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On the Stability of Incompressible MHD Modes in Magnetic Cylinder with Twisted Magnetic Field and Flow

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Abstract

In this work, we studied MHD modes in a magnetically twisted flux tube with a twisted flow that is embedded in the uniform magnetic field. We consider when the azimuthal magnetic field and velocity are linear functions of radius (case i) and also more generally when they are arbitrary functions of radius (case ii). Under these assumptions, we obtain the dispersion equation in the incompressible limit. This solution can also be used to describe the MHD perturbations in plasma pinches and vortices. The dispersion equation is simplified by implementing the thin flux tube approximation. It is shown that sausage modes ($m = 0$) become unstable for large enough azimuthal flow speeds. Also, we obtained the unstable modes for $m > 0$. It is shown that the stability criterion of the $m = 1$ mode (for case i) is independent of the background azimuthal components of the plasma velocity and magnetic field. These criteria fully coincide with the result that was previously obtained by Syrovatskiy for a plane interface. Moreover, this result even remains valid when the azimuthal magnetic field and velocity have an arbitrary dependence on radius (case ii). A criterion for the stability of the $m \geqslant 2$ modes is also obtained. It was found that instability of these modes is determined by both longitudinal and azimuthal flows. It is shown that if there is sufficient azimuthal background flow, then all modes with $m \geqslant 2$ will become unstable.

Key words: instabilities – magnetohydrodynamics (MHD) – plasmas – Sun: atmosphere – waves

1. Introduction

It is well known that plasma flows play an important role in the Sun's magnetic atmosphere (see, e.g., Priest 2003; Kallenrode 2004; Filippov 2007, and references therein). It is difficult to obtain an exact analytical solution in general cases that describe, e.g., how wave modes depend on stationary but arbitrary plasma flows. However, for some special cases, e.g., when there is axial symmetry, it is possible to obtain magnetohydrostatic balance equations and to find analytical solutions and stability criteria. These solutions can be used for both qualitative and quantitative analyses of the variety of wave processes observed in the solar magnetic loops, spicules and other magnetic configurations in the solar atmosphere.

The cylindrically symmetric flux tube with a twisted magnetic field is a well-known model for theoretical study of MHD perturbations (see, e.g., Goossens 1991; Bennett et al. 1999; Erdélyi & Fedun 2006, 2007, 2010; Giagkiozis et al. 2015). For a long time, this basic but instructive model was also used to study plasma processes in space (see, for example, Roberts 1991; Ladikov-Roev et al. 2013; Cheremnykh et al. 2014, 2018; Klimushkin et al. 2017) and laboratory high-temperature plasmas (Suydam 1958; Shafranov 1970; Bateman 1978; Galeev & Sudan 1989; Burdo et al. 1994; Andrushchenko et al. 1999). This model is also useful for study fundamental plasma physics problems (see, e.g., Solov'ev 1967; Cheremhykh & Revenchuk 1992; Andrushchenko et al. 1993; Cheremnykh et al. 1994, 1994; Filippov 2007; Cheremnykh 2008, to name but a few).

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This analytical model can be applied to describe behavior of twisted jet-like plasma structures in the solar atmosphere that have been observed in X-ray, EUV, H_α, and other spectral lines. These structures, such as type I and II spicules may appear as a result of magnetic reconnection (see, e.g., Shibata et al. 2007) and have been widely observed by, e.g., the *Hinode* satellite and the Swedish Solar Telescope (SST; see, e.g., De Pontieu et al. 2007, 2012; Kosugi et al. 2007; Sharma et al. 2018). Recently, oscillations in spicules have been interpreted as kink and sausage MHD wave modes (see, e.g., Jess et al. 2012, 2015, and references therein). Morton et al. (2012) have also reported on signatures of transverse oscillations in the fibril structures. The proposed model of a magnetic flux tube in the presence of magnetic twist and twisted flow can provide us with a better understanding of plasma processes and wave generation by photospheric rotational motion, such as inter-granular vortices (see, e.g., Bonet et al. 2008, 2010; Wedemeyer-Böhm & Rouppe van der Voort 2009; Giagkiozis et al. 2017; Kato & Wedemeyer 2017). Such types of magnetic configurations and plasma flows are also frequently observed in solar tornadoes (see, e.g., Li et al. 2012; Su et al. 2012; Wedemeyer-Böhm et al. 2012) and naturally appear in numerical MHD simulations of various regions of the solar atmosphere (see, e.g., Fedun et al. 2011a, 2011b; Shelyag et al. 2011, 2012, 2013; González-Avilés et al. 2017, 2018; Murawski et al. 2018; Snow et al. 2018).

In the second half of the 19th century, Helmholtz (1868) and Kelvin (1910) discovered that the plane interface between two moving liquids with different velocities is unstable. Much later, in the middle of the twentieth century, Landau & Lifshitz (1959), based on equations of motion of ideal fluid in the approximation of a zero thickness interface between two moving liquids, showed a simple derivation of this type of instability. However, this approach does not cover all possible scenarios of instability development between separate moving media. For example, the Kelvin–Helmholtz (KH) instability may be substantially affected by the presence of magnetic field. Syrovatskiy (1953) has found that a sufficiently strong magnetic field parallel to the interface leads to quenching of the KH instability. On the other hand, if the magnetic field is perpendicular to the media interface, the instability development is not hindered (see, e.g., Chandrasekhar 1961).

A detailed analytical study of the effect of longitudinal and azimuthal flows in the context of tokamaks on MHD waves in a circular plasma cylinder in the presence of a twisted magnetic field was carried out in Solov'ev (1967). For a thin plasma cylinder with a homogeneous current, which is rotating as a whole around its axis and surrounded by a disturbed intermediate and ideally conducting casing, it was shown that current and rotation do not affect the stability criterion of the mode $m = 1$. In the case of a plasma tube without a casing, it was found that the azimuthal rotation of the cylinder leads to the instability of MHD modes. If the boundary of the plasma tube is 'fixed' on an ideally conducting wall, then there is an interval of azimuthal rotation speeds at which stability with respect to arbitrary helical modes takes place. In the same paper, for the case when magnetic field and velocity vectors are parallel, a particular Suydam criterion (see, e.g., Suydam 1958) was obtained for perturbations that are independent of the boundary conditions and localized in the neighborhood of points at which the longitudinal wave vector is very small. It follows from the necessary stability criterion that speed can have a destabilizing effect on the Suydam modes (see, e.g., Cheremhykh & Revenchuk 1992). Solov'ev (1967) has found a local stability condition for the special case of axially symmetric oscillations, i.e., sausage modes ($m = 0$), of the magnetic flux tube when a longitudinal magnetic field is not present. The zero azimuthal magnetic field approximation coincides with the Rayleigh criterion (Rayleigh 1916). Some results of Solov'ev (1967) were refined in later papers (Bondeson et al. 1987; Bodo et al. 1989, 1996). For example, Bondeson et al. (1987) have shown analytically and numerically that the plasma flow modifies the Suydam criterion and, at some critical velocity, destabilizes the Suydam modes. Also in this paper, the behavior of the Suydam modes was analyzed at velocities above and below the critical velocity.

Waves and instabilities in solar magnetic tubes with background flow have been investigated in a number of papers (see, e.g., Goossens et al. 1992; Soler et al. 2010; Zaqarashvili et al. 2010, 2015, and references there in). In particular, Goossens et al. (1992) obtained a dispersion relation for MHD modes in a plasma cylinder with a longitudinal magnetic field and flow, which in the long-wave approximation coincides with the dispersion equation obtained by Syrovatskiy (1953) and describes KH instability. Soler et al. (2010) have found that azimuthal plasma flow generates an instability of the KH type in a plasma cylinder with a longitudinal magnetic field. The destabilizing influence of longitudinal flow on the Suydam modes (see, e.g., Suydam 1958) in a plasma cylinder with a helical magnetic field was studied by Zaqarashvili et al. (2010). In all of these works, it was proposed that the presence of flow significantly modifies the dispersion equations and, accordingly, the propagation conditions of MHD waves. It is necessary to mention the general result of all of these works: similar to the KH instability, there exists a plasma velocity limit above which MHD modes become unstable. At the same time, in these studies, the stability analysis of the perturbations was carried out in plasma tubes with either a nontwisted magnetic field or with a nontwisted velocity flow. This was primarily due to the mathematical intractability of consistently dealing with both small oscillations of the equilibrium twisted magnetic field and the equilibrium twisted velocity field at the same time. In the present paper, we solved this problem as follows. Following Appert et al. (1974), usually the two first-order differential equations for radial displacement ξ_r and perturbed total plasma pressure δp_1 are used to describe MHD perturbations in cylindrically symmetric magnetic flux tubes. Analysis shows that this approach can be effectively used for magnetohydrostatic equilibria.

Goossens et al. (1992) has shown that perturbed velocity $\delta \boldsymbol{v}$ is connected with displacement $\boldsymbol{\xi}$ as

$$\delta \boldsymbol{v} = \frac{\partial \boldsymbol{\xi}}{\partial t} + \nabla \times (\boldsymbol{\xi} \times \boldsymbol{V}) - \boldsymbol{\xi}\, \mathrm{div}\, \boldsymbol{V} + \boldsymbol{V}\, \mathrm{div}\, \boldsymbol{\xi} = \frac{\partial \boldsymbol{\xi}}{\partial t} + (\boldsymbol{V} \cdot \nabla)\boldsymbol{\xi} - (\boldsymbol{\xi} \cdot \nabla)\boldsymbol{V}. \quad (1)$$

Here $\boldsymbol{V}$ is the equilibrium velocity. For this analysis, we will use the cylindrical coordinate system (r, φ, z) and assume that magnetic flux tube is axially symmetric. Also, in our model, we assume that the structure of the background plasma and magnetic field depend only on the radial distance r (see also Goossens et al. 1992). Therefore, the equilibrium velocity $\boldsymbol{V}$ and magnetic $\boldsymbol{B}$ fields can be represented in the form:

$$\boldsymbol{V} = V_\varphi(r)\boldsymbol{e}_\varphi + V_z(r)\boldsymbol{e}_z, \quad \boldsymbol{B} = B_\varphi(r)\boldsymbol{e}_\varphi + B_z(r)\boldsymbol{e}_z. \quad (2)$$

One can see from Equation (1) that the most convenient variable for linearization of the MHD equations in the moving plasma is $\delta \boldsymbol{v}$. We will show in Section 3 that if twisted magnetic field and flow are present, the most convenient approach to obtain boundary conditions is to use $\delta \boldsymbol{v}$. These boundary conditions can be obtained in the framework of classical hydrodynamic theory (see, e.g., Landau & Lifshitz 1959). We will show that under necessary approximations the obtained results lead to the results of other works on this topic.

In the present study, we will simplify the equations of small oscillations obtained for the case of a plasma cylinder with a homogeneous current along the cross section of a plasma cylinder, with a longitudinal uniform flow that is also rotating as a whole around the flux tube axis. The equations of small oscillations obtained for such an equilibrium can be used to analyze the stability of MHD modes with arbitrary azimuthal wave numbers. We confine ourselves to considering only incompressible perturbations of the Alfvén type Miyamoto (1997), since it is known that these perturbations are the most unstable (see, e.g., Kadomtsev 1966; Bateman 1978). We will then go on to focus on realization of the KH instability by analyzing the stability of a plasma pinch with a constant external longitudinal magnetic field, a zero external azimuthal field, and a twisted magnetic and velocity field inside the the magnetic flux tube. Importantly, we will demonstrate that in the case of the kink mode ($m = 1$) the instability criterion is independent of the background azimuthal components of the magnetic field and velocity flow.

2. Initial Equations

Let us start with MHD equations that describe the flow of an ideally conducting plasma (see, e.g., Kadomtsev 1966; Priest 2003).

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \rho \boldsymbol{v} = 0, \tag{3}$$

$$\rho\left[\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v}\right] = -\nabla\left(p + \frac{B^2}{2}\right) + (\boldsymbol{B} \cdot \nabla)\boldsymbol{B}, \tag{4}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} = (\boldsymbol{B} \cdot \nabla)\boldsymbol{v} - (\boldsymbol{v} \cdot \nabla)\boldsymbol{B} - \boldsymbol{B} \operatorname{div} \boldsymbol{v}, \tag{5}$$

$$\frac{d}{dt}\left(\frac{p}{\rho^{\gamma}}\right) = 0, \tag{6}$$

where $\boldsymbol{v}$ is the velocity of the elementary plasma volume, ρ is the plasma density, p is the kinetic pressure, and γ is the ratio of specific heats. Here and later in the text we use normalized magnetic field, i.e., $\boldsymbol{B}/\sqrt{4\pi} \to \boldsymbol{B}$.

By using this initial system of equations, we obtain Equation (55), which describes a steady-state plasma flow. Also, from these equations, assuming that every perturbation is proportional to $\exp(-i\omega t + im\varphi + ik_z z)$, we derive the governing equations for small oscillations shown in Equations (64)–(66). From these equations, we obtain Equations (71)–(72)), which are two first-order differential equations coupling the perturbed total pressure δp_1 and perturbed radial velocity component δv_r. These equations can be applied to both the internal $r < a$ and external $r > a$ regions of the magnetic plasma column, where a is its radius (see Figure 1). We will denote the quantities that correspond to each of the internal and external regions with the indices i and e, respectively. By assuming that the equilibrium plasma density profile is piece-wise constant, i.e., $\frac{d\rho}{dr} = 0$ inside and outside of the cylinder, the differential equation for the radial velocity perturbation can be obtained by substituting Equation (72) into (71) (see Appendices A and B for details):

$$\frac{d}{dr}\left[\frac{a_{11}}{k_z^2 + m^2/r^2}\frac{1}{r}\frac{d}{dr}\left(r\frac{\delta v_r}{\omega_1}\right)\right] + 2r\frac{\delta v_r}{\omega_1}\frac{d}{dr}\left[\frac{m}{k_z^2 + m^2/r^2}\frac{a_{12}}{r^2}\right] = \frac{\delta v_r}{\omega_1}\left[a_{11} - \frac{4a_{12}^2k_z^2}{a_{11}(k_z^2 + m^2/r^2)} + r\frac{d}{dr}\left(\frac{B_\varphi^2 - \rho V_\varphi^2}{r^2}\right)\right], \tag{7}$$

where the frequency ω_1 is given by

$$\omega_1 = \omega - \frac{m}{r}V_\varphi - k_z V_z.$$

Equation (7) coincides with Equation (14.36), given in Miyamoto (1997), and is equivalent to the Hain–Lüst equation (Hain & Lüst 1958). If $V_z = V_\varphi = 0$ and $B_\varphi = 0$, then Equation (7) in the incompressible limit is obtained from the equations of small amplitude perturbations given by Goedbloed & Hagebeuk (1972), Choe et al. (1977), and Edwin & Roberts (1983). For $V_z = V_\varphi = 0$ and $B_z = 0$ Equation (7) becomes the equation obtained by Cheremnykh et al. (2014).

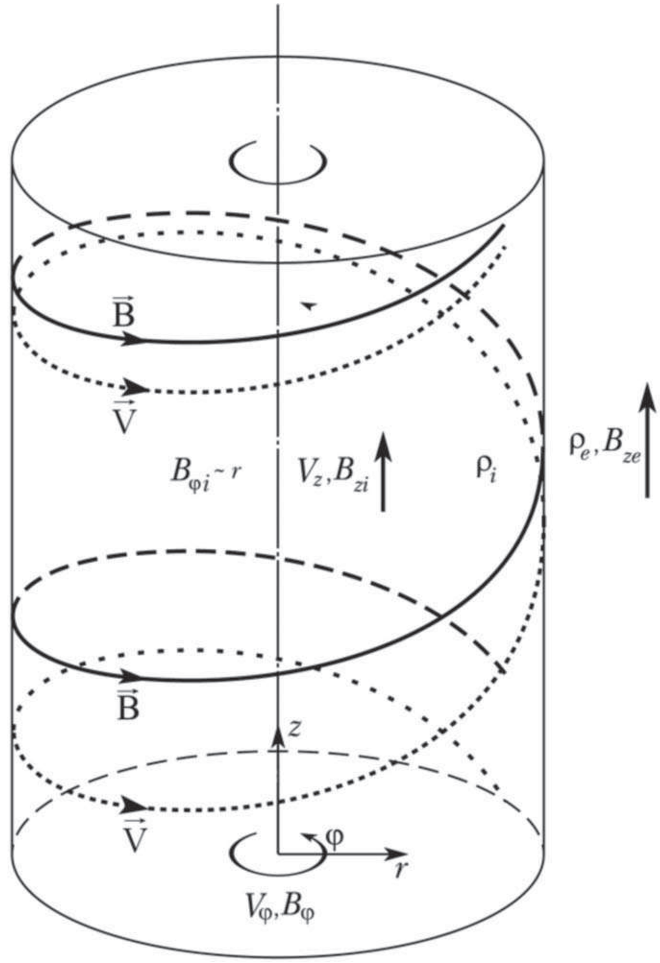

Figure 1. Magnetohydrodynamic equilibrium of the twisted magnetic flux tube in presence of helical velocity flow.

For the case under consideration here, i.e., stationary flow in the plasma cylinder with a twisted magnetic field, the radial components of displacement ξ_r and velocity δv_r are related to each other as:

$$\xi_r = i\frac{\delta v_r}{\omega_1}.$$

Equation (7) can be obtained from Equation (16) in Goossens et al. (1992) in the incompressible limit and after replacing $\xi_r \to \delta v_r/\omega_1$.

From Equations (71)–(72), differential Equation (73) is obtained for δp_1. For the case $V_\varphi = V_z = 0$, Equation (73) coincides with Equation (12) in Bennett et al. (1999). For the following analysis, we will use Equation (7) as well as Equation (73).

3. Boundary Conditions

To obtain a dispersion relation, we will use the differential Equation (73), but this must be supplemented with physical boundary conditions. The first boundary condition can be found by assuming that the flux is not changing through a time-varying interface. If the plasma is moving in the presence of a perturbed discontinuity surface, from the continuity equation Equation (3), we obtain (Landau & Lifshitz 1959),

$$\{\rho(v_n - D_n)\} = 0,$$

where $v_n = \boldsymbol{v} \cdot \boldsymbol{n}$ is the normal velocity component (to the discontinuity surface) of the elementary plasma volume, D_n is the interface velocity, directed, by definition, along the normal to it. The symbol $\{x\} = x_i - x_e$ indicates the difference between the value of x before discontinuity (index i) and after (index e). Since ρ_i and ρ_e are not equal to zero, the condition

$$v_n = D_n$$

must be satisfied both inside and outside of the discontinuity surface.

To obtain v_n and D_n, we use the following equation for the interface

$$Z = r - a - \zeta(\varphi, z, t) = 0, \quad (8)$$

where a is the radius of the plasma cylinder (the nonperturbed surface interface), $\zeta(\varphi, z, t)$ is a small displacement of the interface along r. By taking into account that, $\zeta \sim \exp(-i\omega t + im\varphi + ik_z z)$ from Equation (8), we find that the normal $\boldsymbol{n}$ to the perturbed discontinuity surface has the following components,

$$\boldsymbol{n} = \frac{1}{|\nabla Z|}\left(1, -\frac{im}{r}\zeta, -ik_z\zeta\right).$$

Now, let us consider an arbitrary small volume near the interface inside of the plasma cylinder (i.e., at $r = a - 0$). Since this volume will always be adjacent to the interface, its coordinates r, φ, and z will always satisfy Equation (8). Then the normal component of the velocity of this elementary plasma volume is

$$v_n = \boldsymbol{v} \cdot \boldsymbol{n} = \frac{1}{|\nabla Z|}\left[\delta v_r - i\zeta\left(\frac{mV_\varphi}{r} + k_z V_z\right)\right]\Bigg|_{r=a-0}, \quad (9)$$

where $\boldsymbol{v} = \boldsymbol{V} + \delta\boldsymbol{v}$. To obtain Equation (9), we neglected second-order terms. For the points x^i of the perturbed interface, the following conditions are satisfied

$$Z(x^i, t) = 0, \frac{dx^i}{dt} = D^i,$$

and, therefore,

$$\frac{\partial Z}{\partial t} + D^i\frac{\partial Z}{\partial x^i} = 0.$$

Then the normal component of the velocity D_n of the surface in the presence of perturbations is

$$D_n = \boldsymbol{D} \cdot \boldsymbol{n} = D_i\frac{\partial Z/\partial x^i}{|\nabla Z|} = -\frac{\partial Z/\partial t}{|\nabla Z|} = -\frac{i\omega\zeta}{|\nabla Z|}. \quad (10)$$

Equating Equations (9) and (10), we obtain

$$\delta v_r|_{r=a-0} = -i\left(\omega - \frac{mV_\varphi}{r} - k_z V_z\right)\zeta|_{r=a-0}, \quad (11)$$

and, therefore

$$\zeta|_{a-0} = \frac{i\delta v_r}{\omega_1}\Bigg|_{a-0}. \quad (12)$$

Since Equation (8) is the same for the internal ($r < a$) and external ($r > a$) sides of the interface, for a small volume of plasma near the surface of the discontinuity, beyond the boundary of the plasma cylinder, the following condition must be satisfied,

$$\zeta|_{a+0} = \frac{i\delta v_r}{\omega_1}\Bigg|_{a+0}. \quad (13)$$

Let us assume that thickness of the discontinuity is negligibly small in comparison with the radius of the plasma cylinder. Then, it is obvious that $\zeta_i(a + 0) = \zeta_e(a - 0)$ or

$$\{\zeta\} = \zeta|_{a+0} - \zeta|_{a-0} = 0. \quad (14)$$

From Equations (12) to (14), we obtain the first boundary condition,

$$\left\{\frac{\delta v_r}{\omega_1}\right\} = \frac{\delta v_r}{\omega_1}\Bigg|_{a+0} - \frac{\delta v_r}{\omega_1}\Bigg|_{a-0} = 0, \quad (15)$$

which is usually called the kinematic boundary condition (Whitham 1974). Note that in the absence of a plasma flow through the discontinuity surface this is automatically satisfied.

There is also a dynamic condition (Whitham 1974) that must be satisfied on the interface. This condition can be derived by integrating Equation (7) with respect to the radius from $a - \varepsilon$ to $a + \varepsilon$ and taking the limit $\varepsilon \to 0$. By taking into account the continuity of $\delta v_r/\omega_1$ on the interval $(a - 0, a + 0)$, and also the Equation (72) for δp_1, we have the second boundary condition,

$$\left\{\delta p_1 - \frac{i\delta v_r}{\omega_1}\left(\frac{B_\varphi^2 - \rho V_\varphi^2}{r}\right)\right\} = 0. \quad (16)$$

The quantity $\frac{\delta v_r}{\omega_!}$ in Equations (15) and (16), according to Equation (71) can also be expressed in terms of δp_1 and the plasma equilibrium parameters.

4. Dispersion Relation and Solutions when V_φ and B_φ are Linearly Dependent on r

In this section, we will derive a dispersion relation for MHD perturbations in a plasma cylinder with uniform current. To achieve this, it is necessary to simplify Equations (72), (74), (16), and (17). The simplification of these equations requires specification of the background magnetic and velocity fields, and the chosen conditions at the boundary of the plasma cylinder. In this case, an internal twisted magnetic field can be represented as

$$\boldsymbol{B}_i = B_{\varphi i}(r)\boldsymbol{e}_\varphi + B_{zi}\boldsymbol{e}_z,$$

where $B_{\varphi i} = B_\varphi(a)r/a$ and $B_{zi} = \text{const}$ are the azimuthal and vertical components of the magnetic field. We assume that the magnetic flux tube is surrounded by a constant and nontwisted ($B_{\varphi e} = 0$) magnetic field $B_{ze} \neq 0$. We also assume (similarly to Zaqarashvili et al. 2015) that the twisted plasma flow inside the cylinder takes the form

$$\boldsymbol{v} = V_\varphi(r)\boldsymbol{e}_\varphi + V_z\boldsymbol{e}_z,$$

where we choose $V_z = \text{const}$ and $V_\varphi = \Omega r$. Here, $\Omega = V_\varphi(a)/a = \text{const}$ is the vortex intensity.

By taking into account all of these assumptions, we can consider a particular case of a stationary plasma flow in the presence of homogeneous twisted background magnetic and velocity fields, i.e., $B_\varphi(r)/rB_{zi} = \text{const}$ and $V_\varphi(r)/rV_z = \text{const}$.

For these choices of the magnetic and velocity fields, according to Equation (55), the plasma pressure outside the cylinder p_e is constant. Inside the plasma pressure $_{pi}$, from Equation (55), is given by

$$p_i(r) = p_i(0) + (\rho V_\varphi^2(a)/2 - B_\varphi^2(a))\frac{r^2}{a^2}, \quad (17)$$

where $p_i(0)$ is the plasma pressure on the cylinder axis. For the plasma pinch condition, $B_\varphi^2(a) > \rho V_\varphi^2(a)/2$, which results in the pressure value decreasing at the cylinder boundary (Miyamoto 1997). If $\rho V_\varphi^2(a)/2 > B_\varphi^2(a)$, then pressure radially increases and reaches the maximum value at the boundary. This type of behavior is usually observed in *vortex cylinders* (Batchelor 1970). Hence, the following derived equations will be valid for both plasma pinch and plasma vortex scenarios. To further simplify matters in the derivation, we take the background plasma densities inside and outside of the cylinder to be constants. Similar to Bennett et al. (1999), we consider a "homogeneous plasma" for which

$$\frac{B_{\varphi i}}{r} = \text{const}, \; B_{\varphi e} = 0, \; \frac{V_{\varphi i}}{r} = \text{const},$$
$$V_{\varphi e} = 0, \; B_{zi} = \text{const}, \; B_{ze} = \text{const},$$
$$V_{zi} = \text{const}, \; V_{ze} = 0. \quad (18)$$

Then

$$\omega_1 = \text{const}, \; \rho\omega_A^2 = \text{const},$$
$$D = \text{const}, \; C_1 r = \text{const},$$
$$C_2 = -(k_z^2 + m^2/r^2), \; C_3 = \text{const}. \quad (19)$$

From Equations (73) and (19), we obtain the Bessel equation for δp_1 (Dwight 1947), i.e.,

$$\frac{d^2}{dr^2}\delta p_1 + \frac{1}{r}\frac{d}{dr}\delta p_1 - \left(\frac{m^2}{r^2} + m_0^2\right)\delta p_1 = 0, \quad (20)$$

where

$$m_0^2 = k_z^2\left(1 - \frac{4a_{12}^2}{a_{11}^2}\right),$$
$$a_{11} = \text{const}t, \; a_{12} = \text{const}.$$

Since the coefficients a_{11} and a_{12} in Equation (20) have a different form when $r < a$ and $r > a$, the internal and external solutions can each be determined with their own arbitrary constants. The boundary conditions given by Equations (15) and (72) allow us to find an equation connecting these two arbitrary constants. Another equation for these constants can be found from Equation (16). As a result, we obtain two equations for two unknown quantities. Finding solutions to this system of equations will then allow us to derive the governing dispersion equation.

In the region where $r < a$, the solution (20) has no singularities at $r = 0$ for $m_0^2 > 0$

$$\delta p_1 = C_i I_m(m_0 r), \; r < a, \quad (21)$$

where $I_m(x)$ is a modified Bessel function of the first kind, and C_i is an arbitrary constant. For the case when $m_0^2 < 0$, the solution of Equation (20) for $r < a$ is

$$\delta p_1 = C_i J_m(n_0 r), \; r < a, \quad (22)$$

where $n_0 = -m_0^2 > 0$ and $J_m(x)$ is a Bessel function of the first kind. Outside of the cylinder, the required solution of Equation (20), tending to zero as $r \to \infty$, is

$$\delta p_1 = C_e K_m(|k_z|\, r), \; r > a, \quad (23)$$

where $K_m(x)$ is a modified Bessel function of the second kind and C_e is an arbitrary constant. To obtain the dispersion equation for the case $m_0^2 > 0$, from Equation (15) and (71), we have

$$C_e = C_i \frac{a_{11e}}{a_{11i}^2 - 4a_{12i}^2} \frac{\left[a_{11i} m_0 \frac{dI_m}{dx} - 2a_{12i}\frac{m}{a} I_m(x)\right]\Big|_{x=m_0 a}}{|k_z| \left.\frac{dK_m}{dy}\right|_{y=|k_z|a}}. \quad (24)$$

Here, the subscript i indicates that the value is taken inside the cylinder, and e is outside. Using the resulting relation for constants C_i and C_e, from Equation (16), we derive the dispersion equation,

$$\frac{a_{11i}\frac{x}{I_m(x)}\frac{dI_m}{dx}\Big|_{x=m_0 a} - 2a_{12i} m}{a_{11i}^2 - 4a_{12i}^2} = \frac{\frac{y}{K_m(y)}\frac{dK_m}{dy}\Big|_{y=|k_z|a}}{a_{11e} + \frac{(B_\varphi^2(a) - \rho_i V_\varphi^2(a))}{a^2}\frac{y}{K_m(y)}\frac{dK_m}{dy}\Big|_{y=|k_z|a}}. \quad (25)$$

When the background flow velocities are set to zero, Equation (25) reduces to Equation (23) in Bennett et al. (1999). If we assume $V_\varphi = V_z = 0$ and $B_\varphi = 0$, Equation (25) reduces to Equation 8(a) in Edwin & Roberts (1983).

In the particular case when $B_\varphi = 0$ and $V_\varphi = 0$, we obtain from Equation (25),

$$a_{11e}\frac{1}{I_m(x)}\frac{dI_m}{dx}\bigg|_{x=|k_z|a} = a_{11i}\frac{1}{K_m(y)}\frac{dK_m}{dy}\bigg|_{y=|k_z|a},$$

which is the same as the dispersion relation derived by Goossens et al. (1992) under the assumption that $V_{ze} = 0$. This condition can always be satisfied depending on the choice of reference frame.

The dispersion Equation (25) is a transcendental equation of rather complicated form and therefore it is more convenient to solve it with numerical methods (see, e.g., a numerical study for the cases when $m = \pm 2$ and $m = \pm 3$ in Zaqarashvili et al. 2015). In the present work, for analytical insight, we will only consider this equation in the long wavelength (or thin tube) approximation.

5. ($m \geqslant 1$) Modes in the Long Wavelength Approximation when V_φ and B_φ Are Linearly Dependent on r

For perturbations with a large characteristic wavelength $k_z a \sim m_0 a \ll 1$ and for $m \geqslant 1$ the following relations between

Bessel functions can be applied (see, e.g., Dwight 1947):

$$\left.\frac{x}{I_m(x)}\frac{dI_m}{dx}\right|_{x\ll 1} = \left.\frac{x}{J_m(x)}\frac{dJ_m}{dx}\right|_{x\ll 1} = -\left.\frac{x}{K_m(x)}\frac{dK_m}{dx}\right|_{x\ll 1} = m. \tag{26}$$

These relations significantly simplify dispersion Equation (25) to

$$\rho_i(\omega - k_z V_z)^2 + \rho_e\omega^2 - k_z^2(B_{zi}^2 + B_{ze}^2) + m(m-1)\left(\frac{\rho_i V_\varphi^2(a) - B_\varphi^2(a)}{a^2}\right) - 2\rho_i\frac{V_\varphi(a)}{a}(m-1)(\omega - k_z V_z) - \frac{2k_z}{a}(m-1)B_\varphi(a)B_{zi} = 0. \tag{27}$$

This dispersion relation is valid both for $m_0^2 > 0$ and $m_0^2 < 0$. The plasma perturbations may become unstable if frequency ω, which is determined by Equation (27), has an imaginary part.

For $V_\varphi = V_z = 0$, Equation (27) becomes,

$$\omega^2(\rho_i + \rho_e) = k_z^2(B_{zi}^2 + B_{ze}^2) + m(m-1)\frac{B_\varphi^2(a)}{a^2} + \frac{2k_z}{a}(m-1)B_\varphi(a)B_{zi}, \tag{28}$$

which, as shown previously by Cheremnykh et al. (2017), describes stable modes for all m values. If $B_\varphi(a) = 0$ or $m = 1$ from Equation (28), we obtain the dispersion equation for kink mode in the long wavelength approximation (see, e.g., Roberts 1991):

$$\omega^2 = k_z^2\frac{B_{zi}^2 + B_{ze}^2}{\rho_i + \rho_e}.$$

Note that this equation can also be easily derived from Equation 8(a) of Edwin & Roberts (1983; using our corresponding notations). In the particular case in which $B_{\varphi e} \sim r^{-1}$, $V_\varphi = V_z = 0$ the existence of unstable $m = 1$ modes strongly depends on the value and sign of k_z (see, e.g., Cheremnykh et al. 2018).

For $m = 1$ from Equation (27), it follows that

$$\rho_i(\omega - k_z V_z)^2 + \rho_e\omega^2 = k_z^2(B_{zi}^2 + B_{ze}^2)^2. \tag{29}$$

It can be seen that the dispersion of the kink mode $m = 1$ is completely independent of the azimuthal components of the background flow and magnetic field. From Equation (29),

$$\omega = \frac{\rho_i}{(\rho_i + \rho_e)}k_z V_z \pm \frac{\sqrt{(\rho_i + \rho_e)k_z^2(B_{zi}^2 + B_{ze}^2) - \rho_i\rho_e k_z^2 V_z^2}}{(\rho_i + \rho_e)}. \tag{30}$$

If the inequality

$$V_z^2 > \left(\frac{1}{\rho_i} + \frac{1}{\rho_e}\right)(B_{zi}^2 + B_{ze}^2) \tag{31}$$

is satisfied, from Equation (30) it follows that the frequency has a positive imaginary part and the $m = 1$ mode will be unstable. When the internal and external densities are the same ($\rho_i = \rho_e$), the instability development criterion (31) coincides with the criterion of Syrovatskiy (1953) for a plane plasma interface. From Equation (31), it follows that if a magnetic field is both strong enough and is parallel to the flow velocity field, it will quench the instability. For very weak magnetic fields, Equation (30) shows that the increment of instability, γ, is

$$\gamma \approx k_z V_z\frac{\sqrt{\rho_i\rho_e}}{(\rho_i + \rho_e)}. \tag{32}$$

In the case when $\rho_e \ll \rho_i$, the increment is much less than in the case $\rho_i \sim \rho_e$.

From Equation (27), for modes with $m \geqslant 2$, we obtain

$$\omega = \frac{1}{\rho_i + \rho_e}[\Delta_1 \pm \sqrt{\Delta_2}],$$
$$\Delta_1 = \rho_i\left[k_z V_z + \frac{(m-1)}{a}V_\varphi(a)\right],$$
$$\Delta_2 = \left\{(\rho_i + \rho_e)\left[k_z^2 B_{ze}^2 + \left(k_z B_{zi} + \frac{m-1}{a}B_\varphi(a)\right)^2 + \frac{m-1}{a^2}(B_\varphi^2(a) - \rho_i V_\varphi^2(a))\right] - \rho_i\rho_e\left(k_z V_z + \frac{m-1}{a}V_\varphi(a)\right)^2\right\}. \tag{33}$$

From Equation (33), it follows that instabilities of the modes $m \geqslant 2$ occur when $\Delta_2 < 0$, i.e.,

$$\left(k_z V_z + \frac{m-1}{a}V_\varphi(a)\right)^2 > \left(\frac{1}{\rho_i} + \frac{1}{\rho_e}\right) \times \left[k_z^2 B_{ze}^2 + \left(k_z B_{zi} + \frac{m-1}{a}B_\varphi(a)\right)^2 + \frac{m-1}{a^2}(B_\varphi^2(a) - \rho_i V_\varphi^2(a))\right]. \tag{34}$$

Note, that inequality (34), in contrast to inequality (31), depends on k_z, i.e., it depends on the longitudinal wavelength of the perturbation. It is also seen that both components of the background flow velocity field may contribute to the development of an instability. However, if the longitudinal and azimuthal components of the background magnetic field are strong enough, this could act against the growth of the instability, as could the presence of sufficient plasma density in the system. For very weak magnetic fields, from Equation (33), it follows that the increment is

$$\gamma \approx \frac{1}{\rho_i + \rho_e}\left[\rho_i\rho_e\left(k_z V_z + \frac{m-1}{a}V_\varphi(a)\right)^2 + \frac{m-1}{a}\rho_i(\rho_i + \rho_e)V_\varphi^2(a)\right]^{1/2}. \tag{35}$$

It can be seen from Equations (34) and (35) that the instability of modes $m \geqslant 2$ can develop even in cases when $V_z = 0$ and there is a small $V_\varphi(a)$ flow component. It follows from

Equations (33) and (34) that a twisted magnetic cylinder is stable if $V_z = V_\varphi = 0$ which is consistent with the previous result of Cheremnykh et al. (2017).

Now, let us compare the results of this present work with relevant results obtained previously by other authors. Soler et al. (2010) analyzed the development of the KH instability in a thin magnetic flux tube that was excited due to an azimuthal velocity component. The authors of this work made the following background variable choices,

$$P = 0,\ B_{zi} = B_{ze} = \text{const},\ B_\varphi(a) = 0,\ V_z = 0,\ V_\varphi \ll \frac{B_{zi}}{\sqrt{\rho_i}}.$$

For these choices of background variables to satisfy the condition of instability in Equations (33) and (34), it is necessary to have $m \gg 1$. For such values of m Equation (33) we have that,

$$\omega \approx \frac{\rho_i}{\rho_i + \rho_e}\frac{mV_\varphi(a)}{a} \pm \left[\frac{k_z^2(\rho_i C_{Ai}^2 + \rho_e C_{Ae}^2)}{(\rho_i + \rho_e)} - \frac{\rho_i \rho_e}{(\rho_i + \rho_e)^2} m^2 \frac{V_\varphi^2(a)}{a^2}\right]^{\frac{1}{2}},$$

where

$$C_{Ai}^2 = \frac{B_{zi}^2}{\rho_i},\ C_{Ae}^2 = \frac{B_{ze}^2}{\rho_e}.$$

The inequality (34) in this case is approximately

$$\frac{V_\varphi^2(a)}{C_{Ai}^2} > 2\frac{k_z^2 a^2}{m^2}\left(1 + \frac{\rho_i}{\rho_e}\right).$$

These equations coincide with Equations (18) and (19) of Soler et al. (2010).

Zaqarashvili et al. (2010) studied the influence of longitudinal flow ($V_z \neq 0, V_\varphi = 0$) on normal modes in a uniformly twisted magnetic cylinder ($B_{\varphi e} = 0$ and $B_{\varphi i} \sim r$; see also Kadomtsev 1966; Shafranov 1970; Miyamoto 1997), which satisfy the condition $\boldsymbol{k} \cdot \boldsymbol{B} \approx 0$, which results in a longitudinal wavenumber,

$$k_z \approx -\frac{m}{a}\frac{B_\varphi(a)}{B_{zi}}.$$

For this type of perturbation, Equation (33) can be written as,

$$\omega = \frac{|m|}{1 + \frac{\rho_e}{\rho_i}}\frac{B_\varphi(a)}{a\sqrt{\rho_i}}\left[-M_A \pm \frac{1}{\sqrt{m}}\sqrt{1 + \frac{\rho_e}{\rho_i} - |m|\frac{\rho_e}{\rho_i}M_A^2}\right],$$

where $M_A = \frac{\rho_i V_z^2}{B_{zi}^2}$. This equation is equivalent to equation (27) of Zaqarashvili et al. (2010). In this case, the instability condition (34) is

$$mM_A^2 > 1 + \frac{\rho_i}{\rho_e},$$

which coincides with inequality (28) from Zaqarashvili et al. (2010).

The case of internal twisted plasma flow was considered by Goossens et al. (1992). In this paper, the authors considered the case when $B_{zi} \neq 0$, $B_{ze} \neq 0$, $B_{\varphi i} \sim r$, $B_{\varphi e} = 0$, $V_{zi} \neq 0$, $V_{ze} = 0$, $V_{\varphi i} = V_{\varphi e} = 0$. With $\text{sgn}(m) = 1$, and longitudinal external background flow set to zero Equation (33) takes the form

$$\omega = k_z\frac{\rho_i V_z}{\rho_i + \rho_e} \pm \left\{\frac{1}{\rho_i + \rho_e}\left[k_z^2(B_{zi}^2 + B_{ze}^2) + \frac{B_\varphi^2(a)}{a^2}m(m-1) + 2k_z B_{zi}\frac{B_\varphi(a)}{a}(m-1)\right] - \frac{\rho_i \rho_e}{(\rho_i + \rho_e)^2}k_z^2 V_z^2\right\}^{\frac{1}{2}}.$$

This equation is consistent with Equation (71) from Goossens et al. (1992) when $B_\varphi = V_\varphi = 0$ and Equation (93) from Goossens et al. (1992) when $B_\varphi \neq 0$ and $V_\varphi = 0$.

6. General Case: ($m = 1$) Mode in the Long Wavelength Approximation for an Arbitrary V_φ and B_φ Dependence on r

When obtaining Equations (7), (15), (16), and

$$\delta p_1 = \frac{i\omega_1}{k_z^2 + m^2/r^2}\left[\frac{a_{11}}{r}\frac{d}{dr}\left(r\frac{\delta v_r}{\omega_1}\right) + \frac{2a_{12}}{r}m\frac{\delta v_r}{\omega_1}\right], \tag{36}$$

we took into account that the equilibrium plasma flow and magnetic field were independent of time but dependent on r.

In the approximation of a "thin" plasma cylinder, i.e., $k_z \ll m/r$, Equation (7) takes the form

$$\frac{1}{r}\frac{d}{dr}\left[a_{11}r^3\frac{d}{dr}\left(\frac{\delta v_r}{\omega_1}\right)\right] + (1 - m^2)a_{11}\frac{\delta v_r}{\omega_1} = 0. \tag{37}$$

For the case under consideration, the boundary condition given by Equation (15) remains unchanged and Equation (16) is modified to:

$$\left\{a_{11}r\frac{d}{dr}\left(r\frac{\delta v_r}{\omega_1}\right) + 2a_{12}rm\frac{\delta v_r}{\omega_1} - \frac{m^2}{r}(B_\varphi^2 - \rho V_\varphi^2)\frac{\delta v_r}{\omega_1}\right\} = 0. \tag{38}$$

Now let us remove the background linear dependence imposed on azimuthal magnetic field and velocity components shown previously in Equation (18) so that inside the cylinder V_φ and B_φ are now arbitrary functions of r, i.e., $B_{\varphi i} = B_\varphi(r)$ and $V_{\varphi i} = V_\varphi(r)$. In the case for $m = 1$ from Equation (37), we obtain

$$\frac{\delta v_r}{\omega_1} = \begin{cases} A = \text{const}, & r \leqslant a \\ A\left(\frac{a}{r}\right)^2, & r > a. \end{cases} \tag{39}$$

The solution given by Equation (39) is finite for $r = 0$, vanishes at $r \to \infty$, and satisfies the boundary condition Equation (15). Assuming in Equation (38) $m = 1$ and substituting solution (39) into this equation, as a result, we obtain the dispersion equation

$$\rho_i(\omega - k_z V_z)^2 + \rho_e \omega^2 = k_z^2(B_{zi}^2 + B_{ze}^2). \tag{40}$$

In the case $V_z = 0$, Equation (40) becomes the dispersion equation for the kink mode in the long wavelength limit without axial background flow.

$$\omega^2 = k_z^2\left(\frac{B_{zi}^2 + B_{ze}^2}{\rho_i + \rho_e}\right). \quad (41)$$

From Equation (40), we have:

$$\omega = \frac{\rho_i}{\rho_i + \rho_e}k_z V_z \pm \sqrt{k_z^2(B_{zi}^2 + B_{ze}^2) - \frac{\rho_i\rho_e}{(\rho_i + \rho_e)^2}k_z^2 V_z^2}. \quad (42)$$

Therefore, we obtain that the mode $m = 1$ will be unstable if the inequality

$$V_z^2 > \left(\frac{1}{\rho_i} + \frac{1}{\rho_e}\right)(B_{zi}^2 + B_{ze}^2) \quad (43)$$

is satisfied. It can be seen that the instability condition Equation (43) is independent of $B_r(r)$ and $V_\varphi(r)$.

7. Sausage Modes ($m = 0$) when V_φ and B_φ Linearly Depend on r

In this section, we will consider the sausage mode ($m = 0$) (Kadomtsev 1966; Miyamoto 1997), which plays an important role in the dynamics of the solar magnetic tubes (Erdélyi & Fedun 2006, 2007). For this mode, the dispersion Equation (25) takes the form

$$\frac{a_{11i}\frac{x}{I_0(x)}\frac{dI_0}{dx}\Big|_{x=m_1 a}}{(a_{11i}^2 - 4a_{12i}^2)} = \frac{\frac{y}{K_0(y)}\frac{dK_0}{dy}\Big|_{y=|k_z|a}}{a_{11e} + \frac{(B_\varphi^2(a) - \rho_i V_\varphi^2(a))}{a^2}\frac{y}{K_0(y)}\frac{dK_0}{dy}\Big|_{y=|k_z|a}}, \quad (44)$$

where

$$a_{11i} = \rho_i(\omega - k_z V_z)^2 - k_z^2 B_{zi}^2,$$
$$a_{11e} = \rho_e\omega^2 - k_z^2 B_{ze}^2,$$
$$a_{12i} = \rho_i(\omega - k_z V_z)\frac{V_\varphi(a)}{a} + k_z B_{zi}\frac{B_\varphi(a)}{a},$$
$$m_1 = \frac{k_z\sqrt{a_{11i}^2 - 4a_{12i}^2}}{a_{11i}}. \quad (45)$$

Equation (44) is valid for $m_1^2 > 0$. If $m_0^2 < 0$ the dispersion equation for sausage modes is modified as:

$$\frac{a_{11i}\left(\frac{x}{J_0(x)}\frac{dJ_0}{dx}\right)\Big|_{x=n_1 a}}{(a_{11i}^2 - 4a_{12i}^2)} = \frac{\frac{y}{K_0(y)}\frac{dK_0(y)}{dy}\Big|_{y=|k_z|a}}{a_{11e} + \frac{(B_\varphi^2(a) - \rho_i V_\varphi^2(a))}{a^2}\frac{y}{K_0(y)}\frac{dK_0(y)}{dy}\Big|_{y=|k_z|a}}, \quad (46)$$

where

$$n_1^2 = -m_1^2 = -\frac{k_z^2}{a_{11i}^2}(a_{11i}^2 - 4a_{12i}^2) > 0.$$

To simplify Equations (44) and (46) in the long wavelength approximation ($k_z a \ll 1$), we use the following relations for Bessel functions:

$$\frac{1}{I_0(x)}\frac{dI_0}{dx}\Big|_{x\ll 1} = -\frac{1}{J_0(x)}\frac{dJ_0}{dx}\Big|_{x\ll 1} \approx \frac{x}{2}, \; \frac{1}{K_0(x)}\frac{dK_0}{dx}\Big|_{x\ll 1} \approx \frac{1}{x(C - 1)}, \quad (47)$$

where $C \approx 0.5772$ is the Euler–Mascheroni constant. To obtain the last relation in (47) (see, e.g., Dwight 1947), we took into account that with accuracy up to the first terms:

$$K_0(x) = -\ln\frac{x}{2} - C + \frac{x^2}{4}\left(1 - \ln\left(\frac{x}{2}\right) - C\right),$$
$$\frac{dK_0}{dx} = -K_1(x) = -\frac{1}{x} + \frac{x}{4} - \frac{x}{2}\left(\ln\left(\frac{x}{2}\right) + C\right).$$

Then, for $x \ll 1$ by using L'Hôpital's rule, we can derive the second approximation in (47).

Equations (44) and (46) together with relations (47) simplify and take the same form

$$\rho_i(\omega - k_z V_z)^2 + \rho_e\omega^2\frac{k_z^2 a^2}{2}(1 - C) = k_z^2 B_{zi}^2 + \frac{k_z^2}{2}(B_\varphi^2(a) - \rho_i V_\varphi^2(a)) + \frac{k_z^4 a^2}{2}(1 - C)B_{ze}^2. \quad (48)$$

If in the Equation (48), we assume $V_z = V_\varphi(a) = 0$ and $B_\varphi(a) = 0$, we obtain the dispersion equation:

$$\omega^2 = k_z^2 C_{Ai}^2\left[1 - \frac{k_z^2 a^2}{2}\frac{\rho_e}{\rho_i}(1 - C)\left(1 - \frac{C_{Ae}^2}{C_{Ai}^2}\right)\right], \quad (49)$$

where $C_{Ai}^2 = B_{zi}^2/\rho_i$, $C_{Ae}^2 = B_{ze}^2/\rho_e$. Since for $x \ll 1$ the relation $xK_0(x) = (1 - C)x$ is valid, Equation (49) completely coincides with Equation 10(a) of Edwin & Roberts (1983), which is obtained under the same assumptions. From Equation (48), we find

$$\omega = \frac{1}{\left[\rho_i + \rho_e\frac{k_z^2 a^2}{2}(1 - C)\right]}[\Delta_3 \pm \sqrt{\Delta_4}],$$
$$\Delta_3 = \rho_i k_z V_z,$$
$$\Delta_4 = \left[\rho_i + \rho_e\frac{k_z^2 a^2}{2}(1 - C)\right] \times \left[k_z^2 B_{zi}^2 + \frac{k_z^2}{2}B_\varphi^2(a) + \frac{k_z^4 a^2}{2}(1 - C)B_{ze}^2\right] - \rho_i\rho_e\frac{k_e^4 a^2}{2}(1 - C)V_z^2 - \frac{k_z^2\rho_i V_\varphi^2(a)}{2}\left[\rho_i + \rho_e\frac{k_z^2 a^2}{2}(1 - C)\right]. \quad (50)$$

It follows from Equation (50) that instability is realized when $\Delta_4 < 0$ or in physical variables

$$V_\varphi^2(a) + V_z^2 \frac{k_z^2 a^2 (1 - C)}{\left[\frac{\rho_i}{\rho_e} + \frac{k_z^2 a^2}{2}(1 - C)\right]} > \frac{1}{\rho_i}(2B_{zi}^2 + B_\varphi^2(a) + k_z^2 a^2 (1 - C) B_{ze}^2). \quad (51)$$

In the limit that $k_z a \to 0$, from Equation (51), we see that the instability would primarily depend on the azimuthal plasma flow speed, i.e.,

$$V_\varphi^2(a) > \frac{1}{\rho_i}(2B_{zi}^2 + B_\varphi^2(a)). \quad (52)$$

Therefore, in the thin tube approximation, an unstable sausage mode can develop in condition (52) if the azimuthal flow speed is strong enough. Previously, Solov'ev (1967) derived a local stability criterion $V_\varphi^2 > 0$ for the sausage mode in a rotating plasma cylinder ($V_\varphi \sim r$ and $V_z = 0$) without a longitudinal magnetic field ($B_{zi} = B_{ze} = 0$). Importantly, this criterion was shown to be independent of the boundary conditions at the surface of the plasma cylinder. Therefore, from Equation (51), we can obtain an even stronger criterion for the existence of unstable sausage modes, i.e.,

$$V_\varphi^2(a) > \frac{B_\varphi^2(a)}{\rho_i}. \quad (53)$$

8. Summary of Main Results

Recent high resolution observation and detection of MHD modes in solar magnetic flux tubes show a variety of possible scenarios of their excitation, development, and propagation. For accurate MHD mode identification in observational data, it is crucially important to understand from a theoretical point of view which magnetic and velocity field components are important or could effect MHD wave mode generation and stability.

In this paper, we presented a detailed analysis of incompressible modes for a cylindrical magnetic flux tube with a uniformly twisted background flow and magnetic field. In such a configuration, we found that fast MHD modes exist only in the presence of vertical and azimuthal flows. It was shown that in comparison with the case of a magnetic cylinder with no twisted background flow and magnetic field, the frequency could be modified substantially by the inclusion of these effects, see, e.g., Equations (33) and (34).

We also obtained the equation of small amplitude perturbations, see, e.g., Equations (64)–(67), which allow us to investigate MHD modes of any m value in the presence of background flow. We restricted ourselves to the consideration of incompressible perturbations only. The resulting dispersion relation (25) is transcendental and can be most fully studied only by numerical methods. This relation generalizes the dispersion equation obtained previously by Bennett et al. (1999).

The main attention was paid to finding eigenfrequencies. To determine these frequencies, the dispersion Equation (25) was analyzed in the long wavelength approximation, i.e., $k_z a \ll 1$. This simplification led to dispersion Equation (48) for sausage modes $m = 0$ and dispersion Equation (27)—for modes with $m \geqslant 1$.

For the sausage mode dispersion relation, Equation (48) describes unstable perturbations in the presence of sufficiently high azimuthal flow speeds. Equation (48) coincided with the previous result of Bennett et al. (1999) for $V_\varphi = V_z = 0$.

To describe modes $m \geqslant 1$ in the long wavelength approximation, we derived dispersion Equation (27). From this equation, it follows that kink mode $m = 1$ can be unstable if the longitudinal background flow is large enough, but stability is independent of azimuthal magnetic field and flow components, regardless of their radial profiles. Hence, it was demonstrated that the condition for the transition of this mode from the stable to the unstable regime is determined by the KH criterion. Also, we found that modes with $m \geqslant 2$ can be unstable for sufficiently large longitudinal and azimuthal background flows. To counter these instabilities, the presence of sufficient longitudinal and azimuthal magnetic field components is required. Furthermore, it was found that the larger the m value, the more susceptible the mode is to instability in the presence of twisted background flow.

Regarding future work, since it is well known that the compressibility of a plasma is a further destabilizing factor in itself (see, e.g., Kadomtsev 1966; Miyamoto 1997), this effect should certainly be included to improve upon the current model and to allow for more realistic applications to the solar atmosphere.

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Appendix A
Governing Equations of Small Oscillations

By taking into account the unit vector differentiation rules, e.g.,

$$\frac{\partial \boldsymbol{e}_r}{\partial \varphi} = \boldsymbol{e}_\varphi, \; \frac{\partial \boldsymbol{e}_\varphi}{\partial \varphi} = -\boldsymbol{e}_r, \quad (54)$$

from Equations (4) and (2), we obtain an equation for steady-state plasma flow (see, e.g., Chandrasekhar 1961; Goossens et al. 1992):

$$\frac{dp_1}{dr} + \frac{B_\varphi^2 - \rho V_\varphi^2}{r} = 0, \; p_1 = p + \frac{B^2}{2}. \quad (55)$$

This equation shows a dependence both on the equilibrium magnetic field and azimuthal velocity. Equations (3), (5), and (6) under the above assumptions are satisfied automatically. To obtain the equations for perturbed quantities (which we will denote by symbol δ), let us linearize Equations (3)–(5) by assuming that the perturbed quantities depend on time as $\exp(-i\omega t)$.

From Equations (3) and (6), the expressions for the perturbed density and pressure, after some algebra, can be expressed as follows:

$$\delta\rho = -\frac{i}{\omega_1}\left(\delta v_r \frac{d\rho}{dr} + \rho \,\mathrm{div}\, \delta\boldsymbol{v}\right),$$
$$\delta p = -\frac{i}{\omega_1}\left(\delta v_r \frac{dp}{dr} + \gamma p \,\mathrm{div}\, \delta\boldsymbol{v}\right), \quad (56)$$

and

$$\operatorname{div}\delta\boldsymbol{v} = \frac{1}{r}\frac{\partial}{\partial r}(r\delta v_r) + \frac{1}{r}\frac{\partial}{\partial\varphi}\delta v_\varphi + \frac{\partial}{\partial z}\delta v_z.$$

The frequency in Equation (56) ω_1 is given by

$$\omega_1 = \omega - \frac{m}{r}V_\varphi - k_z V_z. \tag{57}$$

After linearization, Equation (4) can be written as

$$\begin{aligned}-i\omega\rho\delta\boldsymbol{v} + \rho(\boldsymbol{V}\cdot\nabla)\delta\boldsymbol{v} + \rho(\delta\boldsymbol{v}\cdot\nabla)\boldsymbol{V} + \delta\rho(\mathbf{V}\cdot\nabla)\boldsymbol{V}\\ = -\nabla\delta p_1 + (\delta\boldsymbol{B}\cdot\nabla)\boldsymbol{B} + (\boldsymbol{B}\cdot\nabla)\delta\boldsymbol{B},\end{aligned} \tag{58}$$

where δp_1 corresponds to the perturbation of the total plasma pressure:

$$\delta p_1 = \delta p + \boldsymbol{B}\cdot\delta\boldsymbol{B}. \tag{59}$$

By taking the scalar product of Equation (58) with $\boldsymbol{e}_r$, $\boldsymbol{e}_\varphi$, and $\boldsymbol{e}_z$, and taking into account Equation (54), we obtain

$$\begin{aligned}&\rho\omega_1\delta v_r - 2i\rho\frac{V_\varphi}{r}\delta v_\varphi - i\delta\rho\frac{V_\varphi^2}{r} + i\frac{d\delta p_1}{dr}\\ &+ \frac{2iB_\varphi\delta B_\varphi}{r} + k_\parallel B\delta B_r = 0,\\ &\rho\omega_1\delta v_\varphi + i\rho\delta v_r\left(\frac{V_\varphi}{r} + \frac{dV_\varphi}{dr}\right) - \frac{m}{r}\delta p_1\\ &- \frac{i\delta B_r}{r}\frac{d}{dr}(rB_\varphi) + k_\parallel B\delta B_\varphi = 0,\\ &\rho\omega_1\delta v_z + i\rho\frac{dV_z}{dr}\delta v_z - k_z\delta p_1 - i\delta B_r\frac{dB_z}{dr} + k_\parallel B\delta B_z = 0.\end{aligned} \tag{60}$$

In the derivation of Equations (60) the dependence of the perturbed quantities on the coordinates φ and z is of the form $\exp(im\varphi + ik_z z)$. Here $k_\parallel$ is longitudinal (in the direction of the equilibrium magnetic field) wave vector, i.e.,

$$k_\parallel = \frac{1}{B}\left(\frac{m}{r}B_\varphi + k_z B_z\right). \tag{61}$$

From the linearized Equation (5), we obtain

$$\begin{aligned}-i\omega\delta\boldsymbol{B} = (\delta\boldsymbol{B}\cdot\nabla)\boldsymbol{V} + (\boldsymbol{B}\cdot\nabla)\delta\boldsymbol{v}\\ - (\delta\boldsymbol{v}\cdot\nabla)\boldsymbol{B} - (\boldsymbol{V}\cdot\nabla)\delta\boldsymbol{B} - \boldsymbol{B}\operatorname{div}\boldsymbol{\delta v}\end{aligned} \tag{62}$$

and from Equations (2) to (54), we find the components of the perturbed magnetic field, i.e.,

$$\begin{aligned}\delta B_r &= -k_\parallel B\frac{\delta v_r}{\omega_1},\\ \delta B_\varphi &= -\frac{1}{\omega_1}[k_z(B_z\delta v_\varphi - B_\varphi\delta v_z)\\ &\quad + i\frac{d}{dr}(B_\varphi\delta v_r) + i\frac{k_\parallel B}{\omega_1}\left(\frac{dV_\varphi}{dr} - \frac{V_\varphi}{r}\right)\delta v_r\Bigg],\\ \delta B_z &= \frac{1}{\omega_1}\left[\frac{m}{r}(B_z\delta v_\varphi - B_\varphi\delta v_z)\right.\\ &\quad\left. - \frac{i}{r}\frac{d}{dr}(rB_z\delta v_r) - \frac{ik_\parallel B}{\omega_1}\frac{dV_z}{dr}\delta v_r\right].\end{aligned} \tag{63}$$

Eliminating from (60) the perturbation of the magnetic field (63), we obtain the governing equations of small oscillations, where the perturbed quantities are the velocity vector components and the total plasma pressure only,

$$\begin{aligned}&(\rho\omega_1^2 - k_\parallel^2B^2)\delta v_r - 2i\omega_1\rho\frac{V_\varphi}{r}\delta v_\varphi\\ &- i\delta\rho\frac{\omega_1}{r}V_\varphi^2 + i\omega_1\frac{d\delta p_1}{dr}\\ &- \frac{2B_\varphi}{r}[ik_z(B_z\delta v_\varphi - B_\varphi\delta v_z)\\ &- \frac{d}{dr}(B_\varphi\delta v_r) - \frac{k_\parallel B}{\omega_1}\left(\frac{dV_\varphi}{dr} - \frac{V_\varphi}{r}\right)\delta v_r\Bigg] = 0,\end{aligned} \tag{64}$$

$$\begin{aligned}&(\rho\omega_1^2 - k_\parallel^2B^2)\delta v_\varphi + \frac{i\omega_1\rho}{r}\frac{d}{dr}(rV_\varphi)\delta v_r - \frac{m\omega_1}{r}\delta p_1\\ &+ \frac{2ik_\parallel BB_\varphi}{r}\delta v_r - ik_\parallel BB_\varphi\operatorname{div}\delta\boldsymbol{v}\\ &- \frac{ik_\parallel^2B^2}{\omega_1}\left(\frac{dV_\varphi}{dr} - \frac{V_\varphi}{r}\right)\delta v_r = 0,\end{aligned} \tag{65}$$

$$\begin{aligned}&(\rho\omega_1^2 - k_\parallel^2B^2)\delta v_z + i\omega_1\rho\frac{dV_z}{dr}\delta v_r - \omega_1k_z\delta p_1\\ &- ik_\parallel BB_z\,div\delta\boldsymbol{v} - \frac{ik_\parallel^2B^2}{\omega_1^2}\frac{dV_z}{dr}\delta v_r = 0.\end{aligned} \tag{66}$$

Appendix B
Equations of Incompressible Small Amplitude Perturbations

Let us reduce the system of Equations (64)–(66) to two equations for the small amplitude perturbations δv_r and δp_1. Furthermore, we restrict ourselves to incompressible perturbations only, i.e., $c_S \to \infty$, $(\gamma \to \infty)$, where c_S is the sound speed, for which

$$\operatorname{div}\delta\boldsymbol{v} = 0. \tag{67}$$

Let us also assume that equilibrium plasma density satisfies the condition $\frac{d\rho}{dr} = 0$ inside and outside of the cylinder. Under this assumption, from the first Equation of (56) there is no perturbed plasma density, i.e., $\delta\rho = 0$, and Equations (64)–(66) take the form

$$\begin{aligned}&a_{11}\delta v_r - 2ia_{12}\delta v_\varphi - \frac{2B_\varphi(r)}{r}\delta v_r\\ &\times\left[\frac{B_\varphi}{r} - \frac{dB_\varphi}{dr} - \frac{k_\parallel B}{\omega_1}\left(\frac{dV_\varphi}{dr} - \frac{V_\varphi}{r}\right)\right] + i\omega_1\frac{d}{dr}\delta p_1 = 0,\end{aligned} \tag{68}$$

$$a_{11}\delta v_\varphi + 2ia_{12}\delta v_r + \frac{ia_{11}}{\omega_1}\delta v_r\left(\frac{dV_\varphi}{dr} - \frac{V_\varphi}{r}\right) - \frac{m}{r}\omega_1\delta p_1 = 0, \tag{69}$$

$$a_{11}\delta v_z = k_z\omega_1\delta p_1 - \frac{i}{\omega_1}a_{11}\delta v_r\frac{dV_z}{dr}. \tag{70}$$

Here we have introduced the notation,

$$a_{11} = \rho\omega_1^2 - k_\parallel^2B^2,\; a_{12} = \rho\omega_1\frac{V_\varphi(r)}{r} + k_\parallel B\frac{B_\varphi(r)}{r}.$$

Multiplying Equation (68) by a_{11}, Equation (69) by $2ia_{12}$, and adding the resulting equations together, we obtain the following equation, which shows the coupling between δp_1 and δv_r,

$$a_{11}\frac{\partial}{\partial r}\delta p_1 - 2a_{12}\frac{m}{r}\delta p_1 = \frac{i\delta v_r}{\omega_1} \times\left[a_{11}^2 - 4a_{12}^2 + a_{11}r\frac{d}{dr}\left(\frac{B_\varphi^2 - \rho V_\varphi^2}{r^2}\right)\right]. \quad (71)$$

From Equations (56) to (67), we have that

$$\frac{1}{r}\frac{d}{dr}(r\delta v_r) + \frac{im}{r}\delta v_\varphi + ik_z\delta v_z = 0$$

and from Equations (68) to (70), we obtain the following relation between δp_1 and δv_r,

$$\delta p_1 = \frac{i}{k_z^2 + m^2/r^2}\left[\frac{a_{11}}{r}\frac{d}{dr}\left(r\frac{\delta v_r}{\omega_1}\right) + \frac{2a_{12}}{r}m\frac{\delta v_r}{\omega_1}\right]. \quad (72)$$

Eliminating $\delta v_r/\omega_1$ from Equations (71) and (72), we obtain a differential equation in terms of the total perturbed pressure only,

$$\frac{d^2}{dr^2}\delta p_1 + \frac{d}{dr}\delta p_1\left[\frac{C_3}{Dr}\frac{d}{dr}\left(\frac{Dr}{C_3}\right)\right] + \delta p_1\left[\frac{C_3}{Dr}\frac{d}{dr}\left(\frac{rC_1}{C_3}\right) + \frac{1}{D^2}(C_2C_3 - C_1^2)\right] = 0. \quad (73)$$

Here we have introduced the following notations,

$$D = a_{11},$$
$$C_1 = -2a_{12}\frac{m}{r},$$
$$C_2 = -\left(k_z^2 + \frac{m^2}{r^2}\right),$$
$$C_3 = a_{11}^2 - 4a_{12}^2 + a_{11}r\frac{d}{dr}\left(\frac{B_\varphi^2 - \rho V_\varphi^2}{r^2}\right). \quad (74)$$

Equation (73) can be obtained from Equation (18) in Goossens et al. (1992) in the incompressible limit.

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References

Andrushchenko, Z. N., Cheremnykh, O. K., & Edenstrasser, J. W. 1999, PhPl, 6, 2462
Andrushchenko, Z. N., Revenchuk, S. M., & Cheremnykh, O. K. 1993, PlPhR, 19, 124
Appert, K., Gruber, R., & Vaclavik, J. 1974, PhFl, 17, 1471
Batchelor, G. K. 1970, An Introduction to Fluid Dynamics (Cambridge: The Univ. Press)
Bateman, G. 1978, MHD Instabilities (Cambridge, MA: MIT Press), 270
Bennett, K., Roberts, B., & Narain, U. 1999, SoPh, 185, 41
Bodo, G., Rosner, R., Ferrari, A., & Knobloch, E. 1989, ApJ, 341, 631
Bodo, G., Rosner, R., Ferrari, A., & Knobloch, E. 1996, ApJ, 470, 797
Bondeson, A., Iacono, R., & Bhattacharjee, A. 1987, PhFl, 30, 2167
Bonet, J. A., Márquez, I., Sánchez Almeida, J., et al. 2010, ApJL, 723, L139
Bonet, J. A., Márquez, I., Sánchez Almeida, J., Cabello, I., & Domingo, V. 2008, ApJL, 687, L131
Burdo, O. S., Cheremnykh, O. K., Revenchuk, S. M., & Pustovitov, V. D. 1994, PPCF, 36, 641
Chandrasekhar, S. 1961, Hydrodymamic and Hydromagnetic Stability (Cambridge: Cambridge Univ. Press)
Cheremnykh, O. K. 2008, Nelineinaya Dinamika [Russian Journal of Nonlinear Dynamics], 4, 417
Cheremnykh, O. K., Andrushchenko, Z. M., Edenstrasser, J. W., & Taranov, V. B. 1994, PhPl, 1, 2525
Cheremnykh, O. K., Fedun, V., Kryshtal, A. N., & Verth, G. 2017, A&A, 604, A62
Cheremnykh, O. K., Klimushkin, D. Y., & Kostarev, D. V. 2014, KPCB, 30, 209
Cheremnykh, O. K., Kryshtal, A. N., & Tkachenko, A. A. 2018, AdSpR, 61, 603
Cheremhykh, O. K., & Revenchuk, S. M. 1992, PPCF, 34, 55
Choe, J., Tataronis, J. A., & Grossmann, W. 1977, PlPh, 19, 117
De Pontieu, B., Carlsson, M., Rouppe van der Voort, L. H. M., et al. 2012, ApJL, 752, L12
De Pontieu, B., McIntosh, S. W., Carlsson, M., et al. 2007, Sci, 318, 1574
Dwight, H. B. 1947, Tables of Integrals and Other Mathematical Data (New York: MacMillan)
Edwin, P. M., & Roberts, B. 1983, SoPh, 88, 179
Erdélyi, R., & Fedun, V. 2006, SoPh, 238, 41
Erdélyi, R., & Fedun, V. 2007, SoPh, 246, 101
Erdélyi, R., & Fedun, V. 2010, SoPh, 263, 63
Fedun, V., Shelyag, S., Verth, G., Mathioudakis, M., & Erdélyi, R. 2011b, AnGeo, 29, 1029
Fedun, V., Verth, G., Jess, D. B., & Erdélyi, R. 2011a, ApJL, 740, L46
Filippov, B. P. 2007, Eruptive Process on the Sun (Moscow: Fizmatlit)
Galeev, A. A., & Sudan, R. N. (ed.) 1989, Basic Plasma Physics. Handbook of Plasma Physics, Vol. 1 (North-Holland: Amsterdam), 585
Giagkiozis, I., Fedun, V., Erdélyi, R., & Verth, G. 2015, ApJ, 810, 53
Giagkiozis, I., Fedun, V., Scullion, E., & Verth, G. 2017, arXiv:1706.05428
Goedbloed, J. P., & Hagebeuk, H. J. L. 1972, PhFl, 15, 1090
González-Avilés, J. J., Guzmán, F. S., & Fedun, V. 2017, ApJ, 836, 24
González-Avilés, J. J., Guzmán, F. S., Fedun, V., et al. 2018, ApJ, 856, 176
Goossens, M. 1991, Geophysical and Astrophysical MHD, 137
Goossens, M., Hollweg, J. V., & Sakurai, T. 1992, SoPh, 138, 233
Hain, K., & Lüst, R. 1958, ZNatA, 13, 936
Helmholtz, H. 1868, Über Discontinuirliche Flüssigkeitsbewegungen (On Discontinuous Liquid Motion), 23, Monats. Königl. Preuss. Akad. Wiss. Berlin, 215. Engl. trans.: F. Guthrie: On discontinuous movements of fluids. Phil. Mag., 36, 337
Jess, D. B., Morton, R. J., Verth, G., et al. 2015, SSRv, 190, 103
Jess, D. B., Pascoe, D. J., Christian, D. J., et al. 2012, ApJL, 744, L5
Kadomtsev, B. B. 1966, in Reviews of Plasma Physics, 2 ed. M. A. Leontovich (New York, NY: Consultants Bureau), 132
Kallenrode, M.-B. 2004, in Space Physics: an Introduction to Plasmas and Particles in the Heliosphere and Magnetospheres, 3rd, Enlarged, ed. M.-B. Kallenrode (Springer: Berlin)
Kato, Y., & Wedemeyer, S. 2017, A&A, 601, A135
Kelvin, T. W. 1910, Mathematical and Physical Papers (Cambridge: Cambridge Univ. Press), 4
Klimushkin, D. Y., Nakariakov, V. M., Mager, P. N., & Cheremnykh, O. K. 2017, SoPh, 292, 184
Kosugi, T., Matsuzaki, K., Sakao, T., et al. 2007, SoPh, 243, 3
Ladikov-Roev, Y. P., Cheremnykh, S. O., & Yatsenko, V. A. 2013, Journal of Automations and Informations Sciences, 6, 45
Landau, L. D., & Lifshitz, E. M. 1959, Course of Theoretical Physics, Vol. 6 (Oxford: Pergamon)
Li, X., Morgan, H., Leonard, D., & Jeska, L. 2012, ApJL, 752, L22
Miyamoto, K. 1997, Fundamentals of Plasma Physics and Controlled Fusion (Tokyo: Iwanami Book Service Centre)
Morton, R. J., Verth, G., Jess, D. B., et al. 2012, NatCo, 3, 1315
Murawski, K., Kayshap, P., Srivastava, A. K., et al. 2018, MNRAS, 474, 77
Priest, E. R. 2003, in Dynamic Sun, ed. B.N. Dwivedi (Cambridge: Cambridge Univ. Press), 217
Rayleigh, L. 1916, Phil. Mag. Ser., 32, 529
Roberts, B. 1991, Advances in Solar System Magnetohydrodynamics (Cambridge: Cambridge Univ. Press)
Shafranov, V. D. 1970, JETP, 40, 241
Sharma, R., Verth, G., & Erdélyi, R. 2018, ApJ, 853, 61
Shelyag, S., Cally, P. S., Reid, A., & Mathioudakis, M. 2013, ApJL, 776, L4

Shelyag, S., Fedun, V., Erdélyi, R., Keenan, F. P., & Mathioudakis, M. 2012, in ASP Conf. Proc. 463, Second ATST-EAST Meeting: Magnetic Fields from the Photosphere to the Corona, ed. T. Rimmele et al. (San Francisco, CA: ASP), 107

Shelyag, S., Fedun, V., Keenan, F. P., Erdélyi, R., & Mathioudakis, M. 2011, AnGeo, 29, 883

Shibata, K., Nakamura, T., Matsumoto, T., et al. 2007, Sci, 318, 1591

Snow, B., Fedun, V., Gent, F. A., Verth, G., & Erdélyi, R. 2018, ApJ, 857, 125

Soler, R., Terradas, J., Oliver, R., Ballester, J. L., & Goossens, M. 2010, ApJ, 712, 875

Solov'ev, S. 1967, in Reviews of Plasma Physics, Vol. 3, ed. M. A. Leontovich (New York, NY: Consultants Bureau), 277

Su, Y., Wang, T., Veronig, A., Temmer, M., & Gan, W. 2012, ApJL, 756, L41

Suydam, B. R. 1958, in Proc. 2nd UN International Conf. on the Peaceful Uses of Atomic Energy 31, 157, http://www-naweb.iaea.org/napc/physics/2ndgenconf/data/Proceedings%201958/NG900088.pdf

Syrovatskiy, S. 1953, JETP, 24, 622

Wedemeyer-Böhm, S., & Rouppe van der Voort, L. 2009, A&A, 507, L9

Wedemeyer-Böhm, S., Scullion, E., Steiner, O., et al. 2012, Natur, 486, 505

Whitham, G. B. 1974, Linear and Nonlinear Waves (New York: John Wiley & Sons Inc.)

Zaqarashvili, T. V., Díaz, A. J., Oliver, R., & Ballester, J. L. 2010, A&A, 516, A84

Zaqarashvili, T. V., Zhelyazkov, I., & Ofman, L. 2015, ApJ, 813, 123