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Descending congruences of theta lifts on GSp_4

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Abstract

We study the question of when a congruence between two theta lifts on $\mathrm{GSp}_{4/\mathbb{Q}}$ descends to a congruence on modular forms on GL_2 over a quadratic field. In order to accomplish that, we use the theory of the local theta correspondence between similitude orthogonal groups and the similitude symplectic group GSp_4, together with a classification for the degeneration modulo a prime of conductors for the L-parameters of irreducible admissible representations of GSp_4 over a non-archimedean local field. We explain that this is unlikely to be used in conjunction with existing results on congruences for $\mathrm{GSp}_{4/\mathbb{Q}}$ to deduce a theory of congruences over imaginary quadratic fields. On the other hand, we prove a result which does give some such congruence results by twisting.

Keywords: Siegel modular forms, congruences, theta lifts,
2000 MSC: 11F27,
2000 MSC: 11F46

1. Introduction

Congruences between modular forms have been at the heart of many recent breakthroughs within number theory. For modular forms over $\mathbb{Q}$, the theory is now very well established, notably thanks to work of Carayol, Mazur, Ribet, Edixhoven and Coleman-Voloch. Over totally real fields, generalisations of these approaches were given by Fujiwara, Rajaei and the second author; these results are now subsumed within the deformation-theoretic work of Gee ([11]).

Attaching Galois representations to modular forms over imaginary quadratic fields is considerably more indirect than for totally real fields, since there is no

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natural algebro-geometric object in which to find the representations. Recent work of Harris-Lan-Taylor-Thorne ([14]) and Scholze ([23]) use p-adic methods to give a general construction of such Galois representations (and, indeed, Scholze's work shows that Galois representations are even associated to torsion classes); prior to this, Galois representations had only been attached to automorphic representations whose central characters were Galois invariant. The earlier method originates with ideas of Taylor, as explained in the paper of Harris-Soudry-Taylor ([15]), and it is refined by Berger-Harcos ([2]) and Mok ([19]) where local-global compatibility[3] within the local Langlands correspondence is proved (up to Frobenius-semisimplicity). This method is of interest to us since it uses the theta correspondence between orthogonal and symplectic similitude groups, and the local theory is very explicit in terms of the L-parameters.

It is presumably still be of interest to understand congruences. However, work of Calegari and Mazur ([4]) suggests that one should not expect the full range of congruences as in the totally real case, and indeed one should expect congruences to be between p-adic objects, rather than simply classical automorphic representations. Nevertheless, it seems clear from looking at tables in the theses of Cremona ([7]) and Lingham ([18]) that some congruences do exist, although the tables are limited to a rather small number of Hecke eigenvalues. For example, the modular forms f_4 and f_{11} on p.97 of [18] appear to be congruent modulo 2; the forms f_2 and f_9 appear to be congruent modulo 3 (except for coefficients dividing the level, unsurprisingly), and there are many other examples to be seen from these tables (see [27]). Perhaps there are results stating that p-adic objects of small weight and level are necessarily classical, or alternatively perhaps the examples one can see are base changes of classical congruences.

This paper arose from a possible, almost entirely speculative programme to construct congruences. We restrict our attention to those automorphic representations with Galois invariant central character, so that we can use the methods of [15]. In this direction, we are seeking level lowering congruences for GL_2 over imaginary quadratic fields, like the situation for classical modular forms over $\mathbb{Q}$, via a detailed analysis of the local representation theory of GSp_4 (see [9] and [10]). In particular, we consider the L-parameters of all posible local representations of GSp_4 which are obtained as theta lifts from GL_2 in the sense of Harris-Soudry-Taylor. We study these L-parameters at the level of the inertia group and we seek all possible conductor lowering congruences. Also, a twisting result supporting our analysis is proved at the end of the paper.

We expect that everything in this paper will generalise to CM fields since the main study is a purely local study of theta lifts and their congruences, but work over imaginary quadratic fields for ease of exposition. In the rest of this section, we summarise the idea.

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[3] Note that Mok in [19] is assuming distinct Satake parameters in order to prove crystallinity at places above ℓ; nevertheless, Varma announced a crystallinity result for GL_n, not yet available, without this assumption.

Berger for useful discussions.

1.1. Galois representations associated to automorphic representations on GL_2 *over an imaginary quadratic field*

Let us first summarise Taylor's ideas ([15]). Let K be an imaginary quadratic field. Write $\mathcal{O}_K$ for its ring of integers.

Suppose that π is a regular algebraic cuspidal automorphic representation on $\mathrm{GL}_{2/K}$, i.e., a representation of $\mathrm{GL}_2(\mathbb{A}_K)$ with Langlands parameter $W_{\mathbb{C}} = \mathbb{C}^\times \to \mathrm{GL}_2(\mathbb{C})$ at the archimedean place, given by $z \mapsto \mathrm{diag}(z^{1-k}, \bar{z}^{1-k})$ for some integer $k \geq 2$. We require that the central character of π factors through $N : \mathbb{A}_K^\times \longrightarrow \mathbb{A}^\times$.

It is relatively straightforward to give a bijection between such automorphic representations on $\mathrm{GL}_{2/K}$ with specific extra choices of data and automorphic representations on a certain 4-dimensional orthogonal group: see [15], Proposition 2, for the statement. Making such a choice of additional data, we get a representation $\hat{\pi}$ of this orthogonal group, and [15] shows that there is a theta correspondence with a representation Π of $\mathrm{GSp}_{4/\mathbb{Q}}$.

Work of Taylor, Weissauer and Laumon (see [2]) allows us to attach 4-dimensional Galois representations to cuspidal holomorphic[4] automorphic representations on $\mathrm{GSp}_{4/\mathbb{Q}}$, via the ℓ-adic cohomology of the corresponding Siegel 3-folds; some extra work involving twisting the automorphic representations by quadratic characters from a dense set ([2]) shows that these are induced from 2-dimensional representations of $\mathrm{Gal}(\overline{K}/K)$, and these are the desired Galois representations associated to π.

This method suggests an idea for constructing congruences between automorphic representations on K; one starts with a suitable automorphic representation π on $\mathrm{GL}_{2/K}$, takes the theta lift Π to $\mathrm{GSp}_{4/\mathbb{Q}}$, then tries to find another theta lift Π' on $\mathrm{GSp}_{4/\mathbb{Q}}$ with a congruent 4-dimensional Galois representation, lifting from an automorphic representation π' on $\mathrm{GL}_{2/K}$, and check whether the corresponding 2-dimensional Galois representations of $\mathrm{Gal}(\overline{K}/K)$ are congruent.

In fact, we will mention later that just about none of the results are yet known in the generality needed to get a good theory of congruences. Further, it seems certain from the discussion on pp.101–104 of [4] that we should not expect a good theory.

1.2. Theta lifting to GSp_4 *over* $\mathbb{Q}$

Galois representations are attached by taking the global theta lift $\Pi = \bigotimes_v \Pi_v$ of $\pi = \bigotimes_w \pi_w$ (or rather $\hat{\pi}$ on the orthogonal group) to $\mathrm{GSp}_{4/\mathbb{Q}}$. Then work of Weissauer and Laumon (see Theorem 3.1 of [2]) associates a 4-dimensional Galois representation to Π.

Given the original automorphic representation π on $\mathrm{GL}_{2/K}$, we can understand the representation Π on $\mathrm{GSp}_{4/\mathbb{Q}}$ in terms of the local components of the

[4] At the archimedean place, a holomorphic limit of discrete series representation of weight $(k, 2)$.

automorphic representation; these calculations are done in Appendix A of Gan-Ichino ([8]). When $p\mathcal{O}_K = \mathfrak{p}\overline{\mathfrak{p}}$, the theta lift Π_p depends on the pair $(\pi_\mathfrak{p}, \pi_{\overline{\mathfrak{p}}})$; on the other hand, when we have either $p\mathcal{O}_K = \mathfrak{p}$ or $p\mathcal{O}_K = \mathfrak{p}^2$, the local theta lift Π_p depends only on the representation $\pi_\mathfrak{p}$. The calculations are summarised explicitly in [16] (and [27]).

At this stage, we have lifted the original π on $\mathrm{GL}_{2/K}$ to a cuspidal automorphic representation Π on $\mathrm{GSp}_{4/\mathbb{Q}}$, and understand the local behaviour explicitly. We note that there is an L-function criterion for a cuspidal automorphic representation Π on $\mathrm{GSp}_{4/\mathbb{Q}}$ to arise in this way as a theta lift (see Theorem 7.1 of [17]).

We will discuss later which are the difficulties in finding congruent automorphic representations on $\mathrm{GSp}_{4/\mathbb{Q}}$, but for the moment, we suppose that we can find a congruent automorphic representation Π' on $\mathrm{GSp}_{4/\mathbb{Q}}$. We would like to reverse the process, and descend it to an automorphic representation π' on $\mathrm{GL}_{2/K}$.

It turns out that there are two issues. The first is to ensure that it arises as the result of some theta lift from the orthogonal group mentioned earlier, but, as already mentioned, there is an L-function criterion for this, due to Kudla and Rallis. Indeed, if Π' is a theta lift from $\mathrm{GSO}(V)$ for some 4-dimensional quadratic space V, then V has a discriminant, given by a separable quadratic algebra L (either a quadratic field, or $\mathbb{Q} \times \mathbb{Q}$). In fact, Π' is a theta lift from some $\mathrm{GSO}(V)$ with discriminant algebra L if and only if the twisted partial (degree 5) standard L-function $L^S(s, \Pi' \times \chi_L)$ has a pole at $s = 1$, where χ_L denotes the quadratic Hecke character associated to L, which is trivial when $L = \mathbb{Q} \times \mathbb{Q}$.

Assuming this criterion, Π' descends to an orthogonal group. We will suppose that this happens, so that we now have two congruent theta lifts Π and Π' on $\mathrm{GSp}_{4/\mathbb{Q}}$. Given a theta lift of this form, it then descends to a representation π' on $\mathrm{GL}_{2/L}$ for some quadratic field L. The second problem is to ensure that $L = K$.

It is with this second problem that this paper is principally concerned. Although we phrase results below in terms of a more general programme, we could rephrase our results below in terms of the following situation: Suppose that $\pi_{/\mathrm{GL}_{2/K}}$ and $\pi'_{/\mathrm{GL}_{2/L}}$ are two automorphic representations with theta lifts Π and Π' to $\mathrm{GSp}_{4/\mathbb{Q}}$ respectively, which are congruent. Then we determine a number of explicit situations where $K = L$, at least assuming full local-global compatibility.

It would presumably be of interest also to study situations where K and L may be different, and we would get a notion of congruence between automorphic representations on GL_2 over two different quadratic fields, perhaps even between a real quadratic field (where congruences are well understood) and an imaginary quadratic field, which might have some interesting modularity applications.

1.3. Notation

Before we proceed to our analysis, we explain some notation and terminology that we are going to use.

Firstly, if F is a non-archimedean local field and $x \in F^{\times}$, we denote by $|x|$ the normalized absolute value of x. For the irreducibile admissible representations of GSp_4 over a non-archimedean local field, we are going to follow the notation from the book [22], where the representations belong to one of the eleven types of table A.1 of [22]. For the infinite dimensional irreducible admissible representations of GL_2 over a non-archimedean local field, we denote by $\chi_1 \times \chi_2$ the principal series representations parabolically induced from the pair (χ_1, χ_2), by $(\mu| \,|^{1/2})St_{\mathrm{GL}_2}$ the Steinberg representation for GL_2 twisted by the character μ, and by $BC(L/F, \psi)$ the supercuspidal representation which is obtained as a base change from the character ψ of $L^{\times}$, where L/F is a quadratic extension.

In this paper, we extensively use the terms "regular", "invariant", "distinguished" for representations[5] of GSO over some 4-dimensional quadratic space. We recall the definitions briefly. Let V be a 4-dimensional quadratic space defined over some local field F of characteristic different from 2; moreover, let d be its discriminant. An irreducible admissible representation $\tilde{\pi}$ of $\mathrm{GSO}(V)$ will be called *regular* if the induced representation $\hat{\pi}^+$ of $\tilde{\pi}$ to $\mathrm{GO}(V)$ is irreducible. If $\tilde{\pi}$ is not regular, we will say it is *invariant*; in this latter case, the induced representation of $\tilde{\pi}$ to $\mathrm{GO}(V)$ will be of the form $\hat{\pi}^+ \oplus \hat{\pi}^-$, where $\hat{\pi}^+$ and $\hat{\pi}^-$ are irreducible admissible representations of $\mathrm{GO}(V)$. Lastly, $\tilde{\pi}$ will be called *distinguished* if it is invariant and there is an anisotropic vector $w \in V$ such that $\mathrm{Hom}_{\mathrm{SO}(W)}(\tilde{\pi}, 1) \neq 0$, where W is the orthogonal complement of Fw in V, or if $d \neq 1$ and $\tilde{\pi}$ is invariant and 1-dimensional ("boundary" case). For a more comprehensive discussion on these terms one is advised to see also Section 3 of [21], or Section 5.2 of [27].

As far as the local theta correspondence for the similitude pair $\mathrm{GO}(V)$ and GSp_4 is concerned[6], if σ is a representation of $\mathrm{GO}(V)$, then we denote its local theta lift to GSp_4 by $\Theta(\sigma)$, and the unique non-zero irreducible quotient of $\Theta(\sigma)$ by $\theta(\sigma)$.

In addition, we will be utilizing the matrices $N_1, N_2, N_3, N_4, N_5, N_6$ defined in Appendix A.5 in [22], which represent the nilpotent elements of the L-parameters associated to each non-supercuspidal representation of GSp_4 over F. The symbol "$*$" will denote a non-zero entry of a matrix, and the blank entries of a matrix are just zeros. Finally, if we have a quadratic extension of fields L/F, we will denote the non-trivial element of the corresponding Galois group by c.

[5]Representations denoted by $\tilde{\pi}$, will be representations of the orthogonal group GSO over some particular 4-dimensional quadratic space.

[6]The strong Howe duality in this case has been proved by Roberts (see Theorem 1.8 in [21]).

2. Conductor-lowering congruences between theta lifts

We now begin the study of congruences between theta lifts. For simplicity, we will restrict attention to those which reduce the conductor. Throughout, our notation and terminology for local representations of GSp_4 over p-adic fields follows the book of Roberts and Schmidt ([22]); in particular, we refer the reader to the tables in Appendix A of [22] for more details on the classification of non-supercuspidal representations of GSp_4 over a local field into types. In this paper, we will use the notion of inertial types in order to work with the congruences at the level of the inertia group; this is so that the reduction of the conductor can be clearly seen.

Suppose that F is a non-archimedean local field (in what follows, we will generally only need to take the case $F = \mathbb{Q}_p$). Recall that τ is an *inertial type* for G_F if τ is a representation of I_F with open kernel which extends to a representation of G_F. We say that an ℓ-adic representation of G_F *has inertial type* τ if the corresponding Weil-Deligne representation restricted to I_F is equivalent to τ. Also, when we say that an inertial type τ extends to a Weil-Deligne representation, we will mean that there exists a Weil-Deligne representation which is equivalent to τ when restricted to the inertia subgroup.

In the following, we will say that the conductor of a representation *degenerates* modulo ℓ to mean that the conductor of the L-parameter becomes strictly smaller when the L-parameter is considered modulo ℓ. For a detailed description of the conductor of an L-parameter one may consult Sections 4.1 and 4.2 of [27]. In general, congruences will be considered under the reciprocity map.

Our strategy is to list the possible theta lifts, and then to try to identify the possible level lowering congruences via the inertial types. All theta lifts are listed in Table 1 of [16]. We are only interested in those which come from global automorphic representations on GL_2 over quadratic fields, and we therefore consider only those where the local components over GL_2 are principal series, Steinberg or supercuspidal. In this section we will list the possible congruences between inertial types, and in the next we will recall how these may arise as a theta lift.

We will work with an imaginary quadratic field K and a rational prime p. In Subsections 2.1, 2.2, 2.3, we consider the cases where p splits in K, where p does not split in K but is ramified, and where p does not split in K but is unramified, respectively. For each of these cases, we consider all possible local components of an automorphic representation for GL_2 over K, and we see how they lift via the theta correspondence to irreducible admissible representations for GSp_4. Since we are interested in congruences which lower the conductor of the L-parameters of the latter, we note that these congruences are mostly inherited from conductor-lowering congruences in the GL_2 situation (see [5]).

In other words, given the global theta lift $\Pi = \bigotimes_v \Pi_v$, we are seeking, for each local L-parameter, possible congruences (via inertial types) which we may use in order to produce all potential congruences which are also local theta lifts.

Fix an odd rational prime ℓ, and consider a prime p for which $p \neq \ell$. Recall that when $p\mathcal{O}_K = \mathfrak{p}\overline{\mathfrak{p}}$, the local theta lift Π_p depends on the pair $(\pi_{\mathfrak{p}}, \pi_{\overline{\mathfrak{p}}})$; on the other hand, when we have either $p\mathcal{O}_K = \mathfrak{p}$ or $p\mathcal{O}_K = \mathfrak{p}^2$, the local theta lift Π_p depends only on the representation $\pi_\mathfrak{p}$. Below, we write down the L-parameters associated to the local theta lifts Π_p, and their possible congruences mod ℓ for which the conductor degenerates to something smaller. For a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we write A' for $\begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$.

Definition 2.1. We will say that a representation of $\mathrm{GSp}_4(\mathbb{Q}_p)$ is a local theta lift in the

1. *split case*, when it arises as a theta lift from a representation of $\mathrm{GO}(V)$, where V is a 4-dimensional quadratic space with corresponding discriminant algebra $\mathbb{Q}_p \times \mathbb{Q}_p$;
2. *ramified case*, when it arises as a theta lift from a representation of $\mathrm{GO}(V)$, where V is a 4-dimensional quadratic space with corresponding discriminant algebra a ramified quadratic extension of $\mathbb{Q}_p$;
3. *inert case*, when it arises as a theta lift from a representation of $\mathrm{GO}(V)$, where V is a 4-dimensional quadratic space with corresponding discriminant algebra an unramified quadratic extension of $\mathbb{Q}_p$.

As already remarked, we do not need all the theta lifts occurring in Table 1 of [16]; we are only interested in the cases where the local components $\pi_\mathfrak{p}$ on GL_2 are principal series, Steinberg or supercuspidal; these are listed in § 5.3.3 in [27].

From Table 1 of [16] or § 5.3.3 of [27] we see the following:

- The (non-supercuspidal) representations of GSp_4 of types I, Va and VIIIa can occur as theta lifts in both the split and non-split cases.
- However, representations of types IIa, IIIb, IVc, VIa, X and XIa occur only as theta lifts in the split case, and representations of types IIIa, VII, VIIIb, IXa and IXb in the nonsplit case.
- The remaining types do not occur as theta lifts from local representations on GL_2 which are principal series, Steinberg or supercuspidal.

This means that if there is a congruence between Π_p and some representation of the first group, we cannot infer anything about the splitting behaviour of p in L; however, if Π_p is congruent to some representation of one of the types from the second group, we can determine the splitting behaviour of p in L precisely; if the congruence falls in the last group, then it is not a local theta lift. We will discuss this classification in much more detail in the next section.

2.1. Split case

Suppose we have a prime p such that $p\mathcal{O}_K = \mathfrak{p}\overline{\mathfrak{p}}$. Fix isomorphisms $K_\mathfrak{p} \cong \mathbb{Q}_p$ and $K_{\overline{\mathfrak{p}}} \cong \mathbb{Q}_p$. Suppose that we have a pair $(\pi_\mathfrak{p}, \pi_{\overline{\mathfrak{p}}})$ of irreducible admissible representations of $\mathrm{GL}_2(K_\mathfrak{p})$ and $\mathrm{GL}_2(K_{\overline{\mathfrak{p}}})$ respectively, with equal central characters. In this case, an invariant irreducible representation of $\mathrm{GSO}(V)$ is a pair

$(\pi_{\mathfrak{p}}, \pi_{\overline{\mathfrak{p}}})$ with $\pi_{\mathfrak{p}} \cong \pi_{\overline{\mathfrak{p}}}$, and it is necessarily distinguished (see Propositions 3.1 and 4.1 of [20] respectively); that is, the representation $(\pi_{\mathfrak{p}}, \pi_{\overline{\mathfrak{p}}})^-$ does not participate in the theta correspondence (see Theorem 6.8 in [20]). We consider the following cases:

1. $\pi_{\mathfrak{p}} \cong \pi_{\overline{\mathfrak{p}}}$ is a supercuspidal representation; then

$$\Theta((\pi_{\mathfrak{p}}, \pi_{\overline{\mathfrak{p}}})^+) = \theta((\pi_{\mathfrak{p}}, \pi_{\overline{\mathfrak{p}}})^+) = \tau(S, \pi),$$

i.e., a generic representation of type VIIIa. Such a representation has L-parameter Φ_p with semisimple part

$$\rho_0 : w \mapsto \begin{pmatrix} \phi_{\pi_{\mathfrak{p}}}(w)' & \\ & \phi_{\pi_{\mathfrak{p}}}(w) \end{pmatrix},$$

and nilpotent part $N = 0$. Here $\phi_{\pi_{\mathfrak{p}}} : W_{\mathbb{Q}_p} \to \mathrm{GL}_2(\mathbb{C})$ is the L-parameter of $\pi_{\mathfrak{p}}$.
In this case, the congruent inertial types of smaller level can be read off from the corresponding situation for GL_2-supercuspidals, which only happen when the defining character of the supercuspidal is unramified, the types which occur are the trivial case $\begin{pmatrix} 1 &&& \\ & 1 && \\ && 1 & \\ &&& 1 \end{pmatrix}$, and $\begin{pmatrix} 1 & * && \\ & 1 && \\ && 1 & -* \\ &&& 1 \end{pmatrix}$.
In the next section, we will analyse these two; the first can occur in both the split and nonsplit cases, but the second can only occur in the inert case (it lifts to a representation of type IIIa, and it does not include a nontrivial quadratic character of $\mathbb{Z}_p^\times$). They occur when the L-parameter ϕ is congruent on inertia to a reducible representation, and these correspond to congruences between supercuspidal representations on GL_2 and Steinberg or unramified principal series. These congruences can only occur under the condition $N(\mathfrak{p}) \equiv -1 \pmod{\ell}$ (see [5]).
2. If $\pi_{\mathfrak{p}} \not\cong \pi_{\overline{\mathfrak{p}}}$ are both supercuspidal, then $\Theta((\pi_{\mathfrak{p}}, \pi_{\overline{\mathfrak{p}}})^+) = \theta((\pi_{\mathfrak{p}}, \pi_{\overline{\mathfrak{p}}})^+)$ is a generic supercuspidal representation of $\mathrm{GSp}_4(\mathbb{Q}_p)$ with L-parameter

$$\Phi_p : w \mapsto \phi_{\pi_{\mathfrak{p}}}(w) \oplus' \phi_{\pi_{\overline{\mathfrak{p}}}}(w).$$

This is a symplectic direct sum, defined as follows: if $\phi_{\pi_{\mathfrak{p}}}(w) = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$ and $\phi_{\pi_{\overline{\mathfrak{p}}}}(w) = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$ have the same determinant, define

$$\phi_{\pi_{\mathfrak{p}}}(w) \oplus' \phi_{\pi_{\overline{\mathfrak{p}}}}(w) = \begin{pmatrix} a_1 &&& b_1 \\ & a_2 & b_2 & \\ & c_2 & d_2 & \\ c_1 &&& d_1 \end{pmatrix} \in \mathrm{GSp}_4.$$

Because the two L-parameters are different, either or both may have L-parameters congruent to something fixing a line when restricted to inertia;

again, this will require $N(\mathfrak{p}) \equiv -1 \pmod{\ell}$. In this case, we may have congruent inertial types of the following forms:

$$\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}; \quad \begin{pmatrix} 1 & & & * \\ & 1 & * & \\ & & 1 & \\ & & & 1 \end{pmatrix};$$

$$\begin{pmatrix} 1 & & & \\ & 1 & * & \\ & & 1 & \\ & & & 1 \end{pmatrix} \cong \begin{pmatrix} 1 & & & * \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}; \quad \begin{pmatrix} 1 & & & \\ & a & b & \\ & c & d & \\ & & & 1 \end{pmatrix}; \quad \begin{pmatrix} 1 & & & * \\ & a & b & \\ & c & d & \\ & & & 1 \end{pmatrix}.$$

The first two inertial types extend to representations which arise as theta lifts in both the split and the non-split case; the last three can only be theta lifts in the split case.

3. If $\pi_{\mathfrak{p}} \cong \pi_{\overline{\mathfrak{p}}} \cong (\mu|\ |^{1/2})St_{\mathrm{GL}_2}$, then

$$\Theta((\pi_{\mathfrak{p}}, \pi_{\overline{\mathfrak{p}}})^+) = \theta((\pi_{\mathfrak{p}}, \pi_{\overline{\mathfrak{p}}})^+) = \tau(S, \mu);$$

i.e., of type VIa. Its L-parameter Φ_p is given by

$$\rho_0 : w \mapsto \begin{pmatrix} |w|\mu(w) & & & \\ & |w|\mu(w) & & \\ & & \mu(w) & \\ & & & \mu(w) \end{pmatrix}$$

and $N = N_3$. That is,

$$\Phi_p \sim \begin{pmatrix} |\ |\mu & & & * \\ & |\ |\mu & * & \\ & & \mu & \\ & & & \mu \end{pmatrix}.$$

We may have congruent inertial types of the following forms:

$$\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}; \quad \begin{pmatrix} 1 & & & * \\ & 1 & * & \\ & & 1 & \\ & & & 1 \end{pmatrix}.$$

The latter, when $\ell \neq 2$, extends to a representation of type VIa; then we have a congruence between a ramified Steinberg $(\mu|\ |^{1/2})St_{\mathrm{GL}_2}$ and an unramified Steinberg representation, and such a congruence implies that $N(\mathfrak{p}) \equiv 1 \pmod{\ell}$, as in [5]. Representations of type VIa are theta lifts in the split case. The assumption $\ell \neq 2$ is required in order to avoid the situation where the inertial type extends to representations of type Va which can be theta lifts in both the split and the non-split case; note that a quadratic character reduces to a trivial character mod ℓ only when $\ell = 2$.

4. If $\pi_{\mathfrak{p}} = (|\ |^{1/2}\mu_1)St_{\mathrm{GL}_2}$ and $\pi_{\overline{\mathfrak{p}}} = (|\ |^{1/2}\mu_2)St_{\mathrm{GL}_2}$, with $\mu_1 \neq \mu_2$ but $\mu_1^2 = \mu_2^2$, then

$$\Theta((\pi_{\mathfrak{p}}, \pi_{\overline{\mathfrak{p}}})^+) = \theta((\pi_{\mathfrak{p}}, \pi_{\overline{\mathfrak{p}}})^+) = \delta(\left[\frac{\mu_1}{\mu_2}, |\ |\frac{\mu_1}{\mu_2}\right], \mu_2);$$

i.e., of type Va. Such a representation has L-parameter Φ_p with semisimple part

$$\rho_0 : w \to \begin{pmatrix} |w|\mu_2(w) & & & \\ & |w|\mu_1(w) & & \\ & & \mu_1(w) & \\ & & & \mu_2(w) \end{pmatrix},$$

and nilpotent part $N = N_3$. That is,

$$\Phi_p \sim \begin{pmatrix} |\ |\mu_2 & & & * \\ & |\ |\mu_1 & * & \\ & & \mu_1 & \\ & & & \mu_2 \end{pmatrix}.$$

We have the following inertial types with monodromy operator of even rank:

$$\begin{pmatrix} 1 & & & * \\ & \chi & * & \\ & & \chi & \\ & & & 1 \end{pmatrix}; \quad \begin{pmatrix} \chi & & & * \\ & 1 & * & \\ & & 1 & \\ & & & \chi \end{pmatrix}; \quad \begin{pmatrix} 1 & & & * \\ & 1 & * & \\ & & 1 & \\ & & & 1 \end{pmatrix}; \quad \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix};$$

and with monodromy operator of rank 1:

$$\begin{pmatrix} 1 & & & \\ & \chi & * & \\ & & \chi & \\ & & & 1 \end{pmatrix} \cong \begin{pmatrix} \chi & & & * \\ & 1 & & \\ & & 1 & \\ & & & \chi \end{pmatrix}; \quad \begin{pmatrix} 1 & & & \\ & 1 & * & \\ & & 1 & \\ & & & 1 \end{pmatrix} \cong \begin{pmatrix} 1 & & & * \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix},$$

where χ is a character, non-trivial on the inertia group. The inertial types of even rank monodromy operator extend to representations which can be local theta lifts in both the split and the non-split case. The inertial types with monodromy operator of rank 1 extend to representations of type IIa, IVc, Vb, or VIc. If such a representation is a local theta lift, then it is necessarily a theta lift in the split case (types IIa, IVc).

5. Let $\pi_{\mathfrak{p}}$ be a supercuspidal and $\pi_{\overline{\mathfrak{p}}} = (|\ |^{1/2}\mu)St_{\mathrm{GL}_2}$. Then

$$\Theta((\pi_{\mathfrak{p}}, \pi_{\overline{\mathfrak{p}}})^+) = \theta((\pi_{\mathfrak{p}}, \pi_{\overline{\mathfrak{p}}})^+) = \delta(\mu^{-1}\pi_{\mathfrak{p}}, \mu),$$

which is a representation of type XIa. The L-parameter Φ_p is given by the semisimple part

$$\rho_0 : w \mapsto \begin{pmatrix} |w|\mu(w) & & \\ & |w|^{1/2}\mu(w)\phi_{\pi_{\mathfrak{p}}}(w) & \\ & & \mu(w) \end{pmatrix}$$

and the monodromy operator $N = N_2$; here $\phi_{\pi_\mathfrak{p}}$ is the L-parameter associated to $\pi_\mathfrak{p}$. That is,

$$\Phi_p \sim \begin{pmatrix} |\ |\mu & & * \\ & |\ |^{1/2}\mu\phi_{\pi_\mathfrak{p}} & \\ & & \mu \end{pmatrix}.$$

Note that if a character and a supercuspidal representation have conductors that degenerate mod ℓ, then $\ell = 2$; thus, we are excluding a situation like this. We consider three cases:

Let μ above, be an unramified character. Then we can choose the following inertial types:

$$\begin{pmatrix} 1 & & & \\ & a & b & \\ & c & d & \\ & & & 1 \end{pmatrix}; \quad \begin{pmatrix} 1 & & & \\ & 1 & * & \\ & & 1 & \\ & & & 1 \end{pmatrix} \cong \begin{pmatrix} 1 & & & * \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix};$$

$$\begin{pmatrix} 1 & & & * \\ & 1 & * & \\ & & 1 & \\ & & & 1 \end{pmatrix}; \quad \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}.$$

The last of these does not extend to a representation which is a theta lift. The first two can extend to representations coming from theta lifts, but only in the split case. The first congruence is obtained via a degeneration of the matrix N_2 to the zero matrix mod ℓ; while the second one is obtained via the reduction of the supercuspidal representation to an unramified Steinberg, and this requires $N(\mathfrak{p}) \equiv -1 \pmod{\ell}$.

Let μ be a ramified character that stays ramified mod ℓ. In this case, we consider the inertial types:

$$\begin{pmatrix} \chi & & & \\ & a & b & \\ & c & d & \\ & & & \chi \end{pmatrix}; \quad \begin{pmatrix} \chi & & & \\ & 1 & * & \\ & & 1 & \\ & & & \chi \end{pmatrix};$$

$$\begin{pmatrix} \chi & & & * \\ & 1 & * & \\ & & 1 & \\ & & & \chi \end{pmatrix}; \quad \begin{pmatrix} \chi & & & \\ & 1 & & \\ & & 1 & \\ & & & \chi \end{pmatrix}; \quad \begin{pmatrix} \chi & & & * \\ & 1 & & \\ & & 1 & \\ & & & \chi \end{pmatrix},$$

where χ is a non-trivial character of the inertia subgroup. The first two inertial types extend to representations which can be theta lifts only in the split case. The former is obtained via a degeneration of N_2 mod ℓ, and the second when the supercuspidal reduces to unramified Steinberg, that is $N(\mathfrak{p}) \equiv -1 \pmod{\ell}$.

Finally, if μ is a tamely ramified character with unramified reduction, then the inertial type $\begin{pmatrix} 1 & & & * \\ & a & b & \\ & c & d & \\ & & & 1 \end{pmatrix}$ extends to a representation of type XIa, and this can be a theta lift in the split case only. Moreover, we get $N(\mathfrak{p}) \equiv 1 \pmod{\ell}$.

6. Let $\pi_\mathfrak{p}$ be a supercuspidal representation and $\pi_{\overline{\mathfrak{p}}} = \chi_1 \times \chi_2$. Then

$$\theta((\pi_\mathfrak{p}, \pi_{\overline{\mathfrak{p}}})^+) = \chi_1^{-1}\pi_\mathfrak{p} \rtimes \chi_1,$$

i.e., a representation of type X. This has L-parameter Φ_p with monodromy operator $N = 0$, and semisimple part

$$\rho_0 : w \mapsto \begin{pmatrix} \chi_1^{-1}\omega_{\pi_\mathfrak{p}}(w) & & \\ & \phi_{\pi_\mathfrak{p}}(w) & \\ & & \chi_1(w) \end{pmatrix}.$$

That is,

$$\Phi_p \sim \begin{pmatrix} \chi_1^{-1}\omega_{\pi_\mathfrak{p}} & & \\ & \phi_{\pi_\mathfrak{p}} & \\ & & \chi_1 \end{pmatrix} = \begin{pmatrix} \chi_2 & & \\ & \phi_{\pi_\mathfrak{p}} & \\ & & \chi_1 \end{pmatrix}.$$

First consider the case where the supercuspidal representation degenerates mod ℓ; thus, we have $N(\mathfrak{p}) \equiv -1 \pmod{\ell}$. If χ_1 is a ramified representation, we may consider the inertial types

$$\begin{pmatrix} \chi_2' & & & \\ & 1 & * & \\ & & 1 & \\ & & & \chi_1' \end{pmatrix}; \qquad \begin{pmatrix} \chi_2' & & & \\ & 1 & & \\ & & 1 & \\ & & & \chi_1' \end{pmatrix},$$

where χ_1', χ_2' are non-trivial characters of the inertia group; the first one extends to a representation which can be a theta lift only in the split case. If χ_1 is an unramified character, then the inertial types of lower conductor are

$$\begin{pmatrix} 1 & & & \\ & 1 & * & \\ & & 1 & \\ & & & 1 \end{pmatrix}; \qquad \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}.$$

Again, the first one extends to a representation which can be a theta lift only in the split case.

Now consider the case where the character χ_1 is tamely ramified with unramified reduction mod ℓ; hence, $N(\mathfrak{p}) \equiv 1 \pmod{\ell}$. The inertial types one considers in this case are

$$\begin{pmatrix} \chi_2' & & & \\ & a & b & \\ & c & d & \\ & & & 1 \end{pmatrix}; \qquad \begin{pmatrix} 1 & & & \\ & a & b & \\ & c & d & \\ & & & 1 \end{pmatrix},$$

with χ_2' a non-trivial character of the inertia group. These inertial types extend to type X representations, and if a type X representation is a local theta lift, it is a theta lift in the split case.

7. Let $\pi_{\mathfrak{p}} = (|\ |^{1/2}\mu)St_{\mathrm{GL}_2}$ and $\pi_{\overline{\mathfrak{p}}} = \chi_1 \times \chi_2$. Then $\theta((\pi_{\mathfrak{p}}, \pi_{\overline{\mathfrak{p}}})^+)$ is of type IIa or IVc. In fact, it is either (if it is of type IIa)

$$\theta((\pi_{\mathfrak{p}}, \pi_{\overline{\mathfrak{p}}})^+) = \left(|\ |^{1/2}\frac{\mu}{\chi_1}\right) St_{\mathrm{GL}_2} \rtimes \chi_1$$

or (if it is of type IVc, which occurs when $\mu/\chi_1 = |\ |$)

$$\theta((\pi_{\mathfrak{p}}, \pi_{\overline{\mathfrak{p}}})^+) = L(|\ |^{3/2} St_{\mathrm{GL}_2}, \chi_1).$$

The local theta lift has L-parameter Φ_p with nilpotent part $N = N_1$, and semisimple part

$$\rho_0 : w \mapsto \begin{pmatrix} |w|\frac{\mu^2}{\chi_1}(w) & & & \\ & |w|\mu(w) & & \\ & & \mu(w) & \\ & & & \chi_1(w) \end{pmatrix}.$$

That is,

$$\Phi_p \sim \begin{pmatrix} |\ |\frac{\mu^2}{\chi_1} & & & \\ & |\ |\mu & * & \\ & & \mu & \\ & & & \chi_1 \end{pmatrix} = \begin{pmatrix} \chi_2 & & & \\ & |\ |\mu & * & \\ & & \mu & \\ & & & \chi_1 \end{pmatrix}.$$

Suppose that μ is unramified. Then, we have the inertial types:

$$\begin{pmatrix} 1 & & & \\ & 1 & * & \\ & & 1 & \\ & & & 1 \end{pmatrix}; \quad \begin{pmatrix} \chi_2' & & & \\ & 1 & & \\ & & 1 & \\ & & & \chi_1' \end{pmatrix}; \quad \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}.$$

The first inertial type extends to a representation which if it is a theta lift, is necessarily in the split case.
If μ is ramified and stays ramified mod ℓ, we have the following inertial types:

$$\begin{pmatrix} \chi_2' & & & \\ & \mu' & & \\ & & \mu' & \\ & & & \chi_1' \end{pmatrix}; \quad \begin{pmatrix} 1 & & & \\ & \mu' & & \\ & & \mu' & \\ & & & \chi_1' \end{pmatrix}; \quad \begin{pmatrix} 1 & & & \\ & \mu' & & \\ & & \mu' & \\ & & & 1 \end{pmatrix};$$

$$\begin{pmatrix} 1 & & & \\ & \mu' & * & \\ & & \mu' & \\ & & & \chi_1' \end{pmatrix}; \quad \begin{pmatrix} 1 & & & \\ & \mu' & * & \\ & & \mu' & \\ & & & 1 \end{pmatrix}.$$

The last two inertial types extend to representations which are theta lifts only in the split case.
If μ is tamely ramified with unramified mod ℓ reduction, we get

$$\begin{pmatrix} \chi_2' & & & \\ & 1 & * & \\ & & 1 & \\ & & & \chi_1' \end{pmatrix}.$$

If the extension of this inertial type is a local theta lift, we must be in the split case.

8. If $\pi_{\mathfrak{p}} = \chi_1 \times \chi_2$ and $\pi_{\overline{\mathfrak{p}}} = \chi_1' \times \chi_2'$, then

$$\theta((\pi_{\mathfrak{p}}, \pi_{\overline{\mathfrak{p}}})^+) = \frac{\chi_2'}{\chi_1} \times \frac{\chi_1'}{\chi_1} \rtimes \chi_1,$$

i.e., of type I, or (when $\chi_1'/\chi_1 = |\ |$)

$$\theta((\pi_{\mathfrak{p}}, \pi_{\overline{\mathfrak{p}}})^+) = \frac{\chi_2'}{\chi_1} \rtimes |\ |^{1/2} \chi_1 \mathbf{1}_{\mathrm{GSp}_2},$$

i.e., of type IIIb. This has L-parameter

$$\Phi_p \sim \begin{pmatrix} \chi_2 & & & \\ & \chi_2' & & \\ & & \chi_1' & \\ & & & \chi_1 \end{pmatrix}.$$

An inertial type in this case extends to a representation of type I, IIb, IIIb, IVd, Vd, VId. The only representations which are theta lifts are the ones of type I and IIIb. If the extension is of type I, we cannot say if it is a theta lift in the split case or the non-split case.

2.2. Non-split, ramified case

Suppose we have a prime p such that $p\mathcal{O}_K = \mathfrak{p}^2$, and an irreducible admissible representation $\pi_{\mathfrak{p}}$ of $\mathrm{GL}_2(K_{\mathfrak{p}})$ with central character that factors through the norm map via the character $\chi_{\mathfrak{p}}$. Denote by $\tilde{\pi}_{\mathfrak{p}} = (\pi_{\mathfrak{p}}, \chi_{\mathfrak{p}})$ the corresponding irreducible admissible representation of $\mathrm{GSO}(V)$ over $\mathbb{Q}_p$. Note that in this case, $K_{\mathfrak{p}}/\mathbb{Q}_p$ is a ramified extension; this implies that the quadratic character $\epsilon_{K_{\mathfrak{p}}/\mathbb{Q}_p}$ is ramified. We consider the following cases:

1. Suppose that $\pi_{\mathfrak{p}}$ is a supercuspidal representation of $\mathrm{GL}_2(K_{\mathfrak{p}})$. Then we have:
   (a) If $\tilde{\pi}_{\mathfrak{p}}$ is a regular representation, that is, $\pi_{\mathfrak{p}}$ is not a base change from $\mathrm{GL}_2(\mathbb{Q}_p)$, then $\theta(\tilde{\pi}_{\mathfrak{p}}^+)$ is a generic supercuspidal representation of $\mathrm{GSp}_4(\mathbb{Q}_p)$. The L-parameter of such a representation is not given in an explicit enough form for us to write down possible inertial types.

(b) If $\tilde{\pi}_{\mathfrak{p}}$ is invariant and distinguished, then $\pi_{\mathfrak{p}}$ is a base change from some supercuspidal representation π_p of $\mathrm{GL}_2(\mathbb{Q}_p)$ with central character $\chi_{\mathfrak{p}}\epsilon_{K_{\mathfrak{p}}/\mathbb{Q}_p}$. We have that

$$\tilde{\theta}(\hat{\pi}_{\mathfrak{p}}^{+}) = \epsilon_{K_{\mathfrak{p}}/\mathbb{Q}_p} \rtimes \pi_p,$$

i.e., it is a representation of type VII. The L-parameter Φ_p of such a representation has nilpotent part $N = 0$ and semisimple part

$$\rho_0 : w \mapsto \begin{pmatrix} \epsilon_{K_{\mathfrak{p}}/\mathbb{Q}_p}(w)\phi_p(w)' & \\ & \phi_p(w) \end{pmatrix}.$$

That is, the L-parameter is of the form

$$\Phi_p \sim \begin{pmatrix} \epsilon_{K_{\mathfrak{p}}/\mathbb{Q}_p}\begin{pmatrix} a & -b \\ -c & d \end{pmatrix} & \\ & \begin{pmatrix} a & b \\ c & d \end{pmatrix} \end{pmatrix},$$

where $\phi_p \sim \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is the L-parameter associated to π_p.

We consider the following inertial types:

$$\begin{pmatrix} \epsilon_{K_{\mathfrak{p}}/\mathbb{Q}_p} & -* & & \\ & \epsilon_{K_{\mathfrak{p}}/\mathbb{Q}_p} & & \\ & & 1 & * \\ & & & 1 \end{pmatrix}; \quad \begin{pmatrix} \epsilon_{K_{\mathfrak{p}}/\mathbb{Q}_p} & & & \\ & \epsilon_{K_{\mathfrak{p}}/\mathbb{Q}_p} & & \\ & & 1 & \\ & & & 1 \end{pmatrix}.$$

The former extends to a representation of type IIIa, which can be a local theta lift only in the ramified case. The latter extends to a representation of type I or IIIb, and thus we cannot distinguish whether it can be a theta lift in the split or the non-split case. Also, we mention that such congruences occur when a supercuspidal representation has degenerating conductor, and as a result $N(\mathfrak{p}) \equiv -1 \pmod{\ell}$.

Note that if we have an inertial type not containing the quadratic character $\epsilon_{K_{\mathfrak{p}}/\mathbb{Q}_p}$, it means that this character is unramified mod ℓ; such a situation can only occur if $\ell = 2$.

(c) If $\tilde{\pi}_{\mathfrak{p}}$ is invariant non-distinguished, that is, $\pi_{\mathfrak{p}}$ is a base change from a supercuspidal representation π_p of $\mathrm{GL}_2(\mathbb{Q}_p)$ with central character $\chi_{\mathfrak{p}}$, then $\tilde{\theta}(\hat{\pi}_{\mathfrak{p}}^{+})$ and $\tilde{\theta}(\hat{\pi}_{\mathfrak{p}}^{-})$ are both supercuspidal representations of $\mathrm{GSp}_4(\mathbb{Q}_p)$. They lie in the same L-packet, with L-parameter

$$\Phi_p \sim \begin{pmatrix} a & & b \\ & \epsilon_{K_{\mathfrak{p}}/\mathbb{Q}_p}\begin{pmatrix} a & b \\ c & d \end{pmatrix} & \\ c & & d \end{pmatrix},$$

where $\phi_p \sim \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is the L-parameter of the supercuspidal representation π_p.

We consider the inertial types

$$\begin{pmatrix} 1 & & & * \\ & \epsilon_{K_\mathfrak{p}/\mathbb{Q}_p} & * & \\ & & \epsilon_{K_\mathfrak{p}/\mathbb{Q}_p} & \\ & & & 1 \end{pmatrix}; \qquad \begin{pmatrix} 1 & & & \\ & \epsilon_{K_\mathfrak{p}/\mathbb{Q}_p} & & \\ & & \epsilon_{K_\mathfrak{p}/\mathbb{Q}_p} & \\ & & & 1 \end{pmatrix}.$$

If the extension of either inertial type is a local theta lift, it arises in both the non-split and the split case. Again, we have $N(\mathfrak{p}) \equiv -1 \pmod{\ell}$, as a supercuspidal degenerates.

2. Let $\pi_\mathfrak{p} = (\mu|\ |_\mathfrak{p}^{1/2})St_{\mathrm{GL}_2}$. As the central character of $\pi_\mathfrak{p}$ factors through the norm map, there is a (possibly trivial) quadratic character η of $\mathbb{Q}_p^\times$ such that $\mu^c/\mu = \eta \circ N_{K_\mathfrak{p}/\mathbb{Q}_p}$. We have the following:
   (a) $\tilde{\pi}_\mathfrak{p}$ is regular; then $\mu^c \neq \mu$, and η is neither trivial nor the quadratic character $\epsilon_{K_\mathfrak{p}/\mathbb{Q}_p}$. Then

$$\tilde{\theta}(\hat{\pi}_\mathfrak{p}^+) = \delta(|\ |\eta\epsilon_{K_\mathfrak{p}/\mathbb{Q}_p}, |\ |^{-1/2}BC(K_\mathfrak{p}/\mathbb{Q}_p, \mu|\ |_\mathfrak{p}^{1/2})),$$

   i.e., of type IXa. Such a representation has L-parameter with semisimple part

$$\rho_0 : w \mapsto \begin{pmatrix} |w|^{1/2}\eta\epsilon_{K_\mathfrak{p}/\mathbb{Q}_p}(w)\phi'(w) & \\ & |w|^{-1/2}\phi(w) \end{pmatrix},$$

   and monodromy operator $N = N_6$. Here ϕ is the L-parameter of the supercuspidal representation $BC(K_\mathfrak{p}/\mathbb{Q}_p, \mu|\ |_\mathfrak{p}^{1/2})$.
   Note that this supercuspidal representation does not have conductor that degenerates mod ℓ, as $K_\mathfrak{p}/\mathbb{Q}_p$ is a ramified quadratic extension, and a degenerating supercuspidal can only occur when $K_\mathfrak{p}/\mathbb{Q}_p$ is unramified (see [5], or Remark 4.2.4 in [27] and the discussion before). Thus we cannot find inertial types with decreasing conductor.
   (b) $\tilde{\pi}_\mathfrak{p}$ is invariant distinguished; then $\mu = \mu' \circ N_{K_\mathfrak{p}/\mathbb{Q}_p}$ for μ' a character of $\mathbb{Q}_p^\times$, $\eta = \epsilon_{K_\mathfrak{p}/\mathbb{Q}_p}$, and $\chi_\mathfrak{p} = |\ |(\mu')^2\epsilon_{K_\mathfrak{p}/\mathbb{Q}_p}$. Such a representation lifts to

$$\tilde{\theta}(\hat{\pi}_\mathfrak{p}^+) = \epsilon_{K_\mathfrak{p}/\mathbb{Q}_p} \rtimes (\mu'|\ |^{1/2})St_{\mathrm{GL}_2},$$

   i.e., a representation of $\mathrm{GSp}_4(\mathbb{Q}_p)$ of type IIIa. It has L-parameter Φ_p with nilpotent part $N = N_4$, and semisimple part

$$\rho_0 : w \mapsto \begin{pmatrix} |w|\epsilon_{K_\mathfrak{p}/\mathbb{Q}_p}\mu'(w) & & & \\ & \epsilon_{K_\mathfrak{p}/\mathbb{Q}_p}\mu'(w) & & \\ & & |w|\mu'(w) & \\ & & & \mu'(w) \end{pmatrix}.$$

   That is,

$$\Phi_p \sim \begin{pmatrix} |\ |\epsilon_{K_\mathfrak{p}/\mathbb{Q}_p}\mu' & -* & & \\ & \epsilon_{K_\mathfrak{p}/\mathbb{Q}_p}\mu' & & \\ & & |\ |\mu' & * \\ & & & \mu' \end{pmatrix}.$$

One can use the inertial types

$$\begin{pmatrix} \epsilon_{K_\mathfrak{p}/\mathbb{Q}_p} & -* & & \\ & \epsilon_{K_\mathfrak{p}/\mathbb{Q}_p} & & \\ & & 1 & * \\ & & & 1 \end{pmatrix}; \qquad \begin{pmatrix} \epsilon_{K_\mathfrak{p}/\mathbb{Q}_p} & & & \\ & \epsilon_{K_\mathfrak{p}/\mathbb{Q}_p} & & \\ & & 1 & \\ & & & 1 \end{pmatrix}.$$

The first one is obtained if the character μ' is tamely ramified with unramified reduction mod ℓ; this implies that $N(\mathfrak{p}) \equiv 1 \pmod{\ell}$. It can be a local theta lift only in the ramified case, as it extends to a representation of type IIIa. The second one is obtained when the nilpotent part N_4 is congruent to the zero matrix mod ℓ after a conjugation with a particular symplectic matrix, and we cannot tell if it extends to a theta lift in the split or the non-split case.

(c) $\tilde{\pi}_\mathfrak{p}$ is invariant non-distinguished; then $\mu = \mu' \circ N_{K_\mathfrak{p}/\mathbb{Q}_p}$ for μ' a character of $\mathbb{Q}_p^\times$, $\eta = 1$, and $\chi_\mathfrak{p} = |\ |(\mu')^2$. Such a representation lifts to

$$\tilde{\theta}(\hat{\pi}_\mathfrak{p}^+) = \delta([\epsilon_{K_\mathfrak{p}/\mathbb{Q}_p}, |\ |\epsilon_{K_\mathfrak{p}/\mathbb{Q}_p}], \mu'),$$

i.e., a representation of type Va; in addition, we also have the non-zero theta lift $\tilde{\theta}(\hat{\pi}_\mathfrak{p}^-)$ which lies in the same L-packet with $\tilde{\theta}(\hat{\pi}_\mathfrak{p}^+)$, but it is supercuspidal[7] (say, of type Va*). This L-packet corresponds to an L-parameter Φ_p with monodromy operator $N = N_3$, and semisimple part

$$\rho_0 : w \mapsto \begin{pmatrix} |w|\mu'(w) & & & \\ & |w|\epsilon_{K_\mathfrak{p}/\mathbb{Q}_p}\mu'(w) & & \\ & & \epsilon_{K_\mathfrak{p}/\mathbb{Q}_p}\mu'(w) & \\ & & & \mu'(w) \end{pmatrix}.$$

That is,

$$\Phi_p \sim \begin{pmatrix} |\ |\mu' & & & * \\ & |\ |\epsilon_{K_\mathfrak{p}/\mathbb{Q}_p}\mu' & * & \\ & & \epsilon_{K_\mathfrak{p}/\mathbb{Q}_p}\mu' & \\ & & & \mu' \end{pmatrix}.$$

In that case, we have the inertial types

$$\begin{pmatrix} 1 & & & * \\ & \epsilon_{K_\mathfrak{p}/\mathbb{Q}_p} & * & \\ & & \epsilon_{K_\mathfrak{p}/\mathbb{Q}_p} & \\ & & & 1 \end{pmatrix}; \qquad \begin{pmatrix} 1 & & & \\ & \epsilon_{K_\mathfrak{p}/\mathbb{Q}_p} & & \\ & & \epsilon_{K_\mathfrak{p}/\mathbb{Q}_p} & \\ & & & 1 \end{pmatrix};$$

[7] Note that one may have L-packets containing a non-supercuspidal and a supercuspidal; the two such L-packets for GSp_4 are the ones containing representations of type Va and XIa respectively (see discussion on p.60 of [22], or Table 1 in [16]).

$$\begin{pmatrix} 1 & & & * \\ & \epsilon_{K_{\mathfrak{p}}/\mathbb{Q}_p} & & \\ & & \epsilon_{K_{\mathfrak{p}}/\mathbb{Q}_p} & \\ & & & 1 \end{pmatrix}; \qquad \begin{pmatrix} 1 & & & \\ & \epsilon_{K_{\mathfrak{p}}/\mathbb{Q}_p} & * & \\ & & \epsilon_{K_{\mathfrak{p}}/\mathbb{Q}_p} & \\ & & & 1 \end{pmatrix}.$$

The first one is obtained when the character μ' is tamely ramified with unramified reduction mod ℓ, and the rest when the matrix N_3 degenerates to zero, N_2, or N_1 mod ℓ, respectively. Note that if the last inertial type extends to a representation which is a local theta lift, it is of type IIa, and this only occurs in the split case. The remaining inertial types extend to representations of the Weil-Deligne group where one cannot distinguish whether they belong in the split or the non-split case in terms of Definition 2.1 if they are local theta lifts.

3. Let $\pi_{\mathfrak{p}} = \chi_1 \times \chi_2$ be a principal series representation of $\mathrm{GL}_2(K_{\mathfrak{p}})$. Then we have the following cases:
   (a) Let $\tilde{\pi}_{\mathfrak{p}}$ be regular; then $\chi_2 \neq \chi_2^c$, and χ_1 is not equal to $\chi_{\mathfrak{p}}$ or $\chi_{\mathfrak{p}}\epsilon_{K_{\mathfrak{p}}/\mathbb{Q}_p}$. Then

$$\tilde{\theta}(\hat{\pi}_{\mathfrak{p}}^+) = \left(\frac{\chi_1}{\chi_{\mathfrak{p}}}\right)^{-1} \epsilon_{K_{\mathfrak{p}}/\mathbb{Q}_p} \rtimes \frac{\chi_1}{\chi_{\mathfrak{p}}} BC(K_{\mathfrak{p}}/\mathbb{Q}_p, \chi_2^c),$$

   i.e., a representation of type VII, unless $\frac{\chi_1}{\chi_{\mathfrak{p}}} = |\ |^{-1}$ or $|\ |^{-1}\epsilon_{K_{\mathfrak{p}}/\mathbb{Q}_p}$, in which case we have

$$\tilde{\theta}(\hat{\pi}_{\mathfrak{p}}^+) = L(|\ |\epsilon_{K_{\mathfrak{p}}/\mathbb{Q}_p}, |\ |^{-1/2}(|\ |^{-1/2} BC(K_{\mathfrak{p}}/\mathbb{Q}_p, \chi_2^c))),$$

   i.e., a representation of type IXb. The corresponding L-parameter is given by

$$\Phi_p \sim \begin{pmatrix} \epsilon_{K_{\mathfrak{p}}/\mathbb{Q}_p}\phi' & \\ & \left(\frac{\chi_1}{\chi_{\mathfrak{p}}}\right)\phi \end{pmatrix},$$

   where ϕ is the L-parameter of the supercuspidal representation $BC(K_{\mathfrak{p}}/\mathbb{Q}_p, \chi_2^c)$. This supercuspidal representation does not have conductor that degenerates mod ℓ, as $K_{\mathfrak{p}}/\mathbb{Q}_p$ is a ramified quadratic extension, and a degenerating supercuspidal is one with $K_{\mathfrak{p}}/\mathbb{Q}_p$ an unramified extension. Thus there are no inertial types with decreasing conductor.
   (b) $\tilde{\pi}_{\mathfrak{p}}$ is invariant distinguished with $\chi_2 = \chi_2^c$, in which case we write $\chi_2 = \chi_2' \circ N_{K_{\mathfrak{p}}/\mathbb{Q}_p}$ and $\chi_1 = \chi_1' \circ N_{K_{\mathfrak{p}}/\mathbb{Q}_p}$, where χ_1' and χ_2' are characters of $\mathbb{Q}_p^\times$. Then

$$\tilde{\theta}(\hat{\pi}_{\mathfrak{p}}^+) = \chi_1'^{-2}\chi_{\mathfrak{p}}\epsilon_{K_{\mathfrak{p}}/\mathbb{Q}_p} \times \epsilon_{K_{\mathfrak{p}}/\mathbb{Q}_p} \rtimes \chi_1',$$

   i.e., a representation of type I. The L-parameter of this representation is given by

$$\Phi_p \sim \begin{pmatrix} \chi_2' & & & \\ & \epsilon_{K_{\mathfrak{p}}/\mathbb{Q}_p}\chi_2' & & \\ & & \epsilon_{K_{\mathfrak{p}}/\mathbb{Q}_p}\chi_1' & \\ & & & \chi_1' \end{pmatrix}.$$

The possible congruences for which the conductor degenerates are obtained via a degeneration mod ℓ of at least one of the characters χ_1', χ_2'; that is, $N(\mathfrak{p}) \equiv 1 \pmod{\ell}$. Thus the inertial types that we consider are the following:

$$\begin{pmatrix} \chi & & & \\ & \epsilon_{K_\mathfrak{p}/\mathbb{Q}_p}\chi & & \\ & & \epsilon_{K_\mathfrak{p}/\mathbb{Q}_p} & \\ & & & 1 \end{pmatrix}; \qquad \begin{pmatrix} 1 & & & \\ & \epsilon_{K_\mathfrak{p}/\mathbb{Q}_p} & & \\ & & \epsilon_{K_\mathfrak{p}/\mathbb{Q}_p} & \\ & & & 1 \end{pmatrix},$$

where χ is a non-trivial character of $\mathbb{Z}_p^\times$. These inertial types extend to representations which can be local theta lifts in both the split and the non-split case.

(c) $\tilde{\pi}_\mathfrak{p}$ is invariant distinguished with $\chi_2 \neq \chi_2^c$ and $\chi_1 = \chi_\mathfrak{p}$; in this case we have

$$\tilde{\theta}(\hat{\pi}_\mathfrak{p}^+) = \epsilon_{K_\mathfrak{p}/\mathbb{Q}_p} \rtimes BC(K_\mathfrak{p}/\mathbb{Q}_p, \chi_2^c),$$

i.e., of type VII. Such a representation has L-parameter given by

$$\Phi_p \sim \begin{pmatrix} \epsilon_{K_\mathfrak{p}/\mathbb{Q}_p}\phi' & \\ & \phi \end{pmatrix},$$

where ϕ is the L-parameter of the supercuspidal representation $BC(K_\mathfrak{p}/\mathbb{Q}_p, \chi_2^c)$. Note that this supercuspidal representation does not have conductor that degenerates mod ℓ, as $K_\mathfrak{p}/\mathbb{Q}_p$ is a ramified quadratic extension. Thus there are no inertial types with decreasing conductor for this L-parameter.

(d) $\tilde{\pi}_\mathfrak{p}$ is invariant non-distinguished; here we have $\chi_2 \neq \chi_2^c$ and $\chi_1 = \epsilon_{K_\mathfrak{p}/\mathbb{Q}_p}$. In this case, we get that

$$\tilde{\theta}(\hat{\pi}_\mathfrak{p}^+) = \tau(S, BC(K_\mathfrak{p}/\mathbb{Q}_p, \chi_2^c))$$

and

$$\tilde{\theta}(\hat{\pi}_\mathfrak{p}^-) = \tau(T, BC(K_\mathfrak{p}/\mathbb{Q}_p, \chi_2^c)).$$

That is, $\tilde{\theta}(\hat{\pi}_\mathfrak{p}^+)$ and $\tilde{\theta}(\hat{\pi}_\mathfrak{p}^-)$ form a single L-packet consisting of representations of type VIIIa and VIIIb. The corresponding L-parameter is of the form

$$\Phi_p \sim \begin{pmatrix} \phi' & \\ & \phi \end{pmatrix},$$

where ϕ is the L-parameter associated to the supercuspidal representation $BC(K_\mathfrak{p}/\mathbb{Q}_p, \chi_2^c)$.

Again, this supercuspidal representation does not have conductor that degenerates mod ℓ, as $K_\mathfrak{p}/\mathbb{Q}_p$ is ramified. Thus we cannot find inertial types with decreasing conductor for this L-parameter.

2.3. Non-split, unramified case

Suppose we have a prime p such that $p\mathcal{O}_K = \mathfrak{p}$ (i.e., is inert in K), and an irreducible admissible representation $\pi_\mathfrak{p}$ of $\mathrm{GL}_2(K_\mathfrak{p})$, with central character that factors through the norm map via the character $\chi_\mathfrak{p}$. Denote by $\tilde{\pi}_\mathfrak{p} = (\pi_\mathfrak{p}, \chi_\mathfrak{p})$ the corresponding irreducible admissible representation of $\mathrm{GSO}(V)$ over $\mathbb{Q}_p$. In this case, $K_\mathfrak{p}/\mathbb{Q}_p$ is an unramified quadratic extension; this implies that the quadratic character $\epsilon_{K_\mathfrak{p}/\mathbb{Q}_p}$ is unramified. We consider the following cases:

1. Suppose that $\pi_\mathfrak{p}$ is a supercuspidal representation of $\mathrm{GL}_2(K_\mathfrak{p})$. Then we have:
   (a) If $\tilde{\pi}_\mathfrak{p}$ is a regular representation, that is, $\pi_\mathfrak{p}$ is not a base change from $\mathrm{GL}_2(\mathbb{Q}_p)$, then $\theta(\hat{\pi}_\mathfrak{p}^+)$ is a generic supercuspidal representation of $\mathrm{GSp}_4(\mathbb{Q}_p)$. The L-parameter of such a representation is not given in an explicit enough form for us to write down possible inertial types.
   (b) If $\tilde{\pi}_\mathfrak{p}$ is invariant and distinguished, then $\pi_\mathfrak{p}$ is a base change from some supercuspidal representation π_p of $\mathrm{GL}_2(\mathbb{Q}_p)$ with central character $\chi_\mathfrak{p}\epsilon_{K_\mathfrak{p}/\mathbb{Q}_p}$. We have that

$$\tilde{\theta}(\hat{\pi}_\mathfrak{p}^+) = \epsilon_{K_\mathfrak{p}/\mathbb{Q}_p} \rtimes \pi_p,$$

i.e., it is a representation of type VII. The L-parameter Φ_p of such a representation has nilpotent part $N = 0$ and semisimple part

$$\rho_0 : w \mapsto \begin{pmatrix} \epsilon_{K_\mathfrak{p}/\mathbb{Q}_p}(w)\phi_p(w)' & \\ & \phi_p(w) \end{pmatrix}.$$

That is, the L-parameter is of the form

$$\Phi_p \sim \begin{pmatrix} \epsilon_{K_\mathfrak{p}/\mathbb{Q}_p}\begin{pmatrix} a & -b \\ -c & d \end{pmatrix} & \\ & \begin{pmatrix} a & b \\ c & d \end{pmatrix} \end{pmatrix},$$

where $\phi_p \sim \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is the L-parameter associated to π_p.

The inertial types we may consider in this case are:

$$\begin{pmatrix} 1 & -* & & \\ & 1 & & \\ & & 1 & * \\ & & & 1 \end{pmatrix}; \qquad \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}.$$

These are obtained via a degenerating supercuspidal representation, i.e., $N(\mathfrak{p}) \equiv -1 \pmod{\ell}$. The first one extends to a representation of type IIIa or IVb. If this representation is a local theta lift, it corresponds to a representation of type IIIa, which can be a local theta lift only in the inert case.

(c) If $\tilde{\pi}_{\mathfrak{p}}$ is invariant non-distinguished, that is, $\pi_{\mathfrak{p}}$ is a base change from a supercuspidal representation π_p of $\mathrm{GL}_2(\mathbb{Q}_p)$ with central character $\chi_{\mathfrak{p}}$, then $\tilde{\theta}(\hat{\pi}_{\mathfrak{p}}^+)$ and $\tilde{\theta}(\hat{\pi}_{\mathfrak{p}}^-)$ are both supercuspidal representations of $\mathrm{GSp}_4(\mathbb{Q}_p)$. They lie in the same L-packet, with L-parameter

$$\Phi_p \sim \begin{pmatrix} a & & b \\ & \epsilon_{K_{\mathfrak{p}}/\mathbb{Q}_p}\begin{pmatrix} a & b \\ c & d \end{pmatrix} & \\ c & & d \end{pmatrix},$$

where $\phi_p \sim \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is the L-parameter of the supercuspidal representation π_p.
The inertial types that can be considered are

$$\begin{pmatrix} 1 & & & * \\ & 1 & * & \\ & & 1 & \\ & & & 1 \end{pmatrix}; \quad \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}.$$

These both extend to representations which can be theta lifts in both the split and the non-split cases.

2. Let $\pi_{\mathfrak{p}} = (\mu| \,|_{\mathfrak{p}}^{1/2})St_{\mathrm{GL}_2}$. As the central character of $\pi_{\mathfrak{p}}$ factors through the norm map, there is a (possibly trivial) quadratic character η of $\mathbb{Q}_p^\times$ such that $\mu^c/\mu = \eta \circ N_{K_{\mathfrak{p}}/\mathbb{Q}_p}$. We have the following:

(a) $\tilde{\pi}_{\mathfrak{p}}$ is regular; then $\mu^c \neq \mu$, and η is neither trivial nor the quadratic character $\epsilon_{K_{\mathfrak{p}}/\mathbb{Q}_p}$. Then

$$\tilde{\theta}(\hat{\pi}_{\mathfrak{p}}^+) = \delta(|\,|\eta\epsilon_{K_{\mathfrak{p}}/\mathbb{Q}_p}, |\,|^{-1/2}BC(K_{\mathfrak{p}}/\mathbb{Q}_p, \mu|\,|_{\mathfrak{p}}^{1/2})),$$

i.e., of type IXa. Such a representation has L-parameter with semisimple part

$$\rho_0 : w \mapsto \begin{pmatrix} |w|^{1/2}\eta\epsilon_{K_{\mathfrak{p}}/\mathbb{Q}_p}(w)\phi'(w) & \\ & |w|^{-1/2}\phi(w) \end{pmatrix},$$

and monodromy operator $N = N_6$. Here ϕ is the L-parameter of the supercuspidal representation $BC(K_{\mathfrak{p}}/\mathbb{Q}_p, \mu|\,|_{\mathfrak{p}}^{1/2})$.
The following inertial types are obtained via a degenerating supercuspidal, thus $N(\mathfrak{p}) \equiv -1 \pmod{\ell}$. We consider two cases:
Suppose η is a ramified quadratic character (it does not degenerate mod ℓ, when $\ell \neq 2$). Then we consider inertial types of the form

$$\begin{pmatrix} \eta' & -* & & \\ & \eta' & & \\ & & 1 & * \\ & & & 1 \end{pmatrix}; \quad \begin{pmatrix} \eta' & & & \\ & \eta' & & \\ & & 1 & \\ & & & 1 \end{pmatrix},$$

where η' is a non-trivial quadratic character of $\mathbb{Z}_p^\times$. The first one extends to a representation of type IIIa, and can be a local theta lift only in the inert case.
If η is unramified, we consider the inertial types

$$\begin{pmatrix} 1 & -* & & \\ & 1 & & \\ & & 1 & * \\ & & & 1 \end{pmatrix}; \qquad \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}.$$

The former extends to a representation of type IIIa or IVb, and can be a local theta lift only in the inert case.

(b) $\tilde{\pi}_{\mathfrak{p}}$ is invariant distinguished; then $\mu = \mu' \circ N_{K_{\mathfrak{p}}/\mathbb{Q}_p}$ for μ' a character of $\mathbb{Q}_p^\times$, $\eta = \epsilon_{K_{\mathfrak{p}}/\mathbb{Q}_p}$, and $\chi_{\mathfrak{p}} = |\ |(\mu')^2\epsilon_{K_{\mathfrak{p}}/\mathbb{Q}_p}$. Such a representation lifts to

$$\tilde{\theta}(\hat{\pi}_{\mathfrak{p}}^+) = \epsilon_{K_{\mathfrak{p}}/\mathbb{Q}_p} \rtimes (\mu'|\ |^{1/2})St_{\mathrm{GL}_2},$$

i.e., a representation of $\mathrm{GSp}_4(\mathbb{Q}_p)$ of type IIIa. It has L-parameter Φ_p with nilpotent part $N = N_4$, and semisimple part

$$\rho_0 : w \mapsto \begin{pmatrix} |w|\epsilon_{K_{\mathfrak{p}}/\mathbb{Q}_p}\mu'(w) & & & \\ & \epsilon_{K_{\mathfrak{p}}/\mathbb{Q}_p}\mu'(w) & & \\ & & |w|\mu'(w) & \\ & & & \mu'(w) \end{pmatrix}.$$

That is,

$$\Phi_p \sim \begin{pmatrix} |\ |\epsilon_{K_{\mathfrak{p}}/\mathbb{Q}_p}\mu' & -* & & \\ & \epsilon_{K_{\mathfrak{p}}/\mathbb{Q}_p}\mu' & & \\ & & |\ |\mu' & * \\ & & & \mu' \end{pmatrix}.$$

The inertial types below are obtained when the character μ' is tamely ramified, with unramified reduction mod ℓ. We have

$$\begin{pmatrix} 1 & -* & & \\ & 1 & & \\ & & 1 & * \\ & & & 1 \end{pmatrix}; \qquad \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}.$$

Again, the first one extends to a representation of type IIIa or IVb, which can be a local theta lift only in the inert case.

(c) $\tilde{\pi}_{\mathfrak{p}}$ is invariant non-distinguished; then $\mu = \mu' \circ N_{K_{\mathfrak{p}}/\mathbb{Q}_p}$ for μ' a character of $\mathbb{Q}_p^\times$, $\eta = 1$, and $\chi_{\mathfrak{p}} = |\ |(\mu')^2$. Such a representation lifts to

$$\tilde{\theta}(\hat{\pi}_{\mathfrak{p}}^+) = \delta([\epsilon_{K_{\mathfrak{p}}/\mathbb{Q}_p}, |\ |\epsilon_{K_{\mathfrak{p}}/\mathbb{Q}_p}], \mu'),$$

i.e., a representation of type Va; in addition, we also have the non-zero theta lift $\tilde{\theta}(\hat{\pi}_{\mathfrak{p}}^-)$ which lies in the same L-packet with $\tilde{\theta}(\hat{\pi}_{\mathfrak{p}}^+)$, but it is supercuspidal (say, of type Va*). This L-packet corresponds to an L-parameter Φ_p with monodromy operator $N = N_3$, and semisimple part

$$\rho_0 : w \mapsto \begin{pmatrix} |w|\mu'(w) & & & \\ & |w|\epsilon_{K_\mathfrak{p}/\mathbb{Q}_p}\mu'(w) & & \\ & & \epsilon_{K_\mathfrak{p}/\mathbb{Q}_p}\mu'(w) & \\ & & & \mu'(w) \end{pmatrix}.$$

That is,

$$\Phi_p \sim \begin{pmatrix} |\ |\mu' & & & * \\ & |\ |\epsilon_{K_\mathfrak{p}/\mathbb{Q}_p}\mu' & * & \\ & & \epsilon_{K_\mathfrak{p}/\mathbb{Q}_p}\mu' & \\ & & & \mu' \end{pmatrix}.$$

The following inertial types are obtained either because μ' is a tamely ramified character with unramified reduction mod ℓ (in this case $N(\mathfrak{p}) \equiv 1 \pmod{\ell}$), or after a degeneration of N_3 to zero, N_2, or N_1 mod ℓ. We consider

$$\begin{pmatrix} 1 & & & * \\ & 1 & * & \\ & & 1 & \\ & & & 1 \end{pmatrix}; \quad \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix};$$

$$\begin{pmatrix} 1 & & & * \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}; \quad \begin{pmatrix} 1 & & & \\ & 1 & * & \\ & & 1 & \\ & & & 1 \end{pmatrix}.$$

The first and the second can be theta lifts in both the split and the non-split case. The third does not extend to a theta lift, and the last one can only extend to a theta lift in the split case (it extends to type IIa and Vb, and Vb is not a theta lift).

3. Let $\pi_\mathfrak{p} = \chi_1 \times \chi_2$ be a principal series representation of $\mathrm{GL}_2(K_\mathfrak{p})$. Then we have the following cases:
   (a) Let $\tilde{\pi}_\mathfrak{p}$ be regular; then $\chi_2 \neq \chi_2^c$, and χ_1 is not equal to $\chi_\mathfrak{p}$ or $\chi_\mathfrak{p}\epsilon_{K_\mathfrak{p}/\mathbb{Q}_p}$. Then

$$\tilde{\theta}(\hat{\pi}_\mathfrak{p}^+) = \left(\frac{\chi_1}{\chi_\mathfrak{p}}\right)^{-1}\epsilon_{K_\mathfrak{p}/\mathbb{Q}_p} \rtimes \frac{\chi_1}{\chi_\mathfrak{p}} BC(K_\mathfrak{p}/\mathbb{Q}_p, \chi_2^c),$$

i.e., a representation of type VII, unless $\frac{\chi_1}{\chi_\mathfrak{p}} = |\ |^{-1}$ or $|\ |^{-1}\epsilon_{K_\mathfrak{p}/\mathbb{Q}_p}$, in which case we have

$$\tilde{\theta}(\hat{\pi}_\mathfrak{p}^+) = L(|\ |\epsilon_{K_\mathfrak{p}/\mathbb{Q}_p}, |\ |^{-1/2}(|\ |^{-1/2}BC(K_\mathfrak{p}/\mathbb{Q}_p, \chi_2^c))),$$

i.e., a representation of type IXb. The corresponding L-parameter is given by

$$\Phi_p \sim \begin{pmatrix} \epsilon_{K_\mathfrak{p}/\mathbb{Q}_p}\phi' & \\ & \left(\frac{\chi_1}{\chi_\mathfrak{p}}\right)\phi \end{pmatrix},$$

where ϕ is the L-parameter of the supercuspidal representation $BC(K_\mathfrak{p}/\mathbb{Q}_p, \chi_2^c)$. To get possible congruences mod ℓ, the supercuspidal representation associated to ϕ must degenerate, i.e., $N(\mathfrak{p}) \equiv -1 \pmod{\ell}$. We consider two cases:
If $\frac{\chi_1}{\chi_\mathfrak{p}}$ is unramified, we have the inertial types

$$\begin{pmatrix} 1 & -* & & \\ & 1 & & \\ & & 1 & * \\ & & & 1 \end{pmatrix}; \qquad \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}.$$

If $\frac{\chi_1}{\chi_\mathfrak{p}}$ is ramified and does not degenerate mod ℓ, we have the inertial types

$$\begin{pmatrix} 1 & -* & & \\ & 1 & & \\ & & \chi & * \\ & & & \chi \end{pmatrix}; \qquad \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \chi & \\ & & & \chi \end{pmatrix},$$

where χ is a non-trivial character of $\mathbb{Z}_p^\times$.
In both cases, only the first inertial type extends to a representation which, if it is a theta lift, it is a theta lift only in the inert case.

(b) $\tilde{\pi}_\mathfrak{p}$ is invariant distinguished with $\chi_2 = \chi_2^c$, in which case we write $\chi_2 = \chi_2' \circ N_{K_\mathfrak{p}/\mathbb{Q}_p}$ and $\chi_1 = \chi_1' \circ N_{K_\mathfrak{p}/\mathbb{Q}_p}$, where χ_1' and χ_2' are characters of $\mathbb{Q}_p^\times$. Then

$$\tilde{\theta}(\hat{\pi}_\mathfrak{p}^+) = \chi_1'^{-2}\chi_\mathfrak{p}\epsilon_{K_\mathfrak{p}/\mathbb{Q}_p} \times \epsilon_{K_\mathfrak{p}/\mathbb{Q}_p} \rtimes \chi_1',$$

i.e., a representation of type I. The L-parameter of this representation is given by

$$\Phi_p \sim \begin{pmatrix} \chi_2' & & & \\ & \epsilon_{K_\mathfrak{p}/\mathbb{Q}_p}\chi_2' & & \\ & & \epsilon_{K_\mathfrak{p}/\mathbb{Q}_p}\chi_1' & \\ & & & \chi_1' \end{pmatrix}.$$

The possible congruences for which the conductor degenerates are obtained via a degeneration mod ℓ of at least one of the characters χ_1', χ_2'; that is, $N(\mathfrak{p}) \equiv 1 \pmod{\ell}$. Thus the inertial types that one may consider are the following:

$$\begin{pmatrix} \chi & & & \\ & \chi & & \\ & & 1 & \\ & & & 1 \end{pmatrix}; \qquad \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix},$$

where χ is a non-trivial character of $\mathbb{Z}_p^\times$. These inertial types extend to representations which can be local theta lifts in both the split and the non-split case.

(c) $\tilde{\pi}_\mathfrak{p}$ is invariant distinguished with $\chi_2 \neq \chi_2^c$ and $\chi_1 = \chi_\mathfrak{p}$; in this case we have

$$\tilde{\theta}(\hat{\pi}_\mathfrak{p}^+) = \epsilon_{K_\mathfrak{p}/\mathbb{Q}_p} \rtimes BC(K_\mathfrak{p}/\mathbb{Q}_p, \chi_2^c),$$

i.e., of type VII. Such a representation has L-parameter given by

$$\Phi_p \sim \begin{pmatrix} \epsilon_{K_\mathfrak{p}/\mathbb{Q}_p}\phi' & \\ & \phi \end{pmatrix},$$

where ϕ is the L-parameter of the supercuspidal representation $BC(K_\mathfrak{p}/\mathbb{Q}_p, \chi_2^c)$. In order to get a conductor lowering congruence, we need the supercuspidal representation $BC(K_\mathfrak{p}/\mathbb{Q}_p, \chi_2^c)$ to degenerate mod ℓ (i.e., $N(\mathfrak{p}) \equiv -1 \pmod{\ell}$), so we have the inertial types

$$\begin{pmatrix} 1 & -* & & \\ & 1 & & \\ & & 1 & * \\ & & & 1 \end{pmatrix}; \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

The first one extends to a representation which can be a local theta lift in the inert case.

(d) $\tilde{\pi}_\mathfrak{p}$ is invariant non-distinguished; here we have $\chi_2 \neq \chi_2^c$ and $\chi_1 = \epsilon_{K_\mathfrak{p}/\mathbb{Q}_p}$. In this case, we get that

$$\tilde{\theta}(\hat{\pi}_\mathfrak{p}^+) = \tau(S, BC(K_\mathfrak{p}/\mathbb{Q}_p, \chi_2^c))$$

and

$$\tilde{\theta}(\hat{\pi}_\mathfrak{p}^-) = \tau(T, BC(K_\mathfrak{p}/\mathbb{Q}_p, \chi_2^c)).$$

That is, $\tilde{\theta}(\hat{\pi}_\mathfrak{p}^+)$ and $\tilde{\theta}(\hat{\pi}_\mathfrak{p}^-)$ form a single L-packet consisting of representations of type VIIIa and VIIIb. The corresponding L-parameter is of the form

$$\Phi_p \sim \begin{pmatrix} \phi' & \\ & \phi \end{pmatrix},$$

where ϕ is the L-parameter associated to the supercuspidal representation $BC(K_\mathfrak{p}/\mathbb{Q}_p, \chi_2^c)$.

We have the same inertial types as in the previous case, since in order to lower the conductor of an L-parameter of the above form, we need the supercuspidal representation $BC(K_\mathfrak{p}/\mathbb{Q}_p, \chi_2^c)$ to degenerate mod ℓ (that is, $N(\mathfrak{p}) \equiv -1 \pmod{\ell}$). The inertial type $\begin{pmatrix} 1 & -* & & \\ & 1 & & \\ & & 1 & * \\ & & & 1 \end{pmatrix}$ extends to a representation of type IIIa, or IVb. If it is a local theta lift, it is of type IIIa and it arises in the inert case.

3. Inertial types for GSp_4 and theta lifts

In the previous section, we identified the possible congruent inertial types, and now we wish to recall whether or not they can come from theta lifts, and, if so, the possible forms of the original representation. Most of the types involved arise in several situations, as we have seen.

The first group are those which are trivial on the diagonal on inertia, but which may or may not have some off-diagonal entries. The rank of the monomdromy matrix, which governs the off-diagonal entries on inertia, can be 0, 1, 2 or 3, although representations with monodromy of rank 3 are of type IVa, and do not arise as theta lifts. Let's enumerate the remaining cases:

- The trivial inertial type, $\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$, extends to a representation of Type I (amongst others), which can be a local theta lift in both the split and non-split cases; in the split case, it arises as a theta lift of a pair of principal series representations, and in the nonsplit case as a theta lift of a single principal series representation, where the defining characters must satisfy certain conditions in each case.
- The inertial type $\begin{pmatrix} 1 & & & \\ & 1 & * & \\ & & 1 & \\ & & & 1 \end{pmatrix}$, isomorphic to the type $\begin{pmatrix} 1 & & & * \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$, can extend to a variety of representations of GSp_4 with corresponding monodromy matrix N_1 or N_2 in the notation of [22]. These are of types IIa, IVc, Vb, Vc and VIc. As mentioned above, only the first two of these are theta lifts, and both can only arise in the split case, both arising as theta lifts of a pair (π_1, π_2) where π_1, say, is principal series, and π_2 is Steinberg. (Type IIa is the general case, and type IVc is a degenerate example.)
- The inertial type $\begin{pmatrix} 1 & * & & \\ & 1 & & \\ & & 1 & -* \\ & & & 1 \end{pmatrix}$ also occurred several times above. This is the inertial type of representations of types IIIa and IVb; the latter does not occur as a theta lift in our situation, and the former can only occur as a theta lift in the nonsplit case, arising as the theta lift of certain Steinberg representations on GL_2.
- The inertial type $\begin{pmatrix} 1 & & & * \\ & 1 & * & \\ & & 1 & \\ & & & 1 \end{pmatrix}$ corresponds to representations with monodromy N_3 in the notation of [22]; these are of types Va and VIa/VIb.

The latter type can only occur in the split case, as a theta lift of a pair (π_1, π_2) where $\pi_1 \cong \pi_2$ are Steinberg representations; type Va arises as a theta lift from a pair of non-isomorphic Steinberg representations in the split case, but can also arise as a theta lift of a single Steinberg in the nonsplit case under certain conditions.

There are some essentially diagonal inertial types, possibly with monodromy as above, but non-trivial on the diagonal:

- The inertial types

$$\begin{pmatrix} 1 & & & \\ & \chi & & \\ & & \chi & \\ & & & 1 \end{pmatrix}, \quad \begin{pmatrix} \chi_1 & & & \\ & 1 & & \\ & & 1 & \\ & & & \chi_2 \end{pmatrix}, \quad \begin{pmatrix} \chi & & & \\ & \chi_1 & & \\ & & \chi_2 & \\ & & & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & & & \\ & \chi & & \\ & & \chi & \\ & & & \chi_2 \end{pmatrix}, \quad \begin{pmatrix} \chi_1 & & & \\ & \chi & & \\ & & \chi & \\ & & & \chi_2 \end{pmatrix},$$

  where χ, χ_1, χ_2 are non-trivial characters of the inertia subgroup, extend to representations of type I. These occur in both the split and the non-split case.

  Nevertheless, an inertial type of the form $\begin{pmatrix} \chi & & & \\ & \chi & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$, with χ a non-trivial character on the inertia, extends to a representation of type I or IIIb. Even though representations of type I can be theta lifts in both the split and the non-split case, representations of type IIIb can be theta lifts only in the split case.

- Inertial types of the form

$$\begin{pmatrix} 1 & & & * \\ & \chi_1 & & \\ & & \chi_2 & \\ & & & 1 \end{pmatrix} \cong \begin{pmatrix} \chi_1 & & & \\ & 1 & * & \\ & & 1 & \\ & & & \chi_2 \end{pmatrix}, \begin{pmatrix} 1 & & & \\ & \chi_1 & * & \\ & & \chi_2 & \\ & & & 1 \end{pmatrix} \cong \begin{pmatrix} \chi_1 & & & * \\ & 1 & & \\ & & 1 & \\ & & & \chi_2 \end{pmatrix},$$

  with χ_1 and χ_2 being (possibly equal) non-trivial characters of the inertia group, extend to representations of type IIa, Vb, and Vc. Representations of type IIa occur as theta lifts only in the split case, while Vb and Vc are not local theta lifts. We also used the inertial type $\begin{pmatrix} 1 & & & \\ & \chi & * & \\ & & \chi & \\ & & & \chi_1 \end{pmatrix}$, with χ, χ_1 non-trivial characters of the inertia subgroup; this extends to a representation of type IIa, which is a local theta lift in the split case.

- The inertial types $\begin{pmatrix} 1 & & & * \\ & \chi & * & \\ & & \chi & \\ & & & 1 \end{pmatrix}$ or $\begin{pmatrix} \chi & & & * \\ & 1 & * & \\ & & 1 & \\ & & & \chi \end{pmatrix}$, with χ being a non-trivial character of the inertia group, extend to a representation of type Va which occurs as a theta lift in both the split and non-split case.
- We also have inertial types with monodromy operator N_4 and non-trivial diagonal, namely, $\begin{pmatrix} \chi & -* & & \\ & \chi & & \\ & & 1 & * \\ & & & 1 \end{pmatrix}$ or $\begin{pmatrix} 1 & -* & & \\ & 1 & & \\ & & \chi & * \\ & & & \chi \end{pmatrix}$, where χ is a non-trivial character of the inertia subgroup. These extend to representations of type IIIa, which can only be a theta lift in the non-split case; it is obtained by lifting particular Steinberg representations on GL_2.

There are also inertial types with an irreducible 2×2-block:

- $\begin{pmatrix} 1 & & & \\ & a & b & \\ & c & d & \\ & & & 1 \end{pmatrix}$, $\begin{pmatrix} \chi & & & \\ & a & b & \\ & c & d & \\ & & & \chi \end{pmatrix}$, or $\begin{pmatrix} \chi & & & \\ & a & b & \\ & c & d & \\ & & & 1 \end{pmatrix}$, with χ a non-trivial character of the inertia group; these extend to representations of type X or XIb, but the latter doesn't occur as a theta lift, and the former only occurs in the split case, as the theta lift from a pair (π_1, π_2) where one is unramified principal series and the other is supercuspidal.
- $\begin{pmatrix} 1 & & & * \\ & a & b & \\ & c & d & \\ & & & 1 \end{pmatrix}$; this occurs only as the inertial type of a representation of type XIa, and this is again a theta lift only in the split case, arising from a pair (π_1, π_2) where one is Steinberg with unramified defining character and the other supercuspidal.

4. Descending congruences

Let K be an imaginary quadratic field, and $\ell \neq 2$ a rational prime unramified in K. Let $\Sigma = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3$ be a finite set of finite places not above ℓ, where Σ_1 contains only primes lying above rational primes which ramify in K, Σ_2 contains primes which lie above rational primes that split in K, and Σ_3 contains primes which lie above rational primes inert in K. Also, denote by S the finite set of rational primes lying below primes in Σ.

Suppose we have a regular algebraic cuspidal automorphic representation $\pi = \bigotimes_w \pi_w$ of $\mathrm{GL}_2(\mathbb{A}_K)$ which is not a base change from $\mathbb{Q}$ with Galois invariant central character, such that π is unramified outside Σ, and for the places in Σ we have the following:

1. If $\mathfrak{p} \in \Sigma_1$, suppose that $\pi_\mathfrak{p}$ is one of the following:
   (a) $\pi_\mathfrak{p}$ is supercuspidal, with $\tilde{\pi}_\mathfrak{p}$ invariant and distinguished;
   (b) $\pi_\mathfrak{p} = (\mu|\,|_\mathfrak{p}^{1/2})St_{\mathrm{GL}_2}$, with $\tilde{\pi}_\mathfrak{p}$ invariant distinguished.
2. If $\mathfrak{p}, \overline{\mathfrak{p}} \in \Sigma_2$, suppose that $(\pi_\mathfrak{p}, \pi_{\overline{\mathfrak{p}}})$ are one of the following:
   (a) $\pi_\mathfrak{p} \ncong \pi_{\overline{\mathfrak{p}}}$ both supercuspidal;
   (b) $\pi_\mathfrak{p} \cong \pi_{\overline{\mathfrak{p}}} \cong (\mu|\,|^{1/2})St_{\mathrm{GL}_2}$;
   (c) $\pi_\mathfrak{p} = (|\,|^{1/2}\mu_1)St_{\mathrm{GL}_2}$, $\pi_{\overline{\mathfrak{p}}} = (|\,|^{1/2}\mu_2)St_{\mathrm{GL}_2}$, with $\mu_1 \neq \mu_2$ but $\mu_1^2 = \mu_2^2$;
   (d) $\pi_\mathfrak{p}$ supercuspidal, $\pi_{\overline{\mathfrak{p}}} = (|\,|^{1/2}\mu)St_{\mathrm{GL}_2}$;
   (e) $\pi_\mathfrak{p}$ supercuspidal, $\pi_{\overline{\mathfrak{p}}} = \chi_1 \times \chi_2$;
   (f) $\pi_\mathfrak{p} = (\mu|\,|^{1/2})St_{\mathrm{GL}_2}$, $\pi_{\overline{\mathfrak{p}}} = \chi_1 \times \chi_2$.
3. If $\mathfrak{p} \in \Sigma_3$, suppose $\pi_\mathfrak{p}$ is one of the following:
   (a) $\pi_\mathfrak{p}$ is supercuspidal, with $\tilde{\pi}_\mathfrak{p}$ invariant and distinguished;
   (b) $\pi_\mathfrak{p} = (\mu|\,|_\mathfrak{p}^{1/2})St_{\mathrm{GL}_2}$, with $\tilde{\pi}_\mathfrak{p}$ either regular or invariant distinguished;
   (c) $\pi_\mathfrak{p} = \chi_1 \times \chi_2$, with $\tilde{\pi}_\mathfrak{p}$ either regular, or invariant distinguished with $\chi_2 \neq {}^c\chi_2$ and $\chi_1 = \chi_\mathfrak{p}$, or invariant and non-distinguished.

As π has Galois invariant central character and it is not a base change from $\mathbb{Q}$, we may attach to it an irreducible ℓ-adic Galois representation $\rho : G_K \to \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$; we assume that the reduction $\overline{\rho}$ of ρ is irreducible. The theta lift $\Pi = \bigotimes_v \Pi_v$ of π, is a cuspidal automorphic representation of $\mathrm{GSp}_4(\mathbb{A})$, which is unramified outside $S \cup \{\ell\}$. The local components Π_p, for $p \in S$, are described explicitly in Theorems 5.3.11 and 5.3.12 of [27]. For each Π_p, denote by Φ_p and R_p the associated L-parameters and Galois representations respectively. If

$$R : G_\mathbb{Q} \to \mathrm{GSp}_4(\overline{\mathbb{Q}}_\ell)$$

is the associated Galois representation to Π, we will assume full local-global compatibility[8]; thus, for all p, we have

$$R_p \cong R|_{D_p},$$

where D_p is the decomposition group at p.

The following definition will simplify our exposition:

Definition 4.1. For a prime $p \in S$, an inertial type τ_p is called *admissible* if one of the following holds:

1. if p is split in K and there exists a representation R_p' extending the inertial type τ_p such that it corresponds to an irreducible admissible representation of $\mathrm{GSp}_4(\mathbb{Q}_p)$ that can be a local theta lift only in the split case;

[8]For cohomological cuspidal automorphic representations for GSp_4 with irreducible associated Galois representation, this is a result of Sorensen (see [24]) and Mok (see Theorem 3.1 of [19]). For a non-cohomological representation which is a theta lift, the required local-global compatibility result exists up to semisimplification; this is Theorem 4.11 of [19].

2. if p is ramified in K and there exists a representation R'_p extending the inertial type τ_p such that it corresponds to an irreducible admissible representation of $\mathrm{GSp}_4(\mathbb{Q}_p)$ that can be a local theta lift only in the ramified case;
3. if p is inert in K and there exists a representation R'_p extending the inertial type τ_p such that it corresponds to an irreducible admissible representation of $\mathrm{GSp}_4(\mathbb{Q}_p)$ that can be a local theta lift only in the inert case.

Suppose now that Π' is an automorphic representation of $\mathrm{GSp}_4(\mathbb{A})$ unramified outside S such that the partial standard L-function $\zeta^S(\Pi', \chi_0, s)$ has a pole at $s = 1$; denote by R' the corresponding Galois representation. Here χ_0 is a quadratic Dirichlet character of $\mathbb{A}^\times$. Moreover, assume that R' is congruent mod ℓ to the Galois representation R, such that for all $p \in S$, R'_p has admissible inertial type.

Theorem 4.2. *Under the assumptions above, the congruence between* Π *and* Π' *descends to a congruence between the cuspidal automorphic representation* π*, and another automorphic representation* π' *for* GL_2 *over the imaginary quadratic field* K*.*

Proof. As the partial standard L-function $\zeta^S(\Pi', \chi_0, s)$ of Π' has a pole at $s = 1$, by Theorem 7.1 of [17] and Section 5 of [28], Π' is a global theta lift from a representation of $GO(V)$, where V is a 4-dimensional quadratic space over a corresponding discriminant algebra L; this is either a quadratic extension of $\mathbb{Q}$ or $\mathbb{Q} \times \mathbb{Q}$. The first thing we need to prove is that the discriminant algebra L is in fact the imaginary quadratic field K.

Let $p \neq \ell$ be in S, and R'_p be the associated Galois representation to Π'_p. Firstly, Π'_p is a local theta lift since Π' is a non-zero global theta lift (see also Theorem 1.3 of [25]). The fact that R'_p has an admissible inertial type ensures that the splitting behaviour of p in L is the same as the one in K. As Π' is a theta lift and Π'_ℓ is unramified (as $\ell \notin S$), we see that ℓ is unramified in L (Proposition 4.2 of [16]). This implies that L is a quadratic extension of $\mathbb{Q}$, and also that the discriminant of L is the same as the discriminant of K. That is, Π' is a non-zero global theta lift from an automorphic representation for GL_2 over the imaginary quadratic field K, to which we associate a Galois representation

$$\rho' : G_K \to \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell).$$

The latter Galois representation induces to R'.

We now restrict the representations R and R' to G_K, and we obtain

$$R|_{G_K} = \rho \oplus \rho^c$$

and

$$R'|_{G_K} = \rho' \oplus \rho'^c.$$

Note that since we have assumed $\overline{\rho}$ to be irreducible, we have that $\overline{R}|_{G_K} = \overline{R}'|_{G_K}$ is the direct sum $\overline{\rho}' \oplus \overline{\rho}'^c$ of irreducible representations. As $\overline{\rho}'$ is irreducible, we get that ρ' is irreducible. Since R and R' are congruent modulo ℓ, we get that either ρ and ρ' have isomorphic mod ℓ Galois representations, or ρ and ρ'^c have isomorphic mod ℓ Galois representations. □

Our goal is not only to construct congruences, but also to try to lower the level, in the same way as in known for $\mathbb{Q}$ and for totally real fields. We would ideally like to choose a global representation R' whose local behaviour forces it to have lower level in a similar way to Gee's constructions over totally real fields, and this is the subject of the next section. Before we do that, let us make some further remarks about our local analysis.

At the beginning of this section, we listed the possible local components of π (for $p \in S$) so that their theta lifts are congruent to representations which always have at least one admissible inertial type; that is, for each Π_p with $p \in S$, we may always find a Π'_p, such that Π_p and Π'_p are both theta lifts in the same case (i.e., split, ramified, or inert case). This was derived from our analysis of the possible inertial types, in Sections 2 and 3.

The same analysis implies that for some local components of π, we have inertial types which force the splitting behaviour to change. Namely, for $\mathfrak{p} \in \Sigma$, we have the following:

1. If $\mathfrak{p} \in \Sigma_1$, let $\pi_{\mathfrak{p}}$ be such that $\tilde{\pi}_{\mathfrak{p}}$ is an invariant non-distinguished representation. When the theta lift Π_p is congruent mod ℓ to some theta lift Π'_p with corresponding Weil-Deligne representation of inertial type $\begin{pmatrix} 1 & & & \\ & \epsilon_{K_{\mathfrak{p}}/\mathbb{Q}_p} & * & \\ & & \epsilon_{K_{\mathfrak{p}}/\mathbb{Q}_p} & \\ & & & 1 \end{pmatrix}$, then Π'_p is a theta lift in the split case.
2. If $\mathfrak{p} \in \Sigma_2$, let $(\pi_{\mathfrak{p}}, \pi_{\overline{\mathfrak{p}}})$ be such that $\pi_{\mathfrak{p}} \cong \pi_{\overline{\mathfrak{p}}}$ is a supercuspidal representation. When the theta lift Π_p is congruent mod ℓ to some theta lift Π'_p with corresponding Weil-Deligne representation of inertial type $\begin{pmatrix} 1 & * & & \\ & 1 & & \\ & & 1 & -* \\ & & & 1 \end{pmatrix}$, then Π'_p is a theta lift in the inert case.
3. If $\mathfrak{p} \in \Sigma_3$, let $\pi_{\mathfrak{p}}$ be such that $\tilde{\pi}_{\mathfrak{p}}$ is an invariant non-distinguished representation. When the theta lift Π_p is congruent mod ℓ to some theta lift Π'_p with corresponding Weil-Deligne representation of inertial type $\begin{pmatrix} 1 & & & \\ & 1 & * & \\ & & 1 & \\ & & & 1 \end{pmatrix}$, then Π'_p is a theta lift in the split case.

Our analysis on the possible local congruences for GSp_4 also gives the following.

Remark 4.3. Suppose we have a cuspidal automorphic representation $\Pi = \bigotimes_v \Pi_v$ of $\mathrm{GSp}_4(\mathbb{A})$, which is a global theta lift from some cuspidal automorphic representation $\pi = \bigotimes_w \pi_w$ for GL_2 over an imaginary quadratic field K. Let $\mathfrak{p}$ be a prime that lies above a rational prime p that ramifies in K. If $\pi_{\mathfrak{p}}$ falls in one of the following cases:

1. $\pi_{\mathfrak{p}} = (\mu| |_{\mathfrak{p}}^{1/2}) St_{\mathrm{GL}_2}$ with $\tilde{\pi}_{\mathfrak{p}}$ regular,
2. $\pi_{\mathfrak{p}} = \chi_1 \times \chi_2$ with $\tilde{\pi}_{\mathfrak{p}}$ regular, or invariant distinguished with $\chi_2 \neq \chi_2^c$ and $\chi_1 = \chi_{\mathfrak{p}}$, or invariant non-distinguished,

then one cannot lower the level of Π at the prime p.

5. Congruences between automorphic representations on $\mathrm{GSp}_{4/\mathbb{Q}}$

We have classified above inertial types that lower the level. Inspired by work of Diamond-Taylor, Gee (e.g., [11]) has proved a number of theorems about congruences of Galois representations associated to automorphic representations on various algebraic groups. The general form of Gee's results are as follows:

> Begin with a mod ℓ representation $\overline{\rho}$ of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ which is the reduction of a representation ρ associated to a (global) cuspidal automorphic representation on the given group. Choose a finite set of finite places S (we shall assume that none divide ℓ in this note), and for each $v \in S$, choose some inertial type τ_v such that τ_v is a lift of $\overline{\rho}|_{I_v}$.
>
> Then there is a modular ℓ-adic representation ρ' which is again a lift of $\overline{\rho}$, and where $\rho'|_{I_v} \cong \tau_v$ for each $v \in S$.

(Generally, there will be several additional hypotheses; for example, there is generally an assumption that $\overline{\rho}|_{G_{\mathbb{Q}(\zeta_\ell)}}$ is absolutely irreducible, which corresponds to the "big" terminology of §2.5 of [6].)

For congruences between forms on GSp_4, the best results on congruences along the lines of Gee's general results, is due to Gee and Geraghty ([12]); the main theorem is Theorem 7.6.6 of their paper. Given a modular Galois representation R, one chooses local inertial types at finitely many places S as above, and recovers a modular Galois representation R' congruent to R. If R' is associated to a representation Π', then we would like to be able to combine this approach with the results of the previous section to deduce level lowering congruences for $\mathrm{GL}_{2/K}$.

Unfortunately, we see no way to force a congruent representation on GSp_4 to come from a theta lift as we need in the previous section; this sort of global condition will surely be difficult to force with finitely many local conditions. Perhaps one might hope for a version of Greenberg's conjecture which might guarantee that the global 4-dimensional representation is induced given suitable local conditions above the residue characteristic of the Galois representation.

There are a few other issues in using the main result of Gee and Geraghty as it is presented in their paper. Let us indicate that some of these have been addressed by subsequent work.

As stated, the main theorem of Gee and Geraghty only applies to globally generic representations. The reason is that Gee and Geraghty deduce their main result from a similar result for GL_4, and then use a transfer between GSp_4 and GL_4; this transfer is only known in the globally generic case. However, when we take the global theta lift to a cuspidal automorphic representation Π of $\mathrm{GSp}_4(\mathbb{A}_\mathbb{Q})$, we have that at the archimedean place ∞, Π_∞ is holomorphic and non-generic, so that Π is not globally generic. For our global theta lift, which is non-generic, one can get functoriality in the sense that we require using the endoscopic classification of Arthur; in particular, the lifting $\tilde{\Pi}$ (an automorphic representation of $\mathrm{GL}_4(\mathbb{A}_\mathbb{Q})$) of Π is given by Arthur's global parameter, since Π and $\tilde{\Pi}$ share the same global parameter. This lift is described by Mok in [19]; in particular see the proof of Theorem 3.1 of [19].

The next issue arises because our representation R does not have regular Hodge-Tate weights (see Remark 5.4.9 of [27]), whereas the main result of Gee and Geraghty requires the Hodge-Tate weights to be regular. The referee points out that in forthcoming work of Allen, Calegari, Caraiani, Gee, Helm, Le Hung, Newton, Scholze, Taylor, and Thorne, and of Boxer, Calegari, Gee and Pilloni, modularity lifting is accomplished for Galois representations like the ones that we use for GSp_4, but at the time of this writing the authors have not seen this result.

Another issue arises with the first condition in Theorem 7.6.6 of [12], that the image is "big". This assumption asserts that the image of the mod ℓ Galois representation $\overline{R}$ is big enough to contain $\mathrm{Sp}_4(k)$ for some finite field k such that $\mathbb{F}_\ell \subset k$. This does not happen for representations which we consider, since $R = \mathrm{ind}_{G_K}^{G_\mathbb{Q}} \rho$. Results of Thorne (see [26]) imply that the "big" image condition can be relaxed to a so-called "adequate" image condition; for this notion see Definition 2.3 of [26]. In addition, Guralnick-Herzig-Taylor-Thorne in Theorem 9 of [13], prove that if $\overline{R}$ is (absolutely) irreducible, then it is adequate for ℓ big enough; in our case certainly for $\ell \geq 11$.

6. Congruences by twisting

While we hope that the tabulation of possible congruences between theta lifts is useful in other situations, at this stage, applications to congruences seem unlikely.

However, as Carayol ([5]) does in the classical case, one can get congruences by twisting by characters, and can use this to get some results on level lowering in the imaginary quadratic case. In this final section we will prove two level lowering results for cuspidal automorphic representations over an imaginary quadratic field K, by adjusting an argument of Carayol that lowers the level by twisting the automorphic representation by a character. In particular, we prove the following two theorems, for an inert prime and for a split prime in K respectively.

Theorem 6.1. *Suppose we have a modular mod ℓ Galois representation*

$$\overline{\rho} : G_K \to \mathrm{GL}_2(\overline{\mathbb{F}}_\ell),$$

i.e., it has a lift ρ which arises from a regular algebraic cuspidal automorphic representation π. Assume that the component $\pi_\mathfrak{p}$ of π, at a prime $\mathfrak{p}$ which lies above a rational prime p that stays inert in K with $p \neq \ell$, is one of the following types:

1. *it is a principal series representation $\pi_\mathfrak{p} = \mu \times \nu$, with μ tamely ramified with unramified reduction such that it factors through the norm map, and ν ramified;*
2. *it is a twisted Steinberg representation $\pi_\mathfrak{p} = (\mu|\ |^{1/2})St_{\mathrm{GL}_2}$, with μ tamely ramified with unramified reduction such that it factors through the norm map.*

Then $\overline{\rho}$ is modular of conductor lower than the conductor of π.

Proof. In both cases, μ is assumed to be a tamely ramified character with unramified reduction. Such a character can be decomposed as $\mu = \mu_{nr}\mu_r$, where μ_{nr} is an unramified character of $K_\mathfrak{p}^\times$, and μ_r is a tamely ramified character of $K_\mathfrak{p}^\times$ with trivial reduction (i.e., $a(\mu_r) = 1$ but $a(\overline{\mu}_r) = 0$, where $a(-)$ denotes the Artin conductor), such that $\mu_r(\varpi_\mathfrak{p}) = 1$. Moreover, in both cases we have assumed that μ factors through the norm map and since μ_{nr} is unramified (i.e., its kernel contains the kernel of the norm map) we have that μ_r factors through the norm map too. This fact will enable us to extend μ_r to a grössencharacter $\tilde{\mu}_r$. As any element $x \in K_\mathfrak{p}^\times$ can be written as $x = \varpi_\mathfrak{p}^n u$ for some $n \in \mathbb{Z}$ and $u \in \mathcal{O}_{K_\mathfrak{p}}^\times$ (and as μ_r is trivial on $\varpi_\mathfrak{p}$), we get that μ_r is a character of $\mathcal{O}_{K_\mathfrak{p}}^\times$ which (as a tamely ramified character) is trivial on $(1 + \varpi_\mathfrak{p}\mathcal{O}_{K_\mathfrak{p}})$; this means that μ_r is a character of $\mathcal{O}_{K_\mathfrak{p}}^\times/(1+\varpi_\mathfrak{p}\mathcal{O}_{K_\mathfrak{p}}) \cong (\mathcal{O}_{K_\mathfrak{p}}/\varpi_\mathfrak{p}\mathcal{O}_{K_\mathfrak{p}})^\times$. As $\mathfrak{p} = p\mathcal{O}_K$ is a principal ideal, we have $(\mathcal{O}_{K_\mathfrak{p}}/\varpi_\mathfrak{p}\mathcal{O}_{K_\mathfrak{p}})^\times \cong (\mathcal{O}_K/\mathfrak{p})^\times$, so that

$$\mu_r : (\mathcal{O}_K/\mathfrak{p})^\times \to \mathbb{C}^\times.$$

That is, μ_r extends to a Dirichlet character for K of conductor $\mathfrak{p}$. Since μ_r factors through the norm map, and since $\mathfrak{p}$ has norm p^2 (as p is inert in K), we have that μ_r factors through $(\mathbb{Z}/p^2\mathbb{Z})^\times$; that is, if $N_{K/\mathbb{Q}}$ is the norm map of the extension $K/\mathbb{Q}$, we have

$$\mu_r : (\mathcal{O}_K/\mathfrak{p})^\times \xrightarrow{N_{K/\mathbb{Q}}} (\mathbb{Z}/p^2\mathbb{Z})^\times \xrightarrow{\phi} \mathbb{C}^\times.$$

By Proposition 3.1.2 of [3], the Dirichlet character ϕ of conductor p^2, extends to a grössencharacter

$$\tilde{\phi} : \mathbb{Q}^\times \backslash \mathbb{A}_\mathbb{Q}^\times \to \mathbb{C}^\times,$$

which we may compose with the idèle norm map

$$\tilde{N}_{K/\mathbb{Q}} : K^\times \backslash \mathbb{A}_K^\times \to \mathbb{Q}^\times \backslash \mathbb{A}_\mathbb{Q}^\times$$

to get a grössencharacter

$$\tilde{\mu}_r : K^\times \backslash \mathbb{A}_K^\times \to \mathbb{C}^\times$$

that extends μ_r. Now we are able to proceed to the twisting argument.

1. Firstly we consider π with local component $\pi_\mathfrak{p} = \mu \times \nu$ at $\mathfrak{p}$, and attached Galois representation ρ. We twist π with $\tilde{\mu}_r^{-1}$ to get a cuspidal automorphic representation $\tilde{\mu}_r^{-1}\pi$ that has local component at $\mathfrak{p}$ the representation

$$\pi'_\mathfrak{p} = \mu_r^{-1}(\mu_{nr}\mu_r \times \nu) = \mu_{nr} \times \mu_r^{-1}\nu.$$

   As μ factors through the norm map, the central character of $\tilde{\mu}_r^{-1}\pi$ factors through the norm map, and as a result we may attach to it a Galois representation $\tilde{\mu}_r^{-1}\rho$. Then the conductor of $\pi'_\mathfrak{p}$ is

$$a(\pi'_\mathfrak{p}) = a(\mu_{nr}) + a(\mu_r^{-1}\nu)$$

   with $a(\mu_{nr}) = 0 < 1 = a(\mu)$ and $a(\mu_r^{-1}\nu) \leq a(\nu)$, since $a(\mu_r^{-1}) = 1$ and $a(\nu) \geq 1$. That is, $a(\pi'_\mathfrak{p}) < a(\pi_\mathfrak{p})$.
   Moreover, the conductor of π at the other places is not getting bigger under the twisting since μ_r, as a Dirichlet character, has conductor $\mathfrak{p}$. Therefore, the power of $\mathfrak{p}$ dividing the conductor of $\tilde{\mu}_r^{-1}\rho$ is smaller than the power of $\mathfrak{p}$ dividing the conductor of ρ. The congruence occurs as $\tilde{\mu}_r^{-1}$ has trivial mod ℓ reduction, i.e., ρ and $\tilde{\mu}_r^{-1}\rho$ are congruent mod ℓ.
2. Now we consider π with local component $\pi_\mathfrak{p} = (\mu|\ |^{1/2})St_{\mathrm{GL}_2}$, and attached Galois representation ρ. We twist π with $\tilde{\mu}_r^{-1}$ to get a cuspidal automorphic representation $\tilde{\mu}_r^{-1}\pi$ with local component at $\mathfrak{p}$ the representation

$$\pi'_\mathfrak{p} = (\mu_{nr}|\ |^{1/2})St_{\mathrm{GL}_2}.$$

   Again $\tilde{\mu}_r^{-1}\pi$ has central character that factors through the norm map, so that we may attach to it a Galois representation $\tilde{\mu}_r^{-1}\rho$. The conductor of $\pi'_\mathfrak{p}$ is $a(\pi'_\mathfrak{p}) = 1$ while $a(\pi_\mathfrak{p}) = 2a(\mu) = 2$. That is

$$a(\pi'_\mathfrak{p}) < a(\pi_\mathfrak{p}).$$

   For the same reasons as before the conductors at the other places do not get bigger under twisting, and we have a level lowering congruence between ρ and $\tilde{\mu}_r^{-1}\rho$.

□

Theorem 6.2. *Suppose we have a modular mod ℓ Galois representation*

$$\overline{\rho} : G_K \to \mathrm{GL}_2(\overline{\mathbb{F}}_\ell),$$

i.e., it has a lift ρ which arises from a regular algebraic cuspidal automorphic representation π. Let $p\mathcal{O}_K = \mathfrak{p}\overline{\mathfrak{p}}$ with $p \neq \ell$, such that for the components $\pi_\mathfrak{p}$ and $\pi_{\overline{\mathfrak{p}}}$ of π at $\mathfrak{p}$ and $\overline{\mathfrak{p}}$ respectively, we have that $a(\pi_{\overline{\mathfrak{p}}}) > 1$ and that $\pi_\mathfrak{p}$ is one of the following types:

1. *principal series representation $\mu \times \nu$, with μ tamely ramified with unramified reduction, and ν ramified;*
2. *twisted Steinberg representation $(\mu|\ |^{1/2})St_{\mathrm{GL}_2}$, with μ tamely ramified with unramified reduction.*

Then $\overline{\rho}$ is modular of lower conductor than the conductor of π.

Proof. Note that $\pi_{\mathfrak{p}}$ and $\pi_{\overline{\mathfrak{p}}}$ have equal central characters in this situation.

Next we notice that in the case where $p\mathcal{O}_K = \mathfrak{p}\overline{\mathfrak{p}}$, we may fix isomorphisms $K_{\mathfrak{p}} \cong \mathbb{Q}_p$ and $K_{\overline{\mathfrak{p}}} \cong \mathbb{Q}_p$. That is, a character of $K_{\mathfrak{p}}^\times$ can essentially be thought of as a character of $\mathbb{Q}_p^\times$. As in the proof of Theorem 6.1, we may write $\mu : \mathbb{Q}_p^\times \to \mathbb{C}^\times$ as a product $\mu = \mu_{nr}\mu_r$, such that μ_{nr} is unramified, μ_r is tamely ramified with trivial reduction and $\mu_r(p) = 1$. Then, as before, we write μ_r as a character

$$\mu_r : \mathbb{Z}_p^\times/1 + p\mathbb{Z}_p \to \mathbb{C}^\times,$$

and considering also that $\mathbb{Z}_p^\times/1 + p\mathbb{Z}_p \cong (\mathbb{Z}_p/p\mathbb{Z}_p)^\times \cong (\mathbb{Z}/p\mathbb{Z})^\times$, μ_r becomes a Dirichlet character of conductor p. As in the earlier case, μ_r extends to a grössencharacter, which we compose with the idèle norm map $\tilde{N}_{K/\mathbb{Q}}$ to get

$$\tilde{\mu}_r : K^\times\backslash\mathbb{A}_K^\times \xrightarrow{\tilde{N}_{K/\mathbb{Q}}} \mathbb{Q}^\times\backslash\mathbb{A}_{\mathbb{Q}}^\times \to \mathbb{C}^\times.$$

Now we consider the two cases of the theorem.

1. Suppose that $\pi_{\mathfrak{p}} = \mu \times \nu$, with ρ being the Galois representation attached to π. We consider the twist $\tilde{\mu}_r^{-1}\pi$, which has local component at $\mathfrak{p}$ the representation
$$\pi'_{\mathfrak{p}} \cong \mu_r^{-1}(\mu \times \nu) = \mu_{nr} \times \mu_r^{-1}\nu$$
with $a(\mu_{nr}) + a(\mu_r^{-1}\nu) < a(\mu) + a(\nu)$. As $\tilde{\mu}_r^{-1}$ factors through the norm map, our new cuspidal automorphic representation $\tilde{\mu}_r^{-1}\pi$ has Galois invariant central character, and so we may attach to it a Galois representation $\tilde{\mu}_r^{-1}\rho$ which is congruent to ρ since μ_r is trivial modulo ℓ. Therefore, if $\mathfrak{p}^\kappa\overline{\mathfrak{p}}^\lambda$ divides exactly the conductor of ρ, then $\mathfrak{p}^{\kappa'}\overline{\mathfrak{p}}^{\lambda'}$ divides exactly the conductor of $\tilde{\mu}_r^{-1}\rho$, with $\kappa' + \lambda' < \kappa + \lambda$.
2. Now suppose that $\pi_{\mathfrak{p}} = (\mu|\ |^{1/2})St_{\mathrm{GL}_2}$, with ρ the Galois representation attached to π. The twist $\tilde{\mu}_r^{-1}\pi$ now has local component at $\mathfrak{p}$ the representation
$$\pi'_{\mathfrak{p}} \cong \mu_r^{-1}(\mu|\ |^{1/2})St_{\mathrm{GL}_2} = (\mu_{nr}|\ |^{1/2})St_{\mathrm{GL}_2}$$
with
$$a((\mu_{nr}|\ |^{1/2})St_{\mathrm{GL}_2}) = 1 < 2 = a((\mu|\ |^{1/2})St_{\mathrm{GL}_2}).$$
The cuspidal automorphic representation $\tilde{\mu}_r^{-1}\pi$ has attached a Galois representation $\tilde{\mu}_r^{-1}\rho$ which is congruent to ρ modulo ℓ, for the same reasons as above. So again, the level is getting lower by twisting by $\tilde{\mu}_r^{-1}$.

Note that in both cases, the conductor of $\pi_{\overline{\mathfrak{p}}}$ cannot be raised by twisting with μ_r^{-1}, as we have assumed that $a(\pi_{\overline{\mathfrak{p}}}) > 1$. □

Remark 6.3. The assumption that the conductor of $\pi_{\overline{\mathfrak{p}}}$ is greater than 1 in Theorem 6.2 excludes the following phenomenon. Let $\pi_{\mathfrak{p}} = \mu_r^{-1}\mu_{nr} \times \mu_r\nu_{nr}$ and $\pi_{\overline{\mathfrak{p}}} = \mu_{nr} \times \nu_{nr}$ (which have equal central characters), where μ_r is tamely ramified with trivial mod ℓ reduction, and μ_{nr}, ν_{nr} are unramified characters. After twisting the automorphic representation with $\tilde{\mu}_r^{-1}$ as in the Theorem, we get local components

$$\pi'_{\mathfrak{p}} = \mu_r^{-2}\mu_{nr} \times \nu_{nr}$$

and

$$\pi'_{\overline{\mathfrak{p}}} = \mu_r^{-1}\mu_{nr} \times \mu_r^{-1}\nu_{nr}.$$

This not only lowers the conductor of $\pi_{\mathfrak{p}}$, but at the same time might raise the conductor of $\pi_{\overline{\mathfrak{p}}}$. Something similar can take place when we have, for example, $\pi_{\overline{\mathfrak{p}}} = (\chi|\ |^{1/2})St_{\mathrm{GL}_2}$ with χ an unramified character.

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