* $\phi(u_1, ..., u_n)$ *(with* $FV(\phi)$ *among* $u_1, ..., u_n$*) in intuitionistic predicate logic with equality. Then, there is a closed application term* $r_\mathcal{P}$ *independent of* $\mathcal{A}$ *such that:*

$$r_\mathcal{P} \Vdash \forall u_1 ... \forall u_n \phi(u_1, ..., u_n).$$

Proof. First, we find realizers for the following logical principles that relate bounded and unbounded quantification:

$$\forall u \in a\phi(u) \leftrightarrow \forall u[u \in a \rightarrow \phi(u)]$$

$$\exists u \in a\phi(u) \leftrightarrow \exists u[u \in a \wedge \phi(u)].$$

We have: $r \Vdash \forall u[u \in a \rightarrow \phi(u)]$

$$\begin{aligned}
&\Leftrightarrow \quad \forall x \in V(\mathcal{A})\, r \Vdash x \in a \rightarrow \phi(x)\\
&\Leftrightarrow \quad \forall x \in V(\mathcal{A})\, \forall e \in \mid \mathcal{A} \mid [e \Vdash x \in a \rightarrow re \Vdash \phi(x)]\\
&\Leftrightarrow \quad \forall x \in V(\mathcal{A})\, \forall e \in \mid \mathcal{A} \mid [\exists c(\langle (e)_0, c\rangle \in a \wedge (e)_1 \Vdash x = c) \rightarrow re \Vdash \phi(x)]\\
&\Rightarrow \quad \forall c \forall e \in \mid \mathcal{A} \mid [(\langle (e)_0, c\rangle \in a \wedge (e)_1 \Vdash c = c) \rightarrow re \Vdash \phi(c)]\\
&\Rightarrow \quad \forall \langle f, c\rangle \in a \quad r(\mathbf{p}f\mathbf{i}_r) \Vdash \phi(c)\\
&\Rightarrow \quad \lambda f.r(\mathbf{p}f\mathbf{i}_r) \Vdash \forall u \in a\phi(u).
\end{aligned}$$

Conversely, if $r \Vdash \forall u \in a\phi(u)$, then, equivalently $\forall \langle k, h\rangle \in a \quad rk \Vdash \phi(h)$, and this implies that $\forall x \in \mathtt{V}(\mathcal{A}) \forall f \in |\mathcal{A}|[\exists c(\langle (f)_0, c\rangle \in a \wedge (f)_1 \Vdash x = c) \rightarrow \mathbf{i}_\phi(\mathbf{p}(r(f)_0)(f)_1) \Vdash \phi(x)]$.
Now, let $\mathbf{R} := \mathbf{p}(\lambda r.\lambda f.r(\mathbf{p}f\mathbf{i}_r))(\lambda r.\lambda f.\mathbf{i}_\phi(\mathbf{p}(r(f)_0)(f)_1)))$.
Then $\mathbf{R} \Vdash \forall \vec{q} \forall u(\forall v \in u\phi(v) \leftrightarrow \forall v[v \in u \rightarrow \phi(v)])$, where $\forall \vec{q}$ quantifies over the remaining $FV(\phi)$. Similarly, one can find $\mathbf{R}'$ such that:
$\mathbf{R}' \Vdash \forall \vec{q} \exists u(\exists v \in u\phi(v) \leftrightarrow \exists v[v \in u \wedge \phi(v)])$.
We skip the remaining laws of intuitionistic predicate logic. □

Theorem 2.7 (**The Soundness Theorem for CZF**). *For each axiom* ϕ *of* **CZF**, *there is a closed application term* t *such that* **CZF** *proves that*

$$t \Vdash_\mathcal{A} \phi$$

holds for every applicative structure $\mathcal{A}$.

Proof. This is shown in [19, Theorem 5.1]. □

Definition 2.8. With each natural number n we associate an application term $\underline{n}$ by letting $\underline{0} := \mathbf{0}$ and $\underline{n+1} = \mathbf{s}_N\underline{n}$.

Proposition 2.9. *For any* $n, m \in \omega$ *we have,* $\mathbf{PCA}^+ \vdash \underline{n} \downarrow\ \wedge N(\underline{n})$ *and*

$$n = m \Longleftrightarrow \mathbf{PCA}^+ \vdash \underline{n} = \underline{m}.$$

Proof. $\mathbf{P}CA^+ \vdash n \downarrow\ \wedge N(\underline{n})$ is obvious by the axioms. To show the second part let $n \neq m$. Then either one of them is 0 and the other is a successor or both are successors.

(i) Suppose that $n = 0$ and $m = k + 1$ for some $k \in \omega$. Then, $\underline{n} = \mathbf{0}$ and $\underline{m} = \mathbf{s}_N \underline{k}$ and thus (by Axiom (7) of $\mathbf{PCA}^+$) $\mathbf{PCA}^+ \vdash \mathbf{s}_N \underline{k} \neq \mathbf{0}$. Hence $\mathbf{PCA}^+ \vdash \mathbf{s}_N \underline{k} \neq \mathbf{0}$ which implies that $\mathbf{PCA}^+ \vdash \underline{n} \neq \underline{m}$.

(ii) Suppose that $n = k + 1$ and $m = l + 1$ for some $k, l \in \omega$. By axioms of $\mathbf{PCA}^+$ we have $\mathbf{p}_N \underline{n} \downarrow$ and $\mathbf{p}_N \underline{m} \downarrow$, $\mathbf{p}_N(\mathbf{s}_N \underline{k}) = \underline{k}$ and $\mathbf{p}_N(\mathbf{s}_N \underline{l}) = \underline{l}$ provably in $\mathbf{PCA}^+$, and, since $n \neq m$ entails $k \neq l$ we can inductively assume that $\mathbf{PCA}^+ \vdash \underline{k} \neq \underline{l}$. Therefore, $\mathbf{PCA}^+ \vdash \underline{n} = \mathbf{s}_N \underline{k} \neq \mathbf{s}_N \underline{l} = \underline{m}$.

This yields what we want. $\square$

Definition 2.10 (**Representing ω in $\mathtt{V}(\mathcal{A})$**). ω is represented in $\mathtt{V}(\mathcal{A})$ by $\overline{\omega}$ given by an injection of ω into $\mathtt{V}(\mathcal{A})$ defined as follows:

$$\begin{aligned} \overline{n} &:= \{\langle \underline{m}, \overline{m} \rangle \mid m \in n\} \\ \overline{\omega} &:= \{\langle \underline{n}, \overline{n} \rangle \mid n \in \omega\}. \end{aligned}$$

Strictly speaking, by $\underline{n}$ above we mean the interpretation of the application term $\underline{n}$ in $\mathcal{A}$. Note also that therefore the sets $\overline{n}$ and $\overline{\omega}$ depend on $\mathcal{A}$.

The rationale for this representation of ω in $\mathtt{V}(\mathcal{A})$ is of course that it provides the right witness for the existential quantifier in the axiom of infinity to render it realizable.

To verify that $\overline{\omega} \in \mathtt{V}(\mathcal{A})$ note that $\overline{\omega} \subseteq |\mathcal{A}| \times \mathtt{V}(\mathcal{A})$, which implies that $\overline{\omega} \in \mathtt{V}(\mathcal{A})$ by (2.2).

We shall sometimes write $V(\mathcal{A}) \models \phi$ to convey that there exists $a \in |\mathcal{A}|$ such that $a \Vdash_{\mathcal{A}} \phi$. The actual realizer a can always be retrieved from the proofs.

Proposition 2.11. *Membership and equality on $\overline{\omega}$ are realizably absolute. In other words, for all $n, m \in \omega$ we have:*

(i) $n = m \iff V(\mathcal{A}) \models \overline{n} = \overline{m}$.

(ii) $n \in m \iff V(\mathcal{A}) \models \overline{n} \in \overline{m}$.

Proof. We prove (i) and (ii) simultaneously by induction on $n+m$ (also exhibiting the realizers).

(i) If $n = m$ then $\mathbf{i}_r \Vdash \overline{n} = \overline{m}$. Now, suppose that $r \Vdash \overline{n} = \overline{m}$. Then we have:

$$(1) \qquad \begin{aligned} \forall f, d[(\langle f, d \rangle \in \overline{n} \to (r)_0 f \Vdash d \in \overline{m}) \\ \wedge (\langle f, d \rangle \in \overline{m} \to (r)_1 f \Vdash d \in \overline{n})]. \end{aligned}$$

Since $\langle \underline{k}, \overline{k} \rangle \in \overline{n}$ holds for all $k \in n$, applying the induction hypothesis to (1) we get $\forall k \in n\ k \in m$. By symmetry we also deduce $\forall i \in m\ i \in n$. Hence $n = m$.

(ii) If $n \in m$, then $\langle \underline{n}, \overline{n} \rangle \in \overline{m}$, and hence $\mathbf{p}\underline{n}\mathbf{i}_r \Vdash \overline{n} \in \overline{m}$.

Now, suppose that $e \Vdash \overline{n} \in \overline{m}$. Then there exists c such that $\langle e_0, c \rangle \in \overline{m} \wedge e_1 \Vdash \overline{n} = c$. This implies that $c = \overline{k}$ for some $k \in m$. So the induction hypothesis from part (i) yields $n = k$, and therefore $n \in m$.

$\square$

2.1 Absoluteness Properties

An important notion that we need to represent in $V(\mathcal{A})$ is that of an ordered pair of its elements. We need to internalize this basic part of set theory in $V(\mathcal{A})$. This will be done in such a way that the crucial properties of the ordered pair become realizable (for further explanations see [16, p. 104-105] or [19, Section 8]).

Definition 2.12. *For $a, b \in V(\mathcal{A})$, define $\{a, b\}_{\mathcal{A}} := \{\langle \mathbf{0}, a\rangle, \langle \mathbf{1}, b\rangle\}$ and let*

$$\langle a, b\rangle_{\mathcal{A}} := \{\langle \mathbf{0}, \{a, a\}_{\mathcal{A}}\rangle, \langle \mathbf{1}, \{a, b\}_{\mathcal{A}}\rangle\}.$$

Lemma 2.13 (Internal Pairing in $V(\mathcal{A})$). *If $a, b, x \in V(\mathcal{A})$ then:*

(i) $\mathtt{V}(\mathcal{A}) \models x \in \{a, b\}_{\mathcal{A}} \leftrightarrow x = a \vee x = b.$

(ii) $\mathtt{V}(\mathcal{A}) \models x \in \langle a, b\rangle_{\mathcal{A}} \leftrightarrow x = \{a, a\}_{\mathcal{A}} \vee x = \{a, b\}_{\mathcal{A}}.$

Proof. (i): $e \Vdash x \in \{a, b\}_{\mathcal{A}}$. Then there exists c such that $\langle (e)_0, c\rangle \in \{a, b\}_{\mathcal{A}}$ and $(e)_1 \Vdash x = c$. But, $\langle (e)_0, c\rangle \in \{a, b\}_{\mathcal{A}}$ implies that $\langle (e)_0, c\rangle = \langle \mathbf{0}, a\rangle$ or $\langle (e)_0, c\rangle = \langle \mathbf{1}, b\rangle$, and hence we obtain:

$$[(e)_0 = \mathbf{0} \wedge (e)_1 \Vdash x = a] \vee [(e)_0 = \mathbf{1} \wedge (e)_1 \Vdash x = b].$$

Thus, $e \Vdash x = a \vee x = b$.

Conversely, suppose that $e \Vdash x = a \vee x = b$. Then retracing the steps of the foregoing proof backwards shows that $e \Vdash x \in \{a, b\}_{\mathcal{A}}$. And therefore $\mathbf{p}(\lambda x.x)(\lambda x.x)$ provides a realizer for (i).

(ii): First assume that $e \Vdash x \in \langle a, b\rangle_{\mathcal{A}}$. Then there exists c such that $\langle (e)_0, c\rangle \in \langle a, b\rangle_{\mathcal{A}}$ and $(e)_1 \Vdash x = c]$. But $\langle (e)_0, c\rangle \in \langle a, b\rangle_{\mathcal{A}}$ implies that either $\langle (e)_0, c\rangle = \langle \mathbf{0}, \{a, a\}_{\mathcal{A}}\rangle$ or $\langle (e)_0, c\rangle = \langle \mathbf{1}, \{a, b\}_{\mathcal{A}}\rangle$, and hence:

$$[(e)_0 = \mathbf{0} \wedge (e)_1 \Vdash x = \{a, a\}_{\mathcal{A}}] \vee [(e)_0 = \mathbf{1} \wedge (e)_1 \Vdash x = \{a, b\}_{\mathcal{A}}].$$

Thus, $e \Vdash x = \{a, a\}_{\mathcal{A}} \vee x = \{a, b\}_{\mathcal{A}}$.

Conversely, if $e \Vdash x = \{a, a\}_{\mathcal{A}} \vee x = \{a, b\}_{\mathcal{A}}$, then we have

$$[(e)_0 = \mathbf{0} \wedge (e)_1 \Vdash x = \{a, a\}_{\mathcal{A}}] \vee [(e)_0 = \mathbf{1} \wedge (e)_1 \Vdash x = \{a, b\}_{\mathcal{A}}].$$

Thus, $\langle (e)_0, c\rangle \in \langle a, b\rangle_{\mathcal{A}} \wedge (e)_1 \Vdash x = c$ for some $c \in V(\mathcal{A})$. So, $e \Vdash x \in \langle a, b\rangle_{\mathcal{A}}$.
Therefore $\mathbf{p}(\lambda x.x)(\lambda x.x)$ is also a realizer for (ii). □

2.2 Axioms of choice and $V(\mathcal{A})$

It follows from [16] and [19] that arguing in **CZF**, the principles **DC**, **RDC**, and **PAx** hold in $V(\mathcal{K}_1)$ assuming their validity in the background universe V. Moreover, $\mathbf{AC}^{\omega,\omega}$ holds in $V(\mathcal{K}_1)$ regardless of whether it holds in V. Here we show that $\mathcal{K}_1$ can actually be replaced by any applicative structure $\mathcal{A}$.

Theorem 2.14. *Let $\mathcal{A}$ be any applicative structure. Then:*

(i) (**CZF**) $V(\mathcal{A}) \models \mathbf{AC}^{\omega,\omega}$.

(ii) ($\mathbf{CZF} + \mathbf{AC}_\omega$) $V(\mathcal{A}) \models \mathbf{AC}_\omega$.

(iii) ($\mathbf{CZF} + \mathbf{DC}$) $V(\mathcal{A}) \models \mathbf{DC}$.

(iv) ($\mathbf{CZF} + \mathbf{RDC}$) $V(\mathcal{A}) \models \mathbf{RDC}$.

(v) $(\mathbf{CZF}+\mathbf{PAx})$ $\quad V(\mathcal{A}) \models \mathbf{PAx}$.

Proof. (i) Suppose that

$$e \Vdash \forall i \in \bar{\omega} \exists j \in \bar{\omega}\, \theta(i,j).$$

Then, $\forall \langle a, x\rangle \in \bar{\omega}\ ea \Vdash \exists j \in \bar{\omega}\, \theta(x,j)$, and hence

$$\forall \langle a, x\rangle \in \bar{\omega}\, \exists y\, [\langle (ea)_0, y\rangle \in \bar{\omega} \,\wedge\, (ea)_1 \Vdash \theta(x,y)].$$

Now, because for $\langle r, s\rangle \in \bar{\omega}$, s is uniquely determined by r, the above entails that there exists a function $f : \omega \to \omega$ such that for all $n \in \omega$,

$$\langle (e\underline{n})_0, \overline{f(n)}\,\rangle \in \bar{\omega} \quad \text{and} \quad (e\underline{n})_1 \Vdash \theta(\overline{n}, \overline{f(n)}\,). \tag{2}$$

Now define

$$g \;:=\; \{\langle \underline{n}, \langle \overline{n}, \overline{f(n)}\,\rangle_{\mathcal{A}}\rangle \mid n \in \omega\}.$$

Clearly, $g \in V(\mathcal{A})$. We first prove that g picks the right things and care about its functionality later. As

$$\mathbf{p}\underline{n}\mathbf{i}_r \Vdash \langle \overline{n}, \overline{f(n)}\,\rangle_{\mathcal{A}} \in g \tag{3}$$

it follows from (2) and (3) that with

$$h \;:=\; \lambda u.\mathbf{p}(\mathbf{p}(\mathbf{p}((eu)_0\mathbf{i}_r), \mathbf{p}u\mathbf{i}_r), (eu)_1)$$

we have for all $n \in \omega$ that

$$h\underline{n} \Vdash \exists y\, [y \in \bar{\omega} \,\wedge\, \langle \overline{n}, y\rangle_{\mathcal{A}} \in g \,\wedge\, \theta(\overline{n}, y)]. \tag{4}$$

As for functionality of g, assume that $x, y, z \in V(\mathcal{A})$ and

$$d \Vdash \langle x, y\rangle_{\mathcal{A}} \in g \,\wedge\, \langle x, z\rangle_{\mathcal{A}} \in g.$$

Then there exist $y', z' \in \mathcal{A}$ such that $\langle d_{0,0}, y'\rangle \in g$, $\langle d_{1,0}, z'\rangle \in g$, and

$$d_{0,1} \Vdash y' = \langle x, y\rangle_{\mathcal{A}} \quad\wedge\quad d_{1,1} \Vdash z' = \langle x, z\rangle_{\mathcal{A}}. \tag{5}$$

Moreover, there exist $n, m \in \omega$ such that $y' = \langle \overline{n}, \overline{f(n)}\,\rangle_{\mathcal{A}}$ and $z' = \langle \overline{m}, \overline{f(m)}\,\rangle_{\mathcal{A}}$. Thus it follows from Lemma 2.13 that $V(\mathcal{A}) \models \overline{n} = \overline{m}$, and therefore $n = m$ by Lemma 2.11. As a result, one can effectively construct a realizer $d' \in \mathcal{A}$ from d such that $d' \Vdash y = z$, showing functionality of g.

(ii) Validating $\mathbf{AC}_\omega$ in $V(\mathcal{A})$ is very similar to the proof of (i). Suppose that

$$e \Vdash \forall i \in \bar{\omega} \exists y\, \theta(i,y).$$

Then, $\forall \langle a, x\rangle \in \bar{\omega}\ ea \Vdash \exists y\, \theta(x,y)$, and hence

$$\forall \langle a, x\rangle \in \bar{\omega}\, \exists z \in V(\mathcal{A})\, ea \Vdash \theta(x,z)].$$

Now, invoking $\mathbf{AC}_\omega$ in V that there exists a function $F : \omega \to V(\mathcal{A})$ such that for all $n \in \omega$,

$$e\underline{n} \Vdash \theta(\overline{n}, F(n)). \tag{6}$$

Now define

$$G \;:=\; \{\langle \underline{n}, \langle \overline{n}, F(n)\rangle_{\mathcal{A}}\rangle \mid n \in \omega\}.$$

Clearly, $G \in V(\mathcal{A})$. The rest of the proof proceeds similarly as in (i).

(iii) Let $t, u \in V(\mathcal{A})$, and suppose the following:

(7)
$$e \Vdash \forall x \in t\, \exists y \in t \quad \phi(x, y)$$

and

(8)
$$e^* \Vdash u \in t$$

Then, (by the definition of realizability) (7) is equivalent to:

$$\forall \langle a, x\rangle \in t\, \exists y[\langle (ea)_0, y\rangle \in t \wedge (ea)_1 \Vdash \phi(x, y)].$$

Thus, for all $a \in |\mathcal{A}|$ and for all z in $V(\mathcal{A})$, if $\langle a, z\rangle \in t$, then $ea\downarrow$ and there is a q' in $V(\mathcal{A})$ such that $\langle (ea)_0, q'\rangle \in t \wedge (ea)_1 \Vdash \phi(z, q')$.

From (8) we conclude that there exists u_0 such that

$$\langle (e^*)_0, u_0\rangle \in t \wedge (e^*)_1 \Vdash u = u_0.$$

Externally, define $\phi^{\Vdash}$ by:

$$\phi^{\Vdash}(\langle a, z\rangle, \langle b, q\rangle) \quad \Leftrightarrow \quad b = (ea)_0 \wedge (ea)_1 \Vdash \phi(z, q).$$

By the validity of **DC** in V, there exists a function $F : \omega \longrightarrow t$ with:

$$F(0) = \langle (e^*)_0, u_0\rangle \text{ and for each } n \in \omega, \quad \phi^{\Vdash}(F(n), F(n+1)).$$

Next, we need to internalize F and show that it provides the function required for the validity of **DC** in $V(\mathcal{A})$. If x is an ordered pair $\langle u, v\rangle$, we use $(x)^s_0$ and $(x)^s_1$ to denote its standard set-theoretic projections, i.e., $(x)^s_0 = u$ and $(x)^s_1 = v$.

Let $\overline{F}$ (the internalization of F) be defined as follows:

$$\overline{F} := \{\langle \mathbf{p}(\underline{n}, (F(n))^s_0), \langle \overline{n}, (F(n))^s_1\rangle_{\mathcal{A}}\rangle : n \in \omega\}.$$

Clearly, $\overline{F} \in V(\mathcal{A})$ as $\mathbf{p}(\underline{n}, (F(n))^s_0) \in |\mathcal{A}|$ and $\langle \overline{n}, (F(n))^s_1\rangle_{\mathcal{A}} \in V(\mathcal{A})$ (by internal pairing properties).

Now, we need to check that $\overline{F}$ is internally a function from $\overline{\omega}$ to t.

Firstly, we show that $V(\mathcal{A})$ thinks that $\overline{F}$ is a binary relation with domain $\overline{\omega}$ and range a subset of t using properties of internal pairing in $V(\mathcal{A})$.

To prove that $\overline{F}$ is realizably functional, suppose that:

(9)
$$h \Vdash \langle \overline{n}, x\rangle_{\mathcal{A}} \in \overline{F}$$

and

(10)
$$k \Vdash \langle \overline{n}, y\rangle_{\mathcal{A}} \in \overline{F}.$$

Then, (9) is equivalent to the existence of an element $c \in V(\mathcal{A})$ such that:

(11)
$$\langle (h)_0, c\rangle \in \overline{F} \wedge (h)_1 \Vdash \langle \overline{n}, x\rangle_{\mathcal{A}} = c.$$

$\langle (h)_0, c\rangle \in \overline{F}$ yields that $(h)_0$ must have the form $\mathbf{p}(\underline{m}, (F(m))^s_0)$ and c be of the form $\langle \overline{m}, (F(m))^s_1\rangle_{\mathcal{A}}$ for some $m \in \omega$. But (11) entails that $V(\mathcal{A}) \models \overline{n} = \overline{m}$ using (2.13) and hence $n = m$ by Lemma 2.11. Hence, $h_{00} = \underline{n}$ and $c = \langle \overline{n}, (F(n))^s_1 \rangle_{\mathcal{A}}$, where h_{00} is an abbreviation for $((h)_0)_0$. Thus,

$$(h)_1 \Vdash \langle \overline{n}, x \rangle_{\mathcal{A}} = \langle \overline{n}, (F(n))^s_1 \rangle_{\mathcal{A}}. \tag{12}$$

Likewise (10) yields

$$(k)_1 \Vdash \langle \overline{n}, y \rangle_{\mathcal{A}} = \langle \overline{n}, (F(n))^s_1 \rangle_{\mathcal{A}}. \tag{13}$$

Using Lemma 2.13 it follows from (12) and (13) that we get a realizer e' such that $e' \Vdash x = y$ and e' can be computably obtained from e, h, k, showing that $\overline{F}$ is realizably functional.

Next, to verify that $V(\mathcal{A}) \models \overline{F} \subseteq \overline{\omega} \times t$, suppose that $h \Vdash \langle x, y \rangle_{\mathcal{A}} \in \overline{F}$. Then, using arguments as before:

$$(h)_1 \Vdash \langle x, y \rangle_{\mathcal{A}} = \langle \overline{n}, (F(n))^s_1 \rangle_{\mathcal{A}}$$

where $h_{00} = \underline{n}$. with $h_{00} = \underline{n}$. We also have $\mathbf{p}(h_{00}, \mathbf{i}_r) \Vdash \overline{n} \in \bar{\omega}$ and, thanks to the definition of $\overline{F}$, $\mathbf{p}(h_{01}, \mathbf{i}_r) \Vdash (F(n))^s_1 \in t$. Thus we can computably obtain h^* from h such that $h^* \Vdash x \in \bar{\omega} \ \wedge \ y \in t$.

Finally, we need to show the realizability of $\overline{F}(0) = u$ (where 0 stands for the empty set in the sense of $V(\mathcal{A})$ which really can be taken to be the empty set) and of $\forall u \in \overline{\omega}\phi(\overline{F}(u), \overline{F}(u+1))$.
As for the realizability of $\overline{F}(0) = u$, suppose $r \Vdash \langle 0, u_0 \rangle_{\mathcal{A}} \in \overline{F}$. Then, there exists c such that $\langle (r)_0, c \rangle \in \overline{F} \wedge (r)_1 \Vdash \langle 0, u_0 \rangle_{\mathcal{A}} = c$. $\langle (r)_0, c \rangle \in \overline{F}$ entails that $(r)_0$ has the form $\mathbf{p}(\underline{n}, (F(n))^s_0)$ and c has the form $\langle \overline{n}, (F(n))^s_1 \rangle_{\mathcal{A}}$. Hence, $(r)_1 \Vdash \langle 0, u_0 \rangle_{\mathcal{A}} = \langle \overline{n}, (F(n))^s_1 \rangle_{\mathcal{A}}$. As the latter implies $\mathtt{V}(\mathcal{A}) \models 0 = \overline{n}$ by the internal pairing properties, this forces $n = 0$ by (2.11), and hence $\mathtt{V}(\mathcal{A}) \models u_0 = (F(0))^s_1$, so that $(r)_1 = \mathbf{i}_r$ and $\mathbf{p}(\mathbf{p}(\mathbf{0}, (e^*)_0), \mathbf{i}_r) \Vdash \langle 0, u_0 \rangle_{\mathcal{A}} \in \overline{F}$ *i.e.* $\Vdash \overline{F}(0) = u_0$ and since $(e^*)_1 \Vdash u = u_0$, there is a realizer e' that can be computably obtained from r and e^* such that $e' \Vdash \overline{F}(0) = u$.
Next, we deal with the realizability of $\forall u \in \overline{\omega}\, \phi(\overline{F}(u), \overline{F}(u+1))$. Since for all $n \in \omega$ we have $[\phi^{\Vdash}(F(n), F(n+1))]$,

$$((F(n+1))^s_0 = (e(F(n))^s_0)_0 \ \text{ and} \tag{14}$$

$$(e(F(n))^s_0)_1 \Vdash \phi((F(n))^s_1, (F(n+1))^s_1). \tag{15}$$

Using the recursion theorem for applicative structures, we computably obtain $\rho \in |\mathcal{A}|$ from e, e^* such that

$$\rho\underline{0} = (e^*)_0 \text{ and } \rho(\underline{n+1}) = (e\rho(\underline{n}))_0.$$

Using induction on n, it follows that $\rho\underline{n} = (F(n))^s_0$ for all $n \in \omega$. Further, by induction on n, it can be shown that:

(a) $\mathbf{p}(\mathbf{p}(\underline{n}, \rho n), \mathbf{i}_r) \Vdash \langle \overline{n}, (F(n))^s_1 \rangle_{\mathcal{A}} \in \overline{F}$.

(b) $(e(\rho n))_1 \Vdash \phi((F(n))^s_1, (F(n+1))^s_1)$.

To verify this, first let $n = 0$. We have

(i) $\mathbf{p}(\mathbf{p}(\mathbf{0})(\rho(\mathbf{0})))(\mathbf{i}_r) \Vdash \langle \overline{0}, (F(0))^s_1 \rangle_{\mathcal{A}} \in \overline{F}$, *i.e* $\mathbf{p}(\mathbf{p}(\mathbf{0})((e^*))_0)(\mathbf{i}_r) \Vdash \langle \overline{0}, u_0 \rangle_{\mathcal{A}} \in \overline{F}$ by the above argument.

(ii) That $(e\rho(\mathbf{0}))_1 \Vdash \phi((F(0))_1^s, (F(1))_1^s)$, *i.e* $(e(e^*)_0))_1 \Vdash \phi(u_0, (F(1))_1^s)$ holds can be seen as follows. Since $\forall n \in \omega[\phi^{\Vdash}(F(n), F(n+1))]$, we have:

$$\begin{array}{rcl} (e(F(n))_0^s)_1 & \Vdash & \phi((F(n))_1^s, (F(n+1))_1^s) \\ (e(F(0))_0^s)_1 & \Vdash & \phi((F(0))_1^s, (F(1))_1^s) \\ (e(e^*)_0)_1 & \Vdash & \phi(u_0, (F(1))_1^s) \end{array}$$

Next, we do the induction step, so assume the result for n and we have to verify that:

(1) $\mathbf{p}(\mathbf{p}(\underline{n+1})(\rho(\underline{n+1})))(\mathbf{i}_r) \Vdash \langle\overline{n+1}, (F(n+1))_1^s\rangle_{\mathcal{A}} \in \overline{F}$.
To show this, assume that $r \Vdash \langle\overline{n+1}, (F(n+1))_1^s\rangle_{\mathcal{A}} \in \overline{F}$. Then,

$$r \Vdash \langle\overline{n+1}, (F(n+1))_1^s\rangle_{\mathcal{A}} \in \overline{F} \Leftrightarrow \exists c[\langle (r)_0, c\rangle \in \overline{F} \wedge (r)_1 \Vdash \langle\overline{n+1}, (F(n+1))_1^s\rangle_{\mathcal{A}} = c]$$

and hence, $(r)_0$ must have the form $(\underline{m}, (F(m))_0^s)$ and c must have the form $\langle\overline{m}, (F(m))_1^s\rangle_{\mathcal{A}}$ for some $m \in \omega$. So, $(r)_1 \Vdash \langle\overline{n+1}, (F(n+1))_1^s\rangle_{\mathcal{A}} = \langle\overline{m}, (F(m))_1^s\rangle_{\mathcal{A}}$ which implies that a realizer $\hat{r}$ can be calculated such that $\hat{r} \Vdash \overline{n+1} = \overline{m}$ which, by (2.11), yields that $n+1 = m$ and by (2.9) we obtain that $\underline{n+1} = \underline{m}$. So, by the induction hypothesis we have $\mathbf{p}(\mathbf{p}(\underline{n+1})(\rho(n+1)))(\mathbf{i}_r) \Vdash \langle\overline{n+1}, (F(n+1))_1^s\rangle_{\mathcal{A}} \in \overline{F}$.

(2) We claim that $(e\rho(n+1))_1 \Vdash \phi((F(n+1))_1^s, (F(n+2))_1^s)$. We know by (15) that

$$(e(F(n+1))_0^s)_1 \Vdash \phi((F(n+1))_1^s, (F(n+2))_1^s)$$

holds. Note that:

$$\begin{array}{rcl} \rho(\underline{0}) = (e^*)_0 & = & (F(0))_0^s \\ \rho(\underline{1}) & = & (e(F(0))_0^s)_0 \\ \rho(\underline{2}) & = & (e(e(F(0))_0^s)_0)_0 \\ \rho(\underline{n+1}) & = & \underbrace{(e(e...(e}_{\text{n-times}}(F(0))_0^s)_0)_0)_0 \end{array}$$

and

$$\begin{array}{rcl} (F(1))_0^1 & = & (e(F(0))_0^s)_0 \\ (F(2))_0^1 & = & (e(e(F(0))_0^s)_0)_0 \\ (F(3))_0^1 & = & (e(e(e(F(0))_0^s)_0)_0)_0 \\ (F(n+1))_0^1 & = & \underbrace{(e(e...(e}_{\text{n-times}}(F(0))_0^s)_0)_0)_0 \end{array}$$

Thus, clearly $\rho(\underline{n+1}) = (F(n+1))_0^1$. Therefore,

$$(e(\rho(\underline{n+1})))_1 \Vdash \phi((F(n+1))_1^s, (F(n+2))_1^s).$$

(iv) Given part *(iii)* of this theorem, working in $\mathbf{CZF} + \mathbf{RDC}$, it is enough to find a realizer for the following schema:

$$\forall x(\phi(x) \to \exists y[\phi(y) \wedge \psi(x,y)]) \wedge \phi(a_0) \longrightarrow \exists s(a_0 \in s \wedge \forall x \in s \exists y \in s[\phi(y) \wedge \psi(x,y)]).$$

So, let $a_0 \in V(\mathcal{A})$ and suppose the following hold:

(16) $$e \Vdash \forall x(\phi(x) \to \exists y[\phi(y) \wedge \psi(x,y)]) \text{ and}$$

$$(17) \qquad r \Vdash \phi(a_0)$$

Now, we have:

$$\begin{aligned}
(16) \;\Leftrightarrow\; & \forall x \in V(\mathcal{A})\; e \Vdash \phi(x) \to \exists y[\phi(y) \wedge \psi(x,y)]. \\
\Leftrightarrow\; & \forall f \in \mid \mathcal{A} \mid \forall x \in V(\mathcal{A})[f \Vdash \phi(x) \to ef \Vdash \exists y(\phi(y) \wedge \psi(x,y))]. \\
\Leftrightarrow\; & \forall f \in \mid \mathcal{A} \mid \forall x \in V(\mathcal{A})[f \Vdash \phi(x) \to \exists y \in V(\mathcal{A}) ef \Vdash \phi(y) \wedge \psi(x,y)].
\end{aligned}$$

Thus, for all f in $\mid \mathcal{A} \mid$ and for all $x \in V(\mathcal{A})$ we have

$$f \Vdash \phi(x) \to \exists y \in V(\mathcal{A})\, [(ef)_0 \Vdash \phi(y) \wedge (ef)_1 \Vdash \psi(x,y)].$$

Let $N = \{\underline{n} \mid n \in \omega\}$. By applying **RDC** to the above, we conclude that there are functions $i : \mathbf{N} \longrightarrow \mathcal{A}$, $j : \mathbf{N} \longrightarrow \mathcal{A}$ and $l : \omega \longrightarrow V(\mathcal{A})$ with $i(\underline{0}) = r$, $l(0) = a_0$ and for all n in ω, we have:

$$i(\underline{n}) \Vdash \phi(l(n)) \text{ and } j(\underline{n}) \Vdash \psi(l(n), l(n+1)),$$

$$i(\underline{n+1}) = (ei(\underline{n}))_0 \text{ and } j(\underline{n}) = (ei(\underline{n}))_1.$$

Using the recursion theorem for $\mathcal{A}$, one can explicitly calculate $t_i, t_j \in |\mathcal{A}|$ from e and r such for all $n \in \omega$, $i(\underline{n}) = t_i\underline{n}$ and $j(\underline{n}) = t_j\underline{n}$. And thus the function $h : \mathbf{N} \longrightarrow \mathcal{A}$ defined by $h(\underline{n}) = \mathbf{p}(\underline{n}, \mathbf{p}(i(\underline{n}), j(\underline{n})))$ for some $n \in \omega$ is representable in $|\mathcal{A}|$ via an element t_h computable from e and r as well, i.e., $h(\underline{n}) = t_h\underline{n}$ for all $n \in \omega$.

Now, set

$$B := \{\langle h(\underline{n}), l(n)\rangle : n \in \omega\}.$$

$B \in V(\mathcal{A})$, since $h(\underline{n}) \in \mid \mathcal{A} \mid$ and $l(n) \in V(\mathcal{A})$.
Now, we need to find a realizer e^* such that:

$$(18) \qquad e^* \Vdash a_0 \in B.$$

As

$$e^* \Vdash a_0 \in B \Leftrightarrow \exists c[\langle (e^*)_0, c\rangle \in B \wedge (e^*)_1 \Vdash a_0 = c]$$

and $\langle (e^*)_0, c\rangle \in B$ iff $\langle (e^*)_0, c\rangle = \langle h(\underline{n}), l(n)\rangle$, we arrive at $(e^*)_0 = h(\underline{n})$ and $c = l(n) = a_0$. Since l is a function, n must be 0 and thus $e^* = \mathbf{p}(h(\underline{0}), \mathbf{i}_r)$. So, $(h(\underline{0}), \mathbf{i}_r) \Vdash a_0 \in B$.
Furthermore, for $\langle k, u\rangle \in B$ we have $k = h(\underline{n})$ for some $n \in \omega$. Consequently, $(h(\underline{n}))_0 = \underline{n} = (k)_0$ and $u = l((k)_0)$, hence $\langle h(\mathbf{S_N}((k)_0)), l((k)_0 + 1)\rangle \in B$. Moreover, since $(k)_1 = \mathbf{p}(i(\underline{n}), j(\underline{n}))$ we have $k_{1,0} = i(\underline{n})$, thus $k_{1,0} \Vdash \phi(l(n))$, so $k_{1,0} \Vdash \phi(u)$ and $k_{1,1} = j(\underline{n})$ which implies $k_{1,1} \Vdash \psi(u, l((k)_0 + 1))$.
Therefore,

$$(19) \qquad \forall \langle k, u\rangle \in B\, \exists v\, [\langle h(\mathbf{S_N}((k)_0)), v\rangle \in B \;\wedge\; k_{1,0} \Vdash \phi(u) \wedge k_{1,1} \Vdash \psi(u,v)].$$

From (18) and (19), it is clear that there exists a realizer $\grave{e}$ computed from e and r such that:

$$\grave{e} \Vdash a_0 \in B \wedge \forall x \in B \exists y \in B[\phi(x) \wedge \psi(x,y)]$$

and hence,

$$\grave{e} \Vdash \exists s(a_0 \in s \wedge \forall x \in s \exists y \in s[\phi(x) \wedge \psi(x,y)])$$

which completes the proof of (iv).

(v) Let $s \in V(\mathcal{A})$. We are aiming to find a set $B^* \in V(\mathcal{A})$ such that $V(\mathcal{A})$ believes that B^* is a base that maps onto s.
Since **PAx** holds in the background model V, we can choose a base B and a surjective map $j : B \longrightarrow s$. As s is a set of ordered pairs, we may define:

$$j_0 : B \longrightarrow \mathcal{A} \text{ and } j_1 : B \longrightarrow V(\mathcal{A})$$

by

$$j_0(u) = 1^{st}(j(u)) \text{ and } j_1(u) = 2^{nd}(j(u))$$

where these functions denote the standard projections of ordered pairs in set theory. Using transfinite recursion, for any set x, we define:

$$x^{st} = \{\langle \mathbf{0}, y^{st} \rangle : y \in x\}.$$

$x^{st} \in V(\mathcal{A})$ is straightforwardly proved by $\in$-induction as follows:
Inductively assume that $y^{st} \in V(A)$ for all $y \in x$. As $\mathbf{0} \in \mathcal{A}$ this implies that

$$\{\langle \mathbf{0}, y^{st} \rangle \mid y \in x\} \subseteq |\mathcal{A}| \times V(\mathcal{A})$$

and thus $x^{st} \in V(\mathcal{A})$ by (2.2) part (ii).
To complete the proof we need the following facts.

Proposition 2.15.

(i)

$$x = y \quad \textit{iff} \quad V(\mathcal{A}) \models x^{st} = y^{st}.$$

(ii)

$$x \in y \quad \textit{iff} \quad V(\mathcal{A}) \models x^{st} \in y^{st}.$$

Proof. We show (i) and (ii) simultaneously by $\in$-induction as follows:

(i): The implication from left to right is immediate. As for the other direction, suppose that $e \Vdash x^{st} = y^{st}$. Then,

$$\forall \langle f, u \rangle \in x^{st} \; ((e)_0 f \Vdash u \in y^{st}) \wedge \forall \langle f, u \rangle \in y^{st} \; ((e)_1 f \Vdash u \in x^{st}).$$

If $z \in x$ then $\langle \mathbf{0}, z^{st} \rangle \in x^{st}$, thus $V(\mathcal{A}) \models z^{st} \in y^{st}$ and thus inductively $z \in y$. By a symmetric argument, $z \in y$ yields $z \in x$. Hence $x = y$.

(ii): Again the left to right direction is obvious. Suppose $V(\mathcal{A}) \models x^{st} \in y^{st}$. Then there exists $z \in y$ such that $V(\mathcal{A}) \models x^{st} = z^{st}$, and therefore inductively $x = z$, and hence $x \in y$.

□

As a result, the map taking x to x^{st} is an injection from V to $V(\mathcal{A})$.
Next, let

$$B^* := \{\langle j_0(u), \langle (j_0(u))^{st}, u^{st} \rangle_{\mathcal{A}} \rangle : u \in B\}.$$

Note that the map $u \longmapsto \langle j_0(u), \langle (j_0(u))^{st}, u^{st} \rangle_{\mathcal{A}} \rangle$ injects B onto B^* and hence B^* is a base (in the sense of the ground universe).
Define

$$l : B \longrightarrow V(\mathcal{A})$$

such that $l(u) = \langle (j_0(u))^{st}, u^{st} \rangle_{\mathcal{A}}$ and let

$$j^* := \{\langle j_0(u), \langle l(u), j_1(u) \rangle_{\mathcal{A}} \rangle : u \in B\}.$$

As $j_0(u) \in \mathcal{A}$ and $l(u), j_1(u) \in V(\mathcal{A})$, it follows that $j^* \in V(\mathcal{A})$. Now, we claim that:

$$V(\mathcal{A}) \models j^* \text{ maps } B^* \text{ onto } s. \tag{20}$$

Firstly, we verify that $V(\mathcal{A}) \models j^* \subseteq B^* \times s$. Towards this goal, assume that $e \Vdash \langle b, c \rangle_{\mathcal{A}} \in j^*$. Then there exists f such that $\langle (e)_0, f \rangle \in j^*$ and $(e)_1 \Vdash \langle b, c \rangle_{\mathcal{A}} = f$.
However, $\langle (e)_0, f \rangle \in j^*$ means that there exists $u \in B$ with $(e)_0 = j_0(u)$ and $f = \langle l(u), j_1(u) \rangle_{\mathcal{A}}$, and hence $(e)_1 \Vdash \langle b, c \rangle_{\mathcal{A}} = \langle l(u), j_1(u) \rangle_{\mathcal{A}}$.
So, we need to find realizers $r \Vdash l(u) \in B^*$ and $r^* \Vdash j_1(u) \in s$.
Well, $r \Vdash l(u) \in B^* \Leftrightarrow \exists c[\langle (r)_0, c \rangle \in B^* \wedge (r)_1 \Vdash l(u) = c]$. However, $\langle (r)_0, c \rangle \in B^*$ implies $(r)_0 = j_0(u)$ and $c = \langle (j_0(u))^{st}, u^{st} \rangle_{\mathcal{A}} = l(u)$ for some $u \in B$, and hence letting $r := \mathbf{p}(j_0(u), \mathbf{i}_r)$ we get get $r \Vdash l(u) \in B^*$.
$r^* \Vdash j_1(u) \in s$ is equivalent to the existence of a c such that $\langle (r^*)_0, c \rangle \in s \wedge (r^*)_1 \Vdash j_1(u) = c$. It follows from $\langle (r^*)_0, c \rangle \in s$ that $(r^*)_0 = j_0(u)$ and $c = j_1(u)$ and hence, with $r^* := \mathbf{p}(j_0(u), \mathbf{i}_r)$ we have $r^* \Vdash j_1(u) \in s$.
Therefore, a realizer e^* can be computed from e such that

$$e^* \Vdash b \in B^* \wedge c \in s$$

which shows that $V(\mathcal{A}) \models j^* \subseteq B^* \times s$.
To verify that j^* is realizably total on B^*, assume $e \Vdash \langle c, d \rangle_{\mathcal{A}} \in B^*$. Then there exists f such that $\langle (e)_0, f \rangle \in B^* \wedge (e)_1 \Vdash \langle c, d \rangle_{\mathcal{A}} = f$. But $\langle (e)_0, f \rangle \in B^*$ has the form $\langle j_0(u), \langle (j_0(u))^{st}, u^{st} \rangle_{\mathcal{A}} \rangle$ for some $u \in B$. So, $(e)_0 = j_0(u)$ and $f = \langle (j_0(u))^{st}, u^{st} \rangle_{\mathcal{A}} = l(u)$ for some $u \in B$. Thus, for some $u \in B$ we have, $(e)_0 = j_0(u)$ and $(e)_1 \Vdash \langle c, d \rangle_{\mathcal{A}} = l(u)$.
Since $\mathbf{p}(j_0(u), \mathbf{i}_r) \Vdash j_1(u) \in s$ and

$$\mathbf{p}(j_0(u), \mathbf{i}_r) \Vdash \langle l(u), j_1(u) \rangle_{\mathcal{A}} \in j^* \tag{21}$$

a realizer $\hat{e}$ can be computed from e such that:

$$\hat{e} \Vdash \langle c, d \rangle_{\mathcal{A}} \textit{ is in the domain of } j^*.$$

To verify (21) let $r \Vdash \langle l(u), j_1(u) \rangle_{\mathcal{A}} \in j^*$. Then,
$r \Vdash \langle l(u), j_1(u) \rangle_{\mathcal{A}} \in j^* \Leftrightarrow \exists c[\langle (r)_0, c \rangle \in j^* \wedge (r)_1 \Vdash \langle l(u), j_1(u) \rangle_{\mathcal{A}} = c]$.
From $\langle (r)_0, c \rangle \in j^*$, it follows that $(r)_0 = j_0(u)$ and $c = \langle l(u), j_1(u) \rangle_{\mathcal{A}}$ for some $u \in B$, and hence $\mathbf{p}(j_0(u), \mathbf{i}_r) \Vdash \langle l(u), j_1(u) \rangle_{\mathcal{A}} \in j^*$.
Hence, since we have already verified that $V(\mathcal{A}) \models j^* \subseteq B^* \times s$, we can infer that $V(\mathcal{A}) \models B^*$ is the domain of j^*.
Next, we need to show that j^* is realizably functional. To this end, assume that:

$$f \Vdash \langle b, c \rangle_{\mathcal{A}} \in j^* \tag{22}$$

$$h \Vdash \langle b, d \rangle_{\mathcal{A}} \in j^*. \tag{23}$$

So by (22) there is a q such that $\langle (f)_0, q \rangle \in j^* \wedge (f)_1 \Vdash \langle b, c \rangle_{\mathcal{A}} = q$ which entails that $(f)_0$ has the form $j_0(u)$ and q has the form $\langle l(u), j_1(u) \rangle_{\mathcal{A}}$ for some u in B, so that $(f)_1 \Vdash \langle b, c \rangle_{\mathcal{A}} = \langle l(u), j_1(u) \rangle_{\mathcal{A}}$.

And from (23) we get that there is a q' such that $\langle (h)_0, q'\rangle \in j^* \wedge (h)_1 \Vdash \langle b, d\rangle_{\mathcal{A}} = q'$, and similarly we obtain that $(h)_0$ has the form $j_0(v)$ and q' has the form $\langle l(v), j_1(v)\rangle_{\mathcal{A}}$ for some $v \in B$ and hence, $(h)_1 \Vdash \langle b, d\rangle_{\mathcal{A}} = \langle l(v), j_1(v)\rangle_{\mathcal{A}}$.
Therefore a realizer r can be extracted such that $r \Vdash l(u) = l(v)$, so $V(\mathcal{A}) \models l(u) = l(v)$. By the definition of l, we have:

$$\begin{array}{rcl} V(\mathcal{A}) & \models & l(u) = l(v) \\ V(\mathcal{A}) & \models & \langle (j_0(u))^{st}, u^{st}\rangle_{\mathcal{A}} = \langle (j_0(v))^{st}, v^{st}\rangle_{\mathcal{A}} \\ & \Leftrightarrow & \mathtt{V}(\mathcal{A}) \models u^{st} = v^{st} \\ & \Leftrightarrow & u = v \quad \text{by (2.15).} \end{array}$$

Therefore, there is a realizer $\grave{e}$ computable from f, h such that $\grave{e} \Vdash c = d$.

Next, we need to show that j^* is realizably surjective. To this end, suppose that $e \Vdash x \in s$. Then there exists a c such that $\langle (e)_0, c\rangle \in s \wedge (e)_1 \Vdash x = c$.
$\langle (e)_0, c\rangle \in s$ implies that $\langle (e)_0, c\rangle$ has the form $\langle j_0(u), j_1(u)\rangle$ for some $u \in B$ because $j : B \longrightarrow s$ maps B onto s. Furthermore, since $\mathbf{p}(j_0(u), \mathbf{i}_r) \Vdash l(u) \in B^*$ and $\mathbf{p}(j_0(u), \mathbf{i}_r) \Vdash \langle l(u), j_1(u)\rangle_{\mathcal{A}} \in j^*$, it follows that a realizer $\tilde{e}$ can be calculated such that

$$\tilde{e} \Vdash x \textit{ in the range of } \; j^*.$$

This completes the proof of (20).

Finally, we need to verify that $V(\mathcal{A})$ believes that B^* is a base. To verify this, suppose that:

$$e \Vdash \forall x \in B^* \exists y \phi(x, y) \text{ for some formula } \phi. \tag{24}$$

Now, we are aiming to compute a realizer e^{**} calculable from e satisfying:

$$e^{**} \Vdash \exists H[\mathbf{Fun}(H) \wedge \mathbf{dom}(H) = B^* \wedge \forall x \in B^* \phi(x, H(x))] \tag{25}$$

$$\begin{array}{rcl} \text{Note that } e \Vdash \forall x \in B^* \exists y \phi(x, y) & \Leftrightarrow & \forall \langle q, c\rangle \in B^* \quad eq \Vdash \exists y \phi(c, y) \\ & \Leftrightarrow & \forall \langle q, c\rangle \in B^* \exists d \in V(\mathcal{A}) \quad eq \Vdash \phi(c, d). \end{array}$$

Hence, from (24) it follows that:

$$\forall \langle q, c\rangle \in B^* \exists y \in V(\mathcal{A}) \quad eq \Vdash \phi(c, y).$$

Now, because B^* is a base in the ground universe, there is a function

$$F : B^* \longrightarrow V(\mathcal{A})$$

such that

$$\forall \langle q, c\rangle \in B^* \quad eq \Vdash \phi(c, F(\langle q, c\rangle)).$$

Next, we need an internalization of F namely $\tilde{F}$, defined by:

$$\tilde{F} := \{\langle \mathbf{p}(eq, q), \langle c, F(\langle q, c\rangle)\rangle_{\mathcal{A}}\rangle : \langle q, c\rangle \in B^*\}.$$

Since, $eq \in \mathcal{A}$, $\mathbf{p}(eq, q) \in \mathcal{A}$, $c \in V(\mathcal{A})$, $\langle q, c\rangle \in B^* \in V(\mathcal{A})$ and also $F(\langle q, c\rangle) \in V(\mathcal{A})$, we can deduce that $\tilde{F} \in V(\mathcal{A})$.
First, we need to show that $V(\mathcal{A}) \models \mathbf{dom}(\tilde{F}) = B^*$. Towards this goal, suppose that

$h \Vdash x \in B^*$. Then there exist c such that $\langle (h)_0, c\rangle \in B^*$ and $(h)_1 \Vdash x = c$. $\langle (h)_0, c\rangle \in B^*$ yields that

$$\mathbf{p}(\mathbf{p}(e(h)_0, (h)_0), \mathbf{i}_r) \Vdash \langle c, F(\langle (h)_0, c\rangle)\rangle_{\mathcal{A}} \in \tilde{F},$$

from which we can effectively construct a realizer $\hat{h}$ such that $\hat{h} \Vdash x \in \mathbf{dom}(\tilde{F})$.

Conversely, assume that $d \Vdash \langle x, y\rangle_{\mathcal{A}} \in \tilde{F}$. Then there exists $\langle q, c\rangle \in B^*$ such that $\langle (d)_0, \langle c, F(\langle q, c\rangle)\rangle_{\mathcal{A}}\rangle \in \tilde{F}$ where $q = ((d)_0)_1$ and $(d)_1 \Vdash \langle x, y\rangle_{\mathcal{A}} = \langle c, F(\langle q, c\rangle)\rangle_{\mathcal{A}}$. Consequently, $\mathbf{p}(((d)_0)_1, \mathbf{i}_r) \Vdash c \in B^*$. Therefore we can calculate an index d^* from d such that $d^* \Vdash x \in B^*$.

Finally, we have all the pieces to construct $\hat{e}$ from e such that

$$\hat{e} \Vdash \mathbf{dom}(\tilde{F}) = B^*.$$

Next, it remains to show that $\tilde{F}$ is realizably functional. To this end, suppose:

$$f \Vdash \langle b, c\rangle_{\mathcal{A}} \in \tilde{F} \tag{26}$$

$$h \Vdash \langle b, d\rangle_{\mathcal{A}} \in \tilde{F}. \tag{27}$$

(26) and (27) provide $\langle q, x\rangle, \langle q', y\rangle \in B^*$ such that $((f)_0)_1 = q$, $((h)_0)_1 = q'$, and

$$(f)_1 \Vdash \langle b, c\rangle_{\mathcal{A}} = \langle x, F(\langle q, x\rangle)\rangle_{\mathcal{A}} \quad\wedge\quad (h)_1 \Vdash \langle b, d\rangle_{\mathcal{A}} = \langle y, F(\langle q', y\rangle)\rangle_{\mathcal{A}}. \tag{28}$$

The latter yields $V(\mathcal{A}) \models x = y$. Since $\langle q, x\rangle, \langle q', y\rangle \in B^*$ there exist $u, v \in B$ satisfying

$$x = \langle (j_0(u))^{st}, u^{st}\rangle_{\mathcal{A}} \text{ and } q = j_0(u)$$

as well as

$$y = \langle (j_0(v))^{st}, v^{st}\rangle_{\mathcal{A}} \text{ and } q' = j_0(v).$$

As $V(\mathcal{A}) \models x = y$, the above implies

$$V(\mathcal{A}) \models q^{st} = (q')^{st} \wedge u^{st} = v^{st},$$

and so by Proposition 2.15, we arrive at $q = q'$ and $u = v$, which also yields $x = y$ and $F(\langle q, x\rangle) = F(\langle q', y\rangle)$. Thus, also taking (28) into account, we can construct a realizer ν such that $\nu f h \Vdash c = d$. This verifies the functionality of $\tilde{F}$, so $V(\mathcal{A}) \models \tilde{F}$ *is a function.*

In sum, taking all the foregoing together, we can calculate in $\mathcal{A}$ a realizer e^{**} from e such that (25) holds.

This completes the proof.

□

The upshot of the preceding proofs is that there exist uniform realizers for the choice principles discussed in this paper in that a closed application term can be exhibited for each choice principle P such that its interpretation in any applicative structure $\mathcal{A}$ furnishes a realizer in $\mathcal{A}$ for P in the universe $V(\mathcal{A})$. Preservation of these choice principles was first shown for the first Kleene algebra where realizers are codes for partial recursive functions. This paper shows that no particular properties of partial recursive functions are required.

References

[1] P. Aczel: *The type theoretic interpretation of constructive set theory.* In: MacIntyre, A. and Pacholski, L. and Paris, J, editor, *Logic Colloquium '77* (North Holland, Amsterdam 1978) 55–66.

[2] P. Aczel: *The type theoretic interpretation of constructive set theory: Choice principles.* In: A.S. Troelstra and D. van Dalen, editors, *The L.E.J. Brouwer Centenary Symposium* (North Holland, Amsterdam 1982) 1–40.

[3] P. Aczel: *The type theoretic interpretation of constructive set theory: Inductive definitions.* In: R.B. et al. Marcus, editor, *Logic, Methodology and Philosophy of Science VII* (North Holland, Amsterdam 1986) 17–49.

[4] P. Aczel, M. Rathjen: *Notes on constructive set theory*, Technical Report 40, Institut Mittag-Leffler (The Royal Swedish Academy of Sciences, 2001). *https://events.math.unipd.it/3wftop/pdf/AczelRathjen.pdf*

[5] P. Aczel, M. Rathjen: *Notes on constructive set theory*, Preprint (2010) 243 pages. http://www1.maths.leeds.ac.uk/ rathjen/book.pdf

[6] J. Barwise: *Admissible Sets and Structures* (Springer-Verlag, Berlin, Heidelberg, New York, 1975).

[7] M. Beeson: *Foundations of Constructive Mathematics.* (Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1985).

[8] M. Beeson: *Continuity in intuitionistic set theories*, in: M Boffa, D. van Dalen, K. McAloon (eds.): *Logic Colloquium '78* (North-Holland, Amsterdam, 1979).

[9] M. Beeson: *Foundations of Constructive Mathematics.* (Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1985).

[10] E. Bishop and D. Bridges: *Constructive Analysis.* (Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1985).

[11] S. Feferman: *A language and axioms for explicit mathematics*, in: J.N. Crossley (ed.): *Algebra and Logic*, Lecture Notes in Math. 450 (Springer, Berlin 1975) 87–139.

[12] S. Feferman: *Constructive theories of functions and classes* in: Boffa, M., van Dalen, D., McAloon, K. (eds.), *Logic Colloquium '78* (North-Holland, Amsterdam 1979) 159–224.

[13] H. Friedman: *Some applications of Kleene's method for intuitionistic systems.* In: A. Mathias and H. Rogers (eds.): *Cambridge Summer School in Mathematical Logic*, volume 337 of *Lectures Notes in Mathematics* (Springer, Berlin, 1973) 113–170.

[14] S.C. Kleene: *On the interpretation of intuitionistic number theory.* Journal of Symbolic Logic 10 (1945) 109–124.

[15] G. Kreisel and A.S. Troelstra: *Formal systems for some branches of intuitionistic analysis.* Annals of Mathematical Logic 1 (1970) 229–387.

[16] D.C. McCarty: *Realizability and recursive mathematics*, PhD thesis, Oxford University (1984), 281 pages.

[17] J. Myhill: *Some properties of Intuitionistic Zermelo-Fraenkel set theory.* In: A. Mathias and H. Rogers (eds.): *Cambridge Summer School in Mathematical Logic*, volume 337 of *Lectures Notes in Mathematics* (Springer, Berlin, 1973) 206–231.

[18] J. Myhill: *Constructive set theory.* Journal of Symbolic Logic, 40:347–382, 1975.

[19] M. Rathjen: *Realizability for constructive Zermelo-Fraenkel set theory.* In: J. Väänänen, V. Stoltenberg-Hansen (eds.): *Logic Colloquium '03.* Lecture Notes in Logic 24 (A.K. Peters, 2006) 282–314.

[20] M. Rathjen: *Models of intuitionistic set theories over partial combinatory algebras.* (Springer, 2006) 68–78.

[21] M. Rathjen: *Remarks on Barr's Theorem: Proofs in geometric theories.* In: Probst, Schuster (eds.): Concepts of Proof in Mathematics, Philosophy, and Computer Science, Ontos Mathematical Logic (De Gruyter, Berlin, 2016) 347–374.

[22] A.S. Troelstra and D. van Dalen: *Constructivism in Mathematics, Volumes I, II.* (North Holland, Amsterdam, 1988).