Semigroups of I-quotients

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A Thesis Submitted for the Degree of PhD

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August 2011

Abstract

Let Q be an inverse semigroup. A subsemigroup S of Q is a *left I-order* in Q or Q is a semigroup of *left I-quotients* of S, if every element in Q can be written as $a^{-1}b$ where $a, b \in S$ and a^{-1} is the inverse of a in the sense of inverse semigroup theory. If we insist on a and b being $\mathcal{R}$-related in Q, then we say that S is *straight* in Q and Q is a semigroup of *straight left I-quotients* of S.

We give a theorem which determines when two semigroups of straight left I-quotients of given semigroup are isomorphic.

Clifford has shown that, to any right cancellative monoid with the (LC) condition, we can associate an inverse hull. By saying that a semigroup S has the (LC) condition we mean for any $a, b \in S$ there is an element $c \in S$ such that $Sa \cap Sb = Sc$. According to our notion, we can regard such a monoid as a left I-order in its inverse hull. We extend this result to the left ample case where we show that, if a left ample semigroup has the (LC) condition, then it is a left I-order in its inverse hull.

The structure of semigroups which are semilattices of bisimple inverse monoids, in which the set of identity elements forms a subsemigroup, has been given by Gantos. We prove that such semigroups are strong semilattices of bisimple inverse monoids. Moreover, they are semigroups of left I-quotients of semigroups with the (LC) condition, which are strong semilattices of right cancellative monoids with the (LC) condition. We show that a strong semilattice S of left ample semigroups with (LC) and such that the connecting homomorphisms are (LC)-preserving, itself has the (LC) condition and is a left I-order in a strong semilattice of inverse semigroups.

We investigate the properties of left I-orders in primitive inverse semigroups. We give necessary and sufficient conditions for a semigroup to be a left I-order in a primitive inverse semigroup. We prove that a primitive inverse semigroup of left I-quotients is unique up to isomorphism.

We study left I-orders in a special case of a bisimple inverse ω-semigroup, namely, the bicyclic monoid. Then, we generalise this to any bisimple inverse ω-semigroup. We characterise left I-orders in bisimple inverse ω-semigroups.

Contents

1 Introduction **1**

2 Preliminaries **6**
2.1 Notation and Relations 6
2.2 Inverse semigroups 9
2.3 Ample semigroups 11
2.4 Some special classes of inverse semigroups 13
2.4.1 Bisimple inverse semigroups 13
2.4.2 Clifford semigroups 16
2.4.3 E-unitary inverse semigroups 17
2.4.4 Primitive inverse semigroups 18
2.5 Categories 20

3 Left I-orders **25**
3.1 Definitions and examples 25
3.2 Extension of homomorphisms 29

4 Inverse hulls of left I-quotients of left ample semigroups **36**
4.1 Preliminaries 37
4.2 The main theorem 38
4.3 Some special cases 41

5 Left I-orders in semilattices of inverse semigroups **50**
5.1 Left I-quotients of semilatices of right cancellative semigroups 51
5.2 Left I-quotients of semilattices of left ample semigroups 72
5.3 The category of semilattices of certain bisimple inverse monoids 76

6 Primitive inverse semigroups of left I-quotients **81**
6.1 Preliminaries 82
6.2 The main theorem 85
6.3 Uniqueness 93
6.4 Primitive inverse semigroups of I-quotients 94
6.5 The abundant case 96

7 Bicyclic semigroups of left I-quotients **99**
7.1 Preliminaries 100
7.2 Upper subsemigroups 104
7.3 Lower subsemigroups 106
7.4 Two-sided subsemigroups 107

8 Bisimple inverse ω-semigroups of left I-quotients **110**
8.1 Preliminaries 111
8.2 The main theorem 113
8.3 Proper bisimple inverse ω-semigroups of left I-quotients 127
8.4 Uniqueness 128

IN THE NAME OF ALLAH, THE BENEFICENT, THE MERCIFUL.

Chapter 1

Introduction

Many authors are interested in the theory of semigroups of quotients. We recall that a subsemigroup S of a group G is a *left order* in G or G is a *group of left quotients* of S if any element in G can be written as $a^{-1}b$ where $a, b \in S$. Ore and Dubreil [1] showed that a semigroup S has a group of left quotients if and only if S is right reversible and cancellative. By saying that a semigroup S is right reversible we mean for any $a, b \in S, Sa \cap Sb \neq \emptyset$. Murata in 1950 [35] extended the notion of a group of left quotients to a semigroup of quotients. He insisted on a semigroup of quotients being a monoid and that inverses were inverses lying in the group of units. A different definition proposed by Fountain and Petrich in 1985 [6] was restricted to completely 0-simple semigroups of left quotients. Gould in 1985 [17] extended this concept to left orders in arbitrary semigroup. The idea is that a subsemigroup S of a semigroup Q is a *left order* in Q if every element in Q can be written as $a^{\sharp}b$ where $a, b \in S$ and $a^{\sharp}$ is an inverse of a in a subgroup of Q and if, in addition, every square-cancellable element of S (an element a of a semigroup S is square-cancellable if $a\mathcal{H}^*a^2$) lies in a subgroup of Q. In this case we say that Q is a semigroup of *left quotients* of S. *Right orders* and *semigroups of right quotients* are defined dually. If S is both a left and a right order in Q, then S is an *order* in Q and Q is a *semigroup of quotients* of S.

McAlister introduced two concepts of quotients for an inverse semigroup Q with a subsemigroup S in [28]. The first one says that Q is a *semigroup of quotients* of S if every element in Q can be written as $ab^{-1}c$ where $a, c, b \in S$. The second one is that Q is a semigroup of *strong quotients* of S if every element in Q can be written as $ab^{-1}c$ where $a, c, b \in S$ and $b \in Sa \cap cS$.

Our new definition is that a subsemigroup S of an inverse semigroup Q is a *left I-order* in Q and Q is a semigroup of *left I-quotients* of S if every element in Q can be written as $a^{-1}b$ where $a, b \in S$ and a^{-1} is the inverse of a in the sense of an inverse semigroup theory. *Right I-orders* and *semigroups of right I-quotients* are defined dually. If S is both a left I-order and a right I-order in Q, then S is an *I-order* in Q and Q is a *semigroup of I-quotients* of S.

Let S be a left I-order in Q. Then S is *straight* in Q if every $q \in Q$ can be written as $q = a^{-1}b$ where $a, b \in S$ and $a \mathcal{R} b$ in Q; we also say that Q is a semigroup of *straight left quotients* of S.

The main purpose of this thesis is to study the theory of quotients in inverse semigroups. We investigate the structure of quotients in some classes of inverse semigroups.

In addition to this introduction, this thesis comprises seven chapters. In Chapter 2 we begin by providing brief accounts of the basic ideas concerning inverse semigroups. As we are interested in inverse semigroups we devote the second section to introducing some well known concepts for inverse semigroups. The class of all inverse semigroups is properly contained in the class of all ample semigroups; in Section 2.3 we give some useful descriptions of such semigroups. In Section 2.4 we introduce some special classes of inverse semigroups; we will investigate their left I-orders in the next chapters. In the final section we introduce some definitions from category theory and we explain the relationship between categories and monoids.

In Chapter 3 we give the formal definitions of a left (right) I-order and a straight left (right) I-order. Also, we give a number of examples of left I-orders. In the final section of this chapter we concentrate on the uniqueness of straight left I-quotients, in the following sense. We ask a question: if a semigroup S is a straight left I-order in an inverse semigroup Q and embedded in another inverse semigroup P under a homomorphism φ, when can φ be extended to a homomorphism from Q to P? In order to answer this question we introduce a ternary relation on S and then the answer is given in Theorem 3.2.9, which gives necessary and sufficient conditions for straight left I-quotients of a given semigroup to be isomorphic. Consequently, we have solved the problem of the uniqueness of Q. The result of this section can be found in [12].

In Chapter 4 we show that starting with a left ample semigroup S with the (LC) condition, the inverse hull of S is a semigroup of left I-quotients of S. By saying that a semigroup S has the (LC) condition we mean for any $a, b \in S$ there is an element $c \in S$ such that $Sa \cap Sb = Sc$. The main theorem of this chapter gives a necessary and sufficient condition for a left ample semigroup to be a left I-order in its inverse hull. Right cancellative monoids are precisely left ample semigroups possessing a single idempotent. Our result generalises and extends Clifford's result for a right cancellative monoid. It is worth pointing out that the inverse hull of the left ample semigroup need not be bisimple. In Section 4.3 of this chapter we give a necessary and sufficient condition to make it bisimple, in the case that S satisfies (LC). The results of this chapter have already appeared in [12].

In Chapter 5 we investigate left I-orders in certain semilattices of bisimple inverse semigroups. Since a Clifford semigroup is certainly a semilattice of bisimple inverse semigroups, our result will significantly extend that for left orders in Clifford semigroups. Gantos has shown that, if S is a semilattice of right cancellative monoids with the (LC) condition and certain further conditions, then we can associate it with a semilattice of bisimple inverse semigroups. In the main theorem of Chapter 5, we show that one of Gantos's conditions is equivalent to S itself having the (LC) condition. We use this equivalence to define a simple form for the multiplication which is easier to deal with than the form which Gantos used. We link this with Clifford's result and our definition of left I-order to introduce a new aspect for such semigroups which we can read as follows: If S is a semilattice of right cancellative monoids with (LC) and S has (LC), then S is a left I-order in a semilattice of inverse hull semigroups. Moreover, we prove that such S is a left I-order in a strong semilattice of inverse hull semigroups. We end this chapter by extending these ideas to the following:

Let S be a strong semilattice Y of left ample semigroups $S_\alpha, \alpha \in Y$, such that each S_α has (LC). Using the general results of the previous chapter, each S_α is a left I-order in its inverse hull $\Sigma(S_\alpha)$. We show that S is a left I-order in Q where Q is a strong semilattice Y of the inverse hulls $\Sigma(S_\alpha)$ of the semigroups $S_\alpha, \alpha \in Y$, if and only if the connecting morphisms are *(LC)-preserving*, and this is equivalent to S having (LC). In this case, Q is isomorphic to the inverse hull of S. The result of this section can be found in [12]. In Section 5.3, we define a category **SR** of semilattices of right cancellative monoids with the (LC) condition, which themselves have the (LC) condition, and a category **SB** of semilattices of bisimple inverse monoids, in which the set of identity elements forms a subsemigroup, and show that **SR** and **SB** are equivalent. This result may be specialised in section 2.5[1] to show that the corresponding category of right cancellative monoids with (LC) is equivalent to the category of bisimple inverse monoids.

In Chapter 6 we consider left I-orders in primitive inverse semigroups. We show that any left I-order in a primitive inverse semigroup is straight, a result which will play a significant role in Section 6.3. The main theorem in this chapter, Theorem 6.2.1 gives necessary and sufficient conditions for a semigroup to have a primitive inverse semigroup of left I-quotients. Brandt semigroups, that is, inverse completely 0-simple semigroups, are precisely 0-simple primitive inverse semigroups. We specialise our result to left I-orders in Brandt semigroups[2] a result which may be regarded as generalisation of the main theorem in [11]. As a consequence of our work, any left I-order S in a primitive inverse semigroup Q is straight, and showing that S satisfies conditions in Theorem 3.2.9, we prove that primitive inverse semigroups of left I-quotients of S are isomorphic. We recall that a semigroup S is an abundant semigroup if each $\mathcal{R}^*$-class and each $\mathcal{L}^*$-class of S contains an idempotent. In the last section of this chapter we consider abundant left I-orders in primitive inverse semigroups.

In Chapter 7 we concentrate on a left I-order in one of the most fundamental semigroups; the bicyclic monoid. In order to study left I-orders in the bicyclic monoid, we need to know more about its subsemigroups. A description of the subsemigroups of the bicyclic monoid was obtained in [3]. By using this description we study left

[1] I wish to thank Professor D. B. McAlister for pointing out the idea of this section to me.

[2] We understand that Cegarra has an equivalent result.

I-orders in the bicyclic monoid. It is well-known that the bicyclic monoid plays a very important role in the study of inverse ω-semigroups.

In Chapter 8 we characterise left I-orders in bisimple inverse ω-semigroups. A bisimple inverse ω-semigroup Q is isomorphic to a Bruck-Reilly semigroup $\mathbf{BR}(\mathbf{G},\theta)$, where $\mathbf{G}$ is the group of units of Q and θ is a group homomorphism of $\mathbf{G}$. We show that any left I-order in a bisimple inverse ω-semigroup is straight. The main theorem of this chapter gives necessary and sufficient conditions for a semigroup to be a left I-order in a bisimple inverse ω-semigroup. We study the special case when a semigroup is a left I-order in a proper bisimple inverse ω-semigroup $\mathbf{BR}(\mathbf{G},\theta)$, that is, θ is one-to-one. In the final section of this chapter we are concerned with the uniqueness of bisimple inverse ω-semigroup left I-quotients of a given semigroup.

Some of the work from Chapters 6, 7 and 8 form the subject matter of several papers [13], [14] and [15], recently submitted for publication.

Chapter 2

Preliminaries

In this chapter we will present all of the basic elementary concepts of semigroup theory used throughout this thesis. All of the definitions and results are standard and can be found in [1, 23] and [38].

2.1 Notation and Relations

A *semigroup* $S = (S, .)$ is a set S together with an associative binary operation on S. Some semigroups contain an *identity* element, that is, an element $1 \in S$ such that $a1 = 1a = a$, for all $a \in S$. Such semigroups are called *monoids*. If a semigroup contains an element 0 that satisfies $x0 = 0x = 0$ for all $x \in S$, then 0 is called a *zero* element of the semigroup. Note that identity and zero elements, if they exist, are unique.

We use S^1 and S^0 to denote the semigroup S with an identity or a zero adjoined if necessary, respectively. That is,

$$S^1 = \begin{cases} S & \text{if } S \text{ has an identity element,} \\ S \cup \{1\} & \text{otherwise,} \end{cases}$$

with the multiplication extended by defining $a1 = 1a = a$ for all $a \in S^1$. And

$$S^0 = \begin{cases} S & \text{if } S \text{ has a zero element,} \\ S \cup \{0\} & \text{otherwise,} \end{cases}$$

with the multiplication extended by defining $a0 = 0a = 0$ for all $a \in S^0$.

An element $e \in S$ is an *idempotent* if $e^2 = e$. The set of all idempotents of S is denoted by $E = E(S)$. A *band* is a semigroup such that every element is an idempotent and a commutative band is a *semilattice.*

Let $(X, \leq)$ be a partially ordered set. If $x, y, z \in X$ and $z \leq x, y$, then z is a *lower bound* of x and y. If z lies above all other lower bounds of x and y it is the *greatest lower bound* and z is denoted by $x \wedge y$. If every pair of elements in X has a greatest lower bound, then X is a *meet semilattice.*

On any semigroup S, we define a partial order relation $\leq$ on $E(S)$ by

$$e \leq f \text{ if and only if } ef = fe = e.$$

If S is a commutative semigroup, then $(E(S), \leq)$ forms a meet semilattice. If S contains 0, then $0 \leq e$ for every e in $E(S)$. An idempotent element e of S is called *primitive* if $e \neq 0$ and $f \leq e$ implies that $f = 0$ or $e = f$.

A non-empty subset T of a semigroup S is a *subsemigroup* of S if it is closed under the operation of S. A subsemigroup of S which is a group with respect to the multiplication inherited from S will be called a *subgroup* of S. If A is a non-empty subset of S, then the intersection of all subsemigroups of S containing A, is *the subsemigroup* of S *generated by the set* A; and is denoted by $\langle A \rangle$. The subsemigroup $\langle A \rangle$ consists of all elements in S that can be written as a finite product of elements of A.

A non-empty subset A of a semigroup S is called a *right ideal* if $AS \subseteq A$, a *left ideal* if $SA \subseteq A$ and an *ideal* if it is both a left and a right ideal. If a is an element in a semigroup S the smallest left ideal containing a is denoted by $Sa \cup \{a\}$ which we may conveniently write as S^1a, and which we call the *principal left ideal generated by* a. The *principal right ideal generated by* a is defined dually and *two-sided ideal generated by* a is S^1aS^1.

Let S and T be semigroups. A function $\varphi : S \longrightarrow T$ is called a *homomorphism* of S into T if for all $a, b \in S$ we have $(a\varphi)(b\varphi) = (ab)\varphi$. If S and T are monoids then, to be called a *monoid homomorphism*, φ must also satisfy $1_S\varphi = 1_T$. If φ is one-to-one, then φ is a *monomorphism* or *embedding* of S into T. If it is surjective it will be called an *epimorphism.* Also, a homomorphism is called an *isomorphism* if it is bijective. If there is an epimorphism from S onto T, we say that T is a *homomorphic image* of S. We say that S and T are *isomorphic* if there is an isomorphism between S and T and we write $S \cong T$. A homomorphism from S to itself is called an *endomorphism* and an isomorphism from S to itself is called an *automorphism.*

The equivalence relation $\mathcal{R}$ on a semigroup S is defined by the rule that $a \,\mathcal{R}\, b$ if and only if $aS^1 = bS^1$. The relation $a \,\mathcal{L}\, b$ is defined dually. Also, we say that $a \,\mathcal{J}\, b$ if they generate the same principal two-sided ideal. The intersection of $\mathcal{L}$ and $\mathcal{R}$ is denoted by $\mathcal{H}$. It is easy to see that $\mathcal{R}$ and $\mathcal{L}$ are, respectively, a left congruence and a right congruence. It is a significant fact that $\mathcal{L}$ and $\mathcal{R}$ are commute, so that, consequently $\mathcal{D} = \mathcal{R} \circ \mathcal{L}$ is an equivalence and is the join of $\mathcal{L}$ and $\mathcal{R}$ in the lattice of equivalence relations on S. We recall that these equivalence relations are called *Green's relations.* The $\mathcal{R}$-class containing the element a will be written as R_a. Similarly for L_a, H_a, D_a and J_a.

We say that a semigroup S is *simple* if it consists of a single $\mathcal{J}$-class (0-simple if $S^2 \neq 0$ and $\{0\}$ and $S \setminus \{0\}$ are the only $\mathcal{J}$-classes). A *completely 0-simple* semigroup is a 0-simple semigroup with a primitive idempotent. A semigroup S is said to be *bisimple* if it consists of a single $\mathcal{D}$-class (a 0-bisimple semigroup is a semigroup with two $\mathcal{D}$-classes, $\{0\}$ and $S \setminus \{0\}$).

The next lemma gives an elementary characterization of $\mathcal{R}$ and $\mathcal{L}$.

Lemma 2.1.1. *Let S be a semigroup and let $a, b \in S$. Then:*

(1) $(a, b) \in \mathcal{R}$ *if and only if there exist $x, y \in S^1$ such that $ax = b,\ by = a$.*

(2) $(a, b) \in \mathcal{L}$ *if and only if there exist $x, y \in S^1$ such that $xa = b,\ yb = a$.*

There is a generalization of Green's relation $\mathcal{R}$ which is defined by the rule that $a \,\mathcal{R}^*\, b$ if and only if the elements a, b of S are related by Greens relation $\mathcal{R}$ in some oversemigroup of S. The relation $\mathcal{L}^*$ is defined dually. The join of the relations $\mathcal{R}^*$ and $\mathcal{L}^*$ is denoted by $\mathcal{D}^*$ and their intersection by $\mathcal{H}^*$. An alternative definition for $\mathcal{R}^*$ is given by the next lemma.

Lemma 2.1.2. *[31] Let S be a semigroup and let $a, b \in S$. Then the following conditions are equivalent:*

(1) $(a, b) \in \mathcal{R}^*$;

(2) *for all* $x, y \in S^1$, $xa = ya$ *if and only if* $xb = yb$.

It is easy to see from this lemma that $\mathcal{R}^*$ is a left congruence and $\mathcal{L}^*$ is a right congruence. This lemma can be simplified when one of the elements involved is an idempotent as follows:

Corollary 2.1.3. *[8] Let a be an element of a semigroup S, and $e \in E(S)$. Then the following are equivalent:*

(1) $a\,\mathcal{R}^*\,e$;

(2) $ea = a$ *and for all* $x, y \in S^1$, $xa = ya$ *implies* $xe = ye$.

A semigroup S in which each $\mathcal{L}^*$-class and each $\mathcal{R}^*$-class contains an idempotent will be called an *abundant* semigroup. If the idempotents commute in an abundant semigroup we call it an *adequate* semigroup and in this case the $\mathcal{R}^*$-class ($\mathcal{L}^*$-class) of $a \in S$ contains a unique idempotent, denoted by a^+ (a^*). If the idempotents of S commute and we insist only that each $\mathcal{R}^*$-class ($\mathcal{L}^*$-class) contains an idempotent, we call S a *left adequate* (*right adequate*) semigroup.

2.2 Inverse semigroups

An element a of a semigroup S is called a *regular* element if there exists an element x in S such that $axa = a$. The semigroup S is *regular* if its elements are regular. An *inverse semigroup* is a regular semigroup in which all the idempotents commute. Equivalently, an inverse semigroup is a semigroup S such that for all $a \in S$ there is a *unique* $b \in S$ such that $aba = a$ and $bab = b$. The element b is *the inverse* of a and is denoted by a^{-1}. It is worth noting that $(a^{-1})^{-1} = a$ and $(ab)^{-1} = b^{-1}a^{-1}$ for all $a, b \in S$.

If X is a non-empty subset of an inverse semigroup S, then the intersection X^* of all inverse subsemigroups of S containing X, is *the inverse subsemigroup* of S

generated by X. It is easy to see that $X^* = \langle X \cup X^{-1} \rangle$ where X^{-1} is the set of all inverses of elements of X.

For an inverse semigroup, each $\mathcal{R}$-class and each $\mathcal{L}$-class contains exactly one idempotent, that is, $aa^{-1} \in R_a$ and $a^{-1}a \in L_a$. Consequently, for every $a, b \in S$,

$$a \,\mathcal{R}\, b \text{ if and only if } aa^{-1} = bb^{-1},$$

$$a \,\mathcal{L}\, b \text{ if and only if } a^{-1}a = b^{-1}b$$

where a^{-1}, b^{-1} is the inverse of a, b in S, respectively.

An inverse semigroup S, possesses a natural partial order relation $\leq$ which is defined as follows. If $a, b \in S$, then

$$a \leq b \text{ if and only if } a = eb \text{ for some idempotent } e.$$

With respect to the natural partial order $E(S)$ is a meet semilattice with $e \wedge f = ef$.

A *transformation* on a set X is a function from X into itself. The set of all transformations on X is a semigroup under composition (from left to right) and denoted by $\mathcal{T}_X$ and we call it the *full transformation semigroup* on X. For any semigroup S and non-empty set X, a homomorphism ϕ from S into $\mathcal{T}_X$ is called a *representation* of S and ϕ is called a *faithful* representation if it is one to one. For any element a in a semigroup S the transformation $\rho_a\,[\lambda_a]$ defined by $x\rho_a = xa\,[x\lambda_a = ax]$ for all $x \in S^1$ is the (*inner*) *right* [*left*] *translation* of S corresponding to the element a of S, and the mapping $a \longrightarrow \rho_a$ of S into $\mathcal{T}_{S^1}$ is a faithful representation.

A *partial transformation* on a set X is a function α mapping a subset A of X into a subset B of X. The partial transformation semigroup $\mathcal{PT}_X$ consists of all partial maps of X and the operation is composition of partial mappings, where for $\alpha, \beta \in \mathcal{PT}_X$,

$$\operatorname{dom} \alpha\beta = (\operatorname{im} \alpha \cap \operatorname{dom} \beta)\alpha^{-1} \text{ and } \forall x \in \operatorname{dom} \alpha\beta,\ x(\alpha\beta) = (x\alpha)\beta.$$

The *symmetric inverse* semigroup on a set X, denoted by $\mathcal{I}_X$, is the set of all one-to-one partial transformations of a set X with multiplication as above. Note that $\alpha\beta = 0$ if $\operatorname{im} \alpha \cap \operatorname{dom} \beta = \emptyset$ where 0 is the empty transformation.

The natural partial order on $\mathcal{I}_X$ is easily seen to be the domain restriction of a partial map, that is:

$$\alpha \leq \beta \text{ if and only if } \operatorname{dom}(\alpha) \subseteq \operatorname{dom}(\beta) \text{ and } \alpha = \beta\,|_{\operatorname{dom}\alpha}.$$

We can look at any inverse semigroup as a subsemigroup of a symmetric inverse semigroup as we will note in the next theorem.

Theorem 2.2.1. (***Vagner-Preston***) *[25] Every inverse semigroup embeds in a suitable symmetric inverse semigroup.*

Lemma 2.2.2. *[23] In the symmetric inverse semigroup;*

(1) $\alpha\,\mathcal{L}\,\beta$ *if and only if* $\operatorname{im}\alpha = \operatorname{im}\beta$,

(2) $\alpha\,\mathcal{R}\,\beta$ *if and only if* $\operatorname{dom}\alpha = \operatorname{dom}\beta$.

2.3 Ample semigroups

Let $\mathcal{I}_X$ be the symmetric inverse monoid on a set X. We can define three unary operations $^{-1}$,$^{+}$ and * as follows: for each $\alpha \in \mathcal{I}_X$,

$$\alpha^{-1} \text{ is the inverse of } \alpha; \alpha^{+} = \alpha\alpha^{-1} \text{ and } \alpha^{*} = \alpha^{-1}\alpha.$$

Let S be a subsemigroup of $\mathcal{I}_X$. If S closed under $^{+}$, then S is a *left ample semigroup* which may be regarded as a (2,1)-subalgebra of $\mathcal{I}_X$, with unary operation $^{+}$. *Right ample semigroups* may be defined in a similar way by considering closure under *. An *ample semigroup* is one which is both left and right ample. It is worth pointing out that an ample semigroup S may not be embeddable in an inverse semigroup in way that respects both $^{+}$ and * (see [16]).

Alternatively, we can define a semigroup S to be left (right) ample if and only if every $\mathcal{R}^*$-class ($\mathcal{L}^*$-class) contains an idempotent, $E(S)$ is subsemilattice of S and S satisfies the left (right) ample condition which is:

$$(ae)^{+}a = ae \quad (a(ea)^{*} = ea) \text{ for all } a \in S \text{ and } e \in E(S)$$

where, for $x \in S$, x^{+} (resp. x^{*}) is the unique idempotent in the $\mathcal{R}^*$-class (resp. $\mathcal{L}^*$-class) of x.

A semigroup is ample if it is both left and right ample. We can note easily that, any inverse semigroup is ample and right cancellative monoids are left ample. By a right (left) cancellative semigroup we mean, a semigroup S such that for all $x, y, z \in S$

$$xz = yz \text{ implies } x = y \ (zx = zy \text{ implies } x = y).$$

It is clear that every right cancellative monoid contains exactly one idempotent.

Following [18], for any left ample semigroup S we can construct an embedding of S into the symmetric inverse semigroup $\mathcal{I}_S$ as follows. For each $a \in S$ we let $\rho_a \in \mathcal{I}_S$ be given by

$$\operatorname{dom} \rho_a = Sa^+ \text{ and } \operatorname{im} \rho_a = Sa$$

and for any $x \in \operatorname{dom} \rho_a$,

$$x\rho_a = xa.$$

Then the map $\theta_S : S \to \mathcal{I}_S$ is a $(2,1)$ embedding.

Following [16], let S and T be two semigroups. We say that a homomorphism $\phi : S \longrightarrow T$ *preserves* $\mathcal{R}^*$, if for any $a, b \in S$,

$$a \,\mathcal{R}^*\, b \text{ implies that } a\phi \,\mathcal{R}^*\, b\phi.$$

From Proposition 1.2 of [5] we have that θ_S preserves $\mathcal{R}^*$.

The inverse hull of a left ample semigroup S is the inverse subsemigroup of $\mathcal{I}_S$ generated by $S\theta_S$. If S is a right cancellative monoid, then for any $a \in S$ we have $a^+ = 1$. Then $\rho_a : S \longrightarrow Sa$ is defined by

$$x\rho_a = xa \quad \text{for each } x \text{ in } S.$$

Hence $\operatorname{dom} \rho_a = S = \operatorname{dom} I_S$, giving that $\operatorname{im} \theta_S \subseteq R_1$ where R_1 is the $\mathcal{R}$-class of I_S in $\mathcal{I}_S$.

We should mention that, these representations are taken from [1] and [18] respectively.

The next lemma gives some elementary properties of left ample semigroups. Proofs can be found in [8] or [18].

Lemma 2.3.1. *Let S be a left ample semigroup. If $a, b \in S$, then*

(1) $(a^+b)^+ = a^+b^+$,

(2) $a^+a = a$,

(3) *for any idempotent e in S and every element a in S, $Sa \cap Se = Sae$.*

2.4 Some special classes of inverse semigroups

The purpose of this section is to introduce some classes of inverse semigroups, for which we consider their left I-orders in the next chapters. We refer the reader into [23] and [1] for more information about such semigroups.

2.4.1 Bisimple inverse semigroups

We recall that a semigroup S (without zero) having a single $\mathcal{D}$-class is said to be bisimple. A semigroup S with a zero element 0 is called 0-bisimple if it contains two $\mathcal{D}$-classes, namely $S \setminus \{0\}$ and $\{0\}$.

Let S be a semigroup with an identity 1. If a and b are elements of S such that $ab = 1$, then we call a a *right unit* and b a *left unit* of S. An element which is both a left and a right unit is called a *unit.*

Let S be an inverse semigroup. The idempotent $e = aa^{-1}$ $(f = a^{-1}a)$ will be called the *left* (*right*) *unit* of a.

Lemma 2.4.1. *[1] Let S be an inverse semigroup. Then S is bisimple if and only if for any two idempotents e, f in S there exists an element of S with left unit e and right unit f.*

Clifford [1] in 1953, showed that every bisimple inverse monoid may be constructed from a right cancellative monoid satisfying the condition that the intersection of any two principal left ideals is again a principal left ideal. We shall call such a condition the (LC) condition. He proved the following theorem [1]:

Theorem 2.4.2. *(Clifford, 1953) Let Q be a bisimple inverse monoid with identity 1 and let $R = R_1$, the $\mathcal{R}$-class of 1. Then R is a right cancellative monoid and the intersection of any two principal left ideals is a principal left ideal, i.e. for each $a, b \in R$, there exists $c \in R$ such that $Ra \cap Rb = Rc$. Conversely, let S be a right cancellative monoid in which the intersection of any two principal left ideals is a principal left ideal and let $\Sigma(S)$ be the inverse hull of S. Then $\Sigma(S)$ is bisimple inverse monoid such that the $\mathcal{R}$-class of 1 is isomorphic to S.*

We should point out that Reilly [41] has shown that the structure of any bisimple inverse semigroup with or without identity is determined by any of its $\mathcal{R}$-classes regarded as RP-system, that is, an ordered pair (R, P) where R is a partial semigroup and P is a subsemigroup of R satisfying certain conditions.

The inverse hull of a right cancellative monoid is an important tool in describing the structure of bisimple inverse monoids. Let S be a right cancellative monoid with the (LC) condition. The mapping $\rho : S \longrightarrow \Sigma(S)$ given by

$$a\rho = \rho_a$$

is an embedding of S into $\Sigma(S)$ and $S\rho = R_1$ (see [1]), that is, $S\rho$ is the $\mathcal{R}$-class of the identity of $\Sigma(S) = \langle S\rho \cup (S\rho)^{-1} \rangle$. In order to show that $\Sigma(S) = (S\rho)^{-1}(S\rho)$ we show that every element in $\Sigma(S)$ can be written in the form $\rho_a^{-1}\rho_b$ for some $a, b \in S$. In other words, for any $\rho_c, \rho_d \in S\rho$ the element $\rho_c\rho_d^{-1} = \rho_a^{-1}\rho_b$ for some $\rho_a, \rho_b \in S\rho$. Now let $\rho_c\rho_d^{-1}$ be in $\Sigma(S)$. Then $Sc \cap Sd = Sw$ for some $w \in S$ as S has (LC). Then $w = tc = rd$ for some $t, r \in S$ and $\rho_w = \rho_t\rho_c = \rho_r\rho_d$ in $S\rho$. As $\rho_c^{-1}\rho_c, \rho_d^{-1}\rho_d$ and $\rho_w^{-1}\rho_w$ are the identity mappings on Sc, Sd and Sw respectively. We have

$$\rho_c^{-1}\rho_c\rho_d^{-1}\rho_d = I_{Sc \cap Sd} = I_{Sw} = \rho_w^{-1}\rho_w.$$

Since $\rho_t = \rho_w\rho_c^{-1}$ and $\rho_r = \rho_w\rho_d^{-1}$, we get

$$\rho_t^{-1}\rho_r = \rho_c\rho_w^{-1}\rho_w\rho_d^{-1} = \rho_c\rho_c^{-1}\rho_c\rho_d^{-1}\rho_d\rho_d^{-1} = \rho_c\rho_d^{-1}.$$

Thus $\rho_c\rho_d^{-1} = \rho_t^{-1}\rho_r$ as we desired.

The idea for the above argument came from the next Proposition due to Fountain and Kambites [10]. They have done some modification on Theorem 2.4.2 as follows:

Proposition 2.4.3. *The following are equivalent for a right cancellative monoid S:*

(1) *$\Sigma(S)^0$ is 0-bisimple;*

(2) *The domain of each non-zero element of $\Sigma(S)^0$ is a principal left ideal;*

(3) *Every non-zero element of $\Sigma(S)^0$ can be written in the form $\rho_a^{-1}\rho_b$ for some $a, b \in S$;*

(4) *for any $a, b \in S$, either $Sa \cap Sb = \emptyset$ or $Sa \cap Sb = Sc$ for some $c \in S$.*

A bisimple inverse ω-semigroup is a bisimple inverse semigroup whose idempotents form an ω-chain, that is, $E(S) = \{e_m : m \in \mathbb{N}^0\}$ where $e_0 \geq e_1 \geq e_2 \geq \ldots$. Thus if S is a bisimple inverse ω-semigroup, on $E(S)$ we have

$$e_m \leq e_n \text{ if and only if } m \geq n.$$

Reilly [40] determined the structure of all bisimple inverse ω-semigroups as follows: Let G be a group and let θ be an endomorphism of G. Let $\mathbf{BR}(G, \theta)$ be the semigroup on $\mathbb{N}^0 \times G \times \mathbb{N}^0$ with multiplication

$$(m, g, n)(p, h, q) = \big(m - n + t, (g\theta^{t-n})(h\theta^{t-p}), q - p + t\big)$$

where t=max$\{n, p\}$ and θ^0 is interpreted as the identity map of G. As was shown in [40] (cf [23]), $\mathbf{BR}(G, \theta)$ is a bisimple inverse ω-semigroup and every bisimple inverse ω-semigroup is isomorphic to some $\mathbf{BR}(G, \theta)$. In the case where G is trivial, then $\mathbf{BR}(G, \theta) = \mathcal{B}$ where $\mathcal{B}$ the *bicyclic semigroup*, which we identify with $\mathbb{N}^0 \times \mathbb{N}^0$. The set of idempotents of $\mathbf{BR}(G, \theta)$ is $\{(m, 1, m); m \in \mathbb{N}^0\}$ and for any (m, g, n) in $\mathbf{BR}(G, \theta)$,

$$(m, g, n)^{-1} = (n, g^{-1}, m).$$

For any $(m, a, n), (p, b, q) \in \mathbf{BR}(G, \theta)$,

$$(m, a, n)\,\mathcal{R}\,(p, b, q) \text{ if and only if } m = p,$$

$$(m, a, n)\,\mathcal{L}\,(p, b, q) \text{ if and only if } n = q,$$

and, consequently,

$$(m, a, n)\,\mathcal{H}\,(p, b, q) \text{ if and only if } m = p \text{ and } n = q.$$

It follows that $\mathcal{H}$ is the identity relation on $\mathcal{B}$.

2.4.2 Clifford semigroups

A *Clifford semigroup* is an inverse semigroup with central idempotents. Before introducing an alternative definition for such semigroup, we need the following definitions from [1].

Definition 2.4.4. Let Y be a semilattice. A semigroup S is called a *semilattice* Y of semigroups $S_\alpha, \alpha \in Y$, if $S = \bigcup_{\alpha \in Y} S_\alpha$ where $S_\alpha S_\beta \subseteq S_{\alpha\beta}$ and $S_\alpha \cap S_\beta = \emptyset$ if $\alpha \neq \beta$ and we write $S = \mathcal{S}(Y; S_\alpha)$.

If each S_α is a right cancellative monoid, we say that S is a semilattice of right cancellative monoids.

The above structure is a gross structure of S but not fine. For, if $x \in S_\alpha, y \in S_\beta$, then we know that $xy \in S_{\alpha\beta}$, but we do not know what it looks like. We can cope with this by the following fine structure:

Definition 2.4.5. Let Y be a semilattice. To each $\alpha \in Y$ associate a semigroup S_α and assume that $S_\alpha \cap S_\beta = \emptyset$ if $\alpha \neq \beta$. For each pair $\alpha, \beta \in Y, \alpha \geq \beta$, let
$\varphi_{\alpha,\beta} : S_\alpha \longrightarrow S_\beta$ be a homomorphism such that the following conditions hold:

(1) $\varphi_{\alpha,\alpha} = \iota_{S_\alpha}$,

(2) $\varphi_{\alpha,\beta}\varphi_{\beta,\gamma} = \varphi_{\alpha,\gamma}$ if $\alpha \geq \beta \geq \gamma$.

On the set $S = \bigcup_{\alpha \in Y} S_\alpha$ define a multiplication by

$$a * b = (a\varphi_{\alpha,\alpha\beta})(b\varphi_{\beta,\alpha\beta})$$

if $a \in S_\alpha, b \in S_\beta$

With respect to this multiplication S is a semigroup, called a *strong semilattice* of semigroups S_α and denote $S = \mathcal{S}(Y; S_\alpha; \varphi_{\alpha,\beta})$.

The next theorem from [23] gives an alternative description of Clifford semigroups.

Theorem 2.4.6. *Let S be a semigroup with set E of idempotents. Then the following statements are equivalent:*

(1) *S is a Clifford semigroup;*

(2) *S is a semilattice of groups;*

(3) *S is a strong semilattice of groups;*

(4) *S is regular, and the idempotents of S are central;*

(5) *S is regular, and $\mathcal{D}^S \cap (E \times E) = 1_E$.*

We should point out that not every semilattice of semigroups is a strong semilattice of semigroups (see for example [43]).

2.4.3 E-unitary inverse semigroups

We say that an inverse semigroup S is *E-unitary* (or *proper*) if $E(S)$ is a *unitary* subsemigroup, that is, for all a in S and e in $E(S)$,

$$ae \in E(S) \text{ implies that } a \in E(S).$$

Equivalently,

$$ea \in E(S) \text{ implies that } a \in E(S).$$

Thus S is proper if an element above an idempotent in the natural partial order, is also an idempotent.

We now describe the well known *McAlister's P-Theorem.* Let $\mathcal{X}$ be a partially ordered set and let $\mathcal{Y}$ be a subset of $\mathcal{X}$ with the following properties.

(1) $\mathcal{Y}$ is a lower semilattice, that is, every pair of elements A, B in $\mathcal{Y}$ has a greatest lower bound $A \wedge B$ in $\mathcal{Y}$;

(2) $\mathcal{Y}$ is an order ideal of $\mathcal{X}$, that is, for A, B in $\mathcal{X}$, if A is in $\mathcal{Y}$ and $B \leq A$, then B is in $\mathcal{Y}$.

Now let G be a group which acts on $\mathcal{X}$ (on the left), by order automorphisms. This means that there exists a function $G \times \mathcal{X} \longrightarrow \mathcal{X}$, in notation $(g, \alpha) \longrightarrow g\alpha$, such that

(i) for all A, B in $\mathcal{X}$, $A \leq B$ if and only if $gA \leq gB$;

(ii) for all A in $\mathcal{X}$, $1A = A$;

(*iii*) for all g, h in G and all A in $\mathcal{X}$, $g(hA) = (gh)A$.

The triple $(G, \mathcal{X}, \mathcal{Y})$ is also assumed to have the following properties:

(3) for every X in $\mathcal{X}$, there exists g in G and an A in $\mathcal{Y}$ such that $gA = X$;
(4) for all $g \in G$, $g\mathcal{Y} \cap \mathcal{Y} \neq \emptyset$.

The triple $(G, \mathcal{X}, \mathcal{Y})$ have the above properties is a *McAlister triple.* Moreover,

$$\mathcal{M}(G, \mathcal{X}, \mathcal{Y}) = \{(A, g) \in \mathcal{Y} \times G : g^{-1}A \in \mathcal{Y}\}$$

is an inverse semigroup with the multiplication $(A, g)(B, h) = (A \wedge gB, gh)$, such that $(A, g)^{-1} = (g^{-1}A, g^{-1})$. Semigroups of the form $\mathcal{M}(G, \mathcal{X}, \mathcal{Y})$ are called *P-semigroups.* One of the main results in the study of E-unitary inverse semigroups is McAlister's P-Theorem:

Theorem 2.4.7. *[23, 32, 33] Let $(G, \mathcal{X}, \mathcal{Y})$ be a McAlister triple. Then $\mathcal{M}(G, \mathcal{X}, \mathcal{Y})$ is an E-unitary inverse semigroup. Conversely, every E-unitary inverse semigroup is isomorphic to one of this kind.*

Munn [34] showed that the relation

$$\sigma = \{(a, b) \in S \times S : ea = eb \text{ for some } e^2 = e \in S\}$$

is the *minimum group congruence* on any inverse semigroup S, that is, σ is the smallest congruence on S such that S/σ is a group.

We now give some an alternative condition for an inverse semigroup to be proper.

Proposition 2.4.8. *[33] The following are equivalent for an inverse semigroup S:*
(1) *S is proper;*
(2) *$\sigma \cap \mathcal{R} = I_S$, where I_S is the identity relation on S.*

2.4.4 Primitive inverse semigroups

An inverse semigroup S with zero such that $S \neq \{0\}$ is a *primitive inverse* semigroup if all its nonzero idempotents are primitive.

Lemma 2.4.9. *[21] Let S be a primitive inverse semigroup. Then for $e \in E(S)$ and $s \in S$, if $es \neq 0$, then $es = s$. Similarly, $se \neq 0$ implies $se = s$.*

Let I, Λ be non-empty sets and let G be a group. Suppose that $P = (p_{\lambda i})$ is a $\Lambda \times I$ matrix over the 0-group $G^0(= G \cup \{0\})$. We call P a sandwich matrix. We say that P is regular if no row or column of P consists entirely of zeros, that is,

$$\text{for all } i \in I \text{ there exists } \lambda \in \Lambda \text{ such that } p_{\lambda i} \neq 0$$

$$\text{for all } \lambda \in \Lambda \text{ there exists } i \in I \text{ such that } p_{\lambda i} \neq 0.$$

We assume that P is regular.
Now let $S = (I \times G \times \Lambda) \cup \{0\}$, and define a multiplication on S by

$$(i, a, \lambda)(j, b, \mu) = \begin{cases} (i, ap_{\lambda j}b, \mu) & \text{if } p_{\lambda j} \neq 0 \\ 0 & \text{if } p_{\lambda j} = 0, \end{cases}$$

$$(i, a, \lambda)0 = 0(i, a, \lambda) = 00 = 0 \quad ....(*).$$

The semigroup S is defined by $\mathcal{M}^0(G; I, \Lambda; P)$. This structure is due to Rees (1940) and we call such a semigroup a *Rees matrix semigroup.*

Theorem 2.4.10. *[1] Let G^0 be a $\Lambda \times I$ matrix with entries in G^0 be a 0-group, let I, Λ be non-empty sets and let $P = (p_{\lambda i})$ be a $\Lambda \times I$ matrix with entries in G^0. Suppose that P is regular. Let $S = (I \times G \times \Lambda) \cup \{0\}$, and define a multiplication on S by (*). Then S is a completely 0-simple semigroup. Conversely, every completely 0-simple semigroup is isomorphic to one constructed in this way.*

Let $S = \mathcal{M}^0(G; I, \Lambda; P)$ be a Rees matrix semigroup. If $I = \Lambda$ and every row and every column of P contains exactly one non-zero entry, then S is inverse. In this case $S \cong \mathcal{M}^0(G; I, I; \Delta)$ where Δ is the $I \times I$ identity matrix (see [1]). We define a *Brandt semigroup* to be a completely 0-simple inverse semigroup. In [1], it is pointed out that a semigroup will be a Brandt semigroup if and only if it is isomorphic to some $\mathcal{M}^0(G; I, I; \Delta)$.

The next lemma [1] links many concepts together.

Lemma 2.4.11. *A semigroup S with zero is completely 0-simple if and only if it satisfies the following three conditions:*

(i) S is regular;

(ii) every non-zero idempotent of S is primitive;

(iii) if e and f are non-zero idempotents of S, then $eSf \neq 0$.

Note that every Brandt semigroup is a primitive inverse semigroup.

Let $\{S_i : i \in I\}$ be a family of disjoint semigroups with zero, and put $S_i^* = S \setminus \{0\}$. Let $S = \bigcup_{i \in I} S_i^* \cup \{0\}$ with the multiplication

$$a * b = \begin{cases} ab \text{ if } a, b \in S_i \text{ for some } i \text{ and } ab \neq 0 & \text{in } S_i; \\ 0, \qquad\qquad \text{otherwise.} \end{cases}$$

With this multiplication S is a semigroup called *a 0-direct union* of the S_i. In [25], it is shown that every primitive inverse semigroup is a 0-direct union of Brandt semigroups.

2.5 Categories

This section is divided into two parts. The first part contains definitions in category theory and the relationship between monoids and categories. In the second part we use category theory to find the relationship between the class of right cancellative monoids with the (LC) condition and the class of bisimple inverse monoids by supplying both classes with suitable homomorphisms. The main source for the definitions and results of this section is [38].

A *set* is a class which is a member of another class, a *proper class* is a class which is not a set. An example is the class of all sets that are not members of themselves:

$$\mathcal{U} = \{S : S \text{ is a set and } S \notin S\}.$$

If $\mathcal{U}$ is a set, then $\mathcal{U}$ is either a member of itself or not.

If $\mathcal{U} \in \mathcal{U}$, then by the defining condition of $\mathcal{U}$ we have $\mathcal{U} \notin \mathcal{U}$, a contradiction. Therefore, $\mathcal{U} \notin \mathcal{U}$, but now by the defining condition we have $\mathcal{U} \in \mathcal{U}$, a contradiction. Thus $\mathcal{U}$ is *not* a set.

Definition 2.5.1. A category $\mathbf{C}$ is given by

(1) a class $\mathrm{Ob}\,\mathbf{C}$ of objects;

(2) a class $\mathrm{Hom}\,\mathbf{C}$ of homomorphisms; for each pair (u,v) of objects, a set $\mathrm{Hom}\,(u,v)$ of homomorphisms (arrows) such that $\mathrm{Hom}\,\mathbf{C} = \bigcup_{u,v\in \mathrm{Ob}\,\mathbf{C}} \mathrm{Hom}\,(u,v)$;

(3) for each triple (u,v,w) of objects, a mapping from $\mathrm{Hom}\,(u,v) \times \mathrm{Hom}\,(v,w)$ into $\mathrm{Hom}\,(u,w)$ which associates to each $p \in \mathrm{Hom}\,(u,v)$ and $q \in \mathrm{Hom}\,(v,w)$ the composition $pq \in \mathrm{Hom}\,(u,w)$. Composition is assumed to be associative.

(4) for each object u, there exists a homomorphism $1_u \in \mathrm{Hom}\,(u,u)$, called the *identity homomorphism* at u, such that for each pair (u,v) of objects, for each $p \in \mathrm{Hom}\,(u,v)$ and $q \in \mathrm{Hom}\,(v,u)$, $1_u p = p$ and $q1_u = q$.

We write $q : u \longrightarrow v$ as an alternative to $q \in \mathrm{Hom}\,(v,u)$.

A *subcategory* $\mathbf{S}$ of a category $\mathbf{C}$ is a category $\mathbf{S}$ whose objects are objects in $\mathbf{C}$ and whose homomorphisms are homomorphisms in $\mathbf{C}$ with the same identities and composition of homomorphisms, satisfying the following conditions;

(1) if $a \in \mathrm{Ob}\,\mathbf{S}$, then $1_a \in \mathrm{Hom}\,\mathbf{S}$;

(2) if $\alpha, \beta \in \mathrm{Hom}\ \mathbf{S}$ and $\alpha\beta$ is defined in $\mathbf{C}$, then $\alpha\beta \in \mathrm{Hom}\,\mathbf{S}$;

(3) if $\alpha \in \mathrm{Hom}\,\mathbf{S}$ and $\alpha : a \longrightarrow b$, then $a, b \in \mathrm{Ob}\,\mathbf{S}$.

Let S be a monoid and A any object, for each $r \in S$ we associate an arrow $A \xrightarrow{r} A$. The arrows of the category correspond to the elements of S. The composition of arrows is just the binary operation of S. The identity arrow $1_A = 1$ is just the unit of S. This construction does produce a category. Thus A monoid $(S, ., 1)$ can be viewed as a category with just one object.

We can look at the category as algebraic structure in it is own right as we will see in the following definition from [38].

Definition 2.5.2. Let $\mathbf{C}$ be a set equipped with a partial binary operation. If $x, y \in \mathbf{C}$ and the product xy is defined we write $\exists xy$. An element $e \in \mathbf{C}$ is called an *identity* if $\exists ex$ implies $ex = x$ and $\exists xe$ implies $xe = x$. The set of identities of $\mathbf{C}$ is denoted by $\mathbf{C}_0$. The set $\mathbf{C}$ is said to be a *category* if the following three axioms hold:

(1) $x(yz)$ exists if and only if $(xy)z$ exists in which case they are equal.

(2) $x(yz)$ exists if and only if xy and yz exist.

(3) For each $x \in \mathbf{C}$ there exist identities $e, f \in \mathbf{C}$ such that $\exists xe$ and $\exists fx$.

Before we give the definition of an equivalence of categories, we first need to introduce the following definitions.

Definition 2.5.3. A functor F from a category $\mathbf{C}$ to a category $\mathbf{D}$ is a pair of mappings, one from $\operatorname{Ob}\mathbf{C}$ to $\operatorname{Ob}\mathbf{D}$ and the another from $\operatorname{Hom}\mathbf{C}$ to $\operatorname{Hom}\mathbf{D}$ and satisfying the following conditions ;

(i) $F1_a = 1_{Fa}$ for all $a \in \operatorname{Ob}\mathbf{C}$,

(ii) if $\alpha : a \longrightarrow b$, then $F\alpha : Fa \longrightarrow Fb$ and if $\beta : b \longrightarrow c$, then $F(\alpha\beta) = (F\alpha)(F\beta)$.

Definition 2.5.4. Let F and G be functors from a category $\mathbf{C}$ to a category $\mathbf{D}$. A natural transformation $\tau : F \longrightarrow G$ of functors F and G is a function mapping $\operatorname{Ob}\mathbf{C}$ into $\operatorname{Hom}\mathbf{D}$ such that

1. $\tau(a) : Fa \longrightarrow Ga$ where $a \in \operatorname{Ob}\mathbf{C}$, $\tau(a) \in \operatorname{Hom}\mathbf{D}$

2. if $\alpha : a \longrightarrow b$ is a homomorphism, then the diagram

$$\begin{tikzcd} Fa \arrow[r, "F\alpha"] \arrow[d, "\tau(a)"'] & Fb \arrow[d, "\tau(b)"] \\ Ga \arrow[r, "G\alpha"] & Gb \end{tikzcd}$$

commutes.

If also $\tau(a)$ is an isomorphism for each $a \in \operatorname{Ob}\mathbf{C}$, then τ is a *natural equivalence* of the functors F and G.

Definition 2.5.5. A quadruple (F,G,σ, τ) is an equivalence of the categories $\mathbf{C}$ and $\mathbf{D}$ if

(i) F is a functor from $\mathbf{C}$ to $\mathbf{D}$,

(ii) G is a functor from $\mathbf{D}$ to $\mathbf{C}$,

(iii) $\sigma : I_{\mathbf{C}} \longrightarrow GF$ is a natural equivalence,

(iv) $\tau : I_{\mathbf{D}} \longrightarrow FG$ is a natural equivalence.

Theorem 2.4.2, due to Clifford, in Section 2.4.2, shows that every bisimple inverse monoid gives rise to a right cancellative monoid with the (LC) condition (its $\mathcal{R}$-class containing the identity element). Conversely from such a monoid we can construct a bisimple inverse monoid. We will supply these classes of semigroups with suitable homomorphisms to allow us to build categories and we will show these categories are equivalent.

Notation 2.5.6. *Let a category* $\mathbf{B}$ *be given by*

$\mathrm{Ob}\,\mathbf{B}$ *are bisimple inverse monoids,*

$\mathrm{Hom}\,\mathbf{B}$ *are monoid homomorphisms of* $\mathrm{Ob}\,\mathbf{B}$.

Notation 2.5.7. *Let a category* $\mathbf{R}$ *be given by*

$\mathrm{Ob}\,\mathbf{R}$ *are right cancellative monoids with the (LC) condition,*

$\mathrm{Hom}\,\mathbf{R}$ *are monoid homomorphisms of* $\mathrm{Ob}\,\mathbf{R}$ *say* $\varphi : R \longrightarrow R'$, *such that if* $Ra \cap Rb = Rc$ *for some* $a, b, c \in R$, *then* $R'(a\varphi) \cap R'(b\varphi) = R'(c\varphi)$.

It is worth pointing out that in Chapter 4, homomorphisms in $\mathrm{Hom}\,\mathbf{R}$, will be called (LC)-preserving and we define this notation in a more general context.

From Theorem 2.4.2, if $S \in \mathrm{Ob}\,\mathbf{B}$, then its $\mathcal{R}$-class of the identity is in $\mathrm{Ob}\,\mathbf{R}$ and in fact this is the first part of a proof of the following lemma from [38].

Lemma 2.5.8. *For every* $S \in \mathrm{Ob}\,\mathbf{B}$, *let* $U(S)$ *be the* $\mathcal{R}$*-class of identity of* S, *and for every* $\varphi \in \mathrm{Hom}\,\mathbf{B}$ *say* $\varphi : S \longrightarrow S'$ *let* $U(\varphi) = \varphi\,|_{U(S)}$. *Then* U *is a functor from* $\mathbf{B}$ *to* $\mathbf{R}$.

Now our aim is to define a functor from $\mathbf{R}$ to $\mathbf{B}$ and we know in advance from Theorem 2.4.2, that if $S \in \mathrm{Ob}\,\mathbf{R}$, then its inverse hull is a bisimple inverse monoid $\Sigma(S)$. Let $R \in \mathrm{Ob}\,\mathbf{R}$. Then on $R \times R$ define a relation τ by

$$(a, b)\tau(c, d) \Leftrightarrow a = uc, b = ud \text{ for some unit } u \text{ of } R.$$

Then τ is an equivalent relation. We denote the $\tau-$class containing (a, b) by $[a, b]$ and let

$$R^{-1} \circ R = \{[a, b]; a, b \in R\}.$$

Since $R \in \mathrm{Ob}\,\mathbf{R}$, for any $a, b \in R$ we have that $Ra \cap Rb = Rc$ for some $c \in R$, then let $a \vee b$ denote the representative of the $\mathcal{L}$-class of c and we define $a * b$ by

$$(a * b)b = a \vee b = b \vee a = (b * a)a.$$

Now we can define a multiplication on $R^{-1} \circ R$ by

$$[a, b][c, d] = [(c * b)a, (b * c)d].$$

In [38] it is shown that $R^{-1} \circ R$ is isomorphic to $\Sigma(R)$. Hence we get the first part of the next lemma.

Lemma 2.5.9. *Let $R \in \mathrm{Ob}\,\mathbf{R}$ and put $V(R) = R^{-1} \circ R$. For any $\varphi \in \mathrm{Hom}\,\mathbf{R}$, say $\varphi : R \longrightarrow R'$, let $V(\varphi) : [a, b] \longrightarrow [a\varphi, b\varphi]$ ($[a, b] \in R^{-1} \circ R$). Then V is a functor from $\mathbf{R}$ to $\mathbf{B}$.*

Let $S \in \mathrm{Ob}\,\mathbf{B}$, with $\mathcal{R}$-class of the identity $U(S)$. Then in view of left order as in the argument precedes Proposition 2.4.3, we have that $S = U(S)^{-1}U(S)$ and we can say that the mapping in the next lemma is defined on all of S.

Lemma 2.5.10. *For any $S \in \mathrm{Ob}\,\mathbf{B}$, define $\xi(S) : S \longrightarrow VU(S)$ such that $\xi(S) : a^{-1}b \longmapsto [a, b]$ where $a, b \in U(S)$. Then ξ is a natural equivalence of the functor $I_{\mathbf{B}}$ and VU.*

$$\begin{tikzcd} S \arrow[r, "\varphi"] \arrow[d, "\xi(S)"'] & S' \arrow[d, "\xi(S')"] \\ VU(S) \arrow[r, "VU(\varphi)"] & VU(S') \end{tikzcd}$$

Lemma 2.5.11. *For any $R \in \mathrm{Ob}\,\mathbf{R}$, define $\eta(R) : R \longrightarrow UV(R)$ such that $\eta(R) : r \longmapsto [1, r]$ ($r \in R$). Then η is a natural equivalence of the functor $I_{\mathbf{R}}$ and UV.*

$$\begin{tikzcd} R \arrow[r, "\varphi"] \arrow[d, "\eta(R)"'] & R' \arrow[d, "\eta(R')"] \\ UV(R) \arrow[r, "UV(\varphi)"] & UV(R') \end{tikzcd}$$

Theorem 2.5.12. *[38] The quadruple (U, V, ξ, η) is an equivalence of categories $\mathbf{B}$ and $\mathbf{R}$.*

Chapter 3

Left I-orders

In this chapter we will introduce the basic definitions and give some examples of left I-orders. We concentrate on straight left I-orders. We give some properties of such left I-orders which are used in the proof of Theorem 3.2.9. One consequence of this theorem is that we can determine necessary and sufficient conditions for two semigroups of straight left I-quotients of a given semigroup to be isomorphic. The point of working with straight left I-quotients is that many cases of left I-quotients are straight, as we will see in the following chapters.

3.1 Definitions and examples

Ore and Dubreil [1] showed that, if S is a subsemigroup of a group G, then G is a group of left quotients of S in the following sense: every element of G may be expressed as $a^{-1}b$ for $a, b \in S$, if and only if S is a right reversible, that is, for all $a, b \in S$, we have that $Sa \cap Sb \neq \emptyset$, and S is cancellative. Fountain and Petrich [6] have extended this to a special class of semigroup and they provided the first formal definition of a semigroup of left quotients, but they restricted their attention to orders in completely 0-simple semigroups. The idea is that for a semigroup S to be a left order in a completely 0-simple semigroup Q, every element in Q can be written as $a^{\sharp}b$ where $a, b \in S$ and $a^{\sharp}$ is the inverse of a in a subgroup of Q. This definition has been extended to the class of all semigroups [17]. The idea is that a subsemigroup S of a semigroup Q is a *left order* in Q or Q is a *semigroup of left quotients* of S if every element of Q can be written as $a^{\sharp}b$ where $a, b \in S$ and $a^{\sharp}$

is the inverse of a in a subgroup of Q and if, in addition, every *square-cancellable* element of S (an element a of a semigroup S is square-cancellable if $a\,\mathcal{H}^*\,a^2$) lies in a subgroup of Q. *Semigroups of right quotients* and *right orders* are defined dually. If S is both a left order and a right order in a semigroup Q, then S is an *order* in Q and Q is a semigroup of *quotients* of S. This definition and its dual were used in [17] to characterize semigroups which have bisimple inverse ω-semigroups of left quotients. In the case of a completely 0-simple semigroup of quotients Q, if $a\,\mathcal{H}^*\,a^2$, that is, a is a square-cancellable element, then as $H_0^* = \{0\}$, we have that if $a \neq 0$, then $a^2 \neq 0$, so that from the structure of Q, we know $a\,\mathcal{H}\,a^2$ and so a must be in a subgroup of Q.

In the case of inverse semigroups, the notion of quotients has been effectively defined by a number of authors without being made fully explicit. The first one was introduced by Clifford [2] where he showed that from any right cancellative monoid S with (LC) there is a bisimple inverse monoid Q such that $Q = S^{-1}S$; that is, every element q in Q can be written as $a^{-1}b$ where $a, b \in S$, and a^{-1} is the inverse of a in Q in the sense of inverse semigroup theory. By saying that a semigroup S has the (LC) *condition* we mean that for any $a, b \in S$ there is an element $c \in S$ such that $Sa \cap Sb = Sc$. Thus, (LC) is a rather stronger condition than right reversibility.

McAlister has introduced two concepts of quotients in [28] as follows:

Concept 1: Let Q be an inverse semigroup and S a subsemigroup of Q. Call Q a *semigroup of quotients* of S if every element in Q can be written as $ab^{-1}c$ where $a, b, c \in S$.

Concept 2: With Q and S as above, say Q is a *semigroup of strong quotients* of S if every element in Q can be written as $ab^{-1}c$ where $a, b, c \in S$ and $b \in Sa \cap cS$.

We now give the formal definition of a semigroup of left I-quotients. Let S be a subsemigroup of an inverse semigroup Q. Then we say that S is a *left I-order* in Q or Q is a semigroup of *left I-quotients* of S, if every element of Q can be written as $a^{-1}b$ where a and b are elements of S. We stress that a^{-1} is the inverse of a in the sense of inverse semigroup theory. *Right I-orders* and *semigroups of right I-quotients* are defined dually. If S is both a left I-order and a right I-order in Q, then we say that S is an *I-order* in Q, and Q is a semigroup of *I-quotients* of S.

We remark that if a semigroup S is a left order in Q in the sense of [11, 17] (in particular Q is inverse), then it is certainly a left I-order. For, if S is a left order in Q, then we insist that any $q \in Q$ can be written as $q = a^{\sharp}b$ where $a, b \in S$ and $a^{\sharp}$ is the inverse of a *in a subgroup of* S so that certainly $a^{\sharp} = a^{-1}$, but however the converse is not true as we will see in an example in this section.

If S is a left I-order in Q and S has a right identity e, then this must be an identity of Q (and hence of S). For any $q \in Q$ we have that $q = a^{-1}b$ where $a, b \in S$, so that

$$qe = a^{-1}be = a^{-1}b = q$$

and

$$eq = ea^{-1}b = (ae)^{-1}b = a^{-1}b = q.$$

A left I-order S in an inverse semigroup Q is *straight* in Q if every element in Q can be written as $a^{-1}b$ where $a, b \in S$ and $a \,\mathcal{R}\, b$ in Q; we also say that Q is a *semigroup of straight left I-quotients* of S. If S is straight in Q, then we have the advantage of controlling products in Q.

The rest of this section is devoted to illustrative examples. The first is typical of the class of examples in which a semigroup S is a left I-order in an inverse semigroup Q, where the idempotents form a chain. We will study one of such classes in Chapter 8.

Example 3.1.1. *The bicyclic semigroup $\mathcal{B}$ consists of all pairs of non-negative integers with multiplication given by*

$$(k, l)(m, n) = (k - l + t, n - m + t) \text{ where } t = \max\{l, m\}.$$

It is easy to see that $\mathcal{B}$ is an inverse semigroup with identity $(0, 0)$ and $(i, j)^{-1} = (j, i)$.

Green's relations $\mathcal{R}$ and $\mathcal{L}$ are given by

$$(i, j)\, \mathcal{L}\, (k, l) \text{ if and only if } j = l,$$

and

$$(i,j)\,\mathcal{R}\,(k,l) \text{ if and only if } i=k.$$

It is clear that the $\mathcal{R}$-class of the identity $R_{(0,0)}$ is a subsemigroup of $\mathcal{B}$. If (i,j) is any element of $\mathcal{B}$, then

$$(i,j)=(i,0)(0,j)=(0,i)^{-1}(0,j)$$

so that $R_{(0,0)}$ is a left I-order in $\mathcal{B}$. On the other hand, the only element of $R_{(0,0)}$ lying in a subgroup is $(0,0)$, and $(0,0)^\sharp(0,n)=(0,n)$ for any $(0,n)\in R_{(0,0)}$. Thus $R_{(0,0)}$ is not a left order in $\mathcal{B}$.

The fact that $R_{(0,0)}$ is a left I-order in $\mathcal{B}$ is a very special case of the result of [2] mentioned in the Introduction, which we shall revisit. The semigroup $\mathcal{B}$ is bisimple and we shall see that bisimple inverse semigroups play an important role in this theory. Suppose that Q is bisimple inverse semigroup and pick an $\mathcal{R}$-class $R=R_e$ of Q, with $e\in E(Q)$. Let $q\in Q$. As Q is bisimple we can find mutually inverse elements $x,x^{-1}\in Q$ such that $xx^{-1}=e$ and $x^{-1}x=qq^{-1}$. Then $q=x^{-1}xq$ and $xq\,\mathcal{R}\,xqq^{-1}=x\,\mathcal{R}\,e$. Thus any element of Q can be written as a quotient of elements *chosen from any $\mathcal{R}$-class.*

In fact, by the argument preceding Proposition 2.4.3, due to Clifford, the $\mathcal{R}$-class of the identity of a bisimple inverse monoid is a left I-order in it. Hence every element q in a bisimple inverse monoid Q can be written as $a^{-1}b$ where $a,b\in R_1$. Thus

$$q=a^{-1}b=cc^{-1}a^{-1}b=c(ac)^{-1}b$$

where $a\,\mathcal{R}\,c\,\mathcal{R}\,1$. That is, Q is a semigroup of quotients of R_1 in sense of the first concept by McAlister.

The next example gives a left I-order in a Brandt semigroup. We will introduce necessary and sufficient conditions for any semigroup to have a Brandt semigroup of left I-quotients in Chapter 6.

Example 3.1.2. *Let H be a left order in a group G, and let $\mathcal{B}^0=\mathcal{B}^0(G,I)$ be a Brandt semigroup over G where $|I|\geq 2$. Fix $i\in I$ and let*

$$S_i=\{(i,h,j):h\in H,j\in I\}\cup\{0\}.$$

Then S_i is a straight left I-order in $\mathcal{B}^0$.

To see this, notice that S_i is a subsemigroup, $0 = 0^{-1}0$, and for any $(j,g,k) \in \mathcal{B}^0$, we may write $g = a^{-1}b$ where $a, b \in H$. Then

$$(j,g,k) = (i,a,j)^{-1}(i,b,k)$$

where $(i,a,j),(i,b,k) \in S_i$. It is easy to see that S_i is not a left order in $\mathcal{B}^0$.

3.2 Extension of homomorphisms

Fountain and Petrich in [6], gave an example of a semigroup having two non-isomorphic semigroups of right quotients. In the classical case Easdown and Gould [4] showed that a semigroup can have non-isomorphic inverse semigroups of left quotients. This will also happen in our case. In this section we prove a theorem concerning a straight left I-order S in an inverse semigroup Q with S embedded in another inverse semigroup P. Under what conditions is Q isomorphic to P ? This question led us to determine when two straight left I-quotients of a given semigroup are isomorphic. We begin by introducing the following notion.

Definition 3.2.1. *Let S be a subsemigroup of Q and let $\phi : S \to P$ be a homomorphism from S to a semigroup P. If there is a homomorphism $\overline{\phi} : Q \to P$ such that $\overline{\phi}|_S = \phi$, then we say that ϕ* lifts to Q. *If ϕ lifts to an isomorphism, then we say that Q and P are* isomorphic over S.

To achieve our goal, we must first examine when two quotients $a^{-1}b$ and $c^{-1}d$ are equal, where $a,b,c,d \in S$ and S is a left I-order in Q. In Lemma 3.2.4 below, we give conditions on S such that $a^{-1}b = c^{-1}d$; the use of Green's relations in Q in our conditions will be 'internalised' to S at a later point.

Lemma 3.2.2. *Let b,c,x,y be elements of an inverse semigroup Q such that $x \,\mathcal{R}\, y$. If $bc^{-1} = x^{-1}y$, then $xb = yc$.*

Proof. We have that

$$bc^{-1}cb^{-1} = (bc^{-1})(bc^{-1})^{-1} = (x^{-1}y)(x^{-1}y)^{-1} = x^{-1}yy^{-1}x = x^{-1}x$$

as $x\,\mathcal{R}\,y$. Hence

$$bc^{-1}c = bb^{-1}bc^{-1}c = bc^{-1}cb^{-1}b = x^{-1}xb$$

and so $xbc^{-1}c = xb$. From $y = xbc^{-1}$ we have

$$xb = xbc^{-1}c = yc,$$

as required. □

If S is a left I-order in an inverse semigroup Q, then $\mathcal{R}$ and $\mathcal{L}$ will be relations on Q, unless otherwise stated. To emphasise that $\mathcal{R}$ and $\mathcal{L}$ are relations on Q, we may write $\mathcal{R}^Q$ or $\mathcal{R}$ in Q and $\mathcal{L}^Q$ or $\mathcal{L}$ in Q. The relation $\mathcal{R}^*$ will always refer to S.

Lemma 3.2.3. *Let S be a left I-order in Q. Let $q = a^{-1}b$ in Q where $a, b \in S$. Then $a\,\mathcal{R}^Q\,b$ if and only if $b\,\mathcal{L}^Q\,q\,\mathcal{R}^Q\,a^{-1}$.*

Proof. Suppose that $a\,\mathcal{R}^Q\,b$. Then, $aq = aa^{-1}b = b$ and $qb^{-1} = a^{-1}bb^{-1} = a^{-1}$, so that $q\,\mathcal{L}^Q\,b$ and $q\,\mathcal{R}^Q\,a^{-1}$. On the other hand, let $q = a^{-1}b$ where $q\,\mathcal{R}^Q\,a^{-1}$ and $q\,\mathcal{L}^Q\,b$. Then $q \in R_{a^{-1}} \cap L_b$ in Q. From [1, Theorem 2.17], this means there exists $e \in E(Q)$ such that $e \in L_{a^{-1}} \cap R_b$ in Q. The conclusion is that $e = aa^{-1} = bb^{-1}$, so $a\,\mathcal{R}^Q\,b$. □

Notice that if S is a straight left I-order in an inverse semigroup Q, then S *intersects every $\mathcal{L}$-class of Q.*

Lemma 3.2.4. *Let S be a straight left I-order in Q. Let $a, b, c, d \in S$ with $a\,\mathcal{R}^Q\,b$ and $c\,\mathcal{R}^Q\,d$. Then $a^{-1}b = c^{-1}d$ if and only if there exist $x, y \in S$ with $xa = yc$ and $xb = yd$ and such that $a\,\mathcal{R}^Q\,x^{-1}$, $x\,\mathcal{R}^Q\,y$ and $y\,\mathcal{L}^Q\,c^{-1}$.*

Proof. Suppose first that $a^{-1}b = c^{-1}d$. By Lemma 3.2.3, $b\,\mathcal{L}^Q\,a^{-1}b = c^{-1}d\,\mathcal{L}^Q\,d$ and $a^{-1}\,\mathcal{R}^Q\,a^{-1}b = c^{-1}d\,\mathcal{R}^Q\,c^{-1}$ so that $a\,\mathcal{L}^Q\,c$. Let $x, y \in S$ be such that $ac^{-1} = x^{-1}y$ and $x\,\mathcal{R}^Q\,y$. Then

$$b = ac^{-1}d = x^{-1}yd,$$

so that $xb = yd$. From Lemma 3.2.2, $xa = yc$. As $a\,\mathcal{L}^Q\,c$ we have that $a^{-1}\,\mathcal{R}^Q\,c^{-1}$ hence

$$a\,\mathcal{R}^Q\,aa^{-1}\,\mathcal{R}^Q\,ac^{-1} = x^{-1}y\,\mathcal{R}^Q\,x^{-1},$$

and therefore $x\,\mathcal{L}^Q\,a^{-1}$. On the other hand, $a^{-1}\,\mathcal{R}^Q\,c^{-1}$ implies that $a^{-1}a = c^{-1}c$ and then

$$y = xac^{-1}\,\mathcal{L}^Q\,a^{-1}ac^{-1} = c^{-1}cc^{-1} = c^{-1},$$

as required.

Conversely, if $xa = yc$ and $xb = yd$ for some $x, y \in S$ with $x\,\mathcal{R}^Q\,y$, $a\,\mathcal{R}^Q\,x^{-1}$ and $y\,\mathcal{L}^Q\,c^{-1}$, then $a = x^{-1}yc$, $b = x^{-1}yd$ and in view of the fact that

$$b\,\mathcal{R}^Q\,a\,\mathcal{R}^Q\,x^{-1}\,\mathcal{R}^Q\,x^{-1}x,$$

we have

$$\begin{aligned} a^{-1}b &= (x^{-1}yc)^{-1}(x^{-1}yd) \\ &= c^{-1}y^{-1}xx^{-1}yd \\ &= c^{-1}y^{-1}yd && \text{as } x\,\mathcal{R}^Q\,y \\ &= c^{-1}cc^{-1}d && \text{as } y\,\mathcal{L}^Q\,c^{-1} \\ &= c^{-1}d. \end{aligned}$$

□

Lemma 3.2.5. *Let Q be an inverse monoid. Let $a, b, c, d \in R_1$. Then*

$$a^{-1}b = c^{-1}d \text{ if and only if } a = uc \text{ and } b = ud$$

for some unit u.

Proof. Suppose that $a^{-1}b = c^{-1}d$ where $a, b, c, d \in R_1$. We have

$$a^{-1}\,\mathcal{R}\,a^{-1}b = c^{-1}d\,\mathcal{R}\,c^{-1} \text{ in } Q.$$

Then $a\,\mathcal{L}\,c$ in Q. Since $a\,\mathcal{R}\,b$, it follows that $b = aa^{-1}b = ac^{-1}d$. We claim that ac^{-1} is a unit. As $a\,\mathcal{L}\,c$, it follows that $ac^{-1}\,\mathcal{L}\,cc^{-1} = 1$. Since $a^{-1}\,\mathcal{R}\,c^{-1}$ we have that $1 = aa^{-1}\,\mathcal{R}\,ac^{-1}$ and hence $u = ac^{-1}$ is a unit, and we obtain $b = ud$. Since $u = ac^{-1}$ and $a\,\mathcal{L}\,c$ we have that $uc = ac^{-1}c = a$. The converse is clear. □

Now we give the following well known Lemma from [25].

Lemma 3.2.6. *Let $\theta : Q \longrightarrow P$ be a homomorphism between inverse semigroups.*

(1) $(s^{-1})\theta = (s\theta)^{-1}$ *for all $s \in Q$.*

(2) *If e is an idempotent, then $e\theta$ is an idempotent.*

(3) *$Q\theta$ is an inverse subsemigroup of P.*

(4) *If* $a, b \in Q$ *and* $a \leq_{\mathcal{L}} b$ *in* Q, *then* $a\theta \leq_{\mathcal{L}} b\theta$ *in* P.

(5) *If* $a\theta$ *is idempotent, then* $a\theta = e\theta$ *for some* $e = e^2 \in Q$.

Lemma 3.2.7. *Let* S *be a left I-order in* Q *and let* φ *be a homomorphism from* S *to a semigroup* P. *Then* φ *can be extended to at most one homomorphism* $\psi : Q \longrightarrow P$.

Proof. If $\psi : Q \longrightarrow P$ is any homomorphism extending φ, then if $q = a^{-1}b \in Q$ we have

$$q\psi = (a^{-1}b)\psi = (a\psi)^{-1}b\psi = (a\varphi)^{-1}(b\varphi).$$

□

Let S be a subsemigroup of an inverse semigroup Q. We use Green's relations on Q to define binary relations $\leq^Q_{\mathcal{R},S}, \mathcal{R}^Q_S, \leq^Q_{\mathcal{L},S}$ and $\mathcal{L}^Q_S$ and a ternary relation $\mathcal{T}^Q_S$ on S by the rules:

$$\leq^Q_{\mathcal{R},S} = \leq^Q_{\mathcal{R}} \cap (S \times S) \text{ and } \leq^Q_{\mathcal{L},S} = \leq^Q_{\mathcal{L}} \cap (S \times S),$$

so that $\leq^Q_{\mathcal{R},S}$ and $\leq^Q_{\mathcal{L},S}$ are, respectively, left and right compatible quasi-orders. We then define $\mathcal{R}^Q_S$ and $\mathcal{L}^Q_S$ to be the associated equivalence relations, so that

$$\mathcal{R}^Q_S = \mathcal{R}^Q \cap (S \times S) \text{ and } \mathcal{L}^Q_S = \mathcal{L}^Q \cap (S \times S).$$

Consequently, $\mathcal{R}^Q_S$ and $\mathcal{L}^Q_S$ are left and right compatible. For any $a, b, c \in S$,

$$(a, b, c) \in \mathcal{T}^Q_S \text{ if and only if } ab^{-1}Q \subseteq c^{-1}Q.$$

The following Lemma is an application of the $\mathcal{T}$ relation and we will use it to prove our main theorem.

Lemma 3.2.8. *Let* S *and* T *be subsemigroups of inverse semigroups* Q *and* P *respectively, and let* $\phi : S \to T$ *be a homomorphism. If for all* $a, b, c \in S$,

$$(a, b, c) \in \mathcal{T}^Q_S \Rightarrow (a\phi, b\phi, c\phi) \in \mathcal{T}^P_T,$$

then for all $u, v \in S$,

$$u \leq^Q_{\mathcal{R}} v^{-1} \Rightarrow u\phi \leq^P_{\mathcal{R}} (v\phi)^{-1}.$$

Proof. Suppose that $u, v \in S$ and $uQ \subseteq v^{-1}Q$. Then $uu^{-1}Q \subseteq v^{-1}Q$, so that $(u, u, v) \in \mathcal{T}_S^Q$. By assumption, $(u\phi, u\phi, v\phi) \in \mathcal{T}_T^P$, so that

$$u\phi P = u\phi(u\phi)^{-1}P \subseteq (v\phi)^{-1}P$$

and $u\phi \leq_{\mathcal{R}}^P (v\phi)^{-1}$ as required. □

We use the relation $\mathcal{T}_S^Q$ to prove our rather general result below. *As in the classical case, $\mathcal{T}_S^Q$ can be avoided in some special cases of interest.*

Theorem 3.2.9. *Let S be a straight left I-order in Q and let T be a subsemigroup of an inverse semigroup P. Suppose that $\phi : S \to T$ is a homomorphism. Then ϕ lifts to a (unique) homomorphism $\overline{\phi} : Q \to P$ if and only if for all $a, b, c \in S$:*

(i) $(a, b) \in \mathcal{R}_S^Q \Rightarrow (a\phi, b\phi) \in \mathcal{R}_T^P$;

(ii) $(a, b, c) \in \mathcal{T}_S^Q \Rightarrow (a\phi, b\phi, c\phi) \in \mathcal{T}_T^P$.

If (i) and (ii) hold and $S\phi$ is a left I-order in P, then $\overline{\phi} : Q \to P$ is onto.

Proof. If ϕ lifts to a homomorphism $\overline{\phi}$, then as homomorphisms between inverse semigroups preserve inverses and Green's relations, it is easy to see that (i) and (ii) hold.

Conversely, suppose that (i) and (ii) hold. We define $\overline{\phi} : Q \to P$ by the rule that

$$(a^{-1}b)\overline{\phi} = (a\phi)^{-1}b\phi$$

where $a, b \in S$ and $a \,\mathcal{R}^Q\, b$.

To show that $\overline{\phi}$ is well defined, suppose that

$$a^{-1}b = c^{-1}d$$

where $a, b, c, d \in S$, $a \,\mathcal{R}^Q\, b$ and $c \,\mathcal{R}^Q\, d$. Then by Lemma 3.2.4, there exist $x, y \in S$ with $xa = yc$ and $xb = yd$ and such that $a \,\mathcal{R}^Q\, x^{-1}$, $x \,\mathcal{R}^Q\, y$ and $y \,\mathcal{L}^Q\, c^{-1}$. Applying ϕ, we have that $x\phi\, a\phi = y\phi\, c\phi$ and $x\phi\, b\phi = y\phi\, d\phi$. By (i) we also have that $x\phi \,\mathcal{R}^P\, y\phi, a\phi \,\mathcal{R}^P\, b\phi$ and $c\phi \,\mathcal{R}^P\, d\phi$. Also, since $b \,\mathcal{R}^Q\, a \,\mathcal{R}^Q\, x^{-1}$ and $d \,\mathcal{R}^Q\, c \,\mathcal{R}^Q\, y^{-1}$, it follows from Lemma 3.2.8 that $b\phi \leq_{\mathcal{R}}^P (x\phi)^{-1}$ and $d\phi \leq_{\mathcal{R}}^P (y\phi)^{-1}$.

From $x\phi\, b\phi = y\phi\, d\phi$ we can now deduce that $b\phi = (x\phi)^{-1}y\phi\, d\phi$ so that

$$\begin{aligned}(a\phi)^{-1}b\phi &= (a\phi)^{-1}(x\phi)^{-1}y\phi\, d\phi\\ &= (x\phi\, a\phi)^{-1}y\phi\, d\phi\\ &= (y\phi\, c\phi)^{-1}y\phi\, d\phi\\ &= (c\phi)^{-1}(y\phi)^{-1}y\phi\, d\phi\\ &= (c\phi)^{-1}d\phi,\end{aligned}$$

so that $\overline{\phi}$ is well defined.

To see that $\overline{\phi}$ lifts ϕ, let $h \in S$; then $h = k^{-1}\ell$ for some $k, \ell \in S$ with $k\,\mathcal{R}^Q\,\ell$. We have that $kh = \ell$ and $h \leq^Q_{\mathcal{R}} k^{-1}$, so that $k\phi h\phi = \ell\phi$ and by (ii) and Lemma 3.2.8, $h\phi \leq^P_{\mathcal{R}} (k\phi)^{-1}$. It follows that $h\phi = (k\phi)^{-1}\ell\phi = h\overline{\phi}$.

We need to show that $\overline{\phi}$ is a homomorphism. To this end, let $a^{-1}b, c^{-1}d \in Q$ where $a, b, c, d \in S, a\,\mathcal{R}^Q\, b$ and $c\,\mathcal{R}^Q\, d$. By (i) we have that $c\phi\,\mathcal{R}^P\, d\phi$. Now $bc^{-1} = u^{-1}v$ for some $u, v \in S$ with $u\,\mathcal{R}^Q\, v$. By Lemma 3.2.2, $ub = vc$, so that $u\phi\, b\phi = v\phi\, c\phi$. Further, $(b, c, u) \in \mathcal{T}^Q_S$, so by assumption (ii), we have that $(b\phi, c\phi, u\phi) \in \mathcal{T}^P_T$. Then from $u\phi\, b\phi(c\phi)^{-1} = v\phi\, c\phi(c\phi)^{-1}$ we obtain $b\phi(c\phi)^{-1} = (u\phi)^{-1}v\phi\, c\phi(c\phi)^{-1}$.

Multiplying, we have

$$(a^{-1}b)(c^{-1}d) = a^{-1}bc^{-1}d = a^{-1}u^{-1}vd = (ua)^{-1}vd,$$

and

$$ua\,\mathcal{R}^Q\, ub = vc\,\mathcal{R}^Q\, vd.$$

Hence

$$\begin{aligned}((a^{-1}b)(c^{-1}d))\overline{\phi} &= ((ua)^{-1}vd)\overline{\phi}\\ &= ((ua)\phi)^{-1}(vd)\phi\\ &= (a\phi)^{-1}(u\phi)^{-1}v\phi\, d\phi\\ &= (a\phi)^{-1}(u\phi)^{-1}v\phi\, c\phi\,(c\phi)^{-1}d\phi\\ &= (a\phi)^{-1}b\phi(c\phi)^{-1}d\phi\\ &= (a^{-1}b)\overline{\phi}(c^{-1}d)\overline{\phi},\end{aligned}$$

so that $\overline{\phi}$ is a homomorphism as required.

If (i) and (ii) hold and $S\phi$ is a left I-order in P, then for any $p \in P$ we have $p = (a\phi)^{-1}b\phi$ for some $a, b \in S$, so that $p = (a^{-1}b)\overline{\phi}$. $\square$

Corollary 3.2.10. *Let S be a straight left I-order in Q and let $\phi : S \to P$ be an embedding of S into an inverse semigroup P such that $S\phi$ is a straight left I-order in P. Then Q is isomorphic to P over S if and only if for any $a, b, c \in S$:*

(i) $(a, b) \in \mathcal{R}_S^Q \Leftrightarrow (a\phi, b\phi) \in \mathcal{R}_{S\phi}^P$; *and*

(ii) $(a, b, c) \in \mathcal{T}_S^Q \Leftrightarrow (a\phi, b\phi, c\phi) \in \mathcal{T}_{S\phi}^P$.

Proof. If Q is isomorphic to P over S then (i) and (ii) hold from Theorem 3.2.9.

Suppose now that (i) and (ii) hold. From Theorem 3.2.9, ϕ lifts to a homomorphism $\overline{\phi} : Q \to P$, where $(a^{-1}b)\overline{\phi} = (a\phi)^{-1}b\phi$. Dually, $\phi^{-1} : S\phi \to Q$ lifts to a homomorphism $\overline{\phi^{-1}} : P \to Q$, where $((a\phi)^{-1}b\phi)\overline{\phi^{-1}} = a^{-1}b$. Clearly $\overline{\phi}$ and $\overline{\phi^{-1}}$ are mutually inverse. $\square$

Where S is left ample, and ϕ preserves $^+$, then we note that (i) in Theorem 3.2.9 and Corollary 3.2.10 is redundant. Further redundancies become apparent in the next chapter.

Chapter 4

Inverse hulls of left I-quotients of left ample semigroups

In [2] Clifford showed that a bisimple inverse monoid can be constructed from the $\mathcal{R}$-class of its identity which is a right cancellative monoid satisfying the condition that the intersection of any two principal left ideals is again a principal left ideal. We shall call this condition the (LC) condition. Conversely, from any right cancellative monoid S that has the (LC) condition we can construct a bisimple inverse monoid such that the $\mathcal{R}$-class of the identity is isomomorphic to S. In [41] Reilly has shown that the structure of any bisimple inverse semigroup with or without identity is determined by any of its $\mathcal{R}$-classes; these will not, in general, be a subsemigroups.

In this chapter we extend Clifford's result to a left ample semigroup with the (LC) condition. In Section 4.1 we introduce some characterizations of the embedding of a left ample semigroup into a symmetric inverse monoid. The main theorem in Section 4.2 gives a necessary and sufficient condition for a left ample semigroup to be a left I-order in its inverse hull. The inverse hull of left I-quotients of left ample semigroup with the (LC) condition may not be bisimple. In Section 4.3 we investigate the special cases where the inverse hull of left I-quotients of left ample semigroup with the (LC) condition is bisimple, simple, or proper. We end this section by tackling the left ample semigroups homomorphism extension problem.

4.1 Preliminaries

In Section 2.3, we showed that there is an embedding of a left ample semigroup S into the symmetric inverse semigroup $\mathcal{I}_S$. *The inverse hull* $\Sigma(S)$ of S is the inverse subsemigroup of $\mathcal{I}_S$ generated by $\operatorname{im}\theta_S$, where θ_S is the embedding of S into the symmetric inverse semigroup $\mathcal{I}_S$ (as defined in Section 2.3). Where convenient we identify S with its image under θ_S in $\Sigma(S)$. We begin with the following useful lemma.

Lemma 4.1.1. *Let S be a left ample semigroup. Let ρ_a and ρ_b be in $S\rho$. Then*

(1) $\operatorname{dom}\rho_a\rho_b^{-1} = (Sa \cap Sb)\rho_a^{-1}$;

(2) $\operatorname{im}\rho_a^{-1}\rho_b = Sa^+b$

(3) $\operatorname{dom}\rho_a^{-1}\rho_b = Sb^+a.$

Proof. (1) and (2) are straightforward.

(3) We have that $\rho_a : Sa^+ \longrightarrow Sa$, $\rho_b : Sb^+ \longrightarrow Sb$ and $\rho_a^{-1} : Sa \longrightarrow Sa^+$, so that

$$\begin{aligned}
\operatorname{dom}\rho_a^{-1}\rho_b &= (\operatorname{im}\rho_a^{-1} \cap \operatorname{dom}\rho_b)\rho_a \\
&= (\operatorname{dom}\rho_a \cap \operatorname{dom}\rho_b)\rho_a \\
&= (Sa^+ \cap Sb^+)\rho_a \\
&= (Sa^+b^+)\rho_a \\
&= (Sb^+a^+)\rho_a \\
&= Sb^+a.
\end{aligned}$$

□

Notice that if $a\,\mathcal{R}^*\,b$, then $\operatorname{dom}\rho_a^{-1}\rho_b = Sa$ and $\operatorname{im}\rho_a^{-1}\rho_b = Sb$.

Lemma 4.1.2. *Let S be a left ample semigroup. Then, for any $a, b \in S$,*

$$\rho_a\,\mathcal{L}\,\rho_b \text{ in } \Sigma(S) \text{ if and only if } a\,\mathcal{L}\,b \text{ in } S.$$

Proof. Suppose that $a, b \in S$. If $\rho_a\,\mathcal{L}\,\rho_b$ in $\Sigma(S)$, then $\operatorname{im}\rho_a = \operatorname{im}\rho_b$ so that $Sa = Sb$. Also, since for any $x \in S$ we have $x = x^+x$ and so $S^1x = Sx$ so that $a\,\mathcal{L}\,b$ in S. The converse is clear. □

Remark 4.1.3. *Let S be a left ample semigroup. Using the fact that $\mathcal{R}^*$ is a left congruence, and $E(S)$ is a semilattice, we see that*

$$a^+b\,\mathcal{R}^*\,a^+b^+ = b^+a^+\,\mathcal{R}^*\,b^+a.$$

Lemma 4.1.4. *Let S be a left ample semigroup, embedded (as a $(2,1)$-algebra) in an inverse semigroup Q. If S is a left I-order in Q, then S is straight.*

Proof. Let $q = a^{-1}b \in Q$ where $a, b \in S$. Then

$$q = (a^+a)^{-1}(b^+b) = a^{-1}a^+b^+b = a^{-1}b^+a^+b = (b^+a)^{-1}(a^+b).$$

We have

$$a^+b\,\mathcal{R}^*\,a^+b^+ = b^+a^+\,\mathcal{R}^*\,b^+a$$

and so $a^+b\,\mathcal{R}^Q\,b^+a$ and S is straight. □

4.2 The main theorem

The main result of this section is Theorem 4.2.2, which gives a characterisation of left ample semigroups with (LC) which are left I-orders in their inverse hulls. We recall that by saying that a semigroup S satisfies the (LC) condition we mean

for any $a, b \in S$ there exists $c \in S$ such that $Sa \cap Sb = Sc$.

Lemma 4.2.1. *Let S be a left ample semigroup. If $b, c \in S$ with $Sb \cap Sc = Sw$, where $ub = vc = w$ and $ub^+ = u$, $vc^+ = v$, then $\rho_b\rho_c^{-1} = \rho_u^{-1}\rho_v$ for some $u, v \in S$.*

Proof. Let b, c, u, v, w as in the hypothesis. From Lemma 4.1.1, we have

$$\begin{aligned} \operatorname{dom}\rho_b\rho_c^{-1} &= (\operatorname{im}\rho_b \cap \operatorname{dom}\rho_c^{-1})\rho_b^{-1} \\ &= (Sb \cap Sc)\rho_b^{-1} \\ &= (Sw)\rho_b^{-1} \\ &= (Sub)\rho_b^{-1} \\ &= Sub^+ \\ &= Su, \end{aligned}$$

and for any $su \in Su$,

$$(su)\rho_b\rho_c^{-1} = (sub)\rho_c^{-1} = (svc)\rho_c^{-1} = svc^+ = sv.$$

In particular, $\operatorname{im}\rho_b\rho_c^{-1} = Sv$. Notice that

$$u = ub^+\,\mathcal{R}^*\,ub = vc\,\mathcal{R}^*\,vc^+ = v.$$

It is now easy to see that $\rho_b\rho_c^{-1} = \rho_u^{-1}\rho_v$. □

Now we introduce the main Theorem in this chapter.

Theorem 4.2.2. *Let S be a left ample semigroup. Then $S\theta_S$ is a left I-order in its inverse hull if and only if S has the (LC) condition.*

If Condition (LC) holds, then $S\theta_S$ is a union of $R^{\Sigma(S)}$-classes.

Proof. Suppose that S is a left I-order in $\Sigma(S)$. By Lemma 4.1.4, for any $b, c \in S$, $\rho_b\rho_c^{-1} = \rho_u^{-1}\rho_v$, with $u\,\mathcal{R}^*\,v$ where $ub^+ = u$ and $vc^+ = v$. By Lemma 4.1.1, $\operatorname{dom}(\rho_b\rho_c^{-1}) = Su$, so that

$$Su = (\operatorname{im}\rho_b \cap \operatorname{dom}\rho_c^{-1})\rho_b^{-1} = (Sb \cap Sc)\rho_b^{-1}.$$

But $\rho_b^{-1}\rho_b$ is the identity on $Sb = \operatorname{im}\rho_b$, and so

$$Sub = (Su)\rho_b = (Sb \cap Sc)\rho_b^{-1}\rho_b = Sb \cap Sc,$$

and S has the (LC) condition.

Conversely, suppose that S has the (LC) condition. Let

$$Q = \{\rho_a^{-1}\rho_b : a, b \in S\} \subseteq \Sigma(S).$$

Observe that for any $a \in S$, $\rho_a = \rho_{a^+a} = \rho_{a^+}^{-1}\rho_a$, so that $S\theta_S \subseteq Q$.

Consider $b, c \in S$. By Condition (LC), there exist $u, v \in S$ with $Sb \cap Sc = Sub$, and $ub = vc$ with $ub^+ = u$ and $vc^+ = v$. By Lemma 4.2.1, $\rho_b\rho_c^{-1} = \rho_u^{-1}\rho_v$ and $u\,\mathcal{R}^*\,v$.

It follows that if $\rho_a^{-1}\rho_b, \rho_c^{-1}\rho_d \in Q$, then

$$(\rho_a^{-1}\rho_b)(\rho_c^{-1}\rho_d) = \rho_a^{-1}(\rho_b\rho_c^{-1})\rho_d = \rho_a^{-1}(\rho_u^{-1}\rho_v)\rho_d = (\rho_u\rho_a)^{-1}(\rho_v\rho_d) = \rho_{ua}^{-1}\rho_{vd},$$

so that Q is closed under multiplication. Clearly Q is closed under taking inverses, so that as $\Sigma(S) \subseteq Q$ from definition of inverse hull, and so $Q = \Sigma(S)$ as required.

If $e \in E(S)$ and $\rho_e\,\mathcal{R}^{\Sigma(S)}\,\rho_a^{-1}\rho_b$, where $a, b \in S$ and $a\,\mathcal{R}^*\,b$, then $\operatorname{dom}\rho_e = \operatorname{dom}\rho_a^{-1}\rho_b$, so that $Se = Sa$ and a is regular in S. Any inverse c of a in S must be such that ρ_c is the unique inverse of ρ_a in Q, so that $\rho_a^{-1} \in S\theta_S$ and hence $\rho_a^{-1}\rho_b \in S\theta_S$.

Finally, if $s \in S$, then $\rho_s\,\mathcal{R}^*\,\rho_{s^+}$ in $S\theta_S$, so that $\rho_s\,\mathcal{R}\,\rho_{s^+}$ in $\Sigma(S)$. It follows from the previous paragraph that $S\theta_S$ is a union of $\mathcal{R}$-classes of $\Sigma(S)$.

□

Lemma 4.2.3. *Let S be a left ample semigroup with (LC). If $\rho_a^{-1}\rho_b, \rho_c^{-1}\rho_d \in \Sigma(S)$, where $a\,\mathcal{R}^*\,b, c\,\mathcal{R}^*\,d$, then*

$$\rho_a^{-1}\rho_b = \rho_c^{-1}\rho_d \Longleftrightarrow a = uc,\ \ b = ud \text{ and } c = va,\ \ d = vb$$

for some u, v in S.

Proof. Suppose that $\rho_a^{-1}\rho_b, \rho_c^{-1}\rho_d \in \Sigma(S)$ where $a\,\mathcal{R}^*\,b$ and $c\,\mathcal{R}^*\,d$. By Lemma 4.1.1, $\rho_a^{-1}\rho_b : Sa \longrightarrow Sb$ and $\rho_c^{-1}\rho_d : Sc \longrightarrow Sd$.
If $\rho_a^{-1}\rho_b = \rho_c^{-1}\rho_d$, then $Sa = Sc$ and $Sb = Sd$. Hence $a = uc$ and $c = va$ for some $u, v \in S$. Since $a \in \operatorname{dom}(\rho_a^{-1}\rho_b)$, it follows that

$$b = b^+\rho_b = a^+\rho_b = a\rho_a^{-1}\rho_b.$$

We have

$$b = a\rho_c^{-1}\rho_d = uc\rho_c^{-1}\rho_d = ud.$$

Dually, $d = vb$.
On the other hand, if $a = uc,\ \ b = ud,\ \ c = va$ and $d = vb$, then

$$\rho_a^{-1}\rho_b = \rho_{uc}^{-1}\rho_{ud} = \rho_c^{-1}\rho_u^{-1}\rho_u\rho_d \leq \rho_c^{-1}\rho_d$$

and

$$\rho_c^{-1}\rho_d = \rho_{va}^{-1}\rho_{vb} = \rho_a^{-1}\rho_v^{-1}\rho_v\rho_b \leq \rho_a^{-1}\rho_b.$$

Hence $\rho_a^{-1}\rho_b = \rho_c^{-1}\rho_d$, as required. □

It is clear that $\rho_a^{-1}\rho_a$ is an idempotent. The next lemma gives the form of the idempotents in $\Sigma(S)$.

Lemma 4.2.4. *Let S be a left ample semigroup with the (LC) condition. Then the set of idempotents of $\Sigma(S)$ has the form $\{\rho_a^{-1}\rho_a : a \in S\}$.*

Proof. From the proof of Theorem 4.2.2, every element of $\Sigma(S)$ has the form $\rho_a^{-1}\rho_b$ for some $a, b \in S$. By Lemma 4.1.4, we can assume that $a\,\mathcal{R}^*\,b$.

Suppose that $\rho_a^{-1}\rho_b$ is an idempotent where $a\,\mathcal{R}^*\,b$. Then

$$\rho_a^{-1}\rho_b\rho_a^{-1}\rho_b = \rho_a^{-1}\rho_b.$$

Since $a \in \mathrm{dom}\,(\rho_a^{-1}\rho_b)$ we have that

$$a\rho_a^{-1}\rho_b\rho_a^{-1}\rho_b = a\rho_a^{-1}\rho_b,$$

but $a\rho_a^{-1}\rho_b = b$. Thus $b\rho_a^{-1}\rho_b = a\rho_a^{-1}\rho_b$, so that $a = b$, as required. $\square$

We will employ the rest of this section to introduce another structure of $\Sigma(S)$ where S has (LC) by using ordered pairs belonging to $S \times S$.

Let $Z = \{(a, b) : a\,\mathcal{R}^*\,b\}$. On Z define a relation $\sim$ by

$$(a, b) \sim (c, d) \iff a = uc,\ b = ud \text{ and } c = va,\ d = vb$$

for some $u, v \in S$. It is straightforward to check that the relation $\sim$ is an equivalence relation on Z. Put $Q = Z/\sim$ and denote the $\sim$-equivalence class of (a, b) by $[a, b]$. Define multiplication on Q by

$$[a, b][c, d] = [xa, yd] \text{ where } Sb \cap Sc = Sw \text{ and } w = xb = yc$$

for some $x, y \in S$.

By Lemma 4.2.3, we have

$$\rho_a^{-1}\rho_b = \rho_c^{-1}\rho_d \iff [a, b] = [c, d].$$

Hence it is easy to see that the map $[a, b] \longrightarrow \rho_a^{-1}\rho_b$ is an isomorphism from Q into $\Sigma(S)$. Then Q is an inverse semigroup with idempotents having the form $\{[a, a] : a \in S\}$ and $[a, b]^{-1} = [b, a]$. Also, S embeds in Q by the map $\theta : S \longrightarrow Q$ defined by $a\theta = [a^+, a]$. Moreover, $S\theta$ is a left I-order in Q.

4.3 Some special cases

In this section we are interested in investigating under which conditions the inverse hull of left I-quotients of a left ample semigroup with (LC) is a bisimple, simple, or proper semigroup. We end this section by introducing Theorem 4.3.9, which gives a necessary and sufficient condition for a homomorphism between left ample semigroups with (LC) to be lifted to a homomorphism between their inverse hulls. We begin with:

Lemma 4.3.1. *For any semigroup S, $\mathcal{R}^* \circ \mathcal{L} = \mathcal{L} \circ \mathcal{R}^*$.*

Proof. Let $a, b \in S$ with $a\,\mathcal{R}^* \circ \mathcal{L}\, b$. Then there exists an element $c \in S$ with $a\,\mathcal{R}^*\, c\,\mathcal{L}\, b$. Either $c = b$, in which case $a\,\mathcal{L}\, a\,\mathcal{R}^*\, b$, or there exist $u, v \in S$ with $c = ub$ and $b = vc$. Hence $c = ub = uvc$, so that as $a\,\mathcal{R}^*\, c$ we deduce $a = uva$, and thus $a\,\mathcal{L}\, va$. But $va\,\mathcal{R}^*\, vc = b$, so that $a\,\mathcal{L} \circ \mathcal{R}^*\, b$ and $\mathcal{R}^* \circ \mathcal{L} \subseteq \mathcal{L} \circ \mathcal{R}^*$. The proof of the dual inclusion is very similar. □

Lemma 4.3.2. *Let S be a left ample semigroup that is a left I-order in an inverse semigroup Q, such that S is a union of $\mathcal{R}$-classes of Q. Then*

(i) S is a $(2,1)$-subalgebra of Q;

(ii) for $a, b \in S$ with $a\,\mathcal{R}^*\, b$, we have $a^{-1}b$ is idempotent if and only if $a = b$;

(iii) for any $a, b \in S$, we have $Sa \subseteq Sb$ if and only if $Qa \subseteq Qb$;

(iv) for any $a, b, c \in S$, we have $Sa \cap Sb = Sc$ if and only if $Qa \cap Qb = Qc$;

(v) S has the (LC) condition;

(vi) Q is bisimple if and only if

$$\mathcal{L}^S \circ \mathcal{R}^* = S \times S;$$

and

(vii) Q is simple if and only if for all $a, b \in S$ there exists $c \in S$ with

$$a\,\mathcal{R}^*\, c \leq_{\mathcal{L}}^S b.$$

Proof. (*i*) We need only show that if $a \in S$, then $aa^{-1} = a^+$. We have that $a\,\mathcal{R}^Q\, aa^{-1}$ and S is a union of $\mathcal{R}^Q$-classes, giving $aa^{-1} \in S$. As $a\,\mathcal{R}^*\, aa^{-1}$ we must have that $aa^{-1} = a^+$.

(*ii*) If $a^{-1}b$ is idempotent, then as $a^{-1}b\,\mathcal{R}^Q\, a^{-1}a$, we must have that $a^{-1}b = a^{-1}a$. Multiplying with a on the left gives $b = bb^{-1}b = aa^{-1}b = aa^{-1}a = a$. The converse is clear.

(*iii*) If $a, b \in S$ and $Sa \subseteq Sb$, then clearly $Qa \subseteq Qb$. On the other hand, if $Qa \subseteq Qb$, then we have that $a = h^{-1}kb$ for some $h, k \in S$. It follows that $a = ((kb)^+h)^{-1}h^+kb$ and using Remark 4.1.3,

$$((kb)^+h)^{-1}\,\mathcal{R}^Q\,((kb)^+h)^{-1}((kb)^+h)\,\mathcal{R}^Q\,((kb)^+h)^{-1}h^+kb = a,$$

so that as S is a union of $\mathcal{R}^Q$-classes, $((kb)^+h)^{-1} \in S$. It follows that $Sa \subseteq Sb$.

(iv) By Lemma 4.1.4 and (i), S is straight in Q. Suppose that $a, b \in S$ and $Sa \cap Sb = Sc$. Then $c \in Sa \cap Sb \subseteq Qa \cap Qb$, so that $Qc \subseteq Qa \cap Qb$. Conversely, if $h^{-1}ka = u^{-1}vb \in Qa \cap Qb$, where $h, k, u, v \in S$, $h \,\mathcal{R}^Q\, k$ and $u \,\mathcal{R}^Q\, v$, then $ka = hu^{-1}vb$ and $hu^{-1} = s^{-1}t$, say, where $s, t \in S$ and $s \,\mathcal{R}^Q\, t$. This gives that

$$ska = tvb \in Sa \cap Sb = Sc,$$

and so $ska = tvb = xc$, where $x \in S$. Now

$$ka = hu^{-1}vb = s^{-1}tvb = s^{-1}xc$$

and then $h^{-1}ka = h^{-1}s^{-1}xc \in Qc$. Hence $Qa \cap Qb \subseteq Qc$, so that $Qa \cap Qb = Qc$.

Conversely, suppose that $a, b \in S$ and $Qa \cap Qb = Qc$. From $Qc \subseteq Qa$ and $Qc \subseteq Qb$, (iii) gives that $Sc \subseteq Sa \cap Sb$. On the other hand, if $u = xa = yb \in Sa \cap Sb$ for some $x, y \in S$, then $u = qc$ for some $q \in Q$, whence $Qu \subseteq Qc$. Again from (iii), $Su \subseteq Sc$ so that $Sa \cap Sb \subseteq Sc$ and we have $Sa \cap Sb = Sc$ as required.

(v) Let $a, b \in S$. Then

$$Qa \cap Qb = Qa^{-1}a \cap Qb^{-1}b = Qa^{-1}ab^{-1}b = Qab^{-1}b,$$

but $ab^{-1} = s^{-1}t$ for some $s, t \in S$ with $s \,\mathcal{R}^Q\, t$, by Lemma 4.1.4 and (i). We have

$$Qa \cap Qb = Qs^{-1}tb = Qtb.$$

From (iv) we now have that $Sa \cap Sb = Stb$, and S has the (LC) condition.

(vi) From Lemma 4.1.4 and (i), it follows that S is straight in Q. Let $a^{-1}b, c^{-1}d \in Q$, where $a \,\mathcal{R}^*\, b$ and $c \,\mathcal{R}^*\, d$. Then

$$\begin{aligned}
a^{-1}b \,\mathcal{D}\, c^{-1}d \text{ in } Q \;&\Leftrightarrow\; a^{-1}b \,\mathcal{R}^Q\, x^{-1}y \,\mathcal{L}^Q\, c^{-1}d \text{ for some } x, y \in S \text{ with } x \,\mathcal{R}^*\, y \\
&\Leftrightarrow\; a^{-1} \,\mathcal{R}^Q\, x^{-1} \text{ and } y \,\mathcal{L}^Q\, d \text{ for some } x, y \in S \text{ with } x \,\mathcal{R}^*\, y \\
&\Leftrightarrow\; a \,\mathcal{L}^Q\, x \text{ and } y \,\mathcal{L}^Q\, d \text{ for some } x, y \in S \text{ with } x \,\mathcal{R}^*\, y \\
&\Leftrightarrow\; a \,\mathcal{L}^S\, x \,\mathcal{R}^*\, y \,\mathcal{L}^S\, d \text{ for some } x, y \in S.
\end{aligned}$$

It follows that Q is bisimple if and only if $\mathcal{L}^S \circ \mathcal{R}^* \circ \mathcal{L}^S$ is universal. But from Lemma 4.3.1, $\mathcal{L}$ and $\mathcal{R}^*$ commute on S, so that Q is bisimple if and only if $\mathcal{L}^S \circ \mathcal{R}^* = S \times S$.

(vii) Since Q is inverse, it follows from [1, Theorem 8.33] that Q is simple if and only if for any $e, f \in E(Q)$, there is an element $q \in Q$ with $e = qq^{-1}$ and $q^{-1}q \leq f$. By (ii), $e = a^{-1}a$ and $f = b^{-1}b$ for some $a, b \in S$. Then Q is simple if and only if there exists $q = c^{-1}d$ (where $c, d \in S$ and $c\,\mathcal{R}^*\,d$) such that

$$e = qq^{-1} = c^{-1}c \text{ and } d^{-1}d = q^{-1}q \leq f.$$

Suppose now that Q is simple, and let $a, b \in S$. Let $e = a^{-1}a$ and $f = b^{-1}b$. Consequently, there exists $c, d \in S$ with $c\,\mathcal{R}^*\,d$, $a^{-1}a = c^{-1}c$ and $d^{-1}d \leq b^{-1}b$. It follows that $a\,\mathcal{L}^Q\,c$ and $Qd \subseteq Qb$. By (iii), $a\,\mathcal{L}^S\,c$ and $Sd \subseteq Sb$. By Lemma 4.3.1, $\exists u \in S$ with $a\,\mathcal{R}^*\,u\,\mathcal{L}^S\,d$, so that $a\,\mathcal{R}^*\,u \leq_{\mathcal{L}}^S b$.

Conversely, suppose the given condition on S holds. Let $e, f \in E(Q)$ be such that $e = a^{-1}a$ and $f = b^{-1}b$, and let $c \in S$ be such that $a\,\mathcal{R}^*\,c \leq_{\mathcal{L}}^S b$. In Q we have $a\,\mathcal{R}^Q\,c \leq_{\mathcal{L}}^Q b$, so that $c^{-1}c \leq b^{-1}b$. Hence $e = a^{-1}a$ and $c^{-1}c \leq f$, so that Q is simple.

□

Corollary 4.3.3. *The following conditions are equivalent for a left ample semigroup S:*

(i) $\Sigma(S)$ is bisimple;

(ii) S has the (LC)condition and $\mathcal{R}^* \circ \mathcal{L} = S \times S$;

(iii) S is a left I-order in $\Sigma(S)$ and $\mathcal{R}^* \circ \mathcal{L} = S \times S$.

Proof. We recall that the embedding of S into $\Sigma(S)$ is via what, in the terminology of [31], are called one-one *partial right translations*. It follows that $\Sigma(S)$ is an inverse subsemigroup of the inverse semigroup $\hat{S}$ of one-one partial right translations. Thus for any $\alpha \in \Sigma(S)$, $\operatorname{dom}\alpha$ is a left ideal and for any $a \in \operatorname{dom}\alpha$ and $x \in S$, $(xa)\alpha = x(a\alpha)$.

$(ii) \Rightarrow (iii)$ and $(iii) \Rightarrow (i)$ are immediate from Lemma 4.3.2 and Theorem 4.2.2.

$(i) \Rightarrow (ii)$. Suppose that $\Sigma(S)$ is bisimple. For any $\alpha \in \Sigma(S)$, we know that $\alpha\,\mathcal{D}\,\rho_e$, for some $e \in E(S)$, so that $\alpha\,\mathcal{R}\,\beta\,\mathcal{L}\,\rho_e$ in $\Sigma(S)$. Then $\operatorname{dom}\alpha = \operatorname{dom}\beta$ and $\operatorname{im}\beta = Se$. It follows that $\operatorname{dom}\beta = S(e\beta^{-1})$ so that $\operatorname{dom}\alpha$ is principal. Now let $a, b \in S$; then

$$\operatorname{dom}(\rho_a\rho_b^{-1}) = (Sa \cap Sb)\rho_a^{-1} = Sw$$

for some $w \in S$, and so

$$Swa = Sa \cap Sb$$

and S has (LC). From Theorem 4.2.2 and Lemma 4.3.2, $\mathcal{R}^* \circ \mathcal{L}$ is universal on S. $\square$

We recall that a left ample semigroup S is *proper* if $\mathcal{R}^* \cap \sigma = \iota$, where σ is the least right cancellative congruence on S, and where σ is given by the formula that for any $a, b \in S$,

$$a \,\sigma\, b \Leftrightarrow ea = eb \text{ for some } e \in E(S).$$

Clearly, if S is a subsemigroup in an inverse semigroup Q, then if $a \,\sigma\, b$ in S, we have that $a \,\sigma\, b$ in Q, but the converse may not be true. In other words, there is a natural morphism from S/σ to Q/σ, but this may not be an embedding.

Theorem 4.3.4. *Let S be a left ample semigroup such that S is a left I-order in Q where S is a union of $\mathcal{R}$-classes of Q. Then the following conditions are equivalent:*

(i) Q is proper;

(ii) S is proper and S/σ embeds naturally in Q/σ;

(iii) S is proper and S/σ is cancellative.

Proof. $(i) \Rightarrow (ii)$ Suppose that Q is proper, and $a, b \in S$ are such that $a \,\sigma\, b$ *in* Q. Then $ea = eb$ for some $e \in E(Q)$ so that $b^+ea = b^+eb = eb^+b = eb$ and $a^+eb = a^+ea = ea^+aa = ea$. Hence $a^+eb = b^+ea$ and so $ea^+b = eb^+a$, as $E(S)$ is a semilattice. But $b^+a \,\mathcal{R}^Q\, a^+b$, and so $b^+a = a^+b$. This gives that $a \,\sigma\, b$ *in* S.

Clearly, if $a, b \in S$ and $a \,(\mathcal{R}^* \cap \sigma)\, b$ in S, then $a \,(\mathcal{R} \cap \sigma)\, b$ in Q, whence $a = b$ and S is proper.

$(ii) \Rightarrow (iii)$ This is clear.

$(iii) \Rightarrow (i)$ Notice that by Lemmas 4.1.4 and 4.3.2, S is straight and any idempotent of Q has the form $a^{-1}a$ for some $a \in S$. Let $a^{-1}b, c^{-1}d \in Q$, where $a, b, c, d \in S$, $a \,\mathcal{R}^*\, b$ and $c \,\mathcal{R}^*\, d$. Suppose that $a^{-1}b \,(\mathcal{R} \cap \sigma)\, c^{-1}d$ in Q. Then there exists $x \in S$ such that

$$x^{-1}xa^{-1}b = x^{-1}xc^{-1}d$$

and $a^{-1}b \,\mathcal{R}^Q\, c^{-1}d$. From the former, $xa^{-1}b = xc^{-1}d$ and from the latter, $a \,\mathcal{L}^Q\, c$, by Lemma 3.2.3. By Lemma 4.3.2, $a \,\mathcal{L}^S\, c$ and so there exist $u, v \in S$ with $a = uc$ and

$c = va$. We may choose u, v such that $a^+u = u$ and $c^+v = v$. Now $a = uc = uva$ so that $a^+ = uva^+$, whence $u = a^+u = uva^+u = uvu$. Similarly, $v = vuv$, so that u and v are mutually inverse in both S and Q.

From $xa^{-1}b = xc^{-1}d$ we have that

$$xa^{-1}b = x(va)^{-1}d = xa^{-1}v^{-1}d = xa^{-1}ud.$$

But, by Lemmas 3.2.3, 4.1.4 and 4.3.2, $xa^{-1}\,\mathcal{L}^Q\,y$ for some $y \in S$, so that $yb = yud$, and as S/σ is cancellative, $b\,\sigma\,ud$ in S. Also, $b\,\mathcal{R}^*\,a = uc\,\mathcal{R}^*\,ud$ so that as S is proper, $b = ud$. Now

$$a^{-1}b = a^{-1}ud = a^{-1}v^{-1}d = (va)^{-1}d = c^{-1}d$$

and Q is proper as required. □

We remark that if the conditions of Theorem 4.3.4 hold, then for any $q = [a^{-1}b] \in Q/\sigma$, we have that $q = [a]^{-1}[b]$ and so the cancellative monoid S/σ is a left order in the group Q/σ.

The following result is classic; most of it follows from Theorem 4.2.2 and Corollary 4.3.3.

Corollary 4.3.5. *[2, 30, 36] The following conditions are equivalent for a right cancellative monoid S:*

(i) $\Sigma(S)$ is bisimple;

(ii) S has the (LC) condition;

(iii) S is a left I-order in $\Sigma(S)$.

If the above conditions hold, then S is the $\mathcal{R}$-class of the identity of $\Sigma(S)$. Further, $\Sigma(S)$ is proper if and only if S is cancellative.

Conversely, the $\mathcal{R}$-class of the identity of any bisimple inverse monoid is right cancellative with Condition (LC).

Proof. The equivalence of $(i), (ii)$ and (iii) follows from Corollary 4.3.3 and the fact that $\mathcal{R}^*$ is universal on S.

Suppose that $(i), (ii)$ and (iii) hold. Let e be the identity of S. As remarked in Section 2.4.1, $\Sigma(S)$ is a monoid with identity e. Since S is a single $\mathcal{R}^*$-class, and the embedding of S into $\Sigma(S)$ is a $(2,1)$-embedding, we have $S \subseteq R_e^{\Sigma(S)}$. Again by Theorem 4.2.2, we have that $R_e^{\Sigma(S)} \subseteq S$, so that $S = R_e^{\Sigma(S)}$.

Since $\sigma = \iota$ on S, it is clear that S is proper and $S/\sigma \cong S$. From Theorem 4.3.4 $\Sigma(S)$ is proper if and only if S is cancellative.

Conversely, let R be the $\mathcal{R}$-class of the identity of a bisimple inverse monoid Q. By Theorem 2.4.2, R is a right cancellative monoid with the (LC) condition, and from the comments before Example 3.1.2, we have that R is a left I-order in Q. $\square$

To study the case when $\Sigma(S)$ is bisimple inverse ω-semigroup, Clifford obtains the following result [2].

Lemma 4.3.6. *Let Q be a bisimple inverse semigroup with identity. Let S be the right unit subsemigroup, and G the unit group, of Q. Then the partially ordered sets of principal left ideals of Q and S are isomomorphic.*

If $\Sigma(S)$ is a bisimple inverse ω-semigroup, then for any two idempotents $a^{-1}a, b^{-1}b$ in $\Sigma(S)$ we have $a^{-1}a \leq b^{-1}b$ or $b^{-1}b \leq a^{-1}a$. In the former case $Qa \subseteq Qb$, and so $Sa \subseteq Sb$. In the latter case $Qb \subseteq Qa$, and so $Sb \subseteq Sa$. Hence

$$a^{-1}a \leq b^{-1}b \Longleftrightarrow Qa \subseteq Qb \Longleftrightarrow Sa \subseteq Sb$$

or

$$b^{-1}b \leq a^{-1}a \Longleftrightarrow Qb \subseteq Qa \Longleftrightarrow Sb \subseteq Sa.$$

Lemma 4.3.7. *Let S be a right cancellative monoid. If the principal left ideals of S are linearly ordered, then the idempotents of $\Sigma(S)$ form a chain.*

Proof. It is clear that S has the (LC) condition and $\Sigma(S)$ is a bisimple inverse semigroup. We proceed to show that $E(\Sigma(S))$ a chain. For any $a^{-1}a, b^{-1}b$ in $E(\Sigma(S))$ for some $a, b \in S$ we have $a = tb$ or $b = ra$ for some $t, r \in S$. In the former case, $a^{-1}ab^{-1}b = a^{-1}a$. Similarly, in the latter case $b^{-1}ba^{-1}a = b^{-1}b$. $\square$

The proof of the following corollary is clear so we have omitted it.

Corollary 4.3.8. *Let S be a right cancellative monoid. If the principal left ideals of S form an ω-chain, then the following are equivalent:*

(i) S is left cancellative;

(ii) $\Sigma(S)$ is a proper bisimple inverse ω-semigroup.

We now give a promised simplification of Theorem 3.2.9. The following result was first proved in the special case of S and T being right cancellative in [42]. First, we say that a $(2,1)$-homomorphism $\phi : S \to T$, where S and T are left ample semigroups with the (LC) condition, is *(LC)-preserving* if, for any $b, c \in S$ with $Sb \cap Sc = Sw$, we have that

$$T(b\phi) \cap T(c\phi) = T(w\phi).$$

This condition is not new: it appeared originally in [42] for right cancellative monoids with (LC), where it was called an *sl homomorphism* and subsequently (or variations thereof, and under different names) in, for example, [20] and [31]. Using the fact that for idempotents e, f of an inverse semigroup Q, we have that $Qe \cap Qf = Qef$, it is easy to verify that any morphism between inverse semigroups is (LC)-preserving.

The following result was first proved in the special case of S and T being right cancellative in [42].

Theorem 4.3.9. *Let S and T be left ample semigroups with the (LC) condition and let Q and P be their inverse hulls. Suppose that $\phi : S \to T$ is a $(2,1)$-homomorphism. Then ϕ lifts to a homomorphism $\overline{\phi} : Q \to P$ if and only if ϕ is (LC)-preserving.*

Proof. For ease in this proof we identify S and T with $S\theta_S$ and $T\theta_T$, respectively. Since S and T is (2,1)-subalgebra of Q and P respectively, so that ϕ must preserve the unary operation $^+$ and hence the relation $\mathcal{R}^*$. Since $(\mathcal{R}^*)^S = \mathcal{R}^Q \cap (S \times S)$ and $(\mathcal{R}^*)^T = \mathcal{R}^P \cap (T \times T)$ so that (i) of Theorem 3.2.9, holds. It remains to show that (ii) of that theorem holds if and only if ϕ is (LC)-preserving.

Suppose first that ϕ is (LC)-preserving. If $(a, b, c) \in \mathcal{T}_S^Q$, then $ab^{-1}Q \subseteq c^{-1}Q$. Now S has (LC) so that $Sa \cap Sb = Sw$ for some $w \in S$ and $ua = vb = w$ for some $u, v \in S$ with $ua^+ = u$ and $vb^+ = v$. From Lemma 4.2.1, $ab^{-1} = u^{-1}v$ and $u\,\mathcal{R}^*\,v$.

Hence $u^{-1}vQ \subseteq c^{-1}Q$ so that $Su \subseteq Sc$, by Lemma 4.3.2. Clearly,

$$T(u\phi) \subseteq T(v\phi), u\phi a\phi = v\phi b\phi = w\phi, u\phi(a\phi)^+ = u\phi \text{ and } v\phi(b\phi)^+ = v\phi.$$

As ϕ is (LC)-preserving, $T(a\phi) \cap T(b\phi) = T(w\phi)$, whence $(a\phi)(b\phi)^{-1} = (u\phi)^{-1}v\phi$ and it follows that $(a\phi, b\phi, c\phi) \in \mathcal{T}_T^P$.

Conversely, suppose that *(ii)* of Theorem 3.2.9 holds, so that ϕ lifts to a homomorphism $\overline{\phi} : Q \to P$. Supppose that $b, c \in S$ and $Sb \cap Sc = Sw$. We have $ub = vc = w$ for some $u, v \in S$ with $ub^+ = u$, $vc^+ = v$ and $u \, \mathcal{R}^* \, v$. This gives that $bc^{-1} = u^{-1}v$ and so, applying $\overline{\phi}$, we have $b\phi(c\phi)^{-1} = (u\phi)^{-1}v\phi$. As T has (LC), we certainly have that $b\phi(c\phi)^{-1} = h^{-1}k$ for some $h, k \in T$ with $h(b\phi) = k(c\phi) = z, T(b\phi) \cap T(c\phi) = Tz$ and $h \, \mathcal{R}^* \, k$ in T. From Lemmas 4.3.2 and 3.2.3 $u\phi \, \mathcal{L} \, h$ in T, so that $w\phi = (ub)\phi = u\phi \, b\phi \, \mathcal{L} \, h(b\phi) = z$ in T. We now have that

$$T(b\phi) \cap T(c\phi) = Tz = T(w\phi)$$

and ϕ is (LC)-preserving. □

The above result could (via a series of intermediate steps) be deduced from Theorem 2.6 of [31]. For, the ample condition ensures that a left ample semigroup is embedded in the semigroup $\hat{S}$ of one-to-one partial right translations of S, via the right regular representation described in Section 4.1. Further, the image of S is contained in $J(S)$, the set of join irreducible elements of $\hat{S}$. By [31, Proposition 1.14], if S has (LC), then $J(S)$ is an inverse semigroup, which is isomorphic to our $\Sigma(S)$. The restriction of θ in [31, Theorem 2.6] to $J(S)$, with a slight adaptation of the notion of *permissible homomorphism*, will now give our Theorem 4.3.9.

Chapter 5

Left I-orders in semilattices of inverse semigroups

Our aim in this chapter is to find the structure of semigroups of left I-quotients of certain semilattices of left ample semigroups with the (LC) condition.

Clifford [1] showed that any right cancellative monoid S with the (LC) condition is the $\mathcal{R}$-class of the identity of its inverse hull $\Sigma(S)$. Moreover, in our terminology, S is a left I-order in $\Sigma(S)$. Gantos [20] considered semigroups which are semilattices of right cancellative monoids with the (LC) condition and certain further conditions. In Chapter 4 we extended Clifford's result to a left ample semigroup with the (LC) condition. In this chapter we extend Gantos's result to certain strong semilattices of left ample semigroups.

In the first section we prove a lemma concerning properties of a semigroup S which is a semilattice Y of right cancellative monoids $S_\alpha, \alpha \in Y$. Gantos has developed a structure for semigroups Q which are semilattices Y of bisimple inverse monoids Q_α, such that the set of identities elements forms a subsemigroup. His structure is determined by semigroups S which are strong semilattices Y of right cancellative monoids $S_\alpha, \alpha \in Y$ with the (LC) condition and certain homomorphisms. We prove one of these conditions is equivalent to a semigroup S having the (LC) condition. We use this equivalence to define a nice form for the multiplication that is easier to deal with than the form which Gantos used.

In the second section we are concerned with a semigroup of left I-quotients of a left ample semigroup S with the (LC) condition which is a strong semilattice Y of left ample semigroups $S_\alpha, \alpha \in Y$, such that each S_α has the (LC) condition. We study the relationship between the inverse hull of S and the semigroup of left I-quotients of S. In the final section of this chapter we use the results in the previous sections to prove the equivalence between two categories.

5.1 Left I-quotients of semilatices of right cancellative semigroups

Gantos's main theorem states: Let S be a strong semilattice Y of right cancellative monoids $S_\alpha, \alpha \in Y$ with (LC) and connecting homomorphisms $\varphi_{\alpha,\beta}, \alpha \geq \beta$. Suppose in addition that (C_2) holds, where (C_2): if $S_\alpha a_\alpha \cap S_\alpha b_\alpha = S_\alpha c_\alpha$ for all $a_\alpha, b_\alpha, c_\alpha \in S_\alpha$, then

$$S_\beta(a_\alpha\varphi_{\alpha,\beta}) \cap S_\beta(b_\alpha\varphi_{\alpha,\beta}) = S_\beta(c\varphi_{\alpha,\beta})$$

for all $\alpha, \beta \in Y$ with $\alpha \geq \beta$. In the terminology of Chapter 4, (C_2) says that the connecting homomorphisms are (LC)-preserving. He obtained a semigroup Q which is a semilattice Y of bisimple inverse semigroup Q_α, with identity $e_\alpha, \alpha \in Y$ such that $\{e_\alpha : \alpha \in Y\}$ is a subsemigroup of Q. In fact, Q_α is the inverse hull of S_α for each $\alpha \in Y$. We show that (C_2) is equivalent to S having the (LC) condition. We then reprove Gantos's result. In Corollary 5.1.14, we provide a simple proof of Theorem 5.1.3 completely independent of [20], by using Theorem 3.2.9 and Lemma 5.1.2. We start with the following useful lemma.

Lemma 5.1.1. *(cf. [9]) Let S be a semilattice Y of right cancellative monoids $S_\alpha, \alpha \in Y$. Let e_α denote the identity of $S_\alpha, \alpha \in Y$. Then*

(1) *$e_\beta a_\alpha = a_\alpha e_\beta$ if $\alpha \geq \beta$ where $a_\alpha \in S_\alpha$;*

(2) *$e_\alpha e_{\alpha\beta} = e_{\alpha\beta}$;*

(3) *$E(S)$ is a semilattice;*

(4) *the idempotents are central;*

(5) *for any $a, b \in S$, $a \mathcal{R}^* b$ in S if and only if $a, b \in S_\alpha$ for some α in Y;*

(6) *S is a left ample semigroup.*

Proof. (1) Let $e_\beta \in S_\beta$ and $a_\alpha \in S_\alpha$ for some $\alpha, \beta \in Y$, where $\alpha \geq \beta$. Then $e_\beta a_\alpha$ and $a_\alpha e_\beta$ are in $S_{\beta\alpha} = S_\beta$. Hence

$$e_\beta a_\alpha = (e_\beta a_\alpha)e_\beta = e_\beta(a_\alpha e_\beta) = a_\alpha e_\beta.$$

(2) Let $e_\alpha \in S_\alpha$ and $e_\beta \in S_\beta$ be the identities of S_α and S_β respectively. From (1) it follows that

$$e_\alpha e_{\alpha\beta} = e_\alpha e_\alpha e_{\alpha\beta} = e_\alpha e_{\alpha\beta} e_\alpha.$$

Hence $(e_\alpha e_{\alpha\beta})e_\alpha e_{\alpha\beta} = e_\alpha e_{\alpha\beta}$, that is, $e_\alpha e_{\alpha\beta}$ is an idempotent in $S_{\alpha\beta}$. But there is only one idempotent in $S_{\alpha\beta}$, so that $e_\alpha e_{\alpha\beta} = e_{\alpha\beta} = e_{\alpha\beta}e_\alpha$.

(3) Let $e_\alpha \in S_\alpha$ and $e_\beta \in S_\beta$ for some $\alpha, \beta \in Y$. Then $e_\alpha e_\beta \in S_{\alpha\beta}$ and from (2) we have that

$$e_\alpha e_\beta = e_\alpha e_\beta e_{\alpha\beta} = e_\alpha e_{\alpha\beta} = e_{\alpha\beta}.$$

(4) Let $e_\alpha \in S_\alpha$ and $a_\beta \in S_\beta$ for some $\alpha, \beta \in Y$. Then $e_\alpha a_\beta \in S_{\alpha\beta}$ and from (1) and (2) we get

$$e_\alpha a_\beta e_{\alpha\beta} = e_\alpha e_{\alpha\beta} a_\beta = e_{\alpha\beta} a_\beta = a_\beta e_{\alpha\beta} = a_\beta e_\alpha e_{\alpha\beta}.$$

Since $e_{\alpha\beta}$ is the identity of $S_{\alpha\beta}$, we have that $e_\alpha a_\beta = a_\beta e_\alpha$.

(5) Suppose that $a \,\mathcal{R}^*\, b$ in S where $a \in S_\alpha$ and $b \in S_\beta$. Then $e_\beta a = e_\beta e_\alpha a$ and so $e_\beta b = e_\beta e_\alpha b$ which implies that $\beta \leq \alpha$. Dually, $\alpha \leq \beta$ and hence $\alpha = \beta$.

Conversely, suppose that $b \in S_\alpha$ and $xb = yb$ for some $x, y \in S$ where $x \in S_\beta$ and $y \in S_\gamma$. Then $\beta\alpha = \alpha\gamma$ as $xb, yb \in S_{\alpha\beta} = S_{\alpha\gamma}$. Thus $xbe_{\alpha\beta} = ybe_{\alpha\beta}$ so that from (1) we get $xe_{\alpha\beta}b = ye_{\alpha\beta}b$, and so $xe_{\alpha\beta}(be_{\alpha\beta}) = ye_{\alpha\beta}(be_{\alpha\beta})$. Now $xe_{\alpha\beta}, ye_{\alpha\beta}, be_{\alpha\beta}$ all lie in $S_{\alpha\beta}$ which is right cancellative, so that $xe_{\alpha\beta} = ye_{\alpha\beta}$. As in the proof of (3) we have that $e_\alpha e_\beta = e_\beta e_\alpha = e_{\alpha\beta}$. Hence $xe_\beta e_\alpha = ye_\beta e_\alpha = ye_\gamma e_\alpha$ and then $xe_\alpha = ye_\alpha$. Also, if $xb = b$, that is, $xb = e_\alpha b$, then $xe_\alpha = e_\alpha e_\alpha = e_\alpha$. Thus $b \,\mathcal{R}^*\, e_\alpha$ in S. Hence for any $a \in S_\alpha$ we have that $a \,\mathcal{R}^*\, b$ in S as required.

(6) From (3) we have that $E(S)$ is a semilattice. By (5) we deduce that each $\mathcal{R}^*$-class contains an idempotent which must be unique as $E(S)$ is a semilattice. Notice that

if $a \in S_\alpha$, then $a^+ = e_\alpha$. To see that S is left ample, let $a \in S_\alpha$ and $e_\beta \in S_\beta$. We have to show that $ae_\beta = (ae_\beta)^+a$. Using (1) and the fact that $e_\alpha e_\beta = e_\beta e_\alpha = e_{\alpha\beta}$ as in the proof of (3) we get

$$(ae_\beta)^+a = e_{\alpha\beta}a = ae_{\alpha\beta} = ae_\alpha e_\beta = ae_\beta$$

as required. □

In the following lemma we show that the 'strong' in Gantos's result is automatic. The proof of the following is entirely routine, but we provide it for completeness.

Lemma 5.1.2. *Let $P = \mathcal{S}(Y; S_\alpha)$ where each S_α is a monoid with identity e_α, such that $E = \{e_\alpha : \alpha \in Y\}$ is a subsemigroup of P. Then E is a semilattice isomorphic to Y and E is central in P.*

If we define $\phi_{\alpha,\beta} : S_\alpha \to S_\beta$ by $a_\alpha\phi_{\alpha,\beta} = a_\alpha e_\beta$, for all $a_\alpha \in S_\alpha$ where $\alpha \geq \beta$, then each $\phi_{\alpha,\beta}$ is a monoid homomorphism, and $P = \mathcal{S}(Y; S_\alpha; \phi_{\alpha,\beta})$. Conversely[1]*, If $P = \mathcal{S}(Y; S_\alpha; \varphi_{\alpha,\beta})$, then for $\alpha, \beta \in Y$ with $\alpha \geq \beta$ and $a \in S_\alpha$ we have $a\varphi_{\alpha,\beta} = ae_\beta = e_\beta a$.*

Proof. It is easy to see that Y is isomorphic to E under the map $\alpha \longrightarrow e_\alpha$, where e_α, is the identity of S_α. Let $a_\alpha \in S_\alpha$ and suppose first that $\alpha \geq \beta$. Then

$$a_\alpha e_\beta = e_\beta(a_\alpha e_\beta) = (e_\beta a_\alpha)e_\beta = e_\beta a_\alpha.$$

Now, for arbitrary e_γ,

$$a_\alpha e_\gamma = (a_\alpha e_\gamma)e_{\alpha\gamma} = a_\alpha(e_\gamma e_{\alpha\gamma}) = a_\alpha e_{\alpha\gamma} = e_{\alpha\gamma}a_\alpha = (e_{\alpha\gamma}e_\gamma)a_\alpha = e_{\alpha\gamma}(e_\gamma a_\alpha) = e_\gamma a_\alpha,$$

so that E is central in P.

It is easy to see that for $\alpha \geq \beta$, $\phi_{\alpha,\beta} : S_\alpha \to S_\beta$ is a monoid homomorphism, $\phi_{\alpha,\alpha} = I_{S_\alpha}$ and for $\alpha \geq \beta \geq \gamma$, $\phi_{\alpha,\gamma} = \phi_{\alpha,\beta}\phi_{\beta,\gamma}$. Let $Q = \mathcal{S}(Y; S_\alpha; \phi_{\alpha,\beta})$ and denote the binary operation in Q by $*$.

For $a_\alpha, b_\beta \in Q$ we have

$$a_\alpha * b_\beta = (a_\alpha\phi_{\alpha,\alpha\beta})(b_\beta\phi_{\beta,\alpha\beta}) = (a_\alpha e_{\alpha\beta})(b_\beta e_{\alpha\beta}) = (a_\alpha b_\beta)e_{\alpha\beta} = a_\alpha b_\beta,$$

[1]This part of the lemma is folklore and was drawn to my attention by Fountain.

as required.

On the other hand, let $P = \mathcal{S}(Y; S_\alpha; \varphi_{\alpha,\beta})$. First, note that if $\alpha, \beta \in Y$ with $\alpha \geq \beta$, then $e_\alpha \varphi_{\alpha,\beta} = e_\beta$ because $\varphi_{\alpha,\beta}$ is a monoid homomorphism.

Suppose that $\alpha, \beta \in Y$ with $\alpha \geq \beta$ and let $a \in S_\alpha$. Then

$$ae_\beta = a\varphi_{\alpha,\beta} e_\beta = a\varphi_{\alpha,\beta}$$

since $a\varphi_{\alpha,\beta} \in S_\beta$ and e_β is the identity of S_β. □

Now we introduce the main theorem of this section. It is worth pointing out that this theorem is equivalent to the main theorem of [20]. However, we give here another proof. The idea is to use the (LC) condition introduced previously in order to prove such a theorem. This proof will make our result easier to extend to other settings in the next section. Let $\Sigma(S)$ be the inverse hull of left I-quotents of a right cancellative monoid S with (LC). In the rest of this section we identify S with $S\theta_S$, where θ_S is the embedding of S into $\mathcal{I}_S$. We write $a^{-1}b$ for short the element $\rho_a^{-1}\rho_b$ of $\Sigma(S)$ where $a, b \in S$.

Theorem 5.1.3. *Let $S = \mathcal{S}(Y; S_\alpha)$ be a semilattice of right cancellative monoids S_α with identity $e_\alpha, \alpha \in Y$. Suppose that S, and each S_α, has (LC). Then $Q = \mathcal{S}(Y; \Sigma_\alpha)$ is a semilattice of bisimple inverse monoids (where Σ_α is the inverse hull of S_α) and the multiplication in Q is defined by: for $a^{-1}b \in \Sigma_\alpha, c^{-1}d \in \Sigma_\beta$,*

$$a^{-1}bc^{-1}d = (ta)^{-1}(rd)$$

where $S_{\alpha\beta}b \cap S_{\alpha\beta}c = S_{\alpha\beta}w$ and $tb = rc = w$ for some $t, r \in S_{\alpha\beta}$.

Proof. By Corollary 4.3.5, each S_α is a left I-order in Σ_α where each S_α is the $\mathcal{R}$-class of the identity of Σ_α and Σ_α is a bisimple inverse monoid. We prove the theorem by means of a sequence of lemmas. We begin by the following lemma due to Clifford.

Lemma 5.1.4. *(cf. [2, Lemma 4.1]) Let T be a right cancellative monoid. Then for $a, b \in T$ we have*

$$a\,\mathcal{L}\,b \text{ if and only if } a = ub,$$

for some unit u of T.

Lemma 5.1.5. *The multiplication is well-defined.*

Proof. Suppose that we have elements a_1, b_1, a_2, b_2 of S_α, c_1, d_1, c_2, d_2 of S_β such that

$$a_1^{-1}b_1 = a_2{}^{-1}b_2 \text{ in } \Sigma_\alpha \text{ and } c_1^{-1}d_1 = c_2{}^{-1}d_2 \text{ in } \Sigma_\beta.$$

By Lemma 3.2.5,

$$a_1 = u_1 a_2, \ \ b_1 = u_1 b_2$$

for some unit $u_1 \in S_\alpha$ and

$$c_1 = v_1 c_2, \ \ d_1 = v_1 d_2$$

for some unit $v_1 \in S_\beta$. By definition,

$$a_1^{-1}b_1 c_1{}^{-1}d_1 = (t_1 a_1)^{-1}(r_1 d_1)$$

where

$$S_{\alpha\beta}b_1 \cap S_{\alpha\beta}c_1 = S_{\alpha\beta}w_1 \text{ and } t_1 b_1 = r_1 c_1 = w_1$$

for some $t_1, r_1, w_1 \in S_{\alpha\beta}$. Also,

$$a_2^{-1}b_2 c_2{}^{-1}d_2 = (t_2 a_2)^{-1}(r_2 d_2)$$

where

$$S_{\alpha\beta}b_2 \cap S_{\alpha\beta}c_2 = S_{\alpha\beta}w_2 \text{ and } t_2 b_2 = r_2 c_2 = w_2$$

for some $t_2, r_2, w_2 \in S_{\alpha\beta}$.

We have to show that $a_1^{-1}b_1 c_1{}^{-1}d_1 = a_2^{-1}b_2 c_2{}^{-1}d_2$, that is,

$$(t_1 a_1)^{-1}(r_1 d_1) = (t_2 a_2)^{-1}(r_2 d_2)$$

and to do this we need to prove that

$$t_1 a_1 = u t_2 a_2 \text{ and } r_1 d_1 = u r_2 d_2$$

for some unit u in $S_{\alpha\beta}$, using Lemma 3.2.5. We aim to prove that $S_{\alpha\beta}w_1 = S_{\alpha\beta}w_2$. We get this if we prove that $S_{\alpha\beta}b_1 = S_{\alpha\beta}b_2$ and $S_{\alpha\beta}c_1 = S_{\alpha\beta}c_2$.

Since $b_1 = u_1 b_2$, using Lemma 5.1.1(1), we have that

$$e_{\alpha\beta} b_1 = e_{\alpha\beta} u_1 b_2 = (u_1 e_{\alpha\beta}) b_2 = (u_1 e_{\alpha\beta})(e_{\alpha\beta} b_2)$$

so that $S_{\alpha\beta} e_{\alpha\beta} b_1 \subseteq S_{\alpha\beta} e_{\alpha\beta} b_2$ and since $b_2 = u_1^{-1} b_1$ the converse implication also holds. We have

$$S_{\alpha\beta} b_1 = S_{\alpha\beta} e_{\alpha\beta} b_1 = S_{\alpha\beta} e_{\alpha\beta} b_2 = S_{\alpha\beta} b_2.$$

Similarly, $S_{\alpha\beta} e_{\alpha\beta} c_1 = S_{\alpha\beta} e_{\alpha\beta} c_2$. Hence $S_{\alpha\beta} w_1 = S_{\alpha\beta} w_2$ so that $w_1 \mathcal{L} w_2$ in $S_{\alpha\beta}$. By Lemma 5.1.4, $w_1 = l w_2$ for some unit l in $S_{\alpha\beta}$. Then

$$w_1 = t_1 b_1 = l w_2 = l(t_2 b_2) = l t_2 (u_1^{-1} b_1).$$

But, by Lemma 5.1.1 $a_1 \mathcal{R}^* b_1$ in S, it follows that $t_1 a_1 = l t_2 u_1^{-1} a_1 = l t_2 a_2$. Since

$$w_1 = r_1 c_1 = l w_2 = l r_2 c_2 = l r_2 v_1^{-1} c_1$$

and $c_1 \mathcal{R}^* d_1$ in S, again using Lemma 5.1.1, we have

$$r_1 d_1 = l r_2 v_1^{-1} d_1 = l r_2 v_1^{-1} v_1 d_2 = l r_2 d_2$$

as required. □

In order to prove the associative law we need to introduce subsidiary lemmas. The proof of the next lemma depends only on the fact that S_α is right cancellative and the proof can be found in [20].

Lemma 5.1.6. *$(S_\alpha a_\alpha \cap S_\alpha b_\alpha) c_\alpha = S_\alpha a_\alpha c_\alpha \cap S_\alpha b_\alpha c_\alpha$ for all $a_\alpha, b_\alpha, c_\alpha \in S_\alpha$.*

In the following lemma we prove the equivalence between S having the (LC) condition and (C_2) mentioned in the introduction. In the next section we extend its statement to a more general context. We are grateful to Dr. Gould for supplying the proof of this equivalence.

Lemma 5.1.7. *Let $S = \mathcal{S}(Y; S_\alpha)$ be a semilattice Y of right cancellative monoids S_α with the (LC) condition. Then*
S has (LC) if and only if whenever $\beta \leq \alpha$, if $S_\alpha a_\alpha \cap S_\alpha b_\alpha = S_\alpha c_\alpha$ $(a_\alpha, b_\alpha, c_\alpha \in S_\alpha)$, then $S_\beta(a_\alpha e_\beta) \cap S_\beta(b_\alpha e_\beta) = S_\beta(c_\alpha e_\beta)$.

Proof. Suppose that $S_\alpha a_\alpha \cap S_\alpha b_\alpha = S_\alpha c_\alpha$ implies $S_\beta a_\alpha \cap S_\beta b_\alpha = S_\beta c_\alpha$ for all $\beta \leq \alpha$. Let $a \in S_\alpha$ and $b \in S_\beta$ for some $\alpha, \beta \in Y$. Then $ae_{\alpha\beta}, e_{\alpha\beta}b \in S_{\alpha\beta}$ so that as $S_{\alpha\beta}$ has (LC) we know that

$$S_{\alpha\beta}(e_{\alpha\beta}a) \cap S_{\alpha\beta}(e_{\alpha\beta}b) = S_{\alpha\beta}c$$

for some $c \in S_{\alpha\beta}$. Now, let $d \in Sa \cap Sb$, say $d \in S_\gamma$ so that $\gamma \leq \alpha\beta$ and $d = ua = vb$ for some $u, v \in S$. By assumption,

$$S_\gamma(e_{\alpha\beta}a)e_\gamma \cap S_\gamma(e_{\alpha\beta}b)e_\gamma = S_\gamma ce_\gamma.$$

Then $S_\gamma ae_\gamma \cap S_\gamma be_\gamma = S_\gamma ce_\gamma$. Now,

$$d = ua = vb = (e_\gamma u)a = (e_\gamma v)b \in S_\gamma a \cap S_\gamma b = S_\gamma c$$

as $e_\gamma u, e_\gamma v \in S_\gamma$. Then $d \in S_\gamma c$ and so $Sd \subseteq Sc$. Thus $Sa \cap Sb \subseteq Sc$. Also, $c \in S_{\alpha\beta}a \subseteq Sa$ and $c \in S_{\alpha\beta}b \subseteq Sb$. Thus $c \in Sa \cap Sb$. Hence $Sc \subseteq Sa \cap Sb$ and we get $Sc = Sa \cap Sb$.

On the other hand, suppose that S has (LC) and let $S_\alpha a_\alpha \cap S_\alpha b_\alpha = S_\alpha c_\alpha$, so that $c_\alpha = u_\alpha a_\alpha = v_\alpha b_\alpha$ for some $u_\alpha, v_\alpha \in S_\alpha$. We claim that

$$Sa_\alpha \cap Sb_\alpha = Sc_\alpha.$$

As S has the (LC) condition there exists $d \in S_\xi$ such that $Sa_\alpha \cap Sb_\alpha = Sd$. Then $d = ka_\alpha = hb_\alpha$ for some $k, h \in S$ and so $\xi \leq \alpha$. Since $c_\alpha \in Sa_\alpha \cap Sb_\alpha$ we have that $c_\alpha = rd$ for some $r \in S$ so that $\alpha \leq \xi$. Hence $\alpha = \xi$, that is, $d \in S_\alpha$ and we can write $d = d_\alpha$.

From $c_\alpha = rd$ we have that $c_\alpha = (e_\alpha r)d_\alpha \in S_\alpha d_\alpha$ so that $S_\alpha c_\alpha \subseteq S_\alpha d_\alpha$. Since $d_\alpha = ka_\alpha = hb_\alpha = (e_\alpha k)a_\alpha = (e_\alpha h)b_\alpha$, we have that $d_\alpha \in S_\alpha a_\alpha \cap S_\alpha b_\alpha = S_\alpha c_\alpha$, and so $S_\alpha d_\alpha \subseteq S_\alpha c_\alpha$. Thus $S_\alpha d_\alpha = S_\alpha c_\alpha$. Hence $d_\alpha \,\mathcal{L}\, c_\alpha$ in S_α, so that $d_\alpha \,\mathcal{L}\, c_\alpha$ in S. We have

$$Sa_\alpha \cap Sb_\alpha = Sc_\alpha.$$

Hence our claim is established.

Now let $\beta \leq \alpha$. Since S_β has the (LC) condition and $e_\beta a, e_\beta b \in S_\beta$ we have that

$$S_\beta(e_\beta a_\alpha) \cap S_\beta(e_\beta b_\alpha) = S_\beta w_\beta$$

for some $w_\beta \in S_\beta$. We aim to show that $S_\beta(e_\beta c_\alpha) = S_\beta w_\beta$.
Since $w_\beta \in S_\beta a_\alpha \cap S_\beta b_\alpha \subseteq Sa_\alpha \cap Sb_\alpha$ we have that $w_\beta \in Sc_\alpha$ and so $w_\beta = lc_\alpha$ for some $l \in S$, say $l \in S_\eta$ so that $\eta \geq \beta$. Since $w_\beta = e_\beta w_\beta = e_\beta lc_\alpha$ and $\eta \geq \beta$, it follows that $w_\beta = e_\beta w_\beta = le_\beta c_\alpha$, by Lemma 5.1.1. Then $w_\beta = (le_\beta)(e_\beta c_\alpha) \in S_\beta c_\alpha$ so that $S_\beta w_\beta \subseteq S_\beta(e_\beta c_\alpha)$.

Conversely, since $c_\alpha = u_\alpha a_\alpha = v_\alpha b_\alpha$ and $\beta \leq \alpha$, it follows that $e_\beta c_\alpha = e_\beta u_\alpha e_\beta a_\alpha = e_\beta v_\alpha e_\beta b_\alpha$, by Lemma 5.1.1. It follows that $e_\beta c_\alpha \in S_\beta a_\alpha \cap S_\beta b_\alpha = S_\beta w_\beta$. Hence $S_\beta(e_\beta c_\alpha) \subseteq S_\beta w_\beta$. Thus $S_\beta(e_\beta c_\alpha) = S_\beta w_\beta$ as required. $\square$

Lemma 5.1.8. *Let $a^{-1}b, a^{-1}e_\alpha \in \Sigma_\alpha$ and $c^{-1}d, e_\beta d \in \Sigma_\beta$ where $a, b \in S_\alpha, c, d \in S_\beta$ and e_α, e_β are the identity elements in S_α and S_β respectively. Then*

(i) $a^{-1}be_\beta d = (ae_{\alpha\beta})^{-1}(bd)$,

(ii) $(a^{-1}e_\alpha)(c^{-1}d) = (ca)^{-1}(de_{\alpha\beta})$.

Proof. (*i*) We have that $S_{\alpha\beta}e_\beta \cap S_{\alpha\beta}b = S_{\alpha\beta} \cap S_{\alpha\beta}b = S_{\alpha\beta}b$ and

$$e_{\alpha\beta}b = (be_{\alpha\beta})e_\beta = (e_{\alpha\beta}b)e_\beta = be_{\alpha\beta},$$

using Lemma 5.1.1. We have

$$\begin{aligned}
(a^{-1}b)(e_\beta d) &= a^{-1}(be_\beta)d \\
&= a^{-1}(be_\beta e_{\alpha\beta})d \\
&= a^{-1}(be_{\alpha\beta})d \\
&= a^{-1}(e_{\alpha\beta}b)d \\
&= (a^{-1}e_{\alpha\beta}^{-1})(bd) \\
&= (e_{\alpha\beta}a)^{-1}(bd) \\
&= (ae_{\alpha\beta})^{-1}(bd).
\end{aligned}$$

(*ii*) We have that $S_{\alpha\beta}c \cap S_{\alpha\beta}e_\alpha = S_{\alpha\beta}c \cap S_{\alpha\beta} = S_{\alpha\beta}c$ and

$$e_{\alpha\beta}c = (ce_{\alpha\beta})e_\alpha = (e_{\alpha\beta}c)e_\alpha = ce_{\alpha\beta},$$

using Lemma 5.1.1. We have

$$(a^{-1}e_\alpha)(c^{-1}d) = (ca)^{-1}(de_{\alpha\beta})$$

as required. $\square$

Lemma 5.1.9. *Let $a^{-1}b \in \Sigma_\alpha, e_\beta d, d^{-1}e_\beta \in \Sigma_\beta$ and $x^{-1}y \in \Sigma_\gamma$ where e_β is the identity element in S_β where $a, b \in S_\alpha, e_\beta, d \in S_\beta$ and $x, y \in S_\gamma$. Then*

(i) $(a^{-1}be_\beta d)x^{-1}y = a^{-1}b(e_\beta dx^{-1}y)$;

(ii) $(a^{-1}bd^{-1}e_\beta)x^{-1}y = a^{-1}b(d^{-1}e_\beta x^{-1}y)$.

Proof. (i) Let $a^{-1}b, e_\beta d, x^{-1}y$ be as in the hypothesis. Then

$$\begin{aligned}(a^{-1}be_\beta d)x^{-1}y &= (ae_{\alpha\beta})^{-1}(bd)x^{-1}y \text{ by Lemma 5.1.8 } (i),\\ &= (t_1a)^{-1}(r_1y)\end{aligned}$$

where $t_1bd = r_1x = w_1$ and

$$S_{\alpha\beta\gamma}(bde_{\alpha\beta\gamma}) \cap S_{\alpha\beta\gamma}(xe_{\alpha\beta\gamma}) = S_{\alpha\beta\gamma}w_1$$

for some $t_1, r_1, w_1 \in S_{\alpha\beta\gamma}$.

On the other hand, by definition of multiplication,

$$\begin{aligned}a^{-1}b(e_\beta dx^{-1}y) &= a^{-1}b(t_2e_\beta)^{-1}(r_2y)\\ &= (t_3a)^{-1}(r_3r_2y)\end{aligned}$$

where $t_2d = r_2x = w_2$ with

$$S_{\beta\gamma}(de_{\beta\gamma}) \cap S_{\beta\gamma}(xe_{\beta\gamma}) = S_{\beta\gamma}w_2 \tag{5.1}$$

for some $t_2, r_2, w_2 \in S_{\beta\gamma}$ and $t_3b = r_3t_2e_{\alpha\beta\gamma} = w_3$ with

$$S_{\alpha\beta\gamma}be_{\alpha\beta\gamma} \cap S_{\alpha\beta\gamma}t_2e_{\alpha\beta\gamma} = S_{\alpha\beta\gamma}w_3 \tag{5.2}$$

for some $t_3, r_3, w_3 \in S_{\alpha\beta\gamma}$. Using (5.1) and Lemma 5.1.7 gives

$$S_{\alpha\beta\gamma}d \cap S_{\alpha\beta\gamma}x = S_{\alpha\beta\gamma}w_2 \tag{5.3}$$

We must show that $(t_1a)^{-1}(r_1y) = (t_3a)^{-1}(r_3r_2y)$. By using Lemma 3.2.5, we have to show that $t_1a = ut_3a$ and $r_1y = ur_3r_2y$ for some unit u in $S_{\alpha\beta\gamma}$.

Once we know $w_1 \,\mathcal{L}\, w_3d$ in $S_{\alpha\beta\gamma}$, we have that $w_1 = hw_3d$ for some unit h in $S_{\alpha\beta\gamma}$ by Lemma 5.1.4. Hence $t_1bd = ht_3bd$ so that $t_1e_{\alpha\beta\gamma}bd = ht_3e_{\alpha\beta\gamma}bd$. Since t_1, ht_3 and $e_{\alpha\beta\gamma}bd$ are in $S_{\alpha\beta\gamma}$, which is right cancellative we obtain $t_1 = ht_3$ so that $t_1a = ht_3a$. Now,

$$w_1 = r_1x = t_1bd = ht_3bd = hr_3t_2d = hr_3r_2x.$$

As r_1, hr_3r_2 and $e_{\alpha\beta\gamma}x$ are in $S_{\alpha\beta\gamma}$ again by right cancellativity in $S_{\alpha\beta\gamma}$ we have that $r_1 = hr_3r_2$ and so $r_1y = hr_3r_2y$.
Now, as S has (LC)

$$\begin{aligned}
S_{\alpha\beta\gamma}w_1 &= S_{\alpha\beta\gamma}bd \cap S_{\alpha\beta\gamma}x \\
&= S_{\alpha\beta\gamma}bd \cap S_{\alpha\beta\gamma}d \cap S_{\alpha\beta\gamma}x \\
&= S_{\alpha\beta\gamma}bd \cap S_{\alpha\beta\gamma}w_2 && \text{by (5.3)} \\
&= S_{\alpha\beta\gamma}bd \cap S_{\alpha\beta\gamma}t_2d \\
&= S_{\alpha\beta\gamma}bde_{\alpha\beta\gamma} \cap S_{\alpha\beta\gamma}t_2de_{\alpha\beta\gamma} \\
&= (S_{\alpha\beta\gamma}b \cap S_{\alpha\beta\gamma}t_2)de_{\alpha\beta\gamma} && \text{by Lemma 5.1.6} \\
&= S_{\alpha\beta\gamma}w_3d && \text{by (5.2).}
\end{aligned}$$

(ii) Let $a^{-1}b, d^{-1}e_\beta, x^{-1}y$ be as in the hypothesis. Then,

$$\begin{aligned}
(a^{-1}bd^{-1}e_\beta)x^{-1}y &= (t_1a)^{-1}(r_1e_\beta)x^{-1}y \\
&= (t_2t_1a)^{-1}(r_2y)
\end{aligned}$$

where $t_1b = r_1d = w_1$ with

$$S_{\alpha\beta}(be_{\alpha\beta}) \cap S_{\alpha\beta}(de_{\alpha\beta}) = S_{\alpha\beta}w_1 \tag{5.4}$$

for some $t_1, r_1, w_1 \in S_{\alpha\beta}$ and $t_2r_1 = r_2x = w_2$ with

$$S_{\alpha\beta\gamma}r_1 \cap S_{\alpha\beta\gamma}x = S_{\alpha\beta\gamma}w_2. \tag{5.5}$$

for some $t_2, r_2, w_2 \in S_{\alpha\beta\gamma}$. By (5.4) and Lemma 5.1.7 we have

$$S_{\alpha\beta\gamma}b \cap S_{\alpha\beta\gamma}d = S_{\alpha\beta\gamma}w_1. \tag{5.6}$$

On the other hand, by Lemma 5.1.8(*ii*),

$$\begin{aligned}
a^{-1}b(d^{-1}e_\beta x^{-1}y) &= a^{-1}b(xd)^{-1}(ye_{\beta\gamma}) \\
&= (t_3a)^{-1}(r_3ye_{\beta\gamma})
\end{aligned}$$

where

$$t_3b = r_3xd = w_3, S_{\alpha\beta\gamma}(xd) \cap S_{\alpha\beta\gamma}(be_{\alpha\beta\gamma}) = S_{\alpha\beta\gamma}w_3$$

for some $t_3, r_3, w_3 \in S_{\alpha\beta\gamma}$.

We have to show that $(t_2t_1a)^{-1}(r_2y) = (t_3a)^{-1}(r_3ye_{\beta\gamma})$. By using Lemma 3.2.5, we have to show that $t_3a = vt_2t_1a$ and $r_3y = vr_2y$ for some unit v in $S_{\alpha\beta\gamma}$.

Once we know $w_3 \mathcal{L} w_2d$ in $S_{\alpha\beta\gamma}$, we have that $w_3 = kw_2d$ for some unit k in $S_{\alpha\beta\gamma}$, by Lemma 5.1.4. Hence $r_3xd = kr_2xd$ so that $r_3e_{\alpha\beta\gamma}xd = kr_2e_{\alpha\beta\gamma}xd$. Since $r_3, e_{\alpha\beta\gamma}xd$ and kr_2 are in $S_{\alpha\beta\gamma}$ which is right cancellative we obtain $r_3 = kr_2$ so that $r_3y = kr_2y$. Now,

$$w_3 = t_3b = r_3xd = kr_2xd = kt_2r_1d = kt_2t_1b.$$

Hence $t_3e_{\alpha\beta\gamma}b = kt_2t_1e_{\alpha\beta\gamma}b$ where $t_3, e_{\alpha\beta\gamma}b$ and kt_2t_1 are in $S_{\alpha\beta\gamma}$ again by right cancellativity in $S_{\alpha\beta\gamma}$ we have that $t_3 = kt_2t_1$ and so $t_3a = kt_2t_1a$.
Now,

$$\begin{aligned}
S_{\alpha\beta\gamma}w_3 &= S_{\alpha\beta\gamma}b \cap S_{\alpha\beta\gamma}xd \\
&= S_{\alpha\beta\gamma}b \cap S_{\alpha\beta\gamma}xd \cap S_{\alpha\beta\gamma}d \\
&= S_{\alpha\beta\gamma}xd \cap S_{\alpha\beta\gamma}w_1 && \text{by}(5.6) \\
&= S_{\alpha\beta\gamma}xd \cap S_{\alpha\beta\gamma}r_1d \\
&= S_{\alpha\beta\gamma}xde_{\alpha\beta\gamma} \cap S_{\alpha\beta\gamma}r_1de_{\alpha\beta\gamma} \\
&= (S_{\alpha\beta\gamma}x \cap S_{\alpha\beta\gamma}r_1)de_{\alpha\beta\gamma} && \text{by Lemma 5.1.6} \\
&= S_{\alpha\beta\gamma}w_2d && \text{by (5.5)}
\end{aligned}$$

as required. □

Lemma 5.1.10. *The associative law holds in Q.*

Proof. Suppose that $a^{-1}b \in \Sigma_\alpha, c^{-1}d \in \Sigma_\beta$ and $s^{-1}t \in \Sigma_\gamma$ where $a, b \in S_\alpha, c, d \in S_\beta$ and $s, t \in S_\gamma$. From Lemma 5.1.9, we have that

$$\begin{aligned}
a^{-1}b(c^{-1}ds^{-1}t) &= a^{-1}b(c^{-1}e_\beta e_\beta d.s^{-1}t) \\
&= a^{-1}b(c^{-1}e_\beta.e_\beta ds^{-1}t) \\
&= (a^{-1}bc^{-1}e_\beta)(e_\beta d.s^{-1}t) \\
&= (a^{-1}bc^{-1}e_\beta.e_\beta d)s^{-1}t \\
&= (a^{-1}b(c^{-1}e_\beta.e_\beta d))s^{-1}t \\
&= (a^{-1}bc^{-1}d)s^{-1}t.
\end{aligned}$$

□

From Lemmas 5.1.10 and 5.1.5 we get the proof of Theorem 5.1.3. □

Let $a \in S_\alpha$ and $b \in S_\beta$ for some $\alpha, \beta \in Y$. By Lemmas 5.1.8 and 5.1.1,

$$e_\alpha ae_\beta b = e_\alpha^{-1}ae_\beta^{-1}b = (e_\alpha e_{\alpha\beta})^{-1}(ab) = e_{\alpha\beta}(ab) = ab$$

and we get the following lemma;

Lemma 5.1.11. *The multiplication on Q extends the multiplication on S.*

The next corollary is now clear.

Corollary 5.1.12. *The semigroup S defined as above is a left I-order in $Q = \bigcup_{\alpha \in Y} \Sigma_\alpha$.*

Let $S = \mathcal{S}(Y; S_\alpha)$ be a semilattice Y of right cancellative monoids S_α with identity $e_\alpha, \alpha \in Y$ such that each $S_\alpha, \alpha \in Y$ has the (LC) condition. By Lemma 5.1.1, $E = \{e_\alpha : \alpha \in Y\}$ is a subsemigroup of S. Hence S is a strong semilattice Y with connecting homomorphisms $\varphi_{\alpha,\beta} : S_\alpha \longrightarrow S_\beta$ given by $a_\alpha \varphi_{\alpha,\beta} = a_\alpha e_\beta$ where $\alpha \geq \beta$ for any $a_\alpha \in S_\alpha$, by Lemma 5.1.2. In fact, every semilattice of right cancellative monoids is a strong semilattice of right cancellative monoids (see [39, Exercises III.7.12]). If S has the (LC) condition, then by Corollary 5.1.12, S has a semigroup of left I-quotients $Q = \bigcup_{\alpha \in Y} \Sigma_\alpha$ where Σ_α is the inverse hull of $S_\alpha, \alpha \in Y$. It is easy to see that e_α is the identity of Σ_α. From Lemma 5.1.7 and Theorem 4.3.9, the $\varphi_{\alpha,\beta}$'s lift to homomorphisms $\phi_{\alpha,\beta} : \Sigma_\alpha \longrightarrow \Sigma_\beta$ and $\phi_{\alpha,\beta}\phi_{\beta,\gamma} = \phi_{\alpha,\gamma}$ for all $\alpha \geq \beta \geq \gamma$, and $\phi_{\alpha,\alpha}$ is the identity on Σ_α. Hence Q is a strong semilattice of bisimple inverse monoids Σ_α's, $\alpha \in Y$, by Lemma 5.1.2. The following theorem is now clear.

Theorem 5.1.13. *Let $S = \mathcal{S}(Y; S_\alpha, \varphi_{\alpha,\beta})$ and for each α, let S_α be a right cancellative monoid with the (LC) condition and Σ_α be its inverse hull of left I-quotients. Suppose that S has the (LC) condition. Then S is a left I-order in a strong semilattice of monoids $Q = \mathcal{S}(Y; \Sigma_\alpha, \phi_{\alpha,\beta})$ where $\varphi_{\alpha,\beta}$'s lift to $\phi_{\alpha,\beta}$'s, $\alpha \geq \beta$.*

In the next corollary we provide an alternative proof for the above theorem, by using our result in Section 3.2.

Corollary 5.1.14. *Let $S = \mathcal{S}(Y; S_\alpha, \varphi_{\alpha,\beta})$ and for each α, let S_α be a right cancellative monoid with the (LC) condition and Σ_α be its inverse hull of left I-quotients. Suppose that S has the (LC) condition. Then for all $\alpha, \beta \in Y$, $\alpha \geq \beta$ we have that $\varphi_{\alpha,\beta}$ lifts to a homomorphism $\phi_{\alpha,\beta}$ of Σ_α into Σ_β. Moreover, S is a left I-order in $Q = \mathcal{S}(Y; \Sigma_\alpha, \phi_{\alpha,\beta})$.*

Proof. By Corollary 4.3.5, each S_α is a left I-order in its inverse hull Σ_α and S_α is the $\mathcal{R}$-class of the identity of Σ_α. We know that $\varphi_{\alpha,\beta} : S_\alpha \longrightarrow S_\beta$ is given by $a_\alpha \varphi_{\alpha,\beta} = a_\alpha e_\beta$ where $\alpha \geq \beta$ for any $a_\alpha \in S_\alpha$. By Theorem 3.2.9, each $\varphi_{\alpha,\beta}$ $(\alpha \geq \beta)$

lifts to a homomorphism $\phi_{\alpha,\beta} : \Sigma_\alpha \longrightarrow \Sigma_\beta$ if $\varphi_{\alpha,\beta}$ preserves $\mathcal{T}_S^Q$ and $\mathcal{R}^Q$ relations. In other words, if $\varphi_{\alpha,\beta}$ satisfies Conditions (i) and (ii) of this theorem. Let

$$bc^{-1}\Sigma_\alpha \subseteq d^{-1}\Sigma_\alpha$$

for some $b, c, d \in S_\alpha$. Hence

$$bc^{-1} = d^{-1}x^{-1}y = (xd)^{-1}y$$

for some $x, y \in S_\alpha$. Also, $bc^{-1} = h^{-1}k$ for some $h, k \in S_\alpha$ with $S_\alpha b \cap S_\alpha c = S_\alpha w$ where $hb = kc = w$. As $h^{-1}k = (xd)^{-1}y$ we have that

$$xd = uh \text{ and } y = uk$$

for some unit $u \in S_\alpha$, by Lemma 3.2.5. Since S has (LC) and $S_\alpha b \cap S_\alpha c = S_\alpha w$ we have that

$$S_\beta(be_\beta) \cap S_\beta(ce_\beta) = S_\beta(we_\beta),$$

by Lemma 5.1.7. It follows, as $\varphi_{\alpha,\beta}$ preserves units, that

$$b\varphi_{\alpha,\beta}(c\varphi_{\alpha,\beta})^{-1} = (h\varphi_{\alpha,\beta})^{-1}(k\varphi_{\alpha,\beta}) = (xd)\varphi_{\alpha,\beta}^{-1}y\varphi_{\alpha,\beta} = (d\varphi_{\alpha,\beta})^{-1}(x\varphi_{\alpha,\beta})^{-1}y\varphi_{\alpha,\beta}.$$

Hence

$$b\varphi_{\alpha,\beta}(c\varphi_{\alpha,\beta})^{-1}\Sigma_\beta \subseteq (d\varphi_{\alpha,\beta})^{-1}\Sigma_\beta$$

and so $(b\varphi_{\alpha,\beta}, c\varphi_{\alpha,\beta}, d\varphi_{\alpha,\beta}) \in \mathcal{T}_S^Q$, as required. It is clear that $\varphi_{\alpha,\beta}$ preserves $\mathcal{R}^Q$ relation. It is straightforward to show that the multiplication on Q extends that of S. Hence $Q = (Y; \Sigma_\alpha, \phi_{\alpha,\beta})$ is a strong semilattice of bisimple monoids. It is clear that S is a left I-order in Q. □

We aim now to prove the converse of Theorem 5.1.13. Let Q be a strong semilattice Y of bisimple inverse monoids Q_α, (with identity e_α) such that $E = \{e_\alpha : \alpha \in Y\}$ is a subsemigroup of Q. By Lemma 5.1.2, E is central in Q. Further if we define $\phi_{\alpha,\beta} : Q_\alpha \longrightarrow Q_\beta$ by $q_\alpha\phi = q_\alpha e_\beta$ $(\alpha \geq \beta)$, then each $\phi_{\alpha,\beta}$ is a monoid homomorphism and $Q = \mathcal{S}(Y; Q_\alpha; \phi_{\alpha,\beta})$. Let S_α be the $\mathcal{R}$-class of the identity e_α in Q_α. Clearly, $\phi_{\alpha,\beta}|_{S_\alpha} : S_\alpha \longrightarrow S_\beta$ and $S = \mathcal{S}(Y; S_\alpha; \phi_{\alpha,\beta}|_{S_\alpha})$ is a strong semilattice Y of right cancellative monoids S_α. We wish to show that S has the (LC) condition. By

Lemma 5.1.7, to show that S has (LC) condition we have to show that $\phi_{\alpha,\beta}|_{S_\alpha}$ is (LC)-preserving ($\alpha \geq \beta$). We need the following technical lemma from [38] (see, Lemma 3.2 of [2]).

Lemma 5.1.15. *(cf. [38, Lemma X.1.3]) Let Q be a bisimple inverse monoid and let R be the $\mathcal{R}$-class of the identity. For any $a, b, c \in R$,*

$$Ra \cap Rb = Rc \text{ if and only if } a^{-1}ab^{-1}b = c^{-1}c.$$

Returning to our argument before Lemma 5.1.15. Let $S_\alpha a \cap S_\alpha b = S_\alpha c$ where $a, b, c \in S_\alpha$. Then, we have that $a^{-1}ab^{-1}b = c^{-1}c$. We claim that $(e_\beta a)^{-1}(e_\beta a)(e_\beta b)^{-1}(e_\beta b) = (e_\beta c)^{-1}(e_\beta c)$ where $\alpha \geq \beta$.
Since E is central in Q we have

$$\begin{aligned}
(e_\beta a)^{-1}(e_\beta a)(e_\beta b)^{-1}(e_\beta b) &= a^{-1}e_\beta e_\beta ab^{-1}e_\beta b \\
&= a^{-1}e_\beta ab^{-1}e_\beta b \\
&= a^{-1}ae_\beta b^{-1}b \\
&= e_\beta a^{-1}ab^{-1}b \\
&= e_\beta c^{-1}c \\
&= e_\beta c^{-1}e_\beta c \\
&= (e_\beta c)^{-1}(e_\beta c).
\end{aligned}$$

Hence our claim is established. By the above lemma $S_\beta e_\beta a \cap S_\beta e_\beta b = S_\beta e_\beta c$ where $\alpha \geq \beta$. Thus by Lemma 5.1.7, S has the (LC) condition and the following theorem is clear.

Theorem 5.1.16. *Let Q be a strong semilattice Y of bisimple inverse monoids Q_α, (with identity e_α) such that $E = \{e_\alpha : \alpha \in Y\}$ is a subsemigroup of Q. Then there is a subsemigroup S of Q with the (LC) condition which is a strong semilattice of right cancellative monoids S_α where S_α is the $\mathcal{R}^{Q_\alpha}$-class of e_α. Moreover, S is a left I-order in Q.*

Combining Theorem 5.1.3 and Theorem 5.1.16, we get the following corollary.

Corollary 5.1.17. *(cf. [20, Main Theorem]) Let $S = \mathcal{S}(Y; S_\alpha)$ be a semilattice Y of right cancellative monoids S_α with identity e_α, such that each S_α has (LC). Suppose in addition that for any $\alpha \geq \beta$, if $S_\alpha a_\alpha \cap S_\alpha b_\alpha = S_\alpha c_\alpha$, then $S_\beta a_\alpha \cap S_\beta b_\alpha = S_\beta c_\alpha$. For each $\alpha \in Y$, let Q_α be the inverse hull of S_α, so that Q_α is a bisimple inverse* *monoid, and S_α is the $\mathcal{R}^{Q_\alpha}$-class of e_α. Then $Q = \mathcal{S}(Y; Q_\alpha)$ is a semigroup of left I-quotients of S, such that $E = \{e_\alpha : \alpha \in Y\}$ is a subsemigroup.*

Conversely, let $Q = \mathcal{S}(Y; Q_\alpha)$ be a semilattice Y of bisimple inverse monoids Q_α, with identity e_α, such that $E = \{e_\alpha : \alpha \in Y\}$ is a subsemigroup. Then $S = \mathcal{S}(Y; R_{e_\alpha})$ is a semilattice of right cancellative monoids R_{e_α}, such that each R_{e_α} has (LC) and for any $\alpha \geq \beta$, if $R_{e_\alpha}a_\alpha \cap R_{e_\alpha}b_\alpha = R_{e_\alpha}c_\alpha$, then $R_{e_\beta}a_\alpha \cap R_{e_\beta}b_\alpha = R_{e_\beta}c_\alpha$ where R_{e_α} is the $\mathcal{R}$-class of the identity e_α in Q_α for all $\alpha \in Y$.

It was shown in [1] that a semigroup Q which is a semilattice Y of inverse semigroups Q_α is an inverse semigroup, but if each Q_α is proper Q may not be proper (see, Example 5.2 [37]).

Let $S = \mathcal{S}(Y; S_\alpha; \varphi_{\alpha,\beta})$ be a strong semilattice Y of right cancellative monoids $S_\alpha, \alpha \in Y$ with the (LC) condition and S has (LC). In Corollary 5.1.14, we showed that S has a strong semilattice of left I-quotients $Q = \mathcal{S}(Y; \Sigma_\alpha, \phi_{\alpha,\beta})$ where Σ_α is the inverse hull of S_α for each $\alpha \in Y$ and each $\phi_{\alpha,\beta}$ is the extension of $\varphi_{\alpha,\beta}$. We recall that the connecting homomorphism $\varphi_{\alpha,\beta}$ is given by $a\varphi_{\alpha,\beta} = e_\beta a$. We employ the rest of this section to study the case when Q is proper.

Theorem 5.1.18. *Let $S = \mathcal{S}(Y; S_\alpha; \varphi_{\alpha,\beta})$, where each S_α is a right cancellative monoid with the (LC) condition and each $\varphi_{\alpha,\beta}$ is (LC)-preserving. Let Σ_α be the inverse hull of left I-quotients of S_α for each $\alpha \in Y$. Then $Q = \mathcal{S}(Y; \Sigma_\alpha; \phi_{\alpha,\beta})$ is a semigroup of left I-quotients of S. Moreover, each $\phi_{\alpha,\beta}$ is one-to-one and each Σ_α proper if and only if Q is proper and $(*)$ holds where $(*)$ is the following condition: for all $a, b \in S_\alpha$ and for all $\alpha \geq \beta$,*

$$a\phi_{\alpha,\beta}\, \mathcal{L}^{\Sigma_\beta}\, b\phi_{\alpha,\beta} \text{ implies that } a\, \mathcal{L}^{\Sigma_\alpha}\, b.$$

Proof. By Corollary 4.3.5, each S_α is a left I-order in its inverse hull Σ_α and S_α is the $\mathcal{R}$-class of the identity of Σ_α. Since each $\varphi_{\alpha,\beta}$ is (LC)-preserving, it follows that S has the (LC) condition, by Lemma 5.1.7. From Theorem 5.1.13, we have that S is a left I-order in Q. Suppose that Q is proper and $(*)$ holds. To show that each $\phi_{\alpha,\beta}$ is one-to-one, let

$$(a^{-1}b)\phi_{\alpha,\beta} = (c^{-1}d)\phi_{\alpha,\beta}$$

where $a^{-1}b, c^{-1}d \in \Sigma_\alpha$ for some $a, b, c, d \in S_\alpha$. Since $\phi_{\alpha,\beta}$ is the extension of $\varphi_{\alpha,\beta}$ for all $\alpha \geq \beta$ in Y, we have

$$(e_\beta a)^{-1}(e_\beta b) = (e_\beta c)^{-1}(e_\beta d).$$

Hence $a^{-1}e_\beta b = c^{-1}e_\beta d$ and so $a^{-1}be_\beta = c^{-1}de_\beta$ as the identities are central in Q. It follows that $a^{-1}b\,\sigma\,c^{-1}d$ in Q. Using Lemma 3.2.3, we get $a\phi_{\alpha,\beta} = e_\beta a\,\mathcal{L}^{\Sigma_\beta}\,e_\beta c = c\phi_{\alpha,\beta}$ and so $a\,\mathcal{L}^{\Sigma_\alpha}\,c$, by assumption. Since $\mathcal{R}^{\Sigma_\alpha} = \mathcal{R}^Q \cap (\Sigma_\alpha \times \Sigma_\alpha)$ and $\mathcal{L}^{\Sigma_\alpha} = \mathcal{L}^Q \cap (\Sigma_\alpha \times \Sigma_\alpha)$ for each $\alpha \in Y$ (see, Proposition 2.4.2 of [23]) we have that $a\,\mathcal{L}^Q\,c$ and so $a^{-1}\,\mathcal{R}^Q\,c^{-1}$. Again by Lemma 3.2.3,

$$a^{-1}b\,\mathcal{R}^Q\,a^{-1}\,\mathcal{R}^Q\,c^{-1}\,\mathcal{R}^Q\,c^{-1}d,$$

so that $a^{-1}b\,\mathcal{R}^Q\,c^{-1}d$. Since Q is proper and $a^{-1}b\,(\sigma \cap \mathcal{R}^Q)\,c^{-1}d$, it follows that $a^{-1}b = c^{-1}d$, by Proposition 2.4.8. Hence $\phi_{\alpha,\beta}$ is one-to-one. It is clear that if Q is proper, then Σ_α is proper for all $\alpha \in Y$.

On the other hand, suppose that each $\phi_{\alpha,\beta}$ is one-to-one and Σ_α is proper for each $\alpha \in Y$. To show that Q is proper, let

$$a^{-1}bc^{-1}c = c^{-1}c$$

where $a^{-1}b \in \Sigma_\alpha$ and $c^{-1}c \in \Sigma_\beta$ for some $a, b \in S_\alpha$ and $c \in S_\beta$. It is clear that $\beta \leq \alpha$ so that $S_{\alpha\beta} = S_\beta$. By definition of multiplication

$$a^{-1}bc^{-1}c = (xa)^{-1}(yc) = c^{-1}c$$

where $xb = yc$ for some $x, y \in S_\beta$ and as $xa, yc, c \in S_\beta$ we have that $xa = uc = yc$ for some unit u in S_β, by Lemma 3.2.5. Since $xb = yc$ we have that $xe_\beta b = yc = xe_\beta a$, as Σ_β is proper, it follows that S_β is cancellative, by Corollary 4.3.5. Hence $e_\beta b = e_\beta a$ and so $a\phi_{\alpha,\beta} = b\phi_{\alpha,\beta}$. Hence $a = b$ as $\phi_{\alpha,\beta}$ is one-to-one. To show that $(*)$ holds, let $a\phi_{\alpha,\beta}\,\mathcal{L}^{\Sigma_\beta}\,b\phi_{\alpha,\beta}$ for all $\alpha \geq \beta$ in Y and for all $a, b \in S_\alpha$. We have

$$(a^{-1}a)\phi_{\alpha,\beta} = (b^{-1}b)\phi_{\alpha,\beta}$$

so that as $\phi_{\alpha,\beta}$ is one-to-one $a^{-1}a = b^{-1}b$ in Σ_α and so $a\,\mathcal{L}^{\Sigma_\alpha}\,b$ as required. $\square$

Following [22], let P be a semilattice Y of inverse semigroups P_α, and let $\sigma_\alpha = \sigma(P_\alpha)$ be the minimum group congruence on P_α. Define τ on P by

$$p\,\tau\,q \iff p\,\sigma_\alpha\,q \text{ in } P_\alpha \text{ for some } \alpha \in Y.$$

It is shown in [22] that τ is a congruence on P and P/τ is a semilattice Y of groups P_α/σ_α. That is, $P/\tau = \bigcup_{\alpha\in Y}(P_\alpha/\sigma_\alpha)$. For any $a\sigma_\alpha \in P_\alpha/\sigma_\alpha$ and $b\sigma_\beta \in P_\beta/\sigma_\beta$ we have

$$(a\sigma_\alpha)(b\sigma_\beta) = (a\tau)(b\tau) = (ab)\tau = (ab)\sigma_{\alpha\beta}.$$

Lemma 5.1.19. *[22] Let P be a semilattice Y of proper inverse semigroups $P_\alpha, \alpha \in Y$ and let τ be defined as above, so that P/τ is a semilattice Y of groups $G_\alpha = P_\alpha/\sigma_\alpha$ and define the mappings $\psi_{\alpha,\beta} : G_\alpha \longrightarrow G_\beta$ by $g\psi_{\alpha,\beta} = ge_\beta$ where $a \in G_\alpha$ and e_β denotes the identity of G_β. Then the following are equivalent:*

(1) *P is proper;*

(2) *P/τ is proper;*

(3) *$\psi_{\alpha,\beta}$ is one-to-one where $\alpha \geq \beta$.*

Corollary 5.1.20. *Let $S = \mathcal{S}(Y; S_\alpha, \varphi_{\alpha,\beta})$, where each S_α is a right cancellative monoid with the (LC) condition and each $\varphi_{\alpha,\beta}$ is (LC)-preserving. Let $Q = \mathcal{S}(Y; \Sigma_\alpha, \phi_{\alpha,\beta})$ be the semigroup of left I-quotients of S where Σ_α is the inverse hull of S_α for each $\alpha \in Y$ and σ_α be defined as above for each $\alpha \in Y$. Then the following are equivalent:*

(1) *Each S_α is left cancellative and $\phi_{\alpha,\beta}$ is one-to-one for all $\alpha, \beta \in Y$ with $\alpha \geq \beta$;*

(2) *Q is proper and $(*)$ holds;*

(3) *Each Σ_α is proper and $\psi_{\alpha,\beta} : G_\alpha \longrightarrow G_\beta$ is one-to-one for all $\alpha, \beta \in Y$ where $G_\alpha = \Sigma_\alpha/\sigma_\alpha$ for all $\alpha \in Y$ and $(*)$ holds.*

Proof. (1) $\Longrightarrow$ (2). By Corollary 4.3.5, Σ_α is proper for each $\alpha \in Y$. Then (2) follows by Theorem 5.1.18.

(2) $\Longrightarrow$ (3). It is clear that if Q is proper, then Σ_α is proper for all $\alpha \in Y$. From Lemma 5.1.19, we have that $\psi_{\alpha,\beta}$ is one-to-one for all $\alpha, \beta \in Y$ where $\alpha \geq \beta$. Hence (3) holds.

(3) $\Longrightarrow$ (1). By Corollary 4.3.5, each S_α is left cancellative. It remains to show that each $\phi_{\alpha,\beta}$ is one-to-one. Since $\psi_{\alpha,\beta}$ is one-to-one for all $\alpha, \beta \in Y$ where $\alpha \geq \beta$ we have that Q is proper, by Lemma 5.1.19. Hence (1) holds by Theorem 5.1.18. $\square$

The following variation of Corollary 5.1.20 can be considered as a partial generalisation of Corollary 4.3.5.

Lemma 5.1.21. *Let $S = \mathcal{S}(Y; S_\alpha, \varphi_{\alpha,\beta})$, where each S_α is a right cancellative monoid with the (LC) condition and each $\varphi_{\alpha,\beta}$ is (LC)-preserving. Let $Q = \mathcal{S}(Y; \Sigma_\alpha, \phi_{\alpha,\beta})$ be the semigroup of left I-quotients of S where Σ_α is the inverse hull of S_α for each $\alpha \in Y$. Then the following are equivalent:*

(1) *Each S_α is left cancellative and $\phi_{\alpha,\beta}$ is one-to-one for all $\alpha, \beta \in Y$ with $\alpha \geq \beta$;*

(2) *Each Σ_α is proper and $\phi_{\alpha,\beta}$ is one-to-one for all $\alpha, \beta \in Y$ with $\alpha \geq \beta$;*

(3) *Q is proper and $(*)$ holds.*

Proof. (1) $\Longrightarrow$ (2) Since each S_α is left cancellative, it follows that Σ_α is proper for all $\alpha \in Y$, by Corollary 4.3.5. The implication (2) $\Longrightarrow$ (3) follows from Theorem 5.1.18. (3) $\Longrightarrow$ (1) Since Q is proper, it follows that each Σ_α is proper so that (1) follows from Corollary 4.3.5 and Theorem 5.1.18. $\square$

Remark 5.1.22. In the rest of this section we let $S = \mathcal{S}(Y; S_\alpha; \varphi_{\alpha,\beta})$ be a strong semilattice of right cancellative monoids S_α, $\alpha \in Y$ with the (LC) condition, and assume that S has the (LC) condition, and, let $Q = \mathcal{S}(Y; \Sigma_\alpha; \phi_{\alpha,\beta})$ be a semigroup of left I-quotients of S, where each Σ_α is the inverse hull of S_α for each $\alpha \in Y$. By Corollary 4.3.5, each S_α is a left I-order in Σ_α and S_α is the $\mathcal{R}$-class of the identity of Σ_α.

Remark 5.1.23. From Corollary 4.3.5 and Lemma 3.2.5, we deduce that for any $a, b \in S_\alpha$ and for all $\alpha \in Y$ we have

$$a \,\mathcal{L}\, b \text{ in } S_\alpha \text{ if and only if } a \,\mathcal{L}\, b \text{ in } \Sigma_\alpha.$$

By the above Remark, $(*)$ holds if and only if $(*)'$ holds where $(*)'$ is the following condition:
for all $a, b \in S_\alpha$ and for all $\alpha \geq \beta$,

$$a\varphi_{\alpha,\beta} \,\mathcal{L}\, b\varphi_{\alpha,\beta} \text{ in } S_\beta \text{ implies that } a \,\mathcal{L}\, b \text{ in } S_\alpha.$$

If we insisted on Q being proper, then by Lemma 5.1.18, the sufficient conditions are $\phi_{\alpha,\beta}$ is one-to-one for all $\alpha, \beta \in Y$ with $\alpha \geq \beta$ and Σ_α is proper for all $\alpha \in Y$. Such conditions are related to the structure of Q. We shall introduce equivalent conditions on the structure of S in order to do so. We begin with the following lemma.

Lemma 5.1.24. *Let $\phi_{\alpha,\beta}, \varphi_{\alpha,\beta}$ and S be as in the Remark 5.1.22. If $\phi_{\alpha,\beta}$ is one-to-one for all $\alpha, \beta \in Y$ with $\alpha \geq \beta$, then*

(i) $\varphi_{\alpha,\beta}$ is one-to-one for all $\alpha, \beta \in Y$ with $\alpha \geq \beta$;

(ii) $(*)'$ holds.

Proof. (*i*) Since $\phi_{\alpha,\beta}$ is one-to-one for all $\alpha, \beta \in Y$ with $\alpha \geq \beta$. Then as $\phi_{\alpha,\beta}$ is the extension of $\varphi_{\alpha,\beta}$ we have that $\varphi_{\alpha,\beta}$ is one-to-one.
(*ii*) Suppose that $a\varphi_{\alpha,\beta} \,\mathcal{L}\, b\varphi_{\alpha,\beta}$ in S_β for all $\alpha \geq \beta$ in Y and $a, b \in S_\alpha$ so that $a\varphi_{\alpha,\beta} \,\mathcal{L}^{\Sigma_\beta}\, b\varphi_{\alpha,\beta}$, by Remark 5.1.23. Hence $a\phi_{\alpha,\beta} \,\mathcal{L}^{\Sigma_\beta}\, b\phi_{\alpha,\beta}$ so that $a\phi_{\alpha,\beta}^{-1}a\phi_{\alpha,\beta} = b\phi_{\alpha,\beta}^{-1}b\phi_{\alpha,\beta}$. We have that $(a^{-1}a)\phi_{\alpha,\beta} = (b^{-1}b)\phi_{\alpha,\beta}$. As $\phi_{\alpha,\beta}$ is one-to-one we have that $a^{-1}a = b^{-1}b$ so that $a \,\mathcal{L}^{\Sigma_\alpha}\, b$. By Remark 5.1.23, $a \,\mathcal{L}\, b$ in S_α as required. □

Lemma 5.1.25. *Let $\phi_{\alpha,\beta}, \varphi_{\alpha,\beta}, S$ and Q be as in the Remark 5.1.22. Let Q be proper and $(*)'$ holds. Then*

$$\phi_{\alpha,\beta} \text{ is one-to-one if and only if } \varphi_{\alpha,\beta} \text{ is one-to-one}$$

for all $\alpha, \beta \in Y$ with $\alpha \geq \beta$.

Proof. It is clear that if each $\phi_{\alpha,\beta}$ is one-to-one, then each $\varphi_{\alpha,\beta}$ is one-to-one. Conversely, suppose that $\varphi_{\alpha,\beta}$ is one-to-one for all $\alpha, \beta \in Y$ with $\alpha \geq \beta$. Let

$$(a^{-1}b)\phi_{\alpha,\beta} = (c^{-1}d)\phi_{\alpha,\beta}$$

where $a^{-1}b, c^{-1}d \in \Sigma_\alpha$ for some $a, b, c, d \in S_\alpha$. Since $\phi_{\alpha,\beta}$ is the extension of $\varphi_{\alpha,\beta}$ for all $\alpha \geq \beta$ in Y, we have

$$(e_\beta a)^{-1}(e_\beta b) = (e_\beta c)^{-1}(e_\beta d).$$

Hence $a^{-1}e_\beta b = c^{-1}e_\beta d$ and so $a^{-1}be_\beta = c^{-1}de_\beta$ as the identities are central in Q. It follows that $a^{-1}b \,\sigma\, c^{-1}d$ in Q. By Lemma 3.2.3, $e_\beta a \,\mathcal{L}^{\Sigma_\beta}\, e_\beta c$ and so $e_\beta a \,\mathcal{L}\, e_\beta c$ in S_β, by Remark 5.1.23. Then $a\varphi_{\alpha,\beta} \,\mathcal{L}\, c\varphi_{\alpha,\beta}$ in S_β and so $a \,\mathcal{L}\, c$ in S_α, by $(*)'$. Again by Remark 5.1.23, $a \,\mathcal{L}^{\Sigma_\alpha}\, c$. It follows that $a^{-1} \,\mathcal{R}^{\Sigma_\alpha}\, c^{-1}$ and so $a^{-1} \,\mathcal{R}^Q\, c^{-1}$, by Proposition 2.4.2 of [23]. Again by Lemma 3.2.3,

$$a^{-1}b \,\mathcal{R}^Q\, a^{-1} \,\mathcal{R}^Q\, c^{-1} \,\mathcal{R}^Q\, c^{-1}d.$$

Since Q is proper we have that $a^{-1}b = c^{-1}d$, by Proposition 2.4.8. Thus $\phi_{\alpha,\beta}$ is one to one. $\square$

Before giving the conditions which make Q is proper, by using the structure of S we need the following lemma.

Lemma 5.1.26. *Let $\varphi_{\alpha,\beta}$ and Q be as in the Remark 5.1.22. Then Q is proper if and only if Σ_α is proper for each $\alpha \in Y$ and $\varphi_{\alpha,\beta}$ is one-to-one, for all $\alpha, \beta \in Y$ with $\alpha \geq \beta$.*

Proof. Suppose that Q is proper. It is clear that Σ_α is proper for all $\alpha \in Y$. To show that $\varphi_{\alpha,\beta}$ is one-to-one for all $\alpha, \beta \in Y$ with $\alpha \geq \beta$, let

$$a\varphi_{\alpha,\beta} = b\varphi_{\alpha,\beta}$$

where $a, b \in S_\alpha$. Then $e_\beta a = e_\beta b$ so that $a \,\sigma\, b$ in Q. Since $a \,\mathcal{R}^{\Sigma_\alpha}\, b$, it follows that $a \,\mathcal{R}^Q\, b$, by Proposition 2.4.2 of [23]. As Q is proper we have that $a = b$, by Proposition 2.4.8. Thus $\varphi_{\alpha,\beta}$ is one to one.

On the other hand, assume that $\varphi_{\alpha,\beta}$ is one-to-one for all $\alpha, \beta \in Y$ with $\alpha \geq \beta$ and Σ_α is proper for all $\alpha \in Y$. Let

$$a^{-1}bc^{-1}c = c^{-1}c$$

where $a^{-1}b \in \Sigma_\alpha$ and $c^{-1}c \in \Sigma_\beta$ for some $a, b \in S_\alpha$ and $c \in S_\beta$. By definition of multiplication,

$$a^{-1}bc^{-1}c = (xa)^{-1}(yc) = c^{-1}c$$

where $xb = yc$ for some $x, y \in S_{\alpha\beta}$. It is clear that $\beta \leq \alpha$ so that $S_{\alpha\beta} = S_\beta$. Hence $xa, yc, c \in S_\beta$, so that $xa = uc = yc$ for some unit u in S_β, by Lemma 3.2.5. Since $xb = yc$ we have that $xe_\beta b = yc = xe_\beta a$, and as Σ_β is proper, we have that S_β is cancellative, by Corollary 4.3.5. Since $e_\beta a, e_\beta b$ and x are in S_β which is cancellative we obtain $e_\beta b = e_\beta a$. Hence $a\varphi_{\alpha,\beta} = b\varphi_{\alpha,\beta}$ gives $a = b$ as $\varphi_{\alpha,\beta}$ is one-to-one. Thus Q is proper. $\square$

From Corollary 4.3.5 and Lemma 5.1.26, we have

Corollary 5.1.27. *Let S, $\varphi_{\alpha,\beta}$ and Q be as in the Remark 5.1.22. Then S_α is left cancellative for each $\alpha \in Y$ and $\varphi_{\alpha,\beta}$ is one-to-one, for all $\alpha, \beta \in Y$ with $\alpha \geq \beta$ if and only if Q is proper.*

In the next lemma we shed light on the relationships between $\varphi_{\alpha,\beta}, \phi_{\alpha,\beta}$ and $\psi_{\alpha,\beta}$ for all $\alpha \geq \beta$ in Y which play a significant role in making Q proper.

Lemma 5.1.28. *Let S, Q, $\varphi_{\alpha,\beta}, \phi_{\alpha,\beta}$ be as in the Remark 5.1.22 and $\psi_{\alpha,\beta}$ be as in Corollary 5.1.20. If S_α is left cancellative for all $\alpha \in Y$, then the following are equivalent:*

(i) Each $\varphi_{\alpha,\beta}$ is one-to-one and $(*)'$ holds;

(ii) Each $\phi_{\alpha,\beta}$ is one-to-one and $(*)$ holds;

(iii) Each $\psi_{\alpha,\beta}$ is one-to-one and $(*)$ holds.

Proof. As S_α is left cancellative for each $\alpha \in Y$, we have that Σ_α is proper for each $\alpha \in Y$, by Corollary 4.3.5. Hence we can use Lemma 5.1.19.
$(i) \Longrightarrow (ii)$ From Corollary 5.1.27, we have that Q is proper. Then (ii) follows from Theorem 5.1.18, and the equivalence of $(*)$ and $(*)'$.
$(ii) \Longrightarrow (iii)$ By Lemma 5.1.21, we have that Q is proper and so (iii) follows from Lemma 5.1.19. The implication $(iii) \Longrightarrow (i)$ follows from Lemma 5.1.19, Theorem 5.1.18 and Remark 5.1.23. □

The following diagrams may help the reader to visualise the the relationship between the homomorphisms and semigroups which we have considered.

$$\begin{tikzcd} S_\alpha \arrow[r, "\theta_\alpha"] \arrow[d, "\varphi_{\alpha,\beta}"'] & \Sigma_\alpha \arrow[r, "\sigma_\alpha^\sharp"] \arrow[d, "\phi_{\alpha,\beta}"] & G_\alpha \arrow[d, "\psi_{\alpha,\beta}"] \\ S_\beta \arrow[r, "\theta_\beta"] & \Sigma_\beta \arrow[r, "\sigma_\beta^\sharp"] & G_\beta \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} a \arrow[r, "\theta_\alpha"] \arrow[d, "\varphi_{\alpha,\beta}"'] & \rho_a \arrow[r, "\sigma_\alpha^\sharp"] \arrow[d, "\phi_{\alpha,\beta}"] & (\rho_a)\sigma_\alpha \arrow[d, "\psi_{\alpha,\beta}"] \\ e_\beta a \arrow[r, "\theta_\beta"] & \rho_{e_\beta a} \arrow[r, "\sigma_\beta^\sharp"] & (\rho_{e_\beta a})\sigma_\beta \end{tikzcd}$$

It is easy to see that the above diagrams are commutative.

5.2 Left I-quotients of semilattices of left ample semigroups

In this section we extend the results of the previous section to a strong semilattice of left ample semigroups with the (LC) condition. We recall that a $(2,1)$-homomorphism $\phi : S \to T$, where S and T are left ample semigroups with the (LC) condition is *(LC)-preserving* if, for any $b, c \in S$ with $Sb \cap Sc = Sw$, we have that

$$T(b\phi) \cap T(c\phi) = T(w\phi).$$

We show that this condition is equivalent to such a semigroup itself having the (LC) condition and its semigroup of left I-quotents is isomorphic to its inverse hull.

In (ii) of the following theorem we give the promised generalisation of Lemma 5.1.7.

Proposition 5.2.1. *Let $S = \mathcal{S}(Y; S_\alpha; \phi_{\alpha,\beta})$, where each S_α is left ample and the connecting morphisms are $(2,1)$-homomorphisms.*

(i) The semigroup S is left ample, and for any $a, b \in S$, $a\,\mathcal{R}^*\,b$ in S if and only if $a, b \in S_\alpha$ for some $\alpha \in Y$ and $a\,\mathcal{R}^*\,b$ in S_α.

(ii) If each S_α has (LC), then S has (LC) if and only if every $\phi_{\alpha,\beta}, \alpha \geq \beta$, is (LC)-preserving.

Proof. (i) Let $f_\alpha, g_\beta \in E(S)$; then

$$f_\alpha g_\beta = (f_\alpha \phi_{\alpha,\alpha\beta})(g_\beta \phi_{\beta,\alpha\beta}) = (g_\beta \phi_{\beta,\alpha\beta})(f_\alpha \phi_{\alpha,\alpha\beta}) = g_\beta f_\alpha,$$

using the fact that $E(S_{\alpha\beta})$ is a semilattice. Thus $E(S)$ is a semilattice.

Suppose now that $a_\alpha (\mathcal{R}^*)^S b_\beta$. Let f_α be the idempotent in the $(\mathcal{R}^*)^{S_\alpha}$-class of a_α. Then, as $f_\alpha a_\alpha = a_\alpha$ we must also have that $f_\alpha b_\beta = b_\beta$ so that $\beta \leq \alpha$. With the dual we obtain that $\alpha = \beta$; clearly, then $a_\alpha (\mathcal{R}^*)^{S_\alpha} b_\beta$.

Conversely, suppose that $a_\alpha (\mathcal{R}^*)^{S_\alpha} b_\alpha$ and $x_\gamma a_\alpha = y_\delta a_\alpha$. Then $\gamma\alpha = \delta\alpha = \mu$, say, and $(x_\gamma \phi_{\gamma,\mu})(a_\alpha \phi_{\alpha,\mu}) = (y_\delta \phi_{\delta,\mu})(a_\alpha \phi_{\alpha,\mu})$. But $\phi_{\alpha,\mu}$ is a (2,1)-homomorphism, and $a_\alpha (\mathcal{R}^*)^{S_\alpha} b_\alpha$, so that $a_\alpha \phi_{\alpha,\mu} (\mathcal{R}^*)^{S_\mu} b_\alpha \phi_{\alpha,\mu}$. We thus obtain that

$$(x_\gamma \phi_{\gamma,\mu})(b_\alpha \phi_{\alpha,\mu}) = (y_\delta \phi_{\delta,\mu})(b_\alpha \phi_{\alpha,\mu})$$

and hence $x_\gamma b_\alpha = y_\delta b_\alpha$. Making an easy adjustment for $x_\gamma = 1$ yields that $a_\alpha (\mathcal{R}^*)^S b_\alpha$.

Notice that from the above, there is no ambiguity in the use of the superscript $^+$. To see that S is left ample, let $a_\alpha \in S$ and $f_\beta \in E(S)$. Then

$$(a_\alpha f_\beta)^+ a_\alpha = ((a_\alpha \phi_{\alpha,\alpha\beta})(f_\beta \phi_{\beta,\alpha\beta}))^+ (a_\alpha \phi_{\alpha,\alpha\beta}) = (a_\alpha \phi_{\alpha,\alpha\beta})(f_\beta \phi_{\beta,\alpha\beta}),$$

using the fact that $S_{\alpha\beta}$ is left ample and therefore satisfies the left ample condition, so that $(a_\alpha f_\beta)^+ a_\alpha = a_\alpha f_\beta$ and S satisfies the left ample condition.

(ii) Suppose that each S_α has (LC).
Assume first that each $\phi_{\alpha,\beta}$ is (LC)-preserving. Let $a_\alpha, b_\beta \in S$ and let $\gamma = \alpha\beta$. As S_γ has (LC) we know that

$$S_\gamma a_\alpha \cap S_\gamma b_\beta = S_\gamma (a_\alpha \phi_{\alpha,\gamma}) \cap S_\gamma (b_\beta \phi_{\beta,\gamma}) = S_\gamma c_\gamma,$$

for some c_γ. We claim that $Sa_\alpha \cap Sb_\beta = Sc_\gamma$.
Certainly $c_\gamma = x_\gamma a_\alpha = y_\gamma b_\beta$ for some $x_\gamma, y_\gamma \in S_\gamma$, so that $c_\gamma \in Sa_\alpha \cap Sb_\beta$ and so

$$Sc_\gamma \subseteq Sa_\alpha \cap Sb_\beta.$$

On the other hand, let $d \in Sa_\alpha \cap Sb_\beta$; then there are elements $u_\mu, v_\nu \in S$ with $d = u_\mu a_\alpha = v_\nu b_\beta$. Let $\tau = \mu\alpha = \nu\beta$, so that $\tau \leq \gamma$. Then

$$d = d_\tau = d_\tau^+ d_\tau = (d_\tau^+ u_\mu) a_\alpha = (d_\tau^+ v_\nu) b_\beta \in S_\tau a_\alpha \cap S_\tau b_\beta.$$

Now $\phi_{\gamma,\tau}$ is (LC)-preserving, so that $S_\tau a_\alpha \cap S_\tau b_\beta = S_\tau c_\gamma$. This gives that $d \in Sc_\gamma$ so that $Sa_\alpha \cap Sb_\beta = Sc_\gamma$ as required.

Conversely, assume that S has (LC) and suppose that $\alpha \geq \beta$; we must show that $\phi_{\alpha,\beta}$ is (LC)-preserving. We first show that for any $a_\alpha, b_\alpha, c_\alpha \in S$,

$$S_\alpha a_\alpha \cap S_\alpha b_\alpha = S_\alpha c_\alpha \text{ if and only if } Sa_\alpha \cap Sb_\alpha = Sc_\alpha.$$

($\Leftarrow$) If $Sa_\alpha \cap Sb_\alpha = Sc_\alpha$, we have that

$$c_\alpha = ua_\alpha = vb_\alpha = (ua_\alpha^+) a_\alpha = (vb_\alpha^+) b_\alpha \in S_\alpha a_\alpha \cap S_\alpha b_\alpha,$$

for some $u, v \in S$ so that $S_\alpha c_\alpha \subseteq S_\alpha a_\alpha \cap S_\alpha b_\alpha$.

On the other hand, if $x_\alpha, y_\alpha \in S_\alpha$ and

$$x_\alpha a_\alpha = y_\alpha b_\alpha \in S_\alpha a_\alpha \cap S_\alpha b_\alpha \subseteq S a_\alpha \cap S b_\alpha,$$

then

$$x_\alpha a_\alpha = y_\alpha b_\alpha = z c_\alpha = (z c_\alpha^+) c_\alpha \in S_\alpha c_\alpha$$

for some $z \in S$. Thus $S_\alpha a_\alpha \cap S_\alpha b_\alpha \subseteq S_\alpha c_\alpha$ and we have $S_\alpha a_\alpha \cap S_\alpha b_\alpha = S_\alpha c_\alpha$ as desired.

($\Rightarrow$) Conversely, suppose that $S_\alpha a_\alpha \cap S_\alpha b_\alpha = S_\alpha c_\alpha$. Since S has (LC) we have that $S a_\alpha \cap S b_\alpha = S d_\beta$ for some $d_\beta \in S$. As $d_\beta \in S a_\alpha$ we have that $\beta \leq \alpha$, but $c_\alpha \in S d_\beta$, so that $\alpha = \beta$. From ($\Leftarrow$) we have that $S_\alpha a_\alpha \cap S_\alpha b_\alpha = S_\alpha d_\alpha$ so that $c_\alpha \,\mathcal{L}\, d_\alpha$ in S_α and hence in S. Consequently, $S a_\alpha \cap S b_\alpha = S c_\alpha$.

We now return to the argument that $\phi_{\alpha,\beta}$ is (LC)-preserving (for $\alpha \geq \beta$). Suppose that $a_\alpha, b_\alpha, c_\alpha \in S$ and $S_\alpha a_\alpha \cap S_\alpha b_\alpha = S_\alpha c_\alpha$. We know that $c_\alpha = x_\alpha a_\alpha = y_\alpha b_\alpha$ for some x_α, y_α so that $c_\alpha \phi_{\alpha,\beta} = (x_\alpha \phi_{\alpha,\beta})(a_\alpha \phi_{\alpha,\beta}) = (y_\alpha \phi_{\alpha,\beta})(b_\alpha \phi_{\alpha,\beta})$, giving that

$$c_\alpha \phi_{\alpha,\beta} \in S_\beta(a_\alpha \phi_{\alpha,\beta}) \cap S_\beta(b_\alpha \phi_{\alpha,\beta}) = S_\beta d_\beta$$

for some d_β. From the above, $S a_\alpha \cap S b_\alpha = S c_\alpha$. We have that for some u_β, v_β,

$$d_\beta = u_\beta(a_\alpha \phi_{\alpha,\beta}) = v_\beta(b_\alpha \phi_{\alpha,\beta}) = u_\beta a_\alpha = v_\beta b_\alpha \in S a_\alpha \cap S b_\alpha = S c_\alpha$$

so that $d_\beta = z_\gamma c_\alpha = ((z_\gamma c_\alpha)^+ z_\gamma)(c_\alpha \phi_{\alpha,\beta}) \in S_\beta(c_\alpha \phi_{\alpha,\beta})$ for some z_γ. It follows that

$$S_\beta(a_\alpha \phi_{\alpha,\beta}) \cap S_\beta(b_\alpha \phi_{\alpha,\beta}) = S_\beta d_\beta = S_\beta(c_\alpha \phi_{\alpha,\beta})$$

and $\phi_{\alpha,\beta}$ has (LC). $\square$

We now give the main result of this section.

Theorem 5.2.2. *Let $S = \mathcal{S}(Y; S_\alpha; \phi_{\alpha,\beta})$ be a strong semilattice of left ample semigroups S_α, such that the connecting homomorphisms are $(2,1)$-homomorphisms. Suppose that each $S_\alpha, \alpha \in Y$ has (LC) and that S has (LC).*

For each $\alpha \in Y$, let Σ_α be the inverse hull of S_α. Then for any $\alpha, \beta \in Y$ with $\alpha \geq \beta$, we have that $\phi_{\alpha,\beta}$ lifts to a homomorphism $\overline{\phi_{\alpha,\beta}} : \Sigma_\alpha \to \Sigma_\beta$. Further, $Q = \mathcal{S}(Y; \Sigma_\alpha; \overline{\phi_{\alpha,\beta}})$ is a strong semilattice of inverse semigroups, such that S is a straight left I-order in Q. Moreover, Q is isomorphic to the inverse hull of S.

Proof. By Theorem 4.2.2, each $S_\alpha\theta_{S_\alpha}$ is a left I-order in its inverse hull - we identify S_α with $S_\alpha\theta_{S_\alpha}$ and write the inverse hull of S_α as Σ_α. By Lemma 4.1.4, S_α is straight in Q_α.

From Proposition 5.2.1, S is left ample and as S has the (LC) condition, the connecting homomorphisms are (LC)-preserving. By Theorem 4.3.9, each $\phi_{\alpha,\beta}(\alpha \geq \beta)$ lifts to a homomorphism $\overline{\phi_{\alpha,\beta}} : \Sigma_\alpha \longrightarrow \Sigma_\beta$. Clearly $\overline{\phi_{\alpha,\alpha}}$ is the identity map and for any $\alpha \geq \beta \geq \gamma, \overline{\phi_{\alpha,\beta}}\ \overline{\phi_{\beta,\gamma}} = \overline{\phi_{\alpha,\gamma}}$. Thus $Q = S(Y;\Sigma_\alpha;\overline{\phi_{\alpha,\beta}})$ is a strong semilattice of inverse semigroups and S is a straight left I-order in Q.

It remains to show that Q is isomorphic to the inverse hull $P = \Sigma(S)$ of S. First, it is easy to check that S is a union of $\mathcal{R}$-classes of Q. For any $a, b \in S$,

$$\begin{aligned} a\,\mathcal{R}_S^Q\, b &\Leftrightarrow a, b \in S_\alpha \text{ for some } \alpha \text{ and } a\,\mathcal{R}_{S_\alpha}^{\Sigma_\alpha}\, b \\ &\Leftrightarrow a, b \in S_\alpha \text{ for some } \alpha \text{ and } a\,(\mathcal{R}^*)^{S_\alpha}\, b \\ &\Leftrightarrow a\,(\mathcal{R}^*)^S\, b \\ &\Leftrightarrow a\theta_S\,\mathcal{R}^P\, b\theta_S. \end{aligned}$$

Let $a_\alpha, b_\beta \in S$; we show that

$$Sa_\alpha \cap Sb_\beta = S(a_\alpha\phi_{\alpha,\alpha\beta}) \cap S(b_\beta\phi_{\beta,\alpha\beta}).$$

Let

$$x = u_\gamma a_\alpha = v_\delta b_\beta \in Sa_\alpha \cap Sb_\beta;$$

then $\gamma\alpha = \delta\beta = \tau$ say, so that $\tau \leq \alpha\beta$ and

$$\begin{aligned} x = x^+x = (x^+u_\gamma)a_\alpha = (x^+v_\delta)b_\beta = (x^+u_\gamma)(a_\alpha\phi_{\alpha,\tau}) = (x^+v_\delta)(b_\beta\phi_{\beta,\tau}) \\ = (x^+u_\gamma)(a_\alpha\phi_{\alpha,\alpha\beta}) = (x^+v_\delta)(b_\beta\phi_{\beta,\alpha\beta}) \in S(a_\alpha\phi_{\alpha,\alpha\beta}) \cap S(b_\beta\phi_{\beta,\alpha\beta}). \end{aligned}$$

Conversely, if

$$y = h_\gamma(a_\alpha\phi_{\alpha,\alpha\beta}) = k_\delta(b_\beta\phi_{\beta,\alpha\beta}) \in S(a_\alpha\phi_{\alpha,\alpha\beta}) \cap S(b_\beta\phi_{\beta,\alpha\beta})$$

then $\gamma\alpha\beta = \delta\alpha\beta = \kappa$ say and

$$y = (y^+h_\gamma)(a_\alpha\phi_{\alpha,\alpha\beta}) = (y^+k_\delta)(b_\beta\phi_{\beta,\alpha\beta}) = (y^+h_\gamma)a_\alpha = (y^+k_\delta)b_\beta \in Sa_\alpha \cap Sb_\beta,$$

verifying that

$$Sa_\alpha \cap Sb_\beta = S(a_\alpha\phi_{\alpha,\alpha\beta}) \cap S(b_\beta\phi_{\beta,\alpha\beta}).$$

Now let $a, b, c \in S$. Consider $ba^{-1} \in Q$; say $b = b_\beta$ and $a = a_\alpha$. Then

$$ba^{-1} = (b\phi_{\beta,\alpha\beta})(a\phi_{\alpha,\alpha\beta})^{-1} = x^{-1}y$$

where $x, y \in S_{\alpha\beta}, x = x(b\phi_{\beta,\alpha\beta})^+, y = y(a\phi_{\alpha,\alpha\beta})^+, S_{\alpha\beta}(b\phi_{\beta,\alpha\beta}) \cap S_{\alpha\beta}(a\phi_{\alpha,\alpha\beta}) = S_{\alpha\beta}(x(b\phi_{\beta,\alpha\beta}))$ and

$$x(b\phi_{\beta,\alpha\beta}) = y(a\phi_{\alpha,\alpha\beta}) = xb = ya.$$

Also, $x = x(b^+\phi_{\beta,\alpha\beta}) = xb^+$ and similarly, $y = ya^+$. From the proof of Proposition 5.2.1 and the argument above, we have that $Sb \cap Sa = S(xb)$. It follows from Lemma 4.2.1, that in $P, b\theta_S(a\theta_S)^{-1} = (x\theta_S)^{-1}(y\theta_S)$.

Now,

$$\begin{aligned}
(a, b, c) \in \mathcal{T}_S^Q &\Leftrightarrow ab^{-1}Q \subseteq c^{-1}Q \\
&\Leftrightarrow Qba^{-1} \subseteq Qc \\
&\Leftrightarrow Qx^{-1}y \subseteq Qc \\
&\Leftrightarrow Qy \subseteq Qc \\
&\Leftrightarrow Sy \subseteq Sc \\
&\Leftrightarrow S\theta_S(y\theta_S) \subseteq S\theta_S(c\theta_S) \\
&\Leftrightarrow P(y\theta_S) \subseteq P(c\theta_S) \\
&\Leftrightarrow P(x\theta_S)^{-1}(y\theta_S) \subseteq P(c\theta_S) \\
&\Leftrightarrow Pb\theta_S(a\theta_S)^{-1} \subseteq P(c\theta_S) \\
&\Leftrightarrow (a\theta_S)(b\theta_S)^{-1}P \subseteq (c\theta_S)^{-1}P \\
&\Leftrightarrow (a\theta_S, b\theta_S, c\theta_S) \in P.
\end{aligned}$$

From Corollary 3.2.10, Q is isomorphic to P via an isomorphism lifting θ_S. □

5.3 The category of semilattices of certain bisimple inverse monoids

In Section 2.5 we saw that the category of right cancellative monoids with the (LC) condition and homomorphisms which are (LC)-preserving is equivalent to the category of bisimple inverse monoids, and monoid homomorphisms. In this section we look at the same problem in a more general context. We begin by defining the following two categories.

Let a category **SR** be given by

Ob (**SR**) are semilattices of right cancellative monoids with the (LC) condition, which themselves have the (LC) condition,

Hom (**SR**) are (2,1)-homomorphisms of Ob (**SR**) which are (LC)-preserving.

Let a category **SB** be given by

Ob (**SB**) are semilattices of bisimple inverse monoids, in which the set of identity elements forms a subsemigroup,

Hom (**SB**) are monoid homomorphisms of Ob (**SB**).

The objects of **SR** and **SB** are $S = \mathcal{S}(Y; S_\alpha)$ and $Q = \mathcal{S}(Y; Q_\alpha)$ respectively, where the S_α's have the (LC) condition, S has the (LC) condition and the Q_α's are bisimple inverse monoids where $\{e_\alpha : e_\alpha$ is the identity of $Q_\alpha\}$ is a subsemigroup of Q. Notice that in the case that Y is the single element semilattice, the categories **SR** and **SB** are the categories **R** and **B** respectively in Section 2.5. We then obtain the result in such a section. By Lemma 5.1.2, Q is is a strong semilattice. Also, by Lemma 5.1.2 and Lemma 5.1.1, S is a strong semilattice Y with connecting homomorphisms $\varphi_{\alpha,\beta} : S_\alpha \longrightarrow S_\beta$ giving by $a_\alpha \varphi_{\alpha,\beta} = e_\beta a_\alpha$ where $\alpha \geq \beta$ and each $\varphi_{\alpha,\beta}$ is (LC)-preserving.

Let $S \in$ Ob (**SR**), that is, $S = \mathcal{S}(Y; S_\alpha)$ is a semilattice Y right cancellative monoids S_α with identity $e_\alpha, \alpha \in Y$ such that each $S_\alpha, \alpha \in Y$ has the (LC) condition and S has the (LC) condition. By Lemma 5.1.1, S is a left ample semigroup and $\{e_\alpha : \alpha \in Y\}$ is a subsemigroup of S. Since S has the (LC) condition, it follows that S is a left I-order in a semigroup Q which is a semilattice Y of bisimple inverse monoids $Q_\alpha, \alpha \in Y$ and isomorphic to the the inverse hull of S, by Corollary 5.1.17 and 5.2.2. It is easy to see that that e_α is the identity of $Q_\alpha, \alpha \in Y$. Hence $\Sigma(S) \in$ Ob (**SB**), as isomorphic semigroups have the same structure.

Let $\varphi \in$ Hom (**SR**) say, $\varphi : S \longrightarrow S'$. We have already seen that S and S' are left ample semigroups with the (LC) condition. Since φ is (LC)-preserving we can extend φ to $\psi : \Sigma(S) \longrightarrow \Sigma(S')$ where $\Sigma(S)$ and $\Sigma(S')$ are the inverse hulls of S and S' respectively, by Theorem 4.3.9.

Lemma 5.3.1. *Let $S \in Ob(\mathbf{SR})$ and put $V(S) = \Sigma(S)$. For any $\varphi \in Hom(\mathbf{SR})$, say $\varphi : S \longrightarrow S'$, let*

$$V(\varphi) : \rho_a^{-1}\rho_b \longrightarrow \rho_{a\varphi}^{-1}\rho_{b\varphi} \quad (a, b \in S).$$

Then V is a functor from **SR** *to* **SB**.

Let $Q \in \mathrm{Ob}(\mathbf{SB})$. Then $Q = \mathcal{S}(Y; Q_\alpha)$ where each $Q_\alpha, \alpha \in Y$ is a bisimple inverse monoid with identity e_α such that $E = \{e_\alpha : \alpha \in Y\}$ is a subsemigroup of Q. By Corollary 5.1.17, $S = \mathcal{S}(Y; R_{e_\alpha})$ is a semilattice of right cancellative monoids R_{e_α} with the (LC) condition, satisfying (C_2) which is equivalent to S having the (LC) condition, by Lemma 5.1.7. Hence $S \in \mathrm{Ob}(\mathbf{SR})$.

Let $\psi : Q \longrightarrow Q'$ so that $\psi|_S : S \longrightarrow S'$ where $S = \mathcal{S}(Y; R_{e_\alpha})$ and $S' = \mathcal{S}(Z; R_{e_{\alpha'}})$. It is clear that $\psi|_S$ is (2,1)-homomorphism. It remains to show that $\psi|_S$ is (LC)-preserving. Since ψ is an inverse monoid homomorphism, we have that $Qa \cap Qb = Qc$ implies that $Q'(a\psi) \cap Q'(b\psi) = Q'(c\psi)$ for any $a, b, c \in S$. Now, by applying Lemma 4.3.2, it is straightforward to see that $\psi|_S$ is (LC)-preserving. We have

Lemma 5.3.2. *For every $Q \in Ob(\mathbf{SB})$, let $U(Q) = \mathcal{S}(Y; R_{e_\alpha})$ and for every $\psi \in Hom(\mathbf{SB})$ say $\psi : Q \longrightarrow Q'$ let $U(\psi) = \psi|_S$. Then U is a functor from* **SB** *to* **SR**.

Next we construct the first natural equivalence.

Lemma 5.3.3. *For any $S \in Ob(\mathbf{SR})$, define $\eta(S) : S \longrightarrow UV(S)$ such that $\eta(S) : a \longmapsto \rho_a$ $(a \in S)$. Then η is a natural equivalence of the functor $I_{\mathbf{S}}$ and UV.*

Proof. Fix $S \in \mathrm{Ob}(\mathbf{SR})$ we can show that $\eta(S)$ is an isomorphism. The diagram

$$\begin{tikzcd} a \arrow[r, "\varphi"] \arrow[d] & a\varphi \arrow[d] \\ \rho_a \arrow[r, "UV(\varphi)"] & \rho_{a\varphi} \end{tikzcd}$$

is commutative since for all $a \in S$,

$$a\varphi(\eta(S')) = \rho_{a\varphi} = a(\eta(S))(UV(\varphi)).$$

Hence, the diagram

$$\begin{array}{ccc} S & \xrightarrow{\varphi} & S' \\ {\scriptstyle \eta(S)}\downarrow & & \downarrow{\scriptstyle \eta(S')} \\ UV(S) & \xrightarrow{UV(\varphi)} & UV(S') \end{array}$$

is commutative. $\square$

Let $Q \in \mathrm{Ob}\,(\mathbf{SB})$, that is, $Q = \bigcup_{\alpha \in Y} Q_\alpha$ where $Q_\alpha, \alpha \in Y$ is a bisimple inverse monoid with identity e_α such that $\{e_\alpha : \alpha \in Y\}$ is a subsemigroup of Q. Then Q is a semigroup of left I-quotients of $U(Q) = \mathcal{S}(Y; R_{e_\alpha})$ where $U(Q)$ has the (LC) condition, by Theorem 5.1.16. By Theorem 2.4.2, each R_{e_α} has the (LC) condition. Since $U(Q)$ is a left ample semigroup with the (LC) condition, it follows that it is a left I-order in its inverse hull $\Sigma(U(Q)) = VU(Q)$, by Theorem 4.2.2. Using Lemma 5.1.2, $U(Q)$ is a strong semilattice of semigroups. In fact, $U(Q)$ is a strong semilattice of left ample semigroups and connecting homomorphisms are (2,1)-homomorphisms. Since $U(Q)$ has the (LC) condition and as each R_{e_α} has the (LC) condition, it follows that Q is isomorphic to $VU(Q)$, by Theorem 5.2.2. We have the first part of the following lemma which gives the second natural equivalence.

Lemma 5.3.4. *For any $Q \in Ob(\mathbf{SB})$, define $\xi(Q) : Q \longrightarrow VU(Q)$ such that $\xi(Q) : a^{-1}b \longmapsto \rho_a^{-1}\rho_b$ where $a, b \in U(Q)$ with $a, b \in R_{e_\alpha}$ for some $\alpha \in Y$. Then ξ is a natural equivalence of the functor $I_{\mathbf{SB}}$ and VU.*

Proof. Fix $Q \in Ob\,\mathbf{SB}$. It is clear from the above remarks that ξ is well defined and ξ is an isomorphism. The diagram

$$\begin{array}{ccc} a^{-1}b & \xrightarrow{\psi} & (a^{-1}b)\psi \\ \downarrow & & \downarrow \\ \rho_a^{-1}\rho_b & \xrightarrow{VU(\psi)} & \rho_{a\psi}^{-1}\rho_{b\psi} \end{array}$$

is commutative, since for $a, b \in U(Q)$

$$((a^{-1}b)\xi(Q))(VU(\psi)) = \rho_{(a\psi)}^{-1}\rho_{(b\psi)} = (a^{-1}b)\psi(\xi(Q')).$$

Hence, the diagram

$$\begin{tikzcd} Q \arrow[r, "\psi"] \arrow[d, "\xi(Q)"'] & Q' \arrow[d, "\xi(Q')"] \\ VU(Q) \arrow[r, "VU(\psi)"] & VU(Q') \end{tikzcd}$$

is commutative. $\square$

By combining Lemmas 5.3.1, 5.3.2, 5.3.3 and 5.3.4 we have the main result of this section.

Theorem 5.3.5. *The quadruple (U, V, ξ, η) is an equivalence of categories* **SB** *and* **SR**.

Chapter 6

Primitive inverse semigroups of left I-quotients

In this chapter we characterise left I-orders in primitive inverse semigroups. In Section 6.1 we begin by investigating some properties of semigroups which are left I-orders in primitive inverse semigroups. We show that any left I-order in a primitive inverse semigroup is straight; this result enables us to use Theorem 3.2.9 in the next sections. The next section is devoted to the proof of Theorem 6.2.1, which characterises those semigroups that have a primitive inverse semigroup of left I-quotients. This theorem shows such semigroups satisfy certain conditions. By starting with a semigroup S which satisfies such conditions, we construct a semigroup of left I-quotients of S which is primitive inverse. It is well-known that Brandt semigroups are a special class of primitive inverse semigroups. We specialise our result to left I-orders in Brandt semigroups, a result that may be regarded as a generalisation of the main theorem in [11], which characterised left orders in Brandt semigroups.

In Section 6.3 we show that a primitive inverse semigroup of left I-quotients of a given semigroup is unique up to isomorphism. Section 6.4 then concentrates on I-orders (two-sided case) in primitive inverse semigroups. In the final section we give characterisations of abundant semigroups S having primitive inverse semigroups of left I-quotients. As we wish to embed S in an inverse semigroup (as a (2,1)-algebra), clearly S must be adequate. If the embedding respects $^+$, that is a (2,1)-algebra embedding, then S is forced to be ample.

6.1 Preliminaries

In this section we introduce a useful characterization of left I-orders in primitive inverse semigroups. We show that any left I-order S in a primitive inverse semigroup Q has a 0 element and is straight in Q.

A semigroup S with zero is defined to be *categorical at 0* if whenever $a, b, c \in S$ are such that $ab \neq 0$ and $bc \neq 0$, then $abc \neq 0$. We say that S is *0-cancellative* if $b = c$ follows from $ab = ac \neq 0$ and from $ba = ca \neq 0$.

Recall that an inverse semigroup S with zero is a primitive inverse semigroup if all its nonzero idempotents are primitive, where an idempotent e of S is called primitive if $e \neq 0$ and $f \leq e$ implies $f = 0$ or $e = f$.

The set of non-zero elements of a semigroup S will be denoted by S^*; in particular, $E^*(S)$ or just E^* is the set of non-zero idempotents of S. We will use the following facts about primitive inverse semigroups heavily through this chapter. Proofs can be found in [21] and [1].

Lemma 6.1.1. *Let Q be a primitive inverse semigroup.*

(i) Q is categorical at 0.

(ii) If $e, f \in E^*$, then $ef \neq 0$ implies $e = f$.

(iii) If $e \in E^*$ and $s \in Q^*$, then

$$es \neq 0 \text{ implies } es = s \text{ and } se \neq 0 \text{ implies } se = s.$$

(iv) If $a, s \in Q^*$ and $as = a$, then $s = a^{-1}a$. Dually, if $sa = a$, then $s = aa^{-1}$.

(v) If $ab \neq 0$, then $a^{-1}a = bb^{-1}$.

From the above lemma we can notice easily that a primitive inverse semigroup is 0-cancellative.

To investigate the properties of a semigroup S which is a left I-order in a primitive inverse semigroup Q we need the relations λ , ρ and τ which are introduced in [26] on any semigroup with zero as follows:

$$a \,\lambda\, b \text{ if and only if } a = b = 0 \text{ or } Sa \cap Sb \neq 0,$$

$$a \,\rho\, b \text{ if and only if } a = b = 0 \text{ or } aS \cap bS \neq 0,$$

$$\tau = \rho \cap \lambda.$$

Lemma 6.1.2. *Let S be any semigroup with zero.*
(i) If $a \,\mathcal{R}^*\, b$ where $a, b \neq 0$, then

$$xa \neq 0 \text{ if and only if } xb \neq 0.$$

(ii) If S is categorical at 0 and 0-cancellative, then

$$x, xa \neq 0 \text{ implies that } x \,\mathcal{R}^*\, xa,$$

for any $x, a \in S$.

Proof. (i) Lemma 2.2 of [11].
(ii) Let $x, a \in S$ with $xa \neq 0$. If $u, v \in S^1$ and $ux = vx$, then clearly $uxa = vxa$.

Conversely, if $uxa = vxa \neq 0$, then by 0-cancellativity, $ux = vx \neq 0$. On the other hand, if $uxa = vxa = 0$, then by categoricity at 0, $ux = vx = 0$ (note that in this case, $u, v \neq 1$). □

In the next lemma we introduce some properties of a semigroup which has a primitive inverse semigroup of left I-quotients.

We make the convention that if S is a left I-order in Q, then $\mathcal{R}$ and $\mathcal{L}$ will be relations on Q and $\mathcal{R}^*, \mathcal{L}^*, \lambda, \rho$ and τ will refer to S.

Proposition 6.1.3. *Let S be a subsemigroup of a primitive inverse semigroup Q. If S is a left I-order in Q, then:*

(1) *S contains the 0 element of Q;*
(2) *$\mathcal{L} \cap (S \times S) = \lambda$;*
(3) *S is a straight left I-order in Q;*
(4) *$Sa \neq 0$ for all $a \in S^*$;*
(5) *$\mathcal{R} \cap (S \times S) = \mathcal{R}^*$;*
(6) *$\rho \subseteq \mathcal{R}^*$.*

Proof. (1) Since Q is a primitive inverse semigroup, it follows that Q is a 0-direct union of Brandt semigroups. Hence $Q = \bigcup_{i \in J} \mathcal{B}_i$ where each $\mathcal{B}_i$ is Brandt and $\mathcal{B}_i \cap \mathcal{B}_j = \{0\}$ for $i \neq j$.

It is well known that if a, b are elements of a Brandt semigroup and a, b lie in different $\mathcal{H}$-classes, then $ab = 0$. Thus, if $a, b \in Q$ are in different $\mathcal{H}$-classes of Q, then $ab = 0$. Hence, if S intersects more than one $\mathcal{H}$-class of Q, then $0 \in S$. Suppose S is contained in an $\mathcal{H}$-class H. If H is not a group $\mathcal{H}$-class, then $a^2 = 0$ for all $a \in H$, and hence for all $a \in S$, so $0 \in S$ a contradiction. If H is a group $\mathcal{H}$-class, then $0 = a^{-1}b \in H$ for some $a, b \in H$, a contradiction. Thus S cannot be contained in an $\mathcal{H}$-class of Q, and so $0 \in S$.

(2) If $a \,\lambda\, b$, then $a = b = 0$ and certainly $a \,\mathcal{L}\, b$ in Q, or $xa = yb \neq 0$ for some $x, y \in S$. In the latter case, $a = x^{-1}yb$ and $b = y^{-1}xa$, so that $a \,\mathcal{L}\, b$ in Q. Conversely, if $a \,\mathcal{L}\, b$ in Q, then either $a = b = 0$, or $a \neq 0$ and $a = x^{-1}yb$ for some $x^{-1}y \in Q^*$ where $x, y \in S$. Then $xa = yb \neq 0$. Hence $a \,\lambda\, b$ in S. It is worth pointing out that in this case λ is transitive. Moreover, it is an equivalence, and $\{0\}$ is a λ-class.

(3) Suppose that $q \in Q^*$. Then $q = a^{-1}b$ for some $a, b \in S$. Since $a^{-1}b \neq 0$, Lemma 6.1.1 gives $aa^{-1} = bb^{-1}$ so that $a \,\mathcal{R}\, b$ in Q.

(4) Let $a = x^{-1}y \neq 0$ for some $x, y \in S$, where $x \,\mathcal{R}\, y$ in Q. By categoricity at 0 and Lemma 6.1.1 we have that $xa = y \neq 0$. Thus $Sa \neq 0$.

(5) It is clear that $\mathcal{R} \cap (S \times S) \subseteq \mathcal{R}^*$. To show that $\mathcal{R}^* \subseteq \mathcal{R} \cap (S \times S)$. Let $a \,\mathcal{R}^*\, b$ in S; from (4) there exists y in S such that $ya \neq 0$. Hence $yb \neq 0$, by Lemma 6.1.2. By Lemma 6.1.1, $aa^{-1} = y^{-1}y = bb^{-1}$ and we get $a \,\mathcal{R}\, b$ in Q.

(6) Suppose that $a \,\rho\, b$ in S. Then $a = b = 0$ and $a \,\mathcal{R}\, b$ in Q, or $ax = by \neq 0$ for some $x, y \in S$. Then $b = axy^{-1}$ and $a = byx^{-1}$, so that $a \,\mathcal{R}\, b$ in Q. By (5), $a \,\mathcal{R}^*\, b$ in S. $\square$

By Lemma 6.1.1 and Proposition 6.1.3, the following corollaries are clear.

Corollary 6.1.4. *Let S be a left I-order in a primitive inverse semigroup Q. If $a^{-1}b \neq 0$, then $a\,\mathcal{R}\,b$. Also, $a^{-1}\,\mathcal{R}\,a^{-1}b\,\mathcal{L}\,b$.*

Corollary 6.1.5. *Let S be a left I-order in a primitive inverse semigroup Q. Let $a^{-1}b, c^{-1}d$ be non-zero elements of Q where a, b, c and d are in S. Then*

(1) *$a^{-1}b\,\mathcal{R}\,c^{-1}d$ if and only if $a\,\lambda\,c$;*

(2) *$a^{-1}b\,\mathcal{L}\,c^{-1}d$ if and only if $b\,\lambda\,d$.*

Proof. (1) We have that $a^{-1}b\,\mathcal{R}\,c^{-1}d$ if and only if $a^{-1}\,\mathcal{R}\,c^{-1}$ if and only if $a\,\mathcal{L}\,c$. By Proposition 6.1.3, this is equivalent to $a\,\lambda\,c$.

(2) This is dual to (1). $\square$

6.2 The main theorem

The aim of this section is to prove the following theorem;

Theorem 6.2.1. *A semigroup S is a left I-order in a primitive inverse semigroup Q if and only if S satisfies the following conditions:*

(A) S is categorical at 0;

(B) S is 0-cancellative;

(C) λ is transitive;

(D) $Sa \neq 0$ for all $a \in S^*$.

Proof. Suppose that S be a left I-order in a primitive inverse semigroup Q. We have observed in Lemma 6.1.3 that S contains the zero element of Q. Then S inherits Conditions (A) and (B) from Q. By Proposition 6.1.3, Conditions (C) and (D) hold.

Conversely, suppose that S satisfies Conditions (A)-(D). Our aim now is to construct a semigroup Q in which S is embedded as a left I-order. We remark that from (C), λ is an equivalence and from the definition of λ, $\{0\}$ is a λ-class. Let

$$\Sigma = \{(a, b) \in S \times S : a\,\mathcal{R}^*\,b\},$$

and

$$\Sigma^* = \{(a, b) \in \Sigma : a, b \neq 0\}.$$

On Σ define $\sim$ as follows;

$$(a,b) \sim (c,d) \iff a=b=c=d=0, \text{ or there exist } x,y \in S^* \text{ such that}$$

$$xa = yc \neq 0, \; xb = yd \neq 0.$$

Lemma 6.2.2. *The relation $\sim$ is an equivalence.*

Proof. It is clear that $\sim$ is symmetric. Let $(a,b) \in \Sigma^*$. By (D), there exists $h \in S$ such that $ha \neq 0$ and hence as $a\,\mathcal{R}^*\,b$ we have that $hb \neq 0$ and so $\sim$ is reflexive. Let

$$(a,b) \sim (c,d) \sim (p,q),$$

where $(a,b),(c,d)$ and (p,q) in Σ^*. Then there exist $x,y,\bar{x},\bar{y} \in S^*$ such that

$$xa = yc \neq 0, \; xb = yd \neq 0 \text{ and } \bar{x}c = \bar{y}p \neq 0, \; \bar{x}d = \bar{y}q \neq 0.$$

To show that $\sim$ is transitive, we have to show that, there are $z,\bar{z} \in S^*$ such that $za = \bar{z}p \neq 0$ and $zb = \bar{z}q \neq 0$.

By (D), $Sc \neq 0$ and $Syc \neq 0$ and clearly $Sc \cap Syc \neq 0$, so that $c\,\lambda\,yc$. Similarly, $\bar{x}c\,\lambda\,c$; since λ is transitive, we obtain $yc\,\lambda\,\bar{x}c$. Hence $wyc = \bar{w}\bar{x}c \neq 0$ for some $w,\bar{w} \in S$. We have

$$wxa = wyc = \bar{w}\bar{x}c = \bar{w}\bar{y}p \neq 0,$$

that is, $wxa = \bar{w}\bar{y}p \neq 0$. Since $c\,\mathcal{R}^*\,d$ and $wyc = \bar{w}\bar{x}c \neq 0$ we have that $wyd = \bar{w}\bar{x}d \neq 0$ so that similarly, $wxb = \bar{w}\bar{y}q \neq 0$ as required. $\square$

Let $[a,b]$ denote the $\sim$-equivalence class of (a,b). We stress that $[0,0]$ contains only the pair $(0,0)$. On $Q = \Sigma/\sim$ define a multiplication as:

$$[a,b][c,d] = \begin{cases} [xa,yd] & \text{if } b\,\lambda\,c \text{ and } xb = yc \neq 0 \text{ for some } x,y \in S; \\ 0, & \text{else} \end{cases}$$

and $0[a,b] = [a,b]0 = 00 = 0$, where $0 = [0,0]$. We put $Q^* = Q \setminus \{[0,0]\}$.

Before we show that the above multiplication is well-defined we can notice easily that $[xa,yd] \in Q$. For $xa\,\mathcal{R}^*\,xb = yc\,\mathcal{R}^*\,yd$.

Lemma 6.2.3. *The multiplication is well-defined.*

Proof. Suppose that $[a_1, b_1] = [a_2, b_2]$, $[c_1, d_1] = [c_2, d_2]$ are in Q^*. Then there are elements x_1, x_2, y_1, y_2 in S^* such that

$$\begin{aligned} x_1a_1 &= x_2a_2 \neq 0, \\ x_1b_1 &= x_2b_2 \neq 0, \\ y_1c_1 &= y_2c_2 \neq 0, \\ y_1d_1 &= y_2d_2 \neq 0. \end{aligned}$$

Now,

$$[a_1, b_1][c_1, d_1] = \begin{cases} [wa_1, \bar{w}d_1] & \text{if } b_1 \,\lambda\, c_1 \text{ and } wb_1 = \bar{w}c_1 \neq 0 \text{ for some } w, \bar{w} \in S; \\ 0, & \text{else} \end{cases}$$

and

$$[a_2, b_2][c_2, d_2] = \begin{cases} [za_2, \bar{z}d_2] & \text{if } b_2 \,\lambda\, c_2 \text{ and } zb_2 = \bar{z}c_2 \neq 0 \text{ for some } z, \bar{z} \in S; \\ 0, & \text{else.} \end{cases}$$

Notice that as $b_1 \,\lambda\, b_2$ and $c_1 \,\lambda\, c_2$, we have that $b_1 \,\lambda\, c_1$ if and only if $b_2 \,\lambda\, c_2$. Hence $[a_1, b_1][c_1, d_1] = 0$ if and only if $[a_2, b_2][c_2, d_2] = 0$. We now assume that $b_1 \,\lambda\, c_1$ (and so $b_2 \,\lambda\, c_2$ also).

We have to prove that $[wa_1, \bar{w}d_1] = [za_2, \bar{z}d_2]$, that is,

$$xwa_1 = yza_2 \neq 0, \ x\bar{w}d_1 = y\bar{z}d_2 \neq 0, \text{ for some } x, y \in S^*.$$

Since $wb_1 = \bar{w}c_1 \neq 0$, $zb_2 = \bar{z}c_2 \neq 0$ and $a_1 \,\mathcal{R}^*\, b_1$, $a_2 \,\mathcal{R}^*\, b_2$ we have that $wa_1 \neq 0$ and $za_2 \neq 0$, by Lemma 6.1.2. Hence

$$wa_1 \,\lambda\, a_1 \,\lambda\, x_1a_1 = x_2a_2 \,\lambda\, a_2 \,\lambda\, za_2.$$

By (C), $wa_1 \,\lambda\, za_2$, that is, $xwa_1 = yza_2 \neq 0$ for some $x, y \in S$. It remains for us to show that $x\bar{w}d_1 = y\bar{z}d_2 \neq 0$. In order to do this we need a technical lemma.
The following lemma is essentially Lemma 4.8 of [11]. We give it for completeness.

Lemma 6.2.4. *Let a, b, c, d, s, t, x, y be non-zero elements of S which satisfy*

$$sa = tc \neq 0, \ sb = td \neq 0, \ xa = yc \neq 0.$$

Then $xb = yd \neq 0$.

Proof. Since $sa \neq 0$, $xa \neq 0$ and $sa \,\lambda\, a \,\lambda\, xa$, it follows that $sa \,\lambda\, xa$, by (C). Hence there are elements $w, z \in S$ such that $zsa = wxa \neq 0$. Since S is 0-cancellative and $a \neq 0$ we have that $zs = wx \neq 0$. Thus by categoricity at 0,

$$ztc = zsa = wxa = wyc \neq 0.$$

Cancelling c gives $zt = wy \neq 0$. By categoricity at 0 again we have

$$wxb = zsb = ztd = wyd \neq 0.$$

Hence $xb = yd \neq 0$. □

Returning now to the proof of Lemma 6.2.3, we can apply Lemma 6.2.4 as follows: since $x_1a_1 = x_2a_2 \neq 0$, $x_1b_1 = x_1b_2 \neq 0$ and $xwa_1 = yza_2 \neq 0$, it follows that $xwb_1 = yzb_2 \neq 0$. Since $wb_1 = \bar{w}c_1$, $zb_2 = \bar{z}c_2$ we have

$$x\bar{w}c_1 = xwb_1 = yzb_2 = y\bar{z}c_2 \neq 0.$$

Reapply the same lemma to get $x\bar{w}d_1 = y\bar{z}d_2 \neq 0$ as required. □

Lemma 6.2.5. *The multiplication is associative.*

Proof. Let $[a,b],[c,d],[p,q] \in Q^*$ and set

$$X = ([a,b][c,d])[p,q] = \begin{cases} [xa, yd][p,q] & \text{if } b \,\lambda\, c \ \text{ and } \ xb = yc \neq 0 \text{ for some } x, y \in S; \\ 0, & \text{else} \end{cases}$$

and

$$Y = [a,b]([c,d][p,q]) = \begin{cases} [a,b][\bar{x}c, \bar{y}q] & \text{if } d \,\lambda\, p \ \text{ and } \ \bar{x}d = \bar{y}p \neq 0 \text{ for some } \bar{x}, \bar{y} \in S; \\ 0, & \text{else.} \end{cases}$$

Suppose that $X = 0$. If $b \not\lambda\, c$, then either $d \not\lambda\, p$ (in which case $Y = 0$) or, $d \,\lambda\, p$ and $\bar{x}d = \bar{y}p \neq 0$, for some $\bar{x}, \bar{y} \in S$. Then $\bar{x}c \neq 0$ and as $c \,\lambda\, \bar{x}c$, $b \not\lambda\, \bar{x}c$, so that again, $Y = 0$.

On the other hand, if $b \,\lambda\, c$ so that $xb = yc \neq 0$ for some $x, y \in S$, and if $yd \not\lambda\, p$, then $d \not\lambda\, p$ so that $Y = 0$.

Conversely, if $Y = 0$, then if $d \not\lambda\, p$ we have either $b \not\lambda\, c$ (which case $X = 0$) or $b \,\lambda\, c$ and $yd \neq 0$. In this case, $p \not\lambda\, yd$, so that $X = 0$. If $d \,\lambda\, p$, then we must have that

$b \not\lambda \bar{x}c$, so that $b \not\lambda c$ and again $X = 0$. We therefore assume that $X \neq 0$ and $Y \neq 0$. Then

$$X = [xa, yd][p, q] = [sxa, rq], \ syd = rp \neq 0$$

and

$$Y = [a, b][\bar{x}c, \bar{y}q] = [\bar{s}a, \bar{r}\bar{y}q], \ \bar{s}b = \bar{r}\bar{x}c \neq 0.$$

for some $s, r, \bar{s}, \bar{r} \in S$.

We have to show that $X = Y$. That is,

$$wsxa = \bar{w}\bar{s}a \neq 0, \ wrq = \bar{w}\bar{r}\bar{y}q \neq 0$$

for some $w, \bar{w} \in S^*$. By 0-cancellativity this equivalent to $wsx = \bar{w}\bar{s} \neq 0$ and $wr = \bar{w}\bar{r}\bar{y} \neq 0$. Since $xb \neq 0$, $sx \neq 0$ and S categorical at 0 we have that $sxb \neq 0$ also, $\bar{s}b \neq 0$. Hence $sxb \,\lambda\, \bar{s}b$, and so there exist $w, \bar{w} \in S$ such that $wsxb = \bar{w}\bar{s}b \neq 0$. As S 0-cancellative, we have that $wsx = \bar{w}\bar{s} \neq 0$.

Now, since $wsxb = \bar{w}\bar{s}b \neq 0$ and $\bar{s}b = \bar{r}\bar{x}c \neq 0$, $xb = yc \neq 0$ we have that $wsyc = \bar{w}\bar{r}\bar{x}c \neq 0$. As S is 0-cancellative we have that $wsy = \bar{w}\bar{r}\bar{x} \neq 0$. Then $wsyd = \bar{w}\bar{r}\bar{x}d \neq 0$, but $syd = rp \neq 0$ and $\bar{x}d = \bar{y}p \neq 0$ so that $wrp = \bar{w}\bar{r}\bar{y}p \neq 0$. Thus $wr = \bar{w}\bar{r}\bar{y} \neq 0$ as required. □

Let $[a, b] \in Q^*$. By (D), for $a \in S^*$ there exists $x \in S$ such that $xa \neq 0$. Clearly, $[xa, xb] \in Q^*$. Again using (D), there exists $t \in S$ with $txa \neq 0$, so that $(tx)a = t(xa) \neq 0$ and as $a \,\mathcal{R}^*\, b$, $(tx)b = t(xb) \neq 0$. Then the following lemma is clear.

Lemma 6.2.6. *If $[a, b], [xa, xb] \in Q^*$, then $[a, b] = [xa, xb]$.*

Let $a \in S^*$. By (D), there exists $x \in S$ such that $xa \neq 0$. From Lemma 6.1.2, we get $x \,\mathcal{R}^*\, xa$. Hence $[x, xa] \in Q^*$. If $(y, ya) \in \Sigma^*$, then $xa \,\lambda\, a \,\lambda\, ya$ and so $xa \,\lambda\, ya$, by (C). Hence there exist $s, s' \in S$ with $sxa = s'ya \neq 0$. Cancelling a gives $sx = s'y \neq 0$ so that $[x, xa] = [y, ya]$. So we have a well defined function $\theta : S \longrightarrow Q$ defined by $0\theta = 0$ and for $a \in S^*, a\theta = [x, xa]$ where $x \in S^*$ with $xa \neq 0$.

Lemma 6.2.7. *The mapping θ is an embedding of S into Q.*

Proof. Let a, b be non-zero elements of S and let $a\theta = b\theta$ so that $[x, xa] = [y, yb]$ for some $x, y \in S$ with $xa \neq 0$, $yb \neq 0$. Hence there exist $w, \bar{w} \in S^*$ such that

$$wx = \bar{w}y \neq 0, \; wxa = \bar{w}yb \neq 0.$$

Then, $wxa = wxb \neq 0$ and as S is 0-cancellative we have that $a = b$. Thus θ is one-one.

To show that θ is a homomorphism, let $a, b \in S^*$ and $a\theta = [s, sa]$, $b\theta = [t, tb]$ where $s, t \in S$ with $sa \neq 0$ and $tb \neq 0$ for some $s, t \in S$.

Suppose that $ab = 0$. If $sa \,\lambda\, t$, then $usa = vt \neq 0$ for some $u, v \in S$. By categoricity at 0, $usab = vtb \neq 0$, a contradiction. Hence $sa \not\lambda\, t$ and $a\theta b\theta = 0 = (ab)\theta$.

Assume therefore that $ab \neq 0$. Let $(ab)\theta = [x, xab]$ where $x \in S$ with $xab \neq 0$. By categoricity at 0, $sab \neq 0$. Hence $sab \,\lambda\, b \,\lambda\, tb$ so that $sab \,\lambda\, tb$, by (C). It follows that $wsab = \bar{w}tb \neq 0$ for some $w, \bar{w} \in S$.
Since S is 0-cancellative we have that $wsa = \bar{w}t \neq 0$. Hence $sa \,\lambda\, t$ and we have that $a\theta b\theta \neq 0$. Moreover, from $xa \neq 0$ and $sa \neq 0$ we have that $sa \,\lambda\, a \,\lambda\, xa$ and so $sa \,\lambda\, xa$, by (C). It follows that there exist $m, n \in S$ such that $msa = nxa \neq 0$. By cancelling a we have that $ms = nx \neq 0$ and by categoricity at 0, $msab = nxab \neq 0$. Thus

$$\begin{aligned}
a\theta b\theta &= [s, sa][t, tb] \\
&= [ws, \bar{w}tb] \\
&= [ws, wsab] \\
&= [s, sab] && \text{by Lemma 6.2.6,} \\
&= [x, xab] \\
&= (ab)\theta.
\end{aligned}$$

□

Lemma 6.2.8. *The semigroup Q is regular.*

Proof. Let $[a,b] \in Q^*$. Then, since $[b,a] \in Q^*$ we get

$$\begin{aligned} [a,b][b,a][a,b] &= [xa,xa][a,b] && \text{for some } x \in S \text{ with } xb \neq 0 \\ &= [a,a][a,b] && \text{by Lemma 6.2.6,} \\ &= [ya,yb] && \text{for some } y \in S \text{ with } ya \neq 0 \\ &= [a,b] && \text{by Lemma 6.2.6.} \end{aligned}$$

□

For any $a \in S^*$ we have that $[a,a] \in Q^*$ and $[a,a][a,a] = [xa,xa]$ where $x \in S$ with $xa \neq 0$. By Lemma 6.2.6, $[xa,xa] = [a,a]$, that is, $[a,a]$ is idempotent. The next lemma describes the form of $E(Q)$.

Lemma 6.2.9. $E(Q) = \{[a,a], a \in S^*\} \cup \{0\}$ *and forms a semilattice.*

Proof. Let $[a,b] \in E(Q^*)$. Then, $[a,b][a,b] = [a,b]$. Hence $[xa,yb] = [a,b]$ where $xb = ya \neq 0$ for some $x,y \in S$. By definition of $\sim$ there exist $t,r \in S^*$ such that $txa = ra \neq 0$ and $tyb = rb \neq 0$. Since S is 0-cancelltive, $tx = ty = r \neq 0$. Thus $x = y$ and so $a = b$.
For $[a,a],[b,b] \in E(Q^*)$ we have that $[b,b][a,a] = 0 \Longleftrightarrow a \not\lambda\, b \Longleftrightarrow [a,a][b,b] = 0$ and if $a \,\lambda\, b$, then

$$\begin{aligned} [a,a][b,b] &= [\acute{s}a,\acute{t}b] && \text{where } \acute{s}a = \acute{t}b \neq 0 \text{ for some } \acute{s},\acute{t} \in S \\ &= [\acute{t}b,\acute{s}a] \\ &= [b,b][a,a]. \end{aligned}$$

□

By Lemmas 6.2.8 and 6.2.9, Q is inverse, so that $[b,a]$ is the unique inverse of $[a,b]$.

Lemma 6.2.10. *The semigroup Q is primitive.*

Proof. Suppose that $[a,a],[b,b] \in E(Q^*)$ are such that $[a,a] \leq [b,b]$. Then

$$\begin{aligned} [a,a] &= [a,a][b,b] \\ &= [xa,yb] && \text{for some } x,y \in S \text{ where } xa = yb \\ &= [yb,yb] \\ &= [b,b] && \text{by Lemma 6.2.6.} \end{aligned}$$

□

By Lemma 6.2.7, we can regard S as a subsemigroup of Q. Let $[a,b] \in Q^*$ and $a\theta = [x, xa], b\theta = [y, yb]$ where $xa \neq 0$ and $yb \neq 0$ for some $x, y \in S$. By Lemma 6.1.2, $ya \neq 0$ so that $xa \,\lambda\, a \,\lambda\, ya$ and by (C), it follows that $xa \,\lambda\, ya$. Hence $txa = rya \neq 0$ for some $t, r \in S$. By (B), $tx = ry \neq 0$ and so $x \,\lambda\, y$. We have

$$\begin{aligned}(a\theta)^{-1}(b\theta) &= [x, xa]^{-1}[y, yb] \\ &= [xa, x][y, yb] \\ &= [txa, ryb] \\ &= [txa, txb] \qquad \text{by Lemma 6.2.6,} \\ &= [a, b].\end{aligned}$$

Hence S is a left I-order in Q. This completes the proof of Theorem 6.2.1. □

It is worth pointing out that if $e \in E(Q^*)$, then $e = a^{-1}a$ for some $a \in S^*$. For $e = a^{-1}b \in E(Q^*)$ as $a \,\mathcal{R}\, b$ we have that $b = ae$ and $a = be$. Then it is clear that $a = b$.

Corollary 6.2.11. *A semigroup S is a left I-order in a Brandt semigroup Q if and only if S satisfies that conditions in Theorem 6.2.1 and in addition, for all $a, b \in S^*$ there exist $c, d \in S$ such that $ca \,\mathcal{R}^*\, d \,\lambda\, b$.*

Proof. Suppose that S is a left I-order in Q and let $a, b \in S^*$. Since Q has a single non-zero $\mathcal{D}$-class, there exists $q \in Q$ such that $a \,\mathcal{R}\, q \,\mathcal{L}\, b$ in Q. Let $q = c^{-1}d$ where $c, d \in S$ and $c \,\mathcal{R}\, d$ so that $ca \,\mathcal{R}\, cc^{-1}d = d$. By Corollary 6.1.4, $c^{-1} \,\mathcal{R}\, c^{-1}d \,\mathcal{L}\, d$, it follows that $d \,\mathcal{L}\, c^{-1}d \,\mathcal{L}\, b$. Hence $ca \,\mathcal{R}\, d \,\mathcal{L}\, b$ and so $ca \,\mathcal{R}^*\, d \,\lambda\, b$, by Proposition 6.1.3.

On the other hand, if S satisfies the given conditions, then we can show that Q is Brandt. For, if $q = a^{-1}b$ and $p = c^{-1}d \in Q^*$, then $b, d \in S^*$ so there exist $u, v \in S$ with $ub \,\mathcal{R}^*\, v \,\lambda\, d$. In $Q, ub \,\mathcal{R}\, v \,\mathcal{L}\, d$, so that b and d (and hence q and p) lie in the same Brandt subsemigroup of Q (as Q is a 0-direct union of Brandt semigroups). □

Lemma 6.2.12. *Let $Q = \bigcup_{i \in I} Q_i$ be a primitive inverse semigroup where Q_i is a Brandt semigroup. If S is a left I-order in Q, then S is a 0-direct union of semigroups that are left I-orders in the Brandt semigroups $Q_i, i \in I$.*

6.3 Uniqueness

In this section we show that a semigroup S has, up to isomorphism, at most one primitive inverse semigroup of left I-quotients.

Theorem 6.3.1. *Let S be a left I-order in a primitive inverse semigroup Q. If $\phi : S \to T$ is an isomorphism where T is a left I-order in a primitive inverse semigroup P, then ϕ lifts to an isomorphism $\overline{\phi} : Q \to P$*

Proof. Let $\phi : S \to T$ be as given. From Proposition 6.1.3, S and T both contain 0 and clearly ϕ preserves this. Let $a, b \in S^*$ with $a \,\mathcal{R}\, b$ in Q. By Condition (D), there exists $c \in S$ with $ca \neq 0$ and hence $cb \neq 0$. It follows that $(c\phi)(a\phi), (c\phi)(b\phi)$ are non-zero in P, so that $a\phi \,\mathcal{R}\, b\phi$ in P.

We also show that ϕ preserves $\mathcal{L}$; for if $a, b \in S^*$ and $a \,\mathcal{L}\, b$ in Q, then $a \,\lambda\, b$ in S, by Lemma 6.1.3. It follows that $ca = db \neq 0$ for some $c, d \in S$. Then $c\phi a\phi = d\phi b\phi \neq 0$ so that $a\phi \,\mathcal{L}\, b\phi$ in P.

It remains to show that ϕ preserves $\mathcal{T}_S^Q$. Suppose therefore that $a, b, c \in S$ and $ab^{-1}Q \subseteq c^{-1}Q$. Then either $ab^{-1} = 0$, or $ab^{-1} \,\mathcal{R}\, c^{-1}$ in Q. In the former case, either a or b is 0 or a and b are not $\mathcal{L}$-related in Q, it follows that either $a\phi$ or $b\phi$ is 0 or $a\phi$ and $b\phi$ are not $\mathcal{L}$-related in P, giving $(a\phi)(b\phi)^{-1} = 0$ and so $(a\phi)(b\phi)^{-1}P \subseteq (c\phi)^{-1}P$. On the other hand, if $ab^{-1} \neq 0$, then $a, b \neq 0$ and so $a \,\mathcal{L}\, b$ and $a \,\mathcal{R}\, c^{-1}$ in Q. It follows that $ca \neq 0$ and so $a\phi \,\mathcal{L}\, b\phi$ and $(c\phi)(a\phi) \neq 0$ in P. Consequently,

$$0 \neq (a\phi)(b\phi)^{-1}P = (a\phi)P = (c\phi)^{-1}P.$$

Since ϕ (and, dually, ϕ^{-1}) preserve both $\mathcal{R}$ and $\mathcal{T}$, it follows from Corollary 3.2.10, that ϕ lifts to an isomorphism $\overline{\phi} : Q \to P$. $\square$

The following corollary may be deduced from the previous theorem.

Corollary 6.3.2. *If Q_1, Q_2 are primitive inverse semigroups of left I-quotients of a semigroup S, then Q_1, Q_2 are isomorphic by an isomorphism which restricts to the identity map on S.*

Proposition 6.3.3. *If a semigroup S has a primitive inverse semigroup Q of left I-quotients and a primitive inverse semigroup $\acute{Q}$ of right I-quotients, then Q and $\acute{Q}$ are both semigroups of I-quotients of S, so that $Q \cong \acute{Q}$ by Corollary 6.3.2.*

Proof. We show that Q is a semigroup of right I-quotients of S. Let $q \in Q$. If $q = 0$, then $q = 0^{-1}0 = 00^{-1}$. If $q \in Q^*$, then $q = a^{-1}b$ for some $a, b \in S^*$ where $a \mathcal{R} b$ in Q. Then $a \mathcal{R}^* b$ in S. Pick $c \in S$ with $ca \neq 0$ and hence $cb \neq 0$, by Lemma 6.1.2. The semigroup Q' is a union of Brandt semigroups, say $Q' = \bigcup_{i \in I} B_i$. Now, $a, b, c \in Q'$ and ca, cb are non-zero, so $a, b, c \in B_i$ for some $i \in I$. Since $ca \neq 0$ we have $c^{-1}c = aa^{-1}$ in B_i, by Lemma 6.1.1. Similarly, $c^{-1}c = bb^{-1}$ and so $aa^{-1} = bb^{-1}$ giving $a \mathcal{R} b$ in B_i and hence Q'. By the dual of Proposition 6.1.3, $a \rho b$ in S, that is, $ax = by \neq 0$ for some $x, y \in S$, so that $xy^{-1} = a^{-1}b$ in Q (and $\acute{Q}$). Thus S an I-order in Q and similarly, in $\acute{Q}$. □

6.4 Primitive inverse semigroups of I-quotients

In this section we study the case where a semigroup is both a left and a right I-order in a primitive inverse semigroup, that is, an I-order in a primitive inverse semigroup.

Lemma 6.4.1. *Let S have a primitive inverse semigroup Q of I-quotients. Then*

(1) $\mathcal{R}^* = \mathcal{R} \cap (S \times S) = \rho$,

(2) $\mathcal{L}^* = \mathcal{L} \cap (S \times S) = \lambda$,

(3) $\mathcal{H}^* = \mathcal{H} \cap (S \times S) = \tau$.

Proof. (1) By Proposition 6.1.3, $\mathcal{R} \cap (S \times S) = \mathcal{R}^*$. By the dual of Proposition 6.1.3, $\mathcal{R} \cap (S \times S) = \rho$. Hence $\rho = \mathcal{R}^*$.
(2) This is dual to (1).
(3) Immediate from (1) and (2). □

Since $\mathcal{H}$ is a congruence on any primitive inverse semigroup the following corollary is clear.

Corollary 6.4.2. *Let Q be primitive inverse semigroup of I-quotients of a semigroup S. Then $\mathcal{H}^*$ is a congruence relation on S.*

If S is an I-order in a primitive inverse semigroup Q, then S satisfies the conditions of Theorem 6.2.1, and in addition, the duals $(\acute{C})$ and $(\acute{D})$ of (C) and (D). We can reduce these conditions by using the next lemma.

Lemma 6.4.3. *Left S be a left I-order in a primitive inverse semigroup Q and suppose that $aS \neq 0$ for all $a \in S^*$. Then*

$$\mathcal{R}^* \subseteq \rho \text{ if and only if } S \text{ is an I-order in } Q.$$

Proof. Suppose that $\mathcal{R}^* \subseteq \rho$, by Proposition 6.1.3, $\mathcal{R}^* = \rho$, and so ρ is transitive. By the dual of Theorem 6.2.1, S is a right I-order in a primitive inverse semigroup $\acute{Q}$. By Proposition 6.3.3, $Q \cong \acute{Q}$ and S is an I-order in Q. On the other hand, if S is an I-order in Q, then by Lemma 6.4.1, $\mathcal{R}^* = \rho$ as required. □

Now we introduce condition (E) which appeared in [11] for a semigroup with zero as follows:

(E) $a \,\rho\, b$ if and only if $a = b = 0$ or there exists an element x in S such that

$$xa \neq 0 \text{ and } xb \neq 0.$$

Lemma 6.4.4. *For a semigroup S, the following conditions are equivalent:*

(1) *S has a primitive inverse semigroup of I-quotients;*

(2) *S is 0-cancellative, categorical at 0, and S satisfies (D), $(\acute{D})$, (E) and $(\acute{E})$.*

Proof. If (1) holds, then by Theorem 6.2.1, and its dual, we need only to show that S satisfies (E) and its dual. Suppose that $a \,\rho\, b$ and $a, b \neq 0$. Then, $a \,\mathcal{R}^*\, b$, by Lemma 6.4.1. By (D), there exists $x \in S$ such that $xa \neq 0$, and so $xb \neq 0$, by Lemma 6.1.2. Conversely, if $xa \neq 0$ and $xb \neq 0$, then using (D), $xa \,\rho\, x$ and $x \,\rho\, xb$. Since ρ is transitive, $xa \,\rho\, xb$, it follows that $xat = xbr \neq 0$ for some $t, r \in S$. By cancelling x we have that $at = br \neq 0$. Thus $a \,\rho\, b$. Similarly, S satisfies $(\acute{E})$.

Suppose that (2) holds. We show that λ and ρ are transitive. In order to prove this, we show that $\mathcal{R}^* = \rho$ and $\mathcal{L}^* = \lambda$. Let $a \,\mathcal{R}^*\, b$. Then, either $a = b = 0$ or $a, b \neq 0$ and by (D), $xa \neq 0$ for some $x \in S$ and as $a \,\mathcal{R}^*\, b$, we have that $xb \neq 0$, by Lemma 6.1.2. By (E), $a \,\rho\, b$ so that $\mathcal{R}^* \subseteq \rho$.

Conversely, if $a \rho b$, then either $a = b = 0$ (so that $a \mathcal{R}^* b$) or $ah = bk \neq 0$ for some $h, k \in S$. Suppose now that $u, v \in S^1$ and $ua = va$. If $ua = va \neq 0$, then by categoricity at 0, $uah = vah \neq 0$, so that $ubk = vbk \neq 0$ and 0-cancellativity gives $ub = vb \neq 0$. On the other hand, if $ua = va = 0$, then $uah = vah = 0$, so that $ubk = vbk = 0$. By categoricity at 0, $ub = vb = 0$. Similarly, $ub = vb$ implies $ua = va$. Hence $a \mathcal{R}^* b$. □

We now summarise the result of this section.

Proposition 6.4.5. *For a semigroup S, the following conditions are equivalent:*

(1) *S is an I-order in a primitive inverse semigroup;*

(2) *S satisfies conditions (A),(B),(C),(D), $(\acute{C})$ and $(\acute{D})$;*

(3) *S satisfies conditions (A),(B),(C),(D), $(\acute{D})$ and $\mathcal{R}^* \subseteq \rho$;*

(4) *S satisfies conditions (A),(B),(D),$(\acute{D})$,(E) and $(\acute{E})$.*

Proof. The equivalence of (1) and (2) follows from Theorem 6.2.1 and its dual. The equivalence of (1) and (3) is immediate from Theorem 6.2.1 and its dual, and Lemmas 6.4.1 and 6.4.3. Finally, the equivalence of (1) and (4) is given by Lemma 6.4.4. □

6.5 The abundant case

In this final section we give characterizations of abundant semigroups which are left I-orders in primitive inverse semigroups.

Fountain [7] has generalised the Rees theorem to show that every abundant semigroup in which the non-zero idempotents are primitive, is isomorphic to what he calls a *PA-blocked Rees matrix semigroup.* We refer the interested reader to [7] for more details. It is clear that if an abundant semigroup is a left I-order in a primitive inverse semigroup, then it is adequate. More than this, it must be ample, as we now explain.

We recall that a semigroup S is a left (right) ample if and only if S is left (right) adequate and satisfies the left (right) ample condition which is:

$$(ae)^+a = ae \quad (a(ea)^* = ea) \text{ for all } a \in S \text{ and } e \in E(S).$$

A semigroup is an ample semigroup if it is both left and right ample. From [18] a semigroup S is left ample if and only if it is embeds in an inverse semigroup T such that $\mathcal{R} \cap (S \times S) = \mathcal{R}^*$. If a left ample semigroup S has a primitive inverse semigroup of left I-quotients Q, then for any $a \in S$ we have that $a \, \mathcal{R}^* \, a^+$, and by Proposition 6.1.3, $a \, \mathcal{R} \, a^+$, that is, $a^+ = aa^{-1}$. Hence the following lemma is clear.

Lemma 6.5.1. *Let S be a left I-order in a primitive inverse semigroup Q. Then the following are equivalent:*

(1) *S is left adequate;*

(2) *S is left ample.*

In the next lemma we introduce an equivalent condition for categoricity at 0 for any primitive ample semigroup with zero.

Lemma 6.5.2. *Let S be a primitive ample semigroup with zero. Then the following are equivalent:*

(*i*) *S is categorical at 0;*

(*ii*) *$a^* = b^+ \Longleftrightarrow ab \neq 0$ for a and b in S^*.*

Proof. (*i*) $\Longrightarrow$ (*ii*) Let $a, b \in S$. If $ab \neq 0$, then $aa^*b^+b \neq 0$, so $a^*b^+ \neq 0$ and so by primitivity $a^* = b^+$. Conversely, if $a^* = b^+$, then $aa^* \neq 0$ and $a^*b = b^+b \neq 0$, so by categoricity at 0, $ab = aa^*b \neq 0$.
(*ii*) $\Longrightarrow$ (*i*) Suppose that $ab \neq 0$ and $bc \neq 0$ where $a, b, c \in S$. Then $a^* = b^+$ and $b^* = c^+$. Hence $b^+(bc) \neq 0$ gives $b^+(bc)^+ = (b^+(bc))^+ \neq 0$ and so by primitivity $b^+ = (bc)^+$. Thus $a^* = (bc)^+$ so that $a^*(bc) \neq 0$ and thus $a(bc) \neq 0$. $\square$

We can offer some simplification of Theorem 6.2.1, in the case that S is adequate.

Proposition 6.5.3. *[7, Proposition 5.5] For a semigroup S with zero, the following conditions are equivalent:*

(1) *S is categorical at 0, 0-cancellative and satisfies:*
for each element a of S, there is an element e of S such that $ea = a$ and an element f of S such that $af = a$ $(*)$

(2) *S is a primitive adequate semigroup;*

(3) *S is isomorphic to PA-blocked Rees $I \times I$ matrix semigroup $\mathcal{M}(M_{\alpha\beta}; I, I, \Gamma; P)$ where the sandwich matrix P is diagonal and $p_{ii} = e_\alpha$ for each $i \in I_\alpha, \alpha \in \Gamma$.*

From the above lemma and Theorem 6.2.1, the following lemma is clear.

Lemma 6.5.4. *For a semigroup S with zero, the following conditions are equivalent:*

(1) *S is abundant and left I-order in a primitive inverse semigroup Q;*

(2) *S is a primitive adequate semigroup and λ is transitive;*

(3) *S is 0-cancellative, categorical at 0, S satisfies $(*)$ and λ is transitive.*

In the two-sided case we have the following.

Lemma 6.5.5. *For a semigroup with zero, the following conditions are equivalence:*

(1) *S is abundant and an I-order in a primitive inverse semigroup Q;*

(2) *S is primitive adequate and λ, ρ are transitive;*

(3) *S is 0-cancellative, categorical at 0, satisfies $(*)$ and λ, ρ are transitive.*

Proposition 6.5.6. *Let S be a left ample semigroup and a left I-order in a primitive inverse semigroup Q. If S is a union of $\mathcal{R}$-classes of Q, then $Q \cong \Sigma(S)$.*

Proof. By Corollary 6.3.2, it is enough to show that $\Sigma(S)$ is primitive, since by Theorem 4.2.2 and Lemma 4.3.2, S is a left I-order in $\Sigma(S)$. Let $0 \neq e \leq f$ in $\Sigma(S)$. Then, $e = a^{-1}a$ and $f = b^{-1}b$ for some $a, b \in S$ where $e, f \in E(\Sigma(S))$. We have that $0 \neq e = ef$, so that $ab^{-1} \neq 0$ and $ab^{-1} = c^{-1}d$ for some $c, d \in S$ with $c\,\mathcal{R}^*\,d$ in S and so $c\,\mathcal{R}\,d$ in $\Sigma(S)$. Then by Lemma 3.2.2, $ca = db \neq 0$, so that in Q, $a\,\mathcal{L}\,b$ and so $a\,\mathcal{L}\,b$ in S by Lemma 4.3.2. Therefore in $\Sigma(S)$. Hence $e = f$ as required. □

Chapter 7

Bicyclic semigroups of left I-quotients

The first published description of the bicyclic semigroup was given by Evgenii Lyapin in 1953 [27]. The bicyclic semigroup $\mathcal{B}$ is the most straightforward example of a bisimple inverse ω-semigroup. In fact, it is a semigroup with many remarkable properties. A description of the subsemigroups of the bicyclic monoid was given in 2005 [3]. In this chapter we use this description to study left I-orders in the bicyclic monoid. By description of left I-orders in $\mathcal{B}$, we will obtain:

Theorem 7.0.7. *Let S be a subsemigroup of $\mathcal{B}$. If S is a left I-order in $\mathcal{B}$, then it is straight.*

In the preliminaries, after introducing the necessary notation, we give the description of subsemigroups of $\mathcal{B}$ from [3].

Subsemigroups of $\mathcal{B}$ fall into three classes: upper, lower and two-sided. In Sections 3, 4 and 5 we give necessary and sufficient conditions for upper, lower and two-sided subsemigroups of $\mathcal{B}$ to be left I-orders in $\mathcal{B}$, respectively. In each case, such left I-orders are straight, which proves Theorem 7.0.7.

7.1 Preliminaries

The bicyclic semigroup $\mathcal{B}(a,b)$ is defined by the monoid generated by two elements a and b subject only to the condition that $ba = 1$. It follows that the elements can all be written in the standard form $a^i b^j$ where $i, j \geq 0$. We can write out the elements of $\mathcal{B}$ in array.

$$\begin{array}{c|cccccc}
1 & b & b^2 & b^3 & b^4 & \cdots \\
\hline
a & ab & ab^2 & ab^3 & ab^4 & \cdots \\
a^2 & a^2b & a^2b^2 & a^2b^3 & a^2b^4 & \cdots \\
a^3 & a^3b & a^3b^2 & a^3b^3 & a^3b^4 & \cdots \\
a^4 & a^4b & a^4b^2 & a^4b^3 & a^4b^4 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

The multiplication on $\mathcal{B}$ is defined as follows:

$$a^k b^l a^m b^n = \begin{cases} a^{k+m-l} b^n & l \leq m, \\ a^k b^{l-m+n} & l > m. \end{cases}$$

We can put the two cases together as follows:

$$a^k b^l a^m b^n = a^{k-l+t} b^{n-m+t} \text{ where } t = \max\{l, m\}.$$

The monoid $\mathcal{B}$ is thus isomorphic to the monoid $\mathbb{N}^0 \times \mathbb{N}^0$ with multiplication

$$(k,l)(m,n) = (k - l + t, n - m + t) \text{ where } t = \max\{l, m\}.$$

It is easy to see that $\mathcal{B}$ is an inverse semigroup: the element $a^i b^j$ has inverse $a^j b^i$. The idempotents of $\mathcal{B}$ are of the form

$$e_n = a^n b^n \ (n = 0, 1, 2, \ldots) \text{ which satisfy } 1 = e_0 \geq e_1 \geq e_2 \geq \ldots.$$

Green's relations $\mathcal{L}, \mathcal{R}$ and $\mathcal{H}$ are given by

$$a^i b^j \, \mathcal{L} \, a^k b^l \text{ if and only if } j = l,$$

$$a^i b^j \, \mathcal{R} \, a^k b^l \text{ if and only if } i = k,$$

and

$$a^i b^j \, \mathcal{H} \, a^k b^l \text{ if and only if } i = k \text{ and } j = l.$$

In the array, the rows are the $\mathcal{R}$-classes of $\mathcal{B}$, the columns are the $\mathcal{L}$-classes and the $\mathcal{H}$-classes are points. There is only one $\mathcal{D}$-class; that is, $\mathcal{B}$ is a bisimple monoid (hence simple).

Following [3], we start by introducing some basic subsets of $\mathcal{B}$,

$D \;=\; \{a^ib^i : i \geq 0\}$ ....the *diagonal.*
$L^p \;=\; \{a^ib^j : 0 \leq j \leq p, i \geq 0\}$ for $p \geq 0$ ....the *left strip* (determined by p).

For $0 \leq q \leq p$ we define the *triangle*

$$T_{q,p} = \{a^ib^j : q \leq i \leq j < p\}.$$

For $i, m \geq 0$ and $d > 0$ we define the rows

$$\Lambda_i = \{a^ib^j : j \geq 0\},\ \Lambda_{i,m,d} = \{a^ib^j : d|j-i, j \geq m\}$$

and in general for $I \subseteq \{0, \ldots, m-1\}$,

$$\Lambda_{I,m,d} = \bigcup_{i \in I} \Lambda_{i,m,d} = \{a^ib^j : i \in I, d|j-i, j \geq m\}.$$

For $p \geq 0, d > 0, r \in [d] = \{0, \ldots, d-1\}$ and $P \subseteq [d]$ we define the *squares*

$$\Sigma_p = \{a^ib^j : i, j \geq p\},\ \Sigma_{p,d,r} = \{a^{p+r+ud}b^{p+r+vd} : u, v \geq 0\},$$

$$\Sigma_{p,d,P} = \bigcup_{r \in P} \Sigma_{p,d,r} = \{a^{p+r+ud}b^{p+r+vd} : r \in P, u, v \geq 0\}.$$

It is worth pointing out that in [3] it was shown that a subsemigroup of $\mathcal{B}$ is inverse if and only if it has the form $F_D \cup \Sigma_{p,d,P}$ where F_D is a finite subset of the diagonal (which may be empty). The function $\widehat{\ }: \mathcal{B} \longrightarrow \mathcal{B}$ defined by $a^ib^j \to \widehat{a^ib^j} = a^jb^i$ is an anti-isomorphism. Geometrically it is the reflection with respect to the main diagonal.

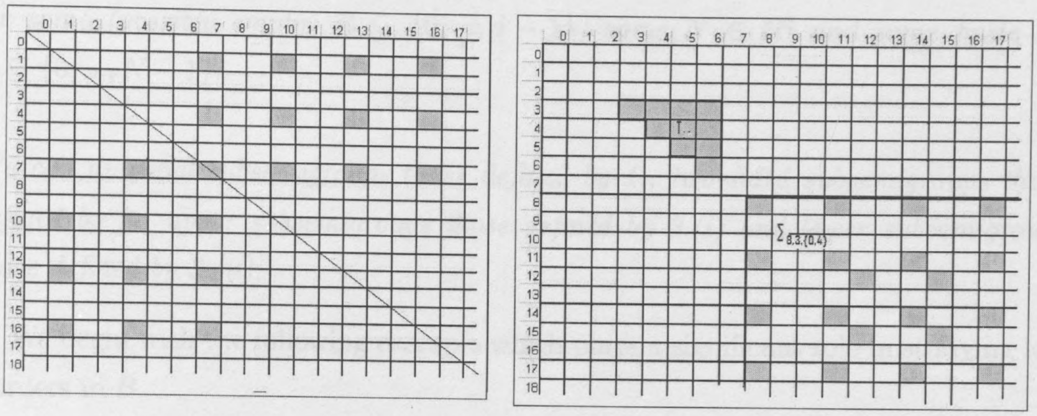

Figure 7.1: The subsets $\Lambda_{\{1,4,7\},7,3}, \widehat{\Lambda}_{\{1,4,7\},7,3}, \Sigma_{8,3,\{0,4\}}$ and $T_{3,7}$

Proposition 7.1.1. *[3] Let S be a subsemigroup of the bicyclic monoid. Then one of the following conditions holds:*

1. S is a subset of the diagonal, $S \subseteq D$.

2. S is a union of a subset of a triangle, a subset of the diagonal above the triangle, a square below the triangle and some lines belonging to a strip determined by the square and the triangle, or the reflection of this union with respect to the diagonal. Formally there exist $q, p \in \mathbb{N}^0$ with $q \leq p, d \in \mathbb{N}, I \subseteq \{q, ..., p-1\}$ with $q \in I, P \subseteq \{0, ..., d-1\}$ with $0 \in P, F_D \subseteq D \cap L^q$, $F \subseteq T_{q,p}$ such that S is of one of the following forms:

$$(i)\ S = F_D \cup F \cup \Lambda_{I,p,d} \cup \Sigma_{p,d,P};$$
$$(ii)\ S = F_D \cup \widehat{F} \cup \widehat{\Lambda}_{I,p,d} \cup \Sigma_{p,d,P}.$$

3. There exist $d \in \mathbb{N}$, $I \subseteq \mathbb{N}^0$, $F_D \subseteq D \cap L_{min(I)}$ and sets $S_i \subseteq \Lambda_{i,i,d}$ $(i \in I)$ such that S is of one of the following forms:

$$(i)\ S = F_D \cup \bigcup_{i \in I} S_i;$$
$$(ii)\ S = F_D \cup \bigcup_{i \in I} \widehat{S}_i;$$

where each S_i has the form

$$S_i = F_i \cup \Lambda_{i,m_i,d}$$

for some $m_i \in \mathbb{N}^0$ and some finite set F_i, and

$$I = I_0 \cup \{r + ud : r \in R, u \in \mathbb{N}^0, r + ud \geq N\}$$

for some (possibly empty) $R \subseteq \{0, ..., d-1\}$, *some* $N \in \mathbb{N}^0$ *and some finite set* $I_0 \subseteq \{0, ..., N-1\}$.

We call diagonal subsemigroups those defined by 1., two-sided subsemigroups those defined by 2., upper subsemigroups those defined by 3.(i) and lower subsemigroups those defined by 3.(ii).

We begin with the following example which plays a significant role in studying left I-orders in $\mathcal{B}$.

Example 7.1.2. *Let* $R_1 = \{a^0 b^j : j \geq 0\}$ *be the* $\mathcal{R}$*-class of the identity element 1 of* $\mathcal{B}$ *and* $q = a^m b^n \in \mathcal{B}$. *Then*

$$q = a^m b^n = (a^0 b^m)^{-1}(a^0 b^n),$$

so that R_1 *is a straight left I-order in* $\mathcal{B}$. *In fact, it is a special case of Clifford's result, mentioned in the Chapter 3.*

Remark 7.1.3. Any subsemigroup of $\mathcal{B}$ that contains R_1 is a straight left I-order in $\mathcal{B}$.

Lemma 7.1.4. *Let* S *be a left I-order in* $\mathcal{B}$. *Then for any* $\mathcal{L}$*-class* L_k *of* $\mathcal{B}$, $S \cap L_k \neq \emptyset$.

Proof. Let $k \in \mathbb{N}^0$. Then

$$\begin{aligned} a^k b^k &= (a^i b^j)^{-1}(a^m b^n) \\ &= a^j b^i a^m b^n \\ &= a^{j-i+t} b^{n-m+t} \end{aligned}$$

where $t = \max\{i, m\}$, for some $a^i b^j, a^m b^n \in S$. Hence $k = j - i + t = n - m + t$, so that either $k = j$ or $k = n$. Thus $S \cap L_k \neq \emptyset$. □

We conclude this section by the following lemma which plays a significant role in the next sections.

Lemma 7.1.5. *Let* S *be a left I-order in* $\mathcal{B}$ *and let* $d \in \mathbb{N}$. *If for all* $a^i b^j \in S$ *we have that* $d | i - j$, *then* $d = 1$.

Proof. Let $a^kb^l \in \mathcal{B}$. Then there exist $a^ib^j, a^mb^n \in S$ with

$$\begin{aligned} a^kb^l &= (a^ib^j)^{-1}(a^mb^n) \\ &= a^jb^ia^mb^n \\ &= a^{j-i+t}b^{n-m+t} \end{aligned}$$

where $t = \max\{i, m\}$. Now

$$\begin{aligned} k - l &= (j - i + t) - (n - m + t) \\ &= (j - i) + (m - n) \equiv 0 (\mathrm{mod}\ d). \end{aligned}$$

It follows that $d = 1$. □

7.2 Upper subsemigroups

In this section we give necessary and sufficient conditions for an upper subsemigroup S of $\mathcal{B}$ to be a left I-order in $\mathcal{B}$. The upper subsemigroups of $\mathcal{B}$ are those having all elements on or above the diagonal; that is, all elements satisfy: $a^ib^j, j \geq i$. Throughout this section S is an upper subsemigroup of $\mathcal{B}$ having the form (3).(i) in Proposition 7.1.1. We have already met one of them, which is the $\mathcal{R}$-class of the identity. By Lemma 7.1.4, we deduce that any left I-order upper subsemigroup is a monoid.

The next example is of a subsemigroup bigger than $\mathcal{R}$-class of the identity. In fact, it is the largest upper subsemigroup of $\mathcal{B}$.

Example 7.2.1. *The upper subsemigroup $\mathcal{B}^+ = \{a^ib^j : j \geq i\}$ of $\mathcal{B}$ is a straight left I-order in $\mathcal{B}$, by Remark 7.1.3 as $R_1 \subseteq \mathcal{B}^+$. In fact, we can write any element a^ib^j of $\mathcal{B}$ as follows*

$$a^ib^j = (a^ib^{i+j})(a^jb^{i+j})^{-1}$$

where $a^ib^{i+j}, a^jb^{i+j} \in \mathcal{B}^+$, that is, $\mathcal{B}^+$ is a right I-order in $\mathcal{B}$. Hence $\mathcal{B}^+$ is an I-order in $\mathcal{B}$. It is worth pointing out that $\mathcal{B}^+$ is a full subsemigroup of $\mathcal{B}$ in the sense that $E(\mathcal{B}) = E(\mathcal{B}^+)$.

Remark 7.2.2. Let S be an upper subsemigroup of $\mathcal{B}$. If $i \notin I$, then S does not contain any element of form a^ib^j for all $j > i$, and only contains a^ib^i if $a^ib^i \in F_D$.

Lemma 7.2.3. *Let S be an upper subsemigroup of $\mathcal{B}$. If S is a left I-order in $\mathcal{B}$, then $d = 1$ and $0 \in I$.*

Proof. Since S an upper subsemigroup, it follows that for all $a^ib^j \in S$ we have that $d|j-i$ for some $d \in \mathbb{N}$. By Lemma 7.1.5, it is clear that $d=1$. It is therefore remains to show that $0 \in I$. By Lemma 7.1.4, we have that $1 \in S$. Let $a^0b^h \in \mathcal{B}$ for some $h \in \mathbb{N}$. Hence

$$\begin{aligned} a^0b^h &= (a^ib^j)^{-1}(a^mb^n) \\ &= a^{j-i+t}b^{n-m+t} \end{aligned}$$

where $t = \max\{i,m\}$, for some $a^ib^j, a^mb^n \in S$. For $0 = j-i+t$ we must have that $t=i$ and $j=0$. As $i \leq j$, it follows that $i=j=0$ and so $a^0b^h = a^mb^n \in S$. $\square$

The following corollary is obvious.

Corollary 7.2.4. *Let S be an upper subsemigroup of $\mathcal{B}$. If S is a left I-order in $\mathcal{B}$, then $R_1 \subseteq S$.*

By Lemma 7.1.4, $S \cap L_1 \neq \emptyset$. As $F_D = D \cap L_{min(I)}$, the following corollary is clear.

Corollary 7.2.5. *Let S be an upper subsemigroup of $\mathcal{B}$. If S is a left I-order in $\mathcal{B}$, then $F_D = \{1\}$ or $F_D = \emptyset$.*

We now come to the main result of this section.

Proposition 7.2.6. *For an upper subsemigroup S of $\mathcal{B}$, the following are equivalent:*

(i) S is a left I-order in $\mathcal{B}$;

(ii) $R_1 \subseteq S$.

Moreover, writing S as $S = F_D \cup \bigcup_{i \in I} S_i$, we have $R_1 \subseteq S$ if and only if $0 \in I, d=1$ and $F_D \cup F_0 = \{1, ..., a^0b^{m_0-1}\}$.

Proof. The equivalence of (i) and (ii) follows from Example 7.1.2 and Corollary 7.2.4. The remaining statement follows from inspection of the description of S as in $3(i)$ of Proposition 7.1.1. $\square$

Corollary 7.2.7. *Let S be an upper subsemigroup of $\mathcal{B}$. If S is a left I-order in $\mathcal{B}$, then it is straight.*

7.3 Lower subsemigroups

In this section we give necessary and sufficient conditions for the lower subsemigroups of $\mathcal{B}$ to be left I-orders in $\mathcal{B}$. Throughout this section S is a lower subsemigroup of $\mathcal{B}$ having the form (3).(ii) in Proposition 7.1.1. We begin with:

Example 7.3.1. *The lower subsemigroup $T = \{a^i b^j : i \geq j, i \geq m\}$ of $\mathcal{B}$ is a straight left I-order in $\mathcal{B}$. Since for any element $q = a^k b^h$ in $\mathcal{B}$ we have*

$$q = a^k b^h = a^k b^{k+h+m} a^{k+h+m} b^h = (a^{k+h+m} b^k)^{-1} (a^{k+h+m} b^h)$$

and it is clear that $a^{k+h+m} b^k$ and $a^{k+h+m} b^h$ are in T.

Remark 7.3.2. Let S be a lower subsemigroup of $\mathcal{B}$. If $j \notin I$, then S contains no element $a^i b^j$ with $i > j$.

Lemma 7.3.3. *Let S be a lower subsemigroup of $\mathcal{B}$. If S is a left I-order in $\mathcal{B}$, then $d = 1$ and $0 \in I$.*

Proof. Since S a lower subsemigroup, it follows that for all $a^i b^j \in S$ we have that $d | j - i$ for some $d \in \mathbb{N}$. By Lemma 7.1.5, it is clear that $d = 1$. Let $a^h b^0 \in \mathcal{B}$ where $h \in \mathbb{N}$. Then

$$a^h b^0 = (a^i b^j)^{-1} (a^m b^n) = a^{j-i+t} b^{n-m+t}$$

where $t = \max\{m, i\}$, so that $0 = n - m + t$. Hence we deduce that $n = 0$ and $t = m$. We also have that $h = j - i + m$ so that $m = h + (i - j) \geq h$ so that $a^m b^0 \in S$. Hence $0 \in I$. □

Since $F_D \subseteq D \cap L_{min(I)}$, the following corollary is clear.

Corollary 7.3.4. *Let S be a lower subsemigroup of $\mathcal{B}$. If S is a left I-order in $\mathcal{B}$, then $F_D = \{1\}$ or $F_D = \emptyset$.*

Suppose that a lower subsemigroup S is a left I-order in $\mathcal{B}$. From Lemma 7.3.3, we have that $d = 1$ and $0 \in I$. We claim that $I = \mathbb{N}^0$. By Corollary 7.3.4, $F_D = \{1\}$ or $F_D = \emptyset$, so that as S intersects every $\mathcal{L}$-class of $\mathcal{B}$, by Lemma 7.1.4, $I = \mathbb{N}^0$. We have one half of the following proposition.

Proposition 7.3.5. *A lower subsemigroup S is a left I-order in $\mathcal{B}$ if and only if $d = 1$ and $I = \mathbb{N}^0$.*

Proof. Suppose that $d = 1$ and $I = \mathbb{N}^0$. Then

$$\widehat{\Lambda}_{i,m_i,1} = \{a^j b^i : j = t + i, j \geq m_i\} = \{a^{t+i} b^i : t + i \geq m_i\}.$$

For any $a^h b^k \in \mathcal{B}$ we have that $a^h b^k = (a^{h+k+t} b^h)^{-1}(a^{h+k+t} b^k)$ where $t = \max\{m_h, m_k\}$ for $i \in \mathbb{N}^0$. It is clear that $a^{h+k+t} b^h, a^{h+k+t} b^k \in S$. $\square$

The following corollary is clear from the proof of Proposition 7.3.5.

Corollary 7.3.6. *Let S be a lower subsemigroup of $\mathcal{B}$. If S is a left I-order in $\mathcal{B}$, then it is straight.*

7.4 Two-sided subsemigroups

In this section we give necessary and sufficient conditions for the two-sided subsemigroups of $\mathcal{B}$ to be left I-orders in $\mathcal{B}$. The two-sided subsemigroups of $\mathcal{B}$ have the forms (2).(*i*) and (2).(*ii*) in Proposition 7.1.1. Throughout this section we shall assume that a two-sided subsemigroup S of $\mathcal{B}$ is proper, in the sense $S \neq \mathcal{B}$.

We divide this section into two parts. We study the first form in the first part, and the second form in the second part.

We begin with the two-sided subsemigroups which have the form (2).(*i*) in Proposition 7.1.1.

Let $a^m b^n \in F \subseteq S = F_D \cup F \cup \Lambda_{I,p,d} \cup \Sigma_{p,d,P}$. Then, $d|(n-m)$. For, as $0 \in P$, $a^p b^{p+d} \in S$ and we have that $a^p b^{p+d} a^m b^n = a^p b^{n-m+p+d} \in \Sigma_{p,d,P}$, so that $d|(m-n-d)$, that is, $m - n = (t+1)d$ for some $t \in \mathbb{N}^0$. Hence for any $a^i b^j \in S$ we have that $d|i-j$. By Lemma 7.1.5, the first part of the following lemma is clear.

Lemma 7.4.1. *If a two-sided subsemigroup $S = F_D \cup F \cup \Lambda_{I,p,d} \cup \Sigma_{p,d,P}$ of $\mathcal{B}$ is a left I-order in $\mathcal{B}$, then $d = 1$ and $q = 0$. Consequently, $R_1 \subseteq S$.*

Proof. Let $a^0b^h \in \mathcal{B}$ where $h \in \mathbb{N}$. Then

$$a^0b^h = (a^ib^j)^{-1}(a^mb^n) = a^{j-i+t}b^{n-m+t}$$

where $t = \max\{m, i\}$, so that $0 = j-i+t$. Hence we deduce that $j = 0$. If $a^ib^j \in \Sigma_{p,d,P}$, then as $\Sigma_{p,d,P}$ is an inverse subsemigoup of $\mathcal{B}$ we have that $a^0b^h \in S$. In the case where $a^ib^j \notin \Sigma_{p,d,P}$ we must have that $a^ib^j \in F_D \cup F \cup \Lambda_{I,p,d}$. Hence $j \geq i$ so that $i = j = 0$. It follows that $a^0b^h = a^mb^n \in S$. Hence $q = 0$. □

Since $F_D \subseteq \{1\}$ we have that $F_D = \{1\}$ or $F_D = \emptyset$. In either case, $S \cap L_1 = \{1\}$. Then the following corollaries are clear.

Corollary 7.4.2. *If a two-sided subsemigroup $S = F_D \cup F \cup \Lambda_{I,p,d} \cup \Sigma_{p,d,P}$ of $\mathcal{B}$ is a left I-order in $\mathcal{B}$, then $F_D = \{1\}$ or $F_D = \emptyset$.*

Corollary 7.4.3. *A two-sided subsemigroup $S = F_D \cup F \cup \Lambda_{I,p,d} \cup \Sigma_{p,d,P}$ of $\mathcal{B}$ is a left I-order in $\mathcal{B}$ iff $R_1 \subseteq S$.*

Corollary 7.4.4. *If $S = F_D \cup F \cup \Lambda_{I,p,d} \cup \Sigma_{p,d,P}$ is a left I-order in $\mathcal{B}$, then it is straight.*

Now, we start studying the second form which has the form (2).(*ii*) in Proposition 7.1.1.

Let $a^mb^n \in \widehat{F} \subseteq S = F_D \cup \widehat{F} \cup \widehat{\Lambda}_{I,p,d} \cup \Sigma_{p,d,P}$. Then, $d|n-m$. For, since $a^pb^{p+d} \in \Sigma_{p,d,P}$, it follows that $a^mb^na^pb^{p+d} = a^{m-n+p}b^{p+d} \in \Sigma_{p,d,P}$. Hence $d|(m-n-d)$, that is, $m-n-d = td$ for some $t \in \mathbb{N}^0$ and so $m-n = (t+1)d$. Hence for any $a^ib^j \in S$ we have that $d|i-j$. By Lemma 7.1.5, the first part of the following lemma is clear.

Lemma 7.4.5. *If a two-sided subsemigroup $S = F_D \cup \widehat{F} \cup \widehat{\Lambda}_{I,p,d} \cup \Sigma_{p,d,P}$ of $\mathcal{B}$ is a left I-order in $\mathcal{B}$, then $d = 1$ and $q = 0$.*

Proof. Suppose that $q \neq 0$, let $a^0b^k \in \mathcal{B}$ where $k \in \mathbb{N}$. Then

$$a^0b^k = (a^ib^j)^{-1}(a^mb^n) = a^{j-i+t}b^{n-m+t}$$

where $t = \max\{m, i\}$, so that $0 = j - i + t$. Hence we can deduce that $j = 0$. If $i = 0$, then $a^0b^k = a^mb^n$ so that $a^0b^k \in S$ a contradiction and so $i > 0$. Hence $a^ib^0 \in S$, but $a^ib^0 \in \widehat{\Lambda}_{0,p,1} \cup \widehat{F}$ as $\widehat{F} \subset T_{0,p}$ a contradiction again. Therefore $q = 0$ as required. $\square$

Remark 7.4.6. In the case where $q = 0$ it is easy to see that $a^pb^0 \in S$. If $m \notin I$, then $a^ub^m \notin \widehat{F}$ for any $0 \leq u < p$. For, if $a^ub^m \in \widehat{F}$, then $a^pb^0a^ub^m = a^{p+u}b^m \in \widehat{\Lambda}_{I,p,d}$ a contradiction.

Proposition 7.4.7. *The subsemigroup $S = F_D \cup \widehat{F} \cup \widehat{\Lambda}_{I,p,d} \cup \Sigma_{p,d,P}$ of $\mathcal{B}$ is a left I-order in $\mathcal{B}$ if and only if $d = 1$ and $I = \{0, ..., p-1\}$.*

Proof. ($\Longrightarrow$) Suppose that S is a left I-order in $\mathcal{B}$. Then any element $q = a^mb^n \in \mathcal{B}$ can be written as $(a^ib^j)^{-1}(a^kb^l)$ for some $a^ib^j, a^kb^l \in S$. By Lemma 7.4.5, $d = 1$ and $0 \in I$. It is remains to show that $I = \{0, ..., p-1\}$.

Let $0 < m < p$. Then

$$\begin{aligned} a^mb^m &= (a^ib^j)^{-1}(a^kb^l) \\ &= a^{j-i+t}b^{l-k+t} \end{aligned}$$

where $t = \max\{i, k\}$, for some $a^ib^j, a^kb^l \in S$. Then $m = j$ or $m = l$; so that $a^ub^m \in S$ for some u. If $m \notin I$, so $u < p$, then $a^pb^0a^ub^m = a^{p+u}b^m \in S$, in contradiction to Remark 7.4.6.

($\Longleftarrow$) Suppose that $d = 1$ and $I = \{0, ..., p-1\}$. Then for any $a^mb^n \in \mathcal{B}$ we have

$$a^mb^n = (a^{p+m+n}b^m)^{-1}(a^{p+m+n}b^n).$$

It is clear that $a^{p+m+n}b^m, a^{p+m+n}b^n \in S$. $\square$

Corollary 7.4.8. *If $S = F_D \cup \widehat{F} \cup \widehat{\Lambda}_{I,p,d} \cup \Sigma_{p,d,P}$ is a left I-order in $\mathcal{B}$, then it is straight.*

From corollaries 7.4.4 and 7.4.8, we have the main result in this section.

Corollary 7.4.9. *Let S be a two-sided subsemigroup of $\mathcal{B}$. If S is a left I-order in $\mathcal{B}$, then it is straight.*

Chapter 8

Bisimple inverse ω-semigroups of left I-quotients

Gould [17] gave a general definition of semigroup left quotients extending the special case of this notion introduced in [6]. She used the extension of such a definition to obtain necessary and sufficient conditions for a semigroup to have a bisimple inverse ω-semigroup of left quotients. As mentioned before, if a semigroup S is a left order in an inverse semigroup Q, then Q is also a semigroup of left I-quotients of S, but the converse is not true, that is, if a semigroup S is a left I-order in an inverse semigroup Q, then it may not be left order in Q. The study of semigroups which have bisimple inverse ω-semigroups of left I-quotients seems a natural next step.

In this chapter we investigate left I-orders in bisimple inverse ω-semigroups. In [19] it was shown that if $\mathcal{H}$ is a congruence on a regular semigroup Q, then every left order S in Q is straight. To prove this, Gould uses the fact that S intersects every $\mathcal{H}$-class of Q. Since $\mathcal{H}$ is congruence on any bisimple inverse ω-semigroup, any left order S in such a semigroup must be straight. In Section 8.1 we give some preliminary results which enable us to show a left I-order semigroup S in a bisimple inverse ω-semigroup Q, intersects every $\mathcal{L}$-class of Q. This is used to show that S is straight in Q.

In the next section we give the main result in this chapter, Theorem 8.2.1, which gives necessary and sufficient conditions for a semigroup to be a left I-order in a bisimple inverse ω-semigroup. This theorem extends the result in [17].

In Section 8.3 we investigate a special case when a semigroup S is a left I-order in a proper bisimple inverse ω-semigroup Q. In Section 8.4 we combine the result in Section 8.1 with Theorem 3.2.9 to determine when two bisimple inverse ω-semigroups of left I-quotients of a given semigroup S are isomorphic over S.

8.1 Preliminaries

We refer the reader to Section 2.4.1 for background on bisimple inverse ω-semigroups.

Let Q be a bisimple inverse ω-semigroup. When convenient we identify Q as $\mathbf{BR}(G,\theta)$ for some group G and endomorphism $\theta : G \longrightarrow G$. Let R_n (L_n) denote the $\mathcal{R}$-class ($\mathcal{L}$-class) of Q containing the idempotent $e_n = (n,1,n)$ $(n \in \mathbb{N}^0)$. From the above,

$$R_m = \{(m,a,n) : a \in G, n \in \mathbb{N}^0\},$$

$$L_n = \{(m,a,n) : a \in G, m \in \mathbb{N}^0\}.$$

Clearly,

$$\begin{aligned} H_{m,n} = R_m \cap L_n &= \{(m,a,n) : a \in G\} \\ &= \{q \in Q : qq^{-1} = e_m, q^{-1}q = e_n\} \end{aligned}$$

and from the multiplication in $\mathbf{BR}(G,\theta)$,

$$H_{m,n}H_{p,q} \subseteq H_{m-n+t,q-p+t},$$

where $t = \max\{n,p\}$.

Let S be any semigroup such that there is a homomorphism $\varphi : S \longrightarrow \mathcal{B}$ where $\mathcal{B}$ is the bicyclic semigroup. We define functions $l, r : S \longrightarrow \mathbb{N}^0$ by $\varphi(a) = \big(r(a), l(a)\big)$. We also put $H_{i,j} = (i,j)\varphi^{-1}$, so that S is a disjoint union of subsets of the $H_{i,j}$ for some $i,j \in \mathbb{N}^0$ and

$$H_{i,j} = \{a \in S : r(a) = i, l(a) = j\}.$$

From the above, $\mathcal{H}$ is a congruence on any bisimple inverse ω-semigroup Q. Moreover, there is a surjective homomorphism $\overline{\varphi} : Q \longrightarrow \mathcal{B}$ with $Ker\overline{\varphi} = \mathcal{H}$ where $\mathcal{B}$ is the bicyclic semigroup. As above we will index the $\mathcal{H}$-classes of Q by putting $H_{i,j} = (i,j)\overline{\varphi}^{-1}$. It is easy to see that for elements $p,q \in Q$, $p\,\mathcal{R}\,q$ if and only if $p\overline{\varphi} = (i,j)$ and $q\overline{\varphi} = (i,k)$ for some $i,j,k \in \mathbb{N}^0$. A dual result holds for the relation $\mathcal{L}$ (see [17]).

Let S be a left I-order in Q. Let $\varphi = \overline{\varphi}|_S$. Then, φ is a homomorphism from S to $\mathcal{B}$. Unfortunately, this homomorphism is not surjective in general, since S need not intersect every $\mathcal{H}$-class of Q. But we can as above index the elements of S.

In Chapter 7 we showed that, if a semigroup S is a left I-order in a bicyclic semigroup $\mathcal{B}$, then S intersects every $\mathcal{L}$-class of $\mathcal{B}$. Moreover, it is straight. In fact, this is true for any left I-order in a bisimple inverse ω-semigroup, as we will see in the next lemmas.

Lemma 8.1.1. *If a semigroup S is a left I-order in a bisimple inverse ω-semigroup Q, then $S \cap L_n \neq \emptyset$ for all $n \in \mathbb{N}^0$.*

Proof. Let $p \in H_{n,n}$. Then, $p = a^{-1}b$ for some $a, b \in S$ with $a \in H_{i,j}$ and $b \in H_{k,l}$. Hence

$$p = a^{-1}b \in H_{j,i}H_{k,l} \subseteq H_{j-i+\max(i,k),l-k+\max(i,k)},$$

and so $n = j - i + \max(i,k) = l - k + \max(i,k)$. As $\max(i,k) = i$ or k, so that either $n = j$ or $n = l$. Hence $S \cap L_n \neq \emptyset$. □

In [17] it was shown that if S a left order in a bisimple inverse ω-semigroup Q, then it is straight. The following lemma due to Gould extends this to the left I-order in bisimple inverse ω-semigroup.

Lemma 8.1.2. *If a semigroup S is a left I-order in a bisimple inverse ω-semigroup Q, then S is straight.*

Proof. Let $(h,q,k) \in Q$. Then,

$$(h,q,k) = (i,a,j)^{-1}(t,b,s) = (j,a^{-1},i)(t,b,s)$$

for some $(i,a,j), (t,b,s) \in S$. Let $n = \max\{i,t\}$; since $S \cap L_n \neq \emptyset$, by Lemma 8.1.1, there exists $(u,c,n) \in S \cap L_n$ and hence $(u,c,n)^{-1}(u,c,n) = (n,1,n)$, so that $(n,1,n)\,\mathcal{R}\,(t,b,s)$ or $(n,1,n)\,\mathcal{R}\,(i,a,j)$. In both cases, we have

$$\begin{aligned}(h,q,k) &= (i,a,j)^{-1}(n,1,n)(t,b,s)\\ &= (i,a,j)^{-1}(u,c,n)^{-1}(u,c,n)(t,b,s)\\ &= \big((u,c,n)(i,a,j)\big)^{-1}\big((u,c,n)(t,b,s)\big).\end{aligned}$$

It is clear that $(u,c,n)(i,a,j)\,\mathcal{R}\,(u,c,n)(t,b,s)$. Hence S is straight. □

The following corollary which recaps earlier facts we have been using, is clear.

Corollary 8.1.3. *Let Q be a bisimple inverse ω-semigroup. Then*

(i) $(m,a,n)^{-1}(m,b,t)\,\mathcal{R}\,(i,c,j)^{-1}(i,d,k)$ *if and only if* $n=j$;

(ii) $(m,a,n)^{-1}(m,b,t)\,\mathcal{L}\,(i,c,j)^{-1}(i,d,k)$ *if and only if* $t=k$.

8.2 The main theorem

This section is entirely devoted to proving Theorem 8.2.1 which gives a characterisation of semigroups which have a bisimple inverse ω-semigroup of left I-quotients.

Theorem 8.2.1. *A semigroup S is a left I-order in a bisimple inverse ω-semigroup Q if and only if S satisfies the following conditions:*

(A) There is a homomorphism $\varphi : S \longrightarrow \mathcal{B}$ such that $S\varphi$ is a left I-order in $\mathcal{B}$;

(B) For $x, y, a \in S$,

$$(i)\ l(x), l(y) \geqslant r(a) \text{ and } xa = ya \text{ implies } x = y,$$

$$(ii)\ r(x), r(y) \geqslant l(a) \text{ and } ax = ay \text{ implies } x = y.$$

(C) For any $b, c \in S$, there exist $x, y \in S$ such that $xb = yc$ where

$$x \in H_{r(x), r(b)-l(b)+\max\big(l(b), l(c)\big)},\quad y \in H_{r(x), r(c)-l(c)+\max\big(l(b), l(c)\big)}.$$

Proof. Let S be a left I-order in a bisimple inverse ω-semigroup Q. For Condition (A), since S is a left I-order in Q and there is a homomorphism $\overline{\varphi} : Q \to \mathcal{B}$, it follows that we can restrict $\overline{\varphi}$ on S to get a homomorphism φ from S to $\mathcal{B}$. Let $(i,j) \in \mathcal{B}$. Then, there is an element q in Q such that $q \in H_{i,j}$ for some $i, j \in \mathbb{N}^0$. Put $q = a^{-1}b$ for some $a, b \in S$ with $a\,\mathcal{R}\,b$ in Q, so that $r(a) = r(b)$. Hence

$$q \in H_{l(a),r(a)}H_{r(a),l(b)} \subseteq H_{l(a),l(b)},$$

then

$$\begin{aligned}(i,j) &= \big(l(a), l(b)\big)\\ &= \big(r(a), l(a)\big)^{-1}\big(r(b), l(b)\big)\\ &= (a\varphi)^{-1}(b\phi).\end{aligned}$$

Hence $S\varphi$ is a left I-order in $\mathcal{B}$.

To see that $(B)(i)$ holds, suppose that $x, y, a \in S$ where $l(x), l(y) \geqslant r(a)$ and $xa = ya$. Since $a^{-1} \in H_{l(a),r(a)}$ and $xaa^{-1} = yaa^{-1}$ we have that $xe_{r(a)} = ye_{r(a)}$ and since $r(a) \leqslant l(x), l(y)$, it follows that $e_{l(x)}, e_{l(y)} \leqslant e_{r(a)}$. Then

$$xe_{l(x)}e_{r(a)} = xx^{-1}xe_{r(a)} = xe_{r(a)} = ye_{r(a)} = yy^{-1}ye_{r(a)} = ye_{l(y)}e_{r(a)}$$

and so $x = xe_{l(x)} = ye_{l(y)} = y$. The proof of $(B)(ii)$ is similar.

Finally, we consider (C). Let $b, c \in S$. Then, $bc^{-1} \in Q$ and

$$bc^{-1} \in H_{r(b),l(b)}H_{l(c),r(c)} \subseteq H_{r(b)-l(b)+\max\left(l(b),l(c)\right),r(c)-l(c)+\max\left(l(b),l(c)\right)}.$$

Since S is a straight left I-order in Q, it follows that $bc^{-1} = x^{-1}y$ where $x \mathcal{R} y$ for some $x, y \in S$, and so $xb = yc$, by Lemma 3.2.2. From $bc^{-1} = x^{-1}y$ we have

$$H_{r(b)-l(b)+\max\left(l(b),l(c)\right),r(c)-l(c)+\max\left(l(b),l(c)\right)} = H_{l(x),l(y)},$$

so that $l(x) = r(b) - l(b) + \max\left(l(b), l(c)\right)$ and $l(y) = r(c) - l(c) + \max\left(l(b), l(c)\right)$.

Conversely, we suppose that S satisfies conditions $(A), (B)$ and (C). Now, our aim is to construct via equivalence classes of ordered pairs of elements of S a bisimple inverse ω-semigroup Q, which is a semigroup of straight left I-quotients of S. First, we let

$$\Sigma = \{(a, b) \in S \times S : r(a) = r(b)\}$$

and on Σ we define the relation $\sim$ as follows:

$$(a, b) \sim (c, d) \Leftrightarrow \text{ there are elements } x, y \text{ in } S \text{ such that } xa = yc \text{ and } xb = yd$$

where $l(x) = r(a)$, $l(y) = r(c)$ and $r(x) = r(y)$.

Notice that if $(a, b) \sim (c, d)$, then $l(a) = l(c)$ and $l(b) = l(d)$. For, $l(a) = l(xa) = l(yc) = l(c)$ and $l(b) = l(xb) = l(yd) = l(d)$.

Lemma 8.2.2. *The relation $\sim$ is an equivalence.*

Proof. It is clear that $\sim$ is symmetric. Let $(a, b) \in \Sigma$. By (C), for any $a \in S$ there exists $x \in S$ with $l(x) = r(a)$, so that $xa = xa$ and $xb = xb$ and $l(x) = r(a) = r(b)$, and hence $\sim$ is reflexive.

Suppose that $(a,b) \sim (c,d) \sim (p,q)$. Then there are elements $x, y, \bar{x}, \bar{y}$ in S such that

$$xa = yc, \quad xb = yd,$$
$$\bar{x}c = \bar{y}p, \quad \bar{x}d = \bar{y}q,$$

where

$$r(x) = r(y),\ l(x) = r(a),\ l(y) = r(c),$$

and

$$r(\bar{x}) = r(\bar{y}),\ l(\bar{x}) = r(c),\ l(\bar{y}) = r(p).$$

By Condition (C), for $y, \bar{x}$ there exist $s, t \in S$ such that $s\bar{x} = ty$ where

$$s \in H_{r(s), r(\bar{x}) - l(\bar{x}) + \max\left(l(\bar{x}), l(y)\right)},\ t \in H_{r(s), r(y) - l(y) + \max\left(l(\bar{x}), l(y)\right)}.$$

Since $l(\bar{x}) = r(c) = l(y)$, it follows that $l(s) = r(\bar{x})$ and $l(t) = r(y) = r(x)$. Now,

$$txa = tyc = s\bar{x}c = s\bar{y}p,$$

and

$$txb = tyd = s\bar{x}d = s\bar{y}q.$$

Hence $txa = s\bar{y}p$ and $txb = s\bar{y}q$ where $tx \in H_{r(s), r(a)},\ s\bar{y} \in H_{r(s), r(p)}$. We have

$$l(tx) = r(a),\ l(s\bar{y}) = r(p) \text{ and } r(tx) = r(s\bar{y}),$$

that is, $(a,b) \sim (p,q)$. Thus $\sim$ is transitive. $\square$

We write the $\sim$-equivalence class of (a,b) as $[a,b]$ and denote by Q the set of all $\sim$-equivalence classes of Σ. If $[a,b], [c,d] \in Q$, then by (C), for b and c there exist x, y such that $xb = yc$ where

$$x \in H_{r(x), r(b) - l(b) + \max\left(l(b), l(c)\right)},\ y \in H_{r(x), r(c) - l(c) + \max\left(l(b), l(c)\right)}$$

and it is easy to see that

$$r(xa) = r(xb) = r(yc) = r(yd) = r(x) = r(y)$$

and we deduce that $[xa, yd] \in Q$. Define a multiplication on Q by

$$[a,b][c,d] = [xa, yd]$$

where $xb = yc$ and $x \in H_{r(x), r(b) - l(b) + \max\left(l(b), l(c)\right)},\ y \in H_{r(x), r(c) - l(c) + \max\left(l(b), l(c)\right)}$.

Lemma 8.2.3. *The given multiplication is well defined.*

Proof. Suppose that $[a_1, b_1] = [a_2, b_2]$ and $[c_1, d_1] = [c_2, d_2]$. Then there are elements x_1, x_2, y_1, y_2 in S such that

$$\begin{aligned} x_1 a_1 &= x_2 a_2, \\ x_1 b_1 &= x_2 b_2, \\ y_1 c_1 &= y_2 c_2, \\ y_1 d_1 &= y_2 d_2, \end{aligned}$$

where

$$l(x_1) = r(a_1),\ l(x_2) = r(a_2),\ r(x_1) = r(x_2)$$

and

$$l(y_1) = r(c_1),\ l(y_2) = r(c_2),\ r(y_1) = r(y_2).$$

Note that, consequently,

$$l(a_1) = l(a_2),\ l(b_1) = l(b_2),\ l(c_1) = l(c_2) \text{ and } l(d_1) = l(d_2).$$

By definition,

$$[a_1, b_1][c_1, d_1] = [xa_1, yd_1]$$

where $xb_1 = yc_1$ and $x \in H_{r(x), r(b_1) - l(b_1) + \max\left(l(b_1), l(c_1)\right)}$, $y \in H_{r(x), r(c_1) - l(c_1) + \max\left(l(b_1), l(c_1)\right)}$. Also,

$$[a_2, b_2][c_2, d_2] = [\bar{x}a_2, \bar{y}d_2]$$

where $\bar{x}b_2 = \bar{y}c_2$ and $\bar{x} \in H_{r(\bar{x}), r(b_2) - l(b_2) + \max\left(l(b_2), l(c_2)\right)}$, $\bar{y} \in H_{r(\bar{x}), r(c_2) - l(c_2) + \max\left(l(b_2), l(c_2)\right)}$.

We must show that $[xa_1, yd_1] = [\bar{x}a_2, \bar{y}d_2]$. That is, we need to show that there are $w, \bar{w} \in S$ such that $wxa_1 = \bar{w}\bar{x}a_2$ and $wyd_1 = \bar{w}\bar{y}d_2$ with

$$r(w) = r(\bar{w}),\ l(w) = r(xa_1) \text{ and } l(\bar{w}) = r(\bar{x}a_2).$$

Before completing the proof of Lemma 8.2.3, we present the following lemma.

Lemma 8.2.4. *Let $a_1, a_2, b_1, b_2 \in S$ be such that*

$$r(a_1) = r(b_1),\ r(a_2) = r(b_2)$$

and suppose that $x_1, x_2, w_1, w_2 \in S$ are such that

$$x_1 a_1 = x_2 a_2,\ x_1 b_1 = x_2 b_2,\ w_1 a_1 = w_2 a_2$$

where $r(x_1) = r(x_2)$, $l(x_1) = r(a_1)$, $l(x_2) = r(a_2)$ *and* $r(w_1) = r(w_2)$. *Then* $w_1 b_1 = w_2 b_2$.

Proof. Let $a_1, a_2, b_1, b_2, x_1, x_2, w_1, w_2$ exist as given. Note that consequently $l(a_1) = l(a_2)$ and $l(b_1) = l(b_2)$. By (C), for w_1, x_1 there exist $x, y \in S$ such that $xw_1 = yx_1$ where

$$x \in H_{r(x),r(w_1)-l(w_1)+\max\left(l(w_1),l(x_1)\right)}, \; y \in H_{r(x),r(x_1)-l(x_1)+\max\left(l(w_1),l(x_1)\right)}.$$

Then $xw_1a_1 = yx_1a_1$, and

$$xw_2a_2 = xw_1a_1 = yx_1a_1 = yx_2a_2.$$

Now,

$$xw_2 \in H_{r(x),l(w_2)-l(w_1)+\max\left(l(w_1),l(x_1)\right)}, \; yx_2 \in H_{r(x),l(x_2)-l(x_1)+\max\left(l(w_1),l(x_1)\right)}$$

and as $l(x_1) = r(a_1)$ and $l(x_2) = r(a_2)$, we have

$$l(yx_2) = r(a_2) - r(a_1) + \max\left(l(w_1), r(a_1)\right) \geqslant r(a_2)$$

and

$$l(xw_2) = l(w_2) - l(w_1) + \max\left(l(w_1), r(a_1)\right).$$

Since $xw_2a_2 = yx_2a_2$ and $l(yx_2) \geqslant r(a_2)$. Then, in order to use Condition (B)(i), we have to show that $l(xw_2) \geqslant r(a_2)$. Since $w_1a_1 = w_2a_2$ and $r(w_1) = r(w_2)$ we obtain that

$$r(w_1) - l(w_1) + \max\left(l(w_1), r(a_1)\right) = r(w_1) - l(w_2) + \max\left(l(w_2), r(a_2)\right) \tag{8.1}$$

so that

$$l(xw_2) = l(w_2) - l(w_1) + \max\left(l(w_1), r(a_1)\right) = \max\left(l(w_2), r(a_2)\right) \geqslant r(a_2)$$

as desired. Then $xw_2 = yx_2$, by (B)(i). Since $xw_1 = yx_1$ and $x_1b_1 = x_2b_2$ we have

$$xw_1b_1 = yx_1b_1 = yx_2b_2 = xw_2b_2.$$

Once we know $r(w_1b_1), r(w_2b_2) \geqslant l(x)$ we have $w_1b_1 = w_2b_2$, by (B)(ii). Now,

$$w_1b_1 \in H_{r(w_1)-l(w_1)+\max\left(l(w_1),r(b_1)\right),l(b_1)-r(b_1)+\max\left(l(w_1),r(b_1)\right)}$$

and

$$w_2b_2 \in H_{r(w_1)-l(w_2)+\max\big(l(w_2),r(b_2)\big),l(b_1)-r(b_2)+\max\big(l(w_2),r(b_2)\big)},$$

so that

$$\begin{aligned} r(w_1b_1) &= r(w_1)-l(w_1)+\max\big(l(w_1),r(a_1)\big) \quad \text{as } l(x_1)=r(a_1)=r(b_1) \\ &= r(w_1)-l(w_1)+\max\big(l(w_1),l(x_1)\big) \\ &= l(x) \end{aligned}$$

and

$$\begin{aligned} r(w_2b_2) &= r(w_1)-l(w_2)+\max\big(l(w_2),r(a_2)\big) \quad \text{as } r(b_2)=r(a_2) \\ &= r(w_1)-l(w_1)+\max\big(l(w_1),r(a_1)\big) \quad \text{by (8.1)} \\ &= r(w_1)-l(w_1)+\max(l(w_1),l(x_1)) \quad \text{as } l(x_1)=r(a_1) \\ &= l(x). \end{aligned}$$

Hence the proof of our Lemma is complete. □

Returning to the proof of Lemma 8.2.3, by (C), for xa_1 and $\bar{x}a_2$ there exist $w, \bar{w} \in S$ such that $wxa_1 = \bar{w}\bar{x}a_2$ where

$$w \in H_{r(w),r(xa_1)-l(xa_1)+\max\big(l(xa_1),l(\bar{x}a_2)\big)} \text{ and } \bar{w} \in H_{r(w),r(\bar{x}a_2)-l(\bar{x}a_2)+\max\big(l(xa_1),l(\bar{x}a_2)\big)}.$$

Using the fact that $l(b_1)=l(b_2)$, $l(c_1)=l(c_2)$ and $l(a_1)=l(a_2)$, it is easy to see that $l(xa_1)=l(\bar{x}a_2)$. Therefore

$$l(w)=r(xa_1)=r(x) \text{ and } l(\bar{w})=r(\bar{x}a_2)=r(\bar{x}).$$

Since $r(w)=r(\bar{w})$ and $wxa_1=\bar{w}\bar{x}a_2$, it remains to show that $wyd_1=\bar{w}\bar{y}d_2$.
It is easy to see that $r(wx)=r(w)=r(\bar{w})=r(\bar{w}\bar{x})$. We know that

$$r(x_1)=r(x_2),\ l(x_1)=r(a_1),\ l(x_2)=r(a_2),\ r(a_1)=r(b_1) \text{ and } r(a_2)=r(b_2).$$

Since $x_1a_1=x_2a_2$, $x_1b_1=x_2b_2$ and $wxa_1=\bar{w}\bar{x}a_2$, we have that $wxb_1=\bar{w}\bar{x}b_2$, by Lemma 8.2.4.
We also have $xb_1=yc_1$ and $\bar{x}b_2=\bar{y}c_2$, so that $wyc_1=wxb_1=\bar{w}\bar{x}b_2=\bar{w}\bar{y}c_2$. Thus

$$y_1c_1=y_2c_2,\ y_1d_1=y_2d_2 \text{ and } wyc_1=\bar{w}\bar{y}c_2.$$

Since $r(y_1)=r(y_2)$, $l(y_1)=r(c_1)$, $l(y_2)=r(c_2)$, $r(c_1)=r(d_1)$, $r(c_2)=r(d_2)$ and $r(wy)=r(\bar{w}\bar{y})$, it follows that $wyd_1=\bar{w}\bar{y}d_2$, by Lemma 8.2.4 again.
Hence $[xa_1,yd_1]=[\bar{x}a_2,\bar{y}d_2]$. This completes the proof of Lemma 8.2.3. □

The next lemma is useful in verifying that the given multiplication is associative. The proof follows immediately from the fact that $l(ab) \geqslant l(b), l(de) \geqslant l(e)$, and (B)($i$).

Lemma 8.2.5. *Let $a, b, c, d, e \in S$. If $abc = dec$ and $l(b) \geqslant r(c)$, $l(e) \geqslant r(c)$, then $ab = de$.*

Lemma 8.2.6. *The given multiplication is associative.*

Proof. Let $[a,b],[c,d],[p,q] \in Q$. Then by using the definition of multiplication in Q we have

$$\big([a,b][c,d]\big)[p,q] = [xa, yd][p,q]$$

where $xb = yc$ and $x \in H_{r(x),r(b)-l(b)+\max\left(l(b),l(c)\right)}$, $y \in H_{r(x),r(c)-l(c)+\max\left(l(b),l(c)\right)}$ for some $x, y \in S$ and then

$$\big([a,b][c,d]\big)[p,q] = [wxa, \bar{w}q]$$

where $wyd = \bar{w}p$ and $w \in H_{r(w),r(yd)-l(yd)+\max\left(l(yd),l(p)\right)}$, $\bar{w} \in H_{r(w),r(p)-l(p)+\max\left(l(yd),l(p)\right)}$ for some $w, \bar{w} \in S$. Similarly,

$$[a,b]\big([c,d][p,q]\big) = [a,b][\bar{x}c, \bar{y}q]$$

where $\bar{x}d = \bar{y}p$ and $\bar{x} \in H_{r(\bar{x}),r(d)-l(d)+\max\left(l(d),l(p)\right)}$, $\bar{y} \in H_{r(\bar{x}),r(p)-l(p)+\max\left(l(d),l(p)\right)}$, and then

$$[a,b]\big([c,d][p,q]\big) = [za, \bar{z}\bar{y}q]$$

where $zb = \bar{z}\bar{x}c$ and $z \in H_{r(z),r(b)-l(b)+\max\left(l(b),l(\bar{x}c)\right)}$, $\bar{z} \in H_{r(z),r(\bar{x}c)-l(\bar{x}c)+\max\left(l(b),l(\bar{x}c)\right)}$.

To complete our proof we have to show that $[wxa, \bar{w}q] = [za, \bar{z}\bar{y}q]$. That is, we need to show that there are $t, h \in S$ such that $twxa = hza$ and $t\bar{w}q = h\bar{z}\bar{y}q$ with

$$r(t) = r(h),\ l(t) = r(wxa) \text{ and } l(h) = r(za).$$

By Condition (C), for wx, z there exist $h, t \in S$ such that $twx = hz$ where

$$t \in H_{r(t),r(wx)-l(wx)+\max\left(l(wx),l(z)\right)},\ h \in H_{r(t),r(z)-l(z)+\max\left(l(wx),l(z)\right)},$$

and so $twxa = hza$ and $twxb = hzb$. Since $xb = yc$ and $zb = \bar{z}\bar{x}c$ we have that $twyc = h\bar{z}\bar{x}c$. But

$$l(y) = r(c) - l(c) + \max\big(l(b), l(c)\big) \geqslant r(c)$$

and

$$l(\bar{x}) = r(d) - l(d) + \max\big(l(d), l(p)\big) = r(c) - l(d) + \max\big(l(d), l(p)\big) \geqslant r(c).$$

By Lemma 8.2.5, $twy = h\bar{z}\bar{x}$ and so $twyd = h\bar{z}\bar{x}d$. Now, $wyd = \bar{w}p$ and $\bar{x}d = \bar{y}p$, so that $t\bar{w}p = h\bar{z}\bar{y}p$. But

$$l(\bar{w}) = r(p) - l(p) + \max\big(l(yd), l(p)\big) \geqslant r(p)$$

and

$$l(\bar{y}) = r(p) - l(p) + \max\big(l(d), l(p)\big) \geqslant r(p),$$

and therefore $t\bar{w} = h\bar{z}\bar{y}$, again by Lemma 8.2.5. Hence $t\bar{w}q = h\bar{z}\bar{y}q$. It remains to prove that

$$l(t) = r(wxa) \text{ and } l(h) = r(za).$$

Since

$$l(t) = r(wx) - l(wx) + \max\big(l(wx), l(z)\big)$$

and

$$l(h) = r(z) - l(z) + \max\big(l(wx), l(z)\big).$$

Calculating, we have

$$r(wx) = r(w) \tag{8.2}$$

$$l(wx) = l(x) - l(yd) + \max\big(l(p), l(yd)\big) \tag{8.3}$$

$$l(z) = r(b) - l(b) + \max\big(l(b), l(\bar{x}c)\big) \tag{8.4}$$

and

$$l(\bar{x}c) = l(c) - l(d) + \max\big(l(d), l(p)\big) \tag{8.5}$$

$$l(yd) = l(d) - l(c) + \max\big(l(b), l(c)\big). \tag{8.6}$$

Since $r(wx) = r(wxa)$ and $r(z) = r(za)$, once we show that $l(z) = l(wx)$, we will have

$$l(t) = r(wx) = r(wxa) \text{ and } l(h) = r(z) = r(za).$$

It is convenient to consider separately two cases.

Case(i): $l(c) \geqslant l(b)$. We have

$$l(yd) = l(d) \text{ and } l(x) = r(b) - l(b) + l(c).$$

If $l(d) \geqslant l(p)$, then from (8.5), $l(\bar{x}c) = l(c)$. From (8.3) and (8.4),

$$l(wx) = l(x) = r(b) - l(b) + l(c) = l(z).$$

If, on the other hand, $l(d) \leqslant l(p)$, then $l(\bar{x}c) = l(c) - l(d) + l(p)$. From (8.3) and (8.4),

$$l(wx) = l(x) - l(d) + l(p)$$

and

$$l(z) = r(b) - l(b) + \max\big(l(b), l(c) - l(d) + l(p)\big).$$

Since $l(c) \geqslant l(b)$ and $l(d) \leqslant l(p)$, it follows that $l(c) - l(d) + l(p) \geqslant l(b)$. Thus

$$\begin{aligned} l(z) &= r(b) - l(b) + l(c) - l(d) + l(p) \\ &= l(x) - l(d) + l(p) \\ &= l(wx). \end{aligned}$$

Case(ii): $l(c) \leqslant l(b)$. We have

$$l(yd) = l(d) - l(c) + l(b) \text{ and } l(x) = r(b).$$

If $l(d) \geqslant l(p)$, then $l(\bar{x}c) = l(c)$. From (8.3) and (8.4),

$$l(wx) = l(x) - l(d) + l(c) - l(b) + \max\big(l(p), l(d) - l(c) + l(b)\big)$$

and

$$l(z) = r(b) - l(b) + \max\big(l(b), l(c)\big) = r(b) = l(x).$$

Since $l(d) \geqslant l(p)$ and $l(c) \leqslant l(b)$ we have that $l(d) - l(c) + l(b) \geqslant l(d) \geqslant l(p)$. Then $l(wx) = l(x)$. Hence $l(z) = l(x) = l(wx)$.

If, on the other hand, $l(d) \leqslant l(p)$, then from (8.5), $l(\bar{x}c) = l(c) - l(d) + l(p)$. From (8.3) and (8.4),

$$l(wx) = l(x) - l(d) + l(c) - l(b) + \max\big(l(p), l(d) - l(c) + l(b)\big)$$

and

$$l(z) = r(b) - l(b) + \max\big(l(b), l(c) - l(d) + l(p)\big).$$

Once again, there are two cases. If $l(c) - l(d) + l(p) \geqslant l(b)$, then

$$l(p) \geqslant l(d) - l(c) + l(b)$$

and so

$$\begin{aligned} l(wx) &= l(x) - l(d) + l(c) - l(b) + l(p) \\ &= r(b) - l(d) + l(c) - l(b) + l(p) \\ &= l(z). \end{aligned}$$

If, on the other hand, $l(c) - l(d) + l(p) \leqslant l(b)$, then $l(p) \leqslant l(d) - l(c) + l(b)$. Hence

$$l(wx) = l(x) = r(b) = l(z).$$

This completes the proof of the lemma. □

Now we aim to show that Q, which we have constructed, is a semigroup of left I-quotients of S. First we show that S is embedded in Q.

As seen earlier, for any $a \in S$ there exists $x \in S$ with $l(x) = r(a)$. Then $xa \in H_{r(x),l(a)}$ and $[x, xa] \in Q$. If $y \in S$ with $l(y) = r(a)$, then $ya \in H_{r(y),l(a)}$ and again $[y, ya] \in Q$. By (C), there exist $s, t \in S$ with $sx = ty$ (and so $sxa = tya$), where $s \in H_{r(s),r(x)}$, $t \in H_{r(s),r(y)}$. Hence $[x, xa] = [y, ya]$. There is therefore a well-defined mapping $\theta : S \longrightarrow Q$ defined by $a\theta = [x, xa]$ where $x \in H_{r(x),r(a)}$.

Lemma 8.2.7. *The mapping θ is an embedding of S into Q.*

Proof. Suppose that $a\theta = b\theta$. Then $[x, xa] = [y, yb]$ where $x \in H_{r(x),r(a)}$ and $y \in H_{r(y),r(b)}$. By definition of $\sim$ there are elements $s, t \in S$ such that $sx = ty$ and $sxa = tyb$ where $l(s) = r(x)$, $l(t) = r(y)$ and $r(s) = r(t)$. We claim that $a = b$.

Since $sxa = tyb = sxb$, once we show that $r(a), r(b) \geqslant l(sx)$ we can use (B)(*ii*) to get $a = b$. Now, it is easy to see that

$$sx \in H_{r(s),r(a)} \text{ and } ty \in H_{r(s),r(b)}$$

and so $l(sx) = r(a)$ and $l(ty) = r(b)$. But $sx = ty$, so that $r(a) = r(b) = l(sx)$. Hence $a = b$ and so θ is one-to-one, our claim is established.

To show that θ is a homomorphism, let $a\theta = [x, xa]$ and $b\theta = [y, yb]$ where $x \in H_{r(x),r(a)}$ and $y \in H_{r(y),r(b)}$. Then

$$a\theta b\theta = [x, xa][y, yb] = [wx, \bar{w}yb]$$

where $wxa = \bar{w}y$ and $w \in H_{r(w),r(xa)-l(xa)+\max\left(l(xa),l(y)\right)}$, $\bar{w} \in H_{r(w),r(y)-l(y)+\max\left(l(xa),l(y)\right)}$. Hence

$$a\theta b\theta = [wx, wxab].$$

Notice that

$$r(xa) = r(x),\ l(xa) = l(a) \text{ and } l(y) = r(b)$$

so that $w \in H_{r(w),r(x)-l(a)+\max\left(l(a),r(b)\right)}$. Then

$$wx \in H_{r(w),r(a)-l(a)+\max\left(l(a),r(b)\right)} = H_{r(w),r(ab)}.$$

It follows that $(ab)\theta = [wx, wxab] = a\theta b\theta$. □

The main purpose of the following is to show that Q is a bisimple inverse ω-semigroup and S is a left I-order in Q. First we need the following simple but useful lemma.

Lemma 8.2.8. *Let $[a, b] \in Q$. Then $[a, b] = [xa, xb]$ for any $x \in S$ with $l(x) = r(a)$.*

Proof. It is clear that $r(xa) = r(x) = r(xb)$, so that $[xa, xb] \in Q$. By (C), for a and xa there exist $t, z \in S$ such that $ta = zxa$ where

$$t \in H_{r(t),r(a)-l(a)+\max\left(l(a),l(xa)\right)},\ z \in H_{r(t),r(xa)-l(xa)+\max\left(l(a),l(xa)\right)}.$$

Since $l(xa) = l(a)$ and $r(xa) = r(x)$ we have that $l(t) = r(a)$ and $l(z) = r(xa) = r(x)$. Also, $l(zx) = r(a)$. Hence by (B)(ii), $t = zx$ and so $tb = zxb$. Thus

$$[a, b] = [xa, xb]$$

as required. □

Lemma 8.2.9. *Let $[a,b],[b,c] \in Q$. Then $[a,b][b,c] = [a,c]$.*

Proof. We have that $r(a) = r(b) = r(c)$. By (C), for b there exists $x \in S$ such that $x \in H_{r(x),r(b)}$. By definition

$$[a,b][b,c] = [xa, xc].$$

Hence Lemma 8.2.8, $[xa, xc] = [a,c]$, as required. □

Let $[a,b] \in Q$. Then $[b,a] \in Q$ and by Lemma 8.2.9,

$$[a,b][b,a][a,b] = [a,b].$$

Then we have

Lemma 8.2.10. *The semigroup Q is regular.*

Let $[a,a] \in Q$. By Lemma 8.2.9, $[a,a][a,a] = [a,a]$, that is, $[a,a]$ is an idempotent in Q. Hence $\{[a,a]; a \in S\} \subseteq E(Q)$. We have one half of the following Lemma.

Lemma 8.2.11. *The set of idempotents of Q is given by $E(Q) = \{[a,a] : a \in S\}$.*

Proof. Let $[a,b] \in E(Q)$. Then,

$$[a,b][a,b] = [a,b]$$

and so $[xa, yb] = [a,b]$ where $xb = ya$ and

$$x \in H_{r(x),r(b)-l(b)+\max\left(l(b),l(a)\right)},\ y \in H_{r(x),r(a)-l(a)+\max\left(l(b),l(a)\right)}$$

so that

$$xa \in H_{r(x),l(a)-l(b)+\max\left(l(b),l(a)\right)},\ yb \in H_{r(x),l(b)-l(a)+\max\left(l(b),l(a)\right)}.$$

Since $[xa, yb] = [a,b]$, it follows that there exist $t, z \in S$ such that $txa = za$ and $tyb = zb$ where $r(t) = r(z)$, $l(t) = r(xa)$ and $l(z) = r(a)$. Moreover, $l(xa) = l(a)$ and $l(yb) = l(b)$. Since $l(t) = r(xa) = r(x) = r(y)$ we have

$$tx \in H_{r(t),r(b)-l(b)+\max\left(l(a),l(b)\right)},\ ty \in H_{r(t),r(a)-l(a)+\max\left(l(b),l(a)\right)}.$$

Hence

$$l(tx) \geqslant r(b) = r(a),\ l(ty) \geqslant r(a) = r(b) \text{ and } l(z) = r(a) = r(b)$$

and so from $tyb = zb$ and $txa = za$ we have that $tx = z = ty$, using (B)(i). As $r(x) = r(y) = l(t)$ and $tx = ty$ we have that $x = y$, by (B)(ii). Then

$$l(b) = l(yb) = l(xb) = l(ya) = l(xa) = l(a)$$

gives $l(x) = r(a) = r(b)$. Since $xb = ya = xa$, it follows that $a = b$, by (B)(ii). Hence $E(Q) \subseteq \{[a, a]; a \in S\}$, and the lemma follows. □

Lemma 8.2.12. *The set $E(Q)$ is ω-chain.*

Proof. Let $[a, a], [b, b] \in E(Q)$. Then,

$$[a, a][b, b] = [xa, yb]$$

where $xa = yb$ and $x \in H_{r(x),r(a)-l(a)+\max(l(a),l(b))}$, $y \in H_{r(x),r(b)-l(b)+\max(l(a),l(b))}$. Hence

$$[a, a][b, b] = [xa, xa] = [yb, yb].$$

If $l(a) \geqslant l(b)$, then $x \in H_{r(x),r(a)}$ and so $xa \in H_{r(x),l(a)}$. By Lemma 8.2.8,

$$[xa, xa] = [a, a].$$

If $l(b) \geqslant l(a)$, then $y \in H_{r(x),r(b)}$ and $yb \in H_{r(x),l(b)}$ so that $[yb, yb] = [b, b]$, by Lemma 8.2.8. □

Notice also from Lemma 8.2.12 that if $l(a) = l(b)$, then $[a, a][b, b] = [a, a] = [b, b]$.

By Lemma 8.2.12, the idempotents of Q form an ω-chain and hence commute, by Lemma 8.2.10, the following Lemma is clear.

Lemma 8.2.13. *The semigroup Q is inverse.*

Lemma 8.2.14. *The semigroup Q is a bisimple inverse semigroup.*

Proof. To show that Q is a bisimple inverse semigroup, we need to prove that, for any two idempotents $[a, a], [b, b]$ in $E(Q)$, there is q in Q such that $qq^{-1} = [a, a]$ and $q^{-1}q = [b, b]$, by Lemma 2.4.1 ([1, Lemma 8.34]).

By (A), $S\varphi$ is a left I-order in $\mathcal{B}$. By Lemma 8.1.2, $S\varphi$ is straight, so that for $(l(a), l(b))$ there exist c, d in S such that

$$(l(a), l(b)) = c\varphi^{-1} d\varphi$$

where $c\varphi \mathcal{R} d\varphi$ in $\mathcal{B}$ so that $c\varphi = (u, l(a))$ and $d\varphi = (u, l(b))$ for some $u \in \mathbb{N}^0$. Hence $q = [c, d] \in Q$. By Lemma 8.2.9,

$$qq^{-1} = [c, d][d, c] = [c, c]$$

and, similarly, $q^{-1}q = [d, d]$. By the argument following Lemma 8.2.12, $[c, c] = [a, a]$ and $[d, d] = [b, b]$, as required. $\square$

The following lemma throws full light on the relationship between S and Q.

Lemma 8.2.15. *Every element of Q can be written as $a^{-1}b$, where a, b are elements of S, $r(a) = r(b)$.*

Proof. Suppose that $q = [a, b] \in Q$ for some $a, b \in S$ with $r(a) = r(b)$. In view of Lemma 8.2.7, we can identify a and b with $[x, xa]$ and $[y, yb]$ respectively, for some $x \in H_{r(x),r(a)}$ and $y \in H_{r(y),r(b)}$. Hence

$$\begin{array}{rcll} a^{-1}b & = & [x, xa]^{-1}[y, yb] & \\ & = & [xa, x][y, yb] & \\ & = & [txa, hyb] & \text{where } tx = hy,\ r(t) = r(h),\ l(t) = r(x) \text{ and } l(h) = r(y) \\ & = & [txa, txb] & \text{where } l(tx) = r(a) \\ & = & [a, b] & \text{by Lemma 8.2.8.} \end{array}$$

$\square$

From Lemmas 8.2.7, 8.2.12, 8.2.13, 8.2.14 and 8.2.15 we deduce that S is a straight left I-order in a bisimple inverse ω-semigroup. $\square$

8.3 Proper bisimple inverse ω-semigroups of left I-quotients

In this section we investigate a special case for a proper bisimple inverse ω-semigroup of left I-quotients $Q = \mathbf{BR}(G,\theta)$. It is shown in [32], that $Q = \mathbf{BR}(G,\theta)$ is proper if and only if θ is one-to-one.

Proposition 8.3.1. *Let S is a left I-order in a bisimple inverse ω-semigroup Q. The following are equivalent for a semigroup S:*

(i) Q is proper;

(ii) S satisfies the following condition:

$$\{(a,b) \in S \times S : ta = tb \text{ for some } t \in S\} \cap \mathcal{R}^Q = I_S$$

where I_S is the identity relation on S.

Proof. $(i) \Longrightarrow (ii)$. Suppose that Q is proper. Let $ta = tb$ for some $t \in S$ and $a \,\mathcal{R}^Q\, b$. Hence $t^{-1}ta = t^{-1}tb$ and as Q is proper, $a = b$, by Proposition 2.4.8.

$(ii) \Longrightarrow (i)$. Let $a^{-1}b$ be an element Q where $a \,\mathcal{R}^Q\, b$ and $c^{-1}c$ be any idempotent in Q such that $a^{-1}bc^{-1}c = c^{-1}c$. Using the proof of Theorem 8.2.1, we have

$$a^{-1}bc^{-1}c = (xa)^{-1}(yc) = c^{-1}c$$

with $xb = yc$ where $x, y \in S$ and $x \in H_{r(x),r(b)-l(b)+\max(l(b),l(c))}$, $y \in H_{r(x),r(c)-l(c)+\max(l(b),l(c))}$. It is clear that $xa \,\mathcal{R}^Q\, yc$. Hence $uxa = vc = uyc$ for some $u, v \in S$, by Lemma 3.2.4. As $yc = xb$ we have that $uxa = uxb$. By assumption, $a = b$ and so Q is proper. $\square$

Corollary 8.3.2. *The following are equivalent for a semigroup S.*

(i) S is a left I-order in a proper bisimple inverse ω-semigroup;

(ii) S satisfies Conditions (A), (B) and (C) of Theorem 8.2.1, and the following condition:

$$\{(a,b) \in S \times S : ta = tb \text{ for some } t \in S \text{ and } r(a) = r(b)\} = I_S$$

where I_S is the identity relation on S.

8.4 Uniqueness

This section will be devoted to proving Theorem 8.4.1, which determines when two bisimple inverse ω-semigroups of left I-quotients of a given semigroup S are isomorphic over S.

Let $B_1 = \mathbf{BR}(G_1, \theta_1)$ and $B_2 = \mathbf{BR}(G_2, \theta_2)$ be bisimple inverse ω-semigroups. We recall that for any $Q = \mathbf{BR}(G, \theta)$ there exist functions $r, l : Q \longrightarrow \mathbb{N}^0$ giving by

$$r(i, g, j) = i, \ l(i, g, j) = j.$$

It is clear that if a semigroup S is a left I-order in Q, then we can restrict these functions on S.

Theorem 8.4.1. *Let S be a left I-order in a bisimple inverse ω-semigroup B_1. Let $\varphi : S \longrightarrow B_2$ be an embedding of S into a bisimple inverse ω-semigroup B_2 such that $S\varphi$ is a left I-order in B_2. Then B_1 and B_2 are isomorphic via an isomorphism extending φ if and only if for all $a \in S$,*

$$r(a) = r(a\varphi) \text{ and } l(a) = l(a\varphi).$$

Proof. Suppose that $r(a) = r(a\varphi)$ and $l(a) = l(a\varphi)$ for all $a \in S$. Define $\psi : B_1 \longrightarrow B_2$ by the rule that

$$(a^{-1}b)\psi = a\varphi^{-1}b\varphi$$

where $a, b \in S$ and $a \,\mathcal{R}^{B_1}\, b$. To show that ψ is well defined, let

$$a^{-1}b = c^{-1}d$$

where $a, b, c, d \in S, a \,\mathcal{R}^{B_1}\, b$ and $c \,\mathcal{R}^{B_1}\, d$. Then by Lemma 3.2.4, there exist $x, y \in S$ such that

$$xa = yc \text{ and } xb = yd$$

where $l(x) = r(a), \ l(y) = r(c)$ and $r(x) = r(y)$. We then have

$$x\varphi a\varphi = y\varphi c\varphi \text{ and } x\varphi b\varphi = y\varphi d\varphi.$$

By assumption, $x\varphi \in H_{r(x\varphi), r(a\varphi)}$, $y\varphi \in H_{r(x\varphi), r(c\varphi)}$. By Lemma 3.2.4, we have

$$a\varphi^{-1}b\varphi = c\varphi^{-1}d\varphi.$$

We show that ψ is a homomorphism. Let $a^{-1}b, c^{-1}d \in B_1$ with $a\,\mathcal{R}^{B_1}\,b$ and $c\,\mathcal{R}^{B_1}\,d$, where $a,b,c,d \in S$. Using the proof of Theorem 8.2.1, we have

$$(a^{-1}bc^{-1}d)\psi = \big((xa)^{-1}(yd)\big)\psi$$

with $xb = yc$ and $x \in H_{r(x),r(b)-l(b)+\max\left(l(b),l(c)\right)}$, $y \in H_{r(x),r(c)-l(c)+\max\left(l(b),l(c)\right)}$.

Since $xb = yc$, it follows that $x\varphi b\varphi = y\varphi c\varphi$ where $x\varphi \in H_{r(x\varphi),r(b\varphi)-l(b\varphi)+\max\left(l(b\varphi),l(c\varphi)\right)}$ and $y\varphi \in H_{r(x\varphi),r(c\varphi)-l(c\varphi)+\max\left(l(b\varphi),l(c\varphi)\right)}$. Then $x\varphi\,\mathcal{L}\,c\varphi b\varphi^{-1}$ and $y\varphi \leq_{\mathcal{L}} (c\varphi)^{-1}$ so that from $x\varphi b\varphi = y\varphi c\varphi$ we have that $x\varphi b\varphi c\varphi^{-1} = y\varphi$. Now,

$$r(b\varphi c\varphi^{-1}) = r(b\varphi) - l(b\varphi) + \max(l(b\varphi), l(c\varphi)) = r(x\varphi^{-1}x\varphi),$$

so that $x\varphi^{-1}y\varphi = b\varphi c\varphi^{-1}$. Hence

$$\begin{aligned}
(a^{-1}bc^{-1}d)\psi &= ((xa)^{-1}(yd))\psi \\
&= (xa)\varphi^{-1}(yd)\varphi \qquad\qquad \text{as } xa\,\mathcal{R}^{B_1}\,xb = yc\,\mathcal{R}^{B_1}\,yd \\
&= a\varphi^{-1}x\varphi^{-1}y\varphi d\varphi \\
&= a\varphi^{-1}b\varphi c\varphi^{-1}d\varphi \\
&= (a^{-1}b)\psi(c^{-1}d)\psi.
\end{aligned}$$

We need to show that ψ is one-to-one. Suppose that

$$(a^{-1}b)\psi = (c^{-1}d)\psi$$

where $a,b,c,d \in S, a\,\mathcal{R}^{B_1}\,b$ and $c\,\mathcal{R}^{B_1}\,d$. Then $a\varphi^{-1}b\varphi = c\varphi^{-1}d\varphi^{-1}$. By assumption, $a\varphi\,\mathcal{R}^{B_2}\,b\varphi$ and $c\varphi\,\mathcal{R}^{B_2}\,d\varphi$ so that in B_2, there exist $t\varphi, s\varphi \in S\varphi$ such that

$$t\varphi a\varphi = r\varphi c\varphi \text{ and } t\varphi b\varphi = s\varphi d\varphi$$

where $l(t\varphi) = r(a\varphi)$, $l(s\varphi) = r(c\varphi)$ and $r(t\varphi) = r(s\varphi)$, by Lemma 3.2.4. Hence in B_1 we have that $ta = rc$ and $tb = rd$ where $t \in H_{r(t),r(a)}$, $r \in H_{r(t),r(c)}$, by assumption. Thus $a^{-1}b = c^{-1}d$, by Lemma 3.2.4.

The converse can be deduced from Theorem 4.1 of [40]. □

In the next corollary we provide an alternative proof for the above theorem, by using our result in Section 3.2.

Corollary 8.4.2. *Let S be a left I-order in a bisimple inverse ω-semigroup B_1. Let $\varphi : S \longrightarrow B_2$ be an embedding of S into a bisimple inverse ω-semigroup B_2 such that $S\varphi$ is a left I-order in B_2. Then following are equivalent:*

(1) *B_1 and B_2 are isomorphic via an isomorphism extending φ;*

(2) *for all $a \in S$, $r(a) = r(a\varphi)$ and $l(a) = l(a\varphi)$;*

(3) *for all $a, b \in S$;*

(*i*) $a \, \mathcal{R}^{B_1} \, b \Longleftrightarrow a\varphi \, \mathcal{R}^{B_2} \, b\varphi$,

(*ii*) $(a, b, c) \in \mathcal{T}_S^{\mathcal{B}_1} \Longleftrightarrow (a\varphi, b\varphi, c\varphi) \in \mathcal{T}_{S\varphi}^{\mathcal{B}_2}$.

Proof. (1) $\Rightarrow$(2) follows from Theorem 4.1 of [40].

(2) $\Rightarrow$ (3) 3(i) is immediate. For any $a, b, c \in S$

$$\begin{aligned}
(a, b, c) \in \mathcal{T}_S^{\mathcal{B}_1} &\Leftrightarrow ab^{-1}\mathcal{B}_1 \subseteq c^{-1}\mathcal{B}_1 \\
&\Leftrightarrow r(ab^{-1}) \geq r(c^{-1}) \\
&\Leftrightarrow r(a) - l(a) + \max(l(a), l(b)) \geq l(c) \\
&\Leftrightarrow r(a\varphi) - l(a\varphi) + \max(l(a\varphi), l(b\varphi)) \geq l(c\varphi) \\
&\Leftrightarrow r(a\varphi b\varphi^{-1}) \geq r(c\varphi^{-1}) \\
&\Leftrightarrow a\varphi b\varphi^{-1}\mathcal{B}_2 \subseteq c\varphi^{-1}\mathcal{B}_2 \\
&\Leftrightarrow (a\varphi, b\varphi, c\varphi) \in \mathcal{T}_{S\varphi}^{\mathcal{B}_2}.
\end{aligned}$$

Hence (3) holds. (3) $\Rightarrow$ (1) follows from Theorem 3.2.9. $\square$

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