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Abstract

The understanding of joint asset return distributions is an important ingredient for managing risks of portfolios. While this is a well-discussed issue in fixed income and equity markets, it is a challenge for energy commodities. In this paper we are concerned with describing the joint return distribution of energy related commodities futures, namely power, oil, gas, coal and carbon.

The objective of the paper is threefold. First, we conduct a careful analysis of empirical returns and show how the class of multivariate generalized hyperbolic distributions performs in this context. Second, we present how risk measures can be computed for commodity portfolios based on generalized hyperbolic assumptions. And finally, we discuss the implications of our findings for risk management analyzing the epxosure of power plants which represent typical energy portfolios.

Our main findings are that risk estimates based on a normal distribution in the context of energy commodities can be statistically improved using generalized hyperbolic distributions. Those distributions are flexible enough to incorporate many characteristics of commodity returns and yield more accurate risk estimates. Our analysis of the market suggests that carbon allowances can be a helpful tool for controlling the risk exposure of a typical energy portfolio representing a power plant.

1 Introduction

Risk management relies on models that reflect a number of important properties of the underlying financial assets that affect the performance of companies' portfolios. What makes this a non-trivial exercise is the complex dynamics of each particular asset in the portfolio on one hand, compounded with the difficult task of simultaneously modeling the interaction between all the portfolio constituents on the other.

The subject matter of risk management is by no means new in the literature, however there are very few risk management applications in the field of energy commodities. Despite the fact that for a number of years there has been an imperative need from companies to understand their overall market exposure to a number of energy commodities, academic excursions in this particular field are yet to come forth. This apparent time lag is perhaps due to the intricacies of this relatively new market, consequence of recent market liberalization (in for instance gas and power), which has proved to exhibit considerably different behavior from that of the more traditional markets such as equity, bonds, etc. Therefore, as a result of the relatively young age of the deregulated power and gas markets, in addition to the birth of the CO_2 emissions market, there is a great deal of catching up to do regarding the understanding and modeling of some energy related commodities.

In recent years the most popular member of the energy commodity class in the academic literature has been electricity. Early models include Lucia and Schwartz (2002) where the authors focus on no-arbitrage models of wholesale electricity prices, futures and forwards in the Scandinavian market. In their work the key building blocks for modeling power prices are mean reverting shocks, long-term shocks (both driven by diffusion) and a seasonal component.

A formal rejection of the Normality assumption in power forward prices can be found in the work of Knittel and Roberts (2005) and Villaplana (2005) who provide ample evidence that the distribution of electricity spot prices is not Normal. Instead fat-tailed and positively skewed distributions are better candidates for power modeling. The distributional properties of electricity prices have also been examined by Cartea and Figueroa (2005) and Weron (2006). The former analyzes the England and Wales market and proposes a mean-reverting jump-diffusion model to capture the main stylized properties of power prices. The latter provides a detailed analysis of power prices for the EEX and Nordpool where the importance of employing alternative distributions to model the dynamics of power prices, for instance the hyperbolic distribution, is pointed out. Further, Eberlein and Stahl (2003) give statistical evidence for modeling electricity spot prices using generalized hyperbolic distributions and show how this distribution can be used for deriving risk capital charges based on value-at-risk. They discuss the superiority of this parametric approach to standardized alternatives that have to be applied to commodities markets in context of the German implementation of the capital adequacy directive.

While research on electricity has grown during the last years, other relevant energy-related commodities such as oil, gas and CO_2 emission allowances are not that well-represented. For example, apart from the work of Manoliu and Tompaidis (2002) who studied the dynamics of natural gas for the US market and Cartea and Williams (2007) who looked at natural gas in the UK market, there is very little literature focusing on gas.

Although there are pressing needs for the more ambitious task of modeling power, coal, gas, oil and emission allowances as one package, there is very little literature on this topic. Multivariate research on statistical properties of energy-related commodities is very limited. Research on the class of commodities can be found in Kat and Oomen (2006) where the authors discuss the role of commodity markets as an alternative investment to bonds and stocks. They focus on different dependence measures for individual commodities and conclude that commodities, in general, seem to be statistically independent from other financial

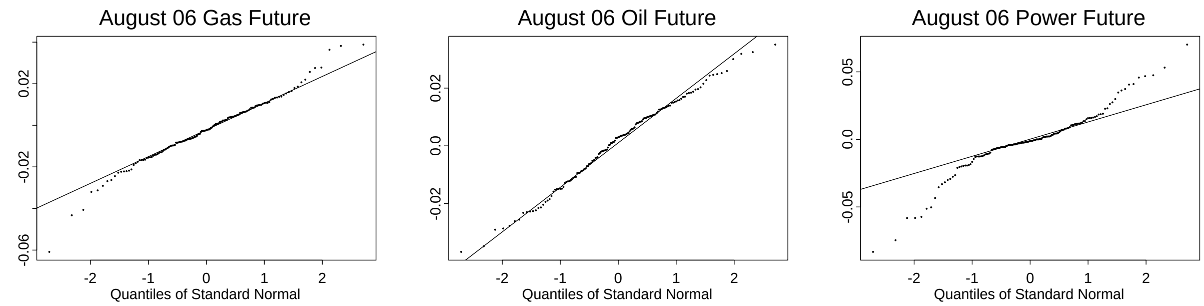

Figure 1: QQPlots of log-returns of gas, oil and power futures with delivery August 2006

markets, but there is strong dependence within a commodity group. Further, the authors state that a multivariate Normal distribution is not appropriate to capture the dependence structure among commodities, stocks, bonds and inflation.

In contrast to the research mentioned above, we will focus on the dependence structure within the group of energy related commodities, which is of most importance to utility companies. We model and estimate the joint return distributions of various commodities and apply the result to risk management issues. This requires multivariate distributions flexible enough to fit the marginal distributions of the single commodities whilst at the same time capturing their dependence structure. The advantage of pursuing a multivariate approach is that besides correlation we will be able to capture dependencies not measured by correlation.

The energy commodities discussed in this article include Brent crude oil, natural gas, coal, power and CO_2 emission allowances with a focus on the European market. These commodities are the most important traded products in the energy sector and are representative of a typical portfolio of a utility company. In order to evaluate the risk of such a portfolio we need to analyze the dependence structure across different commodities as well as the dependence structure across maturity within each commodity class. In addition, we will also report the volatility term-structure for commodities forward contracts. Most importantly, we show that the class of generalized hyperbolic distributions is well-suited to characterize the return distribution of all commodities simultaneously.

Furthermore, we are going to illustrate the consequences of our statistical findings for risk management and compute risk measures such as Value-at-Risk and expected shortfall based on different choices of multivariate distributions and apply the procedure to a given portfolio of energy commodities. As an improvement to the Normal variance-covariance approach for deducing Value-at-Risk in the context of commodities we propose an alternative based on generalized hyperbolic distributions that is straightforward to implement and more flexible.

The distributions under consideration are motivated by univariate findings of Benth and Saltyte-Benth (2004) and Eberlein and Stahl (2003) in energy markets. From a statistical point of view a multivariate normal distribution is not suitable since all marginal distributions would also be Normal and the hypothesis of normally distributed returns has been rejected for assets in many markets; especially energy commodities. Fig. 1 shows the qqplot of univariate log-returns of a typical gas, oil and power futures contract. The oil contract in this example is the only contract with distribution close to a Normal, but for the other commodities this

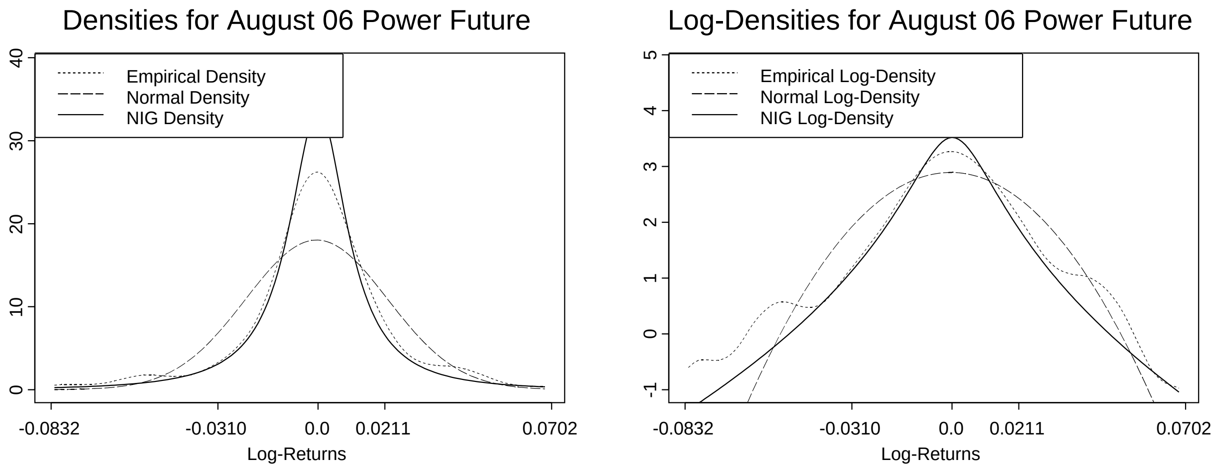

Figure 2: Empirical and estimated densities for log-returns of an electricity future with delivery August 2006

is not true.

Fitting a univariate empirical density to return data using a kernel estimator and comparing it to other probability densities reveals some of the stylized facts of financial markets (cp. Fig. 2). The empirical return density is more peaked at the center than the Normal distribution and also heavier-tailed. These properties have been successfully modeled by the class of univariate generalized hyperbolic distributions (GH) in other financial markets (e. g. Raible (1998)) and in energy related commodity markets (e. g. Weron (2006)). An application of the special case of Normal-Inverse Gaussian distribution (NIG) to fit historical power spot prices and pricing derivatives in this framework can be found in Benth and Saltyte-Benth (2004).

Fig. 2 shows the empirical density of a futures contract on power and compares it to a fitted Normal and NIG density. Transforming the densities to log-scale (Fig. 2 right panel), one can clearly see how the NIG is better equipped to capture the heavy tails. Motivated by these results for the univariate case we analyze the use of GH distributions for the multivariate case. As special cases of the multivariate GH distribution, we will discuss the multivariate t-, hyperbolic and NIG distribution.

The contribution of this article is three-fold. Firstly, we provide a statistical analysis of the most important energy related commodities. Secondly, based on this analysis we examine how and why different risk management decisions may be implemented. Thirdly, we employ our results to give further directions for financial model building, pricing and hedging purposes. As a particular example, we will show how our findings can be used to evaluate the riskiness of a given energy portfolio and illustrate our methodology using portfolios that reflect the typical exposure of power plants.

The rest of this article is organized as follows. In section 2 we briefly introduce the mathematical and statistical tools that are employed. Section 3 gives a description of the data used for the analysis and discusses the data preparation. Section 4 presents the results of the statistical analysis in detail and illustrates the findings. Finally, in section 5 we show the implications of our findings in risk-management.

2 Statistical Tools

The statistical tools that we will use are motivated by the purpose of our study. On one hand, we want to describe correlation and volatility term-structures of commodity futures returns, and on the other we require a description of the entire joint distribution of these returns. Both objectives can be achieved by a variety of statistical procedures. We decided to conduct the correlation-volatility analysis in a non-parametric context. Tests for normality and description of multivariate GH distributions is subsumed under parametric methods and serve the purpose of finding an appropriate joint return distribution. In the following we outline the methods that we apply to the data sets. The construction and characterization of the data itself is covered in Section 3.

2.1 Non-parametric Methods

We will use correlation estimates for two purposes. First, we want to check if two price series are dependent at all. Second, we want to compare the magnitude of dependence for different pairs of time series to establish a term-structure of correlation.

We restrict ourselves to standard pairwise estimators. For two given one-dimensional time series X and Y the Pearson product-moment correlation coefficient P, Spearman's ρ and Kendall's τ are defined by

$$\begin{aligned} P &= \frac{\sum_{i=1}^{n}(X_i-\bar{X})(Y_i-\bar{Y})}{\sqrt{\sum_{i=1}^{n}(X_i-\bar{X})^2\sum_{i=1}^{n}(Y_i-\bar{Y})^2}} \\ \rho &= 1-\frac{6\sum_{i=1}^{n}(R_i^X-R_i^Y)^2}{n(n-1)(n+1)} \\ \tau &= \binom{n}{2}^{-1}\sum_{1\le i<j\le n} sign((X_i-X_j)(Y_i-Y_j)) \end{aligned}$$

where R_i^X and R_i^Y denote the rank order of X_i and Y_i.

Pearson's correlation test is widely-used, despite suffering from many shortcomings such as exhibiting sensitivity to outliers, to estimate correlation and, thus, linear dependence between X and Y. One of the reasons why it is so popular is because it coincides with the correlation estimation for normally distributed random variables in the sense that it is the maximum likelihood estimator.

Spearman's ρ and Kendall's τ are not as restrictive on the dependence structure they can detect. They are based on a more general concept of dependence and can also show values different from zero, when the random variables are uncorrelated, but dependent in another way. Both estimators are known to have a smaller variance than Pearson's estimator and are less sensitive to outliers. We note that Kendall's τ cannot estimate the correlation between X and Y, however, employing the transformation

$$\tau' = \sin \frac{\tau\pi}{2}$$

does allow us to calculate this correlation for the case of elliptically contoured distributions (symmetric GH). This estimator of correlation has proven to perform better in the sense that it has a smaller variance than Pearson's estimator, especially when applied to data that are heavy-tailed. An impressive illustration can be found in Embrechts et al. (2005a, Example 3.31) and a textbook reference is Huber (1981).

Finally, a maximum likelihood fit of any distribution will also yield correlation matrices. In contrast to the correlation coefficients mentioned here, the matrices inferred from distributions are not pairwise estimates. In other words, the pairwise correlation coefficients need not result in a correlation matrix in general.

2.2 Parametric Methods

Since the main objective of our study is the derivation of risk measures for commodity portfolios, the main step in that direction is the specification of an appropriate multivariate distribution for return data. We approach this from two perspectives. First, we must specify a model that can capture the data as precisely as possible, and second, we also require the model to be versatile enough to gain precision, in a statistically significant way, when adding parameters and thus extra levels of complexity to it. In this context it is of importance to start by assessing the quality of a Normal distribution because it is the most popular choice for many practical applications. This distribution allows for straightforward applications in risk management and we can use it as a benchmark to justify the use of alternative multivariate distributions such as the class of generalized hyperbolic distributions.

Tests for normality

There is a large variety of procedures for testing univariate and multivariate normality. The most prominent visual test is a qqplot comparing quantiles of the empirical distribution with quantiles from a Normal distribution. In case of normality this plot should give a straight line. In fact, the method can be applied to multivariate time series as well, since each linear combination of the time series should be Normal, in particular each time series itself. Yet, this is not sufficient for joint normality so that other test statistics will be used, namely Mardia's test for skewness and kurtosis.

Let $X_1, \ldots X_n$ be the realization of a d-dimensional random variable X and $\hat{C} := \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})(X_i - \bar{X})^t$ the empirical covariance matrix. Under the hypothesis of multivariate normality the following asymptotic relations hold:

$$
\begin{aligned}
S_{n,d} &= \frac{1}{6n}\sum_{i=1}^{n}\sum_{j=1}^{n}\Big((X_i-\bar{X})^t\hat{C}^{-1}(X_j-\bar{X})\Big)^3 \sim \chi^2_{d(d+1)(d+2)/6},\\
K_{n,d} &= \frac{\frac{1}{n}\sum_{i=1}^{n}\Big((X_i-\bar{X})^t\hat{C}^{-1}(X_i-\bar{X})\Big)^2 - d(d+2)}{\sqrt{8d(d+2)/n}} \sim \mathcal{N}(0,1),
\end{aligned}
$$

where $S_{n,d}$ is a measure for skewness and $K_{n,d}$ measures excess kurtosis. If the values are greater than a critical value, the hypothesis of joint normality is rejected. For further details on the test (power, robustness etc.) see Mardia (1970) and Mardia et al. (1979).

Distributions

We consider the class of multivariate generalized hyperbolic distributions (GH), in particular we look at the multivariate t-distribution (t), (symmetric) Normal-inverse gaussian (NIG(-S)), and (symmetric) hyperbolic distribution (HYP (-S)). The class of GH distributions has been introduced by Barndorff-Nielsen (1978) and important properties are summarized in Barndorff-Nielsen and Blaesild (1981). Applications can be found in the textbooks by Embrechts et al. (2005a) and Shiryaev (1999) and in the articles by Eberlein and Keller (1995); Eberlein and Prause (1998) and Eberlein and Stahl (2003).

The multivariate GH distribution is given by the joint density

$$
\begin{aligned}
f(x) &= c\frac{K_{\lambda-(d/2)}\Big(\sqrt{(\chi+(x-\mu)^t\Sigma^{-1}(x-\mu))(\psi+\gamma^t\Sigma^{-1}\gamma)}\Big)\, e^{(x-\mu)^t\Sigma^{-1}\gamma}}{\sqrt{(\chi+(x-\mu)^t\Sigma^{-1}(x-\mu))(\psi+\gamma^t\Sigma^{-1}\gamma)}^{\frac{d}{2}-\lambda}},\ x\in\mathbb{R}^d\\
c &= \frac{\sqrt{\chi\psi}^{-\lambda}\psi^{\lambda}(\psi+\gamma^t\Sigma^{-1}\gamma)^{\frac{d}{2}-\lambda}}{(2\pi)^{d/2}|\Sigma|^{1/2}K_\lambda(\sqrt{\chi\psi})}
\end{aligned}
$$

with K_λ denoting the modified Bessel function of the third kind and parameters $\Sigma\in\mathbb{R}^{d\times d}$, $\mu,\gamma\in\mathbb{R}^d$, $\chi,\psi>0$ and $\lambda\in\mathbb{R}$. Thus, a d-dimensional GH is described by $\frac{1}{2}(d(d+5)+4)$ free parameters.

We also mention the representation of GH distributions as Normal mean-variance mixtures, which is useful in many applications. A random variable X is a multivariate Normal mean-variance mixture if

$$X \stackrel{d}{=} \mu + W\gamma + \sqrt{W}AZ \tag{1}$$

with Z a k-dimensional standard Normal, W a non-negative real-valued random variable (mixing variable) independent of Z, $A\in\mathbb{R}^{d\times k}$ and $\mu,\gamma\in\mathbb{R}^d$. If W has a generalized inverse Gaussian distribution with parameters (λ,χ,ψ), then X is GH-distributed with $\Sigma = AA^t$. Often, one can make use of the fact that X, conditional on $W=w$, is Normal with mean $\mu+w\gamma$ and variance $w\Sigma$.

Symmetric distributions are obtained when $\gamma=0$ in Eq. (1). Moreover, a multivariate distribution with hyperbolic marginals is given by $\lambda=1$ and the Normal-Inverse Gaussian becomes a special case when $\lambda = -\frac{1}{2}$. The multivariate t-distribution with ν degrees of freedom can also be obtained either as a limiting case if $\gamma = 0, \ \ \lambda = -\frac{1}{2}\nu, \ \ \chi = \nu, \ \ \psi \to 0$ or by choosing an Inverse Gamma distribution with parameters $(\nu/2, \nu/2)$ as mixture variable (and $\gamma = 0$).

Finally, since we will need to compute portfolios' distribution, we will need the property that GH-distributions are closed under linear operations, i. e. if $X \sim GH_d(\lambda, \chi, \psi, \mu, \Sigma, \gamma)$ and $B \in \mathbb{R}^{k\times d}$, then $BX \sim GH_k(\lambda, \chi, \psi, B\mu, B\Sigma B^t, B\gamma)$.

An alternative definition of the GH is given by a mean-variance mixture. A random variable has d-dimensional GH distribution if it coincides in distribution with

$$m(W) + \sqrt{W}AZ$$

with Z a k-dimensional standard Normal random variable, $W \geq 0$ independent, real-valued generalized inverse gaussian random variable, $A \in \mathbb{R}^{d\times k}$ and m a measurable function with values in $\mathbb{R}^d$.

Estimation of this class of distribution can be done by Maximum-Likelihood procedures, in particular an EM-algorithm, which is an iterative scheme of maximizing conditional likelihoods. For details on the algorithm see Protassov (2004). We use the S-Plus tool provided in Embrechts et al. (2005a).

The goodness of fit of the distributions to a given data set can be assessed by comparing the maximum of the likelihood. The larger the value, the better the fit. Whether a difference is significant and the use of further parameters is statistically justified can be tested by standard asymptotic likelihood ratio tests.

3 Data Set

The data under consideration are daily price series of futures or forwards on power, coal, oil, natural gas and carbon emission allowances. While the portfolio of an energy producing company will most likely include other financial instruments (e.g. FX positions), the energy-related commodities make up a large part of a typical portfolio of a power producing utility.

Since we will establish the joint behavior of commodities across maturity and across underlyings, the analysis is carried out on monthly and yearly futures. Monthly futures are used to determine a term-structure of volatility and correlation within one commodity across maturities, while we use futures for the year 2007 to determine the dependence structure across commodities. In this section we briefly discuss the specifics of each time series. The data analysis starts at the beginning of August 2006.

3.1 Power

The time series of power prices are taken from EEX-traded futures. Available products are power futures with delivery periods being a calendar month, a quarter of a calendar year and

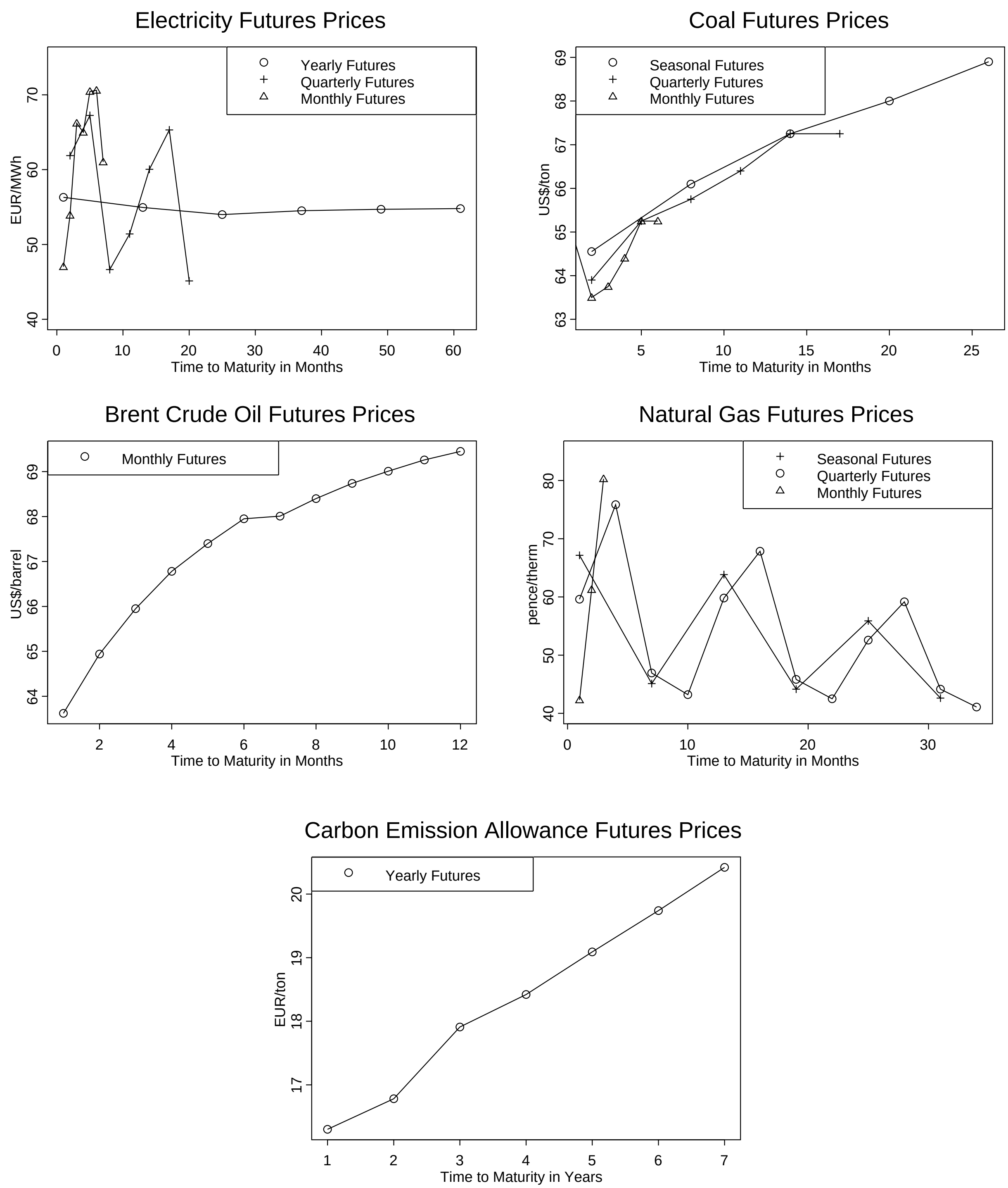

Figure 3: Market Futures prices of different commodities on a given day

the whole calendar year. The EEX market settles the contracts according to the following scheme: When a year-contract comes to delivery, it is split up into the corresponding four quarters. A quarter, which is at delivery, is split up into the corresponding three months and only the month at delivery will be settled either physically or financially by delivering the energy amount continuously during the month. The futures prices are quoted in EUR/MWh.

Fig. 3 (upper left panel) shows the available futures contracts and prices at the EEX on a typical day. One can identify seasonalities in the maturity variable, especially for quarterly contracts: futures during winter months show higher prices than comparable contracts during the summer. This can be explained by an increased demand due to heating and light that can be met only at higher production costs. This has to be taken into account when preparing the data as stationary data are required for our purposes. Details on deseasonalizing and stabilizing volatility are described in Section 3.6.

For the analysis of volatility and correlation term-structures, we will use all price series of monthly futures that have been traded at the EEX in the past covering a period of about 4 years or 1016 trading days. Further, for the cross commodity analysis, we will use the historical price series of the year 2007 future covering about the same time period.

3.2 Coal

For some time now coal futures have been traded at several exchanges, among them the ICE (formerly IPE) in London and the EEX. They offer trades in coal with different points of delivery. In the following we will pick Rotterdam as an example. Similar to power contracts, one can take positions in monthly and quarterly contracts, where a quarterly contract is composed of the corresponding three monthly contracts. Additionally, there are seasonal contracts, covering a delivery period of six months starting in October or April. The futures prices are quoted in US$/t. Fig. 3 (upper right panel) shows futures prices of these contracts. In contrast to power futures, there is no obvious seasonality in the maturity variable.

Unfortunately, monthly historical coal prices is very short. The contracts were introduced at the EEX in May 2006 and at the ICE in March 2006. This does not allow for robust statistical analysis and term-structure of volatility and correlation for coal futures are very difficult to determine.

While the futures market for coal has just started, there is an established market for swaps and forwards on coal. Therefore, we use forward prices, provided by EnBW, for the year 2007 starting in May 2004 covering about 530 trading days. Thus, we can include the commodity coal into the cross commodity analysis. Furthermore, when analyzing coal contracts together with other commodities, we convert all prices to EUR/t.

3.3 Oil

An important location of trade for Brent crude oil is the ICE in London. Available products are monthly futures contracts for at least the next 12 months quoted in US$/barrel. Fig. 3 (mid left panel) shows available futures prices at the ICE. Again, there is no significant seasonality in the maturity variable.

From the available price history of monthly futures we use historical data starting in January 2000. This leaves us with 1647 trading days to assess the term-structure of volatility and correlation.

Since there is no 2007 oil future available at the ICE, we need to construct it artificially as weighted average of monthly futures of the corresponding period. The time series starts in May 2005 and, thus, covers a period of 317 trading days, which can be used for cross commodity analysis. Moreover, when analyzing oil contracts, together with other commodities, we convert all prices to EUR/barrel.

3.4 Gas

Natural gas has gained importance among the energy related commodities as there is an increasing number of gas-fired power plants in Europe. The only exchange in Europe offering sufficient historical price data for futures on natural gas is the ICE in London. Available products are monthly, quarterly and seasonal gas futures. Seasonal contracts start delivery in October and April. Futures prices are quoted in pence per British thermal unit (pence/Btu). Fig. 3 (mid right panel) shows the available futures prices of natural gas at the ICE on a given day.

Considering the seasonal summer and winter contracts separately, one can identify a seasonal behavior of futures price levels. This might be due to relatively small storage capacities and a seasonal demand. As in the case of power this needs special care when analyzing the data set.

The history of monthly natural gas contracts is rather long and we restrict ourselves once again to the last 6 years starting in January 2000. We use four monthly contracts starting with the front month to determine the term-structure behavior of volatility and correlation.

3.5 CO_2 Emission Allowances

Emission allowances, also known as CO_2 or carbon certificates, are the youngest of the energy related commodities which were introduced, in the context of the Kyoto-protocol, by the European Union and exchange based trading started in October 2005. One certificate allows the emission of one ton of CO_2 during a certain time period and companies need to cover their yearly emissions by certificates. The trade in allowances is divided into two periods: Period 1 is up until the end of 2007 and a Period 2 ranges from 2008 to 2012. Although allowances may be transferred within the same period, they cannot be transferred from Period 1 to Period 2. Futures contracts for allowances within one of the two periods but with different maturity dates differ only by a discount factor. In effect, this means that the 2006 and 2007 contracts for allowances in the first period are virtually the same, but different from the 2008 contract for allowances in Period 2. Therefore looking at the term-structure of allowances is not necessary (cp. Fig. 3 lower left panel) hence we concentrate on the joint behavior of the year 2007 price with other commodities.

The introduction in 2005 of emission allowances added another dimension to the problem of managing an energy portfolio for companies that produce power. Since a company's yearly CO_2 production is unknown before the end of the calendar year, it faces a 'quantity' decision. Additionally, it is unclear wether it is advantageous to buy the certificates from the very beginning or excluding them from the portfolio as long as possible; this results in a 'timing' decision. We will place particular emphasis on these questions from a risk point of view in Section 5.

The certificates have been traded at the EEX since October 2005 which constitutes a data set of 190 trading days. Since these data may not lead to statistically significant results, we also employ OTC prices provided by EnBW. These cover a total period of 314 trading days.

3.6 Data Preparation and Construction of Data Sets

Most of the statistical methods that we will employ require at least stationary data sets. This demands that the time series need to have constant mean and autocovariance over time, but none of the prices will satisfy these conditions. In fact, all of them show a behavior, known as the Samuelson effect (see Samuelson (1965)) or volatility term-structure (cp. Fig. 4), where the variance of futures contracts generally increases when the contract approaches delivery. This effect is observable in the price levels as well as in log-returns and usually starts two to three months before maturity. This is why we will not analyze the futures prices directly, but time series which are known as generic time series. Generic time series are artificially constructed time series, that show the prices of futures with (approximately) the same time to maturity. For example, a one-month-ahead generic is the price series of the next-to-delivery contract. For example, in January 2005, this one-month-ahead generic consists of futures prices of the February-2005-future, in February it contains the prices of the March-future and so on. Similarly way we define two-, three- and so on up to six-months-ahead generics. Since these time series are always about at the same time before delivery, they will not show the increasing volatility within one of the time series. Moreover, the volatility (and correlation) term-structure can be read off across the different generics (cp. Fig. 5 right).

After having reduced the time-changing volatility by ruling out the Samuelson effect, we can consider log-returns of the generic time series. For futures on power and gas, which are seasonal in the delivery period variable, we will need to transform the data further. The seasonalities are not observable within one futures price series directly but after transformation to the generics. These will jump at the change of the month to the corresponding month's level. This, in turn, will result in extraordinarily large jumps in return series. Since the jumps do not stem from price formation at exchanges, we will correct them by replacing the jumps by the mean value of the return series which is zero (cp. Fig. 5 left).

Now that the time series can be assumed to be stationary, we have to take care when considering multivariate time series, since each process might come from a different exchange with different trading days. We delete trading days which are a holiday on any of the other exchanges. This results in the longest possible joint time series.

Constructing and deseasonalizing generic time series needs to be done for futures prices with

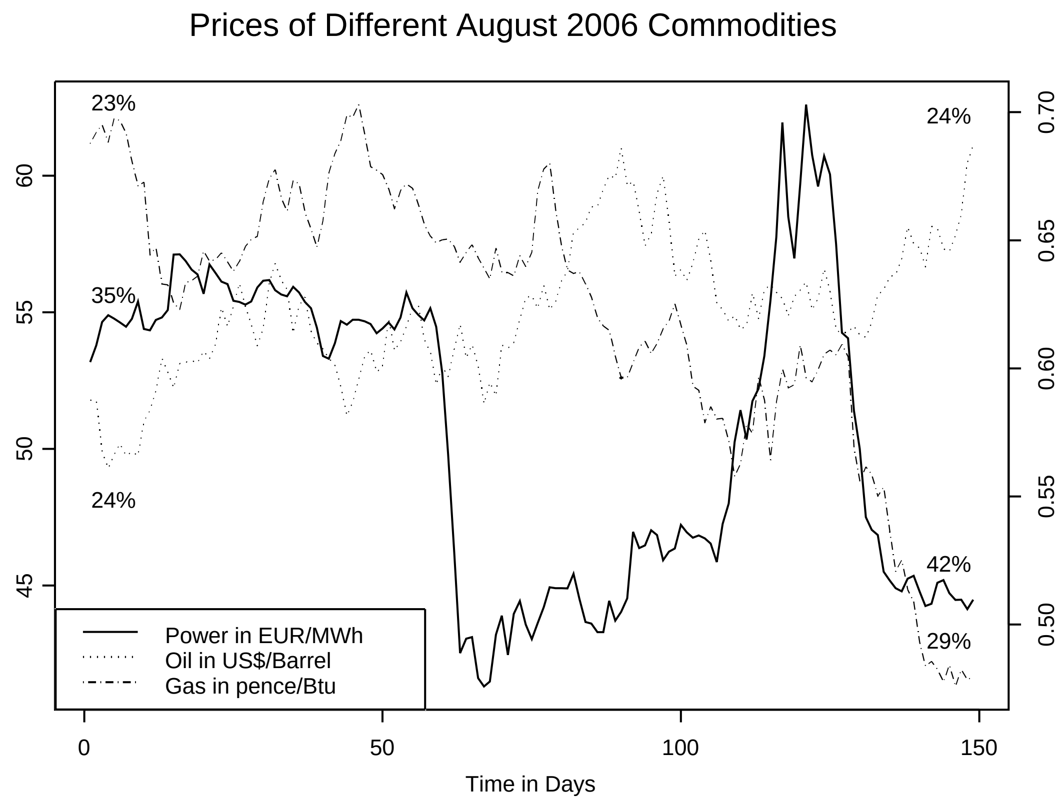

Figure 4: Price paths of oil, gas and power futures for August 2006 during the last 150 trading days with annualized return volatilities over the whole period (left) and the last 50 trading days (right)

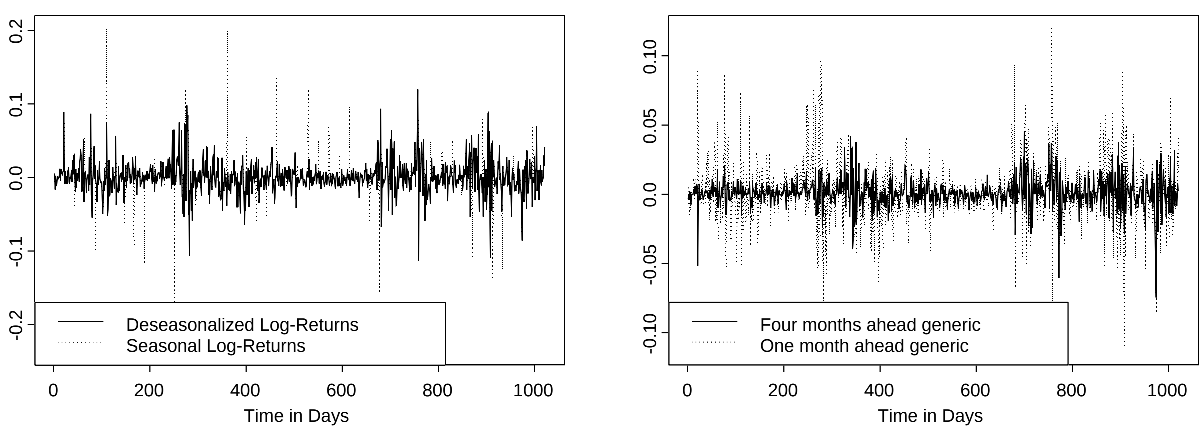

Figure 5: **Left:** Log-returns of generic one month ahead power futures with and without jumps at the end of a month (i. e. original and deseasonalized log-returns) **Right:** Log-Returns of generic one month and four months ahead power futures

delivery period being one month. Contrary, prices of futures contracts with year 2007 as delivery period need not be transformed. At the time of analysis, maturity was more than half a year away and the contracts' volatilities have not increased due to the volatility term-structure effect, which makes the construction of generic time series needless. Evidence for this is given by the empirical volatility term-structure, which is rather flat when there are more than 4 months time to maturity (cp. Fig. 6 lower right panel). Instead, we can work with the original futures price series. For reasons described in Section 3 we do not include natural gas in the analysis.

Using the data material and the preparation method one can construct a large number of possible multivariate data sets. For risk management applications we are interested in answering the following questions which motivates our particular choice:

1. Is there dependence between futures prices of different commodities?
2. Is there dependence between futures prices within one commodity but different maturities (volatility and correlation term-structure)?
3. Is the multivariate Normal distribution a reasonable distribution for modeling commodity returns?
4. How do GH distributions perform? Which subclass is preferable?

The questions can be answered using the following four data sets:

D1 Generic electricity futures with one-month delivery period, time to maturity one to six months, dimension: 6, length: 1016 $\Rightarrow$ Questions 2,3,4

D2 Generic oil futures, time to maturity one to six months, dimension: 6, length: 317 $\Rightarrow$ Questions 2,3,4

D3 Generic natural gas futures with one-month delivery period, time to maturity one to 4 months, dimension: 4, length: 1642 $\Rightarrow$ Questions 2,3,4

D4 Electricity future, coal and oil swaps and CO_2 future with delivery period 2007, dimension: 4, length: 314 $\Rightarrow$ Questions 1,3,4

We point out that it is possible to construct different working data sets from combinations of the available time series. Here we assume that our choice is well-suited to illustrate the statistical properties we are interested in and will enable us to prescribe different risk-management strategies.

4 Results

4.1 Correlation

First, we discuss the dependence structure of the different commodities and futures contracts. We start with correlations within each commodity class, i. e. we use data sets D1-D3.

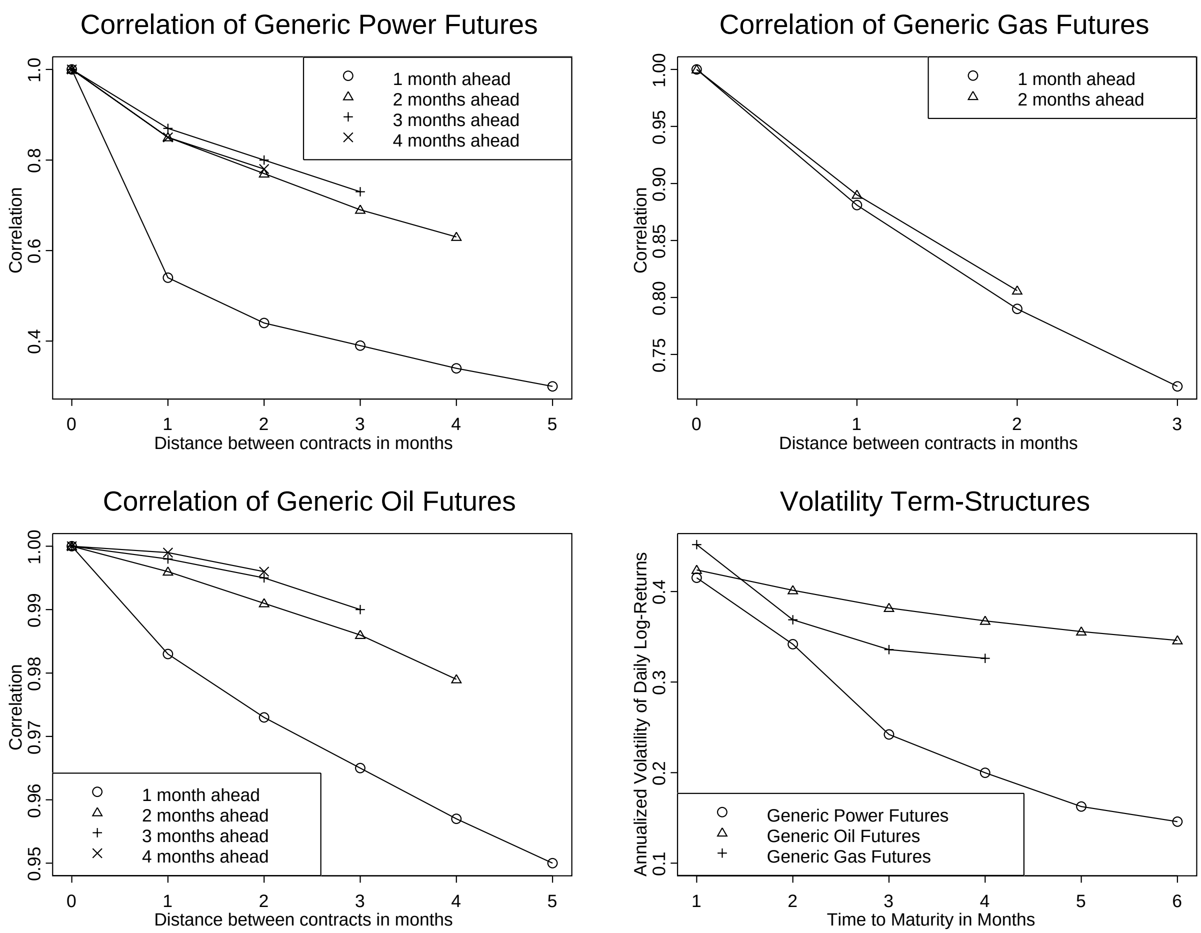

Figure 6: Term-structure of correlation across maturities for different commodities (data sets D1 top left, D2 top right, D3 lower left) and volatility term-structures (lower right)

There is strong correlation among the futures contracts within each commodity group. The correlation is decreasing as time to maturity increases. This is called a term-structure of correlation. The observation holds for all commodities and is confirmed by all means to estimate correlation. In particular, the estimates implied by the best fitting distribution are very close to Pearson's correlation and we omit the different matrices. An illustration of the correlation term-structures is provided in Fig. 6.

In case of power, we notice that correlation of the first generic contract is much smaller than correlation of the other contracts. This can be explained by the fact that the first generic for power is already in the delivery period. This means that it behaves very much like spot, which cannot be used for hedging purposes due to non-storability. This leads to spot price behavior, which usually does not correlate to futures when they are far away from maturity. The other generics show almost identical correlation term-structures. This implies, that the correlation does not depend on the time to maturity (except for the first generic), but only on the difference in maturities between the contracts. For example, the correlation of two consecutive futures is the same when we are 5 months away from delivery or only 2 months.

	Power	Coal	Oil	Carbon	Power	Coal	Oil	Carbon
	Pearsons				Kendall's τ			
Power	1.00	-0.03	0.06	0.67	1.00	-0.06	0.02	0.40
coal	-0.03	1.00	0.15	0.11	-0.06	1.00	0.08	0.09
Oil	0.06	0.15	1.00	0.07	0.02	0.08	1.00	0.07
Carbon	0.67	0.11	0.07	1.00	0.40	0.09	0.07	1.00
	Spearman's ρ				NIG fit			
Power	1.00	-0.09	0.03	0.56	1.00	-0.11	0.00	0.56
coal	-0.09	1.00	0.11	0.13	-0.11	1.00	0.09	0.10
Oil	0.03	0.11	1.00	0.10	0.00	0.09	1.00	0.08
Carbon	0.56	0.13	0.10	1.00	0.56	0.10	0.08	1.00

Table 1: Correlation estimates of year 2007 futures log-returns of power, coal, oil and CO_2 allowances (data set D4)

A similar result can be stated for gas futures, though the basis of different generics is very small (i. e. only the next four months are actively traded). The correlations of the month-ahead generic with the other generics and the two-months ahead generic are close to each other.

Oil futures behave differently. Firstly, the front month seems to be detached from the other generics correlationwise, similar to power futures. Secondly, the correlation term-structure is convex for the front month but concave for other generics. A statement about the identity of the other curves can hardly be made and would be pure speculation.

Additionally, we point out that all futures show a term-structure of volatility, meaning that volatilities of futures contracts increase as time to maturity decreases. We are surprised to see, that oil futures are more volatile than the other commodities and power futures show high volatility compared to other financial assets, but low compared to other commodities. It might be an explanation, that during the period under consideration, oil futures increased from 20US$ up to 75US$ due to political reasons, while power prices stayed at the same level.

Turning the focus to correlation between the different commodities using data set D4, we find that all the pairwise correlation measures (Pearson's product-moment estimator, Spearman's ρ and Kendall's τ) indicate almost no correlation between power and coal and power and oil. The dependence on oil and coal seems to be very small, if not negligible. The linear correlation (Pearson) and the broader dependence (Kendall's τ) yield values close to zero. The correlation matrix implied by the best distributional fit (NIG) contains similar information. The numbers differ only slightly, maybe indicating a negative correlation for power and coal.

The carbon futures reveal that there is a strong relation to power prices; the correlation is about 0.67, as one might expect. The relation between carbon and both oil and coal is not as strong, but all correlation measures indicate a slight positive dependence. The best fitting NIG distribution implies a correlation totally in line with the other dependence measures (cp. Tab. 1).

Summarizing this section, we can state that estimating the correlation in a parametric or

D1: Power		D2: Oil		D3: Natural Gas		D4: Cross Commodity	
S	K	S	K	S	K	S	K
15.48	182.67	928.30	1306.62	22.17	162.59	10.13	84.79
0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

Table 2: Mardia's normality test statistics (p-Values below) for log-returns of generic monthly power, oil and carbon futures (D1-D3) and year-2007 futures (D4)

non-parametric way yields almost identical results within one commodity class and results close to each other across commodities. This means that if we were to base a Normal variance-covariance method on either of the correlation matrices for measuring the risk of a portfolio with these commodities, we are most likely to end up with the same risk estimate. As we will see later, we yield rather different risk estimates when moving away from the Normal distribution (more different than can be explained by the variations in the correlation estimates). This will lead to the conclusion that although correlation can be measured appropriately, the dependence structure is more complex and cannot be summarized by correlation alone.

4.2 Normality Tests and Distributional Fits

Now, we want to report findings about the appropriateness of multivariate Normal distributions in the context of commodity futures.

In all three commodities, the tests for joint normality within a commodity class as well as across commodities clearly reject the multivariate Normal hypothesis (cp. Tab. 2). Large skewness and excess kurtosis are strong indicators for non-normality stressing the importance of the use of skewed distributions to better capture the key features of the behavior of commodities. This result is not surprising in either of the data sets. It has been pointed out in the introduction that univariate findings already indicate that joint normality of commodities is very unlikely. Additionally, monthly futures contracts are already close to spot prices, especially the next-to-delivery futures and they resemble more and more the spot behavior. The features of spot prices are manifold, among them non-normality, for which statistical evidence can be found in e.g. Weron (2006).

Tab. 3 shows that the proposed GH distributions are better capable of modeling the return distribution. The likelihood ratio tests reject the Normal assumption at any confidence level in favor of any other distribution under consideration. This statement is true for all data sets though we present the numbers for data sets D2 (oil) and D4 (cross commodity) only. These two data sets can be taken as representative for all cases. We want to mention that oil data are considered to be "closest-to-Normal" among the data under consideration.

We start with the discussion of the oil data (D2). All symmetric distributions are clearly rejected which can be expected from Mardia's skewness test. The only candidates are NIG and HYP, which have the same number of parameters. Up to this point all other commodities (gas and power) behave similarly in that symmetric distributions are not appropriate. For oil

	D4: Cross Commodity					D2: Oil				
$H_0 \rightarrow$	N	t	NIG-S	HYP-S	NIG	N	t	NIG-S	HYP-S	NIG
vs. t	0.00					0.00				
NIG-S	0.00	1.00				0.00	0.00			
HYP-S	0.00	1.00	$\left(\frac{3.55}{3.53}\right)$			0.00	0.00	$\left(\frac{-96.2}{-59.4}\right)$		
NIG	0.00	1.00	0.23	0.01		0.00	0.00	0.00	0.00	
HYP	0.00	1.00	1.00	0.25	$\left(\frac{3.55}{3.54}\right)$	0.00	0.00	0.00	0.00	$\left(\frac{16.9}{95.3}\right)$

Table 3: Likelihood ratio tests of various multivariate distributions fitted to generic oil (D2) and year-2007 (D4) futures; values below 0.1 (0.05) lead to rejection of H_0 at a 10% (5%) confidence level; Numbers in parenthesis represent maximum log-likelihood values of $H_0 \cdot \frac{1}{1000}$ / $H_1 \cdot \frac{1}{1000}$.

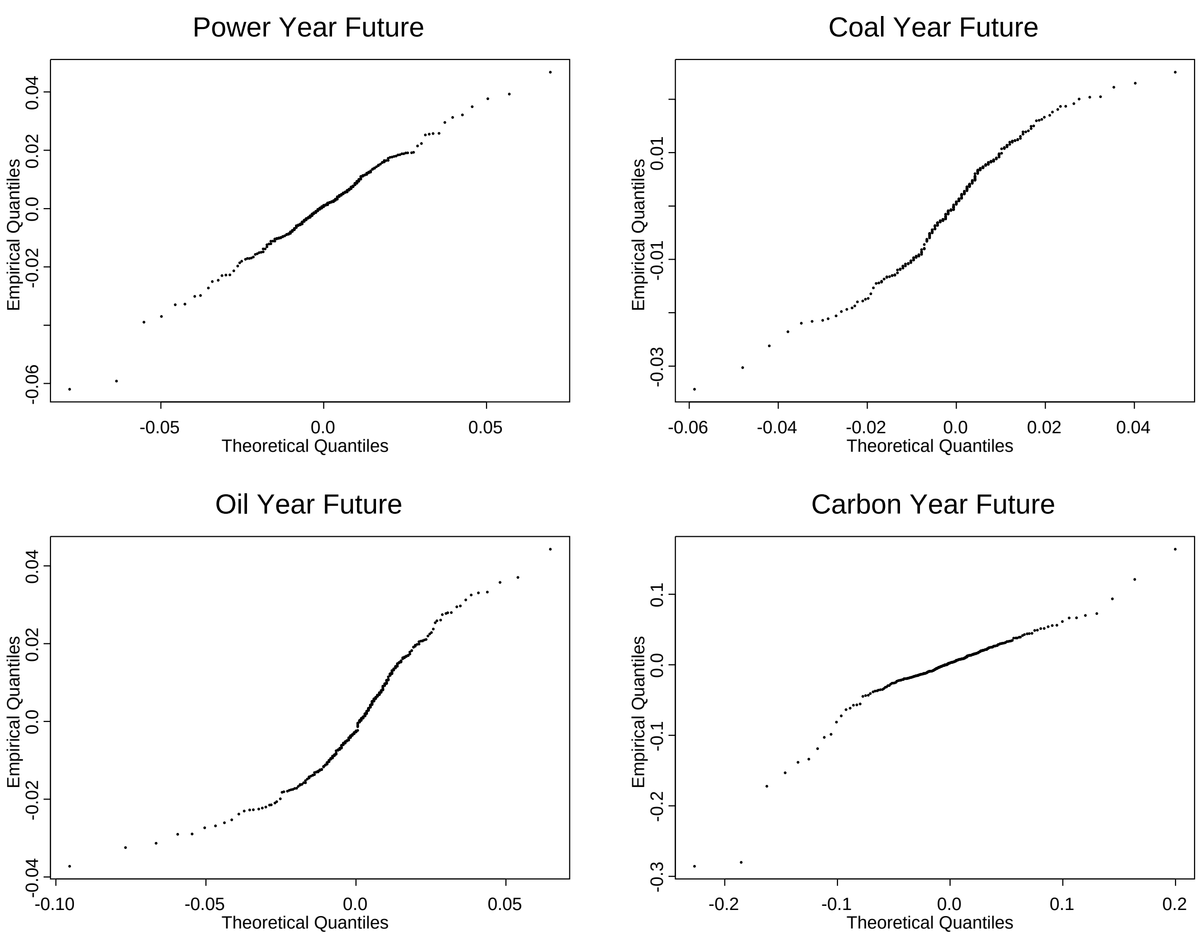

Figure 7: QQplots of commodities against the marginals implied by a joint NIG; power (upper left), coal (upper right), oil (lower left), CO_2 (lower right)

it would be natural to chose HYP as best fitting distribution due to the larger likelihood value. In case of gas and power, NIG gives the best fit from the likelihood point of view. In fact, the choice between these two distributions depends very much on the time horizon we include for estimation. The difference in likelihood is always small but the best fit switches between NIG and HYP, so we cannot recommend one distribution over the other in general. This effect has also been encountered in the analysis of stock data by Eberlein et al. (1998), who argues that the likelihood function is flat in the λ parameter of GH distributions. Deciding between NIG and HYP is changing λ from 1 to $-1/2$ and comparing likelihood values. While we will consider both distributions in the further analysis, we will see that the difference between the two is negligible.

The case of the cross commodity data set D4 is not as clear. Though the Normal distribution is rejected in favor of any other alternative, we find that the likelihood ratio test seems to favors distributions with fewer parameters. In fact, neither t nor NIG-S can be rejected at any reasonable level of confidence. While this seems to contradict the results of Mardia's test for skewness we believe that the data set is too short (317) to appreciate the better fit by the introduction of a further parameter. Thus, from a purely statistical point of view, one should recommend a low parameter family. Since the t distribution requires only one more parameter than the Normal and gives a much better fit at the same time, it seems the logical choice. Yet, in the light of univariate findings and the results for data sets D1-D3 above including Mardia's test for skewness, we believe that there is enough evidence to at least take nonsymmetric distributions into account when basing risk management decisions on such a portfolio.

We want to finish the statistical analysis with an issue that we have encountered concerning the goodness-of-fit. So far we have tested several GH distributions against each other and against the Normal on a multivariate basis. When turning the focus to the implied marginals, GH distributions also perform better than the Normal. Yet, we see that the tail behavior that is governed by the mixture variable W in Eq. 1 is the same for all marginals and can only be a compromise among all commodities. While this issue might not be as prominent in equity markets, the case of commodities is extreme. As we have shown in the introduction (cp. Fig. 1), commodity prices show returns that can range from almost Normal (e. g. oil) to heavy tails (e. g. power). The average heaviness of the tail results in marginals that perform as indicated in Fig. 7, which gives qqplots of each commodity compared to the implied marginal from a joint NIG.

While the fit seems to be very good for the case of electricity (upper left panel), all other commodities show lighter tails than given by the NIG. The worst fit is obtained for oil (lower left panel) which is already well-described by a Normal distribution. Extreme power prices force the mixture variable W in Eq. 1 to overshoot the tail behavior of all other commodities.

In order to secure estimation quality, other adjustments can be made. One can estimate the joint distribution only for those price processes which are actually needed for the analysis later on. Additional processes can lead to biased results. If a main risk driver of a portfolio has been identified, it is possible to weight appropriately the maximum likelihood procedure such that this driver is modeled precisely, of course on account of other price processes. Moreover, if assessing the risk of a portfolio is the objective of analysis, one can think about estimating the distribution based on the empirical portfolio returns directly instead of estimating the joint commodity return distribution first.

Finally, one might think about introducing a mixture variable for each dimension so that all marginals can have a different tail. While this will definitely increase the goodness of fit, we lose properties that are of most importance for risk analysis, i.e. closed under linear transformations. The profit-and-loss distribution as we will use it in the following section is then no longer available.

5 Application to Risk Management

5.1 Computation of Risk Measures

In this section we present the risk measures under consideration, i.e. value-at-risk (VaR) and expected shortfall (ES) and describe their computation for GH distributed losses.

Given a random variable X, representing the loss during a fixed period Δ, with distribution F_X, the value-at-risk is defined by

$$VaR_\alpha[X] = F_X^{\leftarrow}(\alpha)$$

for a specified confidence level α. $F_X^{\leftarrow}$ denotes the quantile function of F_X. The value-at-risk is the smallest number, such that the probability of the loss X exceeding the number is less than $1-\alpha$, where α is typically around 0.9 or higher.

Expected shortfall of an absolute continuous random variable X is defined by

$$\begin{aligned} ES_\alpha[X] &= \frac{1}{1-\alpha}\int_\alpha^1 VaR_u[X]du \\ &= \frac{1}{1-\alpha}\mathbb{E}\left[X\mathbb{1}(X \geq F_X^{\leftarrow}(\alpha))\right] \end{aligned}$$

where $\mathbb{1}(\cdot)$ denotes the indicator function.

While the value-at-risk is the quantile of the loss distribution, the expected shortfall takes the whole tail of the loss distribution into account.

Below we compute expected shortfall assuming that the loss distribution is a member of the GH-class. Remember that if X has a GH distribution, it can be represented as a Normal mean-variance mixture (cp. Eq. 1, $d = 1$). We make use of the fact that, conditionally on the mixture variable $W = w$, X is normally distributed with mean $\mu + w\gamma$ and variance $w\sigma^2$. The expected shortfall can be computed by conditioning on the mixture variable W and using the result for expected shortfall in the Gaussian case, i.e.

$$ES_\alpha\left[N_{\mu,\sigma^2}\right] = \mu + \sigma\frac{\phi(\Phi^{-1}(\alpha))}{1-\alpha}.$$

Here ϕ and Φ denote the standard Normal density and distribution functions, respectively. Further, denoting by f_W the density of the mixing variable W we compute

$$\begin{aligned}
ES_\alpha[X] &= \frac{1}{1-\alpha}\mathbb{E}\left[X\mathbb{1}(X \geq F_X^{\leftarrow}(\alpha))\right] \\
&= \int_{\mathbb{R}} \frac{1}{1-\alpha}\mathbb{E}\left[X\mathbb{1}(X \geq F_X^{\leftarrow}(\alpha))|W=w\right] f_W(w)dw \\
&= \int_{\mathbb{R}} ES_\alpha\left[N_{\mu+w\gamma,w\sigma^2}\right] f_W(w)dw \\
&= \int_{\mathbb{R}} \left(\mu + w\gamma + \sqrt{w}\sigma\frac{\phi(\Phi^{-1}(\alpha))}{1-\alpha}\right) f_W(w)dw \\
&= \mu + \gamma\mathbb{E}[W] + \sigma\frac{\phi(\Phi^{-1}(\alpha))}{1-\alpha}\mathbb{E}\left[\sqrt{W}\right].
\end{aligned}$$

We discuss three special cases:

- Normal: If $\gamma = 0$ and $W = 1$, X is Normal with mean μ and variance σ^2 and the expected shortfall has the well-known form

$$ES_\alpha[X] = \mu + \sigma\frac{\phi(\Phi^{-1}(\alpha))}{1-\alpha}.$$

- t: If $\gamma = 0$ and W is inverse gamma distributed with parameters $\frac{\nu}{2}$ and $\frac{\nu}{2}$, X is t-distributed with ν degrees of freedom. Here, we have

$$\mathbb{E}[W] = \frac{\nu}{\nu-2}, \quad \mathbb{E}\left[\sqrt{W}\right] = \sqrt{2\nu}\frac{\Gamma\left(\frac{\nu-1}{2}\right)}{\Gamma(\nu/2)}.$$

- If W has generalized Inverse Gaussian distribution with parameters λ, χ, ψ, we obtain the class of GH distributions with NIG ($\lambda = -\frac{1}{2}$), HYP ($\lambda = 1$) and symmetric distributions ($\gamma = 0$) as special cases. Here we have

$$\mathbb{E}[W] = \left(\frac{\chi}{\psi}\right)^{1/2}\frac{K_{\lambda+1}(\sqrt{\chi\psi})}{K_\lambda(\sqrt{\chi\psi})}, \quad \mathbb{E}\left[\sqrt{W}\right] = \left(\frac{\chi}{\psi}\right)^{1/4}\frac{K_{\lambda+1/2}(\sqrt{\chi\psi})}{K_\lambda(\sqrt{\chi\psi})},$$

  where K_λ denotes the modified Bessel function of the third kind with index λ.

We still need to determine the loss distribution of the portfolio of a utility company. Considering a portfolio with d assets and ω_i number of contracts of asset i, the value of the portfolio at time t is given by

$$V_t = \sum_{i=1}^{d} \omega_i S_t^{(i)}$$

and the random variable representing the one-period loss of the portfolio is

$$X = -(V_{t+1} - V_t).$$

In order to determine the risk measures discussed above, we need to specify the distribution of X. This can be done either by constructing a historical time series of portfolio returns and fitting a distribution to that series, which can be used in turn to compute the risk measures. This approach lacks flexibility when considering a portfolio with different investment weights but same commodities. In this case one would have to estimate the loss distribution for each portfolio separately. Also, this does not allow to define some portfolio closing strategy since portfolio weights are then changing over time implying different distributions. With the same argument an analysis of the riskiness over different portfolios is also difficult.

Instead, we determine the joint distribution of commodities and derive the loss distribution analytically. Following this approach we have the flexibility to analyze the impact of portfolio weights. For example it is possible to study the impact on risk when changing the weight of some asset, say CO_2, which we will do later on. Also, it is possible to find risk minimizing portfolio weights. We will turn to this issue at the end of the section.

From here, we can follow two paths in specifying the joint commodity distribution: Modeling the joint return distribution and modeling the joint log-return distribution.

- When modeling log-returns, we know the distribution of

  $$x_{t+1}^{(i)} := s_{t+1}^{(i)} - s_t^{(i)}, \; s_t^{(i)} := \log S_t^{(i)}, \quad i = 1, \ldots, d$$

  and we have

  $$X = -(V_{t+1} - V_t) = -\sum \omega_i S_t^{(i)} \left(\exp(x_{t+1}^{(i)}) - 1 \right)$$

  which we can approximate for a small time horizon Δ by

  $$X^{\Delta} = -\sum \omega_i S_t^{(i)} x_{t+1}^{(i)}.$$

  It is clear that the approximation applied here is valid only for small log-returns. Since we are using GH distributions in order to allow for large log-returns the method can be inaccurate. Yet, it is often used in applications since the price processes are forced to stay positive by this approach.

- When modeling returns, we know the distribution of

  $$X_{t+1}^{(i)} := \frac{S_{t+1}^{(i)} - S_t^{(i)}}{S_t^{(i)}}, \quad i = 1, \ldots, d$$

  and obtain

  $$X = -(V_{t+1} - V_t) = -\sum \omega_i S_t^{(i)} X_{t+1}^{(i)}$$

We study the two methods and find that differences are small compared to other uncertainties regarding reliability of data and estimates.

In both cases we need to know the distribution of a linear transformation of a multivariate GH distribution. Here we use the fact that if X has d-dimensional GH distribution with parameters $(\lambda, \chi, \psi, \mu, AA^t, \gamma)$ then for a row vector $a \in \mathbb{R}^d$ aX has a one-dimensional GH distribution with parameters $(\lambda, \chi, \psi, a\mu, aAA^t a^t, a\gamma)$. The analogous result holds true for the t-distribution.

Thus, the distribution of X^Δ in method one and X in method two is given by choosing $a = \left(-\omega_1 S_t^{(1)}, \dots, -\omega_d S_t^{(d)}\right)$, and computation of parameters of the one-dimensional GH distribution is based on multivariate estimates. Now, we can apply the value-at-risk method and the expected shortfall to the loss distribution.

The closed-form solution for the expected shortfall in case of GH distributions allows for a rigorous analysis of the portfolio, e. g. it is possible to determine optimal portfolio weights which minimizes the risk measured by expected shortfall. In the examples below we consider all but one position in the portfolio as fixed and adjust the remaining component in a risk-minimizing way. In order to do so we differentiate $ES_\alpha[X]$ with respect to the portfolio weight under consideration and it is straightforward to compute that a local extreme value in the jth component is attained at

$$a_j = -\frac{1}{\sigma_{jj}} \sum_{i=1, i\neq j}^{d} a_i \sigma_{ji} - \frac{\mu_j + \mathbb{E}[W]\gamma_j}{2\frac{\phi(\Phi^{-1}(\alpha))}{1-\alpha} \mathbb{E}\left[\sqrt{W}\right] \sigma_{jj}} \tag{2}$$

where $a = \left(-\omega_1 S_t^{(1)}, \dots, -\omega_d S_t^{(d)}\right)$ is the vector for the linear transformation and $\mu = (\mu_i) \in \mathbb{R}^d$, $\gamma = (\gamma_i) \in \mathbb{R}^d$ and $AA^t = (\sigma_{ij}) \in \mathbb{R}^{d\times d}$ are parameters of the joint distribution of commodity returns. Since the expected shortfall is a convex function of portfolio weights as can be seen from the second derivative, the local extreme value is a global minimum.

From Eq. 2 we can see that the optimal investment in commodity j is the variance minimizing investment (first term) corrected by risk-adjusted expected returns. We will report these numbers in our numerical study.

The use of such a minimizing portfolio depends on the choice of the parameter that enter the formula. Of course, historical parameter estimates are might not reflect the future development of the markets and an expert's view might differ from statistical findings. In particular, the view on the expected return can be put in question. Historically, one is very likely to find price series moving upwards or downwards, but when the market is efficient one should expect a zero rate of return, which means the mean value of returns should be equal to zero. Using this assumption in Eq. 2 would lead to a zero correction term since $\mu_j + \gamma_j \mathbb{E}[W]$ is the expected return of the jth component of the portfolio. In this case, the shortfall-minimizing portfolio coincides with the variance-minimizing portfolio.

5.2 Numerical Example

In this section, we want to show how to apply the method to a typical energy portfolio and discuss the results. Utility companies as well as financial institutions in the commodity market need to know the riskiness of their portfolio for several reasons. Here we mention some of them:

- They can base trading strategies on the risk numbers in such a way that the financial risk is minimized.

- They can use risk numbers to set limits for traders.
- Banks have to comply to regulatory standards which are based on riskiness of portfolios.

Thus, risk numbers need to be reported on a daily or weekly basis. Of course, the time horizon for the risk number, say the value-at-risk, can be different from application to application. Trading strategies require short-term value-at-risk metrics while the regulatory standards require longer term risk numbers. We will restrict to one-day value-at-risk and one-day expected shortfall, since the statistical analysis is done on daily prices, but the mechanics are straightforward to apply to other time periods. Alternatively, scaling principles can be applied to obtain risk measures for longer holding periods. We want to mention here that the square-root scaling rule is very popular but difficulties can occur when looking at heavy-tailed data. Embrechts et al. (2005b) discuss this issue and present a method to obtain longer-period risks by scaling.

The portfolio of an energy producer depends on many variables. Key components include: size of the company, number and types of power plants, number and type of customers, hedging strategies and many more. While we could assume an arbitrary energy portfolio to analyze the company's exposure to various sources of risk, we focus on the typical building blocks and choose the two most widely used types of power plants in the industry, namely a coal-fired power plant and a gas-fired power plant. We point out that the choice of power plant is arbitrary to a certain extent, but represents a quantity that is easy to interpret. Further, it is a financial position of interest for many utility companies. Finally, banks, that do not own plants but trade in commodity markets, can interpret the portfolio as a spread contract being short electricity and long one of coal, oil or gas.

Although power plants are exposed to several types of risks, such as operational risk, volume risk and many more, in our analysis we will cover the main financial risks in the following way. First, we represent the plant by certain financial futures positions, e.g. long positions of electricity and short positions of coal. Thus, we neglect all optionalities included in the timing of production, i.e. we assume that we run the plant at electricity baseload times and prices and do not incorporate the possibility of larger earnings when producing the energy at peakload times and prices.

Second, risk metrics such as value-at-risk assume a mark-to-market valuation, i.e. compare prices of the assets today with possible prices of the assets at the end of the time period (one day in our case). This relies on the fact that the owner of the portfolio is able to sell the assets at current market prices. This need not be the case as the portfolio size can be large compared to the market and selling all the portfolio would at least influence prices. Then one would have to reduce positions step by step so that the portfolio selling is performed over several periods. Taking this into account, one has to think about an optimal closing strategy, which is not in the scope of this paper. But if the strategy is known, one would have to carry out a value-at-risk analysis for each time step as presented here.

Ignoring the limitations in the volume of trades, risk measures can be applied, strictly speaking, to portfolios whose sizes are small compared to traded volumes in the market. This would be the case, for example, when the production periods (and thus the plant's output) are small. Yet, we illustrate the procedure using typical power plants as portfolios because the economic interpretation is more intuitive than for an abstract portfolio.

The risk analysis presented below is based on multivariate distributions fitting the joint distribution of log-returns of commodity prices. The procedure of fitting and analyzing the goodness of fit has been discussed in Section 4. Similar results are obtained when working on returns instead of log-returns, but allows for exact instead of approximate value-at-risks and expected shortfalls.

Coal-fired Power Plant

A coal-fired power plant burns coal to produce electricity and as byproduct, CO_2 is emitted. One ton of coal contains approximately the same amount of energy as 6.97MWh of electricity. Since energy is not completely transformed when burning the coal, but only 33% (i.e. a heat rate of about 3MWh per ton of coal), 0.33t of coal is required (which is equivalent to 2.3MWh) to produce 1MWh of electricity output. This generates 0.9t of CO_2 output. Scaling this to a plant with typical output capacity of 1,000,000MW during a year, we arrive at the following portfolio:

- Long position: 1,000,000MWh in power contracts. (1 contract=1MW $\hat{=}$ 8760MWh)
- Short position: 330,000t in coal.
- Short position: 900,000t in CO_2.

Usually, the CO_2 position is already partially covered by certificates assigned by the government. That is why we can also think of the plant as coming with additional, say, 800,000t CO_2 in certificates as a long position resulting in a net position of 100,000t CO_2 short. We also consider the case when the CO_2 position is totally hedged (i.e. 0t CO_2). This latter example is comparable to the situation before the introduction of emission allowances. We find the corresponding distributions by fitting a multivariate distribution to power, coal and CO_2 prices.

The expected shortfall for this type of power plant computed by the methods described in the previous section is presented in Tab. 4 for levels $\alpha = 0.99$ and $\alpha = 0.95$ for various numbers of CO_2 contracts. (Standard errors are obtained by bootstrapping and we summarize that none of the errors was above 30,000EUR with highest values for the Normal distribution. Also, the expected shortfalls have higher variation than the value-at-risks, especially at large quantiles α.) The numbers range between 1.5 and 2.2M EUR at a 95% level of confidence (2.0 to 3.5M EUR at 99% level).

We can detect slight differences between the expected shortfall obtained by using an approximation based on the distribution of log-returns and the exact distribution based on relative returns (the latter being usually larger), though differences are within the range of a standard deviation.

Comparing the risk implied by the different distributions, the Normal distribution gives significantly smaller risk estimates than all other candidates, sometimes up to 30% less. This is clearly due to the exponential tail-behavior which is certainly not appropriate for this portfolio (cp. Fig. 10, right panel). Allowing for heavier tails but staying in the class of symmetric distributions (t, NIG-S, HYP-S) we are much closer to the empirical density and the resulting risk is much higher. In terms of expected shortfall, the t distribution seems to be somewhat in the middle between HYP-S and NIG-S, the latter being significantly larger. Adding skewness to the distribution (NIG, HYP) we still have the relation that NIG-risk is higher than HYP-risk, but compared to their symmetric counterparts the numbers are significantly reduced. This leads to the conclusion that the portfolio has a skewed distribution and that large losses (unfavorable movements for the owner of the plant) are less probable than large earnings, so the inclusion of skewness in the distribution is advisable. Choosing between NIG and HYP implies large differences in risk but it is not clear which distribution is closer to the data. Recalling the statistical analysis of Section 4, we came to the conclusion that NIG might be slightly better in capturing the joint behavior, but since we are working on a linear transformation, we can check the goodness-of-fit here as well. While there is no obvious difference between the portfolio loss densities (Fig. 10), we can detect some systematic differences in the expected shortfalls which summarizes the effect of the tail of the distribution functions (cp. Figs. 8 and 9, left panels). The NIG expected shortfall is clearly above the HYP for all reasonable levels of α. The empirical shortfall is in two of three cases (middle and lower panel) in between the two. For low quantiles, the HYP seems to fit the data better, while NIG performs better in the very high quantiles (above 99%).

Next we focus on the value-at-risk of such a power plant. The VaRs range between 1.2 and 2.7M EUR depending on quantiles and portfolio. This is reasonable since it corresponds to a 1.2 to 2.7EUR change in power prices keeping all other quantities constant. Interestingly, the Normal distribution is not quite as far away from the alternatives compared to expected shortfalls. At the 95% level the estimated VaR is even higher than for some alternatives. Only for very large quantiles (99%) the Normal distribution yields the behavior as known from other financial markets implying VaRs lower than heavy-tailed distribution. This is due to the fact that the many extreme values in the data 'force' the Normal distribution to have a large volatility. This makes the spread of the distribution so wide, that the value-at-risk is larger than in the other cases. The t-VaRs are in between NIG-S and HYP-S again, which are in turn much higher than those implied by NIG and HYP, which follows from the same argument as above. Again, it can hardly be decided in favor of one of NIG and HYP, since the VaRs are modeled more precise by one or the other depending on the quantile one is focusing at (cp. Figs. 8 and 9, right panels). The larger the quantile, the better is the fit of NIG.

Finally, we want to discuss the role of the number of CO_2 certificates. It is evident from the data that covering the CO_2 position leads to higher risk, measured by any standard, though the total portfolio size is decreased. This can be easily explained by the positive correlation between power and CO_2 prices and the opposite role in the portfolio, i. e. being long power and short CO_2. From a risk point of view it might be favorable to keep a short position of CO_2 contracts to reduce risk. Moreover we find that the differences in risk between skewed and symmetric distributions seem to reduce when adding CO_2 contracts to the portfolio,

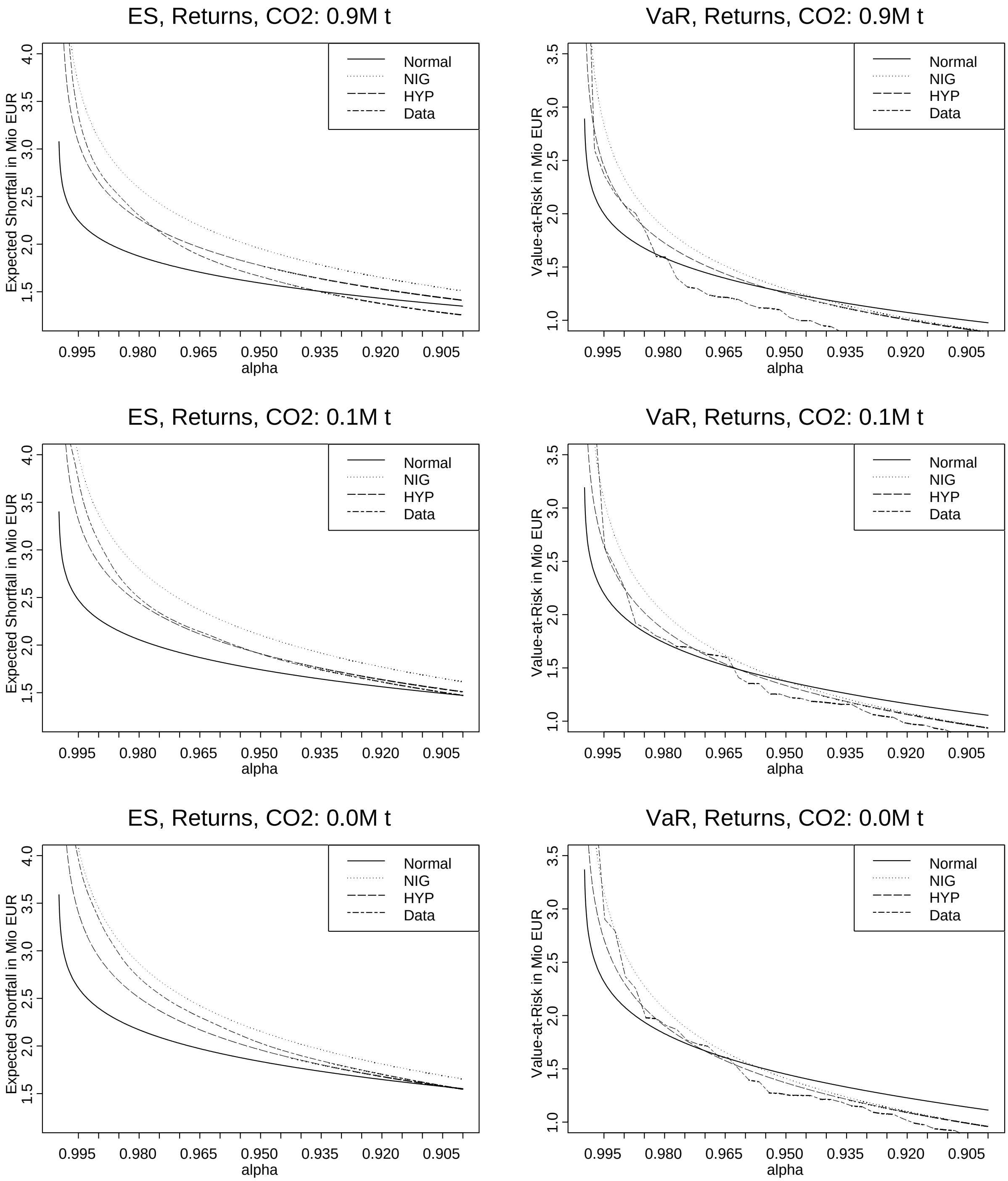

Figure 8: Expected Shortfalls (left) and Value-at-Risks (right) for coal-fired power plant portfolios based on estimates on price returns

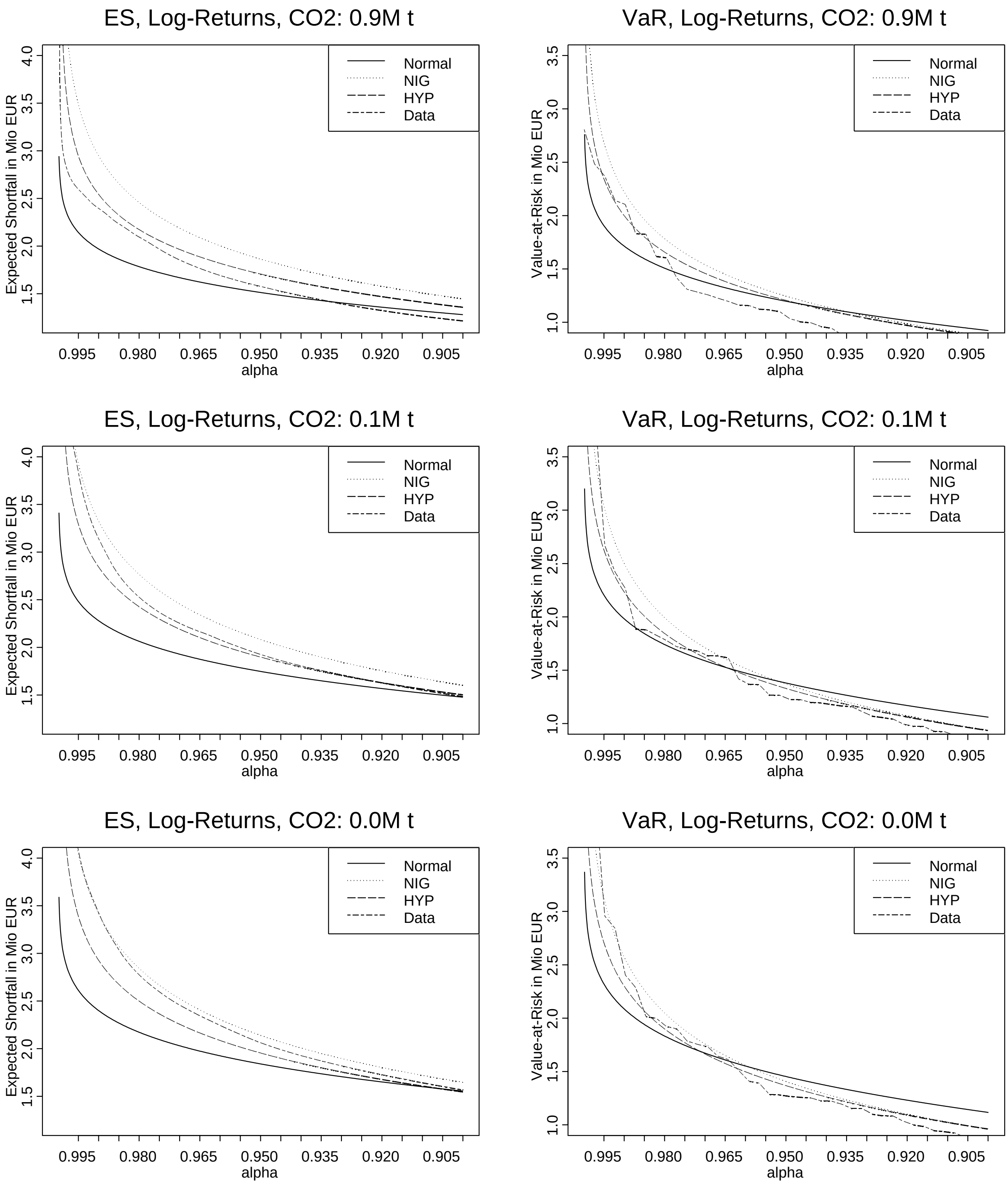

Figure 9: Expected Shortfalls (left) and Value-at-Risks (right) for coal-fired power plant portfolios based on estimates on price log-returns

	Normal		t		NIG-S		HYP-S		NIG		HYP	
	ES_α for a Coal-fired Power Plant based on returns											
0.9M t CO_2	1.59	2.06	1.92	3.19	2.03	3.24	1.84	2.74	1.95	3.10	1.77	2.64
0.1M t CO_2	1.74	2.27	2.04	3.39	2.16	3.46	1.97	2.95	2.10	3.36	1.91	2.86
0.0M t CO_2	1.84	2.39	2.07	3.45	2.20	3.52	2.01	3.02	2.15	3.44	1.96	2.93
	ES_α for a Coal-fired Power Plant based on log-returns											
0.9M t CO_2	1.51	1.97	1.92	3.20	2.00	3.19	1.82	2.72	1.86	2.94	1.71	2.54
0.1M t CO_2	1.75	2.28	2.04	3.40	2.13	3.41	1.95	2.93	2.08	3.32	1.90	2.84
0.0M t CO_2	1.84	2.40	2.08	3.46	2.18	3.48	2.00	3.00	2.14	3.41	1.95	2.92

Table 4: One-Day Expected shortfall in Million EUR of a coal-fired power plant based on returns and log-returns for different confidence levels ($\alpha = 0.95$ left, $\alpha = 0.99$ right)

i. e. covering the short position. The marginal distribution of CO_2 prices has the largest skewness parameter γ among all commodities and, thus, heavily influences the skewness of the portfolio. In the case of a 900,000t CO_2 short position, the portfolio's loss distribution is negatively skewed (i.e. skewed to the left), implying that large negative losses are more likely than large positive losses; the loss distribution is favorable for the owner of the plant. The other extreme of a covered CO_2 position yields a less negatively skewed loss distribution, i.e. a more unfavorable distribution for the owner. Using the relation

$$\gamma_X = \sum_{i=1}^{n} \omega_i S_t^{(i)} \gamma_i$$

with γ_X being the skewness parameter of the loss distribution and γ_i the skewness parameter from the marginal distributions, we can conclude that the loss distribution would be close to symmetric if we had about 1.7M CO_2 contracts in the portfolio (long, based on HYP, log-returns). In this case, there would be almost no difference between the skewed and symmetric distributions. This analysis shows that it is important to include the additional skewness parameters to the multivariate distribution. Depending on the particular portfolio, the derived risk can be substantially different.

While the symmetric distribution would be less favorable than a negatively skewed distribution, it would yield a higher average return since estimates reflect the history of increasing CO_2 prices. Hence, it is not surprising to have an expected shortfall minimizing investment of 840,000 CO_2 contracts long while the minimum variance investment were 771,000 contracts short (based on HYP, log-returns, $\alpha = 95\%$).

We want to emphasize again that the large long position in CO_2 for a shortfall-minimizing portfolio is only due to the positive mean of returns, which is estimated from historical data. Using the parameter estimates for risk management one would have to use parameter which reflect future development, in particular one would correct the expected return. Trusting the efficiency of the market one would set the expected return to zero and obtain the minimum shortfall portfolio equal to the minimum variance portfolio, i.e. a short position in CO_2 contracts as one might expect.

	Normal		t		NIG-S		HYP-S		NIG		HYP	
VaR_α for a Coal-fired Power Plant based on returns												
0.9M t CO_2	1.26	1.80	1.24	2.27	1.34	2.43	1.29	2.16	1.29	2.33	1.25	2.08
0.1M t CO_2	1.38	1.98	1.31	2.43	1.41	2.59	1.37	2.31	1.38	2.52	1.33	2.24
0.0M t CO_2	1.45	2.08	1.33	2.48	1.44	2.65	1.41	2.37	1.41	2.58	1.37	2.30
VaR_α for a Coal-fired Power Plant based on log-returns												
0.9M t CO_2	1.20	1.71	1.24	2.28	1.32	2.40	1.28	2.14	1.24	2.15	1.20	1.99
0.1M t CO_2	1.38	1.98	1.30	2.43	1.40	2.56	1.36	2.30	1.37	2.49	1.33	2.23
0.0M t CO_2	1.45	2.08	1.33	2.48	1.43	2.62	1.40	2.36	1.41	2.56	1.37	2.30

Table 5: One-Day Value-At-Risk in Million EUR of a coal-fired power plant based on returns and log-returns for different confidence levels ($\alpha = 0.95$ left, $\alpha = 0.99$ right)

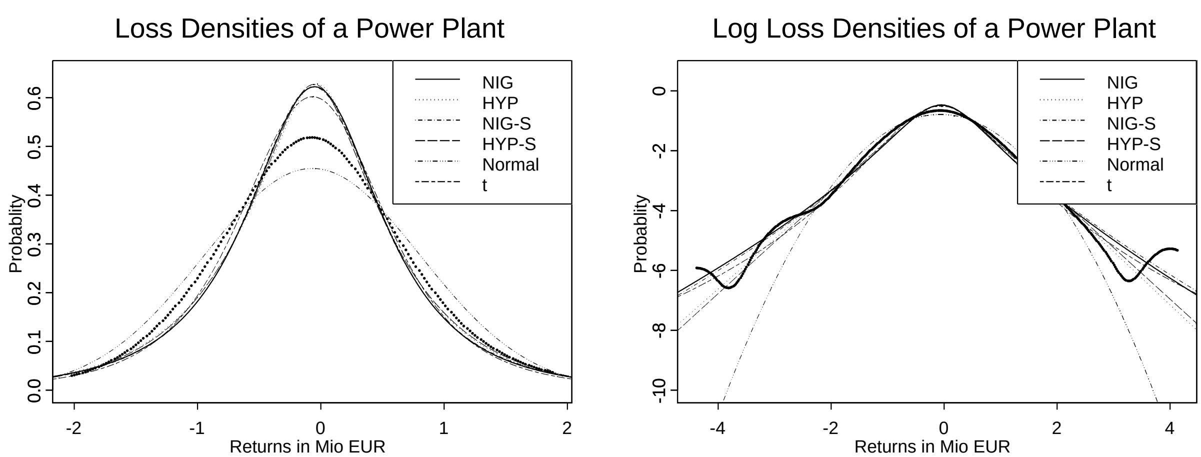

Figure 10: Loss densities computed from multivariate fits representing a coal fired power plant (log-scale). (Fitted to *returns*, 0.1M t CO_2); empirical density bold

	Normal		t		NIG-S		HYP-S		NIG		HYP	
ES_α for a Gas-fired Power Plant based on returns												
0.4M t CO_2	2.34	3.02	2.96	4.65	3.20	4.92	2.87	4.22	3.35	5.11	3.04	4.46
0.1M t CO_2	2.51	3.25	3.01	4.72	3.26	4.99	2.94	4.33	3.42	5.20	3.13	4.59
0.0M t CO_2	2.60	3.36	3.03	4.75	3.29	5.03	2.97	4.37	3.46	5.24	3.17	4.64
ES_α for a Gas-fired Power Plant based on log-returns												
0.4M t CO_2	2.34	3.02	2.96	4.66	3.17	4.86	2.85	4.20	3.28	5.01	3.00	4.40
0.1M t CO_2	2.51	3.24	3.01	4.73	3.23	4.94	2.92	4.30	3.38	5.14	3.11	4.56
0.0M t CO_2	2.59	3.34	3.04	4.76	3.25	4.97	2.96	4.35	3.42	5.18	3.15	4.62

Table 6: One-Day Expected shortfall in Million EUR of a gas-fired power plant based on returns and log-returns for different confidence levels ($\alpha = 0.95$ left, $\alpha = 0.99$ right)

Gas-fired Power Plant

A gas-fired power plant burns gas to produce electricity and in the process CO_2 is released. One thermal unit of gas is energetically equivalent to about 0.0293 MWh. According to a heat rate of about 68.3 btu/MWh we need 2 MWh of natural gas to produce 1 MWh of electricity. Burning the gas emits 0.4 t of CO_2. Often, the delivery price of gas is linked deterministically to the oil price (e.g. in Germany). The exact formula is not standardized and depends also on the type of oil which the price is linked to. An example link formula is

$$\text{Gas in } \frac{EUR}{MWh} = const + 0.5 \cdot \text{Brent Crude Oil in } \frac{EUR}{Barrel}$$

This implies, that a short position of 2 MWh of natural gas is financially equivalent to a short position of 1 Barrel of Brent Crude Oil. Scaling this to a plant that can produce 1,000,000MWh during a year, we face the following positions:

- Long position: 1,000,000MWh in power contracts.
- Short position: 1,000,000 bbl in oil contracts.
- Short position: 400,000t in CO_2 contracts.

Again, the CO_2 position is usually at least partially covered by government issued certificates, so that we will vary the short position of certificates to 100,000t and 0t. We obtain the necessary distributions by estimating the joint distribution of power, oil and CO_2 prices.

The expected shortfalls are summarized in Tab. 6 for $\alpha = 0.95$ and $\alpha = 0.99$ and in Figs. 12 and 13 (left panels) for a continuum of $\alpha \in [0.90, 1.0)$. The VaRs are given in Tab. 7 and Figs. 12 and 13 (right panels). The standard errors are always below 50,000EUR. The loss densities for several distributional assumptions are given in Fig. 11.

Most of the considerations apply to gas-fired power plant as to the coal-fired power plant and we will highlight differences only.

	Normal		t		NIG-S		HYP-S		NIG		HYP	
VaR_α for a Gas-fired Power Plant based on returns												
0.4M t CO_2	1.87	2.63	1.97	3.57	2.14	3.89	2.03	3.38	2.23	4.08	2.14	3.59
0.1M t CO_2	2.01	2.83	2.01	3.64	2.18	3.97	2.07	3.47	2.29	4.18	2.21	3.70
0.0M t CO_2	2.07	2.93	2.03	3.67	2.20	4.00	2.10	3.51	2.31	4.22	2.24	3.75
VaR_α for a Gas-fired Power Plant based on log-returns												
0.4M t CO_2	1.87	2.64	1.97	2.58	2.12	3.84	2.01	3.36	2.19	3.99	2.11	3.53
0.1M t CO_2	2.01	2.83	2.01	3.65	2.16	3.92	2.06	3.45	2.26	4.12	2.19	3.67
0.0M t CO_2	2.07	2.92	2.03	3.68	2.18	3.95	2.09	3.49	2.29	4.17	2.23	3.73

Table 7: One-Day Value-At-Risk in Million EUR of a gas-fired power plant based on returns and log-returns for different confidence levels ($\alpha = 0.95$ left, $\alpha = 0.99$ right)

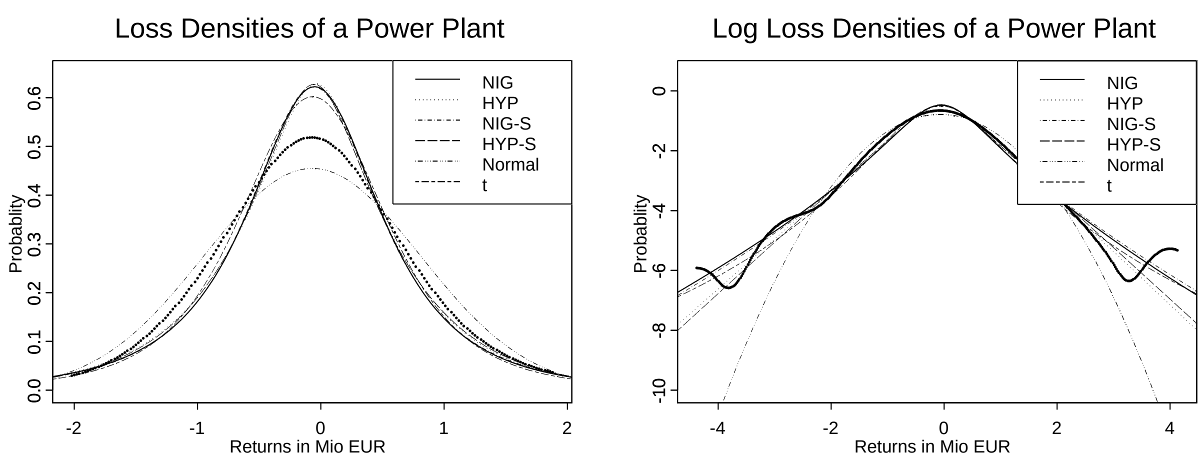

Figure 11: Loss densities computed from multivariate fits representing a gas fired power plant (log-scale). (Fitted to *returns*, 0.1M t CO_2); empirical density bold

First of all, the all risk values are much higher for gas than for coal though the total capacity of the plant is the same. This is largely due to the fact that at the time of analysis the oil prices have risen sharply and the total portfolio value is larger for the gas-fired plant than for the coal-fired one.

Secondly, while we still have the relation that HYP-S implies less risk than NIG-S (and HYP less than NIG) the skewed distributions are far above their symmetric counterparts here. This means that the loss distribution is skewed unfavorable for the owner of the plant. Additionally, the spread between symmetric and skewed distributions seem to widen as the number of CO_2 contracts is increased. Both effects can be explained to a large extent by the reduction of CO_2 shares relative to the total portfolio size. The impact of skewness of CO_2 prices is much smaller now and one needs about 5.5M contracts as a short position to have an unskewed distribution (based on HYP, log-returns). As in the previous case, adjusting the portfolio such that the shape of the distribution is more favorable is only one argument. Since CO_2 prices have risen in the time of analysis, increasing a short position in emissions will decrease profits and thus increase losses. A short position of 744,000 contracts gives a shortfall minimizing compromise which almost coincides with the minimum variance share of 817,000 contracts.

Considering the graphs in Figs. 12 and 13 (left panels) we have to state that the Normal distribution fits the expected shortfall of the portfolio rather well compared to the coal-fired plant. Up to very large quantiles ($< 99.5\%$) the risks are closer to each other than any of the other alternatives. A similar argument holds for the VaR in Figs. 12 and 13 (right panels). While surprising at first glance, we see here the shortcomings of the GH distributions; fitting multivariate distributions to all commodities under consideration the estimated tail behavior summarized by the mixture variable W in Eq. 1 is an average of the empirical tail properties of each commodity, that means the implied tail is the same in all marginals as we have pointed out in Fig. 7. Especially oil prices seem to be described well by a Normal distribution and the empirical tail is lighter than for all other commodities. Yet, the oil position in this portfolio is dominant which results in a good fit of the Normal distribution (because the bad fit for, say, CO_2 has no weight) with a bad fit of GH distribution. Probably one would prefer some member of GH-class over the Normal when fitting to the one-dimensional portfolio directly.

6 Conclusion

In this article we showed that the class of GH distributions is capable of fitting commodity futures prices and statistically clearly outperforms the Normal distribution. We used multivariate techniques to establish a correlation term-structure within each commodity across maturities but also examined the joint behavior of different commodities. In most, but not all, cases we come to the conclusion that a NIG or HYP distribution is necessary to capture the complex behavior of prices and that restricting to special cases is possible only in few circumstances. We proved empirically that the commodities under consideration show correlation which is more pronounced between power and emission allowances. Further, the dependence structure is also captured by the multivariate distributions we chose. Finally, we illustrated the power of the class of GH distributions with a risk analysis of typical commodity

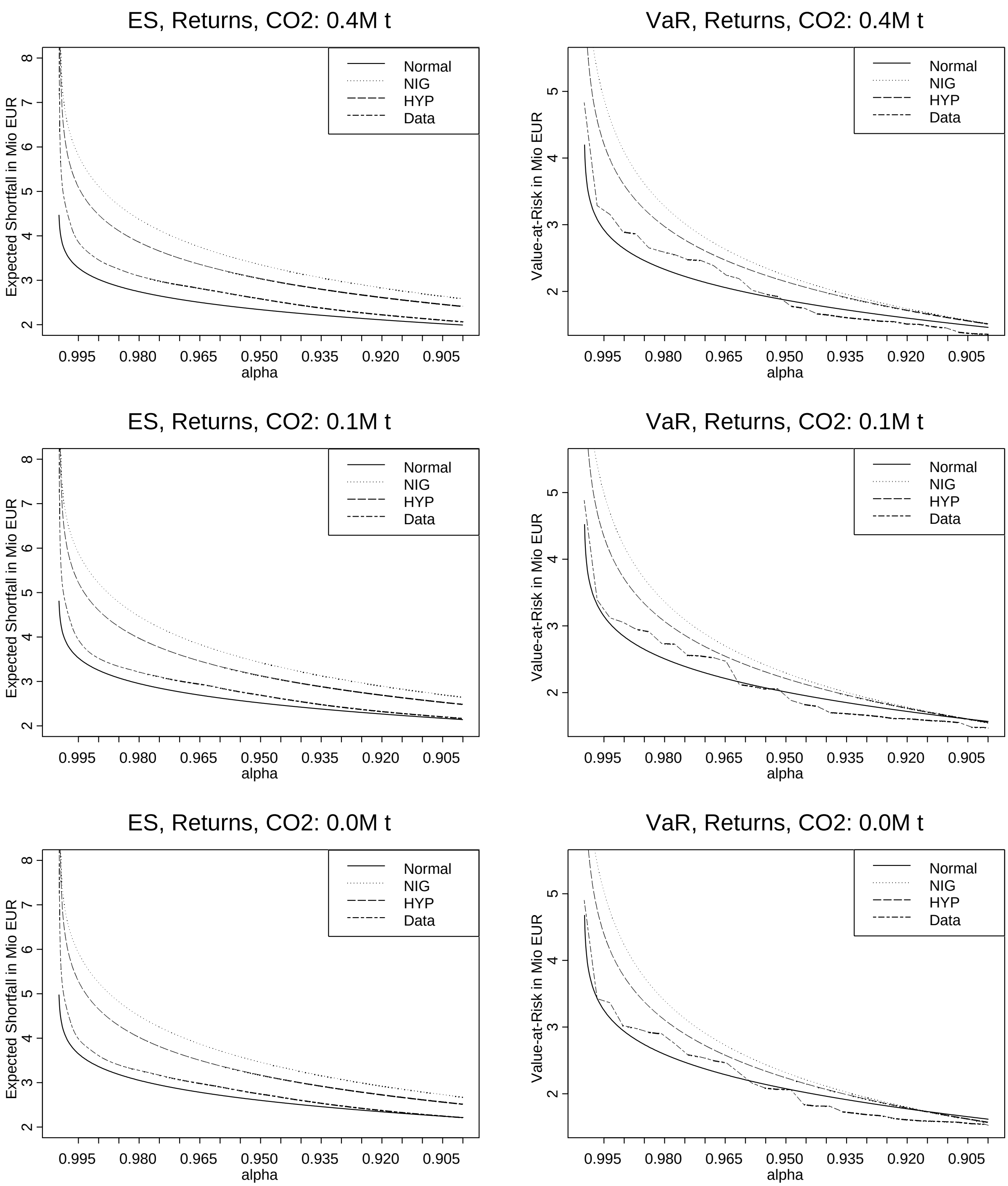

Figure 12: Expected Shortfalls (left) and Value-at-Risks (right) for gas-fired power plant portfolios based on estimates on price returns

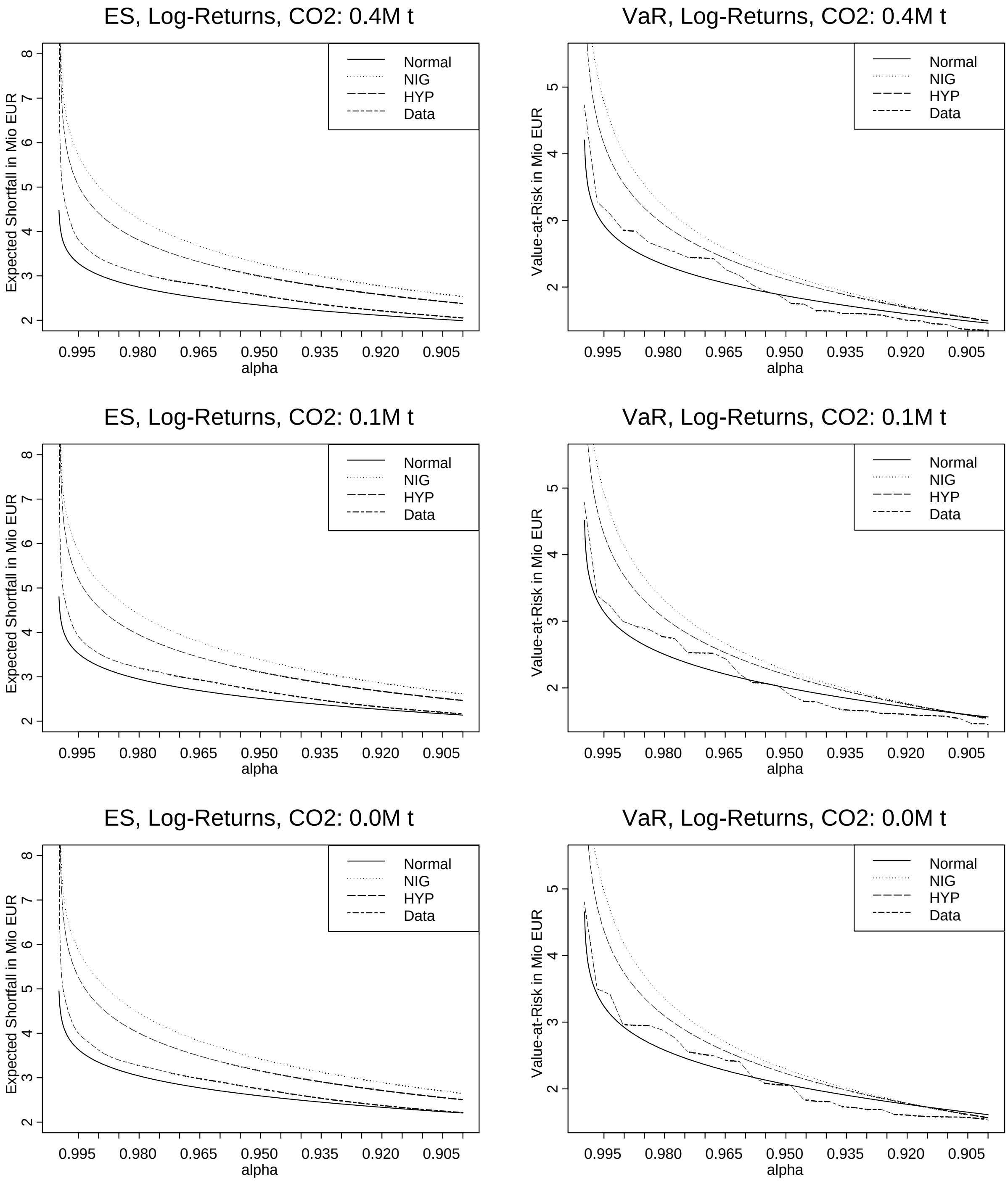

Figure 13: Expected Shortfalls (left) and Value-at-Risks (right) for gas-fired power plant portfolios based on estimates on price log-returns

portfolios.

The *statistical findings* can be used in several applications, we want to mention two of them:

- They can be the basis of financial model building. We claim that a model based on NIG- or HYP- Lévy-processes including a time-changing volatility can be capable of describing the futures market for each single commodity and, in a multivariate setup, a joint market for multiple commodities. Doing so, the pricing of cross commodity contracts such as spread options can be accomplished.

- They can be used to derive an optimal hedging strategy and to improve risk analysis which, in these markets, will be based on multi-period decisions due to the limitations in volume that can be traded at a time.

Additionally we demonstrated how the multivariate fit of the distributions can be applied to risk management issues such as computation of risk measures. This is particularly straightforward with GH distributions due to their normality conditional on the mixture variable. We can state, that the exact choice of the distribution has a major influence on risk measures. The example of two different types of power plants illustrates the mechanics of the heavy-tailed, skewed distributions in connection with the dependent marginals. The most important conclusions for *risk management* are:

- GH distributions give a more realistic view on the riskiness of an energy portfolio than the Normal distribution. Since analytical formulas are available for Value-at-Risk and expected shortfall, the implementation is straightforward.

- Though differences in distributions might be small (or even statistically insignificant) from a maximum-likelihood point of view, it is of most importance to the implied loss distribution for an energy portfolio.

- The introduction of emission trading poses an additional risk factor that has to be integrated into the risk management strategy. The high correlation in the asset movements combined with skewness allows to reduce the financial risk for market participants by taking opposite positions in power and CO_2 certificates.

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