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Stochastic Landau-Lifshitz-Gilbert Equation with Anisotropy Energy Driven by Pure Jump Noise

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Abstract

In this work we study a stochastic three-dimensional Landau-Lifshitz-Gilbert equation with non-zero anisotropy energy, which is drive by pure jump noise. We show existence of weak martingale solutions taking values in a two-dimensional sphere $\mathbb{S}^2$. The construction of the solution is based on the classical Faedo-Galerkin approximation, the compactness method and the Jakubowski version of the Skorokhod Theorem for nonmetric spaces.

Keywords: stochastic Landau-Lifshitz equation, weak martingale solutions, Marcus canonical form, Lévy noise
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1. Introduction

In this work we consider the following stochastic Landau-Lifshitz-Gilbert equation (LLGE) with non-zero anisotropy energy and driven by pure jump noise in “Marcus" canonical sense

$$\begin{cases} du(t) = (\lambda_1 u(t) \times (\Delta u(t) - \nabla\phi(u(t))) - \lambda_2 u(t) \times (u(t) \times (\Delta u(t) - \nabla\phi(u(t))))) \, dt \\ \qquad\qquad + u(t) \times \left(\sum_{i=1}^N e_i \diamond dL_i(t) \right), \\ \frac{\partial u}{\partial n}(t,x) = 0, \text{ on } (0,\infty) \times \partial D, \quad u(0,x) = u_0(x), \text{ on } D, \quad |u_0(x)| = 1 \text{ for all } x \in D, \end{cases} \tag{1.1}$$

where $L(t) := (L_1(t), \cdots, L_N(t))$ is a $\mathbb{R}^N-$ valued Lévy process with pure jump, with the jump intensity measure ν, see e.g. [22], is such that $\operatorname{supp}\nu \subset B$, where B is the closed unit ball in $\mathbb{R}^N$, i.e.

$$L(t) = \int_0^t \int_B l \, \tilde{\eta}(ds, dl). \tag{1.2}$$

Precise definition of $\diamond$ will be stated later. For each $i = 1, 2, \ldots N$, let $e_i : D \to \mathbb{R}^3$ be such that $e_i \in \mathbb{L}^\infty \cap \mathbb{W}^{1,3}$. We assume that the domain $D \subset \mathbb{R}^3$ is bounded with C^1 boundary ∂D. Here we have considered that the total energy $\mathcal{E}$ of the LLGE consists of both the exchange and anisotropy energies, i.e.

$$\mathcal{E}(u) = \mathcal{E}_{an}(u) + \mathcal{E}_{ex}(u) = \int_D \left(\phi(u(x)) + \frac{1}{2} |\nabla u(x)|^2 \right) \, dx.$$

Here $\mathcal{E}_{an}(u) := \int_D \phi(u(x)) \, dx$ stands for the anisotropy energy, and this models the existence of preferred directions for the magnetization (the so-called easy axes), which usually depend on the crystallographic structure of the material. We assume that the anisotropy energy density

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$\phi \in C_b^2(\mathbb{R}^3; \mathbb{R}^+)$. $\mathcal{E}_{ex}(u) := \frac{1}{2}\int_D |\nabla u(x)|^2 \, dx$ stands for the exchange energy which penalizes spatial variations of u. Thus the effective field, denoted by H_{eff}, which is the negative of the gradient (with respect to u) of the total magnetic energy functional $\mathcal{E}$, takes the form $\Delta u - \nabla\phi(u)$, which is present in the first two terms of the right hand side of the equation (1.1).

Let us briefly mention here some physical motivations behind this work. Recall that the stationary solutions of the deterministic version of the equation (1.1) correspond to the equilibrium states of the ferromagnet and are not unique in general. An important problem in the theory of ferromagnetism is to describe phase transitions between different equilibrium states induced by thermal fluctuations of the field H_{eff}. Therefore, the deterministic LLGE needs to be modified in order to incorporate random fluctuations of the field H_{eff} into the dynamics of the magnetization u and to describe noise-induced transitions between equilibrium states of the ferromagnet. A simple way to incorporate the noise into the deterministic LLGE is to perturb the effective field by a Lévy noise in the so called "Marcus" form. It is worth to mention here that the program to analyze noise induced transitions was initiated by Néel [21] and further developed in [3], [14] and others, and most recently by the first named author, Goldys and Jegaraj in [4] and [5], and by both the authors in [9].

Another motivation arises from the mechanism called "Barkhausen effect" (see Chapter 6.7.5 in [2]), which is observed as a series of random discontinuous changes in the size and orientation of ferromagnetic domains during the process of magnetization or demagnetization of a ferromagnetic material. Recently, condensed matter physicists (e.g. Mayergoyz *et al.* [17], [18]) considered jump-noise process in magnetization dynamics equations to account for thermal bath effects. See also Fruchart *et al.* [11] for a different physical motivation. We hope that our detailed mathematical study might contribute in understanding these effects.

In this work, the main technical issue lies in the fact that the noise must preserve the invariance property under coordinate transformation and this plays an important role in preserving the constraint condition. In other words, one needs to find an analogue of Stratonovich integral in the case of stochastic integral with respect to compensated Poisson Random Measure. The work of Marcus [16], developed later by Applebaum and Kunita, (see e.g. Section 6.10 of Applebaum [1] and Kunita [15]) provides a framework to resolve this technical issue. However, to the best of our knowledge, there is no concrete work on stochastic partial differential equations driven by Lévy noise in the "Marcus" canonical form. Our recent work [9] and the current paper are motivated by this question and we believe similar questions are yet unanswered for many other constrained PDEs (e.g. harmonic map flow, nonlinear Schrödinger equation on a compact Riemannian manifold, nematic liquid crystal model etc.) driven by jump noise or Lévy noise. We hope this work will contribute to the understanding of these questions, and open up directions for theoretical and (possibly) numerical study of various constrained PDEs perturbed by jump or Lévy noise. Also, there are some very recent work, see e.g. Chevyrev and Friz [10], where rough differential equations are studied in the spirit of Marcus canonical stochastic differential equations by dropping the assumption of continuity prevalent in the rough path literature. Therefore we hope that rough path theory may be integrated with our approach to gain newer insight into the analysis of constrained SPDEs.

Brzeźniak *et al.* [4] considered the above equation when $\phi = 0$ and is perturbed by a Gaussian noise in the Stratonovich sense, and proved existence of weak martingale solutions taking values in a sphere $\mathbb{S}^2$. In a later paper Brzeźniak and Li [8], the result was generalised in the presence of anisotropy energy. In a very recent work [9], we have addressed existence of weak martingale solution of the stochastic LLGE in the absence of anisotropy energy. In this work we extend this result for a more general LLGE. The presence of anisotropy energy creates additional technical difficulty at various levels as compared to [9], e.g. in order to prove existence of solution we need to consider a special approximation involving projection π_n applied to $\nabla\phi$, see approximate equation (4.12). Also passage to the limit to prove the existence of the martingale solution (see Step 3 of Theorem 4.3; in particular Lemma 4.7 and Lemma 4.10) becomes technically hard. The motivation for generalising our previous work stems from the problem related to magnetisation reversal and as an immediate future direction we would like to study large deviation principles (LDP) for stochastic LLGE with anisotropy energy. We will then apply this LDP to show that small noise can cause magnetisation reversal.

2. The Marcus Mapping

We omit the general discussion on the Marcus canonical stochastic differential equations. We refer interested readers to the work of Marcus [16], Section 6.10 of Applebaum [1] and Kunita [15].

Before we define the Marcus mapping, let us consider that $\big(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}\big)$ is a filtered probability space, where $\mathbb{F} = \big(\mathcal{F}_t\big)_{t\geqslant 0}$ is the filtration, and this probability space satisfies the so called usual conditions. As specified earlier, we further assume that the jump intensity measure ν of the associated Lévy process $L = (L(t))$, $t \geqslant 0$, is such that $\operatorname{supp}\nu \subset B$, where B is the closed unit ball in $\mathbb{R}^N$. We also denote by η the time-homogenous Poissson random measure associated to L. It is known that the intensity measure of η is equal to $\mathrm{Leb}\otimes\nu$. We denote by $\tilde{\eta} := \eta - \mathrm{Leb}\otimes\nu$ the corresponding compensated time-homogeneous Poisson random measure.

Define a bounded linear map

$$g_i : \mathbb{H}^1 \ni u \mapsto u \times e_i \in \mathbb{H}^1. \tag{2.1}$$

The map g_i is indeed bounded due to the Sobolev emebdding $\mathbb{H}^1 \hookrightarrow \mathbb{L}^6$ and $e_i \in \mathbb{L}^\infty \cap \mathbb{W}^{1,3}$. Let us define a generalized Marcus mapping

$$\Phi : \mathbb{R}_+ \times \mathbb{R}^N \times \mathbb{H}^1 \to \mathbb{H}^1$$

such that for each fixed $l \in \mathbb{R}^N$, $u_0 \in \mathbb{H}^1$, the function $t \mapsto \Phi(t, l, u_0)$ is the continuously differentiable solution of the ordinary differential equation

$$\frac{du}{dt}(t) = \sum_{i=1}^{N} l_i g_i(u(t)), \ \ t \geqslant 0, \tag{2.2}$$

with $u(0) = u_0 \in \mathbb{H}^1$, and $l = (l_1, l_2, \dots, l_N) \in \mathbb{R}^N$. Let us observe that since $e_i \in \mathbb{L}^\infty$, the maps g_i are also bounded linear from $\mathbb{L}^2$ to $\mathbb{L}^2$. Hence also the map Φ is well defined as a map $\Phi : \mathbb{R}_+ \times \mathbb{R}^N \times \mathbb{L}^2 \to \mathbb{L}^2$.

Notation: We fix $t = 1$ now onward in this paper and consider Φ as the function of last variable for fixed t and l. Denote $\Phi(l, \cdot) := \Phi(1, l, \cdot)$.

Equation (1.1) with notation $\diamond$ is defined in the integral form as following

$$\begin{aligned} u(t) = u_0 &+ \int_0^t \big(\lambda_1 u(s) \times (\Delta u(s) - \nabla\phi(u(s))) - \lambda_2 u(s) \times (u(s) \times (\Delta u(t) - \nabla\phi(u(t))))\big)\, ds \\ &+ \int_0^t \int_B \left[\Phi(l, u(s)) - u(s)\right] \tilde{\eta}(ds, dl) + \int_0^t \int_B \left\{\Phi(l, u(s)) - u(s) - \sum_{i=1}^{N} l_i g_i(u(s))\right\} \nu(dl) ds. \end{aligned} \tag{2.3}$$

For $z \in \mathbb{H}^1$, we denote

$$G(l, z) := \Phi(l, z) - z, \quad H(l, z) := \Phi(l, z) - z - \sum_{i=1}^{N} l_i g_i(z),$$

$$b(z) := \int_B \left[\Phi(l, z) - z - \sum_{i=1}^{N} l_i g_i(z)\right] \nu(dl) = \int_B H(l, z)\, \nu(dl).$$

With the above notation, equation (2.3) becomes

$$\begin{aligned} u(t) = u_0 &+ \int_0^t \big(\lambda_1 u(s) \times (\Delta u(s) - \nabla\phi(u(s))) - \lambda_2 u(s) \times (u(s) \times (\Delta u(t) - \nabla\phi(u(t))))\big)\, ds \\ &+ \int_0^t \int_B G(l, u(s))\, \tilde{\eta}(ds, dl) + \int_0^t b(u(s))\, ds, \end{aligned} \tag{2.4}$$

with the same initial, boundary and initial saturation conditions as in (1.1).

3. The Operator A

Denote by A the $-$Laplacian with the Neumann boundary conditions acting on $\mathbb{R}^3$-valued functions, i.e.

$$\begin{cases} D(\mathrm{A}) & := \{u \in \mathbb{H}^2 : \frac{\partial u}{\partial n} = 0 \text{ on } \partial D\}, \\ \mathrm{A}u & := -\Delta u, \quad u \in D(\mathrm{A}), \end{cases} \tag{3.1}$$

where $n = (n_1, n_2, n_3)$ is the unit outward normal vector field on ∂D and $\frac{\partial u}{\partial n}$ is the directional derivative of u in the direction n.

It is well known that A is a self-adjoint operator in $\mathbb{L}^2$ and that $(I+\mathrm{A})^{-1}$ is compact. Hence there exists an orthonormal basis $\{\tilde{e}_n\}_{n=1}^{\infty}$ of $\mathbb{L}^2$ consisting of eigenvectors of A. Define $\mathrm{A}_1 := I + \mathrm{A}$. It is also known that

$$D(\mathrm{A}_1^{1/2}) = \mathbb{H}^1. \tag{3.2}$$

We will often denote the space $D(\mathrm{A}_1^{1/2})$ by V. Note that V is a dense subspace of $\mathbb{L}^2$ and V is imbedded in $\mathbb{L}^2$ continuously. Identifying $\mathbb{L}^2$ with its dual and denoting by V' the dual of V we have

$$\mathrm{V} \hookrightarrow \mathbb{L}^2 \hookrightarrow \mathrm{V}'. \tag{3.3}$$

Thus $(\mathrm{V}, \mathbb{L}^2, \mathrm{V}')$ is a Gelfand triple.

Definition 3.1 (Fractional power spaces of $\mathrm{A}_1 = I + \mathrm{A}$)**.** *For any non-negative real number β we define the Hilbert space $X^\beta := D(\mathrm{A}_1^\beta)$, which is the domain of the fractional power operator A_1^β with the norm $|\cdot|_{X^\beta} := |\mathrm{A}_1^\beta \cdot|_{\mathbb{L}^2}$. The space $X^0 = \mathbb{L}^2$ is identified with its dual. For positive real β, the dual of X^β is denoted by $X^{-\beta}$ and the norm $|\cdot|_{X^{-\beta}}$ of $X^{-\beta}$ satisfies $|x|_{X^{-\beta}} = |\mathrm{A}_1^{-\beta} x|_{\mathbb{L}^2}$ when x is in $\mathbb{L}^2$.*

4. Statement of the Main Result

Now we are ready to formulate the definition of a weak martingale solution to problem (2.4). Let us note that our solution is weak in both probabilistic and stochastic senses.

Definition 4.1. *Given $T \in (0, \infty)$, $u_0 \in \mathbb{H}^1$ and a Borel non-negative measure ν on $\mathbb{R}^N$ such that*

$$\operatorname{supp} \nu \subset B \text{ and } \int_B |l|^2 \, \nu(dl) < \infty, \tag{4.1}$$

where B is the unit ball in $\mathbb{R}^N$, a weak martingale solution to equation (2.4) is a system $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{F}}, \bar{\mathbb{P}}, \bar{u}, \bar{\eta})$, such that:

(a) $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{F}}, \bar{\mathbb{P}})$ *is a filtered probability space with a filtration $\bar{\mathbb{F}} := (\bar{\mathcal{F}}_t)_{t \geqslant 0}$;*

(b) $\bar{\eta}$ *is a time homogeneous Poisson random measure on the measurable space $(B, \mathscr{B}(B))$ over $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{F}}, \bar{\mathbb{P}})$ with the intensity measure $\mathrm{Leb} \otimes \nu$;*

(c) $\bar{u} : [0,T] \times \bar{\Omega} \to \mathbb{H}^1$ *is an $\bar{\mathbb{F}}$-progressively measurable weakly càdlàg process, and*

(c1) for $\beta = \frac{1}{2}$, $\bar{\mathbb{P}}$-a.e. $\omega \in \bar{\Omega}$,

$$\bar{u}(\cdot, \omega) \in \mathbb{D}\left([0,T]; X^{-\beta}\right), \tag{4.2}$$

where $\mathbb{D}\left([0,T]; X^{-\beta}\right)$ denotes the Skorokhod space;

(c2) for every $p \geqslant 1$,

$$\bar{\mathbb{E}}\left[\sup_{t \in [0,T]} \left\{ |\nabla \bar{u}(t)|_{\mathbb{L}^2}^2 + \int_D \phi(\bar{u}(t,x)) dx \right\}^p \right] < \infty, \tag{4.3}$$

$$\bar{\mathbb{E}}\left[\left(\int_0^T |\bar{u}(t) \times (\Delta \bar{u}(t) - \nabla \phi(\bar{u}(t)))|_{\mathbb{L}^2}^2 \, dt\right)^p \right] < \infty; \tag{4.4}$$

(d) *the constraint condition is satisfied, i.e.,*

$$|\bar{u}(t,x)|_{\mathbb{R}^3} = 1, \quad \text{for} \quad \textit{Lebesgue a.e. } x \in D \textit{ for all } t \in [0,T], \; \bar{\mathbb{P}} \textit{ a.s.}; \tag{4.5}$$

(e) *for each* $\varphi \in L^4(\bar{\Omega}; X^\beta)$, *we have:*

$$\begin{aligned}
\langle \bar{u}(t), \varphi \rangle_{\mathbb{L}^2} &= \langle u_0, \varphi \rangle_{\mathbb{L}^2} - \lambda_1 \int_0^t \sum_i \int_D \left\langle \frac{\partial \bar{u}}{\partial x_i}(s,x), \frac{\partial \varphi}{\partial x_i}(x) \times \bar{u}(s,x) \right\rangle_{\mathbb{L}^2} dx\, ds \\
&\quad - \lambda_2 \int_0^t \sum_i \int_D \left\langle \frac{\partial \bar{u}}{\partial x_i}(s,x), \frac{\partial (\bar{u} \times \varphi)}{\partial x_i}(s,x) \times \bar{u}(s,x) \right\rangle_{\mathbb{L}^2} dx\, ds \\
&\quad - \lambda_1 \int_0^t \langle \bar{u}(s) \times \nabla \phi(\bar{u}(s)), \varphi \rangle_{\mathbb{L}^2}\, ds + \lambda_2 \int_0^t \langle \bar{u}(s) \times (\bar{u}(s) \times \nabla \phi(\bar{u}(s))), \varphi \rangle_{\mathbb{L}^2}\, ds \\
&\quad + \int_0^t \int_B \langle G(l, \bar{u}(s)), \varphi \rangle_{\mathbb{L}^2}\, \tilde{\bar{\eta}}(ds, dl) + \int_0^t \langle b(\bar{u}(s)), \varphi \rangle_{\mathbb{L}^2} ds,
\end{aligned} \tag{4.6}$$

for all $t \in [0,T]$, $\bar{\mathbb{P}}$ *almost everywhere.*

Remark 4.2. We wish to use the notation $u \times \Delta u$ for our weak martingale solution u even when we do not know that u has weak second order derivatives. First of all let us recall (see Appendix A in [8]) that for $u \in \mathbb{H}^1$ we say $u \times \Delta u \in \mathbb{L}^2$ iff there exists $B \in \mathbb{L}^2$ such that for all $\psi \in \mathbb{W}^{1,3}$,

$$\langle B, \psi \rangle_{\mathbb{L}^2} = \sum_{i=1}^3 \left\langle \frac{\partial u}{\partial x_i}, u \times \frac{\partial \psi}{\partial x_i} \right\rangle_{\mathbb{L}^2}. \tag{4.7}$$

In such a case we put $u \times \Delta u := B$. Note that if $u \in \mathbb{H}^1$ and $u \times \Delta u \in \mathbb{L}^2$ then

$$\langle u \times \Delta u, \psi \rangle_{\mathbb{L}^2} = \sum_{i=1}^3 \left\langle \frac{\partial u}{\partial x_i}, u \times \frac{\partial \psi}{\partial x_i} \right\rangle_{\mathbb{L}^2}, \; \psi \in \mathbb{W}^{1,3}. \tag{4.8}$$

Note that we use the space $\mathbb{W}^{1,3}$ so that the RHS of (4.7) makes sense for $u \in \mathbb{H}^1$. However, if additionally $u \in \mathbb{L}^\infty$, then the RHS of (4.7) makes sense for all $\psi \in \mathbb{W}^{1,2} = \mathbb{H}^1$. Hence, because $\mathbb{W}^{1,3}$ is dense in $\mathbb{W}^{1,2}$, we deduce that if $u \in \mathbb{H}^1 \cap \mathbb{L}^\infty$ and $u \times \Delta u \in \mathbb{L}^2$ then (4.8) holds for all $\psi \in \mathbb{H}^1$.

Note that if ψ and $u, v \in \mathbb{H}^1$ are such that $u \times \Delta u \in \mathbb{L}^2$ and $v \times \psi \in \mathbb{W}^{1,3}$, then, as $\langle a \times b, c \rangle = \langle b, c \times a \rangle$, for all $a, b, c \in \mathbb{R}^3$, we infer that from (4.8) that

$$\begin{aligned}
\langle v \times (u \times \Delta u), \psi \rangle_{\mathbb{L}^2} &= -\langle u \times \Delta u, v \times \psi \rangle_{\mathbb{L}^2} = -\sum_{i=1}^3 \left\langle \frac{\partial u}{\partial x_i}, u \times \frac{\partial (v \times \psi)}{\partial x_i} \right\rangle_{\mathbb{L}^2} \\
&= \sum_{i=1}^3 \left\langle \frac{\partial u}{\partial x_i}, \frac{\partial (v \times \psi)}{\partial x_i} \times u \right\rangle_{\mathbb{L}^2}.
\end{aligned}$$

Suppose now that $u, v \in \mathbb{H}^1$ such that $u \times \Delta u \in \mathbb{L}^2$. Then by the Sobolev embedding $\mathbb{H}^1 \hookrightarrow \mathbb{L}^6$ and the Hölder inequality, $v \times (u \times \Delta u) \in \mathbb{L}^{\frac{3}{2}}$. Moreover, by approximating v by more regular functions and using the previous equality one can prove that for any $\psi \in \mathbb{W}^{1,2} = \mathbb{H}^1$,

$$\langle v \times (u \times \Delta u), \psi \rangle_{\mathbb{L}^2} = = \sum_{i=1}^3 \left\langle \frac{\partial u}{\partial x_i}, \frac{\partial (v \times \psi)}{\partial x_i} \times u \right\rangle_{\mathbb{L}^2}. \tag{4.9}$$

We now state the main result of this paper.

Theorem 4.3. *For every $T > 0$, $u_0 \in \mathbb{H}^1$ and every Borel non-negative measure ν on $\mathbb{R}^N$ satisfying condition (4.1), there exists a weak martingale solution to the problem (2.4).*
Moreover, the solution satisfies the following inequality

$$\bar{\mathbb{E}}\left[\left(\int_0^T |\bar{u}(t)\times(\bar{u}(t)\times(\Delta\bar{u}(t)-\nabla\phi(\bar{u}(t))))|^2_{\mathbb{L}^2}\,dt\right)^{\frac{p}{2}}\right] \quad < \quad \infty. \tag{4.10}$$

as well as the following equality holds in $\mathbb{L}^2$, for all $t\in[0,T]$, $\bar{\mathbb{P}}$-a.s.

$$\begin{aligned}\bar{u}(t) \quad = \quad & u_0+\lambda_1\int_0^t \bar{u}(s)\times(\Delta\bar{u}(s)-\nabla\phi(\bar{u}(s)))\,ds \\ & -\lambda_2\int_0^t \bar{u}(s)\times\big(\bar{u}(s)\times(\Delta\bar{u}(s)-\nabla\phi(\bar{u}(s)))\big)\,ds \\ & +\int_0^t\int_B G(l,\bar{u}(s))\,\tilde{\bar{\eta}}(ds,dl)+\int_0^t b(\bar{u}(s))ds. \end{aligned}\tag{4.11}$$

Proof. Let us first briefly explain how one can deduce the second part of the Theorem. The inequality (4.10) follows from inequality (4.4) and the constraint condition (4.5). The form (4.11) of the stochastic LLGE follows from its weak form (4.6) because the set of testing elements $L^4(\bar{\Omega};X^\beta)$, for $\beta=\frac{1}{2}$ is separable and the following four inequalities hold:

$$\bar{\mathbb{E}}\int_0^T |\bar{u}(t)\times(\Delta\bar{u}(t)-\nabla\phi(\bar{u}(t)))|^2_{\mathrm{V}'}\,dt<\infty;\quad \bar{\mathbb{E}}\int_0^T |\bar{u}(t)\times\big(\bar{u}(t)\times(\Delta\bar{u}(t)-\nabla\phi(\bar{u}(t)))\big)|^2_{\mathrm{V}'}\,dt<\infty;$$
$$\bar{\mathbb{E}}\int_0^T\int_B |G(l,\bar{u}(t))|^2_{\mathbb{L}^2}\,\nu(dl)dt<\infty;\quad \bar{\mathbb{E}}\int_0^T |b(\bar{u}(t))|^2_{\mathrm{V}'}\,dt<\infty.$$

The first inequality is a special case of (4.4) due to the embedding $\mathbb{L}^2\hookrightarrow \mathrm{V}'=X^{-\frac{1}{2}}$. The second inequality is a special case of inequality (4.10) due to the embedding $\mathbb{L}^{\frac{3}{2}}\hookrightarrow\mathrm{V}'$. Using the linear growth and Lipschitz properties of G and b the last two inequalities can be proved.
Let us now give an outline of the proof of the first part of the theorem. We will do it in a few steps.
Step - 1 (Faedo-Galerkin Approximation and A-Priori Estimates) Let us fix $u_0\in\mathbb{H}^1$ and a Borel non-negative measure ν on $\mathbb{R}^N$ satisfying condition (4.1). Then, see e.g. [13] and/or [22], there exists a filtered probability space $(\Omega,\mathcal{F},\mathbb{F},\mathbb{P})$, where $\mathbb{F}=(\mathcal{F}_t)_{t\geqslant 0}$, satisfying the so called usual conditions and there exists a Lévy process $L=(L(t))$, $t\geqslant 0$, whose jump intensity measure is equal to ν. We also denote by η the time-homogenous Poissson random measure associated to L. It is known that the intensity measure of η is equal to Leb $\otimes\nu$. We denote by $\tilde{\eta}:=\eta-\mathrm{Leb}\otimes\nu$ the corresponding compensated time-homogeneous Poisson random measure.

Let π_n denote the orthogonal projection from $\mathbb{L}^2$ onto $\mathbb{H}_n := \mathrm{linspan}\{\tilde{e}_1,\cdots,\tilde{e}_n\}$, where $\{\tilde{e}_k\}_{k=1}^\infty$ is an orthonormal basis of $\mathbb{L}^2$, consisting of eigenvectors of A. Consider the following approximate equation in $\mathbb{H}_n$:

$$\begin{cases} du_n(t) \;=\; \pi_n\Big\{\lambda_1 u_n(t)\times\big[\Delta u_n(t)-\pi_n\nabla\phi(u_n(t))\big] \\ \qquad\qquad -\lambda_2 u_n(t)\times\Big(u_n(t)\times\big[\Delta u_n(t)-\pi_n\nabla\phi(u_n(t))\big]\Big)\Big\}\,dt+\sum_{i=1}^N \pi_n\left(u_n(t)\times e_i\right)\diamond dL_i(t). \\ u_n(0) \;=\; \pi_n u_0. \end{cases}\tag{4.12}$$

Let us define the following maps

$$\begin{aligned} F_n^1 \;&:\; \mathbb{H}_n\ni u\mapsto \pi_n\Big(u\times\big[\Delta u-\pi_n\nabla\phi(u)\big]\Big)\in\mathbb{H}_n \\ F_n^2 \;&:\; \mathbb{H}_n\ni u\mapsto \pi_n\Big[u\times\Big(u\times\big[\Delta u-\pi_n\nabla\phi(u)\big]\Big)\Big]\in\mathbb{H}_n \end{aligned}$$

$$g_i^n \quad : \quad \mathbb{H}_n \ni u \mapsto \pi_n(u \times e_i) \in \mathbb{H}_n.$$

Let $\Phi^n(t, l, u_n(0))$ be a flow on $\mathbb{H}_n$ corresponding to $\sum_{i=1}^N l_i g_i^n$, i.e.

$$\begin{cases} \dfrac{d\Phi_n}{dt}(t, l, u_n(0)) = \sum_{i=1}^N l_i g_i^n(\Phi_n(t, l, u_n(0))), \ t \geqslant 0, \\ \\ \Phi_n(0, l, u_n(0)) = u_n(0) \in \mathbb{H}_n. \end{cases}$$

For $u_n \in \mathbb{H}_n$, we denote

$$G_n(l, u_n) := \Phi_n(l, u_n) - u_n, \quad H_n(l, u_n) := \Phi_n(l, u_n) - u_n - \sum_{i=1}^N l_i g_i^n(u_n),$$

$$b_n(u_n) := \int_B \left[\Phi_n(l, u_n) - u_n - \sum_{i=1}^N l_i g_i^n(u_n) \right] \nu(dl) = \int_B H_n(l, u_n)\, \nu(dl).$$

Hence the problem (4.12) can be written in the following integral form:

$$u_n(t) = u_n(0) + \int_0^t \left(\lambda_1 F_n^1(u_n(s)) - \lambda_2 F_n^2(u_n(s)) \right) ds$$
$$+ \int_0^t \int_B G_n(l, u_n(s))\, \tilde{\eta}(ds, dl) + \int_0^t b_n(u_n(s))\, ds, \quad t \geqslant 0. \tag{4.13}$$

The following result plays an important role in order to prove the existence of a unique solution of the approximated equation (4.13).

Lemma 4.4. *The maps F_n^1 and F_n^2 are Lipschitz on balls, i.e., for every $R > 0$ there exists a constant $K_j = K_j(n, R)$, $j = 1, 2$ such that whenever $u, v \in \mathbb{H}_n$ with $|u|, |v| \leqslant R$, we have*

$$|F_n^j(u) - F_n^j(v)|_{\mathbb{H}_n} \leqslant K_j |u - v|_{\mathbb{H}_n}, \quad j = 1, 2.$$

Furthermore, there exist constants $R_1, R_2 > 0$ such that for any $u, u_1, u_2 \in \mathbb{H}_n$,

$$|b_n(u_2) - b_n(u_1)|^2_{\mathbb{H}_n} + \int_B |G_n(l, u_2) - G_n(l, u_1)|^2_{\mathbb{H}_n}\, \nu(dl) \leqslant R_1 |u_2 - u_1|^2_{\mathbb{H}_n}, \tag{4.14}$$

$$|b_n(u)|^2_{\mathbb{H}_n} + \int_B |G_n(l, u)|^2_{\mathbb{H}_n}\, \nu(dl) \leqslant R_2 |u|^2_{\mathbb{H}_n}. \tag{4.15}$$

Hence the equation (4.13) has a unique global strong solution in $\mathbb{H}_n$ (see [9] and [8] for more details). Moreover, we have the following a-priori estimates for all $n \in \mathbb{N}$.

Lemma 4.5. *Assume that $e_i \in \mathbb{L}^\infty \cap \mathbb{W}^{1,3}$ and $T \in (0, \infty)$. Let $2 \leqslant p < \infty$ and $\beta > \frac{1}{4}$. Then there exists a constant C, which does not depend on n but may depend on $u_n(0)$, e_i, T, p and β such that*

$$|u_n(t)|^2_{\mathbb{L}^2} = |u_n(0)|^2_{\mathbb{L}^2} \quad \mathbb{P}\ a.s., \tag{4.16}$$

$$\bar{\mathbb{E}} \left[\sup_{t \in [0,T]} \left\{ |\nabla u_n(t)|^2_{\mathbb{L}^2} + \int_D \phi(u_n(t, x)) dx \right\}^p \right] \leqslant C, \tag{4.17}$$

$$\mathbb{E} \left[\left(\int_0^T |u_n(t) \times (\Delta u_n(t) - \pi_n \nabla \phi(u_n(t)))|^2_{\mathbb{L}^2}\, dt \right)^p \right] \leqslant C, \tag{4.18}$$

$$\mathbb{E} \left[\left(\int_0^T |u_n(t) \times (u_n(t) \times (\Delta u_n(t) - \pi_n \nabla \phi(u_n(t))))|^2_{\mathbb{L}^{\frac{3}{2}}}\, dt \right)^{\frac{p}{2}} \right] \leqslant C, \tag{4.19}$$

$$\mathbb{E} \left[\int_0^T |\pi_n[u_n(t) \times (u_n(t) \times (\Delta u_n(t) - \pi_n \nabla \phi(u_n(t))))]|^2_{X^{-\beta}}\, dt \right] \leqslant C. \tag{4.20}$$

Step - 2 (Tightness) Let us fix $p \in [2,\infty)$, $q \in [2,6)$ and $\beta > \frac{1}{4}$, and denote

$$\mathscr{Z}_T := L^2_w(0,T;\mathbb{H}^1) \cap L^p(0,T;\mathbb{L}^q) \cap \mathbb{D}([0,T];X^{-\beta}) \cap \mathbb{D}([0,T];\mathbb{H}^1_w),$$

and $\mathcal{T}$ as the supremum of all the four topologies. We first prove that

Lemma 4.6. *The set of measures $\{\mathcal{L}(u_n),\ n \in \mathbb{N}\}$ is tight on $(\mathscr{Z}_T, \sigma(\mathcal{T}))$.*

This result can be proved using the fact that $(u_n)_{n\in\mathbb{N}}$ is a sequence of càdlàg $\mathbb{F}$-adapted $X^{-\beta}$-valued process satisfying $\sup_{n\in\mathbb{N}} \mathbb{E}\left[|u_n|^2_{L^\infty(0,T;\mathbb{H}^1)}\right] < \infty$, and the Aldous condition in $X^{-\beta}$, i.e. for every sequence $(\tau_n)_{n\in\mathbb{N}}$ of $\mathbb{F}$-stopping times with $\tau_n + \theta \leqslant T$ and for every $n \in \mathbb{N}$ and $\theta \geqslant 0$,

$$\mathbb{E}\left[|u_n(\tau_n+\theta) - u_n(\tau_n)|^\alpha_{X^{-\beta}}\right] \leqslant C\theta^\gamma \tag{4.21}$$

for some $\alpha, \gamma > 0$ and some constant $C > 0$. Tightness in a space similar to $\mathscr{Z}_T$, with the càdlàg functions replaced by continuous ones, has been used in a recent paper [7] on the stochastic Schrödinger equation.

By (4.13), we have

$$\begin{aligned} u_n(t) &= u_n(0) + \lambda_1 \int_0^t F^1_n(u_n(s))\,ds - \lambda_2 \int_0^t F^2_n(u_n(s))\,ds \\ &\quad + \int_0^t \int_B G_n(l,u_n(s))\,\tilde{\eta}(ds,dl) + \int_0^t b_n(u_n(s))\,ds, \\ &:= \sum_{i=0}^4 J^i_n(t), \ \text{in } \mathbb{H}_n \ \mathbb{P}-\text{a.s. for all } t \in [0,T]. \end{aligned} \tag{4.22}$$

Let $\theta > 0$. To prove u_n satisfies the condition (4.21), it is sufficient to check that each term $J^i_n,\ i = 1,2,3,4$ satisfies the same.
For the term J^0_n, condition (4.21) is trivially true.
Using the continuous embedding $\mathbb{L}^2 \hookrightarrow X^{-\beta}$, Hölder's inequality and a priori estimate (4.18), we have

$$\begin{aligned} \mathbb{E}\left[|J^1_n(\tau_n+\theta) - J^1_n(\tau_n)|_{X^{-\beta}}\right] &= \lambda_1 \mathbb{E}\left[|\int_{\tau_n}^{\tau_n+\theta} F^1_n(u_n(s))ds|_{X^{-\beta}}\right] \\ &\leqslant c\lambda_1 \mathbb{E}\left[\int_{\tau_n}^{\tau_n+\theta} |\pi_n\big(u_n(s) \times (\Delta u_n(s) - \pi_n \nabla\phi(u_n(s)))\big)|_{\mathbb{L}^2} ds\right] \\ &\leqslant c\lambda_1 \mathbb{E}\left[\int_{\tau_n}^{\tau_n+\theta} |u_n(s) \times (\Delta u_n(s) - \pi_n \nabla\phi(u_n(s)))|_{\mathbb{L}^2} ds\right] \\ &\leqslant c\lambda_1 \theta^{1/2}\ \mathbb{E}\left[\int_{\tau_n}^{\tau_n+\theta} |u_n(s) \times (\Delta u_n(s) - \pi_n \nabla\phi(u_n(s)))|^2_{\mathbb{L}^2} ds\right]^{1/2} \\ &\leqslant c\lambda_1 \theta^{1/2}\ \mathbb{E}\left[\int_0^T |u_n(s) \times (\Delta u_n(s) - \pi_n \nabla\phi(u_n(s)))|^2_{\mathbb{L}^2} ds\right]^{1/2} \leqslant c_1\theta^{1/2}. \end{aligned}$$

Thus J^1_n satisfies condition (4.21) with $\alpha = 1$ and $\gamma = \frac{1}{2}$.
For J^2_n, we proceed similarly using the continuous embedding $\mathbb{L}^{3/2} \hookrightarrow X^{-\beta}$, Hölder's inequality and a priori estimate (4.19) and find that J^2_n satisfies condition (4.21) with $\alpha = 1$ and $\gamma = \frac{1}{2}$.
Now let us observe that in view of the a priori estimate (4.16), for each $p \in [2,\infty)$, we have

$$\sup_{n\in\mathbb{N}} \mathbb{E}\left[\sup_{t\in[0,T]} |u_n(t)|^p_{\mathbb{L}^2}\right] < \infty. \tag{4.23}$$

Therefore using Itô-Lévy isometry, (4.15), Hölder's inequality and (4.23), we infer

$$\begin{aligned}
\mathbb{E}\left[|J_n^3(\tau_n+\theta)-J_n^3(\tau_n)|^2_{X^{-\beta}}\right] &= \mathbb{E}\left[\left|\int_{\tau_n}^{\tau_n+\theta}\int_B G_n(l,u_n(s))\tilde{\eta}(ds,dl)\right|^2_{X^{-\beta}}\right] \\
&\leqslant c\mathbb{E}\left[\left|\int_{\tau_n}^{\tau_n+\theta}\int_B G_n(l,u_n(s))\tilde{\eta}(ds,dl)\right|^2_{\mathbb{L}^2}\right] = c\mathbb{E}\left[\int_{\tau_n}^{\tau_n+\theta}\int_B |G_n(l,u_n(s))|^2_{\mathbb{L}^2}\,\nu(dl)\;ds\right] \\
&\leqslant cR_2\mathbb{E}\left[\int_{\tau_n}^{\tau_n+\theta}|u_n(s)|^2_{\mathbb{L}^2}ds\right] \leqslant cR_2\theta\mathbb{E}\left[\sup_{t\in[0,T]}|u_n(t)|^2_{\mathbb{L}^2}\right] \leqslant c_3\theta.
\end{aligned}$$

So J_n^3 satisfies condition (4.21) with $\alpha=2$ and $\gamma=1$.
Using the similar argument we observe that J_n^4 satisfies condition (4.21) with $\alpha=1$ and $\gamma=\frac{1}{2}$.
This completes the proof of the Lemma.

Step - 3 (Existence of Martingale Solution) Let us fix $p\in[2,\infty), q\in[2,6)$ and $\beta>\dfrac{1}{4}$. In view of the above lemma, the set of measures $\{\mathcal{L}(u_n),\ n\in\mathbb{N}\}$ is tight on the space $\mathscr{Z}_T$, the set $\{\mathcal{L}(u_n,\eta_n),\ n\in\mathbb{N}\}$ is tight on the space $\mathscr{Z}_T\times\mathbb{M}_{\bar{\mathbb{N}}}([0,T]\times B)$, where $\mathbb{M}_{\bar{\mathbb{N}}}(S)$ denotes the set of all $\mathbb{N}\cup\{\infty\}$ valued measures on the measurable space $(S,\mathscr{S})$. Therefore, by the generalised Jakubowski-Skorokhod embedding theorem from Brzeźniak *et al.* [6], there exists a subsequence $(n_k)_{k\in\mathbb{N}}$, a probability space $\left(\bar{\Omega},\bar{\mathcal{F}},(\bar{\mathcal{F}}_t)_{t\geqslant 0},\bar{\mathbb{P}}\right)$ and, on this space, $\mathscr{Z}_T\times\mathbb{M}_{\bar{\mathbb{N}}}([0,T]\times B)$-valued random variables $(u_\star,\eta_\star)$, $(\bar{u}_k,\bar{\eta}_k)$, $k\in\mathbb{N}$, such that

(1) $\mathcal{L}\big((\bar{u}_k,\bar{\eta}_k)\big)=\mathcal{L}\big((u_{n_k},\eta_{n_k})\big)$ for all $k\in\mathbb{N}$,

(2) $(\bar{u}_k,\bar{\eta}_k)\to(u_\star,\eta_\star)$ in $\mathscr{Z}_T\times\mathbb{M}_{\bar{\mathbb{N}}}([0,T]\times B)$, and

(3) $\bar{\eta}_k(\bar{\omega})=\eta_\star(\bar{\omega})$, for all $\bar{\omega}\in\bar{\Omega}$.

We will denote these sequences again by $\big((u_n,\eta_n)\big)_{n\in\mathbb{N}}$ and $\big((\bar{u}_n,\bar{\eta}_n)\big)_{n\in\mathbb{N}}$, respectively. By the definition of the space $\mathscr{Z}_T$, we have

$$\bar{u}_n\to u_\star \text{ in } L^2_w(0,T;\mathbb{H}^1)\cap L^p(0,T;\mathbb{L}^q)\cap\mathbb{D}([0,T];X^{-\beta})\cap\mathbb{D}([0,T];\mathbb{H}^1_w)\quad \bar{\mathbb{P}}-\text{a.s.}. \tag{4.24}$$

For the end of the proof we take special values of the parameters: $p=4, q=4$ and $\beta=\dfrac{1}{2}$. Now, due to Kuratowski theorem, Borel subsets of $\mathbb{D}([0,T];\mathbb{H}_n)$ are Borel subsets of $L^2_w(0,T;\mathbb{H}^1)\cap L^4(0,T;\mathbb{L}^4)\cap\mathbb{D}([0,T];X^{-\frac{1}{2}})\cap\mathbb{D}([0,T];\mathbb{H}^1_w)$, and $\mathbb{P}\{u_n\in\mathbb{D}([0,T];\mathbb{H}_n)\}=1$. Hence, we may assume that $\bar{u}_n$ takes values in $\mathbb{H}_n$ and that the laws on $\mathbb{D}([0,T];\mathbb{H}_n)$ of u_n and $\bar{u}_n$ are equal. Therefore, $\bar{u}_n$ satisfies the a-priori estimates given in Lemma 4.5, and hence there exist measurable processes Y and Z on $[0,T]\times\bar{\Omega}$, such that $Y\in L^{2r}(\bar{\Omega};\ L^2(0,T;\mathbb{L}^2))$, for $r\geqslant 1$, and $Z\in L^2(\bar{\Omega};\ L^2(0,T;\mathbb{L}^{\frac{3}{2}}))$ and

$$\bar{u}_n\times(\Delta\bar{u}_n-\pi_n\nabla\phi(\bar{u}_n))\to Y \text{ weakly in } L^{2r}(\bar{\Omega};L^2(0,T;\mathbb{L}^2)), \tag{4.25}$$

$$\bar{u}_n\times\big(\bar{u}_n\times(\Delta\bar{u}_n-\pi_n\nabla\phi(\bar{u}_n))\big)\to Z \text{ weakly in } L^2(\bar{\Omega};L^2(0,T;\mathbb{L}^{\frac{3}{2}})) \text{ and} \tag{4.26}$$

$$\pi_n(\bar{u}_n\times(\bar{u}_n\times(\Delta\bar{u}_n-\pi_n\nabla\phi(\bar{u}_n))))\to Z \text{ weakly in } L^2(\bar{\Omega};L^2(0,T;X^{-\beta})). \tag{4.27}$$

The following results provide important characterisation of the processes Y and Z. We omit here the proof of the first part and refer the reader to [8, 9] for detailed discussion.

Lemma 4.7. *For any measurable process $\varphi\in L^4(\bar{\Omega};L^4(0,T;\mathbb{W}^{1,4}))$, we have the equalities*

$$\lim_{n\to\infty}\bar{\mathbb{E}}\int_0^T\langle\bar{u}_n(s)\times\Delta\bar{u}_n(s),\,\varphi(s)\rangle_{\mathbb{L}^2}\,ds=\bar{\mathbb{E}}\int_0^T\sum_{i=1}^3\left\langle\frac{\partial u_\star(s)}{\partial x_i}\,,u_\star(s)\times\frac{\partial\varphi(s)}{\partial x_i}\right\rangle_{\mathbb{L}^2}ds, \tag{4.28}$$

$$\lim_{n\to\infty}\bar{\mathbb{E}}\int_0^T\langle\bar{u}_n(s)\times\pi_n\nabla\phi(\bar{u}_n(s)),\,\varphi(s)\rangle_{\mathbb{L}^2}\,ds=\bar{\mathbb{E}}\int_0^T\langle u_\star(s)\times\nabla\phi(u_\star(s)),\,\varphi(s)\rangle_{\mathbb{L}^2}\,ds. \tag{4.29}$$

Proof. In order to prove (4.29), we first write

$$\bar{u}_n \times \pi_n \nabla\phi(\bar{u}_n) - u_\star \times \nabla\phi(u_\star) = \big(\bar{u}_n - u_\star\big) \times \pi_n \nabla\phi(\bar{u}_n) + u_\star \times \big((\pi_n - I)\nabla\phi(u_\star)\big) \\ + u_\star \times \big(\pi_n(\nabla\phi(\bar{u}_n) - \nabla\phi(u_\star))\big). \tag{4.30}$$

We deal with the first sequence on the RHS of (4.30) by applying the Vitali Convergence Theorem, because (i) it is bounded in $L^2(\bar{\Omega};\, L^2(0,T;\mathbb{L}^2))$ by estimate (4.16) from Lemma 4.5 (satisfies, as observed above, also by the sequence $(\bar{u}_n)$) and hence uniformly integrable and (ii) it is a.s, convergent to 0 on $\bar{\Omega} \times (0,T) \times \mathbb{L}^2$. We deal with the second term, because $\nabla\phi(u_\star) \in \mathbb{L}^2$ a.s. on $\bar{\Omega} \times (0,T)$ again by applying the Vitali Convergence Theorem. Finally, we deal with the last term by observing that in view of the Lipschitz property of $\nabla\phi$ and (4.24) we have, $\bar{\mathbb{P}}$ − a.s.,

$$\nabla\phi(\bar{u}_n) \to \nabla\phi(u_\star) \;\text{ in }\; L^2(0,T;\mathbb{L}^2).$$

The proof is thus complete. □

Remark 4.8. As a consequence of the above Lemma, for any measurable process $\varphi \in L^4(\bar{\Omega}; L^4(0,T;\mathbb{W}^{1,4}))$, we have the following result

$$\begin{aligned} &\lim_{n\to\infty} \bar{\mathbb{E}} \int_0^T \langle \bar{u}_n(t) \times (\Delta\bar{u}_n(t) - \pi_n\nabla\phi(\bar{u}_n(t)))\,,\, \varphi(t)\rangle_{\mathbb{L}^2}\, dt \\ &= \bar{\mathbb{E}} \int_0^T \langle Y(t), \varphi(t)\rangle_{\mathbb{L}^2}\, dt \\ &= \bar{\mathbb{E}} \int_0^T \sum_{i=1}^3 \left\langle \frac{\partial u_\star(t)}{\partial x_i}\,, u_\star(t) \times \frac{\partial\varphi(t)}{\partial x_i} \right\rangle_{\mathbb{L}^2} dt - \bar{\mathbb{E}} \int_0^T \langle u_\star(t) \times \nabla\phi(u_\star(t)),\, \varphi(t)\rangle_{\mathbb{L}^2}\, dt. \end{aligned}$$

Remark 4.9. It follows by considering processes of the form $\varphi(t,\omega) := \chi_S(t,\omega)\psi$, where χ_S is the indicator of a measurable subset S of $[0,T] \times \bar{\Omega}$ and ψ is a fixed element of $\mathbb{W}^{1,4}$, that for each $\psi \in \mathbb{W}^{1,4}$

$$\langle Y(t,\omega), \psi\rangle_{\mathbb{L}^2} = \sum_{i=1}^3 \left\langle \frac{\partial u_\star(t,\omega)}{\partial x_i}, u_\star(t,\omega) \times \frac{\partial\psi}{\partial x_i} \right\rangle_{\mathbb{L}^2} - \langle u_\star(t,\omega) \times \nabla\phi(u_\star(t,\omega)),\, \psi\rangle_{\mathbb{L}^2} \tag{4.31}$$

for almost every $(t,\omega) \in [0,T] \times \bar{\Omega}$. Since $\mathbb{W}^{1,4}$ is separable, for (t,ω) outside a set of measure zero, equality (4.31) holds for all $\psi \in \mathbb{W}^{1,4}$.

Lemma 4.10. *For any process $\psi \in L^4(\bar{\Omega}; L^4(0,T;\mathbb{L}^4))$ we have*

$$\begin{aligned} &\lim_{n\to\infty} \bar{\mathbb{E}} \int_0^T \langle \bar{u}_n(s) \times \big(\bar{u}_n(s) \times (\Delta\bar{u}_n(s) - \pi_n\nabla\phi(\bar{u}_n(s)))\big), \psi(s)\rangle_{\mathbb{L}^2}\, ds \\ &= \bar{\mathbb{E}} \int_0^T {}_{\mathbb{L}^{\frac{3}{2}}}\langle Z(s), \psi(s)\rangle_{\mathbb{L}^3}\, ds \qquad (4.32) \\ &= \bar{\mathbb{E}} \int_0^T {}_{\mathbb{L}^{\frac{3}{2}}}\langle u_\star(s) \times Y(s), \psi(s)\rangle_{\mathbb{L}^3}\, ds. \qquad (4.33) \end{aligned}$$

Remark 4.11. Since $L^4(\bar{\Omega}; L^4(0,T;\mathbb{L}^4))$ is dense in $L^2(\bar{\Omega}; L^2(0,T;\mathbb{L}^3))$, we may conclude that $Z = u_\star \times Y$ as elements of $L^2(\bar{\Omega}; L^2(0,T;\mathbb{L}^{\frac{3}{2}}))$.

Now for $t \in [0,T]$ and $v \in L^4(\bar{\Omega}; X^\beta)$, we denote

$$\begin{aligned} M_n(\bar{u}_n, \bar{\eta}_n, v)(t) :&= {}_{X^{-\beta}}\langle \bar{u}_n(0), v\rangle_{X^\beta} + \lambda_1 \int_0^t \langle \bar{u}_n(s) \times (\Delta\bar{u}_n(s) - \pi_n\nabla\phi(\bar{u}_n(s)))\,, v\rangle_{\mathbb{L}^2} ds \\ &\quad - \lambda_2 \int_0^t {}_{X^{-\beta}}\langle \bar{u}_n(s) \times (\bar{u}_n(s) \times (\Delta\bar{u}_n(s) - \pi_n\nabla\phi(\bar{u}_n(s)))), v\rangle_{X^\beta}\, ds \end{aligned}$$

$$+\int_0^t\int_B\langle G_n(l,\bar u_n(s)),v\rangle_{\mathbb{L}^2}\,\tilde{\bar\eta}_n(ds,dl)+\int_0^t\langle b_n(\bar u_n(s)),v\rangle_{\mathbb{L}^2}\,ds,$$

$$M(u_\star,\eta_\star,v)(t):={}_{X^{-\beta}}\langle u_\star(0),v\rangle_{X^\beta}+\lambda_1\int_0^t\langle u_\star(s)\times(\Delta u_\star(s)-\pi_n\nabla\phi(u_\star(s)))\,,v\rangle_{\mathbb{L}^2}ds$$

$$-\lambda_2\int_0^t {}_{X^{-\beta}}\langle u_\star(s)\times(u_\star(s)\times(\Delta u_\star(s)-\pi_n\nabla\phi(u_\star(s)))),v\rangle_{X^\beta}\,ds$$

$$+\int_0^t\int_B\langle G(l,u_\star(s)),v\rangle_{\mathbb{L}^2}\,\tilde\eta_\star(ds,dl)+\int_0^t\langle b(u_\star(s)),v\rangle_{\mathbb{L}^2}\,ds.$$

We can show that

Lemma 4.12. *For all $v\in L^4(\bar\Omega;X^\beta)$,*

$$(a)\ \lim_{n\to\infty}\bar{\mathbb{E}}\left[\int_0^T|\langle\bar u_n(t)-u_\star(t),v\rangle_{\mathbb{L}^2}|^2dt\right]=0, \tag{4.34}$$

$$(b)\ \lim_{n\to\infty}\bar{\mathbb{E}}\left[\int_0^T|M_n(\bar u_n,\bar\eta_n,v)(t)-M(u_\star,\eta_\star,v)(t)|^2dt\right]=0. \tag{4.35}$$

The proof of this Lemma relies on the term by term estimates, the above weak convergences and the characterisation of the processes Y and Z from Lemma 4.7 and Lemma 4.10 and the followup discussions in Remarks 4.8, 4.9, and 4.11.

Finally, putting $\bar u:=u_\star$, and $\bar\eta:=\eta_\star$, we conclude that the system $(\bar\Omega,\bar{\mathcal{F}},\bar{\mathcal{F}}_t,\bar{\mathbb{P}},\bar u,\bar\eta)$ is a martingale solution of the equation (2.4).

Step - 4 (Verification of the Saturation Constraint Condition)

Proposition 4.13. *The process $\bar u$ satisfies*

$$|\bar u(t,x)|_{\mathbb{R}^3}=1,\quad \text{for}\quad \textit{Lebesgue a.e. } x\in D \textit{ for all } t\in[0,T],\ \bar{\mathbb{P}}\ \textit{a.s.} \tag{4.36}$$

In particular, $\bar u$ is càdlàg in $\mathbb{L}^2$.

The proof relies on verification of the conditions of Gyöngy-Krylov [12] Itô lemma and its application. Similar arguments have been provided at the beginning of the proof of this Theorem 4.3. For similar ideas see [4]. □

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