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A General Model of Dynamics on Networks with Graph Automorphism Lumping

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Abstract. In this paper we introduce a general Markov chain model of dynamical processes on networks. In this model, nodes in the network can adopt a finite number of states and transitions can occur that involve multiple nodes changing state at once. The rules that govern transitions only depend on measures related to the state and structure of the network and not on the particular nodes involved. We prove that symmetries of the network can be used to lump equivalent states in state-space. We illustrate how several examples of well-known dynamical processes on networks correspond to particular cases of our general model. This work connects a wide range of models specified in terms of node-based dynamical rules to their exact continuous-time Markov chain formulation.

Keywords: Dynamics on networks, Markov chains, graph automorphisms, lumping, epidemic models, opinion dynamics, social physics.

1 Introduction

Dynamical processes on networks are one of the main topics of network science [5,51,55]. The ubiquity of networks means that a wide range of phenomena have been modelled as dynamical processes on networks, including epidemics [41,52], magnetism [24], opinion dynamics [23, 60, 61], diffusion of innovations [6, 46, 47, 64], rumour spread [17, 31, 38], meme popularity [29], cultural polarisation [1, 10], racial segregation [57, 58], stock market trading [40], cascading failures [27,32,50] and language evolution [4, 9, 14]. For example, the study of epidemics on networks includes research on the effect that network structure has on the epidemic threshold [8, 12], a variety of methods to approximate the dynamics [19,53] and the effectiveness of vaccination strategies [54]. For thorough reviews of epidemic models on networks, see [41, 52]. As another example, the study of opinion dynamics makes use of simple models of voting behaviour based on binary [34], multi-state [62] and continuous [18, 33] opinions, where voters take into account the opinions of one [60] or more [7, 11, 23, 61] neighbours, with the possibility of zealots with fixed opinions [48]. Typically, the effect of network topology on the mean-time to reach consensus [60] and the identification of phase-transitions between ordered (consensus) and disordered (disagreement) states [42,49] are of interest. Of fundamental importance in the study of dynamics on networks is understanding the effect of network topology on dynamics.

General models have been proposed that attempt to capture the wide variety of dynamical processes on networks [2, 3, 21, 25, 26, 41]. From these models, one can derive high-accuracy approximations of the dynamics [21, 25, 26] and identify redundancies in state-space due to network symmetries [2, 41, 44, 59]. However, these models only allow one node in the network to change its state at any instant in time. We call such models *single vertex-state transition* models. In contrast, there are many models where multiple vertices change state at once. Examples include the naming game of language evolution [4], the Schelling model of segregation [57, 58], the Bonabeau model of hierarchy formation [9], the Twitter model of meme propagation [29], and the Sznajd [61] and majority rule [15, 23] models of opinion dynamics. We call such models *multi vertex-state transition* models. Understandably, analytical results are less common for multi vertex-state transition models.

In this paper, after presenting some background material in Sect. 2, we review five models in Sect. 3 that capture the main features of typical dynamical processes on networks. Then in Sect. 4 we introduce a general model, which accounts for multi vertex-state transitions, and in Sect. 5 we prove that the state-space of our model can be lumped using graph automorphisms. In Sect. 6 we connect our general model to those described in Sect. 3 and we conclude with a discussion of our work in Sect. 7.

2 Background

Let $\mathbb{S} = \{S_1, S_2, \ldots, S_n\}$ be the *state-space* of a continuous-time Markov chain whose time-dependent probability distribution over state-space is

$$X(t) = (x_1(t), x_2(t), \ldots, x_n(t))^{\mathrm{T}} ,$$

where $x_i(t)$ is the probability of being in state S_i at time t. The evolution of $X(t)$ is described by the *master equation*

$$\dot{X} = Q^{\mathrm{T}} X ,$$

also known as the forward Kolmogorov or differential Chapman-Kolmogorov equation. The infinitesimal generator Q is an n by n matrix whose ijth component describes the non-negative transition rate from the state S_i to the state S_j for $i \neq j$, and whose diagonal entries ensure the rows sum to zero (i.e. the magnitude of Q_{ii} is the transition rate out of state S_i). See, e.g. [39] for more details about Markov chains. A partition of state-space $\mathbb{L} = \{\mathbb{L}_1, \mathbb{L}_2, \ldots, \mathbb{L}_r\}$ is called a *lumping* if it preserves the Markov property [37, 59]. A necessary and sufficient condition for lumping is that for all $i, j \in \{1, 2, \ldots, r\}$ there exists an R_{ij} that satisfies

$$R_{ij} = \sum_{S_l \in \mathbb{L}_j} Q_{kl} \ , \text{ for all } S_k \text{ in } \mathbb{L}_i.$$

If $y_i = \sum_{S_k \in \mathbb{L}_i} x_k$, and $Y = (y_1, y_2, \ldots, y_r)^{\mathrm{T}}$, then the dynamics of the lumped probability distribution $Y(t)$ are described by

$$\dot{Y} = R^{\mathrm{T}} Y .$$

A *network* or graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ consists of a set of *vertices* or nodes $\mathcal{V}$ and a set of *edges* or links $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$. A pair of vertices $u, v \in \mathcal{V}$ are *neighbours* if $(u, v) \in \mathcal{E}$ and the *neighbourhood* of a vertex is its set of neighbours. An *automorphism* of a network $\mathcal{G}$ is a bijection $g : \mathcal{V} \to \mathcal{V}$ such that $(u, v) \in \mathcal{E}$ if and only if $(g(u), g(v)) \in \mathcal{E}$. We use the shorthand $gv = g(v)$. The set of automorphisms of a network $\mathcal{G}$ form a permutation group $G = \mathrm{Aut}(\mathcal{G})$ called the automorphism group of $\mathcal{G}$ [30].

3 Dynamical Processes on Networks

We now describe several examples of well-known dynamical processes on networks. These examples cover the main model features captured by the general Markov chain model that we introduce in Sect. 4.

The **SIS** (susceptible-infected-susceptible) model [41, 52, 53] is one of the most fundamental of epidemic models. Individuals in a finite population are considered to be in one of two possible states, either **S**usceptible to an infection, or **I**nfected. Each susceptible individual is assumed to contract the infection with a rate proportional to the number of infected individuals they are in contact with. Where such contacts are described by a network, a susceptible node becomes infected at a rate $\beta m(t)$, where β is a positive rate constant and $m(t)$ is the number of infected neighbours they have at time t. In addition, each infected individual is assumed to become susceptible again at a constant rate γ.

The **SIR** (susceptible-infected-recovered) model [41, 52] is another fundamental model of epidemics that is similar in formulation to the SIS model . In the SIR model, individuals can also be in an additional **R**ecovered state. As in the SIS model on a network, a susceptible individual becomes infected at a rate proportional to the number infected neighbours they have. In the SIR model however, individuals recover at a constant rate γ and then remain recovered.

The **voter** model [13, 16, 60] is a simple model of opinion dynamics that is specified in terms of a set of iterated rules. Individuals in a finite population are assumed to have one of two possible opinions, typically denoted by ± 1. Individuals can be influenced by the opinions of others, and the structure of who influences whom is usually represented by a network. At each discrete time-step, an individual is chosen at random and they adopt the opinion of a randomly chosen neighbour. This is repeated until consensus is reached.

The **Twitter** model [29] is a model of meme propagation in on-line social networks such as Twitter. In a particular case of this model, each node in a network has a screen that is either empty or displays a meme. At each time-step, a node is selected at random and if they have a meme on their screen they 'tweet', i.e. the meme replaces whatever is on the screens of their neighbours. This is repeated until all screens display the meme.

The **Schelling** model illustrates the formation of segregation in communities. There are several variations of the original model [57, 58], but we describe the variant in [20], generalised to a network. Nodes in the network correspond to the locations at which people can live and there are edges between neighbouring

locations. Locations can be unoccupied or occupied by two different types of people. A person of one type is unhappy if the proportion of occupied neighbouring locations of the same type is below some threshold. At each time-step, an unhappy person is selected at random and moves to an unoccupied location chosen at random. This is repeated a fixed number of times or until everyone is happy.

Of the models described above, the SIS and SIR models evolve in continuous-time, whereas the voter, Twitter and Schelling model evolve in discrete-time; the nodes in the SIS, voter and Twitter models have two possible states (i.e. binary), whereas nodes in the SIR and Schelling model have three possibilities; in the SIS, SIR and voter models, only one node changes state at any instant in time, whereas in the Twitter and Schelling model, multiple nodes can change state at once.

4 General Markov Chain Model of Dynamics on Networks

In this section we will develop a general model of Markov chain dynamics on networks that captures the main features of the models described in Sect. 3. We assume that the dynamics take place on a network $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with automorphism group G, and that at any point in time, each vertex in the network is associated with one of a finite number of *vertex-states*. We denote the set of all possible vertex-states by $\mathcal{W}$. Thus for the SIR model $\mathcal{W} = \{\mathrm{S}, \mathrm{I}, \mathrm{R}\}$ and for the voter model $\mathcal{W} = \{+1, -1\}$. A *state* of the network $\mathcal{G}$ is a map $S : \mathcal{V} \to \mathcal{W}$, and the *state-space* of $\mathcal{G}$ over $\mathcal{W}$ is the set of all possible states, denoted $\mathbb{S} = \mathcal{W}^{\mathcal{V}}$ (i.e. the set of all functions from $\mathcal{V}$ to $\mathcal{W}$). Let M be the number of vertex-states and N be the number of vertices, then the number of states in state-space is M^N. Since the state-space is finite, we assume that is has been enumerated in some way, i.e. $\mathbb{S} = \{S_1, S_2, \ldots, S_{M^N}\}$. For example, if $\mathcal{V} = \{1, 2, 3, 4\}$ and $\mathcal{W} = \{+1, -1\}$, then one of the $2^4 = 16$ possible states, say $S_i \in \mathbb{S}$, has $S_i(1) = S_i(3) = +1$ and $S_i(2) = S_i(4) = -1$.

Automorphisms of the network permute vertices, so we need to define how they act on state-space. Let G be the automorphism group of $\mathcal{G}$, then we define the *action* of $g \in G$ on a state $S_i \in \mathbb{S}$ to be

$$(gS_i)(v) = S_i(g^{-1}v) \quad \text{for all } v \in \mathcal{V}, \tag{1}$$

i.e. the vertex-state of v in gS_i is the same as the vertex-state of $g^{-1}v$ in S_i. It is easy to prove that this is indeed an action of G on $\mathbb{S}$ in the group theoretic sense (see e.g. [22] for a precise definition of an action of a group).

We focus on dynamical processes that can be represented as Markov chains over the state-space $\mathbb{S}$, where the time-dependent probability distribution over state-space is $X(t) \in \mathbb{R}^{M^N}$, as defined in Sect. 2. Here we assume that the dynamics evolve in continuous-time, but in Sect. 6 we illustrate how to choose the continuous-time transition rates such that the embedded Markov chain corresponds to a discrete-time model of interest.

In our model, transitions will only occur between pairs of states where the differing vertex-states are of particular types.

Definition 1. *For $S_i, S_j \in \mathbb{S}$, the vertices $U \subset \mathcal{V}$ are called* **transition vertices** *if for every $u \in U$, $S_i(u) \neq S_j(u)$ and for every $v \in \mathcal{V} \setminus U$, $S_i(v) = S_j(v)$.*

For notational convenience let $\mathcal{W} = \{W_1, W_2, \ldots, W_M\}$, i.e. we assume that the vertex-states have been ordered in some way. For $1 \leq m \leq M$, a *sub-state* α is a vector in $\{0, 1, \ldots, N\}^M$ whose mth component α_m is the number of transition vertices in the vertex-state W_m. We denote the set of *allowable sub-states* by $\mathbb{A} \subset \{0, 1, \ldots, N\}^M$ and the set of *allowable sub-state transition pairs* by $\mathbb{T} \subset \mathbb{A} \times \mathbb{A}$. Of all possible sub-states, only a few need be allowable, and these must be identified by specific cases of our model. We use the set of allowable sub-state transition pairs to identify pairs of states between which transitions can occur.

Definition 2. *We call $S_i, S_j \in \mathbb{S}$ a* **transition pair**, *denoted*

$$S_i \overset{\alpha,\beta}{\sim} S_j \ , \tag{2}$$

if $(\alpha, \beta) \in \mathbb{T}$ and for each $1 \leq m \leq M$, α_m is the number of vertices u such that $S_i(u) = W_m$ and $S_i(u) \neq S_j(u)$, and β_m is the number of vertices v such that $S_j(v) = W_m$ and $S_j(v) \neq S_i(v)$.

We assume that the transition rate between a transition pair $S_i \overset{\alpha,\beta}{\sim} S_j$ depends only on α, β and a vector of metrics $\mu(t) \in \mathbb{R}^\Lambda$, for integer Λ. This assumption means that the transition rates do not depend on the particular set of transition vertices. We call such models *vertex homogeneous*. Furthermore, we assume that μ is invariant under network automorphisms. In more detail, if G is the automorphism group of the graph $\mathcal{G}$, then we assume there is a function $\eta_\mathcal{G} : \mathbb{S} \times \mathbb{S} \to \mathbb{R}^m$ that is a 'structural measure' [56], i.e. that satisfies $\eta_\mathcal{G}(S_i, S_j) = \eta_\mathcal{G}(gS_i, gS_j)$ for all $S_i, S_j \in \mathbb{S}$ and $g \in G$. As an example, if there is a single transition vertex then $\eta_\mathcal{G}$ might count the number of its neighbours in each vertex-state. We also assume that, for each transition pair $S_i \overset{\alpha,\beta}{\sim} S_j$ with $\mu = \eta_\mathcal{G}(S_i, S_j)$, the transition rate is given by a known function $q_{\alpha,\beta}(\mu)$. Thus the ijth component of the infinitesimal generator is

$$Q_{ij} = \begin{cases} q_{\alpha,\beta}(\mu) \text{ if } S_i \overset{\alpha,\beta}{\sim} S_j \text{ and } \mu = \eta_\mathcal{G}(S_i, S_j), \\ 0 \qquad\qquad\qquad\quad \text{otherwise.} \end{cases} \tag{3}$$

This completes the specification of our general model. To summarise, the model consists of a graph $\mathcal{G}$, a state-space $\mathbb{S}$ over the graph's vertices, a set of allowable sub-state transition pairs $\mathbb{T}$, a structural measure $\eta_\mathcal{G}$ and the transition rates $q_{\alpha,\beta}$ between all allowable sub-state transition pairs $(\alpha, \beta) \in \mathbb{T}$.

5 Graph Automorphism Lumping

In this section we prove that the automorphism group G of the network $\mathcal{G}$ can be used to lump the state-space $\mathbb{S}$ of the general model introduced in Sect. 4. First note that transition pairs persist under automorphisms.

Lemma 1. *If $S_i, S_j \in \mathbb{S}$ and $g \in G$, then $S_i \overset{\alpha,\beta}{\sim} S_j$ if and only if $gS_i \overset{\alpha,\beta}{\sim} gS_j$.*

This lemma is a direct consequence of Definition 2 and the action of the automorphism group on state-space (1), since $S_i(u) = (gS_i)(gu)$ for all $u \in \mathcal{V}$.

It follows from our definition (1) of the action of G on state-space that $\mathbb{S}$ is a G-set [22]. This means that we can partition state-space into equivalence classes where a pair of states $S_i, S_j \in \mathbb{S}$ are equivalent if and only if there is a $g \in G$ such that $S_i = gS_j$. This partition is known as the *orbit partition* of state-space.

Theorem 1. *For the infinitesimal generator Q defined in* (3)*, the orbit partition of state-space under the action of G is a lumping of $\mathbb{S}$.*

The proof of this theorem draws on that of Theorem 2.11 in [41]. Our proof is simplified by the fact that $\eta_{\mathcal{G}}$ is a structural measure, although it is generally easy to prove that particular forms of $\eta_{\mathcal{G}}$ satisfy this requirement.

Proof. Let $\mathbb{L} = \{\mathbb{L}_1, \mathbb{L}_2, \ldots, \mathbb{L}_r\}$ be the orbit partition of the state-space $\mathbb{S}$ under the action of G. Suppose $S_k \in \mathbb{L}_i$ and $S_l \in \mathbb{L}_j$. For any $S_n \in \mathbb{L}_j$, we can find a $g \in G$ such that $S_n = gS_l$. Thus let $S_m = gS_k \in \mathbb{L}_i$. From (3) and Lemma 1, if S_k and S_l do not form a transition pair then $Q_{kl} = Q_{mn} = 0$, otherwise $S_k \overset{\alpha,\beta}{\sim} S_l$ and $S_m \overset{\alpha,\beta}{\sim} S_n$ for $(\alpha, \beta) \in \mathbb{T}$. By definition, $\eta_{\mathcal{G}}(S_k, S_l) = \eta_{\mathcal{G}}(S_m, S_n)$. If $\mu = \eta_{\mathcal{G}}(S_k, S_l)$, then $Q_{kl} = Q_{mn} = q_{\alpha,\beta}(\mu)$. Since $\mathbb{S}$ is a G-set, it follows that g is a bijection on $\mathbb{L}_j$. Thus for all $S_k, S_m \in \mathbb{L}_i$ we have

$$\sum_{S_l \in \mathbb{L}_j} Q_{kl} = \sum_{S_n \in \mathbb{L}_j} Q_{mn},$$

and therefore $\mathbb{L}$ is a lumping. □

6 Example Applications

In this section we show that the dynamical processes on networks introduced in Sect. 3 correspond to particular cases of our general model. We start by describing how to choose the transition rates so that the embedded Markov chain of our general model coincides with a particular discrete-time model of interest. Recall that when Q_{ij} is the transition rate from state S_i to S_j of a continuous-time Markov chain, the transition *probability* of the embedded discrete-time Markov chain is $P_{ij} = -Q_{ij}/Q_{ii}$ [39]. The only part of our general model that is specific to the continuous-time setting is the transition rate function $q_{\alpha,\beta}$. Thus to translate a *discrete-time* model of dynamics on a graph $\mathcal{G}$ with state-space $\mathbb{S}$, allowable sub-state transition pairs $\mathbb{T}$ and structural measure $\eta_{\mathcal{G}}$, as described in Sect. 4,

we must be able to identify transition probabilities $P_{\alpha,\beta}(\mu)$ between allowable sub-state transition pairs that depend only $\mu = \eta_{\mathcal{G}}(S_i, S_j)$. Setting the transition rates to be proportional to the transition probabilities, i.e. $q_{\alpha,\beta} = \tau P_{\alpha,\beta}(\mu)$, results in a continuous-time Markov chain whose embedded discrete-time Markov chain is the discrete-time model of interest. Note that the constant of proportionality τ just sets the time-scale of the transition rates.

As an example, consider the voter model on a graph $\mathcal{G}$ with N vertices. Recall that vertices in the network have opinions ± 1 and at each time-step a vertex is picked at random and adopts the state of a randomly chosen neighbour. Thus the set of vertex states is $\mathcal{W} = \{+1, -1\}$ and let $W_1 = +1$ and $W_2 = -1$. Since only one vertex ever changes vertex-state at any time-step, the set of allowable sub-states is $\mathbb{A} = \{(1,0),(0,1)\}$, i.e. a single $+1$ vertex or a single -1 vertex. If $\alpha = (1,0)$ and $\beta = (0,1)$, then the set of allowable sub-state transition pairs is $\mathbb{T} = \{(\alpha,\beta),(\beta,\alpha)\}$. Transition pairs are then all pairs of states that differ in vertex-state at only one vertex. The probability that a given vertex with vertex-state ± 1 is picked is $1/N$ and if it is picked then it changes vertex-state with probability $n_{\mp}/k$, where $n_{\mp}$ is the number of neighbours it has with vertex-state ∓ 1 and k is its degree. Thus $\eta_{\mathcal{G}}(S_i, S_j)$ should return (n_+, n_-, k). Note that $n_{\pm}$ and k are invariant under automorphisms, since automorphisms do not change the degree of a vertex and the action defined in (1) does not change the vertex-states of neighbours. Thus this choice of $\eta_{\mathcal{G}}$ is a structural measure. If we choose $\tau = N$, then the transition rates are $q_{\alpha,\beta} = n_-/k$ and $q_{\beta,\alpha} = n_+/k$.

The voter model is an example of a binary state-space model with single vertex-state transitions, and is a particular case of the general binary [25, 26] and multi-state [21, 41] models with single vertex-state transitions proposed in the corresponding references. In these models, $S_i, S_j \in \mathbb{S}$ are transitions pairs if they have a single transition vertex. If the set of vertex-states is $\mathcal{W} = \{W_1, W_2, \ldots, W_M\}$, and the vertex-states of the transition vertex u are $S_i(u) = W_k$ and $S_j(u) = W_l$, then the transition rate from from S_i to S_j is given by $f_{W_k,W_l}(n_1, n_2, \ldots, n_M)$, where and n_m is the number of neighbours of u whose vertex-states are W_m. Note that each n_m is invariant under automorphisms. Specific forms of the transition rate f_{W_k,W_l} that correspond to several examples of well known models of dynamics on networks are given in [21, 26]. We can easily identify the features of our general model from those in [21, 25, 26, 41]. The allowable sub-states are $\mathbb{A} = \{\alpha^k \mid 1 \leq k \leq M\}$, where α^k_m is one if $m = k$ and zero otherwise. The set of allowable sub-state transition pairs consists of all distinct pairs of α^k, i.e. $\mathbb{T} = \{(\alpha^k, \alpha^l) \mid 1 \leq k, l \leq M \text{ and } k \neq l\}$. The structural measure $\eta_{\mathcal{G}}$ returns $\mu = (n_1, n_2, \ldots, n_M)$ and the transition rates are $q_{\alpha^k,\alpha^l}(\mu) = f_{W_k,W_l}(n_1, n_2, \ldots, n_M)$.

The general model proposed in this paper also allows for multi vertex-state transitions. For the Twitter model described in Sect. 3, let the set of vertex-states be $\mathcal{W} = \{E, A\}$, where $W_1 = E$ denotes empty and $W_2 = A$ denotes a meme. Recall that a vertex is picked at random and if their vertex-state is A then they change the vertex-state of all of their neighbours to A. Thus the allowable sub-state transitions correspond to cases where groups of vertices all

in state E that share a neighbour in state A all change to state A. If $k_{\max}$ is the largest degree of all vertices in $\mathcal{V}$, then the set of allowable sub-states is $\mathbb{A} = \{\alpha^k, \beta^k \mid 1 \leq k \leq k_{\max}\}$ where $\alpha^k = (k, 0)$ and $\beta^k = (0, k)$. The set of allowable sub-state transition pairs is $\mathbb{T} = \{(\alpha^k, \beta^k) \mid 1 \leq k \leq k_{\max}\}$. The transition vertices of a transition pair must share at least one neighbour that has vertex-state A, and the transition vertices are the only neighbours of these A vertices that have vertex-state E. Thus the transition probability is $P_{\alpha^k,\beta^k} = n_k/N$, where n_k is the number of vertices in vertex-state A whose only neighbours in vertex-state E are the k transition vertices. Since n_k only depends on the neighbourhoods of the transition vertices, it is invariant under automorphisms. If we choose the rate constant $\tau = N$ and structural measure $\eta_{\mathcal{G}}$ to return $\mu = n_k$, then the transition rates are $q_{\alpha^k,\beta^k}(n_k) = n_k$.

For the Schelling model, let the set of vertex-states be $\mathcal{W} = \{E, A, B\}$, where $W_1 = E$ denotes an unoccupied or empty site, and $W_2 = A$ and $W_3 = B$ are the different types of occupied vertices. Recall that a vertex in vertex-state A (B) is unhappy if the proportion of its neighbours in state A (B) is less than a threshold, denoted by ϕ. At each time-step, an unhappy vertex is selected at random and it chooses an empty location at random to move to, leaving its original vertex empty. Note that the total number of vertices in each vertex-state is conserved. The allowable sub-states are $\alpha = (1, 1, 0)$, i.e. one empty and one A vertex, and $\beta = (1, 0, 1)$, i.e. one empty and one B vertex. The set of allowable sub-state transitions are $\mathbb{T} = \{(\alpha, \alpha), (\beta, \beta)\}$. Note that a transition exchanges the vertex-states, but the sub-states remain the same. Note that, by definition, a pair of identical states can not form a transition pair. If an occupied vertex is unhappy, then it is selected at random with probability $1/n_{\mathrm{u}}$, where n_{u} is the number of unhappy vertices. An unoccupied vertex is then selected at random with probability $1/n_E$, where n_E is the number of empty vertices. Let $\eta_{\mathcal{G}}$ return $\mu = (\delta, n_{\mathrm{u}})$, where $\delta = 1$ if the occupied transition vertex is unhappy and $\delta = 0$ otherwise. Note that δ and n_{u} only depend on the vertex-states in neighbourhoods, so are invariant under automorphisms. If we choose the rate constant $\tau = n_E$, then the transition rates are

$$q_{\alpha,\alpha}(\mu) = q_{\beta,\beta}(\mu) = \begin{cases} \frac{1}{n_{\mathrm{u}}} & \text{if } \delta = 1, \\ 0 & \text{otherwise.} \end{cases}$$

The allowable sub-states and sub-state transition pairs, structural metrics and transition rates can be identified in a similar way for other models, such as the naming game, the Bonabeau model, the Sznajd model and majority rule model.

7 Discussion

In this paper we have introduced a general Markov chain model of dynamical processes on networks and connected it to a range of well known models. One of the most important assumptions of our model is vertex homogeneity, i.e. that the dynamical behaviour is independent of the particular vertices involved.

Relaxing this assumption would certainly weaken the state-space compression resulting from network symmetries, in addition to complicating the identification of transition pairs. However, it is our belief that such heterogeneity may not necessarily add qualitatively new dynamics. For example, in traffic modelling, the stability criteria of car-following models is qualitatively the same for both homogeneous and heterogeneous driver behaviour [63]. Our assumption that the transition rates depend only on a structural measure is fundamental to the ability to lump the state-space on the basis of graph automorphisms. While this may seem restrictive, if it were not true, the model would necessarily ignore connectivity structure and negate the point of posing the model on a network in the first place.

The work presented here focuses on the formulation of dynamical processes on networks as Markov chains. It's clear that for networks of even fairly small sizes (e.g 40 vertices) with binary vertex-states, working with the full state-space is prohibitive. However, there has recently been increased interest in the analysis of dynamics on small networks [35, 43] and exploiting symmetries can have a significant impact on the size of networks that can be studied [36]. It would also be interesting to understand what can be said about the dynamics on the full state-space from the 'microscopic' vertex transition rules, e.g. whether the dynamics are ergodic or the nature of any absorbing states. It has recently been shown that real-world networks have a surprising amount of symmetry [45]. In a future publication, we will explore the impact of such symmetries on lumping for the general model introduced here.

Exact analyses of Markov chain dynamics on networks are rare since authors typically resort to mean-field approximations. These can do surprisingly well, but it's unclear what model and network features ensure this [28]. The simplest mean-field approximation corresponds to the 'well-mixed' case, i.e. the fully connected network or complete graph. The simplicity of this case results from the symmetry present in the complete graph [41, 59]. It would be interesting to identify other graphs where symmetry gives rise to significant lumping and better dynamical approximations. However, our experience is that finding such graphs is extremely difficult as they necessarily have to have a large amount of symmetry, and this makes computing the lumped state-space difficult. Alternatively, there is scope to develop more accurate mean-field approximations, similar to those in [21, 25, 26], using the general model proposed here.

In summary, we have presented a general model of dynamics on networks. One of the main purposes of this general model is to show how a wide range of dynamical processes on networks can be formulated as Markov chains. Our hope is that this will help to formalise the analytical treatment of more complex and more realistic models, ultimately improving our understand of the phenomena that they represent.

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