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Inner functions and operator theory

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Abstract

This tutorial paper presents a survey of results, both classical and new, linking inner functions and operator theory. Topics discussed include invariant subspaces, universal operators, Hankel and Toeplitz operators, model spaces, truncated Toeplitz operators, restricted shifts, numerical ranges, and interpolation.

Keywords: inner functions, invariant subspaces, universal operators, Hankel and Toeplitz operators, model spaces, interpolation.

MSC: 47A15, 47B35, 30H10, 30E05.

1 Introduction

Inner functions originally arose in the context of operator theory, via Beurling's theorem on the invariant subspaces of the unilateral shift operator. Since then, they have been seen in numerous contexts in the theory of function spaces. This tutorial paper surveys some of the many ways in which operators and inner functions are linked: these include the invariant subspace problem, the theory of Hankel and Toeplitz operators and the rapidly-developing area of model spaces and the operators acting on them.

The paper is an expanded version of a mini-course given at the Eleventh Advanced Course in Operator Theory and Complex Analysis, held in Seville in June 2014.

1.1 Hardy spaces and shift-invariant subspaces

All our spaces will be complex. We write $\mathbb{D}$ for the open unit disc in $\mathbb{C}$ and $\mathbb{T} = \partial\mathbb{D}$, the unit circle.

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Recall that Hardy space H^2 or $H^2(\mathbb{D})$ is the space of analytic functions on $\mathbb{D}$ with square-summable Taylor coefficients; that is,

$$H^2(\mathbb{D}) = \left\{ f : \mathbb{D} \to \mathbb{C} \text{ analytic}, f(z) = \sum_{n=0}^{\infty} a_n z^n, \|f\|^2 = \sum_{n=0}^{\infty} |a_n|^2 < \infty \right\}.$$

Also $H^2(\mathbb{D})$ embeds isometrically as a closed subspace of $L^2(\mathbb{T})$ via

$$\sum_{n=0}^{\infty} a_n z^n \mapsto \sum_{n=0}^{\infty} a_n \mathrm{e}^{\mathrm{i}nt},$$

where the series converges almost everywhere on $\mathbb{T}$ as well as in the norm of $L^2(\mathbb{T})$. Indeed, $\lim_{r\to 1-} f(r\mathrm{e}^{\mathrm{i}t})$ exists almost everywhere and gives the boundary values of a function f in $H^2(\mathbb{D})$. (See, for example Hoffman (1962).)

It is useful to use the isometric isomorphism $\ell^2(\mathbb{Z}) \to L^2(\mathbb{T})$ given by

$$(a_n)_{n\in\mathbb{Z}} \mapsto \sum_{n=-\infty}^{\infty} a_n \mathrm{e}^{\mathrm{i}nt},$$

which is a consequence of the Riesz–Fischer theorem; this restricts to an isomorphism $\ell_2(\mathbb{Z}_+) \to H^2(\mathbb{D})$.

The first connection between inner functions and operator theory arises on considering the right shift $R : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$. We may ask what its closed invariant subspaces are; that is, the subspaces $\mathcal{M} \subset L^2(\mathbb{T})$ such that $R\mathcal{M} \subset \mathcal{M}$. The answer is to look at the unitarily equivalent operator S of "multiplication by z" on $L^2(\mathbb{T})$.

$$\begin{array}{ccc} \ell^2(\mathbb{Z}) & \xrightarrow{R} & \ell^2(\mathbb{Z}) \\ \downarrow & & \downarrow \\ L^2(\mathbb{T}) & \xrightarrow{S} & L^2(\mathbb{T}) \end{array}$$

There are two cases, for $\mathcal{M}$ a nontrivial closed subspace of $L^2(\mathbb{T})$:

1. $S\mathcal{M} = \mathcal{M}$, if and only if there is a measurable subset $E \subset \mathbb{T}$ such that $\mathcal{M} = \{f \in L^2(\mathbb{T}) : f_{|\mathbb{T}\setminus E} = 0 \text{ a.e.}\}$ (Wiener[4]).
2. $S\mathcal{M} \subsetneq \mathcal{M}$, if and only if there is a unimodular function $\phi \in L^\infty(\mathbb{T})$ such that $\mathcal{M} = \phi H^2$ (Beurling–Helson[5]).

As a sketch proof of item 2, which will be the more important for us, take $\phi \in \mathcal{M} \ominus S\mathcal{M}$ with $\|\phi\|_2 = 1$. One can verify that ϕ is unimodular and that $\mathcal{M} = \phi H^2$.

[4] Wiener, 1988, *The Fourier integral and certain of its applications*, Ch. II.
[5] Helson, 1964, *Lectures on invariant subspaces*.

Corollary 1 (Beurling's theorem[6,7]) – *Let $\mathcal{M}$ be a nontrivial closed subspace of H^2; then $S\mathcal{M} \subset \mathcal{M}$ if and only if $\mathcal{M} = \theta H^2$ where θ is* inner, *that is $\theta \in H^2(\mathbb{D})$ with $|\theta(\mathrm{e}^{\mathrm{i}t})| = 1$ a.e.*

It is easily seen that θ is unique up to multiplication by a constant of modulus 1.

Now, any function $h \in H^2$, apart from the zero function, has a multiplicative factorization $h = \theta u$, where θ is inner, and u is *outer*: Beurling showed that outer functions satisfy

$$\overline{\mathrm{span}}\left\{u, Su, S^2u, S^3u, \dots\right\} = H^2,$$

and they therefore have an operatorial interpretation, as cyclic vectors for the shift S. The inner-outer factorization is unique up to multiplication by a constant of modulus one.

1.2 Examples of inner functions

If $\mathcal{M}$ is a shift-invariant subspace of finite codimension, then θ is a finite Blaschke product,

$$\theta(z) = \lambda \prod_{j=1}^{n} \frac{z - \alpha_j}{1 - \overline{\alpha_j} z},$$

with $|\lambda| = 1$ and $\alpha_1, \dots, \alpha_n \in \mathbb{D}$. Then

$$\mathcal{M} = \left\{f \in H^2 : f(\alpha_1) = \dots = f(\alpha_n) = 0\right\},$$

with the obvious interpretation in the case of non-distinct α_j. We may also form infinite Blaschke products

$$\theta(z) = \lambda z^p \prod_{j=1}^{\infty} \frac{|\alpha_j|}{\alpha_j} \frac{\alpha_j - z}{1 - \overline{\alpha_j} z},$$

where $|\lambda| = 1$, all the α_j lie in $\mathbb{D} \backslash \{0\}$, p is a non-negative integer and $\sum_{j=1}^{\infty}(1 - |\alpha_j|) < \infty$. Recall that the sequences of $\mathbb{D}$ satisfying the last condition are called Blaschke sequences.

There is also a class of inner functions without zeroes, namely the singular inner functions, which may be written as

$$\theta(z) = \exp\left[-\int_{-\pi}^{\pi} \frac{\mathrm{e}^{\mathrm{i}t} + z}{\mathrm{e}^{\mathrm{i}t} - z} \, d\mu(t)\right],$$

[6] Beurling, 1949, "On two problems concerning linear transformations in Hilbert space".

[7] See also Garnett, 2007, *Bounded analytic functions*, theorem II.7.1; Nikolski, 2002, *Operators, functions, and systems: an easy reading. Vol. 1. Hardy, Hankel, and Toeplitz*, section A.1.3.

where μ is a singular positive measure on $[-\pi,\pi)$. For example if μ is a Dirac mass at 0, then $\theta(z) = \exp((z+1)/(z-1))$.

A complete description of inner functions is now available, as they are given as Bs, where B is a Blaschke product and s is a singular inner function. Either factor may be absent.

Note that if θ_1 and θ_2 are inner, then $\theta_1\overline{\theta_2}$ is unimodular on $\mathbb{T}$. These are not all the unimodular functions, but if $\phi \in L^\infty(\mathbb{T})$ is unimodular then for each $\varepsilon > 0$ it can be factorized as $\phi = h_1\overline{h_2}$, with $h_1, h_2 \in H^\infty$ and $\|h_1\|, \|h_2\| < 1+\varepsilon$[8]. Related to this is the Douglas–Rudin theorem that the quotients $\theta_1\overline{\theta_2}$ with θ_1 and θ_2 inner are uniformly dense in the unimodular functions in $L^\infty(\mathbb{T})$[9].

Of particular importance are the interpolating Blaschke products: a Blaschke product B with zeroes (z_j) is interpolating if its zero sequence is an interpolating sequence for H^∞ or, equivalently, there exists $\delta > 0$ such that

$$\inf_k \prod_{j:j\neq k} \left| \frac{z_j - z_k}{1 - \overline{z_k} z_j} \right| = \delta.$$

These Blaschke products play an important role in the study of bounded analytic functions: consider a closed subalgebra B of L^∞ containing H^∞ properly. In establishing a conjecture of R. G. Douglas, Chang; Marshall[10] proved that such algebras (now called Douglas algebras) can be characterized using interpolating Blaschke products: if

$$U_B = \left\{ b : b \text{ interpolating and } b^{-1} \in B \right\},$$

then an algebra is a Douglas algebra if and only if it is the closed algebra generated by H^∞ and the conjugates of the functions in U_B. In other words, $B = [H^\infty, \overline{U_B}]$. Much more is known about interpolating Blaschke products: in particular, Jones[11] showed that one can take the Blaschke products in the Douglas–Rudin theorem to be interpolating. Related work can be found in Marshall and Stray (1996), Garnett and Nicolau (1996), and Garnett (2007). One very interesting question remains open: can every Blaschke product be approximated (uniformly) by an interpolating Blaschke product Hjelle and Nicolau[12] have shown that given a Blaschke product, B, there is an interpolating Blaschke product that approximates B in modulus on $\mathbb{D}$, but this is the best result to date.

[8] Bourgain, 1986, "A problem of Douglas and Rudin on factorization";
Barclay, 2009, "A solution to the Douglas-Rudin problem for matrix-valued functions";
Chalendar and Partington, 2011, *Modern approaches to the invariant-subspace problem*.

[9] Douglas and Rudin, 1969, "Approximation by inner functions".

[10] Chang, 1976, "A characterization of Douglas subalgebras";
Marshall, 1976, "Subalgebras of L^∞ containing H^∞".

[11] Jones, 1981, "Ratios of interpolating Blaschke products".

[12] Hjelle and Nicolau, 2006, "Approximating the modulus of an inner function".

2 Some operators associated with inner functions

2.1 Isometries

1. It is not hard to see that the *analytic Toeplitz operator* or *Laurent operator*, $T_\phi : H^2 \to H^2$, $f \mapsto \phi f$, where $\phi \in H^\infty$, is an isometry if and only if ϕ is inner. Moreover $\operatorname{codim} \phi H^2 < \infty$ if and only if ϕ is a finite Blaschke product.

2. For $\phi : \mathbb{D} \to \mathbb{D}$ holomorphic, we may consider the *composition operator* C_ϕ : $H^2 \to H^2$, $f \mapsto f \circ \phi$. See for example Cowen and MacCluer (1995) for full details on these. In particular, by Littlewood's subordination theorem[13], C_ϕ is automatically continuous.

It is a result of E. A. Nordgren[14] that C_ϕ is an isometry if and only if ϕ is inner and $\phi(0) = 0$. Note that if ϕ is inner, with $\phi(0) = 0$, then for $n > m$ we have

$$\langle \phi^n, \phi^m \rangle = \langle \phi^{n-m}, 1 \rangle = \phi(0)^{n-m} = 0,$$

so that the orthonormal sequence $(z^n)_{n\geq 0}$ in H^2 is mapped to the orthonormal sequence $(\phi^n)_{n\geq 0}$.

Conversely, since $\langle z, 1 \rangle = 0$, we must have $\phi(0) = \langle \phi, 1 \rangle = 0$ if C_ϕ is to be an isometry. Also the condition $\|\phi^n\| = 1$ for all n can be used to check that ϕ is inner.

Bayart[15] shows that C_ϕ is *similar* to an isometry if and only if ϕ is inner and $\phi(p) = p$ for some $p \in \mathbb{D}$.

2.2 Universal operators

An operator U defined on a separable infinite-dimensional Hilbert space $\mathcal{H}$ is said to be *universal* in the sense of Rota, if for every operator T on a Hilbert space $\mathcal{K}$ there is a constant $\lambda \in \mathbb{C}$ and an invariant subspace $\mathcal{M}$ for U such that T is similar to the restriction $\lambda U_{|\mathcal{M}}$.

$$\begin{array}{ccc}
\mathcal{H} & \xrightarrow[\lambda U]{} & \mathcal{H} \\
\uparrow & & \uparrow \\
\mathcal{M} & \xrightarrow[\lambda U]{} & \mathcal{M} \\
{\scriptstyle J}\downarrow & & {\scriptstyle J}\downarrow \\
\mathcal{K} & \xrightarrow[T]{} & \mathcal{K}
\end{array}$$

[13] Littlewood, 1925, "On inequalities in the theory of functions".
[14] E. A. Nordgren, 1968, "Composition operators".
[15] Bayart, 2003, "Similarity to an isometry of a composition operator".

The following theorem provides many examples of universal operators.

Theorem 1 (Caradus[16]) – *If the operator $U : \mathcal{H} \to \mathcal{H}$ is surjective with infinite-dimensional kernel, then it is universal.*

1. Take θ inner, but not a finite Blaschke product. Then using Theorem theorem 1 one can show that the Toeplitz operator $T_{\overline{\theta}} = T_\theta^* : H^2 \to H^2$, with $f \mapsto P_{H^2}(\overline{\theta} f)$ is universal.

   Such an operator T_θ^* is similar to the backward shift A on $L^2(0,\infty)$, given by

   $$Af(t) = f(t+1),$$

   which by the Laplace transform is unitarily equivalent to the adjoint of the operator $M_{e^{-s}}$ of multiplication of e^{-s} on the Hardy space $H^2(\mathbb{C}_+)$ of the right half-plane $\mathbb{C}_+$ (here s is the independent variable). Note that e^{-s} is inner: still, in spite of Beurling's theorem mentioned above, there is no usable characterization of the invariant subspaces of A.

2. Let $\phi : \mathbb{D} \to \mathbb{D}$ be defined by

   $$\phi(z) = \frac{z + 1/2}{1 + z/2};$$

   this is a (hyperbolic) automorphism fixing ± 1. The composition operator C_ϕ has spectrum given by

   $$\sigma(C_\phi) = \left\{ z \in \mathbb{C} : 1/\sqrt{3} \le |z| \le \sqrt{3} \right\}.$$

   For $\lambda \in \operatorname{int} \sigma(C_\phi)$, it can be shown that $C_\phi - \lambda I$ is universal[17]. Note that it has the same invariant subspaces as C_ϕ, and a complete description of them would give a solution to the invariant subspace problem.

These ideas have stimulated studies on cyclic vectors and minimal invariant subspaces for C_ϕ (e.g. Mortini (1995) and Gallardo-Gutiérrez and Gorkin (2011)).

2.3 Hankel and Toeplitz operators

We begin with the orthogonal decomposition

$$L^2(\mathbb{T}) = H^2 \oplus \overline{H_0^2}$$

into closed subspaces spanned by $\{e^{int} : n \ge 0\}$ and $\{e^{int} : n < 0\}$, respectively. Write $P : L^2(\mathbb{T}) \to H^2$ for the orthogonal projection.

[16] Caradus, 1969, "Universal operators and invariant subspaces".

[17] E. Nordgren, Rosenthal, and Wintrobe, 1987, "Invertible composition operators on H^p".

Definition 1 – Let $\phi \in L^\infty(\mathbb{T})$. Then the *Toeplitz operator* $T_\phi : H^2 \to H^2$ is defined by $T_\phi f = P(\phi f)$ for $f \in H^2$. The *Hankel operator* $\Gamma_\phi : H^2 \to \overline{H_0^2}$ is defined by $\Gamma_\phi f = (I-P)\phi f$ for $f \in H^2$.

It is well known that $\|T_\phi\| = \|\phi\|_\infty$[18] and that $\|\Gamma_\phi\| = \operatorname{dist}(\phi, H^\infty)$[19].

2.4 Kernels

1. If $u \in \ker \Gamma_\phi$, then $\phi u \in H^2$, so that $z\phi u \in H^2$ and $zu \in \ker \Gamma_\phi$. Hence, by Beurling's theorem, $\ker \Gamma_\phi = \theta H^2$ for some inner function θ.

   For example, if θ is inner, then $u \in \ker \Gamma_{\overline{\theta}}$ if and only if $\overline{\theta} u \in H^2$, which happens if and only if $u \in \theta H^2$. So all Beurling subspaces occur as Hankel kernels.

2. Suppose that θ is inner. Then $f \in \ker T_{\overline{\theta}}$ if and only if $\langle \overline{\theta} f, g\rangle = 0$ for all $g \in H^2$. This is equivalent to the condition $\langle f, \theta g\rangle = 0$; that is, $f \in H^2 \ominus \theta H^2$. We shall study these spaces in Section section 3 on the following page.

Toeplitz kernels in general have the *near-invariance* property. If $u \in H^2$ and $\theta u \in \ker T_\phi$ for some inner function θ, then $\phi\theta u = \overline{z h}$ for some $h \in H^2$. Hence $\phi u = \overline{\theta}\overline{z h}$ and thus $u \in \ker T_\phi$.

That is, if $v \in \ker T_\phi$ and $v/\theta \in H^2$, then $v/\theta \in \ker T_\phi$.

In particular, if $v \in \ker T_\phi$ and $v/z \in H^2$, then $v/z \in \ker T_\phi$. This property is not the same as being S^*-invariant, even though $S^* v = v/z$ if $v/z \in H^2$.

For example, let $\phi(z) = e^{-z}/z^2$. One may verify that

$$\ker T_\phi = \{(a + bz)e^z : a, b \in \mathbb{C}\}.$$

However $S^* e^z = \frac{e^z - 1}{z}$, which does not lie in $\ker T_\phi$.

Now Hitt[20] showed that a subspace $\mathcal{M} \subset H^2$ is nearly S^*-invariant if and only if it can be written as $\mathcal{M} = f K_\theta$, where θ is inner, $\theta(0) = 0$, $f \in \mathcal{M} \ominus (\mathcal{M} \cap zH^2)$, and K_θ is the model space $H^2 \ominus \theta H^2$, discussed in Section section 3 on the next page.

Moreover, Hayashi; Hayashi[21] showed that such an $\mathcal{M}$ is in fact a Toeplitz kernel if and only if the function f has the property that f^2 is *rigid*, which means that if $g \in H^1$ with $g/f^2 > 0$ a.e., then $g = \lambda f^2$ for some constant $\lambda > 0$. A rigid function is necessarily outer.

[18] Brown and Halmos, 1963, "Algebraic properties of Toeplitz operators".
[19] Nehari, 1957, "On bounded bilinear forms".
[20] Hitt, 1988, "Invariant subspaces of $\mathcal{H}^2$ of an annulus".
[21] Hayashi, 1986, "The kernel of a Toeplitz operator";
Hayashi, 1990, "Classification of nearly invariant subspaces of the backward shift".

3 Model spaces

3.1 Definitions and examples

Since the invariant subspaces for S have the form θH^2, with θ inner, those for S^* have the form $H^2 \ominus \theta H^2$, usually written K_θ. Such spaces are called *model spaces*.

Example 1 –

1. Take $\theta(z) = z^N$, which is inner. Then

$$K_\theta = \operatorname{span}\left\{1, z, z^2, \dots, z^{N-1}\right\}.$$

2. For $\theta(z) = \prod_{k=1}^N \frac{z-\alpha_k}{1-\overline{\alpha_k}z}$ with $\alpha_1, \dots, \alpha_N$ distinct, we have $f \in \theta H^2$ if and only if $f(\alpha_1) = \cdots = f(\alpha_N) = 0$. Then

$$K_\theta = \operatorname{span}\left\{\frac{1}{1-\overline{\alpha_1}z}, \dots, \frac{1}{1-\overline{\alpha_N}z}\right\}.$$

Indeed, for $\alpha \in \mathbb{D}$, $k_\alpha : z \mapsto \frac{1}{1-\overline{\alpha}z}$ is the *reproducing kernel* at α; i.e.,

$$f(\alpha) = \langle f, k_\alpha \rangle \qquad \text{for} \quad f \in H^2,$$

and clearly $f \in \theta H^2$ if and only if f is orthogonal to $k_{\alpha_1}, \dots, k_{\alpha_N}$.

3. For a fixed $\tau > 0$ we write

$$L^2(0,\infty) = L^2(0,\tau) \oplus L^2(\tau,\infty). \tag{1}$$

Under the Laplace transform this maps to the orthogonal decomposition

$$H^2(\mathbb{C}_+) = K_\theta \oplus \theta H^2(\mathbb{C}_+),$$

where $\theta(s) = e^{-s\tau}$; that is, θ is inner. Then K_θ can be written as $e^{s\tau/2} PW_{\tau/2}$, where $PW_{\tau/2}$ is a *Paley–Wiener* space, consisting of entire functions, as considered in signal processing.

In general K_θ is finite-dimensional if and only if θ is a finite Blaschke product.

3.2 Decompositions of H^2 and K_B

Let θ be inner. Then

$$H^2 = K_\theta \oplus \theta K_\theta \oplus \theta^2 K_\theta \oplus \cdots$$

This is an orthogonal direct sum, since if $k_1, k_2 \in K_\theta$ and $0 \le m < n$, then

$$\langle \theta^m k_1, \theta^n k_2 \rangle = \langle k_1, \theta^{n-m} k_2 \rangle = 0,$$

since $k_1 \perp \theta H^2$.

Note that T_θ acts as a shift here, i.e.,

$$\theta(k_1 + \theta k_2 + \theta^2 k_3 + \cdots) = \theta k_1 + \theta^2 k_2 + \theta^3 k_3 + \cdots$$

A special case of this can be identified from equation (1) on page 16, since

$$L^2(0,\infty) = L^2(0,\tau) \oplus L^2(\tau, 2\tau) \oplus \cdots$$

We now look at model spaces corresponding to infinite Blaschke products. If $\alpha_1, \alpha_2, \ldots$ are the zeroes of an infinite Blaschke product B (assumed distinct), then an orthonormal basis of K_B is the Takenaka–Malmquist–Walsh basis given by orthonormalizing the sequence of reproducing kernels associated with the (α_n). We have

$$e_1(z) = \frac{(1-|\alpha_1|^2)^{1/2}}{1-\overline{\alpha_1}z},$$
$$e_2(z) = \frac{(1-|\alpha_2|^2)^{1/2}}{1-\overline{\alpha_2}z}\left(\frac{z-\alpha_1}{1-\overline{\alpha_1}z}\right),$$

and, in general

$$e_n(z) = \frac{(1-|\alpha_n|^2)^{1/2}}{1-\overline{\alpha_n}z}\left(\prod_{k=1}^{n-1}\frac{z-\alpha_k}{1-\overline{\alpha_k}z}\right).$$

It is easily checked that these are orthonormal, and have the same closed span as the reproducing kernels $\frac{1}{1-\overline{\alpha_1}z}, \ldots, \frac{1}{1-\overline{\alpha_n}z}, \ldots$ This closed span is K_B when the (α_n) form a Blaschke sequence, and H^2 otherwise.

3.3 Frostman's theorem and mappings between model spaces

The following result shows that inner functions are not far from Blaschke products, in a precise sense.

Theorem 2 (Frostman[22]) – *Let θ be any inner function. Then, for $\alpha \in \mathbb{D}$, the function $\frac{\theta-\alpha}{1-\overline{\alpha}\theta}$ is also inner; it is a Blaschke product with distinct zeroes for all $\alpha \in \mathbb{D}$ outside an exceptional set E such that for each $0 < r < 1$ the set of real t such that $re^{it} \in E$ has measure zero.*

Note that if ϕ and θ are inner then $\phi \circ \theta$ is also inner (this is not obvious). Here we are considering simply $b \circ \theta$ where b is the inner function with $b(z) = \frac{z-\alpha}{1-\overline{\alpha}z}$.

Frostman gave a stronger version of his theorem, expressed by saying that the exceptional set has logarithmic capacity zero; however, it is beyond the scope of this work.

Theorem 3 – *The* Crofoot transform, *defined for* $\alpha \in \mathbb{D}$ *by*

$$J_\alpha f = \frac{\left(1-|\alpha|^2\right)^{1/2}}{1-\overline{\alpha}\theta} f \qquad (f \in K_\theta),$$

is a unitary mapping from K_θ *onto* $K_{b\circ\theta}$ *for each inner function* θ*.*

In combination with Frostman's theorem, this can be used to construct orthonormal bases for any model space K_θ.

3.4 Truncated Toeplitz and Hankel operators

Truncated Toeplitz operators were introduced by Sarason[23], and have received much attention since then. The idea here is to put finite Toeplitz matrices of the form

$$\begin{pmatrix} a_0 & a_{-1} & \dots & a_{-n} \\ a_1 & a_0 & \dots & a_{-n+1} \\ \dots & \dots & \dots & \dots \\ a_n & a_{n-1} & \dots & a_0 \end{pmatrix} \tag{2}$$

into a more general context. One may also consider finite Hankel matrices of the form

$$\begin{pmatrix} a_{-1} & a_{-2} & \dots & a_{-n-1} \\ a_{-2} & a_{-3} & \dots & a_{-n-2} \\ \dots & \dots & \dots & \dots \\ a_{-n-1} & a_{-n-2} & \dots & a_{-2n-1} \end{pmatrix}. \tag{3}$$

Take θ inner, and $\phi \in L^\infty(\mathbb{T})$; then the truncated Toeplitz operator $A_\phi^\theta : K_\theta \to K_\theta$ is defined by

$$A_\phi^\theta f = P_{K_\theta}(\phi \cdot f) \qquad (f \in K_\theta),$$

where $P : L^2(\mathbb{T}) \to K_\theta$ is the orthogonal projection.

[22] Frostman, 1935, "Potentiel d'équilibre et capacité des ensembles avec quelques applications à la théorie des fonctions".

[23] Sarason, 2007, "Algebraic properties of truncated Toeplitz operators".

The motivating example involves the choice $\theta(z) = z^{n+1}$, and the orthonormal basis $\{1, z, z^2, \dots, z^n\}$ of K_θ, when the matrix of A^θ_ϕ has the form (2), with $(a_n)_{n\in\mathbb{Z}}$ the Fourier coefficients of ϕ.

Similarly for truncated Hankel operators. The operator $B^\theta_\phi : K_\theta \to \overline{zK_\theta}$ is defined by

$$B^\theta_\phi f = P_{\overline{zK_\theta}}(\phi \cdot f) \qquad (f \in K_\theta).$$

Now, if $\theta(z) = z^{n+1}$, then $\overline{zK_\theta}$ has basis $\{\overline{z}, \dots, \overline{z}^{n+1}\}$, and with these bases the operator B^θ_ϕ has a truncated Hankel matrix (3).

4 Restricted shifts

4.1 Basic ideas

We recall that the invariant subspaces of the backwards shift S^* have the form K_θ. We now define $S_\theta : K_\theta \to K_\theta$ by

$$S_\theta = P_{K_\theta} S_{|K_\theta} = (S^*_{|K_\theta})^*.$$

This is the truncated Toeplitz operator with symbol z, and if we take $\theta(z) = z^{n+1}$ it maps as follows: $1 \mapsto z, z \mapsto z^2, \dots, z^{n-1} \mapsto z^n, z^n \mapsto 0$, so that its matrix is given by

$$\begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix}.$$

The restricted shift has a part in the Sz.-Nagy–Foias functional model[24]: if T is a contraction on a Hilbert space H such that $\|(T^*)^n x\| \to 0$ for all $x \in H$ and $\text{rank}(I - T^*T) = \text{rank}(I - TT^*) = 1$, then there is an inner function θ such that T is unitarily equivalent to S_θ.

Proposition 1 – *The invariant subspaces for the restricted shift S_θ are "shifted" model spaces of the form $K_\theta \cap \phi H^2 = \phi K_{\theta/\phi}$, where ϕ is an inner function dividing θ in $H^\infty(\mathbb{D})$.*

Proof. The invariant subspaces for its adjoint, $S^*_{|K_\theta}$ are clearly of the form K_ϕ, where ϕ divides θ in $H^\infty(\mathbb{D})$. Their orthogonal complements are the invariant subspaces for S_θ, and have the required form.

[24] Szökefalvi-Nagy et al., 2010, *Harmonic analysis of operators on Hilbert space*.

It is easy to see that $\operatorname{rank} S_\theta < \infty$ if and only if θ is a finite Blaschke product. We now define the *spectrum* of an inner function θ by

$$\sigma(\theta) = \left\{ w \in \overline{\mathbb{D}} : \liminf_{z \to w} |\theta(w)| = 0 \right\}.$$

For a Blaschke product B, the set $\sigma(B)$ is the closure of the zero set of B in $\overline{\mathbb{D}}$. It can then be shown that in general $\sigma(S_\theta) = \sigma(\theta)$[25].

4.2 Unitary perturbations and dilations

We shall now suppose that $\theta(0) = 0$: this simplifies some of the formulae, but is not a serious restriction. Clark[26] initiated a very fruitful study of unitary perturbations of restricted shifts. In particular, he showed that the set of rank-1 perturbations of S_θ that are unitary can be parametrised as $\{U_\alpha : \alpha \in \mathbb{T}\}$, where

$$U_\alpha f = S_\theta f + \alpha \langle f, S^* \theta \rangle 1, \qquad (f \in K_\theta),$$

noting that the constant function 1 lies in K_θ because $\theta(0) = \langle \theta, 1 \rangle = 0$.

If we consider the case $\theta(z) = z^{n+1}$, as above, we find that the matrix of U_α is now

$$\begin{pmatrix} 0 & 0 & 0 & \dots & 0 & \alpha \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix},$$

so that $1, z, \dots, z^{n-2}, z^{n-1}$ are mapped, respectively, to $z, z^2, \dots, z^{n-1}, \alpha$.

The spectral measure of U_α is called a *Clark measure*, and there are various applications. See, for example, the book Cima, Matheson, and Ross (2006).

For an operator T on a Hilbert space H, we consider the question of finding a unitary operator U on a space containing H, such that its restriction to H is T. In matrix terms we may write

$$U = \begin{pmatrix} T & * \\ * & * \end{pmatrix}.$$

If U is defined on $H \oplus \mathbb{C}$, then we call it a 1-dilation. This is not the same as the standard Sz.-Nagy–Foias dilation as in Szökefalvi-Nagy et al. (2010). In the context of restricted shifts and unitary dilations, there is a connection here with a classical result in geometry, which we now develop.

[25] Helson, 1964, *Lectures on invariant subspaces*, Lec. VIII.
[26] Clark, 1972, "One dimensional perturbations of restricted shifts".

4.3 Numerical ranges

For an integer $n \geq 3$, a closed subset A of $\mathbb{D}$ has the *n-Poncelet property*, if whenever there exists an n-gon P such that P circumscribes A and has its vertices on $\mathbb{T}$, then every point on the unit circle is a vertex of such an n-gon. This was originally studied in the context of an ellipse, as in figure 1. (The figures were produced by an applet written by A. Shaffer.) Associated with the ellipse is a Blaschke product, as we shall explain: its zeroes are denoted by light circles and the zeroes of its derivative by dark circles.

We shall also be considering a generalization of this, namely, an infinite Poncelet property.

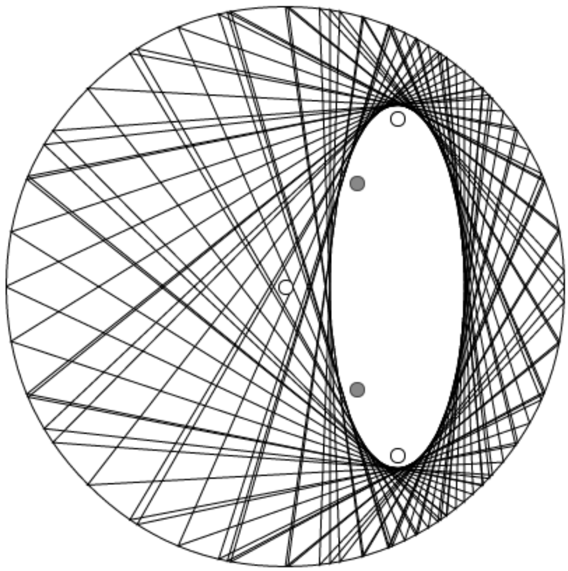

Figure 1: Poncelet ellipse with triangles

Let us suppose first that θ is a finite Blaschke product, and hence K_θ is finite-dimensional. Recall that the *numerical range* of an operator T on a Hilbert space H is defined by

$$W(T) = \{\langle Tx, x\rangle : x \in H, \|x\| = 1\},$$

and, according to the Toeplitz–Hausdorff theorem, is a convex subset of the plane. If T has finite rank, then $W(T)$ is also compact.

Theorem 4 – *For the restricted shift S_θ on a finite-dimensional model space K_θ we have*

$$W(S_\theta) = \bigcap_{\alpha\in\mathbb{T}} W(U_\alpha^\theta),$$

where the U_α^θ are the rank-1 Clark perturbations of $S_{z\theta}$, which are equivalent to unitary 1-dilations of S_θ.

For versions of this results and further developments, see Gau and Wu (1998), Gau and Wu (2003), Gorkin and Rhoades (2008), and Daepp, Gorkin, and Voss (2010).

Note that

$$\sigma(U_\alpha^\theta) = \{z \in \mathbb{T} : z\theta(z) = \alpha\},$$

an $n+1$-point set if the degree of θ is n. Moreover, $W(U_\alpha^\theta)$ is the convex hull of $\sigma(U_\alpha^\theta)$, namely, a polygon. If $\deg\theta = 2$, then it is known that $W(S_\theta)$ is an ellipse, with foci at the eigenvalues of S_θ. Therefore, this ellipse has foci at the zeroes of θ, and it is here expressed as an intersection of triangles.

Figure 2 on the facing page and figure 3 on page 24 show similar examples with $n = 3$ (quadrilaterals) and $n = 4$ (pentagons).

The following more general result was proved in Chalendar, Gorkin, and Partington (2009). Note that numerical ranges no longer need to be closed, so the formulation is slightly different.

Theorem 5 – *Let θ be an inner function. Then*

$$\overline{W(S_\theta)} = \bigcap_{\alpha\in\mathbb{T}} \overline{W(U_\alpha^\theta)},$$

where the U_α^θ are the unitary 1-dilations of S_θ (or, equivalently, the rank-1 Clark perturbations of $S_{z\theta}$).

In general we may regard the numerical ranges of the U_α^θ as convex polygons with infinitely-many sides. Some vectorial generalizations of these results (involving more general contractions) are given in Benhida, Gorkin, and Timotin (2011) and Bercovici and Timotin (2014).

We may now ask how many polygons are needed to determine θ uniquely. Note that the vertices of a polygon are solutions to $z\theta(z) = \alpha$, so we are motivated to consider boundary interpolation by inner functions.

4.4 Interpolation questions

For finite Blaschke products we have the following theorem in Chalendar, Gorkin, and Partington (2011) about identifying two sets of n points. Note that the two sets

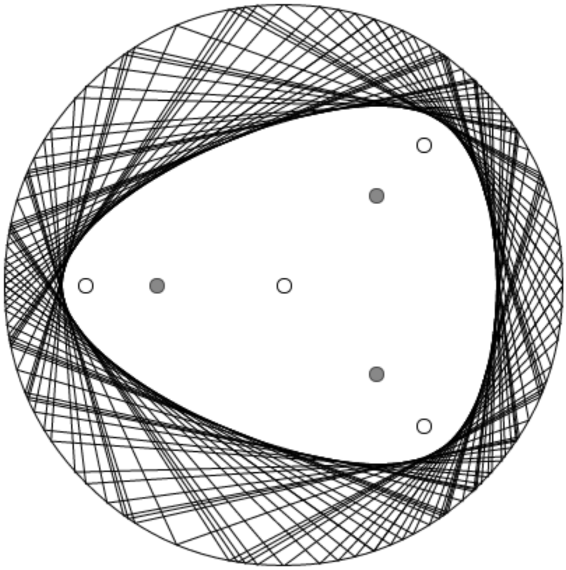

Figure 2: Symmetrical Poncelet curve with quadrilaterals

$\{z_1,\ldots,z_n\}$ and $\{w_1,\ldots,w_n\}$ in the theorem are necessarily interlaced; that is, each z_j lies between two successive w_k and vice-versa.

Theorem 6 – *For a finite Blaschke products θ, ϕ of degree n, suppose that there are distinct points $z_1,\ldots,z_n$ and $w_1,\ldots,w_n$ in $\mathbb{T}$ such that*

$$\theta(z_1) = \cdots = \theta(z_n), \qquad \theta(w_1) = \cdots = \theta(w_n),$$

and

$$\phi(z_1) = \cdots = \phi(z_n), \qquad \phi(w_1) = \cdots = \phi(w_n).$$

Then $\phi = \lambda \frac{\theta - a}{1 - \overline{a}\theta}$ for some $\lambda \in \mathbb{T}$ and $a \in \mathbb{D}$.

We say that ϕ is a *Frostman shift* of θ.

Suppose now that θ is inner with just one singularity on $\mathbb{T}$; that is, it extends analytically across $\mathbb{T}$ except at one point, which we shall take to be $z = 1$. For some such θ, but not all, there will be a sequence $(t_n)_{n\in\mathbb{Z}}$ in $\mathbb{T}$ (necessarily isolated since θ has an analytic extension), accumulating on both sides of the point 1, such that $\theta(t_n) = 1$ for each n. This is called a singularity of type 2 in Chalendar, Gorkin, and Partington (2012): see figure 4 on the following page.

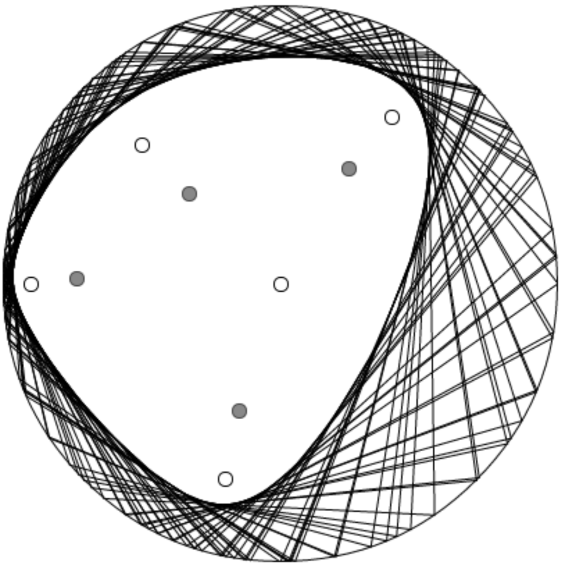

Figure 3: Asymmetrical Poncelet curve with pentagons

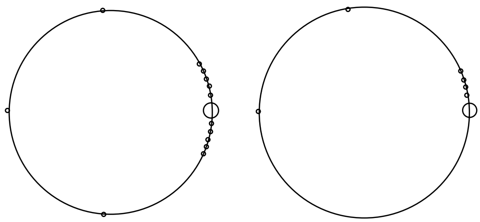

Figure 4: Singularities of type 2 (L) and type 1 (R)

We consider how to determine θ from this data.

We transform to the upper half-plane $\mathbb{C}^+$, using the Möbius mapping

$$\psi(z) = \mathrm{i}\frac{1+z}{1-z}, \qquad \text{with} \quad \psi(1) = \infty.$$

Now consider $F := \psi \circ \theta \circ \psi^{-1}$. Then F is meromorphic on $\mathbb{C}$ with real poles $(b_n)_{n\in\mathbb{Z}}$ accumulating at $\pm\infty$. It maps $\mathbb{C}^+$ to $\mathbb{C}^+$ and $\mathbb{C}^-$ to $\mathbb{C}^-$. Such functions are called *strongly real*. Without loss of generality we may assume that 0 is neither a pole nor a zero of F, in which case we have the following theorem, given in Levin (1980) as the Hermite–Biehler theorem, but attributed to Krein.

Theorem 7 – *For F strongly real with poles (b_n) tending to $\pm\infty$, the zeroes (a_n) and poles (b_n) are interlaced in the sense that we may write $b_n < a_n < b_{n+1}$ for each n, and then*

$$F(z) = c\prod_{n\in\mathbb{Z}} \frac{1 - z/a_n}{1 - z/b_n}, \tag{4}$$

where $c > 0$ unless $a_n b_n < 0$, in which case $c < 0$. There is such a function for each sequence (a_n) interlaced with the (b_n).

Our conclusion is that, given one limit point on $\mathbb{T}$, approached from both sides by solutions to $\theta(z) = 1$, the set $\theta^{-1}(1)$ does not determine θ, whereas the sets $\theta^{-1}(1)$ and $\theta^{-1}(-1)$ together tell us what θ is, to within composition by a Möbius transformation fixing ± 1.

In Chalendar, Gorkin, and Partington (2011) the case of finitely-many singularities is discussed, including cases then some singular points are approached on one side only. Curiously, there is a non-uniqueness case in the Hermite–Biehler expression, apparently missed by Krein. For suppose that $a_n \to 1$ as $n \to -\infty$ and $a_n \to \infty$ as $n \to \infty$. Then, with interlaced (b_n) there is one solution, namely (4), but there is also another possibility, namely

$$F(z) = c(z-1)\prod_{n\in\mathbb{Z}} \frac{1 - z/a_n}{1 - z/b_n}$$

and these are the only possibilities.

On the circle, the corresponding θ has a singularity of type 1 in the terminology of Chalendar, Gorkin, and Partington (2012): see figure 4 on page 24. Thus there are two one-parameter families of inner functions θ for such a choice of $\theta^{-1}(1)$ and $\theta^{-1}(-1)$. A third set, e.g. $\theta^{-1}(\mathrm{i})$, enables one to distinguish between them. Thus one sees that, in a fairly general situation, if $W(S_\theta) = W(S_\phi)$, then θ is a Frostman shift of ϕ and so the restricted shifts are unitarily equivalent.

Some (necessarily less explicit) extensions of these ideas have been given by Bercovici and Timotin, Cor.6.3[27], in the case where the set of singularities of the inner function θ is of measure zero.

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