Infinitely many reducts of homogeneous structures

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This work is dedicated to Tamás E. Schmidt

Abstract. It is shown that the countably infinite dimensional pointed vector space (the vector space equipped with a constant) over a finite field has infinitely many first order definable reducts. This implies that the countable homogeneous Boolean-algebra has infinitely many reducts.

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1. Introduction

We consider structures over finite relational languages (for short, *relational structures*). A relational structure Γ is a *reduct* of the structure Δ if they have the same domain and every relation of Γ has a first-order definition in Δ. Two structures are called first-order equivalent if each is a reduct of the other. The relation "is a reduct of" is clearly transitive, and so induces a partial order on the class of structures on a given domain over a given language.

A relational structure Δ is said to be *homogeneous* if any isomorphism between finite induced substructures can be extended to an automorphism of Δ. Note that a homogeneous structure has the property that the number of isomorphism types of n-element substructure it contains is bounded above by the exponential of a polynomial in n.

A structure Δ is *ω-categorical* if it is (up to isomorphism) the unique countable model of its first-order theory, or equivalently, if its n-types are finite in number and coincide with the orbits of its automorphism group on

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n-tuples. Thus, for example, a homogeneous structure over a finite language is ω-categorical. If Δ is ω-categorical, then there is a bijection between reducts of Δ and closed overgroups of $\mathrm{Aut}(\Delta)$ in the symmetric group on the domain (with the topology of pointwise convergence), see [10]. Thus, finding reducts of Δ is equivalent to finding closed overgroups of $\mathrm{Aut}(\Delta)$. (Note that the ordering of reducts is the reverse of the ordering of closed overgroups.)

Simon Thomas [19] made the intriguing conjecture that a countable homogeneous structure over a finite relational language has only finitely many inequivalent reducts. This conjecture has been verified for several well-known homogeneous structures, but there seems to be no general progress towards proving it. In [8] the reducts of the dense linear order $(\mathbb{Q}, \leq)$ are determined (though the language of reducts is not used there). Thomas himself determined the reducts of the random graph (the unique countable homogeneous universal graph) and of random hypergraphs [19, 20]. Curiously, both the random graph and the dense linear order have 5 reducts. In [11] it is shown that the "pointed" linear order has 116 reducts. Thus adding a constant to $(\mathbb{Q}, \leq)$, the number of reducts can increase significantly. The Henson graphs (the countable homogeneous universal K_n-free graphs for $n \geq 3$ [9]) have no nontrivial reducts [19]. Later, in [4] and [5], a general technique was introduced to investigate first order definable reducts of homogeneous structures on a finite language. Although the strategy works only under very special conditions, it was possible to determine all reducts in some cases. Applying these techniques several structures have been analyzed: the random poset [16], [15], and the random graph revisited [3]. They all have finitely many reducts. For the pointed Henson graphs (H_n, C), Pongrácz [17] showed that for $n = 3$ there are 13, while for $n > 3$ there are 16, reducts of (H_n, C). The proof of this highly non-trivial result requires all known tricks and techniques. Similarly, in [6], the 42 proper reducts of the ordered random graph are determined in 42 pages. In [6] it is mentioned that we do not even know how to show that the lattice of reducts has only finitely many atoms, or no infinite ascending or descending chains.

In order to learn more, it seems to be unavoidable to test the conjecture for more of the classical structures from model theory, independently of whether or not we believe the conjecture. We note that the result of Ahlbrandt and Ziegler [1, Theorem 3.5] addresses the same issue.

The countable dimensional vector spaces over finite fields and the countable atomless Boolean algebra are ω-categorical. They are not homogeneous on a finite relational language – for vector spaces this can be seen by noting that, if the maximum arity of relations in the language is n, then $n+1$ linearly independent vectors and $n+1$ vectors with sum zero and all proper subsets independent are isomorphic as substructures but not of the same type. However, they are of finite signature, and the vector spaces share with homogeneous structures the property that the number of n-types is bounded by the exponential of a polynomial in n. The first-order definable reducts of the countable vector space and the symplectic space over $\mathbb{F}_2$ were determined in [7]. There are finitely many of them. The proper reducts of the vector space are the affine space and the stabilizer of 0 in the symmetric group. In addition the symplectic space, the vector space endowed with a symplectic form $\cdot$ has one additional reduct, the vector space with the ternary relation

$$\{(a,b,c) \mid a \cdot b + a \cdot c + b \cdot c = 0, a, b, c \text{ linearly independent}\}.$$

In this paper we show that the statement of Thomas' conjecture is not true for ω-categorical structures of finite signature. We present infinitely many reducts of the pointed vector spaces over finite fields, and of the homogeneous Boolean algebra.

Our construction for pointed vector spaces over finite fields can be reformulated in terms of infinite analogues of the Reed–Muller codes [14, 18], and are also closely connected with the results of Ahlbrandt and Ziegler [1].

We are grateful to the referee for helpful comments, including the reference to [1].

2. The construction

Let V be a countably infinite dimensional vector space over the 2-element field $\mathbb{F}_2$, and $0 \neq C \in V$. We shall investigate the pointed vector space (V, C) that is obtained by adding C as a constant to V. The automorphism group of V is $\mathrm{Aut}(V) = \mathrm{GL}(V)$ and the automorphism group of (V, C) is the stabilizer of C in $\mathrm{GL}(V)$ so $\mathrm{Aut}(V, C) = \mathrm{GL}(V)_C$.

We are going to consider mappings which interchange the two elements of some cosets of $\langle C \rangle$. In order to specify this, we fix a 1-codimensional subspace $W < V$ not containing C. Then, $V = \langle W, C \rangle$ and $V = \{w, w + C \,|\, w \in W\}$, so any coset of $\langle C \rangle$ contains a unique vector in W. We can specify our maps by giving the set of vectors of W in which the relevant cosets meet W.

For $\varphi \in \mathrm{GL}(V)_C$, let $W_\varphi = W^\varphi \cap W$. We have the following two cases:

- If $W_\varphi = W^\varphi = W$, then $(w + C)^\varphi = w^\varphi + C$ for every $w \in W$. In this case, let $\overline{\varphi} = \varphi$.
- Otherwise, $V = W_\varphi \cup (W \setminus W_\varphi) \cup (W^\varphi \setminus W_\varphi) \cup (V \setminus (W \cup W^\varphi)$. In this case, let $\overline{\varphi}$ be defined in the following way:
  - $v^{\overline{\varphi}} = v^\varphi$ if $v^\varphi \in W_\varphi \cup (W_\varphi + C)$ and
  - $v^{\overline{\varphi}} = v^\varphi + C$ otherwise.

The map $\overline{\varphi}$ is an automorphism of (V, C) mapping W to W. The map $\overline{\varphi}$ is uniquely determined by $\tilde{\varphi} = \overline{\varphi}|_W$. For $\sigma \in \mathrm{Aut}(W)$ let σ^V denote this extension:

- $v^{\sigma^V} = v^\sigma$ if $v \in W$ and
- $v^{\sigma^V} = (v + C)^\sigma + C$ if $v \notin W$

Now, let $U < W$ such that U has codimension 2 in V (that is, $|V : U| = 4$), and let $\sigma \in \mathrm{Aut}\, W$. Then there is a unique $\sigma_U \neq \sigma^V \in \mathrm{Aut}(V, C)$ defined by

- $v^{\sigma_U} = v^{\sigma^V} + C$ if $v^{\sigma^V} \in (W \setminus U) \cup (W \setminus U + C)$.

– $v^{\sigma_U} = v^{\sigma^V}$ otherwise.

Definition 2.1. For $S \subseteq W$ let $h_S \in \mathrm{Sym}(V)$ defined by

$$v^{h_S} = \begin{cases} v + C & \text{if } v \in S \cup (S + C), \\ v & \text{otherwise.} \end{cases}$$

We extend the definiton h_S to arbitrary subsets $S \subset V$ by defining $h_S = h_T$, where T is the image of S under the map taking both w and $w + C$ to w for all $w \in W$. If $v \in S \cup (S + C)$ then we say that h_S *flips* v.

Set $H = \{h_S \mid S \subseteq W\}$. Now H is a subgroup of $\mathrm{Sym}(V)$ isomorphic to Z_2^W. (This follows immediately from Lemma 2.3(dif) below.) With the above notations, $\sigma_U = \sigma^V \circ h_{W \setminus U}$. As noted, the subgroup H is an elementary Abelian 2-group (in particular, it is commutative), and it is normalized by $\mathrm{Aut}(W)^V := \{\sigma^V \mid \sigma \in \mathrm{Aut}(W)\}$. So there is a canonical embedding $\mathrm{Aut}\, W \rtimes Z_2^W \to \langle \mathrm{Aut}\, V, H\rangle$ given by $(\sigma, h) \to \sigma^V \circ h$. Moreover, with the above notations we have $\varphi = \overline{\varphi} \circ h_{W \setminus W_\varphi}$ for every $\varphi \in \mathrm{Aut}\, V$.

Definition 2.2. Given a vector space $\mathbb{F}_2^\lambda$ over $\mathbb{F}_2$ which contains C, and a subspace W' of codimension n in $\mathbb{F}_2^\lambda$, let $H_n = \langle \mathrm{Aut}(\mathbb{F}_2^\lambda), h_{W' \cap W}\rangle$.

Since $\mathrm{Aut}(\mathbb{F}_2^\lambda)$ acts transitively on subspaces of codimension n containing C, we see that H_n does not depend on W_n, only on the codimension n.

Lemma 2.3 (Subspace calculus). *Let V be a vector space over $\mathbb{F}_2$ and let $\dim V = \lambda > n$ where λ is either finite or $\lambda = \omega$*

- (dif) *Let $X, Y \subseteq W$. Then $h_X \circ h_Y = h_{X \triangle Y}$, where $\triangle$ denotes the symmetric difference of subsets.*
- (gen) *For every $h \in H_n \cap H$ there are $W_i < \mathbb{F}_2^\lambda$ ($i = 1, 2, \ldots, k$) such that $h = \prod h_{W_i}$.*
- (aff) *If $W' < \mathbb{F}_2^\lambda$, $\mathrm{codim}\, W' = n$ and $w \in \mathbb{F}_2^\lambda$, then $h_{W'+w} \in H_n$.*
- (trans) *If $h_X \in H_n$ and $w \in \mathbb{F}_2^\lambda$, then $h_{X+w} \in H_n$.*
- (codim) *If $w \in W' < \mathbb{F}_2^\lambda$ and $\mathrm{codim}\, W' < n$, then $h_{W'}, h_{W'+w} \in H_n$.*
- (two) *If $a, b \in U < \mathbb{F}_2^\lambda$ and $\dim U = n + 1$, then there is an element $g \in H_n$ such that $g|_U = h_{\{a,b\}}|_U$.*
- (even) *If $S \subseteq U < \mathbb{F}_2^\lambda$, where $|S|$ is divisible by 2 and $\dim U = n+1$, then there is an element $g \in H_n$ such that $g|_U = h_S|_U$*
- (one) *If $a \in U < \mathbb{F}_2^\lambda$ and $\dim U = n$, then there is an element $g \in H_n$ such that $g|_U = h_{\{a\}}|_U$*
- (odd) *If $S \subseteq U < \mathbb{F}_2^\lambda$ and $\dim U = n$, then there is an element $g \in H_n$ such that $g|_U = h_S|_U$.*

Proof. If we compose the two group elements, the elements of the intersection of X and Y are flipped twice, hence are fixed by the composition. The elements of $X \setminus Y$ and $Y \setminus X$ are flipped once. This gives (dif).

Item (gen) is obvious from the definition of H_n and the fact that the group H is normalized by $\mathrm{Aut}(V)$.

For (aff), if $w \in W'$ then $h_{W'+w} = h_{W'}$ and we are done. If $w \notin W'$, then let $W' = \langle W_1, a\rangle$, where $a \notin W_1$. Then $\langle W_1, w\rangle$ and $\langle W_1, a+w\rangle$ are n-codimensional subspaces of V. Thus $h_{\langle W_1,w\rangle}$ and $h_{\langle W_1,a+w\rangle} \in H_n$. Now, $h_{\langle W_1,w\rangle} \circ h_{\langle W_1,a+w\rangle} = h_{W'+v}$.

For $m > n$ every n-codimensional subspace is the disjoint union of m-codimensional affine subspaces. Hence (codim) follows from (aff) and (dif).

By (gen) let $h_X = \prod h_{W_i}$. Then by item (aff) $h_{W_i+w} \in H_n$ and $h_{X+x} = \prod h_{W_i+x}$ thus (trans) follows.

For (two) let U be an n-codimensional subspace of V such that $U \cap W = \{0, a+b\}$. Then h_{U+a} satisfies the conditions.

Item (even) easily follows from (two).

For (one) let U be an n-codimensional subspace of V such that $U \cap W = \{0\}$. Then h_{U+a} satisfies the conditions.

For (odd) let $S \subset U$ and $g = \prod\limits_{a \in S} h_{U+a}$, where U is the subspace from (odd). Then $U^g = U^{h_S}$.

□

Definition 2.4. Let V be a countably infinite dimensional vector space over the 2-element field, $C \in V$ and let $W < V$ be a 1-codimensional subspace, as above, not containing C. Let $\mathcal{R}_n$ denote the relation consisting of all 2^n-tuples $(x_1, x_2, \ldots, x_{2^n})$ such that $\{x_i, x_i + C \mid 1 \leq i \leq 2^n\}$ is an affine subspace of V and $|\{x_i | \, 1 \leq i \leq 2^n\} \cap W|$ is even.

Proposition 2.5. *Let $n \geq 1$. Then H_n preserves $\mathcal{R}_m$ if and only if $m \geq n+1$.*

Proof. Assume that $m \leq n$. Let $W_{\overline{n}}$ be an n-codimensional subspace and U_m be an m dimensional subspace such that $W_{\overline{n}} \cap U_m = \{0\}$. Such a pair of subspaces exists by the conditions on the dimensions. Let $(x_1, \ldots, x_{2^m})$ be an enumeration of the elements of U_m. Clearly, $(x_1, \ldots, x_{2^m}) \in \mathcal{R}_m$. Let $g = h_{W_n}$. Now, $g \in H_n$ and $v^g = v + C$ holds only for 0 from U_m and for $0 \neq v \in U_m$ we have $v^h = v$. That is $(x_1^g, \ldots, x_{2^m}^g)$ contains $2^m - 1$, in particular odd many elements from W and so g does not preserve $\mathcal{R}_m$.

For the other direction suppose $m \geq n+1$. Let $\varphi \in \mathrm{Aut}(V, C)$ and consider the canonical form $\varphi = \overline{\varphi} \circ h_{W \setminus W_\varphi}$. The map $\overline{\varphi}$ preserves W, hence preserves $\mathcal{R}_k$, as well, for arbitrary k. By item (codim) of Theorem 2.3 we have $h_{W \setminus W_\varphi} \in H_n$, hence it is enough to show that H_{W_1} preserves $\mathcal{R}_m$ for every n-codimensional subspace W_1.

So suppose $(x_1, \ldots, x_{2^m}) \in \mathcal{R}_m$, and let $S = \{x_1, \ldots, x_{2^m}\}$. Then $S \cup (S + C)$ is an affine subspace of V and either $x_i \in W$ or $x_i + C \in W$. Let $S_C = \{x_i \,|\, x_i \notin W\}$. Now, $U = (S_C + C) \cup (S \setminus S_C)$ is an n-dimensional subspace of W and $S = U^{h_{S_C}}$, that is S is obtained from the affine subspace U by applying h_{S_C}. Also, by definition of $\mathcal{R}_m$ we have that $|S \cap U| = |S \cap W|$ is even. Now, let W_1 be an arbitrary n-codimensional subspace of W and $h = h_{W_1}$. Then $\dim(W_1 \cap U) > 1$, hence $|W_1 \cap U|$ is even. Moreover, $S^h = U^{h_{S_C} h}$ so by item (dif) of Theorem 2.3 we have that $|S^h \cap W| = |S^h \cap U| = |U^{h_{S_C} h} \cap U| = |U^{h_{S_C}} \triangle (W_1 \cap U)| = |S \triangle (W_1 \cap U)|$ is even. We have

$\{x_i, x_i + C\} = \{x_i^h, x_i^h + C\}$, hence $(x_1^h, \dots, x_{2^m}^h) \in \mathcal{R}_m$. Thus h_{W_1} preserves $\mathcal{R}_m$ and by item (gen) of Lemma 2.3 H_n preserves $\mathcal{R}_m$, as well.

□

Proposition 2.6. *For the closure of the subgroups H_i we have $\overline{H}_1 \lneq \overline{H}_2 \lneq \dots \lneq \overline{H}_n \lneq \dots$*

Proof. By item (dim) of Lemma 2.3 if $U < V$ and $m = \dim U \geq n$ then $h_U \in H_n$. By item (gen) of Lemma 2.3 these elements generate H_m. Thus $H_m \leq H_n$ and so $\overline{H}_m \leq \overline{H}_n$ holds. By Theorem 2.5 the group H_n preserves the relations $\mathcal{R}_{n+1}$ and so does its closure, $\overline{H}_n$. On the other hand H_{n+1} does not preserve $\mathcal{R}_{n+1}$, hence $\overline{H}_{n+1}$ does not preserve $\mathcal{R}_{n+1}$, either. Thus $H_n \neq H_{n+1}$ and the statement holds.

□

Remark 2.7. By using the observation that each automorphism φ of (V, C) can be written as $\varphi = \overline{\varphi} \circ h_{W \setminus W_\varphi}$, it is easy to see that $H_1 = H_0 = \mathrm{Aut}(V, C) \cup \mathrm{Aut}(V, C) h_W$. In particular $\mathrm{Aut}(V, C)$ is a normal subgroup of index 2 in H_1.

Theorem 2.8. *The lattice of first-order definable reducts of the pointed homogeneous vector space $(\mathbb{F}_2^\omega, C)$ contains an infinite descending chain. In particular $(\mathbb{F}_2^\omega, C)$ has infinitely many first-order definable reducts.*

Proof. The first-order definable reducts of a homogeneous structure are in a one-to-one order-reversing correspondence with the closed supergroups of its automorphism groups. Proposition 2.6 implies the statement. □

Finally, we give the relational description of the groups H_n.

Theorem 2.9. *Let $\pi \in \mathrm{Sym}(V, C)$. Then $\pi \in \overline{H}_n$ if and only if π preserves $\mathcal{R}_{n+1}$.*

Proof. The group H_n preserves $\mathcal{R}_{n+1}$ by Proposition 2.5. Now, let π be a permutation preserving $\mathcal{R}_{n+1}$ and $U \leq V$ be a finite dimensional vector space of V. As automorphisms preserve $\mathcal{R}_m$, we may assume that $\pi = h_S$ for some $S \subseteq W$. It is enough to show that there is an $h \in H_n$ such that $h|U = \pi|U$. Clearly we can assume that $C \in U$. Let $U_1 = U \cap W$. If $\dim U_1 \leq n$ then item (odd) of Lemma 2.3 proves the statement, $H_n|_{U_1} = Z_2^{U_1}$. If $\dim U_1 = n + 1$ then the statement holds by item (even) of Lemma 2.3. Now, let $\dim U_1 = m$, where $m \geq n + 2$. We proceed by induction on $n + m$.

First, let us assume that $n = 1$. We claim that in this case both S and $W \setminus S$ are affine subspaces of W. (This implies that $\operatorname{codim} S = 1$, hence $\pi = h_S \in H_1$.) Suppose first that $u, v, w \in S$. Since π preserve $\mathcal{R}_2$ it follows that $|S \cap \{u, v, w, u + v + w\}|$ is even. Hence $u + v + w \in S$. Similarly if $u, v, w \in W \setminus S$, then $u + v + w \in W \setminus S$.

Now, assume that $n \geq 2$, $m \geq n + 2$, and the statement holds for $n + m - 1$ and $n + m - 2$. Let $\dim U_1 = m$. As automorphisms preserve $\mathcal{R}_m$, we may assume that $\pi|_{U_1} = h_S|_{U_1}$ for some $S \subseteq W$. Let $U_2 \leq U_1 \cap W$ such that $\dim U_2 = m - 1$, and let $w \in U_1 \setminus U_2$. By the induction hypothesis there is an $h_T \in H_n$ such that $h_T|_{U_1} = \pi|_{U_1}$. The permutation $\pi h_T^{-1} = h_{S \triangle T}$ fixes

U_2 elementwise, and it preserves $\mathcal{R}_{n+1}$. Let $k = h_{(S\triangle T)+w}$. Then k preserves $\mathcal{R}_{n+1}$ by the definition of $\mathcal{R}_{n+1}$, and it fixes $U_1 \setminus U_2$ elementwise. By using item (trans) of Lemma 2.3 it is enough to show that there is an element $h \in H_n$ such that $h|_{U_1} = k|_{U_1}$. Let W_2 be a 1-codimensional subspace of W containing U_2, but not containing U_1. We would like to apply the induction hypothesis for the vector space $V_2 = \langle W_2, C\rangle$, the subspace $\langle U_2, C\rangle$, the relation $\mathcal{R}_n$ and the permutation k. For this it is enough to show that k preserves $\mathcal{R}_n$ restricted to $\langle U_2, C\rangle$. So suppose $x_1, x_2 \dots, x_{2^n} \in \langle U_2, C\rangle$ and $(x_1, \dots, x_{2^n}) \in \mathcal{R}_n$ and let $X = \{x_1, \dots, x_{2^n}\}$. We have to show that $|X \cap X^k|$ is even. Let $Y = X \cup (X + w)$. Then $|Y \cap Y^k|$ is even since k preserves $\mathcal{R}_{n+1}$. We know that $k|_{U_1\setminus U_2} = \mathrm{id}_{U_1\setminus U_2}$, hence $Y \cap Y^k = (X \cap X^k) \cup ((X + w) \cap (X + w)^k)) = (X \cap X_k) \cup (X + w)$, therefore $|X \cap X^k| = |Y \cap Y^k| - |X + w| = |Y \cap Y^k| - 2^n$, which is even. So we can apply the induction hypothesis for the vector space V_2, the subspace $\langle U_2, C\rangle$, the relation $\mathcal{R}_n$ and the permutation k. It implies that there are $n-1$-codimensional subspaces $Y_1, Y_2, \dots, Y_t$ of W_2 such that for $h = \prod h_{Y_i}$ we have that $h|_{U_2} = k|_{U_2}$. The subspaces Y_i are n-codimensional subspaces of V, hence $h \in H_n$. Moreover h fixes all elements of $U_1 \setminus U_2 \subset W \setminus W_2$, hence $h|_{U_1} = k|_{U_1}$, and this is what we wanted to show. □

3. Reed–Muller codes

Our construction can be re-formulated in terms of infinite analogues of Reed–Muller codes [14, 18].

Our description of the Reed–Muller codes follows van Lint [12].

A *binary linear code* of length N is a vector subspace of $\mathbb{F}_2^N$. Vectors in this space can be regarded as functions from an N-set to $\mathbb{F}_2$. For this application we take $N = 2^n$, and identify the set of coordinates with $V = \mathbb{F}_2^n$.

The *Reed–Muller code* $\mathrm{RM}(r, n)$ can be described in two different ways:

- it consists of all the functions from V to $\mathbb{F}_2$ which can be represented by polynomials of degree at most r in the coordinates;
- it is spanned by the characteristic functions of subspaces of codimension r in V.

We summarise a few properties of these codes.

- $\mathrm{RM}(r, n)$ has dimension $\sum_{i=0}^{r} \binom{n}{i}$ and minimum weight 2^{n-r};
- $\mathrm{RM}(r, n)^\perp = \mathrm{RM}(n - r - 1, n)$, where orthogonality is with respect to the standard inner product.

Now we return to our reducts. The automorphism group of the pointed vector space $(\mathbb{F}_2^\lambda, C)$ is a semidirect product of the space of linear functions $V \to \mathbb{F}_2$ by the general linear group $\mathrm{GL}(V)$, where $V = \mathbb{F}_2^\lambda/\langle C\rangle$. (For this group acts on the quotient space as the automorphism group of V; the kernel of this action fixes every coset, and so can be represented by maps from V to $\mathbb{F}_2$, the image of a coset being 0 or 1 according as the elements in this coset are fixed or interchanged by the element concerned. To see that the extension splits, choose a complement W for $\langle C\rangle$ in $\mathbb{F}_2^\lambda$; elements of $\mathrm{Aut}(\mathbb{F}_2^\lambda, C)$ form the required complement.) Thus, any closed $\mathrm{GL}(V)$-invariant subspace W of the space of functions $\lambda \to \mathbb{F}_2$ that contains all linear functions will define a closed subgroup $W \rtimes \mathrm{GL}(V)$ containing all automorphisms, and hence a reduct.

Let W_k be the closure of the vector space of functions $f : V \to \mathbb{F}_2$ given by polynomials of degree at most k in the coordinates (these are "infinite RM codes"). Note that, since x^2 and x are equal as functions, we have $W_1 \leq W_2 \leq \cdots$; these subspaces are closed and $\mathrm{GL}(V)$-invariant. The inclusions are strict since, for example, the polynomial of degree k which is the product of k distinct indeterminates cannot be written as a polynomial of smaller degree. (See also the following paragraph). So we have a descending chain of reducts. Note that non-zero vectors in these subspaces all have infinite support.

While there is no inner product defined on the vector space of all functions from V to $\mathbb{F}_2$, we can define the "standard inner product" $u \cdot w$ whenever u is a vector with finite support. Now a function belongs to W_k if and only if it is orthogonal to the characteristic function of every $(k+1)$-dimensional subspace of V. This holds for polynomials of degree k by the same argument as in the finite case. Then, as a convergent sequence of polynomials (in the topology of pointwise convergence) is ultimately constant on any $(k+1)$-dimensional subspace, its limit is also orthogonal to every such subspace. We also see the strict inclusion of the subspaces W_k from this argument: for the product of k distinct indeterminates meets some k-dimensional subspace in a single point, and so fails to be orthogonal to all such subspaces, and cannot lie in W_{k-1}.

So $W_k \rtimes \mathrm{GL}(V)$ is a closed subgroup of the symmetric group containing all automorphisms, and hence a reduct of the pointed vector space.

This argument also verifies the relational definition of the reducts given earlier.

We remark that the two definitions of the finite-dimensional RM codes are no longer equivalent in the infinite case: the space spanned by the characteristic functions of k-dimensional subspaces contains elements of finite support and is not closed.

4. Corollaries

There are two obvious ways to generalize the result of Theorem 2.8. One is to find a similar construction for vector spaces over finite fields of odd characteristic; the other is to find structures that have (V, C) as a first-order definable reduct. We start with the second.

Let $\mathrm{BA} = (B, \wedge, \vee, 0, 1, \neg)$ denote the countable atomless Boolean algebra. It is known that this structure is ω-categorical. For $a, b \in \mathrm{BA}$ let $a + b$ denote the symmetric difference of a and b.

Theorem 4.1. *The lattice of first-order definable reducts of the homogeneous Boolean algebra contains an infinite descending chain. In particular it has infinitely many first-order definable reducts.*

Proof. The vector space $(\mathbb{F}_2^\omega, C)$ is isomorphic to $(B, +, 0, 1)$, hence it is a reduct of the homogeneous Boolean algebra. Being a first-order definable reduct is transitive, so Theorem 2.8 implies the statement. $\square$

For the case of finite fields of odd characteristic, the construction is analogous. Let V be a countably infinite dimensional vector space over the p-element field, $\mathbb{F}_p$ and $0 \neq C \in V$. Let us fix a 1-codimensional subspace $W < V$ not containing C. Then, $V = \langle W, C\rangle$ and $V = \{w, w+\lambda C \,|\, w \in W, \lambda \in \mathbb{F}_p\}$. The automorphism group of V is $\mathrm{Aut}(V) = \mathrm{GL}(V)$ and the automorphism group of (V, C), the pointed vector space is the stabilizer of C in $\mathrm{GL}(V)$ so $\mathrm{Aut}(V, C) = \mathrm{GL}(V)_C$. Let $S \subseteq W$. Let $h_S \in \mathrm{Sym}(V)$ defined by $v^{h_S} = v + C$ for $v \in S + \langle C\rangle$ and $v^{h_S} = v$ otherwise. For a subspace $W' \leq V$, where W' is an n-codimensional subspace containing C, let $H_n = \langle \mathrm{Aut}(V, C), h_{W'}\rangle$. Again, H_n does not depend on the choice of W_n. Let $\mathcal{R}_n$ denote the relation $(x_1, x_2, \ldots, x_{p^n})$, where $\{x_i, x_i + \lambda C \,|\, 1 \leq i \leq p^n, \lambda \in \mathbb{F}_p\}$ is a subspace of V and $\sum x_i \in W$. As in the case of characteristic 2, for the closure of the subgroups H_i we have $\overline{H}_1 \lneq \overline{H}_2 \lneq \cdots \lneq \overline{H}_n \lneq \ldots$. Also, for any $\pi \in \mathrm{Sym}(V, C)$ we have that $\pi \in \overline{H}_n$ if and only if π preserves $\mathcal{R}_{n+1}$. We arrive at the conclusion:

Theorem 4.2. *The lattice of first-order definable reducts of the pointed homogeneous vector space $(\mathbb{F}_p^\omega, C)$ contains an infinite descending chain. In particular $(\mathbb{F}_p^\omega, C)$ has infinitely many first-order definable reducts.*

Corollary 4.3. *The lattice of first-order definable reducts of the pointed homogeneous vector space over a finite field $\mathbb{F}_q$ contains an infinite descending chain. In particular $(\mathbb{F}_q^\omega, C)$ has infinitely many first-order definable reducts.*

Proof. Let p be the characteristic of the field $\mathbb{F}_q$. The structure $(\mathbb{F}_q^\omega, C, +)$, where we consider only the addition as an operation, is a reduct of the pointed vector space, and is isomorphic to the vector space $(\mathbb{F}_p^\omega, C)$. The statement follows from Theorem 2.8 and Theorem 4.2. $\square$

These results can also be shown using an analogue of the Reed–Muller construction; we do not pursue this further.

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