Weak solutions to the Navier–Stokes inequality with arbitrary energy profiles

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Abstract

In a recent paper, Buckmaster & Vicol (arXiv:1709.10033) used the method of convex integration to construct weak solutions u to the 3D incompressible Navier–Stokes equations such that $\|u(t)\|_{L^2} = e(t)$ for a given non-negative and smooth energy profile $e\colon [0,T] \to \mathbb{R}$. However, it is not known whether it is possible to extend this method to construct nonunique solutions *suitable weak solutions* (that is weak solutions satisfying the strong energy inequality (SEI) and the *local energy inequality* (LEI), Leray-Hopf weak solutions (that is weak solutions satisfying the SEI), or at least to exclude energy profiles that are not nonincreasing.

In this paper we are concerned with weak solutions to the Navier–Stokes inequality on $\mathbb{R}^3$, that is vector fields that satisfy both the SEI and the LEI (but not necessarily solve the Navier–Stokes equations). Given $T>0$ and a nonincreasing energy profile $e\colon [0,T]\to[0,\infty)$ we construct weak solution to the Navier–Stokes inequality that are localised in space and whose energy profile $\|u(t)\|_{L^2(\mathbb{R}^3)}$ stays arbitrarily close to $e(t)$ for all $t\in[0,T]$. Our method applies only to nonincreasing energy profiles.

The relevance of such solutions is that, despite not satisfying the Navier–Stokes equations, they satisfy the partial regularity theory of Caffarelli, Kohn & Nirenberg (*Comm. Pure Appl. Math.*, 1982). In fact, Scheffer's constructions of weak solutions to the Navier–Stokes inequality with blow-ups (*Comm. Math. Phys.*, 1985 & 1987) show that the Caffarelli, Kohn & Nirenberg's theory is sharp for such solutions.

Our approach gives an indication of a number of ideas used by Scheffer. Moreover, it can be used to obtain a stronger result than Scheffer's. Namely, we obtain weak solutions to the Navier–Stokes inequality with both blow-up and a prescribed energy profile.

1 Introduction

The Navier–Stokes equations,

$$\begin{aligned} \partial_t u - \nu\Delta u + (u\cdot\nabla)u + \nabla p &= 0,\\ \operatorname{div} u &= 0, \end{aligned}$$

where u denotes the velocity of a fluid, p the scalar pressure and $\nu>0$ the viscosity, comprise a fundamental model for viscous, incompressible flows. In the case of the whole space $\mathbb{R}^3$ the pressure function is given (at each time instant t) by the formula

$$p := \sum_{i,j=1}^3 \partial_{ij}\Psi * (u_i u_j), \tag{1.1}$$

where $\Psi(x) := (4\pi|x|)^{-1}$ denotes the fundamental solution of the Laplace equation in $\mathbb{R}^3$ and "$*$" denotes the convolution. The formula above, which we shall refer to simply as the *pressure function corresponding to u*, can be derived by calculating the divergence of the Navier–Stokes equation.

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The fundamental mathematical theory of the Navier–Stokes equations goes back to the pioneering work of Leray (1934) (see Ożański & Pooley (2017) for a comprehensive review of this paper in more modern language), who used a Picard iteration scheme to prove existence and uniqueness of local-in-time strong solutions. Moreover, Leray (1934) and Hopf (1951) proved the global-in-time existence (without uniqueness) of weak solutions satisfying the energy inequality,

$$\|u(t)\|^2 + 2\nu \int_s^t \|\nabla u(\tau)\|^2 \mathrm{d}\tau \leq \|u(s)\|^2 \tag{1.2}$$

for almost every $s \geq 0$ and every $t > s$ (often called *Leray-Hopf weak solutions*). Here (and throughout the article) $\|\cdot\|$ denotes the $L^2(\mathbb{R}^3)$ norm. Although the fundamental question of global-in-time existence and uniqueness of strong solutions remains unresolved, many significant results contributed to the theory of the Navier–Stokes equations during the second half of the twentieth century. One such contribution is the partial regularity theory introduced by Scheffer (1976*a*, 1976*b*, 1977, 1978 & 1980) and subsequently developed by Caffarelli, Kohn & Nirenberg (1982); see also Lin (1998), Ladyzhenskaya & Seregin (1999), Vasseur (2007) and Kukavica (2009) for alternative approaches. This theory is concerned with so-called *suitable weak solutions*, that is Leray-Hopf weak solutions that are also weak solutions of the *Navier–Stokes inequality* (NSI).

Definition 1.1 (Weak solution to the Navier–Stokes inequality)**.** A divergence-free vector field $u\colon \mathbb{R}^3 \times (0,\infty)$ satisfying $\sup_{t>0} \|u(t)\| < \infty$, $\nabla u \in L^2(\mathbb{R}^3 \times (0,\infty))$ is a *weak solution of the Navier–Stokes inequality* with viscosity $\nu > 0$ if it satisfies the inequality

$$u \cdot (\partial_t u - \nu \Delta u + (u\cdot\nabla)u + \nabla p) \leq 0. \tag{1.3}$$

in a weak sense, that is

$$2\nu \int_0^\infty \int_{\mathbb{R}^3} |\nabla u|^2 \varphi \leq \int_0^\infty \int_{\mathbb{R}^3} \left(|u|^2 (\partial_t \varphi + \nu \Delta \varphi) + (|u|^2 + 2p)(u\cdot\nabla)\varphi \right) \tag{1.4}$$

for all non-negative $\varphi \in C_0^\infty(\mathbb{R}^3 \times (0,\infty))$, where p is the pressure function corresponding to u (recall (1.1)).

The last inequality is usually called the *local energy inequality.* The existence of global-in-time suitable weak solutions given divergence-free initial data $u_0 \in L^2$ was proved by Scheffer (1977) in the case of the whole space $\mathbb{R}^3$ and by Caffarelli et al. (1982) in the case of a bounded domain.

In order to see that (1.4) is a weak form of the NSI (1.3), note that the NSI can be rewritten, for smooth u and p, in the form

$$\frac{1}{2}\partial_t |u|^2 - \frac{\nu}{2}\Delta |u|^2 + \nu |\nabla u|^2 + u\cdot\nabla \left(\frac{1}{2}|u|^2 + p \right) \leq 0, \tag{1.5}$$

where we used the calculus identity $u\cdot\Delta u = \Delta(|u|^2/2) - |\nabla u|^2$. Multiplication by 2φ and integration by parts gives (1.4).

Furthermore, setting

$$f := \partial_t u - \nu\Delta u + (u\cdot\nabla)u + \nabla p,$$

one can think of the Navier–Stokes inequality (1.4) as the inhomogeneous Navier–Stokes equations with forcing f,

$$\partial_t u - \nu \Delta u + (u\cdot\nabla)u + \nabla p = f,$$

where f acts against the direction of the flow u, that is $f\cdot u \leq 0$.

The partial regularity theory gives sufficient conditions for local regularity of suitable weak solutions in space-time. Namely, letting $Q_r(z) := B_r(x) \times (t - r^2, t)$, a space-time cylinder based[1] at $z = (x,t)$, the central result of this theory, proved by Caffarelli et al. (1982), is the following.

[1]Note that here we use the convention of "nonanticipating" cylinders; namely that Q is *based* at a point (x,t) when (x,t) lies on the upper lid of the cylinder

Theorem 1.2 (Partial regularity of the Navier–Stokes equations). *Let $u_0 \in L^2(\mathbb{R}^3)$ be weakly divergence-free and let u be a "suitable weak solution" of the Navier–Stokes equations on $\mathbb{R}^3$ with initial condition condition u_0. There exists a universal constant $\varepsilon_0 > 0$ such that if*

$$\frac{1}{r^2}\int_{Q_r} |u|^3 + |p|^{3/2} < \varepsilon_0 \tag{1.6}$$

for some cylinder $Q_r = Q_r(z)$, $r > 0$, then u is bounded in $Q_{r/2}(z)$.

Moreover there exists a universal constant $\varepsilon_1 > 0$ such that if

$$\limsup_{r\to 0} \frac{1}{r}\int_{Q_r} |\nabla u|^2 < \varepsilon_1 \tag{1.7}$$

then u is bounded in a cylinder $Q_\rho(z)$ for some $\rho > 0$.

Here $\varepsilon_0, \varepsilon_1 > 0$ are certain universal constants (sufficiently small). We note that the proof of the above theorem does not actually use the fact that u is a suitable weak solution, but merely a weak solution to the NSI (which is not the case, however, in the subsequent alternative proofs due to Lin (1998) and Ladyzhenskaya & Seregin (1999)).

The partial regularity theorem (Theorem 1.2) is a key ingredient in the $L_{3,\infty}$ regularity criterion for the three-dimensional Navier–Stokes equations (see Escauriaza, Seregin & Šverák 2003) and the uniqueness of Lagrangian trajectories for suitable weak solutions (Robinson & Sadowski 2009); similar ideas have also been used for other models, such as the surface growth model $\partial_t u = -u_{xxxx} - \partial_x^2 u_x^2$ (Ożański & Robinson 2017), which can serve as a "one-dimensional model" of the Navier–Stokes equations (Blömker & Romito 2009, 2012).

A key fact about the partial regularity theory is that the quantities involved in the local regularity criteria (that is $|u|^3$, $|p|^{3/2}$ and $|\nabla u|^2$), are known to be globally integrable for any vector field satisfying $\sup_{t>0}\|u(t)\| < \infty$, $\nabla u \in L^2(\mathbb{R}^3 \times (0,\infty))$ (which follows by interpolation, see for example, Lemma 3.5 and inequality (5.7) in Robinson et al. (2016)); thus in particular for any Leray-Hopf weak solution. Therefore Theorem 1.2 shows that, in a sense, if these quantities localise near a given point $z \in \mathbb{R}^3 \times (0,\infty)$ in a way that is "not too bad", then z is not a singular point, and thus there cannot be "too many" singular points. In fact, by letting $S \subset \mathbb{R}^3 \times (0,\infty)$ denote the singular set, that is

$$S := \{(x,t) \in \mathbb{R}^3 \times (0,\infty) \colon u \text{ is unbounded in any neighbourhood of } (x,t)\}, \tag{1.8}$$

this can be made precise by estimating the "dimension" of S. Namely, a simple consequence of (1.6) and (1.7) is that

$$d_B(S) \le 5/3, \qquad \text{and} \qquad d_H(S) \le 1, \tag{1.9}$$

respectively[2], see Theorem 15.8 and Theorem 16.2 in Robinson et al. (2016). Here d_B denotes the *box-counting dimension* (also called the *fractal dimension* or the *Minkowski dimension*) and d_H denotes the *Hausdorff dimension*. The relevant definitions can be found in Falconer (2014), who also proves (in Proposition 3.4) the important property that $d_H(K) \le d_B(K)$ for any compact set K.

Very recently, Buckmaster & Vicol (2017) proved nonuniqueness of weak solutions to the Navier–Stokes equations on the torus $\mathbb{T}^3$ (rather than on $\mathbb{R}^3$). Their solutions belong to the class $C([0,T];L^2(\mathbb{T}^3))$, but they do not belong to the class $L^2((0,T);H^1(\mathbb{T}^3))$. Thus in particular these do not satisfy the energy inequality (1.2), and so they are neither Leray-Hopf weak solutions or weak solutions of the NSI. Moreover, the constructions of Buckmaster & Vicol (2017) include weak solutions with increasing energy $\|u(t)\|$.

In this article we work towards the same goal as Buckmaster & Vicol (2017), but from a different direction. Given an open set $W \subset \mathbb{R}^3$ and a nonincreasing, continuous energy profile $e\colon [0,T] \to [0,\infty)$ we construct a weak solution to the NSI such that its energy stays arbitrarily close to e and its support is contained in W for all times. Namely we prove the following theorem.

[2] In fact, (1.7) implies a stronger estimate than $d_H(S) \le 1$; namely that $\mathcal{P}^1(S) = 0$, where $\mathcal{P}^1(S)$ is the *parabolic Hausdorff measure* of S (see Theorem 16.2 in Robinson et al. (2016) for details).

Theorem 1.3 (Weak solutions to the NSI with arbitrary energy profile)**.** *Given an open set $W \subset \mathbb{R}^3$, $\varepsilon > 0$, $T > 0$ and a continuous, nonincreasing function $e\colon [0,T] \to [0,\infty)$ there exist $\nu_0 > 0$ and a weak solution u of the NSI for all $\nu \in [0,\nu_0]$ such that* $\operatorname{supp} u(t) \subset W$ *for all $t \in [0,T]$ and*

$$\big| \|u(t)\| - e(t) \big| \le \varepsilon \qquad \text{for all } t \in [0,T]. \tag{1.10}$$

We point out that the vector field u given by the above theorem does not satisfy the Navier–Stokes equations (but merely the NSI). The above theorem remains valid if the norm $\|u(t)\|$ is replaced by any L^p norm, with $p \in [1,\infty)$ and without the continuity assumption.

Corollary 1.4. *Given $p \in [1,\infty)$, an open set $W \subset \mathbb{R}^3$, $\varepsilon > 0$, $T > 0$ and a nonincreasing function $e\colon [0,T] \to [0,\infty)$ the claim of Theorem 1.3 remains valid with* (1.10) *replaced by*

$$\big| \|u(t)\|_p - e(t) \big| \le \varepsilon \qquad \text{for all } t \in [0,T].$$

Our approach is inspired by some ideas of Scheffer (1985 & 1987), who showed that the bound $d_H(S) \le 1$ is sharp for weak solutions of the NSI (of course, it is not known whether it is sharp for suitable weak solutions of the NSE). His 1985 result is the following.

Theorem 1.5 (Weak solution of NSI with point singularity)**.** *There exist $\nu_0 > 0$ and a function $\mathfrak{u}\colon \mathbb{R}^3 \times [0,\infty) \to \mathbb{R}^3$ that is a weak solution of the Navier–Stokes inequality with any $\nu \in [0,\nu_0]$ such that $\mathfrak{u}(t) \in C^\infty$,* $\operatorname{supp}\mathfrak{u}(t) \subset G$ *for all t for some compact set $G \Subset \mathbb{R}$ (independent of t). Moreover $\mathfrak{u}$ is unbounded in every neighbourhood of (x_0,T_0), for some $x_0 \in \mathbb{R}^3$, $T_0 > 0$.*

It is clear, using an appropriate rescaling, that the statement of the above theorem is equivalent to the one where $\nu = 1$ and $(x_0,T_0) = (0,1)$. Indeed, if $\mathfrak{u}$ is the velocity field given by the theorem then $\sqrt{T_0/\nu_0}\,\mathfrak{u}(x_0 + \sqrt{T_0\nu_0}x, T_0 t)$ satisfies Theorem 1.5 with $\nu_0 = 1$, $(x_0,T_0) = (0,1)$.

In a subsequent paper Scheffer (1987) constructed weak solutions of the Navier–Stokes inequality that blow up on a Cantor set $S \times \{T_0\}$ with $d_H(S) \ge \xi$ for any preassigned $\xi \in (0,1)$.

Theorem 1.6 (Nearly one-dimensional singular set)**.** *Given $\xi \in (0,1)$ there exists $\nu_0 > 0$, a compact set $G \Subset \mathbb{R}^3$ and a function $\mathfrak{u}\colon \mathbb{R}^3 \times [0,\infty) \to \mathbb{R}^3$ that is a weak solution to the Navier–Stokes inequality such that $u(t) \in C^\infty$,* $\operatorname{supp}\mathfrak{u}(t) \subset G$ *for all t, and*

$$\xi \le d_H(S) \le 1,$$

where S is the singular set (recall (1.8)*).*

Ożański (2017) provides a simpler presentation of Scheffer's constructions of $\mathfrak{u}$ from Theorems 1.5 and 1.6 and provides a new light on these constructions. In particular he introduces the concepts of a *structure* (which we exploit in this article, see below), the *pressure interaction function* and the *geometric arrangement*, which articulate the the main tools used by Scheffer to obtain a blow-up, but also describe, in a sense, the geometry of the NSI and expose a number of degrees of freedom available in constructing weak solutions to the NSI. Furthermore, it is shown in Ożański (2017) how can one obtain a blow-up on a Cantor set (Theorem 1.6) by a straightforward generalisation of the blow-up at a single point (Theorem 1.5).

It turns out that the construction from Theorem 1.3 can be combined with Scheffer's constructions to yield a weak solution to the Navier–Stokes inequality with both the blow-up and the prescribed energy profile.

Theorem 1.7 (Weak solutions to the NSI with blow-up and arbitrary energy profile)**.** *Given an open set $W \subset \mathbb{R}^3$, $\varepsilon > 0$, $T > 0$ and a nonincreasing function $e\colon [0,T] \to [0,\infty)$ such that $e(t) \to 0$ as $t \to T$ there exists $\nu_0 > 0$ and a weak solution u of the NSI for all $\nu \in [0,\nu_0]$ such that* $\operatorname{supp} u(t) \subset W$ *for all $t \in [0,T]$*

$$\big| \|u(t)\| - e(t) \big| \le \varepsilon \qquad \text{for all } t \in [0,T],$$

and the singular set S of u is of the form

$$S = S' \times \{T\},$$

where $S' \subset \mathbb{R}^3$ is a Cantor set with $d_H(S') \in [\xi,1]$ for any preassigned $\xi \in (0,1)$.

The structure of the article is as follows. In Section 2 we introduce some preliminary ideas including the notion of a *structure* (v,f,ϕ) on an open subset U of the upper half-plane

$$\mathbb{R}^2_+ := \{(x_1,x_2)\colon x_2>0\}.$$

In Section 3 we briefly sketch how the concept of a structure is used in the constructions of Scheffer (but we will refer the reader to Ożański (2017) for the full proof). We then illustrate some useful properties of structures of the form $(0,f,\phi)$ and we show how they can be used to generate weak solutions to the NSI on arbitrarily long time intervals. In Section 4 we prove our main result, Theorem 1.3, as well as Corollary 1.4. In the final Section 5 we prove Theorem 1.7.

2 Preliminaries

We denote the space of indefinitely differentiable functions with compact support in a set U by $C_0^\infty(U)$. We denote the indicator function of a set U by χ_U. We frequently use the convention

$$h_t(\cdot)\equiv h(\cdot,t),$$

that is the subscript t denotes dependence on t (rather than the t-derivative, which we denote by ∂_t).

We say that a vector field $u\colon\mathbb{R}^3\to\mathbb{R}^3$ is *axisymmetric* if $u(R_\theta x)=R_\theta(u(x))$ for any $\theta\in[0,2\pi)$, $x\in\mathbb{R}^3$, where

$$R_\theta(x_1,x_2,x_3) := (x_1, x_2\cos\phi - x_3\sin\phi, x_2\sin\phi + x_3\cos\phi)$$

is the rotation operation around the x_1 axis. We say that a scalar function $q\colon\mathbb{R}^3\to\mathbb{R}$ is *rotationally invariant* if

$$q(R_\theta x)=q(x)\qquad \text{for } \phi\in[0,2\pi), x\in\mathbb{R}^3.$$

Observe that if a vector field $u\in C^2$ and a scalar function $q\in C^1$ are rotationally invariant then the vector function $(u\cdot\nabla)u$ and the scalar functions

$$|u|^2,\quad \operatorname{div}u,\quad u\cdot\nabla|u|^2,\quad u\cdot\nabla q,\quad u\cdot\Delta u\quad\text{and}\quad \sum_{i,j=1}^3\partial_i u_j\partial_j u_i \tag{2.1}$$

are rotationally invariant, see Appendix A.2 in Ożański (2017) for details.

Let $U\Subset\mathbb{R}^2_+$. Set

$$R(U) := \{x\in\mathbb{R}^3\,:\, x=R_\phi(y,0)\text{ for some }\phi\in[0,2\pi),\ y\in U\}, \tag{2.2}$$

the *rotation of* U.

Given $v=(v_1,v_2)\in C_0^\infty(U;\mathbb{R}^2)$ and $f\colon\overline{U}\to[0,\infty)$ such that $f>|v|$ we define $u[v,f]\colon\mathbb{R}^3\to\mathbb{R}^3$ to be the axisymmetric vector field such that

$$u[v,f](x_1,x_2,0) := \left(v_1(x_1,x_2),v_2(x_1,x_2),\sqrt{f(x_1,x_2)^2-|v(x_1,x_2)|^2}\right).$$

In other words

$$u[v,f](x_1,\rho,\phi) = v_1(x_1,\rho)\widehat{x}_1 + v_2(x_1,\rho)\widehat{\rho} + \sqrt{f(x_1,\rho)^2-|v(x_1,\rho)|^2}\,\widehat{\phi}, \tag{2.3}$$

where the cylindrical coordinates x_1,ρ,ϕ are defined using the representation

$$\begin{cases} x_1 = x_1,\\ x_2=\rho\cos\phi,\\ x_3=\rho\sin\phi\end{cases}$$

and the cylindrical unit vectors $\widehat{x}_1$, $\widehat{\rho}$, $\widehat{\phi}$ are

$$\begin{cases} \widehat{x}_1(x_1,\rho,\phi) := (1,0,0), \\ \widehat{\rho}(x_1,\rho,\phi) := (0,\cos\phi,\sin\phi), \\ \widehat{\phi}(x_1,\rho,\phi) := (0,-\sin\phi,\cos\phi). \end{cases} \tag{2.4}$$

Note that such a definition immediately gives

$$|u[v,f]| = f.$$

Moreover, it satisfies some other useful properties, which we state in a lemma.

Lemma 2.1 (Properties of $u[v,f]$)**.**

(i) *The vector field $u[v,f]$ is divergence free if and only if v satisfies*

$$\operatorname{div}(x_2\, v(x_1,x_2)) = 0 \qquad \text{for all } (x_1,x_2) \in \mathbb{R}^2_+.$$

(ii) *If $v \equiv 0$ then*

$$\Delta u[0,f](x_1,\rho,\phi) = Lf(x_1,\rho)\widehat{\phi},$$

where

$$Lf(x_1,x_2) := \Delta f(x_1,x_2) + \frac{1}{x_2}\partial_{x_2} f(x_1,x_2) - \frac{1}{x_2^2} f(x_1,x_2). \tag{2.5}$$

In particular

$$\Delta u[0,f](x_1,x_2,0) = (0,0,Lf(x_1,x_2)). \tag{2.6}$$

(iii) *For all $x_1,x_2 \in \mathbb{R}$*

$$\partial_{x_3} |u[v,f]|(x_1,x_2,0) = 0. \tag{2.7}$$

Proof. These are easy consequences of the definition (and the properties of cylindrical coordinates), see Lemma 2.1 in Ożański (2017) for details. □

Using part (ii) we can see that the term $u[0,f]\cdot\Delta u[0,f]$ (recall the Navier–Stokes inequality (1.3)), which is axisymmetric, can be made non-negative by ensuring that Lf is non-negative, since

$$u[0,f](x_1,x_2,0)\cdot\Delta u[0,f](x_1,x_2,0) = f(x_1,x_2)Lf(x_1,x_2) \tag{2.8}$$

and f is non-negative by definition. It is not clear how to construct f such that $Lf \geq 0$ everywhere, but there exists a generic way of constructing f which guarantees this property at points sufficiently close to the boundary of U if U is a rectangle. In order to state this construction we denote (given $\eta > 0$) the "η-subset" of U by U_η, that is

$$U_\eta := \{x \in U \colon \operatorname{dist}(x,\partial U) > \eta\}.$$

We have the following result.

Lemma 2.2 (The edge effects)**.** *Let $U \Subset \mathbb{R}^2_+$ be an open rectangle, that is $U = (a_1,b_1)\times(a_2,b_2)$ for some $a_1,a_2,b_1,b_2 \in \mathbb{R}$ with $b_1 > a_1$, $b_2 > a_2 > 0$. Given $\eta > 0$ there exists $\delta \in (0,\eta)$ and $f \in C_0^\infty(\mathbb{R}^2_+;[0,1])$ such that*

$$\operatorname{supp} f = \overline{U}, \quad f > 0 \text{ in } U \text{ with } f = 1 \text{ on } U_\eta,$$

$$Lf > 0 \quad \text{in } U \setminus U_\delta.$$

Proof. See Lemma A.3 in Ożański (2017) for the proof (which is based on Section 5 in Scheffer (1985)). □

In other words, we can construct f that equals 1 on the given η-subset of U such that $Lf>0$ outside of a sufficiently large δ-subset. We will later (in Lemma 4.2) refine this lemma to show that δ can be chosen proportional to η and that f is bounded away from 0 on the δ-subset of U.

We define $p^*[v,f]\colon \mathbb{R}^3\to\mathbb{R}$ to be the pressure function corresponding to $u[v,f]$, that is

$$p^*[v,f](x):=\int_{\mathbb{R}^3}\sum_{i,j=1}^3\frac{\partial_i u_j[v,f](y)\partial_j u_i[v,f](y)}{4\pi|x-y|}\mathrm{d}y, \tag{2.9}$$

and we denote its restriction to $\mathbb{R}^2$ by $p[v,f]$,

$$p[v,f](x_1,x_2):=p^*[v,f](x_1,x_2,0). \tag{2.10}$$

Since $u[v,f]$ is rotationally invariant, the same is true of $p^*[v,f]$ (recall (2.1)). In particular

$$\partial_{x_3}p^*[v,f](x_1,x_2,0)=0 \quad \text{for all } x_1,x_2\in\mathbb{R}, \tag{2.11}$$

as in Lemma 2.1 (iii) above.

2.1 A structure

We say that a triple (v,f,ϕ) is a *structure on* $U\Subset\mathbb{R}^2_+$ if $v\in C_0^\infty(U;\mathbb{R}^2)$, $f\in C_0^\infty(\mathbb{R}^2_+;[0,\infty))$, $\phi\in C_0^\infty(U;[0,1])$ are such that $\operatorname{supp} f=\overline{U}$,

$$\begin{aligned}&\operatorname{supp} v\subset\{\phi=1\}, \qquad \operatorname{div}(x_2\, v(x_1,x_2))=0 \text{ in } U\\ &\text{and} \qquad f>|v| \quad \text{in } U \qquad \text{with} \qquad Lf>0 \text{ in } U\setminus\{\phi=1\}.\end{aligned}$$

Note that, given a structure (v,f,ϕ), we obtain an axisymmetric divergence-free vector field $u[v,f]$ that is supported in $R(U)$ (which is, in particular, away from the x_1 axis), and such that

$$|u[v,f](x,0)|=f(x) \qquad \text{for } x\in\mathbb{R}^2_+.$$

Moreover we note that (av,f,ϕ) is a structure for any $a\in(-1,1)$ whenever (v,f,ϕ) is, and that, given disjoint $U_1,U_2\Subset\mathbb{R}^2_+$ and the corresponding structures (v_1,f_1,ϕ_1), (v_2,f_2,ϕ_2), the triple $(v_1+v_2,f_1+f_2,\phi_1+\phi_2)$ is a structure on $U_1\cup U_2$. Observe that the role of the cutoff function ϕ in the definition of a structure is to cut off the edge effects as well as "cut in" the support of v. Namely, in $R(\{\phi<1\})$ (recall that R denotes the rotation, see (2.2)) we have $Lf\geq 0$ and $v=0$, and so

$$u[v,f]\cdot\Delta u[v,f]\geq 0 \tag{2.12}$$

and

$$u[v,f]\cdot\nabla q=0 \tag{2.13}$$

for any rotationally symmetric function $q\colon\mathbb{R}^3\to\mathbb{R}$. This last property (which follows from (2.11)) is particularly useful when taking $q:=|u[v,f]|^2+2p[v,f]$ as this gives one of the terms in the Navier–Stokes inequality (1.5).

2.2 A recipe for a structure

Using Lemma 2.2 one can construct structures on sets $U\Subset\mathbb{R}^2_+$ in the shape of a rectangle (which is the only shape we will consider in this article) in a generic way. This can be done using the following steps.

- First construct $w\colon U\to\mathbb{R}^2$ that is weakly divergence free (that is $\int_U w\nabla\psi=0$ for every $\psi\in C_0^\infty(U)$) and compactly supported in U.

  For example one can take $w:=(x_2,x_1)\chi_{1<|(x_1,x_2)|<2}$, after an appropriate rescaling and translation (so that $\operatorname{supp} w$ fits inside U); such a w is weakly divergence free due to the fact that $w\cdot n$ vanishes on the boundary of its support, where n denotes the respective normal vector to the boundary.

- Next, set $v := (J_\varepsilon w)/x_2$, where J_ϵ denotes the standard mollification and $\epsilon > 0$ is small enough so that $\operatorname{supp} v \Subset U$.
- Then construct f by using Lemma 2.2 (with any $\eta > 0$) and multiplying by a constact sufficiently large so that $f > |v|$ in U.
- Finally let $\phi \in C_0^\infty(U; [0,1])$ be such that $\{\phi = 1\}$ contains U_δ (from Lemma 2.2) and $\operatorname{supp} v$.

3 Applications of structures

In this section we point out two important applications of the concept of a structure.

3.1 The construction of Scheffer

Here we show how the concept of a structure is used in the Scheffer construction, Theorem 1.5, which we will only use later in proving Theorem 1.7.

We show below how Theorem 1.5 can be proved in a straightforward way using the following theorem.

Theorem 3.1. *There exist a set $U \Subset \mathbb{R}^2_+$, a structure (v, f, ϕ) and $\mathcal{T} > 0$ with the following property: there exist smooth time-dependent extensions v_t, f_t ($t \in [0, \mathcal{T}]$) of v, f, respectively, such that $v_0 = v$, $f_0 = f$, (v_t, f_t, ϕ) is a structure on U for each $t \in [0, \mathcal{T}]$. Moreover, for some $\nu_0 > 0$ the vector field*

$$u(t) := u[v_t, f_t]$$

satisfies the NSI (1.3) *in the classical sense for all $\nu \in [0, \nu_0]$ and $t \in [0, \mathcal{T}]$ as well as a certain large gain in magnitude, namely*

$$|u(\tau x + z, \mathcal{T})| \geq \tau^{-1} |u(x, 0)|, \qquad x \in \mathbb{R}^3, \tag{3.1}$$

for some $\tau \in (0, 1)$, $z \in \mathbb{R}^3$.

Proof. See Sections 1.1 and 3 in Ożański (2017) for a detailed proof. □

In fact, the set U (from the theorem above) is of the form $U = U_1 \cup U_2$ for some disjoint $U_1, U_2 \Subset \mathbb{R}^2_+$ and $(v, f, \phi) = (v_1 + v_2, f_1 + f_2, \phi_1 + \phi_2)$, where (v_1, f_1, ϕ_1), (v_2, f_2, ϕ_2) are some structures on U_1, U_2, respectively. The elaborate part of the proof of Theorem 3.1 is devoted to the careful arrangement of U_1, U_2 and a construction of the corresponding structures and $\mathcal{T} > 0$ which magnify certain interaction between U_1 and U_2 via the pressure function, and thus allows (3.1). We refer the reader to Sections 1.1 and 3 in Ożański (2017) for the full proof of Theorem 3.1. We note, however, that the part of the theorem about the NSI is not that difficult. In fact we show in Lemma 3.3 below that any structure gives rise to infinitely many classical solutions of the NSI (on arbitrarily long time intervals) with $u[v, f]$ as the initial condition.

In order to prove Theorem 1.5 we will make use of an alternative form of the local energy inequality. Namely, the local energy inequality (1.4) is satisfied if the *local energy inequality on the time interval* $[S, S']$,

$$\begin{aligned}\int_{\mathbb{R}^3} |u(x, S')|^2 \varphi \,\mathrm{d}x - \int_{\mathbb{R}^3} |u(x, S)|^2 \varphi \,\mathrm{d}x + 2\nu \int_S^{S'} \int_{\mathbb{R}^3} |\nabla u|^2 \varphi \\ \leq \int_S^{S'} \int_{\mathbb{R}^3} \left(|u|^2 + 2p\right) u \cdot \nabla \varphi + \int_S^{S'} \int_{\mathbb{R}^3} |u|^2 \left(\partial_t \varphi + \nu \Delta \varphi\right),\end{aligned} \tag{3.2}$$

holds for all $S, S' > 0$ with $S < S'$, which is clear by taking S, S' such that $\operatorname{supp} \varphi \subset \mathbb{R}^3 \times (S, S')$. An advantage of this alternative form of the local energy inequality is that it demonstrates how to combine solutions of the Navier–Stokes inequality one after another. Namely, (3.2) shows that a necessary and sufficient condition for two vector fields $u^{(1)} \colon \mathbb{R}^3 \times [t_0, t_1] \to \mathbb{R}^3$, $u^{(2)} \colon \mathbb{R}^3 \times [t_1, t_2] \to \mathbb{R}^3$

satisfying the local energy inequality on the time intervals $[t_0, t_1]$, $[t_1, t_2]$, respectively, to combine (one after another) into a vector field satisfying the local energy inequality on the time interval $[t_0, t_2]$ is that

$$|u^{(2)}(x,t_1)| \leq |u^{(1)}(x,t_1)| \qquad \text{for a.e. } x \in \mathbb{R}^3. \tag{3.3}$$

Using the above property and Theorem 3.1 we can employ a simple switching procedure to obtain Scheffer's construction of the blow-up at a single point (that is the claim of Theorem 1.5). Namely, considering

$$u^{(1)}(x,t) := \tau^{-1} u(\Gamma^{-1}(x), \tau^{-2}(t - \mathcal{T})),$$

where $\Gamma(x) := \tau x + z$, we see that $u^{(1)}$ satisfies the Navier–Stokes inequality (1.3) in a classical sense for all $\nu \in [0, \nu_0]$ and $t \in [\mathcal{T}, (1+\tau^2)\mathcal{T}]$, $\operatorname{supp} u^{(1)}(t) = \Gamma(G)$ for all $t \in [\mathcal{T}, (1+\tau^2)\mathcal{T}]$ and that (3.1) gives

$$\left| u^{(1)}(x,\mathcal{T}) \right| \leq |u(x,\mathcal{T})|, \qquad x \in \mathbb{R}^3 \tag{3.4}$$

(and so u, $u^{(1)}$ can be combined "one after another", recall (3.3)). Thus, since $u^{(1)}$ is larger in magnitude than u (by the factor of τ) and its time of existence is $[\mathcal{T}, (1+\tau^2)\mathcal{T}]$, we see that by iterating such a switching we can obtain a vector field $\mathfrak{u}$ that grows indefinitely in magnitude, while its support shrinks to a point (and thus will satisfy all the claims of Theorem 1.5), see Fig. 1. To be more precise we let $t_0 := 0$,

$$t_j := \mathcal{T} \sum_{k=0}^{j-1} \tau^{2k} \qquad \text{for } j \geq 1, \tag{3.5}$$

$T_0 := \lim_{j\to\infty} t_j = \mathcal{T}/(1-\tau^2)$, $u^{(0)} := u$, and

$$u^{(j)}(x,t) := \tau^{-j} u\left(\Gamma^{-j}(x), \tau^{-2j}(t - t_j)\right), \qquad j \geq 1, \tag{3.6}$$

see Fig. 1. As in (3.4), (3.1) gives that

$$\operatorname{supp} u^{(j)}(t) = \Gamma^j(G) \qquad \text{for } t \in [t_j, t_{j+1}] \tag{3.7}$$

and that the magnitude of the consecutive vector fields shrinks at every switching time, that is

$$\left| u^{(j)}(x,t_j) \right| \leq \left| u^{(j-1)}(x,t_j) \right|, \qquad x \in \mathbb{R}^3, j \geq 1, \tag{3.8}$$

see Fig. 1.

Thus letting

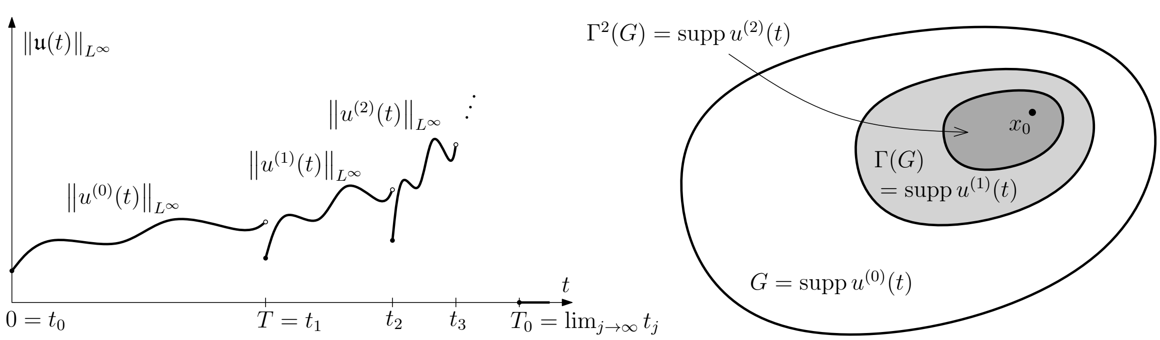

Figure 1: The switching procedure: the blow-up of $\|\mathfrak{u}(t)\|_\infty$ (left) and the shrinking support of $\mathfrak{u}(t)$ (right) as $t \to T_0^-$.

$$\mathfrak{u}(t) := \begin{cases} u^{(j)}(t) & \text{if } t \in [t_j, t_{j+1}) \text{ for some } j \geq 0, \\ 0 & \text{if } t \geq T_0, \end{cases} \tag{3.9}$$

we obtain a vector field that satisfies all claims of Theorem 1.5 with $x_0 := z/(1-\tau)$.

Observe that by construction

$$\|\mathfrak{u}(t)\|_p \to 0 \qquad \text{as } t \to T_0^- \quad \text{for all } p \in [1,3), \tag{3.10}$$

since $\tau \in (0,1)$. Indeed we write for any $t \in [t_j, t_{j+1}]$, $j \geq 0$,

$$\|\mathfrak{u}(t)\|_p = \|u^{(j)}(t)\|_p \leq \sup_{s \in [t_j, t_{j+1})} \|u^{(j)}(s)\|_p = \tau^{-j(1-3/p)} \sup_{s \in [t_0, t_1]} \|u^{(0)}(s)\|_p \to 0 \qquad \text{as } j \to \infty.$$

3.2 Structures of the form $(0, f, \phi)$

Let $U \Subset \mathbb{R}^2_+$. We now focus on the structures on U of the form $(0, f, \phi)$ and, for convenience, we set

$$u[f] := u[0, f].$$

As in (2.13) we see that

$$u[f] \cdot \nabla \left(|u[f]|^2 + 2p[f] \right) = 0 \qquad \text{in } \mathbb{R}^3, \tag{3.11}$$

for any $f \in C_0^\infty(\mathbb{R}^2_+; [0,\infty))$. Using this property we can show that given any structure (v, f, ϕ) on a set $U \Subset \mathbb{R}^2_+$ there exists a time-dependent extension f_t of f such that $(0, f_t, \phi)$ is a structure on U and gives rise to a classical solution to the NSI (for all sufficiently small viscosities) that is almost constant in time. We make this precise in the following lemma, which we will use later.

Lemma 3.2. *Given $\varepsilon > 0$, $T > 0$, $U \Subset \mathbb{R}^2_+$ and a structure (v, f, ϕ) there exists $\nu_0 > 0$ and an axisymmetric classical solution u to the NSI for all $\nu \in [0, \nu_0]$, $t \in [0, T]$ that is supported in $R(\overline{U})$ with $u(0) = u[f]$ and*

$$\|u(t) - u[f]\|_p \leq \varepsilon \qquad \textit{for all } t \in [0,T], p \in [1, \infty]. \tag{3.12}$$

Proof. Let

$$u(t) := u[f_t],$$

where

$$f_t^2 := f^2 - \delta t \phi$$

and $\delta > 0$ is sufficiently small such that $f_t > 0$ in U for all $t \in [0, T]$ (Note this is possible since f is continuous and $\operatorname{supp} \phi \Subset \operatorname{supp} f$). Clearly $u(0) = u[f]$ and (3.12) follows for $p \in \{1, \infty\}$ by taking δ sufficiently small. If $p \in (1, \infty)$ then (3.12) follows using Lebesgue interpolation.

It remains to verify that $u(t)$ satisfies the NSI. To this end let $\nu_0 > 0$ be sufficiently small such that

$$\nu_0 \, |u[f_t](x) \cdot \Delta u[f_t](x)| \leq \frac{\delta}{2} \quad \text{ for } x \in \mathbb{R}^3, t \in [0, T]. \tag{3.13}$$

Due to the axisymmetry of u it is enough to verify the NSI only for points of the form $(x, 0, t)$, for $x \in \overline{U}$, $t \in [0, T]$. Setting q to be the pressure function corresponding to u (that is $q(x,t) := p^*[0, f_t](x)$) we use (3.11) to write

$$\begin{aligned} \partial_t |u(x,0,t)|^2 &= -\delta \phi(x) \\ &= -\delta\phi(x) - u(x,0,t) \cdot \nabla \left(|u(x,0,t)|^2 + 2p(x,0,t) \right) \\ &\leq 2\nu u(x,0,t) \cdot \Delta u(x,0,t) - u(x,0,t) \cdot \nabla \left(|u(x,0,t)|^2 + 2p(x,0,t) \right), \end{aligned}$$

as required, where, in the last step, we used (2.12) for x such that $\phi(x) < 1$ and (3.13) for x such that $\phi(x) = 1$. □

Observe that the proof does not make any use of v. One similarly obtains the same result, but with the claim on the initial condition $u(0) = u[f]$ replaced by a condition at a final time, namely the pointwise inequality $|u(T)| \geq |u[f]|$ everywhere in $\mathbb{R}^3$. We thus obtain the following lemma, which we will use to prove Theorem 1.7.

Lemma 3.3. *Given $\varepsilon > 0$, $T > 0$, $U \Subset P$ and a structure (v, f, ϕ) there exists $\nu_0 > 0$ and an axisymmetric weak solution u to the NSI for all $\nu \in [0, \nu_0]$ that is supported in $\overline{U}$,*

$$|u[f](x)| \leq |u(x, T)| \qquad \text{for all } x \in \mathbb{R}^3 \tag{3.14}$$

and

$$\|u(t) - u[f]\|_p \leq \varepsilon \qquad \text{for all } t \in [0, T], p \in [1, \infty]. \tag{3.15}$$

Proof. The lemma follows in the same way as Lemma 3.2 after replacing "f" in the above proof by "$(1+\epsilon)f$" for sufficiently small $\epsilon > 0$ and then taking $\delta > 0$ (and so also ν_0) smaller. □

Finally, observe that if $f_{1,t}, f_{2,t} \in C_0^\infty(\mathbb{R}^2_+; [0, \infty))$ are disjointly supported (for each t) then

$$p^*[0, f_{1,t} + f_{2,t}] = p^*[0, f_{1,t}] + p^*[0, f_{2,t}]$$

and so

$$u[f_{1,t} + f_{2,t}] \text{ satisfies the NSI in the classical sense} \tag{3.16}$$

whenever each of $u[f_{1,t}]$ and $u[f_{2,t}]$ does. Indeed, this is because the term

$$u \cdot \nabla p(x_1, x_2, 0) = u_3(x_1, x_2, 0)\partial_3 p(x_1, x_2, 0) \tag{3.17}$$

in the NSI vanishes (due to (2.11)). Note that (3.16) does not necessarily hold for structures (v, f, ϕ) with $v \neq 0$, as in this case the term $u \cdot \nabla p$ does not simplify as in (3.17). We will use (3.16) as a substitute for the linearity of the NSI in the proof of Theorem 1.3 in the next section.

4 Proof of Theorem 1.3

In this section we prove Theorem 1.3; namely given an open set $W \subset \mathbb{R}^3$, $\varepsilon > 0$, $T > 0$ and a continuous, nonincreasing function $e\colon [0, T] \to [0, \infty)$ there exist $\nu_0 > 0$ and a weak solution u of the NSI for all $\nu \in [0, \nu_0]$ such that $\operatorname{supp} u(t) \subset W$ for all $t \in [0, T]$ and

$$|\|u(t)\| - e(t)| \leq \varepsilon \qquad \text{for all } t \in [0, T].$$

(Recall that $\|\cdot\|$ denotes the $L^2(\mathbb{R}^3)$ norm.)
We can assume that $e(T) = 0$, as otherwise one could extend e continuously beyond T into a function decaying to 0 in finite time $T' > T$. Moreover, by translation in space we can assume that W intersects the x_1 axis. Let $U \Subset \mathbb{R}^2_+$ be such that $R(\overline{U}) \subset W$ we will construct and axisymmetric weak solution to the NSI (for all sufficiently small viscosities) such that $u(t) \in C_0^\infty(\mathbb{R}^3)$, $\operatorname{supp} u(t) \subset R(\overline{U})$ and

$$|\|u(t)\| - e(t)| \leq \varepsilon.$$

Before the proof we comment on its strategy in an informal manner. Suppose for the moment that we would like to use a similar approach as in the proof of Lemma 3.2, that is define some rectangle $U \Subset \mathbb{R}^2_+$, a structure (v, f, ϕ) on it and $u(t) := u[f_t]$, where

$$f_t^2 := f^2 + (C - De(t)^2)\phi, \tag{4.1}$$

$C, D > 0$, such that

$$\|u(t)\| \approx e(t)$$

at least for small t. In fact we could use the recipe from Section 2.2 to construct (v, f, ϕ). In order to proceed with the calculation (that is to guarantee the NSI) we would need to guarantee that $(e(t)^2)'$ is bounded above by some negative constant, which is not a problem, as the following lemma demonstrates.

Lemma 4.1. *Given $\varepsilon > 0$ and a continuous and nonincreasing function $e\colon [0, T] \to [0, \infty)$ there exist $\zeta > 0$ and $\widetilde{e}\colon [0, T] \to [0, \infty)$ such that $\widetilde{e} \in C^\infty([0, T])$, and*

$$e(t) \leq \widetilde{e}(t) \leq e(t) + \varepsilon, \quad \frac{\mathrm{d}}{\mathrm{d}t}\widetilde{e}^2(t) \leq -\zeta \qquad \text{for } t \in [0, T].$$

Proof. Extend $e(t)$ by $e(T)$ for $t > T$ and by $e(0)$ for $t < 0$. Let $J_\epsilon e^2$ denote a mollification of e^2. Since e^2 is uniformly continuous $J_\epsilon e^2$ converges to e^2 in the supremum norm as $\epsilon \to 0$, and so $\|J_\epsilon e^2 - e^2\|_{L^\infty(\mathbb{R})} < \varepsilon/4$ for sufficiently small ϵ. Then the function

$$\widetilde{e}(t) := \sqrt{J_\epsilon e^2(t) + (\varepsilon/2 - \varepsilon t/4T)}$$

satifies the claim of the lemma with $\zeta := \varepsilon/4T$. □

The problem with (4.1) is that its right-hand side can become negative for small times[3] (so that $(0, f_t, \phi)$ would no longer be a structure, and so $u[f_t]$ would not be well-defined). We will overcome this problem by utilising the property (3.3). Namely, at time t_1 when the right-hand side of (4.1) becomes zero we will "trim" U to obtain a smaller set U^1, on which the right-hand side of (4.1) does not vanish, and we will define a new structure $(0, f_1, \phi_1)$, with $f_1^2 \leq f^2 + (C - De(t_1)^2)\phi$. We will then continue the same way (as in (4.1)) to define $u(t) := u[f_{1,t}]$ for $t \geq t_1$ where

$$f_{1,t}^2 := f^2 - (C_1 - D_1 e(t)^2)\phi_1$$

for an appropriately chosen C_1, D_1. Note that such a continuation satisfies the local energy inequality, since (3.3) is satisfied. We will then continue in the same way to define $U^2, U^3, \ldots$, structures $(0, f_2, \phi_2), (0, f_3, \phi_3), \ldots$, and $u(t) := u[f_{k,t}]$ for $t \in [t_k, t_{k+1}]$, where

$$f_{k,t}^2 := f_k^2 - (C_k - D_k e(t)^2)\phi_k, \tag{4.2}$$

and $C_k, D_K > 0$ are chosen appropriately, until we reach time $t = T$.

Such a procedure might look innocent, but note that there is a potentially fatal flaw. Namely, we need to use an existence result such as Lemma 2.2 in order to construct f_k as well as $\delta_k > 0$; recall that δ_k controls the edge effect (that is $Lf_k \geq 0$ in $U^k \setminus U^k_{\delta_k}$) and that, according to the recipe from Section 2.2, ϕ_k is chosen so that $\phi_k = 1$ on $U^k_{\delta_k}$. However, Lemma 2.2 gives no control of δ_k, and so it seems possible that δ_k shrinks rapidly as k increases, and consequently

$$\inf_{U^k_{\delta_k}} f_k \to 0 \text{ rapidly} \qquad \text{as } k \text{ increases.}$$

Thus (since $\phi_k = 1$ on $U^k_{\delta_k}$) the length of the time interval $[t_k, t_{k+1}]$ would shrink rapidly to 0 as k increases (as the right-hand side of (4.2) would become negative for some x), and so it is not clear whether the union of all intervals,

$$\bigcup_{k\geq 0} [t_k, t_{k+1}],$$

would cover $[0, T]$.

In order to overcome this problem we prove a sharper version of Lemma 2.2 which states that we can choose $\delta = c'\eta$ and f such that $f > c$ in U_δ, where the constants $c, c' \in (0, 1)$ do not depend on the size of U.

Lemma 4.2 (The cut-off function on a rectangle)**.** *Let $a > 0$ and $U \Subset \mathbb{R}^2_+$ be an open rectangle that is at least a away from the x_1 axis, that is $U = (a_1, b_1) \times (a_2, b_2)$ for some $a_1, a_2, b_1, b_2 \in \mathbb{R}$ with $b_1 > a_1$, $b_2 > a_2 > a$. Given $\eta \in (0, 1)$ there exists $f \in C_0^\infty(\mathbb{R}^2_+; [0, 1])$ such that*

$$\operatorname{supp} f = \overline{U}, \quad f > 0 \text{ in } U \text{ with } f = 1 \text{ on } U_\eta,$$

$$Lf > 0 \quad \text{in } U \setminus U_{c'\eta}, \text{ with } f > c \text{ in } U_{c'\eta/2},$$

where $c, c' \in (0, 1/2)$ depend only on a.

Proof. We prove the lemma in Appendix A. □

[3]Note that the point $x \in U$ at which the right-hand side of (4.1) will become negative is located close to the ∂U since only for such x $\phi(x) = 1$ but $f(x) < \max f$.

The above lemma allows us to ensure that the time interval $[0,T]$ can be covered by only finitely many intervals $[t_k, t_{k+1}]$.

We now make the above strategy rigorous.

Proof of Theorem 1.3. Fix $a > 0$ such that $\mathrm{dist}(U, x_1\text{-axis}) \geq a$. By applying Lemma 4.1 we can assume that e^2 is differentiable on $[0,T]$ with $(e^2(t))' \leq -\zeta$ for all $t \in [0,T]$, where $\zeta > 0$. Let K be the smallest positive integer such that

$$(1-c^2)^K e(0)^2 < \varepsilon^2,$$

where $c = c(a)$ is the constant from Lemma 4.2. For $k \in \{1, \ldots, K\}$ let $t_k \in [0,T]$ be such that

$$e(t_k)^2 = (1-c^2)^k e(0)^2. \tag{4.3}$$

(Note t_k is uniquely determined since $(e(t)^2)' \leq -\zeta$.) Let also $t_0 := 0$. Observe that the choice of K implies that

$$e(t)^2 \geq \varepsilon^2/2 \quad \text{for } t \in [t_0, t_K]. \tag{4.4}$$

Indeed, since $e(t)$ is nonincreasing and $c^2 < 1/2$,

$$e(t)^2 \geq e(t_K)^2 = (1-c^2)(1-c^2)^{K-1} e(0)^2 \geq (1-c^2)\varepsilon^2 \geq \varepsilon^2/2,$$

as required.

We set

$$d := \min_{k \in \{0, \ldots, K-1\}} (t_{k+1} - t_k).$$

We will construct a sequence of classical solution $\{u_k\}_{k=0}^{K-1}$ to the NSI for all $\nu \in [0, \nu_0]$ (where ν_0 is fixed in (4.16) below) on the time intervals $[t_k, t_{k+1}]$ such that

$$|u_{k+1}(t_{k+1})| \leq |u_k(t_{k+1})| \quad \text{a.e. in } \mathbb{R}^3 \tag{4.5}$$

and

$$\left| \|u_k(t)\|^2 - e(t)^2 \right| \leq \varepsilon^2/2 \quad \text{for } t \in [t_k, t_{k+1}]. \tag{4.6}$$

Then the claim of the theorem follows by defining

$$u(t) := \begin{cases} u_k(t) & t \in [t_k, t_{k+1}), k \in \{0, \ldots, K-1\} \\ 0 & t \geq t_K. \end{cases}$$

Indeed, (4.5) implies that we can switch from u_k to u_{k+1} at the time t_{k+1} ($k = 0, \ldots, K-1$), so that u is a weak solution of the NSI for all $\nu \in [0, \nu_0]$, $t \in [0,T]$. Moreover (4.6) implies (1.10), since (4.4) gives

$$|\|u(t)\| - e(t)| = \left| \|u(t)\|^2 - e(t)^2 \right| / |\|u(t)\| + e(t)| \leq \varepsilon^2 / 2|e(t)| \leq \varepsilon \quad \text{for } t \in [t_0, t_K), \tag{4.7}$$

and the claim for $t \in [t_K, T]$ follows trivially.

In order to construct u_k (for $k = 0, \ldots, K-1$) we first fix $\mu > 0$ such that

$$\mu \|u[\chi_U]\| = e(0) \tag{4.8}$$

and we set $\eta > 0$ sufficiently small such that

$$\|u[\chi_{U \setminus U_{K\eta}}]\| < \frac{\min\{\varepsilon, \sqrt{d\zeta}\}}{2\mu}. \tag{4.9}$$

Note that (4.3) and (4.8) give

$$e(t_k) = (1-c^2)^k \mu^2 \|u[\chi_U]\|^2. \tag{4.10}$$

We now let $U^k := U_{k\eta}$ and apply Lemma 4.2 to obtain $f_k \in C_0^\infty(P;[0,1])$ such that

$$\operatorname{supp} f_k = \overline{U^k}, \quad f_k > 0 \text{ in } U^k \text{ with } f_k = 1 \text{ on } U^k_\eta,$$

$$Lf_k > 0 \quad \text{ in } U^k \setminus U^k_{c'\eta}, \text{ with } f_k > c \text{ in } U_{c'\eta/2},$$

where $c = c(a)$, $c' = c'(a)$. Let $\phi_k \in C_0^\infty(U^k;[0,1])$ be such that

$$\operatorname{supp} \phi_k \subset U^k_{c'\eta/2} \quad \text{ and } \quad \phi_k = 1 \text{ on } U^k_{c'\eta}.$$

Note that (4.9) implies that

$$\left| \|u[\chi_U]\|^2 - \|u[\phi_k]\|^2 \right|, \left| \|u[f_k]\|^2 - \|u[\phi_k]\|^2 \right| \le \frac{\min\{\varepsilon^2, d\zeta\}}{4\mu^2} \tag{4.11}$$

for all $k = 0, \ldots, K-1$.

We will consider an affine modification $E_k(t)^2$ of $e(t)^2$ on the time interval $[t_k, t_{k+1}]$ such that

$$E_k(t_k)^2 = (1-c^2)^k \mu^2 \|u[\phi_k]\|^2 \quad \text{ and } \quad E_k(t_{k+1})^2 = (1-c^2) E_k(t_k)^2. \tag{4.12}$$

(Recall $e(t)$ satifies the above conditions with $\|u[\phi_k]\|$ replaced by $\|u[\chi_U]\|$, see (4.10)) Namely we let

$$E_k(t)^2 := e(t)^2 - (1-c^2)^k \mu^2 \left(\|u[\chi_U]\|^2 - \|u[\phi_k]\|^2 \right) \left(1 - c^2 \frac{t - t_k}{t_{k+1} - t_k} \right).$$

Roughly speaking, E_k is a convenient modification of e that allows us to satisfy (4.5) while not causing any trouble to either (4.6) or the NSI. For example, we see that

$$\begin{aligned} \left| E_k(t)^2 - e(t)^2 \right| &= (1-c^2)^k \mu^2 \left(\|u[\chi_U]\|^2 - \|u[\phi_k]\|^2 \right) \left(1 - c^2 \frac{t - t_k}{t_{k+1} - t_k} \right) \\ &\le \varepsilon^2/4 \qquad\qquad\qquad\qquad \text{for } t \in [t_k, t_{k+1}] \end{aligned} \tag{4.13}$$

where we used (4.11). This implies in particular that $E_k(t)$ is well-defined (as $e(t)^2 \ge \varepsilon^2/2$, recall (4.4)). Moreover , and so in particular

$$\begin{aligned} (E_k(t)^2)' &= (e(t)^2)' + (1-c^2)^k \mu^2 \left(\|u[\chi_U]\|^2 - \|u[\phi_k]\|^2 \right) \frac{c^2}{t_{k+1} - t_k} \\ &\le -\zeta + \zeta/4 \\ &< -\zeta/2 \qquad\qquad \text{for } t \in [t_k, t_{k+1}]. \end{aligned} \tag{4.14}$$

We can now define u_k by writing

$$u_k(t) := u[f_{k,t}],$$

where

$$f_{k,t}^2 := (1-c^2)^k \mu^2 f_k^2 - \left((1-c^2)^k \mu^2 - \frac{E_k(t)^2}{\|u[\phi_k]\|^2} \right) \phi_k.$$

Observe that, due to the monotonicity of E_k and the choice of η (recall (4.9)), the last term above can be bounded above and below for $t \in [t_k, t_{k+1}]$,

$$0 \le \left((1-c^2)^k \mu^2 - \frac{E_k(t)^2}{\|u[\phi_k]\|^2} \right) \phi_k \le c^2 (1-c^2)^k \mu^2 \phi_k. \tag{4.15}$$

This means, in particular, that $f_{k,t}^2$ is positive in U^k (that is $f_{k,t}$ is well-defined by the above formula). Indeed, this is trivial for $x \in U^k \setminus U^k_{\eta_k/2}$ (as $\phi_k(x) = 0$ in this case), and for $x \in U^k_{\eta_k/2}$ we have $f_k^2(x) > c^2$ and so

$$f_{k,t}^2(x) > (1-c^2)^k \mu^2 c^2 (1 - \phi_k) \ge 0,$$

as required.

Let $\nu_0 > 0$ be sufficiently small such that

$$\nu_0 \left\| u[f_{k,t}] \cdot \Delta u[f_{k,t}] \right\|_\infty \le \frac{\zeta}{4\|u[\chi_U]\|^2} \qquad \text{for } t \in [t_k, t_{k+1}], k = 0, \ldots, K-1. \tag{4.16}$$

Having fixed ν_0 we show that u_k is a classical solution of the NSI with any $\nu \in [0, \nu_0]$ on the time interval $[t_k, t_{k+1}]$. Namely for each such ν we can use the monotonicity of $E_k(t)^2$ (recall (4.14)) to obtain

$$\begin{aligned}
\partial_t |u_k(x,0,t)|^2 &= \partial_t E_k(t)^2 \frac{\phi_k(x)}{\|u[\phi_k]\|^2} \\
&\le -\zeta \frac{\phi_k(x)}{2\|u[\chi_U]\|^2} \\
&= -\zeta \frac{\phi_k(x)}{2\|u[\chi_U]\|^2} - u_k(x,0,t) \cdot \nabla \left(|u_k(x,0,t)|^2 + 2p_k(x,0,t) \right) \\
&\le 2\nu u_k(x,0,t) \cdot \Delta u_k(x,0,t) - u_k(x,0,t) \cdot \nabla \left(|u_k(x,0,t)|^2 + 2p_k(x,0,t) \right),
\end{aligned} \tag{4.17}$$

as required, where, in the last step, we used (2.12) for x such that $\phi_k(x) < 1$ and (4.16) for x such that $\phi_k(x) = 1$.

It remains to verify (4.5) and (4.6). As for (4.5) it suffices to show the claim on $R(\overline{U_{(k+1)\eta}})$ (that is on the support of u_{k+1}). Moreover, since both u_k and u_{k+1} are axially symmetric, it is enough to show the claim at the points of the form $(x, 0)$, where $x = (x_1, x_2) \in \overline{U_{(k+1)\eta}}$. Then $f_k(x) = \phi_k(x) = 1 \ge f_{k+1}(x)$ and so

$$\begin{aligned}
|u_{k+1}(x,0,t_{k+1})|^2 &= (1-c^2)^{k+1} \mu^2 f_{k+1}^2(x) - \left((1-c^2)^{k+1}\mu^2 - \frac{E_{k+1}(t_{k+1})^2}{\|u[\phi_{k+1}]\|^2} \right) \phi_{k+1}(x) \\
&\le (1-c^2)^{k+1}\mu^2 \\
&= (1-c^2)(1-c^2)^k \mu^2 \\
&= (1-c^2)^k \mu^2 f_k^2(x) - c^2 (1-c^2)^k \mu^2 \phi_k(x) \\
&\le (1-c^2)^k \mu^2 f_k^2(x) - \left((1-c^2)^k \mu^2 - \frac{E_k(t_{k+1})^2}{\|u[\phi_k]\|^2} \right) \phi_k(x) \\
&= |u_k(x,0,t_{k+1})|^2
\end{aligned}$$

where we used (4.15) twice.

As for (4.6) we see that

$$\|u_k(t)\|^2 = \|u[f_{k,t}]\|^2 = (1-c^2)^k \mu^2 \|u[f_k]\|^2 - \left((1-c^2)^k \mu^2 \|u[\phi_k]\|^2 - E_k(t)^2 \right)$$

Thus

$$\left| \|u_k(t)\|^2 - E_k(t)^2 \right| = (1-c^2)^k \mu^2 \left| \|u[f_k]\|^2 - \|u[\phi_k]\|^2 \right| \le \varepsilon^2/4,$$

where we used (4.11). This and (4.13) give (4.6), as required. □

4.1 A proof of Corollary 1.4

Here we comment how to modify the proof of Theorem 1.3 to obtain Corollary 1.4.

We first focus on the case when $e(t)$ is not continuous. Since $e(t)$ is not increasing, it has $M \le \lceil 3e(0)/\varepsilon \rceil$ jumps by at least $\varepsilon/3$, where $\lceil w \rceil$ stands for the smallest integer larger or equal $w \in \mathbb{R}$. One can modify Lemma 4.1 to be able to assume that e in Theorem 1.3 is piecewise smooth with $(e(t)^2)' \le \zeta$, and have M jumps. For such e Theorem 1.3 remains valid, by incorporating the jumps into the the choice of t_k's (so that, in particular, the cardinality of $\{t_k\}$ would be $M + K$, rather than K). Corollary 1.4 then follows in the same way as Theorem 1.3.

As for the case when (1.10) is replaced by

$$\left| \|u(t)\|_p - e(t) \right| \le \varepsilon \qquad \text{for all } t \in [0, T],$$

one can also prove an appropriate modification of Theorem 1.3. A short sketch of such a modification is the following.

First, instead of ensuring that $(e^2(t))' \leq -\zeta$ prove an analogue of Lemma 4.1 to ensure that $(e^p(t))' \leq -\zeta$. Then copy the proof of Theorem 1.3 with the following changes.

1. Replace all the terms of the form $\| \cdot \|^2$ by $\| \cdot \|_p^p$, $E_k(t)^2$ by $E_k(t)^p$, $e(t)^2$ by $e(t)^p$ (for each t), μ^2 by μ^p.
2. Replace ε^2 by ε^p, and the calculation (4.7) by

$$\begin{aligned} |\|u(t)\|_p - e(t)| &= \left| \|u(t)\|_p^p - e(t)^p \right| / \sum_{k=0}^{p-1} \|u(t)\|_p^{p-k-1} e(t)^k \\ &\leq \left| \|u(t)\|_p^p - e(t)^p \right| / C(\|u(t)\|_p^{p-1} + e(t)^{p-1}) \\ &\leq \varepsilon^p / Ce(t)^{p-1} \leq C\varepsilon \qquad \text{for } t \in [t_0, t_K), \end{aligned}$$

3. Replace the $L^2(\mathbb{R}^3)$ norm in (4.8) and (4.9) by the $L^p(\mathbb{R}^3)$ norm.

5 Proof of Theorem 1.7

The proof of Theorem 1.7 is similar to the proof of the following weaker result, where the blow-up on a Cantor set is replaced by a blow-up on a single point $x_0 \in \mathbb{R}^3$.

Proposition 5.1. *Given an open set $W \subset \mathbb{R}^3$, $\varepsilon > 0$, $T > 0$ and a nonincreasing function $e\colon [0,T] \to [0,\infty)$ such that $e(t) \to 0$ as $t \to T$ there exists $\nu_0 > 0$ and a weak solution u of the NSI for all $\nu \in [0,\nu_0]$ such that $\operatorname{supp} u(t) \subset W$ for all $t \in [0,T]$*

$$|\|u(t)\| - e(t)| \leq \varepsilon \qquad \text{for all } t \in [0,T],$$

and that u is unbounded in any neighbourhood of (x_0, T) for some $x_0 \in W$.

Proof. By translation we can assume that W intersects the x_1 axis. Since W is open, there exists $\overline{x} = (x_1, 0, 0)$ and $R > 0$ such that $B(\overline{x}, R) \subset W$. Let $\mathfrak{u}$ and T_0 be given by (3.9) and let $T' \in (0,T)$ be the first time such that $e(t) \leq \varepsilon$ for $t \in [T', T]$. Fix $\lambda > 0$ large enough such that $T_0/\lambda^2 < T - T'$ and that

$$\operatorname{diam}(\operatorname{supp} u_0(t)) < R \quad \text{ and } \quad \|u_0(t)\| \leq \varepsilon/3 \qquad \text{for } t \in T_0/\lambda^2, \tag{5.1}$$

where

$$u_0(x,t) := \lambda \mathfrak{u}(\lambda x - a, \lambda^2 t).$$

Here $a \in \mathbb{R}^3$ is chosen such that $\operatorname{supp} u_0(0) \subset B(x_0, R)$. Note that, since $\mathfrak{u}(0)$ is axisymmetric and x_0 lies on the Ox_1 axis, we can assume that $a = (a_1, 0, 0)$ for some $a_1 \in \mathbb{R}$, so that $u_0(0)$ is axisymmetric.

Let $T'' := T - T_0/\lambda^2 \in (T', T)$, and let U_1 and the structure (v_1, f_1, ϕ_1) be such that

$$u[v_1, f_1] = u_0(0), \tag{5.2}$$

namely $U_1 := \lambda U$, $\phi_1 := \phi$, and v_1, f_1 are translations of λv, λf, where U and (v, f, ϕ) were constructed in Theorem 3.1).

Next let u_1 be given by Lemma 3.3 applied with $\varepsilon/3$, T'' and U_1, (v_1, f_1, ϕ_1).

Observe that u_1, $u_0(\cdot - T'')$ can be combined (u_1 for times less than T'' and $u_0(\cdot - T'')$ for times greater or equal T'') by (3.14). Thus if $e(0) \leq \varepsilon$ (that is if $T' = 0$) then

$$u(t) := \begin{cases} u_1(t) & t \in [0, T''], \\ u_0(t - T'') & t \in [T'', T] \end{cases}$$

satisfies all the claims of Theorem 1.3.

If $T' > 0$ then fix a rectangle $U_2 \Subset \mathbb{R}^2_+$ that is disjoint with U_1 and apply Theorem 1.3 with $\varepsilon/3$, T', U_2 and $e_2 := e - \varepsilon$ to obtain u_2. Extend $u_2(t)$ by zero for $t \in [T', T'']$. Then (using (3.16)) we see that

$$u(t) := \begin{cases} u_1(t) + u_2(t) & t \in [0, T''], \\ u_0(t - T'') & t \in [T'', T] \end{cases}$$

satisfies all the claims of Theorem 1.3. □

We now turn to the proof of Theorem 1.7. For this purpose we will need to use Scheffer's construction of a weak solution to the NSI with the singular set S satisfying $d_H(S) \in [\xi, 1]$ (that is Theorem 1.6), similarly as we used the construction with the blow-up at a single point in (5.1) above.

To this end we first introduce some handy notation related to constructions of Cantor sets.

5.1 Constructing a Cantor set

In this section, which is based on Section 5.1 from Ożański (2017), we discuss the general concept of constructing Cantor sets.

The problem of constructing Cantor sets is usually demonstrated in a one-dimensional setting using intervals, as in the following proposition.

Proposition 5.2. *Let $I \subset \mathbb{R}$ be an interval and let $\tau \in (0, 1)$, $M \in \mathbb{N}$ be such that $\tau M < 1$. Let $C_0 := I$ and consider the iteration in which in the j-th step ($j \geq 1$) the set C_j is obtained by replacing each interval J contained in the set C_{j-1} by M equidistant copies of τJ, each of which is contained in J, see for example Fig. 2. Then the limiting object*

$$C := \bigcap_{j \geq 0} C_j$$

is a Cantor set whose Hausdorff dimension equals $-\log M / \log \tau$.

Proof. See Example 4.5 in Falconer (2014) for a proof. □

Thus if $\tau \in (0, 1)$, $M \in \mathbb{N}$ satisfy

$$\tau^\xi M \geq 1 \quad \text{for some } \xi \in (0, 1),$$

we obtain a Cantor set C with

$$d_H(C) \geq \xi. \tag{5.3}$$

Note that both the above inequality and the constraint $\tau M < 1$ (which is necessary for the iteration described in the proposition above, see also Fig. 2) can be satisfied only for $\xi < 1$. In the remainder of this section we extend the result from the proposition above to the three-dimensional setting.

Let $G \subset \mathbb{R}^3$ be a compact set, $\tau \in (0, 1)$, $M \in \mathbb{N}$, $z = (z_1, z_2, 0) \in G$, $X > 0$ be such that

$$\tau^\xi M \geq 1, \qquad \tau M < 1 \tag{5.4}$$

and

$$\begin{aligned} &\{\Gamma_n(G)\}_{n=1,\ldots,M} \text{ is a family of pairwise disjoint subsets of } G, \\ &\text{with } \operatorname{conv}\{\Gamma_n(G) \colon n = 1, \ldots, M\} \subset G, \end{aligned} \tag{5.5}$$

where "conv" denotes the convex hull and

$$\Gamma_n(x) := \tau x + z + (n-1)(X, 0, 0).$$

Equivalently,

$$\Gamma_n(x_1, x_2, x_3) = (\beta_n(x_1), \gamma(x_2), \tau x_3), \tag{5.6}$$

where

$$\begin{cases} \beta_n(x) := \tau x + z_1 + (n-1)X, \\ \gamma(x) := \tau x + z_2, \end{cases} \qquad x \in \mathbb{R}, n = 1, \ldots, M.$$

Now for $j \geq 1$ let

$$M(j) := \{m = (m_1, \ldots, m_j) \colon m_1, \ldots, m_j \in \{1, \ldots, M\}\}$$

denote the set of multi-indices m. Note that in particular $M(1) = \{1, \ldots, M\}$. Informally speaking, each multiindex $m \in M(j)$ plays the role of a "coordinate" which let us identify any component of the set obtained in the j-th step of the construction of the Cantor set. Namely, letting

$$\pi_m := \beta_{m_1} \circ \ldots \circ \beta_{m_j}, \qquad m \in M(j),$$

that is

$$\pi_m(x) = \tau^j x + z_1 \frac{1-\tau^j}{1-\tau} + X \sum_{k=1}^{j} \tau^{k-1}(m_k - 1), \quad x \in \mathbb{R} \tag{5.7}$$

we see that the set C_j obtained in the j-th step of the construction of the Cantor set C (from the proposition above) can be expressed simply as

$$C_j := \bigcup_{m \in M(j)} \pi_m(I),$$

see Fig. 2. Moreover, each $\pi_m(I)$ can be identified by, roughly speaking, first choosing the m_1-th subinterval, then m_2-th subinterval, ... , up to m_j-th interval, where $m = (m_1, \ldots, m_j)$. This is demonstrated in Fig. 2 in the case when $m = (1,2) \in M(2)$.

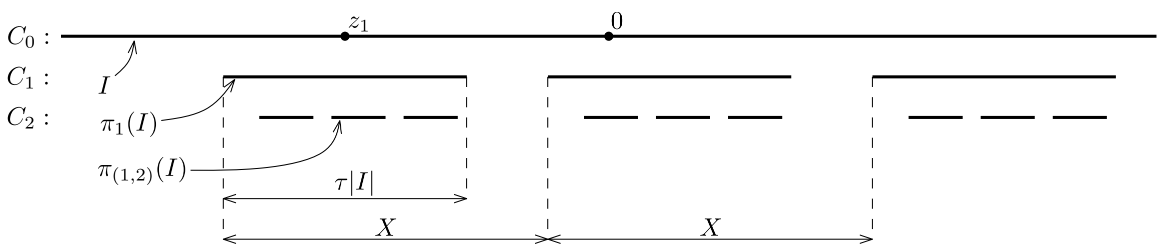

Figure 2: A construction of a Cantor set C on a line (here $M = 3$, $j = 0, 1, 2$).

In order to proceed with our construction of a Cantor set in three dimensions let

$$\Gamma_m(x_1, x_2, x_3) := \left(\pi_m(x_1), \gamma^j(x_2), \tau^j x_3\right). \tag{5.8}$$

Note that such a definition reduces to (5.6) in the case $j = 1$. If $j = 0$ then let $M(0)$ consist of only one element m_0 and let $\pi_{m_0} :=$ id. Moreover, if $m \in M(j)$ and $\overline{m} \in M(j-1)$ is its sub-multiindex, that is $\overline{m} = (m_1, \ldots, m_{j-1})$ ($\overline{m} = m_0$ if $j = 1$), then (5.5) gives

$$\Gamma_m(G) = \Gamma_{\overline{m}}(\Gamma_{m_j}(G)) \subset \Gamma_{\overline{m}}(G), \tag{5.9}$$

which is a three-dimensional equivalent of the relation $\pi_m(I) \subset \pi_{\overline{m}}(I)$ (see Fig. 2). The above inclusion and (5.5) gives that

$$\Gamma_m(G) \cap \Gamma_{\widetilde{m}}(G) = \emptyset \qquad \text{for } m, \widetilde{m} \in M(j),\ j \geq 1, \text{ with } m \neq \widetilde{m}. \tag{5.10}$$

Another consequence of (5.9) is that the family of sets

$$\left\{ \bigcup_{m \in M(j)} \Gamma_m(G) \right\}_j \quad \text{decreases as } j \text{ increases.} \tag{5.11}$$

Moreover, given j, each of the sets $\Gamma_m(G)$, $m \in M(j)$, is separated from the rest by at least $\tau^{j-1}\zeta$, where $\zeta > 0$ is the distance between $\Gamma_n(G)$ and $\Gamma_{n+1}(G)$, $n = 1, \ldots, M-1$ (recall (5.5)).

Taking the intersection in j we obtain

$$S' := \bigcap_{j\geq 0} \bigcup_{m\in M(j)} \Gamma_m(G), \tag{5.12}$$

and we now show that

$$\xi \leq d_H(S') \leq 1. \tag{5.13}$$

Noting that S' is a subset of a line, the upper bound is trivial. As for the lower bound note that

$$S' \supset \bigcap_{j\geq 0} \bigcup_{m\in M(j)} \Gamma_m \left(\mathrm{conv}\{\Gamma_n(G) \colon n = 1, \ldots, M\}\right) =: S''.$$

Thus, letting $I \subset \mathbb{R}$ be the orthogonal projection of $\mathrm{conv}\{\Gamma_n(G) \colon n = 1, \ldots, M\}$ onto the x_1 axis, we see that I is an interval (as the projection of a convex set; this is the reason why we put the extra requirement for the convex hull in (5.5)). Thus the orthogonal projection of S'' onto the x_1 axis is

$$\bigcap_{j\geq 0} \bigcup_{m\in M(j)} \pi_m(I) = C,$$

where C is as in the proposition above. Thus, since the orthogonal projection onto the x_1 axis is a Lipschitz map, we obtain $d_H(S'') \geq d_H(C)$ (as a property of Hausdorff dimension, see, for example, Proposition 3.3 in Falconer (2014)). Consequently

$$d_H(S') \geq d_H(S'') \geq d_H(C) \geq \xi,$$

as required (recall (5.3) for the last inequality).

5.2 Sketch of the Scheffer's construction with a blow-up on a Cantor set

Based on the discussion of Cantor sets above, we now briefly sketch the proof of Theorem 1.6. To this end we fix $\xi \in (0,1)$ and we state the analogue of Theorem 3.1 in the case of the blow-up on a Cantor set.

Theorem 5.3. *There exist a set $U \Subset P$, a structure (v, f, ϕ),$T > 0$, $M \in \mathbb{N}$, $\tau \in (0,1)$, $z = (z_1, z_2, 0) \in G := R(\overline{U})$, $X > 0$, $\nu_0 > 0$ with the following properties: the relations* (5.4) *and* (5.5) *are satisfied and, for each $j \geq 0$ there exist smooth time-dependent extensions $v_t^{(j)}$, $f_t^{(j)}$ ($t \in [0,T]$) of v, f, respectively, such that $v_0^{(j)} = v$, $f_0^{(j)} = f$, $(v_t^{(j)}, f_t^{(j)}, \phi)$ is a structure on U for each $t \in [0,T]$. Moreover*

$$w^{(j)}(x,t) := \sum_{m\in M(j)} u[v_t^{(j)}, f_t^{(j)}]\left(\pi_m^{-1}(\tau^j x_1), x_2, x_3\right) \tag{5.14}$$

satisfies the NSI (1.3) *in the classical sense for all $\nu \in [0, \nu_0]$ and $t \in [0,T]$, is bounded (on $\mathbb{R}^3 \times [0,T]$), and*

$$\left| w^{(j)}(\tau^{-j}\pi_m(y_1), \gamma(y_2), \tau y_3, T)\right| \geq \tau^{-1} |u[v,f](y)| \qquad \text{for } y \in \mathbb{R}^3, m \in M(j+1). \tag{5.15}$$

Proof. See Section 5 in Ożański (2017); there the so-called *geometric arrangement* in the beginning of Section 5.1 gives U, (v, f, ϕ), T_0, M, τ, z and $X > 0$, and Proposition 5.2 constructs $w^{(j)}$ (which is denoted by $v^{(j)}$). □

Observe that the claim of Theorem 3.1 (that is the vector field $u(t)$ in the statement of Theorem 3.1) is recovered by letting $M := 1$ and $u(t) := w^{(0)}(t)$.

Given the theorem above we can easily obtain Scheffer's construction with a blow-up on a Cantor set (that is a solution $\mathfrak{u}$ to Theorem 1.6).

Indeed, let

$$u^{(j)}(x_1,x_2,x_3,t):=\tau^{-j}w^{(j)}(\tau^{-j}x_1,\gamma^{-j}(x_2),\tau^{-j}x_3,\tau^{-2j}(t-t_j)), \tag{5.16}$$

where $t_0:=0$ and $t_j:=T\sum_{k=0}^{j-1}\tau^{2k}$, as in (3.5). Observe that

$$\operatorname{supp} u^{(j)}(t)=\bigcup_{m\in M(j)}\Gamma_m(G),\qquad t\in[t_j,t_{j+1}]$$

(instead of $\Gamma_1^j(G)$, which is the case in the Scheffer's construction with point blow-up; recall (3.7)), which shrinks (as $t\to T_0^-$) to the Cantor set S' (recall (5.12)), whose Hausdorff dimension is greater of equal ξ (recall (5.13)). In fact, generalising the arguments from Section 3.1 we can show that $u^{(j)}$ satisfies the NSI in the classical sense for all $\nu\in[0,\nu_0]$ and $t\in[t_j,t_{j+1}]$,

$$\left|u^{(j)}(x,t_j)\right|\le\left|u^{(j-1)}(x,t_j)\right|,\qquad x\in\mathbb{R}^3, j\ge 1, \tag{5.17}$$

and that consequently the vector field

$$\mathfrak{u}(t):=\begin{cases}u^{(j)}(t) & \text{if } t\in[t_j,t_{j+1}) \text{ for some } j\ge 0,\\ 0 & \text{if } t\ge T_0\end{cases} \tag{5.18}$$

satisfies all the claims of Theorem 1.6. We refer the reader to Section 5.2 in Ożański (2017) to a more detailed explanation. Here we prove merely (5.17), which at least explains the condition (5.15).

It is enough to consider $x\in\bigcup_{m\in M(j)}\Gamma_m(G)$, as otherwise the claim is trivial. Thus suppose that $x=\Gamma_m(y)$ for some $m\in M(j)$ and $y\in G$. We obtain

$$\begin{aligned}\left|u^{(j)}(x,t_j)\right|&=\tau^{-j}\left|w^{(j)}(\tau^{-j}x_1,\gamma^{-j}(x_2),\tau^{-j}x_3,0)\right|\\&=\tau^{-j}\sum_{\widetilde{m}\in M(j)}\left|u[v,f]\left(\Gamma_{\widetilde{m}}^{-1}(x)\right)\right|\\&=\tau^{-j}|u[v,f](y)|\\&\le\tau^{-(j-1)}\left|w^{(j-1)}\left(\tau^{-(j-1)}\pi_m(y_1),\gamma(y_2),\tau y_3,T\right)\right|\\&=\left|u^{(j-1)}\left(\pi_m(y_1),\gamma^j(y_2),\tau^j y_3,t_j\right)\right|\\&=\left|u^{(j-1)}(x,t_j)\right|,\end{aligned}$$

as required, where we used (5.10) (so that $\Gamma_{\widetilde{m}}^{-1}(\Gamma_m(y))=y\,\chi_{\widetilde{m}=m}$) in the third equality and (5.15) in the inequality (recall also the definitions (5.16), (5.14), (5.8) of $u^{(j)}$, $w^{(j)}$, Γ_m, respectively).

Finally, we note that

$$\|\mathfrak{u}(t)\|\to 0\qquad \text{as } t\to T_0^-, \tag{5.19}$$

which can be shown in the same way as (3.10) by using boundedness of $w^{(j)}$ and the property that $M\tau<1$. Indeed $w^{(j)}$ consists of M^j disjointly supported and bounded vector fields (recall (5.14)). Therefore $u^{(j)}$ consists of M^j vector fields that are of order τ^{-j} and disjointly supported on sets of the size of order τ^j. Therefore for $t\in[t_j,t_{j+1}]$,

$$\|u^{(j)}(t)\|^2\le CM^j\tau^{(3-2)j}=C(M\tau)^j,$$

which decays to 0 as $j\to\infty$.

5.3 Proof of Theorem 1.7

Given $\mathfrak{u}$ constructed in the previous section, Theorem 1.7 follows in the same way as Proposition 5.1.

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A A sharpening of Lemma 2.2

Here we prove Lemma 4.2 (the sharpening of the "edge effects" Lemma 2.2), which was used in the proof above.

In order to prove the lemma we will need a certain generalised Mean Value Theorem. For $g\colon \mathbb{R} \to \mathbb{R}$ let $g[a,b]$ denote the finite difference of g on $[a,b]$,

$$g[a,b] := \frac{g(a)-g(b)}{a-b}$$

and let $g[a,b,c]$ denote the finite difference of $g[\cdot,b]$ on $[a,c]$,

$$g[a,b,c] := \left(\frac{g(a)-g(b)}{a-b} - \frac{g(c)-g(b)}{c-b} \right) /(a-c).$$

Lemma A.1 (generalised Mean Value Theorem)**.** *If $a<b<c$, g is continuous in $[a,c]$ and twice differentiable in (a,c) then there exists $\xi \in (a,c)$ such that $g[a,b,c] = g''(\xi)/2$.*

Proof. We follow the argument of Theorem 4.2 in Conte & de Boor (1972). Let

$$p(x) := g[a,b,c](x-b)(x-c) + g[b,c](x-c) + g(c).$$

Then p is a quadratic polynomial approximating g at a,b,c, that is $p(a)=g(a)$, $p(b)=g(b)$, $p(c)=g(c)$. Thus the error function $e(x) := g(x)-p(x)$ has at least 3 zeros in $[a,c]$. A repeated application of Rolle's theorem gives that e'' has at least one zero in (a,c). In other words, there exists $\xi \in (a,c)$ such that $g''(\xi) = p''(\xi) = 2g[a,b,c]$. □

Corollary A.2. *If $g \in C^3(a-\delta, a+\delta)$ is such that $g=0$ on $(a-\delta,a]$ and $g'''>0$ on $(a,a+\delta)$ for some $a\in\mathbb{R}$, $\delta>0$ then*

$$\begin{cases} g''(x) > 0, \\ 0 < g'(x) < (x-a)g''(x), \\ g(x) < (x-a)^2 g''(x) \end{cases} \qquad \textit{for } x \in (a,a+\delta).$$

Proof. Since $g'''>0$ on $(a,a+\delta)$ we see that g'' is positive and increasing on this interval and so the first two claims follow for g from the Mean Value Theorem. The last claim follows from the lemma above by noting that $2a-x \in (a-\delta,a]$, and so

$$\begin{aligned} g(x) &= g(2a-x) - 2g(a) + g(x) = 2(x-a)^2 g[2a-x,a,x] \\ &= (x-a)^2 g''(\xi) < (x-a)^2 g''(x), \end{aligned}$$

where $\xi \in (2a-x, x)$. □

We can now prove Lemma 4.2; that is, given $a>0$, $\eta>0$ and an open rectangle $U \Subset \mathbb{R}^2_+$ that is at least a away from the x_1 axis (i.e. $U = (a_1,b_1)\times(a_2,b_2)$ with $a_2 > a$) we construct $f \in C_0^\infty(\mathbb{R}^2_+;[0,1])$ such that

$$\operatorname{supp} f = \overline{U}, \quad f>0 \text{ in } U \text{ with } f=1 \text{ on } U_\eta,$$

$$Lf > 0 \quad \text{in } U \setminus U_{c'\eta}, \text{ with } f > c \text{ in } U_{c'\eta/2},$$

where $c, c' \in (0,1/2)$ depend only on a.

Proof of Lemma 4.2. Let $h \in C^\infty(\mathbb{R};[0,1])$ be a nondecreasing function such that

$$h(x) = \begin{cases} 0 & x \le 0, \\ \mathrm{e}^{-1/x^2} & x \in (0,1/2), \\ 1 & x \ge 1. \end{cases}$$

Let

$$C_h := \|h\|_{W^{2,\infty}(\mathbb{R})} < \infty$$

Observe that $h''' > 0$ on $(0,1/2)$. Let $h_\eta(x) := h(x/\eta)$ and

$$f(x_1,x_2) := f_1(x_1)f_2(x_2),$$

where

$$f_i(x) := h_\eta(x-a_i)h_\eta(b_i-x), \quad i=1,2,$$

see Fig. 3. Clearly

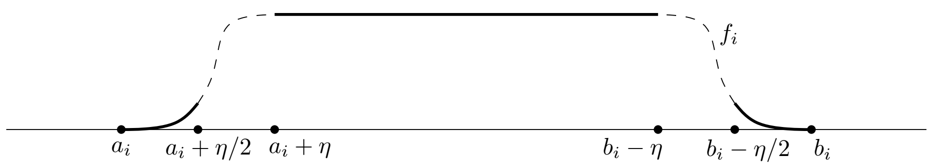

Figure 3: The f_i's, $i=1,2$.

$$f_i''' > 0 \text{ on } (a_i, a_i+\eta/2) \quad \text{and} \quad f_i''' < 0 \text{ on } (b_i-\eta/2, b_i), \quad i=1,2.$$

Moreover $\operatorname{supp} f = \overline{U}$, $f > 0$ in U, and $f = 1$ on U_η. We will show that

$$Lf > 0 \quad \text{on} \quad U \setminus U_{\eta'} \tag{A.1}$$

for

$$\eta' := c'\,\eta, \tag{A.2}$$

where

$$c' := \frac{1}{3}\min\{1,a\}\min\left\{1, \frac{1}{2\sqrt{C_h}}\mathrm{e}^{-9/2\min\{1,a^2\}}\right\}. \tag{A.3}$$

Note that this implies $\eta' \le \eta/3$, and so, by construction, $f > \mathrm{e}^{-8/(c')^2} =: c$ in $U \setminus U_{\eta'}$. Thus the proof of the lemma is complete when we show (A.1).

To this end let

$$\eta'' := \frac{\eta}{3}\min\{1,a\}. \tag{A.4}$$

Obviously $\eta' \le \eta'' \le \eta \le 1$. Letting

$$\begin{aligned} g_1(x_1) &:= f_1''(x_1), \\ g_2(x_2) &:= f_2''(x_2) + f_2'(x_2)/x_2 - f_2(x_2)/x_2^2, \end{aligned}$$

we see that

$$\begin{aligned} Lf(x_1,x_2) &= f_1''(x_1)f_2(x_2) + f_1(x_1)f_2''(x_2) + f_1(x_1)f_2'(x_2)/x_2 - f_1(x_1)f_2(x_2)/x_2^2 \\ &= g_1(x_1)f_2(x_2) + f_1(x_1)g_2(x_2). \end{aligned}$$

We will show that the expression on the right-hand side above is positive in $U \setminus U_{\eta'}$. For this we first show the *claim*:

$$g_2 > f_2''/4 > 0 \qquad \text{on } (a_2, a_2+\eta'') \cup (b_2-\eta'', b_2). \tag{A.5}$$

The claim follows from the corollary of the generalised Mean Value Theorem (see Corollary A.2) by noting that (A.4) gives in particular $\eta'' < a_2/2$, $\eta'' < \eta/2$ and $\eta''/(b_2-\eta'') < 1/2$, and so

$$\begin{aligned} g_2(x_2) &> f_2''(x_2) - f_2(x_2)/x_2^2 > f_2''(x_2)\left(1 - \left(\frac{x_2-a_2}{x_2}\right)^2\right) \\ &> f_2''(x_2)\left(1-\left(\frac{\eta''}{a_2}\right)^2\right) > \frac{3}{4} f_2''(x_2) > \frac{1}{4} f_2''(x_2) > 0 \end{aligned}$$

for $x_2 \in (a_2, a_2+d)$, and

$$\begin{aligned} g_2(x_2) &= f_2''(x_2) + f_2'(x_2)/x_2 - f_2(x_2)/x_2^2 > f_2''(x_2)\left(1 + \frac{x_2-b_2}{x_2} - \left(\frac{x_2-b_2}{x_2}\right)^2\right) \\ &> f_2''(x_2)\left(1 - \frac{\eta''}{b_2-\eta''} - \left(\frac{\eta''}{b_2-\eta''}\right)^2\right) > f_2''(x_2)/4 > 0 \end{aligned}$$

for $x_2 \in (b_2 - \eta'', b_2)$.

Using the claim we see that g_i, f_i are positive on $(a_i, a_i+\eta'') \cup (b_i - \eta'', b_i)$, $i = 1, 2$. Thus

$$Lf > 0 \quad \text{in } \left((a_1, a_1+\eta'') \cup (b_1-\eta'', b_1)\right) \times \left((a_2, a_2+\eta'') \cup (b_2-\eta'', b_2)\right),$$

that is in the "η''-corners" of U, see Fig. 4.

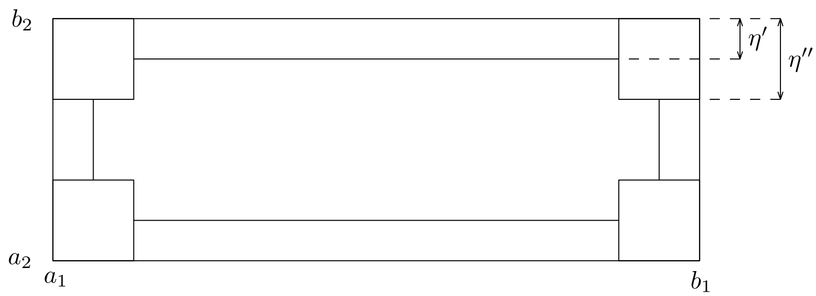

Figure 4: The "η''-corners" and "η'-stripes".

Now let

$$\begin{aligned} m &:= \mathrm{e}^{-9/\min\{1,a^2\}}, \\ M &:= \frac{3\,C_h}{\eta^2 \min\{1,a^2\}}. \end{aligned}$$

A direct calculation gives that

$$f_i \geq m, |g_i| \leq M \quad \text{in } [a_i+\eta'', b_i-\eta''], i = 1,2,$$

and

$$\frac{m}{4} - (\eta')^2 M > 0.$$

(The last property is the consequence of the appearance of the second minimum in (A.3).)

We will show that

$$\begin{aligned} Lf > 0 \quad &\text{in } [a_1+\eta'', b_1-\eta''] \times \left((a_2, a_2+\eta') \cup (b_2-\eta', b_2)\right) \\ &\text{and in } \left((a_1, a_1+\eta') \cup (b_1-\eta', b_1)\right) \times [a_2+\eta'', b_2-\eta''], \end{aligned} \tag{A.6}$$

that is in the "η'-strips" at ∂U between the η''-corners, see Fig. 4. This will finish the proof as the η'-strips together with the η''-corners contain $U \setminus U_{\eta'}$.

In order to prove (A.6) let first $x_1 \in [a_1+\eta'', b_1-\eta'']$ and $x_2 \in (a_2, a_2+\eta')$. Then $g_1(x_1) > -M$, $g_2(x_2) > f_2''(x_2)$ (from (A.5)), $f_2(x_2) < (x_2-a_2)^2 f_2''(x_2)$ (from the generalised Mean Value Theorem, see Corollary A.2), $f_1(x_1) > m$, and so

$$\begin{aligned} Lf(x_1,x_2) &= g_1(x_1)f_2(x_2) + f_1(x_1)g_2(x_2) > -Mf_2(x_2) + f_1(x_1)f_2''(x_2)/4 \\ &> f_2''(x_2)\left(-M(x_2-a_2)^2 + m/4\right) > f_2''(x_2)\left(m/4 - M(\eta')^2\right) > 0. \end{aligned}$$

As for $x_2 \in (b_2-\eta', b_2)$, simply replace a_2 in the above calculation by b_2. The opposite case, that is the case $x_1 \in (a_1, a_1+\eta') \cup (b_1-\eta', b_1)$, $x_2 \in [a_2+\eta'', b_2-\eta'']$, follows in the same way. □

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