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# A Gleason-type theorem for qubits based on mixtures of projective measurements

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## Abstract

We derive Born's rule and the density-operator formalism for quantum systems with Hilbert spaces of dimension two or larger. Our extension of Gleason's theorem only relies upon the consistent assignment of probabilities to the outcomes of projective measurements and their classical mixtures. This assumption is significantly weaker than those required for existing Gleason-type theorems valid in dimension two.

## 1 Introduction

Formulations of quantum theory typically introduce at least three postulates to define *quantum states* and *observables* on the one hand, and to explain how they give rise to *measurable quantities* such as expectation values on the other. One way to set up the necessary machinery (cf. [1], for example) consists of postulating that (i) the states of a quantum system correspond to density operators on a separable, complex Hilbert space  $\mathcal{H}$ ; (ii) measurements of quantum observables are associated with collections of mutually orthogonal projection operators acting on the space  $\mathcal{H}$ ; (iii) the probabilities of measurement outcomes are given by Born's rule.<sup>1</sup> In 1957, Gleason [4] showed that, assuming the second postulate, the other two can be seen

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<sup>1</sup>Normally, these axioms are supplemented by a *measurement postulate* identifying the post-measurement state once a specific outcome has been obtained, by a *dynamical law*, and by a rule how to describe composite quantum systems. Substantially different axiomatic formulations of quantum theory have been proposed in e.g. [2, 3].

as a consequence of a quantum state's most fundamental purpose, that is to assign probabilities to all measurement outcomes in a consistent way.

There is, however, a fly in the ointment: Gleason's result only holds for Hilbert spaces with dimension greater than two. In a two-dimensional space, the requirement of consistency places no restriction on the probabilities that may be assigned to non-orthogonal projections. The resulting surfeit of consistent probability assignments is then too large to be identified with the set of density operators on  $\mathbb{C}^2$ . Hence the question: what modification of the assumptions would be sufficient to recover the probabilistic structure characteristic of quantum theory in a two-dimensional Hilbert space?

Enter *Gleason-type* theorems, which are designed to fill this gap. In 2003, the Born rule for Hilbert spaces of dimension two (or greater) was shown [5, 6] to follow from extending Gleason's idea from projection-valued measures (PVMs) to the more general class of positive operator-valued measures (POMs).

The set of POMs encompasses all quantum measurements, with PVMs being only a small subset thereof. It is, therefore, natural to ask whether there are sets "between" PVMs and POMs from which it is possible to derive a Gleason-type theorem. A first step into this direction was made in 2006 when *three-outcome* POMs were shown to be sufficient for this purpose in the spaces  $\mathbb{C}^d$ , with  $d \geq 2$  [7]. Our contribution will take this reduction even further. We will show that it is possible to derive a Gleason-type theorem upon extending Gleason's probability assignments from PVMs to their convex combinations. The resulting *projective-simulable* measurements [8] represent a particularly simple subset of POMs.

In Sec. 2, we set up our notation and express Gleason's theorem in a form which is suitable for direct comparison with Gleason-type theorems. Sec. 3 describes projective-simulable POMs in order to derive a Gleason-type theorem based on assumptions weaker than those currently known. In the final section, we summarize and discuss our results.

## 2 Known extensions of Gleason's theorem

In this section we review Gleason's theorem and express it in a form which will allow for easy comparison with later variants, including our main result. Let us introduce a number of relevant concepts and establish our notation.

### 2.1 Preliminaries

Let  $\mathcal{H}$  be a finite-dimensional, complex Hilbert space. An *effect* is an operator on  $\mathcal{H}$  occurring in the range of a POM. More explicitly, an effect  $e$  is Hermitian and satisfies

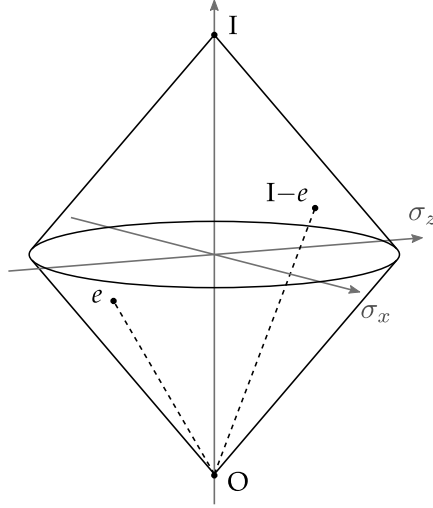


Figure 2.1: Three-dimensional cross section of the four-dimensional qubit effect space, illustrating a generic two-outcome measurement  $\mathbb{D}_e$  in (13), characterized by the effect  $e$ .

$\mathbf{O} \leq e \leq \mathbf{I}$ , where  $\mathbf{O}$  and  $\mathbf{I}$  are the zero and identity operators on  $\mathcal{H}$ , respectively. Here, the ordering of two operators,  $A \leq B$ , say, is defined to hold if the inequality  $\langle \psi|A|\psi \rangle \leq \langle \psi|B|\psi \rangle$  is satisfied for all elements of the Hilbert space,  $|\psi\rangle \in \mathcal{H}$ . We denote the set of all effects  $e$  on the space  $\mathbb{C}^d$  by  $\mathcal{E}(\mathbb{C}^d) \equiv \mathcal{E}_d$ .

It is instructive to visualize the effect space of a qubit. The Pauli operators  $\sigma_x, \sigma_y$  and  $\sigma_z$  with the identity  $\mathbf{I}$  form a basis of Hermitian operators acting on  $\mathbb{C}^2$ . Hence, any qubit effect takes the form

$$e = a\mathbf{I} + b\sigma_x + c\sigma_y + d\sigma_z \in \mathcal{E}_2, \quad (1)$$

where the range of the four parameters  $a, \dots, d \in \mathbb{R}$ , is restricted by the requirement that the operator  $e$  must satisfy  $\mathbf{O} \leq e \leq \mathbf{I}$ .

Fig. 2.1 illustrates three-dimensional cross sections of the four-dimensional effect space obtained upon suppressing the  $y$ -component in (1). The points on the circle in the  $xz$ -plane correspond to rank-1 projection operators and are, in addition to  $\mathbf{O}$  and  $\mathbf{I}$ , the only *extremal* effects in the sense that they cannot be obtained as convex combinations of other effects.

We now turn to the description of measurement outcomes. A fundamental assumption is that the outcome of any measurement performed on a quantum system can be associated with some effect  $e \in \mathcal{E}_d$  [9]. More specifically, we think of a *measurement*  $\mathbb{M}$  as an ordered sequence of effects

$$\mathbb{M} = \llbracket e_1, e_2, \dots, e_n \rrbracket, \quad n \in \mathbb{N}, \quad (2)$$

where

$$\sum_{j=1}^n e_j = \mathbf{I}. \quad (3)$$

We say the effect  $e_j$  is associated with the  $j$ -th outcome of the measurement. It is useful to note that measurements with  $n$  outcomes  $\mathbb{M} = \llbracket e_1, e_2, \dots, e_n \rrbracket$  can be thought of as *vectors* since they are elements of the vector space formed by the Cartesian product of  $n$  copies of the real vector space of Hermitian operators on  $\mathcal{H}$ . Hence, real linear combinations of measurements are well-defined from a mathematical point of view, though only convex combinations necessarily correspond to measurements.

Next, we introduce the concept of a *measurement set*  $\mathbf{M} = \{\mathbb{M}_j, j \in J\}$ , for some (possibly uncountable) indexing set  $J$ , simply consisting of a collection of selected measurements  $\mathbb{M}_j$ . The set  $\mathbf{M}$  is said to define a particular *measurement scenario* if we consider (only) the measurements contained in  $\mathbf{M}$  to be realisable. The measurement set  $\mathbf{PVM}_d$  in a Hilbert space of dimension  $d \geq 2$ , for example, collects all projective measurements; it thus consists of all measurements of the form  $\llbracket P_1, P_2, \dots \rrbracket$ , with at most  $d$  distinct projection operators  $P_j$  on  $\mathbb{C}^d$ , i.e. effects satisfying  $P_j^2 = P_j$ . The set  $\mathbf{PVM}_d$  defines the von Neumann measurement scenario.

The *effect space*  $\mathcal{E}(\mathbf{M}) \subseteq \mathcal{E}_d$  consists of all effects which figure in the measurement set  $\mathbf{M}$ . In a given scenario, not every effect defined on the space  $\mathcal{H}$  necessarily represents a measurement outcome, thus the set  $\mathcal{E}(\mathbf{M})$  may be a proper subset of all effects on the space  $\mathcal{H}$ . For example, in the von Neumann scenario the effect space  $\mathcal{E}(\mathbf{PVM}_d)$  consists solely of projection operators. The largest possible measurement set on a Hilbert space with dimension  $d$  is given by  $\mathbf{POM}_d$ , with the only requirement on a measurement  $\mathbb{M} = \llbracket e_1, e_2, \dots, e_n \rrbracket \in \mathbf{POM}_d$  being that the effects  $e_j$  satisfy Eq. (3), so that indeed  $\mathcal{E}(\mathbf{POM}_d) = \mathcal{E}_d$ . In other words, every effect will figure in some measurement in a scenario which considers all POMs to be realisable. In general, POMs may have infinitely many outcomes. For our considerations, however, those with only finitely many outcomes will be sufficient.

After these preliminaries, we are ready to describe the role of a quantum state in a measurement scenario: it should map each effect  $e \in \mathcal{E}(\mathbf{M})$  in the corresponding effect space to a probability in such a way that the probabilities of all the outcomes in each measurement  $\mathbb{M} \in \mathbf{M}$  sum to one. Such a map is known as a frame function [6].

**Definition 1.** Let  $\mathcal{E}(\mathbf{M})$  be the effect space associated with the measurement set  $\mathbf{M}$ . A *frame function*  $f$  in this measurement scenario is a map  $f : \mathcal{E}(\mathbf{M}) \rightarrow [0, 1]$  such that

$$\sum_{e_j \in \mathbb{M}} f(e_j) = 1, \quad (4)$$

for all measurements  $\mathbb{M}$  in the set  $\mathbf{M}$ .

We will say that the frame function  $f$  respects the measurement set  $\mathbf{M}$  if it consistently assigns probabilities to all effects present in the measurement scenario defined by the set  $\mathbf{M}$ . Structurally, frame functions resemble probability measures which quantify the size of disjoint subsets of a sample space, say, with a relation similar to (4) expressing normalization.

As discussed by Caves et al. [6], this approach is intrinsically non-contextual. When associating outcomes from distinct measurements with the same mathematical object, we are prescribing that they must occur with the same probability for a system in a given state, regardless of context, i.e. which measurements are being performed (see also [10]).

## 2.2 Gleason's theorem

Gleason's theorem conveys a limitation of the form which frame functions may take in a Hilbert space of dimension larger than two. Using the concepts just introduced, the theorem can be expressed as follows.

**Theorem 1** (Gleason [4]). *Any frame function  $f$  respecting the measurement set  $\mathbf{M} = \mathbf{PVM}_d$ ,  $d \geq 3$ , admits an expression*

$$f(e) = \text{Tr}(\rho e), \quad (5)$$

for some density operator  $\rho$  on  $\mathcal{H}$ , and all effects  $e \in \mathcal{E}(\mathbf{PVM}_d)$ .

Originally, Gleason's theorem was stated in terms of measures  $\mu$  acting on closed subspaces of the space  $\mathcal{H}$ . For a countable collection of mutually orthogonal closed subspaces  $\{\mathcal{H}_1, \mathcal{H}_2, \dots, \mathcal{H}_N\}$  which span the entire space, a measure must satisfy

$$\mu(\text{span}\{\mathcal{H}_1, \mathcal{H}_2, \dots, \mathcal{H}_N\}) = \sum_{j=1}^N \mu(\mathcal{H}_j). \quad (6)$$

If the dimension of the space  $\mathcal{H}$  is at least three, then any such measure  $\mu$  with  $\mu(\mathcal{H}) = 1$ , necessarily derives from a density operator  $\rho$  on  $\mathcal{H}$ , via  $\mu(\mathcal{H}_j) = \text{Tr}(\rho P_j)$ ,  $j = 1 \dots N$ , where the operator  $P_j$  is the projection onto the subspace  $\mathcal{H}_j$ . As shown in Appendix A, Theorem 1 is equivalent to the original statement of Gleason's theorem.

If two measurements share an effect we will say—following Gleason—that they *intertwine*. In Hilbert spaces of dimension greater than two the value of a frame function on any two projections is related through measurements which intertwine. This relationship then paves the way for Gleason's theorem. In contrast, projective measurements on  $\mathbb{C}^2$  do not intertwine which means that a frame function may assign probabilities freely to any two non-orthogonal projections. This freedom allows for

frame functions that do not derive from the trace rule, such as Eq. (28) in Sec. 3.3 below. If, however, one considers POMs, measurements also intertwine in dimension two. The consequences of this fact will be seen in the next section.

### 2.3 Gleason-type theorems

Let us now turn to *Gleason-type theorems*, the main topic of this paper. They are variants of Theorem 1 based on measurement sets  $\mathbf{M}$  different from  $\mathbf{PVM}_d$ . The resulting, larger effect spaces  $\mathcal{E}(\mathbf{M})$  allow one to extend Gleason's theorem to the case of a qubit and to derive the result (5) in a simpler way.

The first Gleason-type theorem was obtained by Busch [5], using the measurement set  $\mathbf{M} = \mathbf{POM}_d$ . This is the most general measurement scenario containing all possible POMs, and hence has the largest possible effect space,  $\mathcal{E}(\mathbf{POM}_d)$ .

**Theorem 2** (Busch [5]). *Any frame function  $f$  respecting the measurement set  $\mathbf{M} = \mathbf{POM}_d$ ,  $d \geq 2$ , admits an expression*

$$f(e) = \text{Tr}(e\rho), \quad (7)$$

for some density operator  $\rho$  on  $\mathcal{H}$ , and all effects  $e \in \mathcal{E}(\mathbf{POM}_d) \equiv \mathcal{E}_d$ .

The assumptions of this theorem are indeed stronger than those of Theorem 1 because probabilities are assigned to all *effects*, not just collections of mutually orthogonal projections in the space  $\mathbb{C}^d$ . Busch required that *generalised probability measures*  $v : \mathcal{E}_d \rightarrow [0, 1]$  would need to satisfy the constraints  $v(e_1 + e_2 + \dots) = v(e_1) + v(e_2) + \dots$ , for any sequence of effects which may occur in a POM with any number of outcomes, i.e.  $e_1 + e_2 + \dots \leq I$ . This condition is easily shown to be equivalent to the assumptions in Theorem 2.

The proof of the Theorem 2 differs conceptually from the one given by Gleason. The additivity of frame functions with respect to any two effects  $e_1$  and  $e_2$  occurring in a single measurement  $\mathbb{M}$ ,

$$f(e_1 + e_2) = f(e_1) + f(e_2), \quad (8)$$

forces the frame function to be *homogeneous* for rational numbers,  $f(qe) = qf(e)$ ,  $q \in \mathbb{Q}$ . Combining additivity with positivity,  $f(e) \geq 0$ , a frame function is, furthermore, seen to be homogeneous for *real* numbers,  $f(\alpha e) = \alpha f(e)$ ,  $\alpha \in \mathbb{R}$ , and hence is necessarily linear. Extending this expression linearly from effects to arbitrary Hermitian operators is consistent only with frame functions given by the trace expression (7). The proof also works in separable Hilbert spaces of infinite dimension.

An alternative proof of Busch's Gleason-type theorem was given by Caves et al. [6]. Instead of showing that frame functions must be homogeneous, Caves et al.

establish their *continuity*, first at the effect  $O$ , and then for all effects. This property implies, of course, that frame functions must be linear functions of effects.

Revisiting Gleason's theorem, Granström [7] proceeds along the lines of Busch and Caves et al. when rephrasing the proof. Interestingly, she only uses POMs with at most *three* outcomes: the measurement set is given by  $\mathbf{M} = \mathbf{3POM}_d$  where any

$$\mathbf{M} = \llbracket e_1, e_2, e_3 \rrbracket \in \mathbf{3POM}_d \quad (9)$$

is a collection of at most three effects. Granström's observation is important since her derivation is based on a considerably smaller measurement set than the one required for the earlier Gleason-type theorems.

The following section shows an even smaller measurement set is sufficient to derive a Gleason-type theorem in dimensions  $d \geq 2$ . The reduction is not only quantitative but also represents a conceptual simplification since only POMs arising from classical mixtures of projective measurements, known as projective-simulable measurements, will be required.

### 3 Assigning probabilities to mixtures of projections

#### 3.1 Projective-simulable measurements

*Projective-simulable measurements* (PSMs) are specific POMs which can be realized by performing projective measurements and combining them with classical protocols [8]. The relevant classical procedures are given by probabilistically mixing projective measurements and post-processing of measurement outcomes. Hence, the experimental implementation of projective-simulable—or *simulable*, for brevity—measurements is not more challenging than that of projective measurements. In the following, we will suppress any post-processing since it can always be eliminated by working with suitable mixtures of measurements (see Lemma 1 in [8]). It is important to note that not all POMs are simulable [8]; thus they represent a proper, non-trivial subset of all POMs.

We will now introduce some two- and three-outcome measurements of a qubit which are projective-simulable. These are the only measurements necessary to derive the Gleason-type theorem of Sec. 3.2 when  $d = 2$ . To begin, any (non-trivial) projective qubit measurement with two outcomes takes the form  $\mathbf{M} = \llbracket P_+, P_- \rrbracket$ , with projections  $P_+$  and  $P_- \equiv I - P_+$  on orthogonal one-dimensional subspaces of the space  $\mathbb{C}^2$ . For example, on a spin- $\frac{1}{2}$  particle the measurements implemented by a Stern-Gerlach apparatus oriented along the  $x$ - or the  $z$ - axis would be represented



by

$$\mathbb{M}_x = \frac{1}{2} \llbracket \mathbb{I} + \sigma_x, \mathbb{I} - \sigma_x \rrbracket \quad \text{and} \quad \mathbb{M}_z = \frac{1}{2} \llbracket \mathbb{I} + \sigma_z, \mathbb{I} - \sigma_z \rrbracket, \quad (10)$$

respectively. Now imagine a device which performs  $\mathbb{M}_x$  with probability  $p \in [0, 1]$  and  $\mathbb{M}_z$  with probability  $(1 - p)$ . The statistics produced by this apparatus are, in general, no longer described by a PVM but by a POM, namely by

$$\mathbb{M}_{xz}(p) = p\mathbb{M}_x + (1 - p)\mathbb{M}_z. \quad (11)$$

Consequently, the POM

$$\mathbb{M}_{xz}(p) = \frac{1}{2} \llbracket \mathbb{I} + p\sigma_x + (1 - p)\sigma_z, \mathbb{I} - p\sigma_x - (1 - p)\sigma_z \rrbracket \quad (12)$$

is projective-simulable since only a probabilistic mixture of projective measurements is required to implement it. Mixing the simulable measurement  $\mathbb{M}_{xz}(p)$  with another projective or simulable measurement would result in yet another simulable measurement.

Clearly, this procedure can be lifted to a Hilbert space with dimension  $d$ : mixing any pair of projective or simulable measurements  $\mathbb{M}$  and  $\mathbb{M}'$  with the same number of outcomes, say, produces another simulable measurement represented by  $\mathbb{L}(p) = p\mathbb{M} + (1 - p)\mathbb{M}'$ . In low dimension such as  $d = 2$  and  $d = 3$ , the set of  $n$ -outcome POMs which can be reached in this way has been characterized in terms of semi-definite programs [8].

In our context, the following result for POMs with two outcomes will be important.

**Lemma 1.** *Let  $\mathcal{H}$  be a Hilbert space with finite dimension  $d$ . For any effect  $e \in \mathcal{E}_d$  the two-outcome POM*

$$\mathbb{D}_e = \llbracket e, \mathbb{I} - e \rrbracket \quad (13)$$

*is projective-simulable, i.e.  $\mathbb{D}_e \in \mathcal{E}(\mathbf{PSM}_d)$ .*

*Proof.* Let  $e \in \mathcal{E}_d$  be an effect with eigenvalues  $\lambda_j \in [0, 1]$ ,  $j = 1 \dots d$ , labeled in ascending order, i.e.  $\lambda_j \leq \lambda_{j+1}$ . Being a Hermitian operator, the spectral theorem implies that the effect  $e$  can be written as a linear combination

$$e = \sum_{j=1}^d \lambda_j P_j, \quad (14)$$

where  $P_j \in \mathcal{E}_d$  are rank-1 projections onto mutually orthogonal subspaces of  $\mathcal{H}$ . Defining the projectors  $Q_k = \sum_{j=k}^d P_j$  and letting  $p_k = (\lambda_k - \lambda_{k-1}) \geq 0$  for  $k = 1 \dots d$ ,

where  $\lambda_0 \equiv 0$ , we may rewrite Eq. (14) as

$$e = \sum_{k=1}^d (\lambda_k - \lambda_{k-1}) Q_k = \sum_{k=1}^d p_k Q_k. \quad (15)$$

This expression for the effect  $e$  can be found in [6].

Next, consider the  $(d + 1)$  projective measurements  $\mathbb{P}_j = \llbracket Q_j, I - Q_j \rrbracket$ ,  $j = 0 \dots d$ , where  $Q_0 = O$ . The choice  $p_0 = (1 - \lambda_d)$  ensures that the  $(d + 1)$  non-negative numbers  $p_j$  correspond to probabilities, and satisfy  $\sum_{j=0}^d p_j = 1$ . A mixture of measurements, in which  $\mathbb{P}_j$  is performed with probability  $p_j$ , then simulates the desired POM in (13) since we have

$$\sum_{j=0}^d p_j \mathbb{P}_j = \left\llbracket \sum_{j=0}^d p_j Q_j, \sum_{j=0}^d p_j (I - Q_j) \right\rrbracket = \left\llbracket \sum_{j=1}^d p_j Q_j, I - \sum_{j=1}^d p_j Q_j \right\rrbracket = \llbracket e, I - e \rrbracket, \quad (16)$$

which completes the proof.  $\square$

Any simulable two-outcome measurement such as  $\mathbb{D}_e$  can be used to define simulable three-outcome measurements via  $\llbracket O, e, I - e \rrbracket$  or  $\llbracket e, O, I - e \rrbracket$ , for example, simply by including the effect  $O$  associated with an outcome which will never occur. This observation allows us to easily introduce further simulable three-outcome measurements as probabilistic mixtures.

**Lemma 2.** *Let  $\mathcal{H}$  be a Hilbert space with finite dimension  $d$ . For any effects  $e$  and  $e'$  with  $e + e' \in \mathcal{E}_d$ , the three-outcome POMs*

$$\mathbb{T}_e = \left\llbracket \frac{e}{2}, \frac{e}{2}, I - e \right\rrbracket \quad \text{and} \quad \mathbb{T}_{e,e'} = \left\llbracket \frac{e}{2}, \frac{e'}{2}, I - \frac{(e + e')}{2} \right\rrbracket \quad (17)$$

*are projective-simulable, i.e.  $\mathbb{T}_e$  and  $\mathbb{T}_{e,e'} \in \mathcal{E}(\mathbf{PSM}_d)$ .*

*Proof.* The measurement  $\mathbb{T}_e$  can be obtained from an equal mixture of two three-outcome measurements,

$$\mathbb{T}_e = \frac{1}{2} \llbracket e, O, I - e \rrbracket + \frac{1}{2} \llbracket O, e, I - e \rrbracket, \quad (18)$$

each of which is a padded copy of the simulable two-outcome measurement  $\mathbb{P}_e$ . A slight modification of this argument shows that the measurement  $\mathbb{T}_{e,e'}$  corresponds to an equal probabilistic mixture of two simple simulable three-outcome measurements, viz.,

$$\mathbb{T}_{e,e'} = \frac{1}{2} \llbracket e, O, I - e \rrbracket + \frac{1}{2} \llbracket O, e', I - e' \rrbracket. \quad (19)$$

$\square$

Finally, we would like to point out that in dimension  $d = 2$ , the measurement set  $\mathbf{3PSM}_2$ , which consists of all three-outcome simulable POMs, is an eight-parameter family strictly smaller than  $\mathbf{3POM}_2$ , the set of all all POMs with three outcomes. For example, the three-outcome POM

$$\mathbb{E} = \frac{1}{3} \left[ \left[ \mathbb{I} + \sigma_x, \mathbb{I} - \frac{1}{2}\sigma_x + \frac{\sqrt{3}}{2}\sigma_z, \mathbb{I} - \frac{1}{2}\sigma_x - \frac{\sqrt{3}}{2}\sigma_z \right] \right] \quad (20)$$

is *not* projective-simulable, which can be verified via the semi-definite program provided in [8].

### 3.2 A Gleason-type theorem based on PSMs

We will now state and prove our main result, a Gleason-type theorem derived from projective-simulable measurements.

**Theorem 3** (Projective-simulable measurements). *Any frame function  $f$  respecting the measurement set  $\mathbf{M} = \mathbf{PSM}_d$ ,  $d \geq 2$ , admits an expression*

$$f(e) = \text{Tr}(e\rho), \quad (21)$$

for some density operator  $\rho$  on  $\mathcal{H}$ , and all effects  $e \in \mathcal{E}(\mathbf{PSM}_d) \equiv \mathcal{E}_d$ .

To prove this theorem, we will show that consistently assigning probabilities to projective-simulable measurements entails a probability assignment consistent with all POMs. In other words, a frame function  $f$  respecting the measurement set  $\mathbf{PSM}_d$  necessarily respects the measurement set  $\mathbf{POM}_d$ , at which point we can invoke Theorem 2.

*Proof.* In a first step, we show that the probability assignments to the effects  $e$  and  $e/2$ , for any  $e \in \mathcal{E}_d$ , are not independent. According to Lemmas 1 and 2, the measurements  $\mathbb{D}_e = \llbracket e, \mathbb{I} - e \rrbracket$  and  $\mathbb{T}_e = \llbracket e/2, e/2, \mathbb{I} - e \rrbracket$  are projective-simulable. By the definition of a frame function  $f$  given in (4), the probabilities assigned to the outcomes of these two measurements sum to one,

$$f(e) + f(\mathbb{I} - e) = 1 = f\left(\frac{e}{2}\right) + f\left(\frac{e}{2}\right) + f(\mathbb{I} - e). \quad (22)$$

Hence, for any effect  $e \in \mathcal{E}_d$ , we must have

$$f\left(\frac{e}{2}\right) = \frac{1}{2}f(e). \quad (23)$$

Next, we show that the frame function must be additive for any two effects  $e, e' \in \mathcal{E}_d$  such that  $e + e' \in \mathcal{E}_d$ . Using Lemma 1 again, with  $e = (e + e')/2$ , we find that the two-outcome measurement

$$\mathbb{D}_{\frac{1}{2}(e+e')} = \left[ \left[ \frac{1}{2}(e + e'), I - \frac{1}{2}(e + e') \right] \right] \quad (24)$$

is simulable with projective measurements. Assigning probabilities to the outcomes of the measurements  $\mathbb{D}_{\frac{1}{2}(e+e')}$  and  $\mathbb{T}_{e,e'}$  defined in Eq. (17) is only consistent if the constraint

$$f\left(\frac{1}{2}(e + e')\right) = f\left(\frac{1}{2}e\right) + f\left(\frac{1}{2}e'\right), \quad (25)$$

is satisfied. Due to the (limited) homogeneity of the frame function stated in Eq. (23), we conclude that it must be additive,

$$f(e + e') = f(e) + f(e'), \quad (26)$$

on all effects  $e$  and  $e'$  such that  $e + e' \in \mathcal{E}_d$ .

Now consider any  $n$ -outcome measurement  $\mathbb{M} = \llbracket e_1, e_2, \dots, e_n \rrbracket$  on  $\mathbb{C}^d$ , for  $n \in \mathbb{N}$ . Using (26) repeatedly and recalling the normalization (3) of effects, we find by induction that

$$\sum_{j=1}^n f(e_j) = f(e_1 + e_2) + \sum_{j=3}^n f(e_j) = \dots = f\left(\sum_{j=1}^n e_j\right) = f(I) = 1. \quad (27)$$

Hence, any frame function  $f$  respecting  $\mathbf{PSM}_d$  is seen to respect the measurement set  $\mathbf{POM}_d$ , consisting of all POMs. Therefore, by Theorem 2, the frame function must take the form  $f(e) = \text{Tr}(e\rho)$ , for some density operator  $\rho$  on  $\mathcal{H}$ , and all effects  $e \in \mathcal{E}_d$ , which is the content of Theorem 3.  $\square$

The theorem just proved provides a weakening of the assumptions made by Busch and Caves et al. in Theorem 2: since the set of measurements considered is smaller, fewer restrictions are put on potential frame functions—but exactly the same functions are recovered.

### 3.3 Minimal assumptions for a Gleason-type theorem

We now address the problem of identifying the smallest measurement set in dimension two from which a Gleason-type theorem maybe be derived. Recall that Gleason's theorem does not hold in dimension two; frame functions which respect  $\mathbf{PVM}_2$

but do not stem from a density operator (cf. Eq. (5)) are easy to construct. For instance, assign probabilities to all rank-1 projectors—corresponding to the points of the Bloch sphere—according to the rule

$$g(P) = \begin{cases} 0 & \text{if } P = |0\rangle\langle 0|, \\ 1 & \text{if } P = |1\rangle\langle 1|, \\ \frac{1}{2} & \text{otherwise,} \end{cases} \quad P \in \mathcal{E}(\mathbf{PVM}_2) \quad (28)$$

in addition to  $g(O) = 0$  and  $g(I) = 1$ . Then, for each projective measurement  $\mathbb{P} = \llbracket P, I - P \rrbracket$  we find that the constraint (4) on frame functions is satisfied,

$$g(P) + g(I - P) = 1. \quad (29)$$

Other probability assignments not admitting the desired trace form can be found in [11], for example. These constructions succeed since, for  $d = 2$ , each projector  $P$  occurs *only in one condition* of the form (29) i.e. there are no intertwined measurements.

Similarly frame functions defined on the measurements set  $\mathbf{2POM}_2$ , the four-parameter family (1) of POMs for  $\mathbb{C}^2$  with at most two outcomes, do not yield a Gleason-type theorem. Extending the domain of the function  $g$  in (28) to all effects in  $\mathcal{E}_2$  results in a frame function that respects  $\mathbf{2POM}_2$  but is not of the desired form. Thus, measurements with three or more outcomes are a necessity in a set from which a Gleason-type theorem may be proved. Theorem 3 considers one such case, namely the set of projective-simulable measurements  $\mathbf{PSM}_2$  having  $\mathbf{2POM}_2$  as a proper subset.

Could the measurement set  $\mathbf{PSM}_2$  be the smallest sufficient set? Looking back at the proof of Theorem 3 given in the previous subsection, it becomes clear that only elements of  $\mathbf{PSM}_d$  with at most three outcomes, or those contained in the set  $\mathbf{3PSM}_d$ , are necessary for the result to hold. Furthermore, not all elements of the measurement set  $\mathbf{3PSM}_2$  have been used. While *all* two-outcome POMs  $\mathbb{P}_e \in \mathbf{2POM}_2$  feature, the only simulable three-outcome POMs required are of the form  $\mathbb{T}_e$  or  $\mathbb{T}_{e,e'}$ , defined in (17). However, not all three-outcome simulable POMs fall into one of these categories. For example, the three-outcome measurement

$$\mathbb{T}' = \frac{1}{4} \llbracket I + \sigma_z, I + \sigma_x, 2I - (\sigma_z + \sigma_x) \rrbracket, \quad (30)$$

is simulable but does *not* have the form of either  $\mathbb{T}_e$  or  $\mathbb{T}_{e,e'}$ . Thus, we have actually shown a result slightly stronger than Theorem 3 since, for  $d = 2$ , we can replace the measurement set  $\mathbf{M}$  on which frame functions need to be defined by

$$\mathbf{3PSM}'_2 = \mathbf{2POM}_2 \cup \{ \mathbb{T}_e, \mathbb{T}_{e,e'} \mid e, e' \in \mathcal{E}_d \text{ such that } e + e' \in \mathcal{E}_d \}, \quad (31)$$

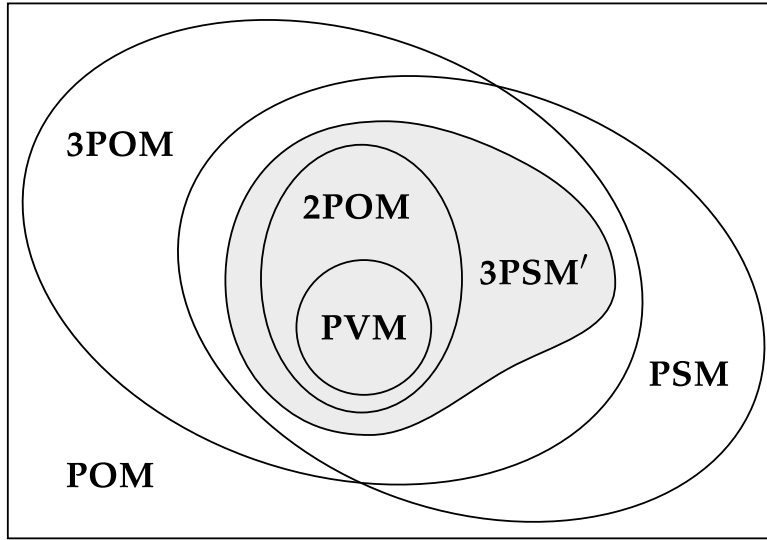


Figure 3.1: Supersets and subsets of the set  $3\text{PSM}'_2$  (grey) given in (31), the smallest measurement set known to entail a Gleason-type theorem for a qubit: it strictly contains the set  $2\text{POM}_2$  of all two-outcome POMs (cf. Eq. (32)) and is strictly contained by the set  $3\text{PSM}_2$  of all simulable three-outcome POMs; for clarity, the index 2 has been dropped from all measurements sets.

which is a proper subset of the measurement set  $3\text{PSM}_2 \equiv 3\text{POM}_2 \cap \text{PSM}_2$ , i.e. all simulable measurements with three outcomes.

We conclude the discussion of “minimal” measurement sets by summarizing the relationship between the sets sufficient to derive a Gleason-type theorem for a qubit,

$$3\text{PSM}'_2 \subset (3\text{POM}_2 \cap \text{PSM}_2) \subset 3\text{POM}_2 \subset \text{POM}_2. \quad (32)$$

Fig. 3.1 also depicts the insufficient subsets of two-outcome projections  $2\text{PVM}_2$  and two-outcome POMs denoted by  $2\text{POM}_2$ .

It is not excluded that measurement sets contained within (or partly overlapping with)  $3\text{PSM}'_2$  exist which would still entail a Gleason-type theorem for qubits. In [6] frame functions respecting the single measurement (20) have been shown to admit an expression as in Eq. (5), but the result depends the assumption that the frame functions be continuous on the set of all effects in  $\mathcal{E}_2$ . Hence this result does not constitute a Gleason-type theorem under our specification.

### 3.4 Mixtures and Boolean lattices

We now consider how Theorem 3 can be interpreted in view of Hall’s discussion [11] of Busch’s Gleason-type theorem, i.e. Theorem 2. Hall reviews the reasons which led

Gleason (following the work of von Neumann and Birkhoff [12] and Mackey [13]) to consider frame functions that respect the measurement set  $\mathbf{PVM}_d$  consisting of *projective* measurements. Namely, a collection of mutually orthogonal projections forms a Boolean lattice, thus making these projections natural candidates to represent disjoint outcomes of an experiment. General collections of effects which sum to the identity, on the other hand, do not have this property (see [14], for example); therefore, a similar justification for considering the measurement set  $\mathbf{POM}_d$  cannot be given.

This reasoning also applies to the setting of Theorem 3 since the measurement set  $\mathbf{PSM}_d$  (or the subset  $\mathbf{3PSM}'_2$ ) contains operators other than projections. Nevertheless, the fact that these measurement sets are made from simulable measurements lends some support to motivating the additivity of frame functions.

Gleason's original argument does not work for a qubit because the constraints (4) on frame functions which result from the measurement set  $\mathbf{PVM}_2$ , are too weak. If one wishes to derive Born's rule in the space  $\mathbb{C}^2$ , it is necessary to consider measurement sets larger than  $\mathbf{PVM}_2$ , thereby invalidating the link between measurements and Boolean lattices. A particularly simple modification of the measurement set consists of including convex combinations of the original projective measurements in  $\mathbf{PVM}_2$ . If one interprets these convex combinations as classical mixtures of projective measurements then one does not make statements about other genuinely quantum mechanical measurements which would lie beyond those of  $\mathbf{PVM}_2$ .

Let us now make explicit all assumptions which are needed so that our main result, Theorem 3, may be used to recover the standard description of states and outcome probabilities of quantum theory. Importantly, similar—if not stronger—assumptions must be made in order to achieve the same goal using the Gleason-type theorems by Busch and Caves et al.

The first assumption is that there exist projective measurements, i.e. measurements whose outcomes may be represented by mutually orthogonal projections on a Hilbert space. Secondly, we assume that it is possible to perform classical mixtures of measurements, that is to say, given a pair of measurements  $\mathbb{M}$  and  $\mathbb{M}'$  then there exists a procedure in which  $\mathbb{M}$  is performed with probability  $p$  and  $\mathbb{M}'$  with probability  $(1 - p)$  for any  $p \in [0, 1]$ . These assumptions alone are *not* sufficient to restrict states to being represented by density operators.

To uncover the additional assumption which is needed to implement our Gleason-type theorem let us consider the procedure just described in the case of a qubit. For example, we may consider an equal mixture  $\mathbb{M}_{xz}(1/2)$  of the measurements

	Probability of outcome 1	Probability of outcome 2
$\mathbb{M}_x, \mathbb{M}_r, \mathbb{M}_s$	1/2	1/2
$\mathbb{M}_z$	0	1

Table 1: The probabilities of the outcomes of measurements  $\mathbb{M}_x$  and  $\mathbb{M}_z$  in Eq. (10) as well as  $\mathbb{M}_r$  and  $\mathbb{M}_s$  in Eq. (33) arising from the probability assignment in Eq. (28).

$\mathbb{M}_x = \llbracket x_+, x_- \rrbracket$  and  $\mathbb{M}_z = \llbracket z_+, z_- \rrbracket$  from Eq. (10) and a mixture  $\mathbb{M}_{rs}(p_+)$  of

$$\begin{aligned} \mathbb{M}_r = \llbracket r_+, r_- \rrbracket & \quad r_{\pm} = \frac{1}{2} \left( \mathbb{I} \pm \frac{1}{2} \left( \sigma_x + \sqrt{3}\sigma_z \right) \right), \\ \mathbb{M}_s = \llbracket s_+, s_- \rrbracket & \quad s_{\pm} = \frac{1}{2} \left( \mathbb{I} \pm \frac{1}{2} \left( \sigma_x - \sqrt{3}\sigma_z \right) \right), \end{aligned} \quad (33)$$

with probabilities  $p_{\pm} = \left(1 \pm 1/\sqrt{3}\right)/2$ , respectively. Now let us work out the probabilities of the first outcomes of the measurements  $\mathbb{M}_{xz}(1/2)$  and  $\mathbb{M}_{rs}(p_+)$  resulting from the probability assignments given in Eq. (28). Using the values given in Table 1, we find that for the mixture  $\mathbb{M}_{xz}(1/2)$ , in which measurements  $\mathbb{M}_x$  and  $\mathbb{M}_z$  are performed with equal probability, outcome one is obtained with probability

$$\frac{1}{2}g(x_+) + \frac{1}{2}g(z_+) = \frac{1}{4}, \quad (34)$$

while the first outcome of  $\mathbb{M}_{rs}(p_+)$  occurs with probability

$$\frac{1}{2} \left( 1 + \frac{1}{\sqrt{3}} \right) g(r_+) + \frac{1}{2} \left( 1 - \frac{1}{\sqrt{3}} \right) g(s_+) = \frac{1}{2}. \quad (35)$$

Not surprisingly, different mixtures of different projective measurements, which correspond to unassociated processes with well-defined outcome probabilities, may result in different outcome probabilities.

According to quantum theory, however, the two mixtures just considered necessarily give rise to the same outcome probabilities for any qubit state and thus may both be represented by the same pair of effects, namely

$$\frac{1}{2} (\mathbb{M}_x + \mathbb{M}_z) = p_+ \mathbb{M}_r + p_- \mathbb{M}_s = \llbracket m, \mathbb{I} - m \rrbracket \equiv \mathbb{D}_m \quad (36)$$

with the effect

$$m = \frac{1}{2} \left( \mathbb{I} + \frac{1}{2} (\sigma_x + \sigma_z) \right), \quad (37)$$

as illustrated in Fig. 3.2.

Thus we see that to exclude  $g(P)$  of (28) as a valid frame function, it is sufficient to assume that a mixture of projective measurements  $\{\mathbb{M}_1, \mathbb{M}_2, \dots\}$ , with probabilities



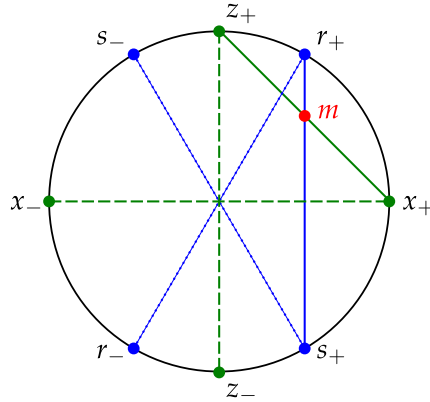


Figure 3.2: Example of an effect which represents an outcome stemming from two different mixing procedures: the effect  $m$  occurs as the first outcome of both  $(\mathbb{M}_x + \mathbb{M}_z) / 2$  and  $p_+ \mathbb{M}_r + p_- \mathbb{M}_s$ ; straight dashed (green) and dotted (blue) lines connect the pairs of effects in the same measurement, and the straight solid lines represent the effects which can be formed by mixing the effects they connect.

$\{p_1, p_2, \dots\}$ , is associated with the convex combination  $(p_1 \mathbb{M}_1 + p_2 \mathbb{M}_2 + \dots)$ . The ensuing assignment of effects from  $\mathcal{E}(\mathbf{POM}_d)$  to represent outcomes of mixtures is our third assumption and results in a theory with effect space  $\mathcal{E}_d$  and measurement set  $\mathbf{PSM}_d$ . When combining this requirement with Theorem 3, frame-function arguments become sufficiently strong to imply Born's rule and the standard density-operator formalism of quantum theory in the space  $\mathbb{C}^d$ .

## 4 Summary and discussion

This paper improves on Gleason-type theorems which aim to extend Gleason's result to Hilbert spaces of dimension  $d = 2$ . The goal is to recover Born's rule and the representation of quantum states as density operators as a product of consistent probability assignments to measurement outcomes. Our main result, given by Theorem 3, shows that any consistent assignment of probabilities to the outcomes of projective-simulable measurements, or the measurement set  $\mathbf{PSM}_d$ , must be associated with a density operator in the desired way. Moreover, we show that a smaller set of measurements  $\mathbf{3PSM}'_d$ , defined in Eq. (31), also has this property.

Our result improves upon existing Gleason-type theorems which are based either on probability assignments to POMs with any number of outcomes (which constitute the set  $\mathbf{POM}_d$ , see [5, 6]) or those with at most three outcomes (which constitute the set  $\mathbf{3POM}_d$ , see [7]). The measurement set we consider,  $\mathbf{3PSM}'_d$ , is a strict subset of  $\mathbf{3POM}_d$ . Fig. 3.1 summarizes the relationship between the sets of measurements.

In addition to these quantitative improvements, Theorem 3 also provides new qualitative insights. Projective-simulable measurements are conceptually simpler than arbitrary POMs because they are just classical mixtures of projective measurements, with an equal level of experimental feasibility. Due to the limitation to simulable measurements, our Gleason-type theorem resembles Gleason’s original theorem more strongly than its predecessors. Furthermore, in Sec. 3.4 we add an explicit assumption to the setting of Gleason’s original theorem in order to extend the result to dimension two. This assumption consists of identifying those measurements which, whilst arising from different mixtures, are known to be indistinguishable in ordinary quantum theory.

Future work will show whether the subset  $\mathbf{3PSM}'_2$  of projective-simulable measurements, on which the proof of Theorem 3 relies, is the smallest possible set from which a Gleason-type theorem may be derived in dimension  $d = 2$ . We cannot exclude that the frame functions respecting  $\mathbf{3PSM}'_2$  are still *overdetermined* in the sense that other sets not containing all of  $\mathbf{3PSM}'_2$  may also entail a Gleason-type theorem for a qubit.

*Acknowledgement.*

Paul Busch (1955-2018) agreed to look at a draft of this paper but it was not meant to be. We dedicate this paper to the memory of our kind colleague and wise friend.

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## A An equivalent form of Gleason's theorem

Theorem 1 is equivalent to Gleason's original theorem. To see this, consider a collection  $\{\mathcal{H}_1, \mathcal{H}_2, \dots\}$  of mutually orthogonal, closed subspaces of  $\mathcal{H}$ . Then there exists a closed subspace  $\mathcal{H}^\perp$  orthogonal to each  $\mathcal{H}_j$  such that  $\text{span}\{\mathcal{H}_1, \mathcal{H}_2, \dots; \mathcal{H}^\perp\} = \mathcal{H}$ . Using projectors  $P_j$  and  $P^\perp$  onto these subspaces, we have, by the definition of a frame function, that

$$\begin{aligned} \sum_j f(P_j) + f(P^\perp) &= 1 = f(P_{\text{span}\{\mathcal{H}_1, \mathcal{H}_2, \dots\}}) + f(P^\perp) \\ &= f\left(\sum_j P_j\right) + f(P^\perp), \end{aligned} \tag{38}$$

which gives

$$\sum_j f(P_j) = f\left(\sum_j P_j\right). \tag{39}$$

This relation implies that any frame function  $f$  respecting  $\mathbf{PVM}_d$  defines a measure  $\mu$  on the closed subspaces  $\mathcal{C}$  of  $\mathcal{H}$  given by  $\mu(\mathcal{C}) = f(P_{\mathcal{C}})$  since

$$\mu(\text{span}\{\mathcal{H}_1, \mathcal{H}_2, \dots\}) = f\left(P_{\text{span}\{\mathcal{H}_1, \mathcal{H}_2, \dots\}}\right) = \sum_j \mu(\mathcal{H}_j). \quad (40)$$

Conversely, if  $\mu$  satisfies Equation (6) and  $\mu(\mathcal{H}) = 1$ , then we have

$$\sum_j \mu(\mathcal{H}_j) + \mu(\mathcal{H}^\perp) = \mu(\text{span}\{\mathcal{H}_1, \mathcal{H}_2, \dots; \mathcal{H}^\perp\}) = \mu(\mathcal{H}) = 1, \quad (41)$$

and hence any such measure  $\mu$  defines a frame function  $f$  respecting  $\mathbf{PVM}_d$ , given by  $f(P_{\mathcal{C}}) = \mu(\mathcal{C})$ .