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Extremes of Locally Stationary Chi-square Processes with Trend

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Abstract: Chi-square processes with trend appear naturally as limiting processes in various statistical models. In this paper we are concerned with the exact tail asymptotics of the supremum taken over $(0, 1)$ of a class of locally stationary chi-square processes with particular admissible trends. An important tool for establishing our results is a weak version of Slepian's lemma for chi-square processes. Some special cases including squared Brownian bridge and Bessel process are discussed.

Key Words: Tail asymptotics; chi-square process; Brownian bridge; Bessel process; fractional Brownian motion; generalized Kolmogorov-Dvoretzky-Erdős integral test; Pickands constant; Slepian's lemma.

AMS Classification: Primary 60G15; secondary 60G70

1 Introduction

Let $\widehat{G}_n(t), t \in (0, 1)$ be the empirical distribution of n independent random variables with uniform distribution on $(0, 1)$ and define the following test statistic

$$L_{n,E}^\nu := \sup_{t \in E} \left(nK(\widehat{G}_n(t), t) - g_\nu(t) \right), \quad E = (0, 1),$$

where $K(s, t) = s \ln \frac{s}{t} + (1-s) \ln \frac{1-s}{1-t}$, and $g_\nu(t), t > 0, \nu > 0$ is a trend function defined by

$$g_\nu(t) = c(t) + \nu \ln(1 + c^2(t)), \quad c(t) = \ln(1 - \ln(4t(1-t))), \quad t \in E.$$

Referring to Theorem 3.2 in [13] we have for any $\nu > 3/4$

$$2L_{n,E}^\nu \xrightarrow{d} L_E^\nu := \sup_{t \in E} \left(\chi^2(t) - 2g_\nu(t) \right), \quad n \rightarrow \infty \quad (1)$$

holds, where $\chi(t) = B(t)/\sqrt{t(1-t)}$ is the normalized standard Brownian bridge. Furthermore, as shown in Theorem 3.4 in [13], the convergence to L_E^ν also holds for another important test statistic. Note that as discussed therein $L_E^\nu < \infty$ almost surely (a.s.) for any $\nu > 3/4$.

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The supremum type test statistics appear naturally in different contexts in statistical problems; see also the recent contributions [2, 5, 18, 19]. In statistical applications, of interest is to determine the critical values of the test statistics, which is usually difficult since the distribution of the statistics is unknown. It is thus important to obtain approximation of them based on the tail asymptotic behavior of the limit of the statistic. For example, in our context it is of interest to know the asymptotic behavior of $\mathbb{P}(L_E^\nu > u)$ as $u \rightarrow \infty$ for any $\nu > 3/4$.

Note that L_E^ν is the supremum of a simple chi-square process (with 1-degree of freedom) with trend. Numerous contributions have been devoted to the study of the tail asymptotics of the supremum of chi or chi-square processes; see e.g., [14, 21, 23, 24, 27, 34] and the references therein. So far in the literature there are no results available which can be used for the derivation of the tail behavior of L_E^ν since it is a supremum taken over an open interval of a (non-Gaussian) chi-square process with trend. Given this significant gap and the fact that chi-square process with trend appears naturally in numerous statistical problems and applications, in this paper we shall focus on the tail asymptotics of the supremum (taken over an open interval) of a large class of chi-square processes with trend. More precisely, define

$$\chi_{\mathbf{b}}^2(t) = \sum_{i=1}^n b_i^2 X_i^2(t), \quad t \in (0, \infty), \quad (2)$$

where

$$1 = b_1 = \dots = b_k > b_{k+1} \geq \dots \geq b_n > 0$$

and X_i 's are independent copies of a centered Gaussian process X with a.s. continuous sample paths. We are interested in the asymptotics of

$$\mathbb{P}\left(\sup_{t \in E} (\chi_{\mathbf{b}}^2(t) - g(t)) > u\right) \quad (3)$$

as $u \rightarrow \infty$, for certain Gaussian processes X and nonnegative continuous trend functions $g(\cdot)$. Restrictions on X and $g(\cdot)$ will be specified to first ensure that

$$\sup_{t \in E} (\chi_{\mathbf{b}}^2(t) - g(t)) < \infty, \quad a.s. \quad (4)$$

holds.

In the special case that $n = 1$ and X is the normalized standard Brownian bridge, then $\chi_1 = X$, the same as χ appearing in (1), is a locally stationary Gaussian process in the notion of [3, 15]. Precisely,

$$\lim_{h \rightarrow 0} \frac{1 - \mathbb{E}(\chi(t)\chi(t+h))}{|h|} = C(t)$$

holds uniformly in $t \in I$, any compact interval in $(0, 1)$, where $C(t) = \frac{1}{2t(1-t)}$ satisfying $C(0) = C(1) = \infty$. Motivated by this fact in this paper we shall consider a large class of centered locally stationary Gaussian processes $\{X(t), t \in (0, 1)\}$ with a.s. continuous sample paths, unit variance and correlation function $r(\cdot, \cdot)$ such that

$$\lim_{h \rightarrow 0} \frac{1 - r(t, t+h)}{K^2(|h|)} = C(t) \quad (5)$$

uniformly in $t \in I$, any compact interval in $(0, 1)$, where $K(\cdot)$ is a positive regularly varying function at 0 with index $\alpha/2 \in (0, 1]$, and $C(\cdot)$ is a positive continuous function satisfying $C(0) = \infty$ or $C(1) = \infty$.

It is noted that condition (5) for the correlation function of X seems natural and is satisfied by many other interesting Gaussian processes. For instance, let $\{B_H(t), t \geq 0\}$ be the standard fractional Brownian motion (fBm) with Hurst index $H \in (0, 1)$ and covariance function given by

$$\text{Cov}(B_H(s), B_H(t)) = \frac{1}{2}(t^{2H} + s^{2H} - |t - s|^{2H}), \quad s, t \geq 0.$$

We can show that the normalized fBm $\{B_H(t)/t^H, t \in (0, \infty)\}$ is a locally stationary Gaussian process. In fact, the correlation of the normalized fBm satisfies

$$\lim_{h \rightarrow 0} \frac{1 - \text{Corr}\left(\frac{B_H(t)}{t^H}, \frac{B_H(t+h)}{(t+h)^H}\right)}{|h|^{2H}} = \frac{1}{2t^{2H}} \quad (6)$$

uniformly in t , for any compact interval in $(0, \infty)$.

In order to derive the asymptotics of (3) for $E = (0, 1)$, we need to impose several other conditions on $g(\cdot)$ and on the locally stationary Gaussian process X ; see Section 2. One natural restriction on $g(\cdot)$ is to ensure that it satisfies (4); functions $g(\cdot)$ satisfying (4) are thus called *admissible functions*. Theorem 2.3 below shows that the admissibility of functions $g(\cdot)$ is related to a generalized Kolmogorov-Dvoretzky-Erdős integral test (or the law of iterated logarithm) of the corresponding processes; see also Appendix. For instance, as discussed in Corollary 2.6 below $2g_\nu(\cdot)$ in (1) is admissible if and only if $\nu > 3/4$; see also [13]. In addition, as an application of our main result given in Theorem 2.4, we obtain the asymptotics of $\mathbb{P}(L_E^\nu > u)$ as $u \rightarrow \infty$ for any $\nu > 3/4$. Other examples related to the fBm and the Bessel process are also discussed.

Organization of the rest of the paper: In Section 2 we present our main results which are illustrated by several examples. Further results are discussed in Section 3 for the case where the Gaussian processes X_i 's are not identically distributed. All the proofs are relegated to Section 4, whereas some technical results are postponed to Appendix.

2 Main Results

We begin with some preliminary notation. We shall use the standard notation for asymptotic equivalence of two functions $f(\cdot)$ and $h(\cdot)$. That is, for any $x_0 \in \mathbb{R} \cup \{\infty\}$, write $f(x) = h(x)(1 + o(1))$ or simply $f(x) \sim h(x)$, if $\lim_{x \rightarrow x_0} f(x)/h(x) = 1$, and write $f(x) = o(h(x))$, if $\lim_{x \rightarrow x_0} f(x)/h(x) = 0$. Denote by $\Gamma(\cdot)$ the Euler Gamma function. Further, denote by $\overleftarrow{K}(\cdot)$ the generalized inverse function of $K(\cdot)$, and set $q(u) = \overleftarrow{K}(u^{-1/2})$ for any $u > 0$. Additionally, write $\mathcal{H}_\alpha, \alpha \in (0, 2]$ for the Pickands constant; see [6, 7, 8, 9, 10, 11, 12, 25, 27] for its definition and generalizations.

Our first result is concerned with the asymptotics of (3) for the case that E is any compact sub-interval in $(0, 1)$. The trend function appears in the asymptotics indirectly through the following constant

$$J_E^g := \int_{t \in E} C^{1/\alpha}(t) e^{-\frac{g(t)}{2}} dt < \infty. \quad (7)$$

Theorem 2.1. *Let $\{X(t), t \in E\}$, with E a compact interval in $(0, 1)$, be a centered locally stationary Gaussian process with a.s. continuous sample paths, unit variance and correlation function $r(\cdot, \cdot)$ that satisfies (5) and $r(s, t) < 1$, $s \neq t \in E$. Then, for any nonnegative continuous trend function $g(\cdot)$ we have*

$$\mathbb{P} \left(\sup_{t \in E} \left(\chi_{\mathbf{b}}^2(t) - g(t) \right) > u \right) \sim \mathcal{H}_\alpha G_{\mathbf{b}} J_E^g \frac{u^{k/2-1} e^{-u/2}}{q(u)} \quad (8)$$

as $u \rightarrow \infty$, where $G_{\mathbf{b}} = \frac{2^{1-k/2}}{\Gamma(k/2)} \prod_{i=k+1}^n (1 - b_i^2)^{-1/2}$ and $q(u) = \overleftarrow{K}(u^{-1/2})$ (with the convention $\prod_{i=1}^m = 1$ if $m < l$).

Remark 2.2. *a) We see from Theorem 2.1 that if X is stationary and $g(t) \equiv 0$ the above result coincides with that derived in [26]. In [29] the authors obtained the tail asymptotics of the supremum of a class of Gaussian random fields with trend indexed on smooth manifolds. It is worth noting that in Theorem 2.1 above the trend function $g(\cdot)$ contributes to the asymptotics through J_E^g , which is quite different from that in the aforementioned paper.*

b) Clearly, if X is well-defined on $(0, 1]$ (e.g., the normalized fBm, see (6)), then under the assumptions of Theorem 2.1 we have (8) holds for any $E = [a, 1]$ with $a \in (0, 1)$. In fact, the set E in the above theorem can be any compact interval in \mathbb{R} provided that everything is well-defined on it.

Our main targets below are to find out under which conditions (4) holds and in such a case to establish (8) by passing from the compact interval E to $(0, 1)$. Of course, conditions should be imposed on the local structures of the process X and $g(\cdot)$ at 0 and 1, separately. According to the proof of Theorem 5.1 in the Appendix (see (35) and (40)), we believe that, under some mild conditions, the sufficient and necessary condition for (4) to hold is given as

$$I_g(S) := \left| \int_{1/2}^S (C(t))^{1/\alpha} \frac{(g(t))^{\frac{k}{2}-1}}{q(g(t))} e^{-\frac{g(t)}{2}} dt \right| < \infty, \quad S \in \{0, 1\}. \quad (9)$$

Furthermore, under these conditions we show that (8) holds with $E = (0, 1)$. Moreover, note that if $\left| \int_{1/2}^S (C(t))^{1/\alpha} dt \right| < \infty$, then (9) holds for all $g(\cdot)$ satisfying $\lim_{t \rightarrow S} g(t) = \infty$. It turns out that in this simpler case the conditions imposed on the correlation function of X and the proof of main results can be significantly simplified. Thus, to simplify argumentation, different scenarios will be discussed according to whether $\int_0^{1/2} (C(s))^{1/\alpha} ds < \infty$ and whether $\int_{1/2}^1 (C(s))^{1/\alpha} ds = \infty$, and different additional assumptions are needed accordingly. For this purpose of crucial importance is the following function

$$f(t) = \int_{1/2}^t (C(s))^{1/\alpha} ds, \quad t \in (0, 1).$$

We denote by $\overleftarrow{f}(t), t \in (f(0), f(1))$ the inverse function of $f(t), t \in (0, 1)$. Further, for any $d > 0$, let $s_{j,d}^{(1)} = \overleftarrow{f}(jd), j \in \mathbb{N} \cup \{0\}$ if $f(1) = \infty$, and let $s_{j,d}^{(0)} = \overleftarrow{f}(-jd), j \in \mathbb{N} \cup \{0\}$ if $f(0) = -\infty$. Denote $\Delta_{j,d}^{(1)} = [s_{j-1,d}^{(1)}, s_{j,d}^{(1)}], j \in \mathbb{N}$ and $\Delta_{j,d}^{(0)} = [s_{j,d}^{(0)}, s_{j-1,d}^{(0)}], j \in \mathbb{N}$, which give a partition of $[1/2, 1)$ in the case $f(1) = \infty$ and a partition of $(0, 1/2]$ in the case $f(0) = -\infty$, respectively.

In addition to the local stationarity of X in (5), we need to impose the following (scenario-dependent) restrictions on the trend $g(\cdot)$ and the correlation function $r(\cdot, \cdot)$. Let therefore $S \in \{0, 1\}$.

Condition A(S): The trend function $g(\cdot)$ is monotone in a neighborhood of S and $\lim_{t \rightarrow S} g(t) = \infty$.

Condition B(S): Suppose that there exists some constant $d_0 > 0$ such that

$$\limsup_{j \rightarrow \infty} \sup_{t \neq s \in \Delta_{j,d_0}^{(S)}} \frac{1 - r(t, s)}{K^2(|f(t) - f(s)|)} < \infty,$$

and when $\alpha = 2$ and $k = 1$, assume further

$$K^2(|t|) = O(t^2), \quad t \rightarrow 0. \quad (10)$$

Condition C(S): With $I_g(S)$ defined in (9) it holds that

$$I_g(S) < \infty. \quad (11)$$

Condition D(S): The following is satisfied

$$\limsup_{\delta \rightarrow 0} \sup_{t \neq s \in (0, \delta)} \frac{1 - r(|S - t|, |S - s|)}{K^2(|f(|S - t|) - f(|S - s|)|)} < \infty.$$

Condition E(S): Suppose that there exists some constant $d_0 > 0$ such that

$$\liminf_{j \rightarrow \infty} \inf_{t \neq s \in \Delta_{j,d_0}^{(S)}} \frac{1 - r(t, s)}{K^2(|f(t) - f(s)|)} > 0.$$

Moreover, there exist $j_0, l_0 \in \mathbb{N}$, $M_0, \beta > 0$, such that for $j \geq j_0$, $l \geq l_0$,

$$\sup_{s \in \Delta_{j+t, d_0}^{(S)}, t \in \Delta_{j, d_0}^{(S)}} |r(s, t)| < M_0 l^{-\beta}. \quad (12)$$

Let

$$\mathcal{E}(0) = (0, 1/2] \quad \text{and} \quad \mathcal{E}(1) = [1/2, 1).$$

The following result is concerned about the almost surely finiteness of the random variable $\sup_{t \in \mathcal{E}(S)} (\chi_{\mathbf{b}}^2(t) - g(t))$.

Theorem 2.3. *Let $\{X(t), t \in E\}$ be given as in Theorem 2.1 with $E = (0, 1)$. If $|f(S)| < \infty$ and conditions **A(S)**, **D(S)** are satisfied, then*

$$\mathbb{P} \left(\sup_{t \in \mathcal{E}(S)} (\chi_{\mathbf{b}}^2(t) - g(t)) < \infty \right) = 1.$$

*If $|f(S)| = \infty$ and conditions **A(S)**, **B(S)** and **E(S)** are satisfied, then*

$$\mathbb{P} \left(\sup_{t \in \mathcal{E}(S)} (\chi_{\mathbf{b}}^2(t) - g(t)) < \infty \right) = 1 \quad \text{or} \quad 0$$

as the integral $I_g(S) < \infty$ or $= \infty$.

The technical proof of Theorem 2.3 is presented in the Appendix.

Next, we present our principle result.

Theorem 2.4. *Let $\{X(t), t \in E\}$ be given as in Theorem 2.1 with $E = (0, 1)$. Then for each of the following scenarios we have that (8) holds for $E = (0, 1)$.*

- (i). $f(0) = -\infty, f(1) = \infty$, and conditions **A(0)**, **B(0)**, **C(0)**, **A(1)**, **B(1)**, **C(1)** are satisfied;
- (ii). $f(0) = -\infty, f(1) < \infty$, and conditions **A(0)**, **B(0)**, **C(0)**, **D(1)** are satisfied;
- (iii). $f(0) > -\infty, f(1) = \infty$, and conditions **D(0)**, **A(1)**, **B(1)**, **C(1)** are satisfied;
- (iv). $f(0) > -\infty, f(1) < \infty$, and conditions **D(0)**, **D(1)** are satisfied.

Remark 2.5. Note that Theorem 2.4 can be easily extended to the case where $E := (0, T)$, with $T \in (0, \infty]$, by using a time-scaling. More precisely, suppose that the process X and the trend function $g(\cdot)$ are well-defined on E . If T is finite, we consider $Y(t) = X(Tt)$, $t \in (0, 1)$; if $T = \infty$, we consider $Z(t) = X(h(t))$, $t \in (0, 1)$, where $h(t)$, $t \in (0, 1)$ is a monotone function with $\lim_{t \rightarrow 0} h(t) = \infty$ and $\lim_{t \rightarrow 1} h(t) = 0$. For example,

$$h(t) = \begin{cases} 1/t, & 0 < t \leq 1/2, \\ 4(1-t), & 1/2 \leq t < 1. \end{cases}$$

Under analogue conditions as in Theorem 2.4 on the processes Y or Z , we can obtain a similar result as in (8) for these two cases.

As an application of Theorem 2.3 and Theorem 2.4, we obtain the following result concerning the supremum of the squared chi-square normalized standard Brownian bridge with trend, L_E^ν defined in (1).

Corollary 2.6. Let L_E^ν be defined in (1) with $E = (0, 1)$. We have:

If $\nu > 3/4$, then, as $u \rightarrow \infty$

$$\mathbb{P}(L_E^\nu > u) \sim \frac{\sqrt{u}e^{-u/2}}{\sqrt{2\pi}} \int_0^1 \frac{1}{t(1-t)} e^{-g_\nu(t)} dt;$$

If $\nu \leq 3/4$, then

$$\mathbb{P}(L_E^\nu = \infty) = 1.$$

Given wide applications of fBm in various fields, we give below a result concerning the tail asymptotics of the squared normalized fBm with trend.

Corollary 2.7. Let $\{X_i(t), t > 0\}$, $i \leq n$ in (2) be independent copies of $\{B_H(t)/t^H, t > 0\}$, with $H \in (0, 1)$, and let $g(\cdot)$ be a nonnegative continuous function on $(0, 1]$ such that $g(t) \uparrow \infty$ as $t \rightarrow 0$. We have:

If $\int_0^1 \frac{1}{t} (g(t))^{k/2+1/(2H)-1} e^{-\frac{g(t)}{2}} dt < \infty$, then, as $u \rightarrow \infty$

$$\mathbb{P}\left(\sup_{t \in (0, 1]} \left(\frac{\chi_{\mathbf{b}}^2(t)}{t^{2H}} - g(t)\right) > u\right) \sim \mathcal{H}_{2H} \prod_{i=k+1}^n (1 - b_i^2)^{-1/2} \frac{u^{\frac{k}{2} + \frac{1}{2H} - 1} e^{-u/2}}{2^{\frac{k}{2} + \frac{1}{2H} - 1} \Gamma(k/2)} \int_0^1 \frac{1}{t} e^{-\frac{g(t)}{2}} dt;$$

If $\int_0^1 \frac{1}{t} (g(t))^{k/2+1/(2H)-1} e^{-\frac{g(t)}{2}} dt = \infty$, then

$$\mathbb{P}\left(\sup_{t \in (0, 1]} \left(\frac{\chi_{\mathbf{b}}^2(t)}{t^{2H}} - g(t)\right) = \infty\right) = 1.$$

Let $\mathbf{W}_n(t) = (W_1(t), W_2(t), \dots, W_n(t))$, $t \geq 0$, $n \geq 1$ be the standard n -dimensional Brownian motion, and let $\|\mathbf{W}_n(t)\| := \sqrt{\sum_{i=1}^n W_i^2(t)}$, $t \geq 0$ be the Bessel process of order n . As a special case of Corollary 2.7, we have the following result.

Corollary 2.8. Let $g(\cdot)$ be a nonnegative continuous function on $(0, 1]$ such that $g(t) \uparrow \infty$ as $t \rightarrow 0$. We have:

If $\int_0^1 \frac{1}{t} (g(t))^{n/2} e^{-\frac{g(t)}{2}} dt < \infty$, then, as $u \rightarrow \infty$

$$\mathbb{P}\left(\sup_{t \in (0, 1]} \left(\frac{\|\mathbf{W}_n(t)\|^2}{t} - g(t)\right) > u\right) \sim \frac{2^{-n/2} u^{n/2} e^{-u/2}}{\Gamma(n/2)} \int_0^1 \frac{1}{t} e^{-\frac{g(t)}{2}} dt; \quad (13)$$

If $\int_0^1 \frac{1}{t} (g(t))^{n/2} e^{-\frac{g(t)}{2}} dt = \infty$, then

$$\mathbb{P} \left(\sup_{t \in (0,1]} \left(\frac{\|\mathbf{W}_n(t)\|^2}{t} - g(t) \right) = \infty \right) = 1. \quad (14)$$

Remark 2.9. It can be shown that the classical law of iterated logarithm for Bessel process follows from Corollary 2.8 or Theorem 5.1. In fact, define

$$g_\rho(t) = 2 \ln \ln(e^2/t) + 2\rho \ln \ln \ln(e^3/t), \quad t \in (0, 1].$$

It follows that

$$\frac{1}{t} (g_\rho(t))^{n/2} e^{-\frac{g_\rho(t)}{2}} \sim \frac{\mathcal{C}}{t \ln(1/t) (\ln \ln(1/t))^{\rho-n/2}}$$

as $t \rightarrow 0$, with \mathcal{C} some positive constant. Elementary calculations show that $\int_0^1 \frac{(g_\rho(t))^{n/2}}{t} e^{-\frac{g_\rho(t)}{2}} dt < \infty$ holds if and only if $\rho > 1 + n/2$. Then, by Corollary 2.8 one can show that, for any $\rho > 1 + n/2$

$$\limsup_{t \rightarrow 0} \frac{\|\mathbf{W}_n(t)\|^2}{t g_\rho(t)} \leq 1,$$

and for any $0 < \rho \leq 1 + n/2$

$$\limsup_{t \rightarrow 0} \frac{\|\mathbf{W}_n(t)\|^2}{t g_\rho(t)} \geq 1.$$

Consequently, we arrive at the classical law of iterated logarithm as follows:

$$\limsup_{t \rightarrow 0} \frac{\|\mathbf{W}_n(t)\|^2}{2t \ln \ln(1/t)} = 1.$$

3 Further Results and Discussions

In this section, we discuss an extension of (8) to the case where X_i 's are not identically distributed. Namely, we assume that $X_i, 1 \leq i \leq n$ are independent locally stationary Gaussian processes on $(0, 1)$ with a.s. continuous sample paths, unit variance and correlation functions $r_i(\cdot, \cdot)$ satisfying for any $1 \leq i \leq k(\leq n)$,

$$\lim_{h \rightarrow 0} \frac{1 - r_i(t, t+h)}{K^2(|h|)} = C_i(t), \quad (15)$$

and, for any $k+1 \leq i \leq n$,

$$\lim_{h \rightarrow 0} \frac{1 - r_i(t, t+h)}{K_i^2(|h|)} = C_i(t) \quad (16)$$

hold uniformly in $t \in I$, any compact interval in $(0, 1)$, where $K(\cdot)$ and $K_i(\cdot), k+1 \leq i \leq n$ are regularly varying functions at 0 with index $0 < \alpha/2 < \alpha_{k+1}/2 \leq \dots \leq \alpha_n/2 < 1$ respectively, and $C_i(\cdot), 1 \leq i \leq n$ are positive continuous functions over $(0, 1)$ satisfying $C_i(0) = \infty$ or $C_i(1) = \infty$.

For simplicity we shall assume $\mathbf{b} = \mathbf{1} = (1, 1, \dots, 1) \in \mathbb{R}^n$, and consider the asymptotics of

$$\mathbb{P} \left(\sup_{t \in E} \left(\chi_{\mathbf{1}}^2(t) - g(t) \right) > u \right)$$

as $u \rightarrow \infty$, for certain admissible functions $g(\cdot)$ and E either $(0, 1)$ or a compact interval in $(0, 1)$.

First, we present the result with E being a compact interval in $(0, 1)$. Recall that $q(u) = \overleftarrow{K}(u^{-1/2})$.

Theorem 3.1. Let $X_i, 1 \leq i \leq n$ be defined as above with correlation functions satisfying (15) and (16). If further $\max\{r_i(s, t), 1 \leq i \leq n\} < 1$ holds for all $s \neq t \in E$, a compact interval of $(0, 1)$, then for any nonnegative continuous trend function $g(\cdot)$ we have, as $u \rightarrow \infty$

$$\begin{aligned} & \mathbb{P} \left(\sup_{t \in E} \left(\chi_1^2(t) - g(t) \right) > u \right) \\ & \sim (2\pi)^{-n/2} \mathcal{H}_\alpha \frac{u^{n/2-1} e^{-u/2}}{q(u)} \\ & \times \int_{(\mathbf{t}, \boldsymbol{\theta}) \in D_E} \left(C_1(t) \prod_{j=2}^n \cos^2(\theta_j) + \sum_{i=2}^k C_i(t) \sin^2(\theta_i) \prod_{j=i+1}^n \cos^2(\theta_j) \right)^{1/\alpha} \prod_{i=3}^n (\cos(\theta_i))^{i-2} e^{-\frac{g(t)}{2}} dt d\boldsymbol{\theta}, \end{aligned} \quad (17)$$

where $\boldsymbol{\theta} = (\theta_2, \dots, \theta_n)$ and $D_E = E \times [-\pi, \pi] \times [-\pi/2, \pi/2]^{n-2}$.

Next, we consider the case $E = (0, 1)$. Similarly as in Section 2, we introduce a crucial function

$$f^*(t) = \int_{1/2}^t C^*(s) ds, \quad t \in (0, 1),$$

with $C^*(t) = \max\{C_i^{1/\alpha}(t), 1 \leq i \leq k, C_i^{1/\alpha_i}(t), k+1 \leq i \leq n\}, t \in (0, 1)$. We denote by $\overleftarrow{f}^*(t), t \in (f^*(0), f^*(1))$ the inverse function of $f^*(t), t \in (0, 1)$. Further, for any $d > 0$, let $s_{j,d}^{(1)} = \overleftarrow{f}^*(jd), j \in \mathbb{N} \cup \{0\}$ if $f^*(1) = \infty$, and let $s_{j,d}^{(0)} = \overleftarrow{f}^*(-jd), j \in \mathbb{N} \cup \{0\}$ if $f^*(0) = -\infty$. Denote $\Delta_{j,d}^{(1)} = [s_{j-1,d}^{(1)}, s_{j,d}^{(1)}], j \in \mathbb{N}$ and $\Delta_{j,d}^{(0)} = [s_{j,d}^{(0)}, s_{j-1,d}^{(0)}], j \in \mathbb{N}$, which give a partition of $[1/2, 1)$ in the case $f^*(1) = \infty$ and a partition of $(0, 1/2]$ in the case $f^*(0) = -\infty$, respectively.

Let $S \in \{0, 1\}$. Additionally to condition **A**(S) and analogously to conditions **B**(S), **C**(S), **D**(S) we impose the following (scenario-dependent) restrictions on the trend function $g(\cdot)$ and the correlation functions $r_i(\cdot, \cdot)$'s.

Condition B'(S): Suppose that there exists some constant $d_0 > 0$ such that

$$\limsup_{j \rightarrow \infty} \sup_{t \neq s \in \Delta_{j,d_0}^{(S)}} \frac{1 - r_i(t, s)}{K^2(|f^*(t) - f^*(s)|)} < \infty, \quad \forall 1 \leq i \leq n.$$

Condition C'(S): It holds that

$$\left| \int_{1/2}^S C^*(t) \frac{(g(t))^{\frac{n}{2}-1}}{q(g(t))} e^{-\frac{g(t)}{2}} dt \right| < \infty.$$

Condition D'(S): The following is satisfied

$$\limsup_{\delta \rightarrow 0} \sup_{t \neq s \in (0, \delta)} \frac{1 - r_i(|S-t|, |S-s|)}{K^2(|f^*(|S-t|) - f^*(|S-s|)|)} < \infty, \quad \forall 1 \leq i \leq n.$$

We present below the main result of this section.

Theorem 3.2. Let $X_i, 1 \leq i \leq n$ be given as in Theorem 3.1. Then, for each of the following scenarios we have that (17) holds for $E = (0, 1)$.

- (i). $f^*(0) = -\infty, f^*(1) = \infty$, and conditions **A**(0), **B'**(0), **C'**(0), **A**(1), **B'**(1), **C'**(1) are satisfied;
- (ii). $f^*(0) = -\infty, f^*(1) < \infty$, and conditions **A**(0), **B'**(0), **C'**(0), **D'**(1) are satisfied;
- (iii). $f^*(0) > -\infty, f^*(1) = \infty$, and conditions **D'**(0), **A**(1), **B'**(1), **C'**(1) are satisfied;
- (iv). $f^*(0) > -\infty, f^*(1) < \infty$, and conditions **D'**(0), **D'**(1) are satisfied.

Remark 3.3. *The problem becomes more difficult when $\mathbf{b} \neq \mathbf{1}$; the difficulty comes from the fact that the expansion of the correlation function of $Y_{\mathbf{b}}(t, \boldsymbol{\theta})$ as in (19) is too complicated.*

As a direct application of Theorem 3.2, we obtain the following result concerning the tail asymptotics of the supremum of a chi-square process with trend.

Corollary 3.4. *Let $\{B(t), t \in [0, 1]\}$ be the standard Brownian bridge, $\{W(t), t \in [0, 1]\}$ be the standard Brownian motion, and $\{B_H(t), t \in [0, 1]\}$ be the standard fBm with Hurst index $H \in (1/2, 1)$. Further, let $g(\cdot)$ be a nonnegative continuous function on $(0, 1)$. If $g(\cdot)$ satisfies $\mathbf{A}(S)$ with $S \in \{0, 1\}$ and $\int_0^1 \frac{1}{t(1-t)} (g(t))^{3/2} e^{-\frac{g(t)}{2}} dt < \infty$, then, as $u \rightarrow \infty$*

$$\mathbb{P} \left(\sup_{t \in (0,1)} \left(\frac{B^2(t)}{t(1-t)} + \frac{W^2(t)}{t} + \frac{B_H^2(t)}{t^{2H}} - g(t) \right) > u \right) \sim \frac{u^{3/2} e^{-u/2}}{3\sqrt{2\pi}} \int_0^1 \frac{2-t}{t(1-t)} e^{-\frac{g(t)}{2}} dt.$$

4 Proofs

For notational simplicity we denote $\varphi(u) = \prod_{i=k+1}^n (1 - b_i^2)^{-1/2} (q(u))^{-1} u^{k/2-1} e^{-u/2}$ (recall $q(u) = \overline{K}(u^{-1/2})$).

In what follows, we use $\mathcal{C}, \mathcal{C}_1, \mathcal{C}_2, \dots$ to denote unspecified positive and finite constants which may not be the same from line to line.

Proof of Theorem 2.1: Using the classical approach when dealing with extremes of chi-square processes we reduce the problem to the study of extremes of Gaussian random fields; see e.g., [19, 27]. To this end, we introduce two particular Gaussian random fields, namely (denote $D = D_E = E \times [-\pi, \pi] \times [-\pi/2, \pi/2]^{n-2}$)

$$\begin{aligned} Y_{\mathbf{b}}(t, \boldsymbol{\theta}) &= \sum_{i=1}^n b_i X_i(t) v_i(\boldsymbol{\theta}), \\ Y_{\mathbf{b}}^{(u)}(t, \boldsymbol{\theta}) &= \sum_{i=1}^n b_i X_i^{(u)}(t) v_i(\boldsymbol{\theta}), \quad (t, \boldsymbol{\theta}) = (t, \theta_2, \dots, \theta_n) \in D, \end{aligned} \tag{18}$$

where $X_i^{(u)}(t) = \frac{X_i(t)}{\sqrt{1+g(t)/u}}$, $t \in E$, and $v_n(\boldsymbol{\theta}) = \sin(\theta_n)$, $v_{n-1}(\boldsymbol{\theta}) = \sin(\theta_{n-1}) \cos(\theta_n)$, \dots , $v_1(\boldsymbol{\theta}) = \cos(\theta_n) \dots \cos(\theta_2)$ are spherical coordinates. In view of [27], for any $u > 0$

$$\mathbb{P} \left(\sup_{t \in E} (\chi_{\mathbf{b}}^2(t) - g(t)) > u \right) = \mathbb{P} \left(\sup_{(t, \boldsymbol{\theta}) \in D} Y_{\mathbf{b}}^{(u)}(t, \boldsymbol{\theta}) > \sqrt{u} \right).$$

Let $A = E \times [-\pi, \pi] \times [-\pi/2, \pi/2]^{k-2}$ and $B_u = [-m(u), m(u)]^{n-k}$ with $m(u) = \ln(u)/\sqrt{u}$. For any $u > 0$

$$\pi(u) \leq \mathbb{P} \left(\sup_{(t, \boldsymbol{\theta}) \in D} Y_{\mathbf{b}}^{(u)}(t, \boldsymbol{\theta}) > \sqrt{u} \right) \leq \pi(u) + \mathbb{P} \left(\sup_{(t, \boldsymbol{\theta}) \in D \setminus (A \times B_u)} Y_{\mathbf{b}}^{(u)}(t, \boldsymbol{\theta}) > \sqrt{u} \right),$$

where

$$\pi(u) := \mathbb{P} \left(\sup_{(t, \boldsymbol{\theta}) \in A \times B_u} Y_{\mathbf{b}}^{(u)}(t, \boldsymbol{\theta}) > \sqrt{u} \right).$$

Since the variance function of $Y_{\mathbf{b}}^{(u)}$ satisfies for $u > 0$

$$\mathbb{E} \left(\left(Y_{\mathbf{b}}^{(u)}(t, \boldsymbol{\theta}) \right)^2 \right) = \frac{u}{u+g(t)} \left(1 - (1 - b_n^2) \sin^2(\theta_n) - \sum_{i=k+1}^{n-1} (1 - b_i^2) \left(\prod_{j=i+1}^n \cos^2(\theta_j) \right) \sin^2(\theta_i) \right),$$

for all u large we have

$$\sup_{(t, \boldsymbol{\theta}) \in D \setminus (A \times B_u)} \left(\mathbb{E} \left((Y_{\mathbf{b}}(t, \boldsymbol{\theta}))^2 \right) \right)^{1/2} \leq (1 - (1 - b_n^2) \sin^2(m(u)))^{1/2} \leq 1 - \mathcal{C}m^2(u).$$

Further, it can be shown from (5) that

$$\mathbb{E} \left(\left(Y_{\mathbf{b}}(t, \boldsymbol{\theta}) - Y_{\mathbf{b}}(s, \boldsymbol{\theta}') \right)^2 \right) \leq \mathcal{C}_1 |t, \boldsymbol{\theta} - (s, \boldsymbol{\theta}')|^{\alpha/2}, \quad (t, \boldsymbol{\theta}), (s, \boldsymbol{\theta}') \in D.$$

Consequently, the Piterbarg inequality (see e.g., Theorem 8.1 in [28] or [27]) implies

$$\begin{aligned} \mathbb{P} \left(\sup_{(t, \boldsymbol{\theta}) \in D \setminus (A \times B_u)} Y_{\mathbf{b}}^{(u)}(t, \boldsymbol{\theta}) > \sqrt{u} \right) &\leq \mathbb{P} \left(\sup_{(t, \boldsymbol{\theta}) \in D \setminus (A \times B_u)} Y_{\mathbf{b}}(t, \boldsymbol{\theta}) > \sqrt{u} \right) \\ &\leq \mathcal{C}_2 u^{\frac{2n}{\alpha}} \Psi(\sqrt{u}/(1 - \mathcal{C}_3 m^2(u))) \\ &= o(\pi(u)), \quad u \rightarrow \infty, \end{aligned}$$

where the last equality is based on the following lower bound

$$\pi(u) \geq \mathbb{P} \left(Y_{\mathbf{b}}^{(u)}(\delta, 0, \dots, 0) > \sqrt{u} \right) = \Psi(\sqrt{u + g(\delta)})(1 + o(1)), \quad u \rightarrow \infty.$$

Consequently,

$$\mathbb{P} \left(\sup_{(t, \boldsymbol{\theta}) \in D} Y_{\mathbf{b}}^{(u)}(t, \boldsymbol{\theta}) > \sqrt{u} \right) = \pi(u)(1 + o(1)), \quad u \rightarrow \infty.$$

To complete the proof, it is thus sufficient to focus on the asymptotics of $\pi(u)$ as $u \rightarrow \infty$. Next, we split the rectangles A and $\tilde{A} = E \times [-\pi + \delta_1, \pi - \delta_1] \times [-\pi/2 + \delta_1, \pi/2 - \delta_1]^{k-2}$ with $\delta_1 \in (0, \pi/2)$, into several subrectangles denoted by $\{A_j\}_{j \in \Upsilon_1}$ and $\{\tilde{A}_j\}_{j \in \Upsilon_2}$ respectively. Further, let L and \tilde{L} represent their maximum length of edges of these subrectangles, respectively. It follows from Bonferroni's inequality that

$$\sum_{j \in \Upsilon_2} \mathbb{P} \left(\sup_{(t, \boldsymbol{\theta}) \in \tilde{A}_j \times B_u} Y_{\mathbf{b}}^{(u)}(t, \boldsymbol{\theta}) > \sqrt{u} \right) - \Lambda(u) \leq \pi(u) \leq \sum_{j \in \Upsilon_1} \mathbb{P} \left(\sup_{(t, \boldsymbol{\theta}) \in A_j \times B_u} Y_{\mathbf{b}}^{(u)}(t, \boldsymbol{\theta}) > \sqrt{u} \right),$$

where

$$\Lambda(u) = \sum_{j < j_1 \in \Upsilon_2} \mathbb{P} \left(\sup_{(t, \boldsymbol{\theta}) \in \tilde{A}_j \times B_u} Y_{\mathbf{b}}^{(u)}(t, \boldsymbol{\theta}) > \sqrt{u}, \quad \sup_{(t, \boldsymbol{\theta}) \in \tilde{A}_{j_1} \times B_u} Y_{\mathbf{b}}^{(u)}(t, \boldsymbol{\theta}) > \sqrt{u} \right).$$

For any fixed j , we have

$$\mathbb{P} \left(\sup_{(t, \boldsymbol{\theta}) \in A_j \times B_u} Y_{\mathbf{b}}^{(u)}(t, \boldsymbol{\theta}) > \sqrt{u} \right) \leq \mathbb{P} \left(\sup_{(t, \boldsymbol{\theta}) \in A_j \times B_u} Y_{\mathbf{b}}(t, \boldsymbol{\theta}) > \sqrt{u + g_j} \right),$$

where $g_j := \min_{t \in h \circ A_j} g(t)$ and h is a projection function defined by

$$h(x_1, \dots, x_k) = x_1.$$

Further,

$$\left(\mathbb{E} \left((Y_{\mathbf{b}}(t, \boldsymbol{\theta}))^2 \right) \right)^{1/2} = 1 - \sum_{i=k+1}^n \frac{1 - b_i^2}{2} \theta_i^2 (1 + o(1)), \quad \theta_i \rightarrow 0, \quad k+1 \leq i \leq n$$

and

$$\begin{aligned} \text{Corr} \left(Y_{\mathbf{b}}(t, \boldsymbol{\theta}), Y_{\mathbf{b}}(s, \boldsymbol{\theta}') \right) &= 1 - C(t_0)K^2(|t - s|)(1 + o(1)) - \sum_{i=2}^{k-1} \frac{1}{2} \left(\prod_{l=i+1}^k \cos^2 \theta_l \right) (\theta_i - \theta'_i)^2 (1 + o(1)) \\ &\quad - \sum_{i=k}^n \frac{b_i^2}{2} (\theta_i - \theta'_i)^2 (1 + o(1)) \end{aligned} \quad (19)$$

as $t, s \rightarrow t_0, |\theta_i - \theta'_i| \rightarrow 0, 2 \leq i \leq k, \theta_i \rightarrow 0, k+1 \leq i \leq n$. Next we introduce some useful notation. Let $\boldsymbol{\theta}_i = (\theta_i, \dots, \theta_k), 2 \leq i \leq k$, and $h_i, 2 \leq i \leq k$ denote projection functions defined by

$$h_i(x_1, \dots, x_k) = (x_i, \dots, x_k), \quad 2 \leq i \leq k.$$

For any $0 < \epsilon < 1$, there exist $u_0 > 0$ and $L_0 > 0$ such that for all $u > u_0$ and $0 < L < L_0$ (recall that L is the maximum length of the edges of subrectangles $\{A_j\}_{j \in \Upsilon_1}$)

$$\mathbb{P} \left(\sup_{(t, \boldsymbol{\theta}) \in A_j \times B_u} Y_{\mathbf{b}}(t, \boldsymbol{\theta}) > \sqrt{u + g_j} \right) \leq \mathbb{P} \left(\sup_{(t, \boldsymbol{\theta}) \in A_j \times B_u} \tilde{Z}_\epsilon(t, \boldsymbol{\theta}) > \sqrt{u + g_j} \right)$$

holds for all $j \in \Upsilon_1$, where $\tilde{Z}_\epsilon(t, \boldsymbol{\theta}) = \frac{Z_\epsilon(t, \boldsymbol{\theta})}{1 + (1 - \epsilon) \sum_{i=k+1}^n \frac{1 - b_i^2}{2} \theta_i^2}$ and $Z_\epsilon(t, \boldsymbol{\theta})$ is a stationary Gaussian process with unit variance and correlation function $r_{Z_\epsilon}(t, \boldsymbol{\theta})$ satisfying

$$r_{Z_\epsilon}(t, \boldsymbol{\theta}) = 1 - (1 + \epsilon)C_j K^2(|t|)(1 + o(1)) - \sum_{i=2}^k \frac{e_{i,j} + \epsilon}{2} \theta_i^2 (1 + o(1)) - (1 + \epsilon) \sum_{i=k+1}^n \frac{b_i^2}{2} \theta_i^2 (1 + o(1))$$

as $(t, \boldsymbol{\theta}) \rightarrow \mathbf{0}$, with $C_j = \max_{t \in h \circ A_j} C(t)$, $e_{k,j} = 1$ and $e_{i,j} = \max_{\boldsymbol{\theta}_{i+1} \in h_{i+1} \circ A_j} \prod_{l=i+1}^k \cos^2(\theta_l), 2 \leq i \leq k-1$. The existence of such correlation function r_{Z_ϵ} can be confirmed by the Assertion on page 265 in [16], see also the reference mentioned therein. Therefore, by using similar arguments as in Theorem 3.2 in [14] (see also Theorem 8.2 in [27]) we can show that

$$\lim_{u \rightarrow \infty} \frac{\mathbb{P} \left(\sup_{(t, \boldsymbol{\theta}) \in A_j \times B_u} \tilde{Z}_\epsilon(t, \boldsymbol{\theta}) > \sqrt{u + g_j} \right)}{\varphi(u)} = a(\epsilon) (2\pi)^{-k/2} \mathcal{H}_\alpha C_j^{1/\alpha} e^{-\frac{g_j}{2}} \prod_{i=2}^k (e_{i,j})^{1/2} \text{mes}(A_j), \quad (20)$$

where $a(\epsilon) \rightarrow 1$ as $\epsilon \rightarrow 0$. Consequently

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \lim_{L \rightarrow 0} \limsup_{u \rightarrow \infty} \frac{\pi(u)}{\varphi(u)} &\leq (2\pi)^{-k/2} \mathcal{H}_\alpha \int_{(t, \boldsymbol{\theta}_2) \in A} (C(t))^{1/\alpha} e^{-\frac{g(t)}{2}} \prod_{i=3}^k (\cos \theta_i)^{i-2} dt d\theta_2 \dots d\theta_k \\ &= \frac{2^{1-k/2}}{\Gamma(k/2)} \mathcal{H}_\alpha J_E^g. \end{aligned} \quad (21)$$

Similarly, with \tilde{L} the maximum length of the edges of the subrectangles $\{\tilde{A}_j\}_{j \in \Upsilon_1}$ we have

$$\lim_{\delta_1 \rightarrow 0} \lim_{\epsilon \rightarrow 0} \lim_{\tilde{L} \rightarrow 0} \liminf_{u \rightarrow \infty} \frac{\sum_{j \in \Upsilon_2} \mathbb{P} \left(\sup_{(t, \boldsymbol{\theta}) \in \tilde{A}_j \times B_u} Y_{\mathbf{b}}^{(u)}(t, \boldsymbol{\theta}) > \sqrt{u} \right)}{\varphi(u)} \geq \frac{2^{1-k/2}}{\Gamma(k/2)} \mathcal{H}_\alpha J_E^g. \quad (22)$$

We consider next the asymptotic of $\Lambda(u) := \Lambda_1(u) + \Lambda_2(u)$ (to be specified later). For $u > 0$ we have

$$\Lambda_1(u) := \sum_{\tilde{A}_j \cap \tilde{A}_{j_1} = \emptyset} \mathbb{P} \left(\sup_{(t, \boldsymbol{\theta}) \in \tilde{A}_j \times B_u} Y_{\mathbf{b}}^{(u)}(t, \boldsymbol{\theta}) > \sqrt{u}, \quad \sup_{(t, \boldsymbol{\theta}) \in \tilde{A}_{j_1} \times B_u} Y_{\mathbf{b}}^{(u)}(t, \boldsymbol{\theta}) > \sqrt{u} \right)$$

$$\leq \sum_{\tilde{A}_j \cap \tilde{A}_{j_1} = \emptyset} \mathbb{P} \left(\sup_{(t,s,\boldsymbol{\theta},\boldsymbol{\theta}') \in \tilde{A}_j \times \tilde{A}_{j_1} \times B_u \times B_u} (Y_{\mathbf{b}}(t, \boldsymbol{\theta}) + Y_{\mathbf{b}}(s, \boldsymbol{\theta}')) > 2\sqrt{u} \right).$$

Further, there exists some $\delta_0 \in (0, 4)$ such that, for all $\tilde{A}_j \cap \tilde{A}_{j_1} = \emptyset$

$$\mathbb{E} \left(\left(Y_{\mathbf{b}}(t, \boldsymbol{\theta}) + Y_{\mathbf{b}}(s, \boldsymbol{\theta}') \right)^2 \right) \leq 4 - \delta_0, \quad (t, \boldsymbol{\theta}, s, \boldsymbol{\theta}') \in \tilde{A}_j \times B_u \times \tilde{A}_{j_1} \times B_u \quad (23)$$

holds for all large u . Consequently, in the light of Borell-TIS inequality (see e.g., [1])

$$\Lambda_1(u) \leq \mathcal{C} e^{\frac{-(2\sqrt{u}-a)^2}{2(4-\delta_0)}} = o(\varphi(u)), \quad u \rightarrow \infty, \quad (24)$$

where $a = \mathbb{E} \left(\sup_{(t,\boldsymbol{\theta}) \in D} Y_{\mathbf{b}}(t, \boldsymbol{\theta}) \right) < \infty$.

Further, we have

$$\begin{aligned} \Lambda_2(u) &:= \sum_{\tilde{A}_j \cap \tilde{A}_{j_1} \neq \emptyset} \mathbb{P} \left(\sup_{(t,\boldsymbol{\theta}) \in \tilde{A}_j \times B_u} Y_{\mathbf{b}}^{(u)}(t, \boldsymbol{\theta}) > \sqrt{u}, \sup_{(t,\boldsymbol{\theta}) \in \tilde{A}_{j_1} \times B_u} Y_{\mathbf{b}}^{(u)}(t, \boldsymbol{\theta}) > \sqrt{u} \right) \\ &\leq \sum_{\tilde{A}_j \cap \tilde{A}_{j_1} \neq \emptyset} \left[\mathbb{P} \left(\sup_{(t,\boldsymbol{\theta}) \in \tilde{A}_j \times B_u} Y_{\mathbf{b}}^{(u)}(t, \boldsymbol{\theta}) > \sqrt{u} \right) + \mathbb{P} \left(\sup_{(t,\boldsymbol{\theta}) \in \tilde{A}_{j_1} \times B_u} Y_{\mathbf{b}}^{(u)}(t, \boldsymbol{\theta}) > \sqrt{u} \right) \right. \\ &\quad \left. - \mathbb{P} \left(\sup_{(t,\boldsymbol{\theta}) \in (\tilde{A}_j \cup \tilde{A}_{j_1}) \times B_u} Y_{\mathbf{b}}^{(u)}(t, \boldsymbol{\theta}) > \sqrt{u} \right) \right]. \end{aligned}$$

Along the same lines of the proof above

$$\limsup_{u \rightarrow \infty} \frac{\Lambda_2(u)}{\varphi(u)} \leq 3^n (a(\epsilon, \tilde{L}) - b(\epsilon, \tilde{L})) \frac{2^{1-k/2}}{\Gamma(k/2)} \mathcal{H}_\alpha J_E^g,$$

where $a(\epsilon, \tilde{L}), b(\epsilon, \tilde{L}) \rightarrow 1$ as $\tilde{L} \rightarrow 0$ and $\epsilon \rightarrow 0$, which implies that

$$\lim_{\epsilon \rightarrow 0} \lim_{\tilde{L} \rightarrow 0} \limsup_{u \rightarrow \infty} \frac{\Lambda_2(u)}{\varphi(u)} = 0. \quad (25)$$

Consequently, we see from (22), (24) and (25) that

$$\lim_{\delta_1 \rightarrow 0} \lim_{\epsilon \rightarrow 0} \lim_{\tilde{L} \rightarrow 0} \liminf_{u \rightarrow \infty} \frac{\pi(u)}{\varphi(u)} \geq \frac{2^{1-k/2}}{\Gamma(k/2)} \mathcal{H}_\alpha J_E^g,$$

which together with (21) establishes the proof. \square

We present next two results which are crucial to the proof of Theorem 2.4.

Lemma 4.1. *Suppose that $V_{\mathbf{b}}^2(t) = \sum_{i=1}^n b_i^2 V_i^2(t), t \in \mathbb{R}$, with $1 = b_1 = \dots = b_k > b_{k+1} \geq \dots \geq b_n > 0$, is a stationary chi-square process, where $V_i, 1 \leq i \leq n$, are independent copies of a stationary Gaussian process $\{V(t), t \in \mathbb{R}\}$ with unit variance, and correlation function $r(\cdot)$ satisfying $r(t) < 1$ for all $t \neq 0$ and $1 - r(t) \sim K^2(|t|)$ as $t \rightarrow 0$, where $K(\cdot)$ is given as in Introduction. Then for any positive constants $a, b \in \mathbb{R}$ satisfying $a < b$ we have*

$$\mathbb{P} \left(\sup_{t \in [a,b]} V_{\mathbf{b}}^2(t) > u \right) \sim \mathcal{H}_\alpha (b - a) \frac{2^{1-k/2}}{\Gamma(k/2)} \varphi(u)$$

as $u \rightarrow \infty$.

Proof. As in the proof of Theorem 2.1 we can choose to work with the supremum of a Gaussian random fields. The proof follows by similar arguments as those in [26], and thus we omit the details here. \square

Lemma 4.2. (weak Slepian's inequality) *Let $V_i(t), W_i(t), t \in A = [c, d] \subset \mathbb{R}, 1 \leq i \leq n$ be independent centered Gaussian processes with a.s. continuous sample paths. If for all $1 \leq i \leq n$,*

$$\mathbb{E}(V_i^2(t)) = \mathbb{E}(W_i^2(t)), t \in A \quad \text{and} \quad \mathbb{E}(V_i(s)V_i(t)) \geq \mathbb{E}(W_i(s)W_i(t)), s, t \in A$$

are satisfied, then, we have, for all $u > 0$,

$$\mathbb{P}\left(\sup_{t \in A} \sum_{i=1}^n V_i^2(t) > u\right) \leq 2^n \mathbb{P}\left(\sup_{t \in A} \sum_{i=1}^n W_i^2(t) > u\right).$$

Proof. If $n = 1$, then a direct application of Slepian's inequality yields

$$\mathbb{P}\left(\sup_{t \in A} V_1^2(t) > u\right) \leq 2\mathbb{P}\left(\sup_{t \in A} V_1(t) > \sqrt{u}\right) \leq 2\mathbb{P}\left(\sup_{t \in A} W_1(t) > \sqrt{u}\right) \leq 2\mathbb{P}\left(\sup_{t \in A} W_1^2(t) > u\right).$$

Below we shall show the claim for $n \geq 2$. Let therefore $\tilde{X}(t, \mathbf{v}) = \sum_{i=1}^n V_i(t)v_i, (t, \mathbf{v}) \in A \times S_{n-1}$ and $\tilde{Y}(t, \mathbf{v}) = \sum_{i=1}^n W_i(t)v_i, (t, \mathbf{v}) \in A \times S_{n-1}$ be the associated Gaussian random fields. Let $K_j, 1 \leq j \leq 2^n$ denote all the quadrants of \mathbb{R}^n and define $\Delta_j = S_{n-1} \cap K_j, 1 \leq j \leq 2^n$. It follows that for any $1 \leq j \leq 2^n$

$$\mathbb{E}\left(\left(\tilde{X}(t, \mathbf{v})\right)^2\right) = \mathbb{E}\left(\left(\tilde{Y}(t, \mathbf{v})\right)^2\right), \quad (t, \mathbf{v}) \in A \times \Delta_j,$$

and

$$\mathbb{E}\left(\tilde{X}(t, \mathbf{v})\tilde{X}(s, \mathbf{v}')\right) \geq \mathbb{E}\left(\tilde{Y}(t, \mathbf{v})\tilde{Y}(s, \mathbf{v}')\right) \quad (t, \mathbf{v}), (s, \mathbf{v}') \in A \times \Delta_j$$

which is due to the fact that the sign of \mathbf{v} is the same as that of \mathbf{v}' in this case. Thus, by applying Slepian's inequality we obtain, for all $u > 0$,

$$\mathbb{P}\left(\sup_{(t, \mathbf{v}) \in A \times \Delta_j} \tilde{X}(t, \mathbf{v}) > u\right) \leq \mathbb{P}\left(\sup_{(t, \mathbf{v}) \in A \times \Delta_j} \tilde{Y}(t, \mathbf{v}) > u\right),$$

hence the proof follows easily. \square

Proof of Theorem 2.4: We present only the proof for the case (ii); the same arguments apply to other cases. First note that for any small $\delta > 0$

$$\begin{aligned} I_{\delta,1}(u) &:= \mathbb{P}\left(\sup_{t \in [\delta, 1-\delta]} (\chi_{\mathbf{b}}^2(t) - g(t)) > u\right) \\ &\leq \mathbb{P}\left(\sup_{t \in (0,1)} (\chi_{\mathbf{b}}^2(t) - g(t)) > u\right) \\ &\leq I_{\delta,1}(u) + I_{\delta,2}(u) + I_{\delta,3}(u), \end{aligned} \tag{26}$$

where

$$I_{\delta,2}(u) := \mathbb{P}\left(\sup_{t \in (0,\delta]} (\chi_{\mathbf{b}}^2(t) - g(t)) > u\right), \quad I_{\delta,3}(u) := \mathbb{P}\left(\sup_{t \in [1-\delta,1)} (\chi_{\mathbf{b}}^2(t) - g(t)) > u\right).$$

Since from Theorem 2.1

$$\lim_{\delta \rightarrow 0} \lim_{u \rightarrow \infty} \frac{I_{\delta,1}(u)}{\varphi(u)} = \mathcal{H}_\alpha \frac{2^{1-k/2}}{\Gamma(k/2)} \int_0^1 (C(t))^{1/\alpha} e^{-\frac{g(t)}{2}} dt$$

it is sufficient to show that

$$\limsup_{\delta \rightarrow 0} \limsup_{u \rightarrow \infty} \frac{I_{\delta,2}(u)}{\varphi(u)} = \limsup_{\delta \rightarrow 0} \limsup_{u \rightarrow \infty} \frac{I_{\delta,3}(u)}{\varphi(u)} = 0. \quad (27)$$

We first consider $I_{\delta,2}(u)$. Without loss of generality, we choose d as a parameter taking values in $\{2^{-m}d_0, m \in \mathbb{N}^+\}$. It is straightforward that

$$\begin{aligned} I_{\delta,2}(u) &\leq \sum_{j=N_{d,\delta}}^{\infty} \mathbb{P} \left(\sup_{t \in \Delta_{j,d}^{(0)}} (\chi_{\mathbf{b}}^2(t) - g(t)) > u \right) \\ &\leq \sum_{j=N_{d,\delta}}^{\infty} \mathbb{P} \left(\sup_{t \in \Delta_{j,d}^{(0)}} \chi_{\mathbf{b}}^2(t) > u + g(s_{j-1,d}^{(0)}) \right), \end{aligned} \quad (28)$$

where $N_{d,\delta} = \lceil -\frac{f(\delta)}{d} \rceil$ with $\lceil \cdot \rceil$ the ceiling function. Further, it follows from condition **B**(0) that there exist some $M_1 > 1, \delta_1 > 0$ and $d_1 > 0$ such that for all $\delta \in (0, \delta_1), d \in (0, d_1), j \geq N_{d,\delta}$,

$$1 - r(t, s) \leq M_1 K^2 (|f(t) - f(s)|) \leq \frac{1}{2} K^2 ((4M_1)^{1/\alpha} |f(t) - f(s)|), \quad t, s \in \Delta_{j,d}^{(0)}$$

hold. Next let $r(\cdot)$ be as in Lemma 4.1. Then for some constant $\delta_2 > 0$

$$1 - r(t) \geq \frac{1}{2} K^2 (|t|), \quad t \in (0, \delta_2). \quad (29)$$

Therefore, for any $d \in (0, \min(d_1, (4M_1)^{-1/\alpha} \delta_2))$ and $\delta \in (0, \delta_1)$

$$\mathbb{E}(X(t)X(s)) \geq \mathbb{E} \left(V((4M_1)^{1/\alpha} f(t)) V((4M_1)^{1/\alpha} f(s)) \right)$$

holds for all $t, s \in \Delta_{j,d}^{(0)}$ and $j \geq N_{d,\delta}$, where V is the stationary Gaussian process given in Lemma 4.1. Consequently, in the light of Lemma 4.2 we have for $\delta \in (0, \delta_1), d \in (0, \min(d_1, (4M_1)^{-1/\alpha} \delta_2))$ and $j \geq N_{d,\delta}$

$$\begin{aligned} \mathbb{P} \left(\sup_{t \in \Delta_{j,d}^{(0)}} \chi_{\mathbf{b}}^2(t) > u + g(s_{j-1,d}^{(0)}) \right) &\leq 2^n \mathbb{P} \left(\sup_{t \in \Delta_{j,d}^{(0)}} V_{\mathbf{b}}^2((4M_1)^{1/\alpha} f(t)) > u + g(s_{j-1,d}^{(0)}) \right) \\ &\leq 2^n \mathbb{P} \left(\sup_{t \in \Delta_d} V_{\mathbf{b}}^2(t) > u + g(s_{j-1,d}^{(0)}) \right) \end{aligned}$$

holds, where $\Delta_d = [0, (4M_1)^{1/\alpha} d]$ and $V_{\mathbf{b}}^2$ is the chi-square process given in Lemma 4.1. Next, by Lemma 4.1 we derive for any $d \in (0, \min(d_1, (4M_1)^{-1/\alpha} \delta_2))$ and $\delta \in (0, \delta_1)$

$$\begin{aligned} &\mathbb{P} \left(\sup_{t \in \Delta_d} V_{\mathbf{b}}^2(t) > u + g(s_{j-1,d}^{(0)}) \right) \\ &\sim H_{\alpha} \frac{2^{1-k/2}}{\Gamma(k/2)} (4M_1)^{1/\alpha} d \prod_{i=k+1}^n (1 - b_i^2)^{-1/2} \frac{(u + g(s_{j-1,d}^{(0)}))^{k/2-1}}{q(u + g(s_{j-1,d}^{(0)}))} e^{-(u + g(s_{j-1,d}^{(0)}))/2} \end{aligned} \quad (30)$$

as $u \rightarrow \infty$. Next, for any $\alpha \in (0, 2)$ or $\alpha = 2, k > 1$, since $\frac{t^{k/2-1}}{q(t)}$ is a regularly varying function at infinity with index $k/2 - 1 + 1/\alpha > 0$, by Karamata's theorem (see, e.g., [33]) we have

$$\frac{(u + g(s_{j-1,d}^{(0)}))^{k/2-1}}{q(u + g(s_{j-1,d}^{(0)}))} \leq C \frac{(g(s_{j-1,d}^{(0)}))^{k/2-1}}{q(g(s_{j-1,d}^{(0)}))} \frac{u^{k/2-1}}{q(u)}$$

holds for all $u > u_0$ and all $j \geq N_{d,\delta}$, with some constant $u_0 > 1$. Moreover, we can write $\frac{t^{k/2-1}}{q(t)} = t^{k/2-1+1/\alpha} \ell(t)$ with $\ell(t)$ a normalized slowly varying function (see e.g., [4]). One can check that $t^{k/2-1+1/\alpha} \ell(t) e^{-\frac{t}{2}}$, $t \geq t_0$ is decreasing for some large t_0 . Therefore, we conclude that for δ and d small enough

$$\begin{aligned} \lim_{u \rightarrow \infty} \frac{I_{\delta,2}(u)}{\varphi(u)} &\leq \mathcal{C} \sum_{j=N_{d,\delta}}^{\infty} \frac{(g(s_{j-1,d}^{(0)})^{k/2-1} e^{-\frac{g(s_{j-1,d}^{(0)})}{2}})}{q(g(s_{j-1,d}^{(0)}))} d \\ &\leq \mathcal{C} \sum_{j=N_{d,\delta}}^{\infty} \int_{-(j-1)d}^{-(j-2)d} \frac{(g(\overleftarrow{f}(t)))^{k/2-1}}{q(g(\overleftarrow{f}(t)))} e^{-\frac{g(\overleftarrow{f}(t))}{2}} dt \\ &\leq \mathcal{C} \int_0^{\overleftarrow{f}(f(\delta)+3d)} (C(t))^{1/\alpha} \frac{(g(t))^{k/2-1}}{q(g(t))} e^{-\frac{g(t)}{2}} dt. \end{aligned} \quad (31)$$

Consequently, letting $\delta \rightarrow 0$, we derive by **C(0)** that (27) holds for $I_{\delta,2}(u)$ when $\alpha \in (0, 2)$ or $\alpha = 2, k > 1$. For $\alpha = 2, k = 1$, we have from (10) that

$$\frac{(u + g(s_{j-1,d}^{(0)})^{k/2-1})}{q(u + g(s_{j-1,d}^{(0)}))} \leq \mathcal{C}_1, \quad \frac{u^{k/2-1}}{q(u)} \geq \mathcal{C}_2.$$

Then, similar arguments as above show that (27) still holds for $I_{\delta,2}(u)$ when $\alpha = 2, k = 1$.

Below we consider $I_{\delta,3}(u)$. Let $Y(t) = X(\overleftarrow{f}(t))$, $t \in [f(1-\delta), f(1)]$. Then correlation function of it is given by $r_Y(t, s) = r(\overleftarrow{f}(t), \overleftarrow{f}(s))$. It follows from **D(1)** that for any δ small enough there exists some $M_3 > 1$ such that

$$\sup_{t \neq s \in (1-\delta, 1)} \frac{1 - r(t, s)}{K^2(|f(t) - f(s)|)} < M_3,$$

which means that

$$1 - r_Y(t, s) \leq M_3 K^2(|t - s|) \leq \frac{1}{2} K^2((8M_3)^{1/\alpha} |t - s|), \quad s \neq t \in [f(1-\delta), f(1)].$$

Furthermore, with the aid of (29) we have for δ small enough

$$1 - r_Y(t, s) \leq 1 - r((8M_3)^{1/\alpha} |t - s|), \quad s \neq t \in [f(1-\delta), f(1)].$$

Thus, similarly as the above case for $I_{\delta,2}(u)$, it follows that

$$\begin{aligned} I_{\delta,3}(u) &\leq 2^n \mathbb{P} \left(\sup_{t \in [f(1-\delta), f(1)]} V_{\mathbf{b}}^2((8M_3)^{1/\alpha} t) > u \right) \\ &\leq 2^n \mathbb{P} \left(\sup_{t \in [0, (f(1)-f(1-\delta))(8M_3)^{1/\alpha}]} V_{\mathbf{b}}^2(t) > u \right) \\ &\leq 2^{n+1} (f(1) - f(1-\delta)) (8M_3)^{1/\alpha} \mathcal{H}_\alpha \frac{2^{1-k/2}}{\Gamma(k/2)} \varphi(u) \end{aligned} \quad (32)$$

holds for δ small enough and u large enough. Consequently, in light of $f(1) < \infty$ we derive that (27) holds for $I_{\delta,3}(u)$. The proof is complete. \square

Proof of Corollary 2.6: By direct calculation, we have

$$1 - \mathbb{E}(\chi(t)\chi(s)) = \sqrt{\frac{s}{t} \frac{1-t}{1-s}} \left(\sqrt{\left(1 + \frac{t-s}{s}\right) \left(1 + \frac{t-s}{1-t}\right)} - 1 \right)$$

$$= \frac{|t-s|}{2t_0(1-t_0)}(1+o(1)), \quad s < t \rightarrow t_0 \in (0, 1),$$

which, in the notation of (5), means that $C(t) = \frac{1}{2t(1-t)}$, $K^2(t) = t$ and $\alpha = 1$. Further

$$f(t) = \int_{1/2}^t \frac{1}{2s(1-s)} ds = \frac{1}{2} \ln \frac{t}{1-t}, \quad f(1) = -f(0) = \infty.$$

Thus, we have

$$s_{j,d}^{(0)} = \overleftarrow{f}(-jd) = \frac{1}{1+e^{2jd}}, \quad s_{j,d}^{(1)} = \overleftarrow{f}(jd) = \frac{1}{1+e^{-2jd}} \quad j \in \mathbb{N} \cup \{0\}, d > 0.$$

Next we verify conditions of Theorem 2.4 and Theorem 2.3. Let therefore in the following $S \in \{0, 1\}$.

Elementary calculations yield that

$$g'_\nu(t) = \frac{1+c^2(t)+2\nu c(t)}{1+c^2(t)} \frac{2t-1}{t(1-t)(1-\ln(4t(1-t)))}.$$

Since $1-\ln(4t(1-t)) \geq 1$ and $c(t) \geq 0$ for $t \in (0, 1)$, we have $g'_\nu(t) < 0, t \in (0, 1/2)$ and $g'_\nu(t) > 0, t \in (1/2, 1)$, implying that $\mathbf{A}(S)$ is satisfied since $g_\nu(t) \uparrow \infty$ as $t \rightarrow 0$ and $t \rightarrow 1$. Now, for any $s < t$ and $s, t \in \Delta_{j,d}^{(0)} = [\frac{1}{1+e^{2jd}}, \frac{1}{1+e^{2(j-1)d}}]$ we have

$$e^{-2d} \leq \frac{s}{t} \leq 1, \quad \frac{1}{2} \leq \frac{1-t}{1-s} \leq 1, \quad \frac{t-s}{s} \leq e^{2d} - 1, \quad \frac{t-s}{1-t} \leq e^{2d} - 1.$$

Thus, there exists $d_0 > 0$ such that for $s < t \in \Delta_{j,d_0}^{(0)} = [\frac{1}{1+e^{2jd_0}}, \frac{1}{1+e^{2(j-1)d_0}}]$

$$\begin{aligned} \frac{1-r(s,t)}{|f(t)-f(s)|} &= \frac{2\sqrt{\frac{s}{t}\frac{1-t}{1-s}} \left(\sqrt{(1+\frac{t-s}{s})(1+\frac{t-s}{1-t})} - 1 \right)}{\ln(1+\frac{t-s}{s}) + \ln(1+\frac{t-s}{1-t})} \\ &\leq 4 \frac{\frac{t-s}{s} + \frac{t-s}{1-t}}{\frac{1}{2}(\frac{t-s}{s} + \frac{t-s}{1-t})} = 8 \end{aligned}$$

and

$$\frac{1-r(s,t)}{|f(t)-f(s)|} \geq e^{-d_0} \frac{\frac{t-s}{s} + \frac{t-s}{1-t}}{2(\frac{t-s}{s} + \frac{t-s}{1-t})} \geq \frac{e^{-d_0}}{2}.$$

Moreover, for all $t \in \Delta_{j,d_0}^{(0)}, s \in \Delta_{j+l,d_0}^{(0)}$, with $l \in \mathbb{N}$ large enough,

$$|r(s,t)| = \frac{s(1-t)}{\sqrt{st(1-t)(1-s)}} \leq 2\sqrt{\frac{s}{t}} \leq 2e^{-(l-1)d_0} \leq Cl^{-2}.$$

Similarly, we can also derive that for $s < t \in \Delta_{j,d_0}^{(1)} = [\frac{1}{1+e^{-2(j-1)d_0}}, \frac{1}{1+e^{-2jd_0}}]$

$$\frac{e^{-d_0}}{2} \leq \frac{1-r(s,t)}{|f(t)-f(s)|} \leq 8,$$

and for all $t \in \Delta_{j,d_0}^{(1)}, s \in \Delta_{j+l,d_0}^{(1)}$, with $l \in \mathbb{N}$ large enough,

$$|r(s,t)| \leq Cl^{-2}.$$

Therefore, conditions $\mathbf{B}(S)$, $\mathbf{E}(S)$ are satisfied. Direct calculations show that

$$c(t) \sim \ln \ln \frac{1}{t}, \quad g_\nu(t) \sim \ln \ln \frac{1}{t}, \quad e^{-g_\nu(t)} \sim \frac{1}{(\ln \ln \frac{1}{t})^{2\nu} \ln \frac{1}{t}}$$

as $t \rightarrow 0$. Consequently,

$$\frac{(g_\nu(t))^{1/2}}{t(1-t)} e^{-g_\nu(t)} \sim \frac{1}{t \ln \frac{1}{t} (\ln \ln \frac{1}{t})^{2\nu-1/2}}$$

as $t \rightarrow 0$. This implies that $\int_{1/2}^1 \frac{(g_\nu(t))^{1/2}}{t(1-t)} e^{-g_\nu(t)} dt = \int_0^{1/2} \frac{(g_\nu(t))^{1/2}}{t(1-t)} e^{-g_\nu(t)} dt < \infty$ if and only if $\nu > 3/4$. Consequently, the claim follows by an application of Theorem 2.4 and Theorem 2.3. This completes the proof. \square

Proof of Corollary 2.7: As discussed in Introduction, the correlation of the fBm satisfies

$$1 - \text{Corr}(B_H(t), B_H(s)) = \frac{|t-s|^{2H}}{2t^{2H}}(1 + o(1)), \quad s, t \rightarrow t_0 \in (0, 1],$$

which means that

$$C(t) = \frac{1}{2t^{2H}}, \quad K^2(t) = t^{2H}, \quad \alpha = 2H.$$

We only need to verify the conditions **B**(0) and **E**(0). It follows that

$$\begin{aligned} f(t) &= 2^{-\frac{1}{2H}} \ln(2t), \quad s_{j,d}^{(0)} = \overleftarrow{f}(-jd) = \frac{1}{2} e^{-2^{\frac{1}{2H}} jd}, \\ \Delta_{j,d}^{(0)} &= \left[\frac{1}{2} e^{-2^{\frac{1}{2H}} jd}, \frac{1}{2} e^{-2^{\frac{1}{2H}} (j-1)d} \right], \quad j \in \mathbb{N}. \end{aligned}$$

Without loss of generality, hereafter we assume that $s < t$. For any $s, t \in \Delta_{j,d}^{(0)}$, $j \in \mathbb{N}$ we have $\frac{t-s}{s} \leq e^{2^{\frac{1}{2H}} d} - 1$, which implies that there exists $d_0 > 0$ such that

$$\frac{t-s}{2s} \leq \left| \ln\left(1 + \frac{t-s}{s}\right) \right| \leq \frac{2(t-s)}{s}$$

holds for all $s, t \in \Delta_{j,d_0}^{(0)}$, $j \in \mathbb{N}$. Moreover, for $s, t \in \Delta_{j,d_0}^{(0)}$, $j \in \mathbb{N}$ with d_0 small enough,

$$\frac{(t^H - s^H)^2}{|t-s|^{2H}} = \frac{\left(1 - \left(\frac{s}{t}\right)^H\right)^2}{\left|\frac{t-s}{t}\right|^{2H}} \leq 2H^2 \left(\frac{t-s}{t}\right)^{2-2H} < 1/2.$$

Thus, we have, for all $s, t \in \Delta_{j,d_0}^{(0)}$, $j \in \mathbb{N}$ with d_0 small enough

$$\frac{1 - r(s, t)}{|f(t) - f(s)|^{2H}} \leq \frac{|t-s|^{2H}}{t^H s^H |\ln(1 + \frac{t-s}{s})|^{2H}} \leq 2^{2H} \frac{s^H}{t^H} \leq 2^{2H},$$

and

$$\frac{1 - r(s, t)}{|f(t) - f(s)|^{2H}} \geq \frac{|t-s|^{2H}}{2t^H s^H |\ln(1 + \frac{t-s}{s})|^{2H}} \geq 2^{-2H-1} \frac{s^H}{t^H} \geq 2^{-2H-1} e^{-2^{\frac{1}{2H}} d_0 H} > 0.$$

In addition,

$$\begin{aligned} |r(s, t)| &= \frac{|t^{2H} + s^{2H} - |t-s|^{2H}|}{t^H s^H} = \frac{\left|\left(\frac{s}{t}\right)^{2H} + 1 - \left(1 - \frac{s}{t}\right)^{2H}\right|}{\left(\frac{s}{t}\right)^H} \leq \frac{\left(\frac{s}{t}\right)^{2H} + 4H \frac{s}{t}}{\left(\frac{s}{t}\right)^H} \\ &\leq C \left(e^{-2^{\frac{1}{2H}} d H l} + e^{-2^{\frac{1}{2H}} d (1-H) l} \right) \leq C l^{-2}, \end{aligned}$$

holds for $t \in \Delta_{j,d_0}^{(0)}$, $s \in \Delta_{j+l,d_0}^{(0)}$, $j \in \mathbb{N}$, with l sufficiently large and with d_0 small enough. Consequently, both conditions **B**(0) and **E**(0) are satisfied, and thus the claim follows by applying Theorem 2.4 and Theorem 2.3. This completes the proof. \square

Proof of Theorem 3.1: The proof is similar to that of Theorem 2.1. The main difference is the expansion of the correlation function of $Y_1(t, \theta)$ defined in (18). Here it is given by (compare with (19))

$$\begin{aligned} \text{Corr}\left(Y_1(t, \boldsymbol{\theta}), Y_1(s, \boldsymbol{\theta}')\right) &= 1 - \left(C_1(t_0) \prod_{j=2}^n \cos^2 \theta_j + \sum_{i=2}^k C_i(t_0) \sin^2 \theta_i \prod_{j=i+1}^n \cos^2 \theta_j \right) K^2(|t-s|)(1+o(1)) \\ &\quad - \sum_{i=2}^{n-1} \frac{1}{2} \left(\prod_{l=i+1}^n \cos^2 \theta_l \right) (\theta_i - \theta'_i)^2 (1+o(1)) - \frac{1}{2} (\theta_n - \theta'_n)^2 (1+o(1)) \end{aligned}$$

as $t, s \rightarrow t_0, |\theta_i - \theta'_i| \rightarrow 0, 2 \leq i \leq n$. The rest of the proof is the same as that in the proof of Theorem 2.1. This completes the proof. \square

Proof of Theorem 3.2: The proof follows with the same arguments as those in the proof of Theorem 2.4, and thus being omitted here. \square

Proof of Corollary 3.4: Denote for any $s, t \in (0, 1)$

$$r_1(s, t) = \text{Corr}(B(t), B(s)), r_2(s, t) = \text{Corr}(W(t), W(s)), r_3(s, t) = \text{Corr}(B_H(t), B_H(s)).$$

In view of the proof of Corollary 2.6 and Corollary 2.7, we have (in the notation of Section 3)

$$C_1(t) = \frac{1}{2t(1-t)}, C_2(t) = \frac{1}{2t}, C_3(t) = \frac{1}{2^{2H} t^{2H}}, K^2(t) = t, K_3^2(t) = 2^{2H-1} t^{2H}, k = 2 < 3 = n.$$

Then, we have for $H \in (1/2, 1)$

$$\begin{aligned} C^*(t) &= \max\left(\frac{1}{2t(1-t)}, \frac{1}{2t}, \frac{1}{2t}\right) = \frac{1}{2t(1-t)}, f^*(t) = \frac{1}{2} \ln \frac{t}{1-t}, \\ s_{j,d}^{(0)} &= \overleftarrow{f^*}(-jd) = \frac{1}{1+e^{2jd}}, s_{j,d}^{(1)} = \overleftarrow{f^*}(jd) = \frac{1}{1+e^{-2jd}} \quad j \in \mathbb{N} \cup \{0\}, d > 0. \end{aligned}$$

Without loss of generality, we assume that $s < t$. Since $\frac{t-s}{s} \leq e^{2d} - 1$ and $\frac{t-s}{1-t} \leq e^{2d} - 1$ hold for $s, t \in \Delta_{j,d}^{(0)}$ or $s, t \in \Delta_{j,d}^{(1)}$, $j \in \mathbb{N}$, we have there exists some $d_0 > 0$ such that

$$\ln\left(1 + \frac{t-s}{s}\right) \geq \frac{t-s}{2s}, \quad \ln\left(1 + \frac{t-s}{1-t}\right) \geq \frac{t-s}{2(1-t)}, \quad 1/2 < \frac{s}{t} \leq 1$$

hold for $s, t \in \Delta_{j,d_0}^{(0)}$ or $s, t \in \Delta_{j,d_0}^{(1)}$, $j \in \mathbb{N}$. This implies, for any $s < t \in \Delta_{j,d_0}^{(0)}$ or any $s < t \in \Delta_{j,d_0}^{(1)}$

$$\frac{1 - r_2(s, t)}{|f^*(t) - f^*(s)|} = \frac{2(t-s)}{\sqrt{t}(\sqrt{t} + \sqrt{s}) |\ln(1 + \frac{t-s}{s}) + \ln(1 + \frac{t-s}{1-t})|} \leq \frac{4s(1-t)}{\sqrt{t}(\sqrt{t} + \sqrt{s})(1-t+s)} \leq 2$$

and

$$\begin{aligned} \frac{1 - r_3(s, t)}{|f^*(t) - f^*(s)|} &= \frac{|t-s|^{2H} - (t^H - s^H)^2}{t^H s^H |\ln(1 + \frac{t-s}{s}) + \ln(1 + \frac{t-s}{1-t})|} \leq \frac{|t-s|^{2H}}{t^H s^H |\ln(1 + \frac{t-s}{s}) + \ln(1 + \frac{t-s}{1-t})|} \\ &\leq \frac{2|t-s|^{2H-1} s(1-t)}{t^H s^H (1-t+s)} \leq \frac{2|t-s|^{2H-1} s^{1-H}}{t^H} \leq \sup_{x \in (1/2, 1]} 2|1-x|^{2H-1} x^{H-1} < \infty. \end{aligned}$$

Thus, condition $\mathbf{B}(S)$ with $S = 0, 1$ holds for $r_2(\cdot, \cdot), r_3(\cdot, \cdot)$. Additionally, it has been shown in proof of Corollary 2.6 that condition $\mathbf{B}(S)$ with $S = 0, 1$ holds for $r_1(\cdot, \cdot)$. Consequently, in the light of Theorem 3.2 we derive that, as $u \rightarrow \infty$

$$\mathbb{P}\left(\sup_{t \in (0, 1)} \left(\frac{B^2(t)}{t(1-t)} + \frac{W^2(t)}{t} + \frac{B_H^2(t)}{t^{2H}} - g(t)\right) > u\right)$$

$$\begin{aligned}
& \sim (2\pi)^{-3/2} u^{3/2} e^{-u/2} \int_0^1 dt \int_{-\pi}^{\pi} d\theta_2 \\
& \times \int_{-\pi/2}^{\pi/2} \left(\frac{1}{2t(1-t)} \cos^2(\theta_2) \cos^2(\theta_3) + \frac{1}{2t} \sin^2(\theta_2) \cos^2(\theta_3) \right) \cos(\theta_3) e^{-\frac{g(t)}{2}} d\theta_3 \\
& = \frac{u^{3/2} e^{-u/2}}{3\sqrt{2\pi}} \int_0^1 \frac{2-t}{t(1-t)} e^{-\frac{g(t)}{2}} dt.
\end{aligned}$$

This completes the proof. \square

5 Appendix

This section is concerned with the proof of Theorem 2.3. We first present, in the following theorem, an asymptotic 0-1 behavior of the chi-square process $\chi_{\mathbf{b}}^2$, which is complementary to the results for Gaussian processes discussed in [30, 31, 35]. Moreover, it can also be viewed as a generalization of the Kolmogorov-Dvoretzky-Erdős integral test theorem; see, e.g., page 163 in [17] or Theorem A in [20]. Recall that, for the Bessel process $\|\mathbf{W}_n(t)\| := \sqrt{\sum_{i=1}^n W_i^2(t)}$, $t \geq 0$ defined in Corollary 2.8, the Kolmogorov-Dvoretzky-Erdős integral test theorem tells that for any positive continuous ultimately increasing $g(\cdot)$ when $t \rightarrow 0$,

$$\mathbb{P}\left(\|\mathbf{W}_n(t)\| \leq \sqrt{tg(t)}, \text{ ultimately as } t \rightarrow 0\right) = 1 \quad \text{or} \quad 0$$

holds according as

$$\int_0^1 \frac{1}{t} (g(t))^{n/2} e^{-\frac{g(t)}{2}} dt < \infty \quad \text{or} \quad = \infty.$$

Theorem 5.1. *Let $\{X(t), t \in E\}$ be given as in Theorem 2.1 with $E = (0, 1)$, and $S \in \{0, 1\}$. Then, for any nonnegative continuous function $g(t)$ satisfying $\mathbf{A}(S)$,*

$$\mathbb{P}\left(\chi_{\mathbf{b}}^2(t) \leq g(t) \text{ ultimately as } t \rightarrow S\right) = 1 \tag{33}$$

holds provided that $I_g(S) < \infty$, $|f(S)| = \infty$ and condition $\mathbf{B}(S)$ is satisfied, or $|f(S)| < \infty$ and condition $\mathbf{D}(S)$ is satisfied, and

$$\mathbb{P}\left(\chi_{\mathbf{b}}^2(t) \leq g(t) \text{ ultimately as } t \rightarrow S\right) = 0 \tag{34}$$

holds provided that $I_g(S) = \infty$ and condition $\mathbf{E}(S)$ is satisfied.

Proof. The idea of the proof comes from [30, 31, 35]. Without loss of generality, we present only the proof for the case $S = 0$. We prove first that (33) holds if $I_g(0) < \infty$, $f(0) = -\infty$ and condition $\mathbf{B}(0)$ is satisfied.

Let

$$E_{j,d} = \left\{ \sup_{t \in \Delta_{j,d}^{(0)}} \chi_{\mathbf{b}}^2(t) > g(s_{j-1,d}^{(0)}) \right\}, \quad j \in \mathbb{N}, d > 0.$$

Similarly to the derivation of (31), we have, for δ, d small,

$$\begin{aligned}
\sum_{j=N_{d,\delta}}^{\infty} \mathbb{P}(E_{j,d}) & \leq 2^n \sum_{j=N_{d,\delta}}^{\infty} H_{\alpha} \frac{2^{1-k/2}}{\Gamma(k/2)} (4M_1)^{1/\alpha} d \varphi(g(s_{j-1,d}^{(0)})) \\
& \leq C \sum_{j=N_{d,\delta}}^{\infty} \frac{(g(s_{j-1,d}^{(0)}))^{\frac{k}{2}-1}}{q(g(s_{j-1,d}^{(0)}))} e^{-\frac{g(s_{j-1,d}^{(0)})}{2}} d
\end{aligned}$$

$$\begin{aligned}
&\leq \mathcal{C} \sum_{j=N_{d,\delta}}^{\infty} \int_{-(j-1)d}^{-(j-2)d} \frac{(g(\overleftarrow{f}(t)))^{k/2-1}}{q(g(\overleftarrow{f}(t)))} e^{-\frac{g(\overleftarrow{f}(t))}{2}} dt \\
&\leq \mathcal{C} \int_0^{1/2} (C(t))^{1/\alpha} \frac{(g(t))^{\frac{k}{2}-1}}{q(g(t))} e^{-\frac{g(t)}{2}} dt < \infty
\end{aligned} \tag{35}$$

holds when $\alpha \in (0, 2)$ or $\alpha = 2, k > 1$, and

$$\sum_{j=N_{d,\delta}}^{\infty} \mathbb{P}(E_{j,d}) \leq \mathcal{C}_1 \int_0^{1/2} (C(t))^{1/\alpha} e^{-\frac{g(t)}{2}} dt < \infty$$

holds when $\alpha = 2, k = 1$. Thus, by Borel-Cantelli lemma

$$\mathbb{P}\left(\exists j_g : \sup_{t \in \Delta_{j,d}^{(0)}} \chi_{\mathbf{b}}^2(t) \leq g(s_{j-1,d}^{(0)}) \text{ for all } j \geq j_g\right) = 1$$

which implies (33) since $g(t) \uparrow$ as $t \downarrow 0$. Next we prove (33) under the conditions that $f(0) > -\infty$ and $\mathbf{D}(0)$ is satisfied. Let for any fixed small $\delta > 0$, $t_n = \delta/n, n \in \mathbb{N}$. Denote

$$H_{n,\delta} = \left\{ \sup_{t \in [t_{n+1}, t_n]} \chi_{\mathbf{b}}^2(t) > g(t_n) \right\}.$$

Similar arguments as in (32) yield that, for N_0 sufficiently large

$$\begin{aligned}
\sum_{n=N_0}^{\infty} \mathbb{P}(H_{n,\delta}) &= \sum_{n=N_0}^{\infty} \mathbb{P}\left(\sup_{t \in [f(t_{n+1}), f(t_n)]} \chi_{\mathbf{b}}^2(\overleftarrow{f}(t)) > g(t_n)\right) \\
&\leq \mathcal{C} \sum_{n=N_0}^{\infty} (f(t_n) - f(t_{n+1})) \varphi(g(t_n)) \\
&\leq \mathcal{C}_1 (f(t_{N_0}) - f(0)) < \infty
\end{aligned}$$

holds. Thus, the proof of (33) is again completed by Borel-Cantelli lemma. Finally, we show (34) when $I_g(0) = \infty$ and condition $\mathbf{E}(0)$ is satisfied. Note that

$$\begin{aligned}
&\mathbb{P}(\chi_{\mathbf{b}}^2(t) \leq g(t) \text{ ultimately as } t \rightarrow 0) \\
&\leq \mathbb{P}\left(\sup_{\mathbf{v} \in \mathcal{S}_{k-1}} Y_k(t, \mathbf{v}) \leq \sqrt{g(t)} \text{ ultimately as } t \rightarrow 0\right) \\
&= \mathbb{P}\left(\sup_{\mathbf{v} \in \mathcal{S}_{k-1}} Y_k(\overleftarrow{f}(-t), \mathbf{v}) \leq \sqrt{g(\overleftarrow{f}(-t))} \text{ ultimately as } t \rightarrow \infty\right),
\end{aligned} \tag{36}$$

where $Y_k(t, \mathbf{v}) = \sum_{i=1}^k X_i(t) v_i$, $(t, \mathbf{v}) \in (0, 1/2] \times \mathcal{S}_{k-1}$, with \mathcal{S}_{k-1} the $(k-1)$ -dimensional unit sphere. By similar arguments as in Lemma 1.4 in [30] or Lemma 3.1 in [31], one can for free assume that

$$2 \ln t \leq g(\overleftarrow{f}(-t)) \leq 3 \ln t, \quad t \in (1, \infty). \tag{37}$$

Denote

$$\begin{aligned}
F_{j,d} &= \left\{ (j-1)d + l q(g(s_{j,d}^{(0)})), \quad l = 0, 1, \dots, \left\lfloor \frac{d}{q(g(s_{j,d}^{(0)}))} \right\rfloor \right\}, \\
G_{j,d} &= \left\{ \left(l_1 h_{j,d}, l_2 h_{j,d}, \dots, l_{k-1} h_{j,d}, \pm \sqrt{1 - \sum_{i=1}^{k-1} l_i^2 h_{j,d}^2} \right), \quad l_1, \dots, l_{k-1} \in \mathbb{Z} \right\},
\end{aligned}$$

with $h_{j,d} = 1/\sqrt{g(s_{j,d}^{(0)})}$. Further, define

$$W_{j,d} = \left\{ \sup_{t \in F_{j,d}} \sup_{\mathbf{v} \in G_{j,d} \cap \mathcal{S}_{k-1}} Y_k(\overleftarrow{f}(-t), \mathbf{v}) \leq \sqrt{g(s_{j,d}^{(0)})} \right\}.$$

In view of (36), to prove (34) it is sufficient that

$$\mathbb{P}(W_{il_0,d}^c \text{ i.o.}) = 1$$

holds, with l_0 as in condition **E**. It is noted that

$$1 - \mathbb{P}(W_{il_0,d}^c \text{ i.o.}) = \lim_{l \rightarrow \infty} \prod_{i=l}^{\infty} \mathbb{P}(W_{il_0,d}) + \lim_{l \rightarrow \infty} \left(\mathbb{P}\left(\bigcap_{i=l}^{\infty} W_{il_0,d}\right) - \prod_{i=l}^{\infty} \mathbb{P}(W_{il_0,d}) \right). \quad (38)$$

It remains to show that both limits above are equal to 0. For the first limit to be zero it is sufficient to show that $\sum_{i=1}^{\infty} \mathbb{P}(W_{il_0,d}^c) = \infty$. For any small $\eta > 0$ denote $\mathcal{S}_{k-1}^\eta = \mathcal{S}_{k-1} \cap ([-\eta, \eta]^{k-1} \times [0, 1])$. Clearly,

$$\mathbb{P}(W_{il_0,d}^c) \geq \mathbb{P}\left(\sup_{t \in F_{j,d}} \sup_{\mathbf{v} \in G_{j,d} \cap \mathcal{S}_{k-1}^\eta} Y_k(\overleftarrow{f}(-t), \mathbf{v}) > \sqrt{g(s_{j,d}^{(0)})}\right).$$

We have from condition **E**(0) that there exist some $M_4 > 0, \delta_3 > 0$ and $d_2 > 0$ such that for any $\delta \in (0, \delta_3)$, $d \in (0, d_2)$,

$$1 - r(\overleftarrow{f}(-t), \overleftarrow{f}(-s)) \geq M_4 K^2(|t-s|) \geq 2K^2((4^{-1}M_4)^{1/\alpha}|t-s|), \quad t, s \in [(j-1)d, jd]$$

holds for all $j \geq N_{d,\delta}$. Let $r_{Y_k}(\cdot, \cdot, \cdot, \cdot)$ be the covariance function of $Y_k(\overleftarrow{f}(-t), \mathbf{v})$. By using similar arguments as in the proof of Lemma 3.3 in [14] we have, for sufficiently small η

$$1 - r_{Y_k}(s, \mathbf{v}, t, \mathbf{v}') \geq 2K^2((4^{-1}M_4)^{1/\alpha}|t-s|) + \frac{3}{8} \sum_{i=1}^{k-1} (v_i - v'_i)^2, \quad t, s \in [(j-1)d, jd], \mathbf{v}, \mathbf{v}' \in \mathcal{S}_{k-1}^\eta$$

holds for all $j \geq N_{d,\delta}$, and in the light of Slepain's inequality

$$\mathbb{P}\left(\sup_{t \in F_{j,d}} \sup_{\mathbf{v} \in G_{j,d} \cap \mathcal{S}_{k-1}^\eta} Y_k(\overleftarrow{f}(-t), \mathbf{v}) > \sqrt{g(s_{j,d}^{(0)})}\right) \geq \mathbb{P}\left(\sup_{t \in \tilde{F}_{j,d}} \sup_{\tilde{\mathbf{v}} \in \tilde{G}_{j,d} \cap [-\eta, \eta]^{k-1}} Z((4^{-1}M_1)^{1/\alpha}t, \tilde{\mathbf{v}}) > \sqrt{g(s_{j,d}^{(0)})}\right),$$

where $\tilde{\mathbf{v}} = (v_1, \dots, v_{k-1}) \in \mathbb{R}^{k-1}$,

$$\begin{aligned} \tilde{F}_{j,d} &= \left\{ l q(g(s_{j,d}^{(0)})), \quad l = 0, 1, \dots, \left\lfloor \frac{d}{q(g(s_{j,d}^{(0)}))} \right\rfloor \right\}, \\ \tilde{G}_{j,d} &= \{(l_1 h_{j,d}, l_2 h_{j,d}, \dots, l_{k-1} h_{j,d}), \quad l_1, \dots, l_{k-1} \in \mathbb{Z}\}, \end{aligned}$$

and Z is a centered homogeneous Gaussian random field with a.s. continuous sample paths, variance 1 and correlation function such that

$$1 - \text{Corr}(Z(s, \tilde{\mathbf{v}}), Z(t, \tilde{\mathbf{v}}')) \sim K^2(|t-s|) + \frac{1}{4} \sum_{i=1}^{k-1} (v_i - v'_i)^2 \quad (39)$$

as $|t-s| \rightarrow 0, |v_i - v'_i| \rightarrow 0, 1 \leq i \leq k-1$. Let $a_0 = (4^{-1}M_1)^{1/\alpha}$, and $a_1 = a_2 = \dots = a_{k-1} = 1/2$. Then, using the same arguments as in Lemma 2.1 in [31] and in Lemma 2.2 in [32], we conclude that

$$\mathbb{P}\left(\sup_{t \in \tilde{F}_{j,d}} \sup_{\tilde{\mathbf{v}} \in \tilde{G}_{j,d} \cap [-\eta, \eta]^{k-1}} Z((4^{-1}M_1)^{1/\alpha}t, \tilde{\mathbf{v}}) > \sqrt{g(s_{j,d}^{(0)})}\right)$$

$$\begin{aligned}
&\geq \frac{(2\eta)^{k-1}d}{q(g(s_{j,d}^{(0)}))(2h_{j,d})^{k-1}} \frac{1}{\sqrt{2\pi}\sqrt{g(s_{j,d}^{(0)})}} e^{-\frac{g(s_{j,d}^{(0)})}{2}} a_0 a_1^{k-1} \left(1 - 2^k \sum_{\substack{0 \leq n_i < \infty, 1 \leq i \leq k \\ \mathbf{n} \neq \mathbf{0}}} \left(1 - \Phi \left((n_1 a_0)^{\alpha/2} / 2 + a_1 / 2 \sum_{i=2}^k n_i \right) \right) \right) \\
&=: \mathcal{C} \frac{(g(s_{j,d}^{(0)}))^{k/2-1}}{q(g(s_{j,d}^{(0)}))} e^{-\frac{g(s_{j,d}^{(0)})}{2}} d
\end{aligned}$$

as $j \rightarrow \infty$, where $\Phi(\cdot)$ is the standard normal distribution function. Therefore, for any δ, d small enough,

$$\begin{aligned}
\sum_{i=N_{d,\delta}}^{\infty} \mathbb{P}(W_{i_0,d}^c) &\geq \mathcal{C} \sum_{i=N_{d,\delta}}^{\infty} \frac{(g(s_{i_0,d}^{(0)}))^{\frac{k}{2}-1}}{q(g(s_{i_0,d}^{(0)}))} e^{-\frac{g(s_{i_0,d}^{(0)})}{2}} d \\
&\geq \mathcal{C} \frac{1}{2l_0} \sum_{i=N_{d,\delta}}^{\infty} \int_{-(i+1)d}^{-(i+2)d} \frac{(g(\overleftarrow{f}(t)))^{k/2-1}}{q((g(\overleftarrow{f}(t))))} e^{-\frac{g(\overleftarrow{f}(t)})}{2}} dt \\
&\geq \mathcal{C} \frac{1}{2l_0} \int_0^{\delta/2} (C(t))^{1/\alpha} \frac{(g(t))^{\frac{k}{2}-1}}{q(g(t))} e^{-\frac{g(t)}{2}} dt = \infty.
\end{aligned} \tag{40}$$

Next, we prove

$$\lim_{l \rightarrow \infty} \left(\mathbb{P} \left(\bigcap_{i=l}^{\infty} W_{i_0,d} \right) - \prod_{i=l}^{\infty} \mathbb{P}(W_{i_0,d}) \right) = 0. \tag{41}$$

In view of Normal Comparison Lemma (see e.g., [22]), we have for any $m > l$

$$\begin{aligned}
&\left| \mathbb{P} \left(\bigcap_{i=l}^m W_{i_0,d} \right) - \prod_{i=l}^m \mathbb{P}(W_{i_0,d}) \right| \\
&\leq \frac{1}{2\pi} \sum_{l \leq i < j \leq m} \sum_{\substack{(t,\mathbf{v}) \in (F_{i_0,d} \times (G_{i_0,d} \cap \mathcal{S}_{k-1})) \\ (t_1,\mathbf{v}_1) \in (F_{j_0,d} \times (G_{j_0,d} \cap \mathcal{S}_{k-1}))}} \frac{|r_{Y_k}(t, \mathbf{v}, t_1, \mathbf{v}_1)|}{\sqrt{1 - r_{Y_k}^2(t, \mathbf{v}, t_1, \mathbf{v}_1)}} e^{-\frac{g^2(s_{i_0,d}^{(0)}) + g^2(s_{j_0,d}^{(0)})}{2(1+r_{Y_k}(t,\mathbf{v},t_1,\mathbf{v}_1))}} =: \Lambda_{l,m}.
\end{aligned}$$

Note that (12) implies that for any $0 < \epsilon < \frac{\beta}{2(2-\beta)}$ with $\beta_1 = \min(\beta, 1)$, there exist $j_0 \in \mathbb{N}$ such that for any $j > i \geq j_0$ and large enough l_0

$$\sup_{\substack{(t,\mathbf{v}) \in (F_{i_0,d} \times (G_{i_0,d} \cap \mathcal{S}_{k-1})) \\ (t_1,\mathbf{v}_1) \in (F_{j_0,d} \times (G_{j_0,d} \cap \mathcal{S}_{k-1}))}} |r_{Y_k}(t, \mathbf{v}, t_1, \mathbf{v}_1)| \leq \sup_{t \in \Delta_{i_0,d}^{(0)}, t_1 \in \Delta_{j_0,d}^{(0)}} |r(t, t_1)| \leq M_0 |(j-i)l_0|^{-\beta} \leq \epsilon.$$

With aid of (37), fundamental calculation yields that for $l \geq j_0$,

$$\begin{aligned}
\Lambda_{l,\infty} &\leq \frac{M_0}{2\pi\sqrt{1-\epsilon^2}} \sum_{l \leq i < j < \infty} |(j-i)l_0|^{-\beta} \frac{d^2}{(q(g(s_{i_0,d}^{(0)})))^2} \left(\sqrt{g(s_{i_0,d}^{(0)})} \right)^{2k} e^{-\frac{g(s_{i_0,d}^{(0)}) + g(s_{j_0,d}^{(0)})}{2(1+M_0|(j-i)l_0|^{-\beta})}} \\
&\leq \mathcal{C} \sum_{i=l}^{\infty} (g(s_{i,d}^{(0)}))^{k+4/\alpha} e^{-\frac{g(s_{i,d}^{(0)})}{2(1+\epsilon)}} \sum_{j=i+l_0}^{\infty} |j-i|^{-\beta} (g(s_{j,d}^{(0)}))^{k+4/\alpha} e^{-\frac{g(s_{j,d}^{(0)})}{2(1+\epsilon)}} \\
&\leq \mathcal{C} \sum_{i=l}^{\infty} (\ln i)^{k+4/\alpha} i^{-\frac{1}{1+\epsilon}} \sum_{j=i+l_0}^{\infty} |j-i|^{-\beta} (\ln j)^{k+4/\alpha} j^{-\frac{1}{1+\epsilon}} \\
&\leq \mathcal{C} \sum_{i=l}^{\infty} i^{-\frac{1}{1+2\epsilon}} \sum_{j=i+1}^{\infty} |j-i|^{-\beta} j^{-\frac{1}{1+2\epsilon}} \\
&\leq \mathcal{C} \sum_{i=l}^{\infty} i^{-\frac{1}{1+2\epsilon}} \left(\sum_{j=i+1}^{2i} |j-i|^{-\beta} j^{-\frac{1}{1+2\epsilon}} + \sum_{j=2i}^{\infty} |j-i|^{-\beta} j^{-\frac{1}{1+2\epsilon}} \right).
\end{aligned}$$

Since

$$\sum_{j=i+1}^{2i} |j-i|^{-\beta} j^{-\frac{1}{1+2\epsilon}} \leq \sum_{j=i+1}^{2i} |j-i|^{-\beta_1} j^{-\frac{1}{1+2\epsilon}} \leq C i^{1-\beta_1-\frac{1}{1+2\epsilon}}$$

and

$$\sum_{j=2i}^{\infty} |j-i|^{-\beta} j^{-\frac{1}{1+2\epsilon}} \leq \sum_{j=2i}^{\infty} 2^{\beta} j^{-\frac{1}{1+2\epsilon}-\beta} \leq C i^{1-\frac{1}{1+2\epsilon}-\beta},$$

then

$$\Lambda_{l,\infty} \leq C \sum_{i=l}^{\infty} i^{1-\beta_1-\frac{2}{1+2\epsilon}} < \infty, \quad l \geq l_0,$$

which implies that (41) holds. This completes the proof. \square

Proof of Theorem 2.3: An immediate application of Theorem 5.1 yields that

$$\mathbb{P} \left(\sup_{t \in \mathcal{E}(S)} (\chi_{\mathbf{b}}^2(t) - g(t)) < \infty \right) = 1$$

provided that $|f(S)| < \infty$ or $I_g(S) < \infty$. For the case $I_g(S) = \infty$, without loss of generality, we focus on $S = 0$. For any such function $g(\cdot)$ satisfying $I_g(0) = \infty$, we can find a nonnegative continuous function $g_1(\cdot)$ (to be determined later) such that $g_1(t) \uparrow \infty$ as $t \rightarrow 0$ and $I_{g+g_1}(0) = \infty$ hold. Then, by Theorem 5.1 we have

$$\mathbb{P} (\chi_{\mathbf{b}}^2(t) \leq g(t) + g_1(t) \text{ ultimately as } t \rightarrow 0) = 0,$$

which implies that

$$\mathbb{P} \left(\sup_{t \in (0, 1/2]} (\chi_{\mathbf{b}}^2(t) - g(t)) = \infty \right) = 1, \quad (42)$$

and the proof will be complete.

Now we give one choice of the function $g_1(\cdot)$. Define $F(s) = \int_s^{1/2} (C(t))^{1/\alpha} \frac{(g(t))^{\frac{k}{2}-1}}{q(g(t))} e^{-\frac{g(t)}{2}} dt$, and let $\overleftarrow{F}(n) = \inf\{s \in (0, 1/2] : F(s) = n\}$, $n \in \mathbb{N}$. Further, we construct a nondecreasing function $w(\cdot)$ such that $w(t) = \frac{t - \overleftarrow{F}(n)}{(n-1)(\overleftarrow{F}(n-1) - \overleftarrow{F}(n))} + \frac{t - \overleftarrow{F}(n-1)}{n(\overleftarrow{F}(n) - \overleftarrow{F}(n-1))}$, $t \in [\overleftarrow{F}(n), \overleftarrow{F}(n-1))$, $n \geq 2$ and $w(t) = 1$, $t \in [\overleftarrow{F}(1), 1/2]$. Let $g_1(t) = \min(-2 \ln w(t), g(t))$, $t \in (0, 1/2]$. If $\alpha \in (0, 2)$ or $\alpha = 2, k > 1$, then by the fact that $\frac{t^{\frac{k}{2}-1}}{q(t)}$ is a regularly varying function at ∞ with index $k/2 - 1 + 1/\alpha > 0$, we have

$$\frac{(g(t) + g_1(t))^{\frac{k}{2}-1}}{q(g(t) + g_1(t))} \geq C \frac{(g(t))^{\frac{k}{2}-1}}{q(g(t))} \quad (43)$$

for any $t \in (0, \overleftarrow{F}(N_0)]$ when N_0 is large enough. In light of **B(0)**, one can easily check that (43) also holds for $\alpha + 2$ and $k = 1$. Consequently,

$$\begin{aligned} & \int_0^{1/2} (C(t))^{1/\alpha} \frac{(g(t) + g_1(t))^{\frac{k}{2}-1}}{q(g(t) + g_1(t))} e^{-\frac{g(t)+g_1(t)}{2}} dt \geq C \int_0^{\overleftarrow{F}(N_0)} (C(t))^{1/\alpha} \frac{(g(t))^{\frac{k}{2}-1}}{q(g(t))} e^{-\frac{g(t)}{2}} w(t) dt \\ & = C \sum_{n=N_0+1}^{\infty} \int_{\overleftarrow{F}(n)}^{\overleftarrow{F}(n-1)} (C(t))^{1/\alpha} \frac{(g(t))^{\frac{k}{2}-1}}{q(g(t))} e^{-\frac{g(t)}{2}} w(t) dt \geq C \sum_{n=N_0+1}^{\infty} \frac{1}{n} = \infty, \end{aligned}$$

which completes the proof. \square

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